Midterm 2020 Macro

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1a)

Homogeneous part of the model:

$$y_t = -0{,}8y_{t-1} - 0{,}1y_{t-2}$$

Challenge-solution: $y_t = A\alpha^t$

$$A\alpha^t = -0{,}8A\alpha^{t-1} - 0{,}1A\alpha^{t-2}$$

$$\Leftrightarrow A\alpha^t + 0{,}8A\alpha^{t-1} + 0{,}1A\alpha^{t-2} = 0$$

Divide by $A\alpha^{t-2}$:

$$\Rightarrow \alpha^2 + 0{,}8\alpha + 0{,}1 = 0$$

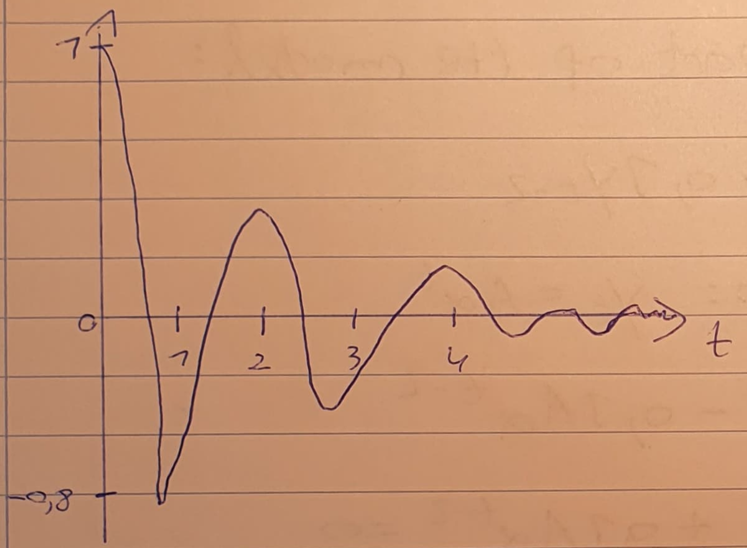
$$\Leftrightarrow \alpha_1 = -0{,}155 \lor \alpha_2 = -0{,}645$$

$$y_t^h = A_1(-0{,}155)^t + A_2(-0{,}645)^t$$

The process is stable as both roots of the characteristic equation are within the unit circle.

To better explain the shape, let us assume $A_1 = A_2 = 1$, and see what happens to y_t^h as t increases:

$t=0 \rightarrow y_0^h = 1$
$t=1 \rightarrow y_1^h = -0,155 - 0,645 = -0,8$
$t=2 \rightarrow y_2^h = 0,44005$
$t=3 \rightarrow y_3^h = -0,27206$
$t=4 \rightarrow y_4^h = 0,173654$



This graph is not accurately drawn but meant to show the shape of the solution.
What we have is an oscillatory convergent solution. It oscillates between positive and negative values of y, but converging rapidly over time. It also never leaves the unit circle.

1b)
The stationarity of an ARMA (2,1) model is connected with the stability of the homogeneous solution. Hence, as the solution is stable (proved in 1a), the model is stationary.

To show inventibility we need to solve for the MA component of the model:

$$y_t = 2 - 0{,}8Ly_t - 0{,}1L^2y_t + \varepsilon_t + 0{,}8L\varepsilon_t$$
$$\Leftrightarrow \; y_t + 0{,}8Ly_t + 0{,}1L^2y_t = 2 + \underline{(1+0{,}8L)}\,\varepsilon_t$$

$$1 + 0{,}8L = 0 \quad \Leftrightarrow \quad \boxed{L = -1{,}25}$$

Since the absolute value of L is outside the unit circle (or, equivalently, $|L^{-1}|$ is inside the unit circle), the process is inventible.

