Math 250A, Fall 2015

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A group G acts on a set S:
$G \times S \to S$
$(g,s) \mapsto g \cdot s$
$e \cdot s = s$
$(gg') \cdot s = g \cdot (g' \cdot s)$
Alternatively,
$\phi : G \to Perm(S)$ is a homomorphism
$(\phi(g))(s) = g \cdot s$
Examples
trivial action: $(\forall g)\ g \mapsto e_{Perm(S)}$
G acting on self by left/right translation, conjugation
G acting on the set of subgroups of G by conjugation: $g \cdot H = gHg^{-1} = \{ghg^{-1} | h \in H\}$
normal subgroup $N \trianglelefteq G$: all $g \in G$ fix N under conjugation
V vector space over a field K, GL(V) acts on V by $L \cdot v = L(v)$

The orbit of s, $O(s) := \{g \cdot s | g \in G\}$
constitutes an equivalence relation on S
The stabilizer (isotropy group) of $s \in S$, $G_s := \{g \in G | g \cdot s = s\}$
G_s is closed under inverses: $g \in G_s \to g \cdot s = s \to g^{-1}gs = g^{-1}s \to s = g^{-1}s$
There exists a natural bijection $\alpha : G/G_s \to O(s)$, $gG_s \mapsto g \cdot s$
well-defined: $g_1G_s = g_2G_s \to \exists g \in G_s, g_1 = g_2g$, $\alpha(g_1G_s) = g_1s = g_2gs = g_2s = \alpha(g_2G_s)$
injective: $\alpha(g_1G_s) = g_1 \cdot s = g_2 \cdot s = \alpha(g_2G_s) \to g_2^{-1}g_1 \cdot s = s$, $g_2^{-1}g_1 \in G_s$, so $g_1G_s = g_2G_s$

Action under conjugation:
the conjugacy classes of a set are the orbits of the action
$O(g) = \{g\} \leftrightarrow g \in Z(G)$ the center of the group
$Z(G) = \{g \in G : xg = gx\ \forall x \in G\}$
in a permutation group, $\sigma(a_1, a_2, a_3, ...a_k)\sigma^{-1} = (\sigma a_1, \sigma a_2, \sigma a_3, ...\sigma a_k)$

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Let Σ be a set of representative elements of the orbits of S.
The index of a subgroup H is $(G : H) = \#(G/H)$
For finite G, $(G : H) = \frac{\#G}{\#H}$ ($g \notin H$, $\exists$ natural bijection $H \to gH$)
$\#S = \sum_{s \in \Sigma} \#O(s) = \sum_s (G : G_s)$
defines a 'mass formula' $\#S = (\sum_s \frac{1}{\#(G_s)})(\#G)$
Let G act on a subgroup H by left translation.
$\#H_s = \#H$ and from the above $\#G = (G : H) \cdot \#H$.
this is a statement of Lagrange's Theorem, $(G : H) = \frac{\#G}{\#H}$.
The kernel of the action $K = \bigcap_{s \in S} G_s$, which is just the kernel of $G \xrightarrow{\phi} Perm(S)$.

We can relate the stabilizers of points in the same orbit.

Let $s' = gs$.

Assume $x \in G_s$.

Since $x \in G_s$, $(gxg^{-1})gs = g(xs) = gs$.

Hence $gxg^{-1} \in G_{gs}$, so $gG_sg^{-1} \subset G_{gs}$.

Apply this relation with $g \to g^{-1}$ and $s \to gs$:

Assume $x \in G_{gs}$.

Then $(g^{-1}xg)(s) = (g^{-1})(xgs) = (g^{-1}gs) = s$.

So $g^{-1}G_{gs}g \subset G_s \to G_{gs} \subset gG_sg^{-1}$

Thus, $gG_sg^{-1} = G_{gs} = G_{s'}$.

