Math 206

Fall 2015

8/26

Definitions

A *norm* on a vector space X (over $\mathbb{F}$) is a function $\|\| : X \to \mathbb{R}^+$ such that
$\|x\| = 0$ iff $x = 0$
$\|\alpha x\| = |\alpha|\|x\|$ (for $\alpha \in F$)
$\|x + y\| \leq \|x\| + \|y\|$
An *algebra* $\mathscr{A}$ over $\mathbb{F}$ is a vector space with distributive $\cdot$ satisfying
$cx \cdot y = c(x \cdot y)$
$x \cdot cy = c(x \cdot y)$ for all $c \in F$
A *normed algebra* over $\mathbb{R}$ or $\mathbb{C}$ is an algebra $\mathscr{A}$ equipped with (vector space) norm satisfying
$\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathscr{A}$
A norm on $\mathscr{A}$ induces a metric
$d(a, b) = \|a - b\|$ on $\mathscr{A}$ and therefore a topology
if $\mathscr{A}$ is complete for this norm, it is a *Banach algebra*

To figure out (use https://www.math.ksu.edu/ nagy/real-an/2-05-b-alg.pdf)

Supposing $\mathscr{A}$ is not necessarily complete
$\|ab\| \leq \|a\|\|b\|$ gives uniform continuity on the product
hence the norm can be extended to the completion $\bar{\mathscr{A}}$ to form a Banach algebra
A metric space M is complete if all Cauchy sequences converge to an element of M
The completion M is all equivalence classes of Cauchy sequences where
$\{a_n\} \sim \{b_n\}$ iff $\lim\limits_{x \to \infty} d(a_n - b_n) = 0$

Examples

For M a compact space, $C(M)$
the set of continuous $\mathbb{R}/\mathbb{C}$-valued functions on M
pointwise operations
$\|f\|_\infty = sup\{|f(x)| : x \in M\}$
For M locally compact, $C_\infty(M)$

the set of continuous $\mathbb{R}/\mathbb{C}$-valued functions on M vanishing at ∞
vanishing at ∞: $\forall \epsilon \exists$ a compact subset of M, outside of which $f < \epsilon$
note that this is non-unital (lacks an identity)
For $\mathscr{O} \subset \mathbb{C}^n$ open
$H^\infty(\mathscr{O})$ the set of all bounded holomorphic functions on $\mathscr{O}$
(M, d) metric space and $f \in C(M)$
Lipschitz constant (which can be $+\infty$) $L_d(f) = sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in M, x \neq y\}$
The *Lipschitz functions* $\mathscr{L}_d(M, d) = \{f : L(f) < \infty\}$
These form a dense subalgebra of C(M) and are in fact a *-subalgebra
$\|f\|_d := \|f\|_\infty + L_d(f)$, can be shown as a normed-algebra norm
$L_d(M, d)$ is complete for this norm
so $L_d(M, d)$ is a Banach algebra
L_d is a seminorm on $\mathscr{L}_d(M, d)$ since it takes value 0 on the constant functions
can recover d from L_d
M a differentiable manifold (e.g. $T = \mathbb{R}/\mathbb{Z}$ the circle)
C(M) $\supseteq C^{(1)}(M)$ the singly-differentiable functions
$f \in C^{(2)}(T) \to Df : T_xM \to \mathbb{R}, \mathbb{C}$
with Df the derivative and T_x the tangent space
If we put on a Riemmannian metric, define $\|f\|^{(1)} = \|f\|_\infty + \|Df\|_\infty$
If $f \in C^{(1)}(T) : \|f\|^{(1)} = \|f\|_\infty + \|f'\|_\infty$
Banach algebra norm, for which this space of functions is complete
For the circle, $C^{(2)}(T) \to \|f\|^{(2)} = \|f\|_\infty + \|f'\|_\infty + \frac{1}{2}\|f''\|_\infty$
the factor $\frac{1}{2}$ ensures that this satisfies the normed algebra condition
$C^{(n)}(T) = \sum_{k=0}^{n} \frac{1}{k!}\|f^{(k)}\|_\infty$
For $C^\infty(T)$ using the collection of norms $\{\|\|^{(n)}\}_{n=1}^\infty$ yields a Fréchet algebra
A Fréchet algebra has a topology defined by a countable family of seminorms
that respect the algebra structure and is complete **(clarify)**
non-commutative algebras
X a Banach space
$\mathscr{B}(X)$ the algebra of bounded operators on X
$\|\|$ operator norm $\to$ Banach algebra
Any closed subalgebra of $\mathscr{B}(X)$ is a Banach algebra