1c)

$$y_2 = 2 - 0{,}8y_1 - 0{,}1y_0 + \varepsilon_2 + 0{,}8\varepsilon_1$$

$$y_3 = 2 - 0{,}8y_2 - 0{,}1y_1 + \varepsilon_3 + 0{,}8\varepsilon_2$$

$$\Leftrightarrow \; y_3 = 2 - 0{,}8(2 - 0{,}8y_1 - 0{,}1y_0 + \varepsilon_2 + 0{,}8\varepsilon_1) - $$
$$- 0{,}1y_1 + \varepsilon_3 + 0{,}8\varepsilon_2$$

$$\Leftrightarrow \; y_3 = 2 - 1{,}6 + 0{,}64y_1 + 0{,}08y_0 - 0{,}8\varepsilon_2 - 0{,}64\varepsilon_1 -$$
$$- 0{,}1y_1 + \varepsilon_3 + 0{,}8\varepsilon_2$$

$$\Leftrightarrow \; y_3 = 2 - 1{,}6 + 0{,}54y_1 + 0{,}08y_0 - 0{,}64\varepsilon_1 + \varepsilon_3$$

Given $y_0 = 1$ and $y_1 = 0{,}2$:

$$\Rightarrow \; y_3 = 2 - 1{,}6 + 0{,}108 + 0{,}08 - 0{,}64\varepsilon_1 + \varepsilon_3$$
$$\Leftrightarrow \; y_3 = 0{,}588 - 0{,}64\varepsilon_1 + \varepsilon_3$$

The solution takes the form of a constant plus past (ε_1) and contemporaneous shocks (ε_3).

y_3 does not depend on ε_2, as by iterating we easily find that ε_2 gets "cut out" of the solution.

1d)

To simplify calculations, I will assume away the intercept (2) of the model, as it does not affect the ACF.

$$var(y_t) = \gamma_0 = E(y_t \cdot y_t) =$$

$$= -0{,}8\,E(y_{t-1} \cdot y_t) - 0{,}1\,E(y_{t-2} \cdot y_t) + E(\varepsilon_t \cdot y_t) + 0{,}8\,E(\varepsilon_{t-1} \cdot y_t)$$

$$= -0{,}8\,\gamma_1 - 0{,}1\,\gamma_2 + \sigma^2 + 0{,}8\,(-0{,}8 + 0{,}8)\,\sigma^2$$

$$\Rightarrow \boxed{\gamma_0 = -0{,}8\,\gamma_1 - 0{,}1\,\gamma_2 + \sigma^2}$$

$$cov(y_t, y_{t-1}) = \gamma_1 = E(y_t \cdot y_{t-1}) =$$

$$= -0{,}8\,E(y_{t-1} \cdot y_{t-1}) - 0{,}1\,E(y_{t-2} \cdot y_{t-1}) + E(\varepsilon_t \cdot y_{t-1}) + 0{,}8\,E(\varepsilon_{t-1} \cdot y_{t-1})$$

$$= -0{,}8\,\gamma_0 - 0{,}1\,\gamma_1 + 0 + 0{,}8\,\sigma^2$$

$$\Rightarrow \gamma_1 = -0{,}8\,\gamma_0 - 0{,}1\,\gamma_1 + 0{,}8\,\sigma^2$$

$$\Leftrightarrow 1{,}1\,\gamma_1 = -0{,}8\,\gamma_0 + 0{,}8\,\sigma^2$$

$$\Leftrightarrow \boxed{\gamma_1 = -\frac{8}{11}\gamma_0 + \frac{8}{11}\sigma^2}$$

$$cov(y_t, y_{t-2}) = \gamma_2 = E(y_t \cdot y_{t-2}) =$$

$$= -0{,}8E(y_{t-1} \cdot y_{t-2}) - 0{,}1E(y_{t-2} \cdot y_{t-2}) +$$
$$+ E(\varepsilon_t \cdot y_{t-2}) + 0{,}8E(\varepsilon_{t-1} \cdot y_{t-2})$$

$$= -0{,}8\gamma_1 - 0{,}1\gamma_0 + 0 + 0$$

$$\Leftrightarrow \boxed{\gamma_2 = -0{,}8\gamma_1 - 0{,}1\gamma_0}$$

Taking the three equations together:

$$\begin{cases} \gamma_0 = -0{,}8\gamma_1 - 0{,}1\gamma_2 + \sigma^2 \\ \gamma_1 = -\frac{8}{11}\gamma_0 + \frac{8}{11}\sigma^2 \\ \gamma_2 = -0{,}8\gamma_1 - 0{,}1\gamma_0 \end{cases}$$