The stabilizer of $s' = gs$ is a conjugate of the stabilizer of s.

p : prime

p-group: a finite group G, $\#G = p^n, n \geq 1$

"A p-group has a non-trivial center"

Notation: S^G is the set of points in S fixed under the group action. ($gs = s\ \forall g \in G$)

Let G act on itself by conjugation ($S = G$). Then $S^G = Z(G)$.

For $s \in S(= G)$, G_s is a subgroup, and its order divides the order of the group, p^n.

Either $O(s)$ is trivial, and $s \in S^G = Z(G)$, otherwise $\#(O(s)) = p^k$ for $k > 0$

$\#S$ = sum of # of elements in the orbits $\equiv_{mod p}$ # of orbits of size $1 = \#(S^G)$.

$\#Z(G) \equiv_{mod p} \#(S^G) \equiv_{mod p} \#S = \#G = p^n \equiv_{mod p} 0$.

$Z(G)$ cannot be 1, since the identity of the group is in the center.

Thus, the order of the center is divisible by p, and must be non-trivial.

$H \leq G$ a finite group, $(G : H) = p$, the smallest prime dividing #G $\to H \trianglelefteq G$

Let $S = G/H$; $\#(S) = (G : H) = p$, and let G act on S by left translation.

This induces $\varphi : G \to S_P$; recall $\#S_p = p!$

The stabilizer of H, $G_H = \{x \in G | xH = H\}$, hence $G_H = H$.

By inspection, we can see that $G_{gH} = gHg^{-1}$.

Let $K = \bigcap_{g \in G} gHg^{-1}$, the largest normal subgroup contained in H.

For each coset gH, K stabilizes that coset, hence K is the kernel of φ.

By the First Isomorphism Theorem $\varphi(G) \leq S_p$.

$(G : K) = \#(G/K) = \#(\varphi(G))$, which divides $\#(S_p) = p!$

Further, since $K \leq H \leq G$, $(G : K) = (G : H)(H : K)$.

Since $(G : K)$ divides $p!$ and $(G : H)$ divides p, $(H : K)$ divides $(p-1)!$.

But p is the smallest prime dividing $\#G$, so $(H : K)$ = 1, $K = H$ and H is normal.

A familiar embedding of a group into a larger group; "Cauchy's Theorem"

$G \hookrightarrow Perm(G)$ by letting G act on itself by left-translation.

Its kernel $K = \{g \in G | gs = s \forall s\} = \{e\}$ (consider $s = e$), so an injection $\to$ an embedding.

Recall $S_n \subset$ group of $n \times n$ invertible matrices. $\sigma \mapsto M(\sigma)$ a permutation matrix.

Need to be careful in the construction to ensure $M(\sigma\tau) = M(\sigma)M(\tau)$!

E.g. $\sigma = (132)$ does $M(\sigma)$ have 1 in the 1st column, 3rd row?
Or in the 1st row, 3rd column? One of these yields $M(\sigma\tau) = M(\tau)M(\sigma)$.

G finite of order n; V the vector space of functions $G \xrightarrow{f} \mathbb{Z}$; note $V \cong \mathbb{Z}^n$
Linear maps $V \to V$ correspond to $n \times n$ matrices over $\mathbb{Z}$: $GL(V) \approx GL(n, \mathbb{Z})$.
Similarly, invertible linear maps correspond to $n \times n$ invertible matrices over $\mathbb{Z}$.
We can embed G in $GL(n, \mathbb{Z})$ by using a left action of G on $GL(n, \mathbb{Z}) = \{\phi : V \to V\}$
Can think of this as an action on $\mathbb{Z}^n \cong V$, whose permutation group is simply $GL(n, \mathbb{Z})$.
Recall that $V = \{f : G \to \mathbb{Z}\}$.
This left action takes the form $L_g \mapsto \phi$ where $\phi(f(x)) = f(xg)$
$L_{gg'} = L_{g'} \circ L_g$ as desired? Verify for yourself.
Yes: $L_{gg'}(\varphi(x)) = \varphi(xgg') = L_{g'}(\varphi(xg)) = L_{g'} \circ L_g(\varphi(x))$
$g \mapsto L_g$ is a homomorphism $G \to GL(V)$
Using $\mathbb{F}_p$ instead of $\mathbb{Z}$, get $G \hookrightarrow GL(n, \mathbb{F}_p)$, an embedding into a finite group.

