Math 250A

Fall 2015

8/27

Group Action

A group G acts on a set S:
$G \times S \to S$
$(g,s) \mapsto g \cdot s$
$e \cdot s = s$
$(gg') \cdot s = g \cdot (g' \cdot s)$
Alternatively,
$\phi : G \to Perm(S)$
ϕ is a homomorphism (gives the corresponding properties)
$(\phi(g))(s) = g \cdot s$

Examples of Group Actions

The trivial action:
$G \to Perm(S)$ where $g \mapsto e_{Perm(S)}$
G acting on self by left/right translation, conjugation
G acting on the set of subgroups of G by conjugation:
$g \cdot H = gHg^{-1} = \{ghg^{-1} | h \in H\}$
Normal subgroup $N \trianglelefteq G$
G acting on N, $g \cdot n := gng^{-1} \in N$
$G = S_3$ where S is the set of subgroups of G of order 2.
S = {{1, (1 2)}, {1, (1 3)}, {1, (2 3)}}
recall $\sigma(a_1, a_2, a_3, ...a_k)\sigma^{-1} = (\sigma a_1, \sigma a_2, \sigma a_3, ...\sigma a_k)$
V vector space over a field K
G = GL(V) = group of invertible linear maps $V \to V$
e.g. if $V = K^n$ then G = GL(n, K)
G acts on V (rather simply) by $L \cdot v = L(v)$

Orbits and Stabilizers

Given G acting on S by $G \times S \to S$ there is an obvious relation on S:

s, s′: $s \sim s' \leftrightarrow \exists g \in G, s' = gs$

the orbit of s is just the equivalence class of s under this relation

i.e., $G \cdot s = \{g \cdot s | g \in G\}$

The conjugacy classes of s are the orbits of S under the group action of G by conjugation

the orbit of s, $O(s) = \{s\} \leftrightarrow s = gsg^{-1} \forall g$

$\leftrightarrow (\forall g) gs = sg$

$\leftrightarrow s \in Z(G)$ the center of the group

Example, for $G = S_3$

the orbit of 1 is $\{1\}$

the orbit of (1 2) = $\{$(1 2), (1 3), (2 3)$\}$

the orbit of (1 2 3) = $\{$(1 2 3), (1 3 2)$\}$

Stabilizer (isotropy group) of a given element $s \in S := G_s$

$G_s = \{g \in G | g \cdot s = s\}$

stabilizer is closed under inverses: $g \in G_s \to g \cdot s = s \to g^{-1}gs = g^{-1}s \to s = g^{-1}s$

large stabilizer$\leftrightarrow$ small orbit

there exists a natural bijection $\alpha : G/G_s \to O(s)$ defined $gG_s \mapsto g \cdot s$

well-definition:

if $g_1 G_s = g_2 G_s$ then $\exists g \in G_s, g_1 = g_2 g$ and $\alpha(g_1 G_s) = g_1 \cdot s = g_2 g s = g_2 s = \alpha(g_2 G_s)$

injectivity:

if $\alpha(g_1 G_s) = g_1 \cdot s = g_2 \cdot s = \alpha(g_2 G_s)$ then $g_2^{-1} g_1 \cdot s = s$, $g_2^{-1} g_1 \in G_s$ and $g_1 G_s = g_2 G_s$

Lang 1.1-1.5