Macroeconometrics - 2168 Joao B. Duarte
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Problem Set 1 - SOLUTIONS

Basic Concepts in Time Series Analysis

Problem 1 Consider the stochastic processes: i) $y_t = 0.5 + 1.2y_{t-1} + \varepsilon_t$, and ii) $y_t = 2.5y_{t-1} - 0.7y_{t-2} + \varepsilon_t$.

a) Find the homogeneous solution for both processes. Are they stable?

i) the homogeneous equation is given by $y_t - 1.2y_{t-1} = 0$.

The homogeneous solution is then $\boxed{y_t^h = A(1.2)^t}$. *Since, the root is outside of the unit circle, the solution is unstable.*

ii) the homogeneous equation is given by $y_t - 2.5y_{t-1} + 0.7y_{t-2} = 0$. *The characteristic equation is then given by:*

$$\alpha^2 - 2.5\alpha + 0.7 = 0$$

$$\alpha = \frac{2.5 \pm \sqrt{(-2.5)^2 - 4 \times 0.7}}{2}$$

$$\alpha_1 = 2.179 \tag{1}$$
$$\alpha_2 = 0.321 \tag{2}$$

Hence, the full set of homogeneous solutions is given by:

$$\boxed{y_t^h = A_1(2.179)^t + A_2(0.321)^t}$$

Since, there is one root that is outside the unit circle, the solution is also unstable.

b) Since i) is unstable, there is no backward looking solution without an initial condition. However, there exists a forward looking solution. Find the particular solution using the method of undetermined coefficients for difference equation i), whereby the challenge solution is given by: $y_t = b_0 + b_1 t - \alpha_0\varepsilon_{t+1} - \alpha_1\varepsilon_{t+2} - \alpha_2\varepsilon_{t+3} - ....$ What is the general solution to i)?

Replace the challenge solution into the difference equation:

$$b_0+b_1t-\alpha_0\varepsilon_{t+1}-\alpha_1\varepsilon_{t+2}-\alpha_2\varepsilon_{t+3}-... = 0.5+1.2(b_0+b_1(t-1)-\alpha_0\varepsilon_t-\alpha_1\varepsilon_{t+1}-\alpha_2\varepsilon_{t+2}-...)+\varepsilon_t$$

Matching coefficients on ε_t we have:

$$\begin{aligned} 0 &= -1.2\alpha_0 + 1 \\ \alpha_0 &= \frac{1}{1.2} \end{aligned}$$

Then matching the remaining coefficients we find:

$$\begin{aligned} -\alpha_0 &= -1.2\alpha_1 \\ -\alpha_1 &= -1.2\alpha_2 \\ &\vdots \\ \alpha_i &= 1.2\alpha_{i+1} \end{aligned}$$

Hence,

$$\boxed{\alpha_i = \left(\frac{1}{1.2}\right)^i \alpha_0 = \left(\frac{1}{1.2}\right)^{i+1}}$$

Then, matching the coefficients on constants we have:

$$\begin{aligned} b_0 &= 0.5 + 1.2b_0 \\ &\boxed{b_0 = \frac{0.5}{1-1.2} = -2.5} \end{aligned}$$

Finally, $b_1 = 0$ in order to match coefficients on time trends. Hence, the particular solution is:

$$y_t^p = -2.5 - \sum_{i=0}^{\infty} (1.2)^{-(i+1)} \varepsilon_{t+1+i}$$

The general solution is the homogeneous solution plus the particular solutions:

$$\boxed{y_t = A(1.2)^t - 2.5 - \sum_{i=0}^{\infty} (1.2)^{-(i+1)} \varepsilon_{t+1+i}}$$

c) Find the particular solution for both difference equations using lag operators. For the particular solution of i) apply the following property of lag operators: for $|a| > 1$, $\frac{\varepsilon_t}{1-aL} = -(aL)^{-1} \sum_{i=0}^{\infty} (aL)^{-i} \varepsilon_t$.

i) $y_t = 0.5 + 1.2Ly_t + \varepsilon_t$

$$y_t = \frac{0.5}{1-1.2} + \frac{\varepsilon_t}{1-1.2L}$$

Applying the lag property we get the same particular solution as in b).

$$\boxed{y_t^p = -2.5 - \sum_{i=0}^{\infty}(1.2)^{-(i+1)}\varepsilon_{t+1+i}}$$

d) Use the initial condition $y_0 = 2$ for i) to solve for the constant of the general forward looking solution.