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Lagrange: If $H \leq G$ then $\#(H) | \#(G)$.
A_4 with $n = 6$: a counterexample to the converse.

If $|G| = p^k \cdot r$, $(p, r) = 1$, a p-Sylow subgroup of G is an $H \leq G$ such that $|H| = p^k$
$\mathbb{Z}_{12}$ has 2-sylow subgroup $\{0,3,6,9\}$ and 3-sylow subgroup $\{0,4,8\}$
D_6 generated by r, s subject to $rs = sr^{-1}$, $r^6 = e$, $s^2 = e$
$\#(D_6) = 12$ so has 3-sylow subgroup $\{1, r^2, r^4\}$
Also has 2-sylow subgroups $\{1, r^3, s, r^3 s\}$, $\{1, r^3, rs, r^4 s\}$, $\{1, r^3, r^2 s, r^5 s\}$

G = $GL_n(\mathbb{F}_p)$, $n \times n$ linear transformations in $\mathbb{F}_p$, equal to $Aut(\mathbb{F}_p^n)$
Approximating the order of $|G|$:
Asserting linear independence in each vector of an $n \times n$ matrix
$|G| = (p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1}) = p^{1+2+3+\cdots+n-1} \cdot r = p^{\frac{n^2-n}{2}} \cdot r$, $(p, r) = 1$
Consider P the set of $n \times n$ upper triangular matrices with 1's on the diagonal.
Then $|P| = p^{1+2+3+\cdots+n-1} = p^{\frac{n^2-n}{2}}$, and P is a p-Sylow subgroup.
Will use this fact in the subsequent proof.

Theorem: (Sylow I) For $|H| = p^k \cdot r$, $(p, r) = 1$, H has a p-Sylow subgroup.
Proof Sketch:
Show $\exists G$, $H \leq G$, such that G has a p-Sylow subgroup
Show that if G has a p-Sylow subgroup and $H \leq G$, then H has a p-Sylow subgroup
Proof:
Cayley's theorem, can embed H (of order n) in S_n by acting on itself by translation.
Additionally $S_n \leq GL_n(\mathbb{F}_p)$ mapping to permutation matrices.
Alternatively, consider $V \cong \mathbb{F}_p^n$, the vector space of functions $\varphi : G \to \mathbb{F}_p$.
Embed H into $GL(V)$ by the action $g \in H \mapsto$ automorphism taking $\varphi(x)$ to $\varphi(xg)$.
$GL_n(\mathbb{F}_p)$ has p-Sylow subgroups. (upper triangular matrices with 1s on diag)

Let P be a p-Sylow subgroup of $G = GL_n(\mathbb{F}_p)$. Let G act on the cosets of P.
Now, $G_{gP} = gPg^{-1}$. Similarly, when H acts on G/P, $G_{gP} = (gPg^{-1} \cap H)$
This intersection is a p-group.
Want to choose $g \in G$ such that $gPg^{-1} \cap H$ is a p-Sylow subgroup.
If $(H : (gPg^{-1} \cap H))$ is coprime to p, then $gPg^{-1} \cap H$ is a p-Sylow subgroup.
By Orbit-Stabilizer, $(H : (gPg^{-1} \cap H)) = O(gP)$.
Note this is an orbit of G/P induced by the action of the group H.
Since P is a p-Sylow subgroup of G, $|G/P| \not\equiv_{mod p} 0$.
The sum of the orbits is $|G/P|$.
Hence there must be some orbit with size coprime to p.
The stabilizer of this orbit $gPg^{-1} \cap H$ is a p-Sylow subgroup H_p.