$$\Rightarrow \gamma_2 = -0{,}8\left(-\frac{8}{11}\gamma_0 + \frac{8}{11}\sigma^2\right) - 0{,}1\gamma_0$$

$$\Leftrightarrow \gamma_2 = \frac{32}{55}\gamma_0 - \frac{32}{55}\sigma^2 - 0{,}1\gamma_0$$

$$\Leftrightarrow \gamma_2 = \frac{53}{110}\gamma_0 - \frac{32}{55}\sigma^2$$

$$\Rightarrow \gamma_0 = -0{,}8\left(-\frac{8}{11}\gamma_0 + \frac{8}{11}\sigma^2\right) - 0{,}1\left(\frac{53}{110}\gamma_0 - \frac{32}{55}\sigma^2\right) +$$
$$+ \sigma^2$$

$$\Leftrightarrow \gamma_0 = \frac{32}{55}\gamma_0 - \frac{32}{55}\sigma^2 - \frac{53}{1100}\gamma_0 + \frac{16}{275}\sigma^2 +$$

$$+ \sigma^2$$

$$\Leftrightarrow \gamma_0 - \frac{32}{55}\gamma_0 + \frac{53}{1100}\gamma_0 = \frac{137}{275}\sigma^2$$

$$\Leftrightarrow \frac{513}{1100}\gamma_0 = \frac{137}{275}\sigma^2 \Leftrightarrow \boxed{\gamma_0 = \frac{524}{513}\sigma^2}$$

$$\boxed{\gamma_1 = -\frac{8}{513}\sigma^2} \quad ; \quad \boxed{\gamma_2 = -\frac{46}{513}\sigma^2}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = -0{,}0153$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = -0{,}087786$$

The MA compoment will affect directly the ACF for lag one. After that, the ACF will decay / converge to zero due to the AR compoment. The moving average compoment does not affect the second lag of the ACF.

2a)

$$E_t(y_{t+2}) = 0{,}1 + 0{,}1\, E_t(y_{t+1}) + 0{,}4\, E_t(z_{t+1}) + 0$$

The two step-ahead conditional forecast for y_t depends on the one step-ahead conditional forecasts for both y_t and z_t. We need to determine those first.

$$E_t(y_{t+1}) = 0{,}1 + 0{,}1\, E_t(y_t) + 0{,}4\, E_t(z_t) + 0$$

$$\Leftrightarrow E_t(y_{t+1}) = 0{,}1 + 0{,}1 y_t + 0{,}4 z_t$$

$$E_t(z_{t+1}) = 0{,}8 - 0{,}3\, E_t(y_t) + 0$$

$$\Leftrightarrow E_t(z_{t+1}) = 0{,}8 - 0{,}3 y_t$$

Plugging these results into the two step-ahead forecast of y_t:

$$E_t(y_{t+2}) = 0{,}1 + 0{,}1(0{,}1 + 0{,}1 y_t + 0{,}4 z_t) + 0{,}4(0{,}8 - 0{,}3 y_t)$$

$$\Leftrightarrow E_t(y_{t+2}) = 0{,}1 + 0{,}01 + 0{,}01 y_t + 0{,}04 z_t + 0{,}32 - 0{,}12 y_t$$

$$\Leftrightarrow E_t(y_{t+2}) = 0{,}43 - 0{,}11 y_t + 0{,}04 z_t$$

$$\frac{\partial E_t(y_{t+2})}{\partial z_t} = 0{,}04$$

Z_t impacts $E_t(Y_{t+2})$ positively, although by a small amount. This, however, is counter-intuitive with regard to the usual economic theory.
We would expect that an increase in Z_t (I'm assuming Z_t as interest rate) would lead to a decrease in Y_{t+2}. This is part of what is commonly referred to as the "price puzzle".