Start with the general solution and apply it at $t = 0$

$$y_0 = 2 = A - 2.5 - \sum_{i=0}^{\infty}(1.2)^{-(i+1)}\varepsilon_{1+i}$$

Now, we just need to solve for A:

$$\boxed{A = 4.5 + \sum_{i=0}^{\infty}(1.2)^{-(i+1)}\varepsilon_{1+i}}$$

e) With an initial condition there exists a backward looking solution even for unstable difference equations. Use the iteration method in i) starting from the same initial condition $y_0 = 2$ to find a solution in terms of past values of ε.

$$\begin{aligned} y_1 &= 0.5 + 1.2y_0 + \varepsilon_1 \\ y_2 &= 0.5 + 1.2y_1 + \varepsilon_2 \\ y_2 &= 0.5 + 1.2(0.5 + 1.2y_0 + \varepsilon_1) + \varepsilon_2 \\ &\vdots \end{aligned}$$

$$\boxed{y_t = \sum_{i=0}^{t-1}(1.2)^i 0.5 + 1.2^t y_0 + \sum_{i=0}^{t-1}(1.2)^i\varepsilon_{t-i}}$$

Problem 2 AR and MA models

a) For the following MA model, $y_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-3}$, show the unconditional mean and variance, the autocovariance γ_1, γ_2, γ_3 and γ_4 as well as the associated autocorrelation coefficients. Interpret your results.

The unconditional mean $E(y_t) = E(\varepsilon_t) - \theta_1 E(\varepsilon_{t-1}) - \theta_2 E(\varepsilon_{t-3}) = 0$
For the variance, multiply y_t *on both sides and take expectations:*

$$Var(y_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2)$$

As for the autocovariances, multiply both sides by y_{t-j} *and take expectations:*

$$E(y_t y_{t-j}) = E(\varepsilon_t y_{t-j}) - \theta_1 E(\varepsilon_{t-1} y_{t-j}) - \theta_2 E(\varepsilon_{t-3} y_{t-j})$$

We get:

$$\begin{aligned}
\gamma_1 &= -\theta_1\sigma^2 \\
\gamma_2 &= \theta_1\theta_2\sigma^2 \\
\gamma_3 &= -\theta_2\sigma^2 \\
\gamma_4 &= 0
\end{aligned}$$

To get the autocorrelations just divide by the variance:

$$\begin{aligned}
\rho_1 &= \frac{-\theta_1}{(1 + \theta_1^2 + \theta_2^2)} \\
\rho_2 &= \frac{\theta_1\theta_2}{(1 + \theta_1^2 + \theta_2^2)} \\
\rho_3 &= \frac{-\theta_2}{(1 + \theta_1^2 + \theta_2^2)} \\
\rho_4 &= 0
\end{aligned}$$

The MA model provides a direct way of modelling autocorrelations. In this case, since we only include up to 3 lags we expect the autocorrelation function to drop to zero after 3 lags. The autocorrelation at 2 steps is not zero even though we do not include a lag 2 MA component. And the reason is that y_{t-2} *indirectly affects* y_t *since shocks at step 3 affect both* y_t *and* y_{t-2}.

b) For the AR(2) model given by $y_t = 0.6y_{t-1} - 0.06y_{t-2} + \varepsilon_t$, find the roots of the characteristic equation, and then sketch the ACF for the first 4 steps. Is it stationary? Using the lag operator, show that the roots of the lag polynomial are the inverse of the roots of the characteristic equation.