Corollary: All p-subgroups of H are contained in a conjugate of P.
Let $J \leq H$ be a p-subgroup. Then $J \cap gPg^{-1}$ is a p-Sylow subgroup of J for some $g \in G$.
A p-group can't contain a proper p-Sylow subgroup, so $J \cap gPg^{-1} = J$ and $J \subset gPg^{-1}$.

Corollary: (Sylow II) All p-Sylow groups are conjugate.
Let $H \leq G$ and $P \leq G$ be p-Sylow subgroups.
By the preceding corollary ($G \leq G$, $H \leq G$, $P \leq G$), $H \subset gPg^{-1}$ for some $g \in G$.
Since $|H| = |P| = |gPg^{-1}|$, $H \cap gPg^{-1} = H$.

Corollary: Every p-subgroup of G is contained in a p-Sylow of G.
By the above, each is contained in a conjugate of P, said conjugate being a p-Sylow.

The p-Sylow subgroups in G are all conjugate, so that:
If P is a p-Sylow of G then $G/N(P) \leftrightarrow$ set of p-Sylows in G.
$N(P)$ the normalizer of P
There are $n_p = (G : N(P))$ p-Sylows in total.

Lemma: If a finite p-group Γ acts on a set X, then $\#(X) \equiv_{mod p} \#(X^\Gamma)$
(X^Γ the fixed points of X under Γ).
Proof:
Each $\frac{|\Gamma|}{|Stab(x_i)|} \equiv_{mod p} 1$ if x_i fixed, else $\frac{|\Gamma|}{|Stab(x_i)|} \equiv_{mod p} 0$.
Hence $\#X = \sum_i \#Orb(x_i) = \sum_i \frac{|\Gamma|}{|Stab(x_i)|} \equiv_{mod p} \#X^\Gamma$.

Let $Syl_p(G)$ describe the p-Sylow subgroups of G and n_p denote its cardinality.
Theorem: (Sylow III) If $|G| = p^k \cdot r$, $k > 0$ then $n_p \equiv_{mod p} 1$. Further, $n_p | r$.
Proof:
Let P act on $Syl_p(G)$ by conjugation.
By the lemma, $\#Syl_p(G) = n_p \equiv_{mod p} (Syl_p(G))^P$.
Suppose Q is fixed under the group action. Then $pQp^{-1} = Q \; \forall p \in P$.
Then $P \leq N(Q)$; similarly $Q \leq N(Q)$.
P, Q are p-Sylow subgroups of N(Q); therefore P, Q are conjugate in N(Q).

However, $Q \trianglelefteq N(Q)$ so that Q is equal to all its conjugates in N(Q), and $P = Q$.
Hence P is the only fixed Sylow-p subgroup so $(Syl_P(G))^P \equiv_{modp} 1$.
G acts on $Syl_p(G)$ as only one orbit since all p-Sylows in G are conjugate.
$(G : P) = n_p$, $n_p = |G| = p^k \cdot r$, $n_p | p^k \cdot r$, but $n_p \nmid p$, so $n_p | r$.

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P, Q p-Sylows and $P \subset N(Q)$ then $P = Q$
reason: $PQ \leq G$ a subgroup of G
HK not necessarily a group, but will be if one normalizes the other ($H \subset N(K)$)

A simple group is a non-trivial group with no non-trivial proper normal subgroups

A finite abelian group G is simple $\leftrightarrow$ G is cyclic of prime order
show this

non-sporadic finite simple groups
$A_n (n \leq 5)$
recall the alternating groups A_n are the even permutations on $\{1, \cdots, n\}$
Lie groups over finite fields, e.g. $\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \subset SL(2, \mathbb{Z}|p\mathbb{Z})$
P = projective; $PSL(2, \mathbb{Z}|p\mathbb{Z}) = SL(2, \mathbb{Z}|p\mathbb{Z})$