2b)
The original B matrix comes from the structural VAR:

$$Y_t + b_{12} Z_t = 0{,}1 + 0{,}1 Y_{t-1} + b_{12} Z_t - b_{12} Z_t + \\ + 0{,}4 Z_{t-1} + \varepsilon_{Yt}$$

$$Z_t + b_{21} \cancel{Y}_t = 0{,}8 + b_{21} Y_t - b_{21} Y_t - 0{,}3 Y_{t-1} + \\ + \varepsilon_{Zt}$$

$$\Rightarrow B X_t = \Gamma_0 + \Gamma_1 X_{t-1} + \varepsilon_t$$

$$B = \begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \qquad X_T = \begin{bmatrix} Y_t \\ Z_t \end{bmatrix}$$

If we assume Z_t not to affect Y_t contemporaneously: $b_{12} = 0$

$$B = \begin{bmatrix} 1 & 0 \\ b_{21} & 1 \end{bmatrix}$$

Given that $e_t = B^{-1} \varepsilon_t$:

$$B^{-1} = \frac{1}{1 - 0 \times b_{21}} \begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix}$$

$$\Leftrightarrow B^{-1} = \begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix}$$

$$e_t = \begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

$$\Leftrightarrow \begin{aligned} e_{1t} &= \varepsilon_{yt} + 0\,\varepsilon_{zt} \\ e_{2t} &= -b_{21} \varepsilon_{yt} + \varepsilon_{zt} \end{aligned}$$

We also know that

var(e_{1t})

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \quad \text{and } \sigma_{21} = \sigma_{12}$$

var(e_{2t})

cov(e_{1t}, e_{2t})

$$\begin{cases} \text{var}(e_{1t}) = \text{var}(\varepsilon_{yt}) \\ \text{var}(e_{2t}) = b_{21}^2 \, \text{var}(\varepsilon_{yt}) + \text{var}(\varepsilon_{zt}) \\ \text{cov}(e_1, e_{2t}) = -b_{21} \, \text{var}(\varepsilon_{yt}) \end{cases}$$

$$\Leftrightarrow \begin{cases} 0,5 = \sigma_y^2 \\ 2 = b_{21}^2 \sigma_y^2 + \sigma_z^2 \\ 0,4 = -b_{21} \sigma_y^2 \end{cases}$$

$$\Rightarrow 0,4 = -b_{21} \times 0,5 \quad \Leftrightarrow b_{21} = -0,8$$

$$\Rightarrow 2 = (-0,8)^2 \times 0,5 + \sigma_z^2 \Leftrightarrow \sigma_z^2 = 1,68$$

$$B = \begin{bmatrix} 1 & 0 \\ -0,8 & 1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 1 & 0 \\ 0,8 & 1 \end{bmatrix}$$

with $x_t = A_0 + A_1 x_{t-1} + e_t$

and

$$x_t = B^{-1}\Gamma_0 + B^{-1}\Gamma_1 x_t + B^{-1}\varepsilon_t$$

we get: $\begin{cases} A_0 = B^{-1}\Gamma_0 \\ A_1 = B^{-1}\Gamma_1 \end{cases}$

$$\Gamma_0 = \begin{bmatrix} 1 & 0 \\ -0,8 & 1 \end{bmatrix} \begin{bmatrix} 0,1 \\ 0,8 \end{bmatrix}$$

$$\Leftrightarrow \Gamma_0 = \begin{bmatrix} 0,1 \\ 0,72 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0,1 & 0,4 \\ -0,3 & 0 \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} 1 & 0 \\ -0,8 & 1 \end{bmatrix} \begin{bmatrix} 0,1 & 0,4 \\ -0,3 & 0 \end{bmatrix}$$

$$\Leftrightarrow \Gamma_1 = \begin{bmatrix} 0,1 & 0,4 \\ 0,38 & -0,32 \end{bmatrix}$$

The structural VAR representation is given by:

$$B X_t = \Gamma_0 + \Gamma_1 X_{t-1} + \varepsilon_t$$

$$B = \begin{bmatrix} 1 & 0 \\ -0,8 & 1 \end{bmatrix} \qquad \Gamma_0 = \begin{bmatrix} 0,1 \\ 0,72 \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} -0,22 & 0,4 \\ 0,3 & 0 \end{bmatrix}$$

$$\sigma_y^2 = 0,5 \;;\; \sigma_z^2 = 1,68$$

All the coefficients of the structural VAR have thus been identified.