The characteristic equation is given by:

$$\alpha^2 - 0.6\alpha + 0.06 = 0$$

The roots are:

$$\alpha_1 = 0.473$$
$$\alpha_2 = 0.127$$

Since both roots are inside the unit root, the process is stationary. Applying the lag operator we get the following:

$$y_t = \frac{\varepsilon_t}{1 - 0.6L + 0.06L^2}$$

The inverse characteristic equation is then given by:

$$1 - 0.6z + 0.06z^2 = 0$$

Which has the following roots:

$$z_1 = 2.113248$$
$$z_2 = 7.886752$$

Problem 3 Let $y_t = 0.6 + 0.4y_{t-2} + \varepsilon_t - 0.1\varepsilon_{t-1}$

a) Find $E[y_t]$, $var(y_t)$ and $cov(y_t, y_{t-1})$. Is it stationary? Is it invertible?

Lets start with the unconditional mean:

$$E(y_t) = 0.6 + 0.4E(y_{t-2}) = 0.6 + 0.4E(y_t)$$

$$\boxed{E(y_t) = \frac{0.6}{1 - 0.4} = 1}$$

For the variance, write the ARMA model in demeaned variables, multiply by $y_t - 1$ both sides and take expectations:

$$Var(y_t) = (0.4)^2 Var(y_{t-2}) + \sigma^2 + (0.1)^2\sigma^2$$

$$\boxed{Var(y_t) = \frac{\sigma^2(1 + (0.1)^2)}{1 - (0.4)^2} = \frac{\sigma^2 1.01}{0.84}}$$

For the autocovariance multiply both sides by $y_{t-1} - 1$ and take expectations:

$$E[y_t - 1][y_{t-1} - 1] = \gamma_1 = 0.4E[y_{t-2} - 1][y_{t-1} - 1] + E[\varepsilon_t][y_{t-1} - 1] - 0.1E[\varepsilon_{t-1}][y_{t-1} - 1]$$

$$\gamma_1 = 0.4\gamma_1 - 0.1\sigma^2$$

$$\boxed{\gamma_1 = -\frac{\sigma^2}{6}}$$

The characteristic equation is given by:

$$\alpha^2 - 0.4 = 0$$

which has root $\alpha = \sqrt{0.4} = 0.632$*. Hence, this ARMA model is stationary since all roots are inside the unit root. Finally, since the parameter on the MA component is less than one in absolute value, this ARMA model is also invertible.*

b) Find the autocorrelation ρ_i for $i = 1, 2, 3, 4, ...$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = -\frac{0.84}{6 \times 1.01} = \boxed{-0.139}$$

Multiply again both sides of the demeaned equation by $y_{t-2} - 1$ *and take expectations:*

$$\gamma_2 = 0.4\gamma_0$$

Hence,

$$\boxed{\rho_2 = 0.4}$$

$$\boxed{\rho_3 = 0.4\rho_2}$$

and finally,

$$\boxed{\rho_4 = 0.4\rho_3}$$

c) Find the forecast $E_t[y_{t+1}]$, $E_t[y_{t+2}]$. How does the MA component affect both steps-ahead forecasts?

$$E_t[y_{t+1}] = 0.6 + 0.4y_{t-1} - 0.1\varepsilon_t$$

$$E_t[y_{t+2}] = 0.6 + 0.4y_t$$

The MA component only affects the one step-ahead forecast and if the shock is positive at time t *it decreases the forecast.*

Problem 4 In this problem set, you will make a forecast of the real GDP annual growth rate for Portugal in 2020. Download the real GDP annual growth rate for Portugal from FRED. The series code is: "NAEXKP01PTA657S".

a) Does the time series look stationary? Why or why not?

b) How do you identify the ARMA model for this series? Identify the model.

c) Estimate the model you have identified in b). Print the output of the estimates and discuss your results. Is the model you estimated stable? Why or why not?

d) Check the model. Is it a good model? Why or why not?

e) Forecast the next two years: 2019 and 2020. Plot your forecasts. How do your predictions compare with the official forecasts of the Bank of Portugal (https://www.bportugal.pt/en/page/projecoes-economicas)?