Simple groups of order ≤ 60.
(a) There are no non-abelian simple groups of order < 60
(b) If G is simple of order 60, then $G \cong A_5$.
($\#A_n = \frac{n!}{2}$)
G simple of order 60.
$H < G$ simple (finite), H proper, $(G : H) = n \geq 2$
G acts on G/H by left translation.
The action is transitive (for each pair xH, yH, $\exists$ permutation taking one to the other)
Therefore, this action is non-trivial.
$\pi : G \to Perm(G/H) = S_n$
$ker(\pi) \neq G$ and is a normal subgroup $\to$ the kernel is trivial.
$\pi : G \hookrightarrow S_n$ and in fact $\pi : G \hookrightarrow A_n$ (if $\#G > 2$)
Why? because $G \cap A_n \trianglelefteq G$
If $G \subset S_n$.
Then $G \to S_n / A_n = \{\pm 1\}$ by the sign map, kernel is $G \cap A_n$.
Recall $sgn : S_n \to \{\pm 1\}$ $sgn(\sigma) = (-1)^t$ given t, num of transpositions
$G/(G \cap A_n) \hookrightarrow S_n / A_n = \{\pm 1\}$
$(G : G \cap A_n) = 1$ or 2.
If G is simple then this cannot be 2 (would be normal subgroup), so =1.
And $G \hookrightarrow A_n$ for that A_n.
G simple, order 60.

H a proper subgroup of G, index n. (consider small values of n)
If $n = 2$ then H is normal in G, a contradiction.
(smallest prime dividing the order of a group)
If $n = 3$ or $n = 4$: $G \hookrightarrow A_3, A_4$ but their orders are too small (3, 12)
If $n = 5$: $G \hookrightarrow A_5$ and they are equal in cardinality $\rightarrow$ done.
Remaining case: $n = 15$.
What is n_5, the number of 5-Sylow subgroups.
$n_5 | 60/5 = 12$, $n_5 = (G : N(P))$ n_5 divides the index
Also, $n_5 \equiv_{mod 5} 1$.
Thus $n_5 = 1$ or $n_5 = 6$.
If $n_5 = 1$ then only one 5-Sylow subgroup of G, must be normal.
This is impossible since G is simple.
Then $n_5 = 6$: tells you there are lots of elements of order 5 in G.
There is no overlap (excepting at the identity) between 5-Sylows.
Hence the number of elements of order 5 is $6 \cdot 4 = 24$
Elements of order 5 in A_5 are 5-cycles (a b c d e).
Need to take all strings of length 5: 120, and divide out by rotations 5.
Thus we get $120/5 = 24$ (check).
Consider n_2 the number of 2-Sylow subgroups.
Then n_2 divides $60/4 = 15$, and $n_2 \neq 1$ because of simplicity.
Also, $n_2 = (G : N(P_2))$, and this can't be 3 since G has no subgroup of index 3.
If $n_2 = 5$ then $N(P_2)$ is the desired index-5 subgroup $\rightarrow$ done.
From divisibility $n_2 = 1, 3, 5, 15$.
Elminate 1 by simplicity, 3 since the index is too small, 5 works, consider 15.

Considering the situation where there are 15 2-Sylow subgroups (of order 4).
These are groups like the Klein 4-group (no elements of order 4).
There are 2 2-Sylow subgroups P and Q where $P \cap Q$ has order 2.
Prove by counting.
Taking intersection, must be proper else they would be the same.
Hence $P \cap Q$ has order 1 or 2.
If there is utterly no overlap, there are $15 \cdot 3 + 1 = 46$ elt's of 2-Sylows.
And these do not have order 5. But there are 24 elements of order 5. Too many.