2c)

The definition of impulse-response function is the effect of a one standard deviation shock to the variables in the model, "shutting down" all the other shocks.

At impact: $\frac{dy_t}{d\varepsilon_{z_t}}$ and $\frac{dz_t}{d\varepsilon_{z_t}}$

As we imposed the restriction that z_t does not contemporaneously affect y_t, it immediately follows that:

$$\frac{dy_t}{d\varepsilon_{z_t}} = 0$$

Turning our VAR into a VMA(∞):

$$x_t = A_0 + A_1 x_{t-1} + e_t$$

$$\Rightarrow x_t = \mu + e_t + A_1 e_{t-1} + A_1^2 e_{t-2} + A_1^3 e_{t-3} + \dots$$

However, this is not very useful as we can't interpret e_t.
We know B, so:

$$x_t = \mu_t + B^{-1}\varepsilon_t + A_1 B^{-1}\varepsilon_{t-1} + A_1^2 B^{-1}\varepsilon_{t-2} + A_1^3 B^{-1}\varepsilon_{t-3} + \dots$$

$$\frac{dx_t}{d\varepsilon_t} = B^{-1} \quad ; \quad \frac{dx_{t+1}}{d\varepsilon_t} = A_1 B^{-1}$$

At impact:

$$\frac{dx_t}{d\varepsilon_t} = \begin{bmatrix} \frac{dy_t}{d\varepsilon_{yt}} & \frac{dy_t}{d\varepsilon_{zt}} \\ \frac{dz_t}{d\varepsilon_{yt}} & \frac{dz_t}{d\varepsilon_{zt}} \end{bmatrix}$$

and $\frac{dx_t}{d\varepsilon_t} = B^{-1}$

$$\Rightarrow \begin{bmatrix} \frac{dy_t}{d\varepsilon_{yt}} & \frac{dy_t}{d\varepsilon_{zt}} \\ \frac{dz_t}{d\varepsilon_{yt}} & \frac{dz_t}{d\varepsilon_{zt}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0{,}8 & 1 \end{bmatrix}$$

We are only concerned with shocks in z_t, so:

$\frac{dy_t}{d\varepsilon_{zt}} = 0$ A one standard deviation shock in z_t at impact has no effect on y_t.

$$\frac{dz_t}{d\varepsilon_{z_t}} = 1$$

A one standard deviation shock in interest rate causes an increase in 1 of the interest rate. This is self-evident.

At step one

$$\frac{dx_{t+1}}{d\varepsilon_t} = A_1 B^{-1}$$

$$A_1 B^{-1} = \begin{bmatrix} 0{,}1 & 0{,}4 \\ -0{,}3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0{,}8 & 1 \end{bmatrix} = \begin{bmatrix} 0{,}42 & 0{,}4 \\ -0{,}3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{dy_{t+1}}{d\varepsilon_{y_t}} & \frac{dy_{t+1}}{d\varepsilon_{z_t}} \\ \frac{dz_{t+1}}{d\varepsilon_{y_t}} & \frac{dz_{t+1}}{d\varepsilon_{z_t}} \end{bmatrix} = \begin{bmatrix} 0{,}42 & 0{,}4 \\ -0{,}3 & 0 \end{bmatrix}$$

$$\frac{dy_{t+1}}{d\varepsilon_{z_t}} = 0{,}4$$

A one standard deviation shock in z_t increases y_{t+1} by 0,4.
Again, I would mention the "price puzzle" here.

$$\frac{dz_{t+1}}{d\varepsilon_{zt}} = 0$$

A one standard deviation shock in z_t has no impact on the one step-ahead variation of z_t.