8/28

Sketch of the course
X a Banach space, $B(X)$ bounded functions on the space
$\mathscr{H}$ a Hilbert space, $\mathscr{B}(\mathscr{H})$ bounded operators on the space
for $T \in \mathscr{B}(\mathscr{H}) \exists$ adjoint operator $T^* \in \mathscr{B}(\mathscr{H})$
$< T\xi, \eta > = < \eta, T^*\xi >$ for $\xi, \eta \in \mathscr{H}$
adjoint is additive, conjugate linear, $T^{**} = T$, $(ST)^* = T^*S^*$
An algebra A over $\mathbb{R}$ or $\mathbb{C}$ is a **-algebra* if it has a $* : A \to A$ satisfying

certain properties (look up)

A **-normal algebra* is a normal *-algebra such that

$(\forall a \in A)\|a^*\| = \|a\|$

A *Banach *-algebra* is a *-normal algebra that is a Banach algebra.

For any $T \in \mathscr{B}(\mathscr{H})$, have $\|T^*T\| = \|T\|^2$ **(check: parse through defns)**

For M a locally compact space, $A = C_\infty(M, \mathbb{C})$, $f^* := \bar{f}$ is a Banach *-algebra

Also have $\|f^*f\| = \|f\|^2$ **(verify: should be easier than the other)**

Little Gelfand-Naimark theorem:

Let A be a commutative Banach *-algebra satisfying $\|a^*a\| = \|a\|^2$.

Then $A \cong C_\infty(M)$ for some locally compact M.

One view of the "spectral theorem"

Let $T \in \mathscr{B}(\mathscr{H})$ with $T^* = T$

Let A be the closed subalgebra of $\mathscr{B}(\mathscr{H})$ generated by T and I (i.e. $p(T) := \Sigma\alpha_k T^K$)

Polynomials closed or stable under *

If $S \in A$ then $S^* \in A$ (i.e. A is a *-subalgebra of $\mathscr{B}(\mathscr{H})$

So A is a Banach *-suubalgebra satisfying $\|S^*S\| = \|S\|^2$

Moreover, A is commutative. (unital, since generated by I)

Then by the Little Gelfand-Naimark theorem, $A \cong C(M)$

Indeed $M \subset \mathbb{R}$, the spectrum of T

If $\mathscr{H}$ is finite dimensional, then M is the set of eigenvalues of T

T is normal if $TT^* = T^*T$

A C*-algebra is a Banach *-algebra over $\mathbb{C}$ satisfying

$\|a^*a\| = \|a\|^2$

Theorem: A commutative C*-algebra is $\cong C_\infty(M)$.

Big Gelfand-Naimark Theorem: (*Math 208, C*-algebras*)

Any C*-algebra is $\cong$ to a closed *-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.

Tangent

algebraic topology, differential geometry, Riemann manifolds, "non-commutative geometry" (Connes)

A *von-Neumann algebra* is a *-subalgebra of $\mathscr{B}(\mathscr{H})$

which is closed under the strong operator topology.

Every commutative von-Neumann algebra is $\cong L^\infty(X, S, \mu)$ (measure spaces) acting on $L^2(X, S, \mu)$ by positive sldkjfalksdjf

For group G, $\alpha : G \to Auto(X) \subseteq \mathscr{B}(X)$

$Auto(X)$ a Banach space

Look at subalgebra of $\mathscr{B}(X)$ generated by $\alpha(G)$.

Leads to considering $l'(G)$ with product $(f \star g)(x) = \Sigma f(y)g(y^{-1}x)$ convolution

$f^*(x) = \overline{f(x^{-1})}$

Banach *-algebra, G commuatative $\to$ Fourier transform