Now we know that some of these 2-Sylow subgroups have non-trivial overlap.
Consider $N(P \cap Q)$ for some such intersection, will be a subgroup of G.
Cannot be all of G, G is simple. (would make $P \cap Q$ normal)
$N(P \cap Q)$ contains P and Q since both are abelian.
Each are normal subgroups of $N(P \cap Q)$, so its order is divisible by 4.
Hence could have order 12, 20, or 60 (divisible by 4, divides 60).
Its index cannot be 1 (G is simple) cannot be 3 (A_n too small), = 5.
QED (**revisit why**).

G finite non-trivial.
If G is simple, $\{e\} \subset G$, $G/\{e\}$ simple.
If G is not simple $G \supset G_1 \supset (e)$, $G_1 \trianglelefteq G$, G_1, G/G_1 smaller than G.
Use principle of strong induction for a full decomposition.

Obtain a successive extension of simple groups.
Given G, such a tower, let $G_i/G_{i+1} = Q_i$ and consider the multiset $\{Q_0, \cdots, Q_{n-1}\}$.
In multiset, order does not matter, and multiplicity does matter.
Jordan-Hölder Theorem: Each composition yields the same multiset up to isomorphism.

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Proposition: Given G, $\exists G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n$, $G = G_0$, $G_{i+1} \trianglelefteq G_i$, G_i/G_{i+1} simple.
This is a normal tower or composition series; the simple quotients are the constituents.
If it is simple, then the filtration is $G \supset \{e\}$.
If G is not simple, $G \supset N \supset \{e\}$, where $G/N, N$ proper in G.
By strong induction, have filtrations for each. To conclude, use:

$\exists$ natural correspondence between subgroups of G/N and subgroups H of G, $N \leq H$
$G \supset L \supset N$, $L/N \subset G/N$
$\pi : G \to G/N$, $K \subset G/N$, $\to \pi^{-1}(K) \leq G$

Jordan-Hölder Theorem:
$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n$
$G_{i+1} \trianglelefteq G_i$, $G_i/G_{i+1} = Q_i$ simple.
The "multiplicity set" $\{Q_0, \cdots, Q_{n-1}\}$ is independent of the filtration.
Where order doesn't count, multiplicity does, and Q_i up to isomorphism.

Related question: can two different groups have the same reduction?
Yes. $S_3 \supset A_3 \supset \{e\}$. Quotients $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.
Also $\mathbb{Z}/6\mathbb{Z} \supset 3\mathbb{Z}/6\mathbb{Z} \supset 6\mathbb{Z}/6\mathbb{Z}$, same quotients but radically different structure.
"Knowing the building blocks does not confer knowledge of the building".

Jordan-Hölder Theorem: Proof.
Base case $n = 1$, $G \supset \{e\}$, $G/\{e\}$ simple and G simple.
Supposing $G \supset G_1 \supset \cdots \supset G_n \supset \{e\} = G_{n+1}$ and $G \supset G'_1 \supset \cdots \supset G'_m \supset \{e\} = G'_{m+1}$.
? $m = n$, $\{G_i/G_{i+1}\} = \{G'_j/G'_{j+1}\}$ ... If $G'_1 = G_1$, then done by induction.
Assume G_1, G'_1 are distinct. Then $G_1 \cap G'_1$ is smaller than G_1 or G'_1.
Also, $G_1G'_1$ is a subgroup since its factors are normal by hypothesis.
Indeed, it is also a normal subgroup since G_1 and G'_1 are invariant under conjugation.
Additionally, $G_1G'_1$ is of size larger than G_1 and G'_1. Thus it must be equal to G.
Can map $G'_1/(G_1 \cap G'_1) \to G_1G'_1/G_1$. Kernel is exactly $G_1 \cap G'_1$, hence injection.
This defines $G'_1/(G_1 \cap G'_1) \hookrightarrow G/G_1$. Symmetrically, $G_1/(G_1 \cap G'_1) = G/G'_1$.
Have $G_1 \supset \cdots \supset G_n \supset \{e\} = G_{n+1}$.
Take $G_1 \supset G_1 \cap G'_1 = H \supset H_1 \supset H_2 \supset \cdots \supset H_k \supset \{e\}$, a Jordan-Hölder filtration of G_1.
Obtained by induction.
Note $G_1/H = G/G'_1$ is the first quotient of this filtration.
By induction, these two filtrations have the same length.
The constituents of G_1 are the constituents of H, with $G_1/H = G/G'_1$ appended.