Assuming the size of the shock doubles:

The system is linear, so I expect impacts to double as well.

$$x_t = \mu + 2B^{-1}\varepsilon_t + 2A_1B^{-1}\varepsilon_{t-1} + \dots$$

$$\frac{dx_t}{d\varepsilon_t} = 2B^{-1} \; ; \; \frac{dx_{t+1}}{d\varepsilon_t} = 2A_1B^{-1}$$

At impact:

$$2B^{-1} = 2\begin{bmatrix} 1 & 0 \\ 0,8 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1,6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{dy_t}{d\varepsilon_{yt}} & \frac{dy_t}{d\varepsilon_{zt}} \\ \frac{dz_t}{d\varepsilon_{yt}} & \frac{dz_t}{d\varepsilon_{zt}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1,6 & 2 \end{bmatrix}$$

$$\frac{dy_t}{d\varepsilon_{zt}} = 0$$

This is still zero, as would be expected because of our identification restriction.

$\frac{dz_t}{d\varepsilon_{z_t}} = 2$ → As earlier stated, this result is self-evident.

Response at step one:

$$\frac{dx_{t+1}}{d\varepsilon_t} = 2A_1B^{-1} = 2\begin{bmatrix} 0{,}42 & 0{,}4 \\ -0{,}3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0{,}84 & 0{,}8 \\ -0{,}6 & 0 \end{bmatrix}$$

$\frac{dy_{t+1}}{d\varepsilon_{z_t}} = 0{,}8$; $\frac{dz_{t+1}}{d\varepsilon_{z_t}} = 0.$

The results are completely coherent with everything I have explained so far.

2d)

FEVD - Forecast error variance decomposition

We can represent our VMA(∞) as:

$$x_t = \mu + \sum_{i=0}^{\infty} A_1^i B^{-1} \varepsilon_{t-i}$$

$$\phi_i = A_1^i B^{-1}$$

$$\Rightarrow x_t = \mu + \sum_{i=0}^{\infty} \phi_i \varepsilon_{t-i}$$

In a general fashion, the m step-ahead forecast error is:

$$x_{t+m} - E_t(x_{t+m}) = \mu - \mu + \sum_{i=0}^{m-1} \phi_i \varepsilon_{t+m-i}$$

For the forecast error variance at impact:

$$\phi_0 = A_1^0 B^{-1} = B^{-1}$$

$$\sigma_y(0)^2 = \sigma_y^2 \times b_{11} + \sigma_z^2 \times b_{12}$$
$$= \sigma_y^2 \times 1 + \sigma_z^2 \times 0$$
$$= \sigma_y^2$$
$$= 0{,}5$$

Note: Here b_{11} and b_{12} refer to the elements of the B^{-1} matrix.

Decomposition

Proportion of $\sigma_y(0)^2$ due to shock in ε_{yt}:

$$\frac{\sigma_y^2 \times 1}{\sigma_y(0)^2} = 1 \rightarrow 100\%$$

Proportion of $\sigma_y(0)^2$ due to shock in ε_{zt}:

$$\frac{\sigma_z^2 \times 0}{\sigma_y^2} = 0 \rightarrow 0\%$$

This shows that shocks in z_t have zero relevance towards explaining changes in y_t contemporaneously.
This matches the conclusions from c).

For the forecast error variance one step-ahead:

$\phi_1 = A_1 B^{-1} = A_1 B^{-1}$

$\sigma_y(1)^2 = \sigma_y^2(1+0{,}42^2) + \sigma_z^2(0+0{,}4^2)$

$\Leftrightarrow \sigma_y(1)^2 = 0{,}5 \times 1{,}1764 + 1{,}68 \times 0{,}16$

$\Leftrightarrow \sigma_y(1)^2 = 0{,}857$

Decomposition

Proportion of $\sigma_y(1)^2$ due to shock in ε_{yt}:

$$\frac{\sigma_y^2(1+0{,}42^2)}{\sigma_y(1)^2} = \frac{0{,}5 \times 1{,}1764}{0{,}857} =$$

$= 0{,}686 \Rightarrow 68{,}6\%$

Proportion of $\sigma_y(1)^2$ due to shock in ε_{zt}:

$$\frac{\sigma_z^2(0+0{,}4^2)}{\sigma_y(1)^2} = \frac{1{,}68 \times 0{,}16}{0{,}857} =$$

$= 0{,}314 \Rightarrow 31{,}4\%$

Again, this matches what I found in d). Z_t shocks have relevance for explaining movements in Y_{t+1} – although shocks in ε_{Y_t} are of a bigger relative importance.