Constituents: G/G_1 + constituents of G_1 = G/G_1 + G/G_1' + constituents of H.
Have $G \supset G_1' \supset H \supset H_1 \supset \cdots \supset H_k = \{e\}$, same length as $G_1' \supset G_2' \supset \cdots \supset G_m' = \{e\}$.
Have related two different filtrations that have are unrelated, by a common filtration, which depends on the intersection of these two filtrations.

Free Groups

S a set, define the free abelian group on S, $\mathbb{Z}^S = \mathbb{Z}\langle S\rangle = \{\sum_{s\in S} n_s \cdot s | n_s \in \mathbb{Z}\}$.
Where all but finitely many of the n_s are 0.
$S = \{1, \cdots, n\}$, $\mathbb{Z}^S = \mathbb{Z}^n = \{(c_1, \cdots, c_n) | c_i \in \mathbb{Z}\}$

$$\sum_{i=0}^{\infty} n_i x^i = \sum_{i=0}^{\infty} n_i \cdot i \in \mathbb{Z}\langle S\rangle$$

where $n_i = 0$ for $i >> 0$.
"To map $\mathbb{Z}\langle X\rangle$ to A in the world of abelian groups is to map S to A in the world of sets."
$S \to \mathbb{Z}\langle S\rangle$ a set map, $s \in S \mapsto 1 \cdot s$.
Given $f : \mathbb{Z}\langle S\rangle A$ homomorphism.
And in fact, $F : Hom(\mathbb{Z}\langle X\rangle, A) \to Maps(S, A)$, F is a bijection.
These elements of the free abelian group are "formal sums".
That is, an $f : S \to \mathbb{Z}$.
Let $f : \mathbb{Z}\langle S\rangle \to A$, $f(\sum n_s s) = \sum_{s\in S} n_s f(s)$
An abelian group A is free of finite rank if $A \cong \mathbb{Z}^n$ for some $n \geq 0$ ($\mathbb{Z} = \mathbb{Z}\langle\emptyset\rangle = 0$).
Define $rank(A) = n$. If $\mathbb{Z}^m \cong A \cong \mathbb{Z}^n$ then n = m.
Why? Take positive integer > 1, e.g. 2. Then $\mathbb{Z}^n/2\mathbb{Z}^n \cong \mathbb{Z}^m/2\mathbb{Z}^m$.
LHS has 2^n elts and RHS has 2^m elts so $n = m$.

A subgroup of a free abelian group of rank n is a free abelian group of rank $\leq n$.
Proof: by induction on n.
$n = 0$: $A = (0) = B$.
$n = 1$: $A = \mathbb{Z} \supset B$. What are the subgroups of $\mathbb{Z}$? $(0), (t) = t\mathbb{Z}, t \geq 1$.
Proof by division algorithm: $\mathbb{Z} \supset B \neq 0$, $t =$smallest positive integer in B.
Division algorithm ensures that all elements are multiples of t.
$B \subset \mathbb{Z}^n \xrightarrow{\pi} \mathbb{Z}$.
$\pi : (c_1, \cdots, c_n) \mapsto c_n \in \mathbb{Z}$.
Cases:
(1) $\pi(B) = (0), B \subset \mathbb{Z}^{n-1}$, free of rank $\leq n-1$
(2) $\pi(B) = t\mathbb{Z}, t \geq 1$
$B \xrightarrow{\pi|_B} t\mathbb{Z} \xrightarrow{surj.} 0$
$ker(\pi)|_B = C$ free of rank $\leq n-1$.
Choose $b \in B$ such that $\pi(b) = t$.
$C \subset \mathbb{Z}^{n-1} : C = ker(\pi)|_B$, free of rank $\leq n-1$.
$C = B \cap \mathbb{Z}^{n-1}$
$C \subset B, \mathbb{Z} \cdot b \subset B$
Missing (pf in Lang)

Simple linear algebra.

$a_1, \cdots, a_n \in A$ corresponds to a homomorphism $\mathbb{Z}^n \to A$, $(c_1, \cdots, c_n) \mapsto \sum_{i=1}^n c_i a_i$.

These are linearly independent if f is 1-to-1, and these span/generate A if f is onto.

A is finitely generated if A is spanned by $a_1, \cdots, a_n$ for some $n \geq 0$, $a_i \in A$

A is finitely generated iff A is a quotient of $\mathbb{Z}^n$ for some n.

Corollary: a subgroup of a finitely generated abelian group is again finitely generated.

$\mathbb{Z}^n \xrightarrow{f} A$ finitely generated, have $B \subset A$, $f^{-1}(B) \leq \mathbb{Z}^n$, and $f^{-1}(B) \cong \mathbb{Z}^k$, $k \leq n$.

A finitely generated, torsion-free.

I.e. given $a \in A$ and $n \cdot a = 0$, $n \geq 1$, then $a = 0$.

Statement: A is free and of finite rank.

Proof: Take a finite set of generators S in which take T lin indep and large as possible.

take $T = a_1, \cdots, a_k$ and $S = a_1, \cdots, a_k, \cdots, a_m$

$\sum_1^{k+1} c_k a_k = 0$, $c_{k+1} \neq 0$

$B = span\{a_1, \cdots, a_k\} \cong \mathbb{Z}^k$.

$a_{k+1}, \cdots, a_m$: some multiple lies on B.

$N \geq 1$; $N \cdot A \subset B$.

Th: NA free, $N : A \to NA$ A torsion free.

Multiplication on A by a positive integer is injective.

A is isomorphic to NA by the mutliplication by n, since NA is free, A is free.

9/15

Abelian group, finitely generated.

Last week:

free group has to do with some correspondence to a $\mathbb{Z}^n$

subgroups of free finitely generated abelian groups are free and finitely generated

subgroups of finitely generated abelian groups are finitely generated

finitely generated, torsion free abelian group is a free abelian group

recall torsion free: for all $n \geq 1$, mult by n, $n \cdot A$ is injective

opposite A torsion: for all $a \in A$, $\exists n \geq 1$ such that $n \times a = 0$

Example of a torsion abelian group: $\mathbb{Q}/\mathbb{Z}$

element $p/q \mod \mathbb{Z}, q \geq 1, p \in \mathbb{Z}$; $q \times \frac{p}{q} = 0$ in $\mathbb{Q}/\mathbb{Z}$

finitely generated abelian groups up to isomorphism

A is a direct sum of a free part $\mathbb{Z}^r$ and a torsion part (a direct sum of cyclic groups)

Direct product of sets A_i indexed by S:

$$\bigoplus_{i \in S} A_i = \{f : S \to \cup_{i \in S} A_i : f(i) \in A_i\}$$

where for all but finitely many i, $f(i) = 0$

this is equivalent to the direct product when S is finite

Image 1: a map from a $\bigoplus_{i \in S} A_i$ to B is determined by the mappings from the A_i

The direct sum is a coproduct.

Image 2: a map into a $\prod_{i \in S} A_i$ is determined by the mappings into the A_i

The direct product is a product (in the categorical sense).
S countably infinite, $A_i = \mathbb{Z}/2\mathbb{Z}$
$\bigoplus_{i \in s} A_i$ is countable, but $\prod_{i \in S} A_i$ is not
Categories: products, coproducts, morphisms
$Mor(?, B) = \prod Mor(A_i, B)$? = co-product
The coproduct of sets is disjoint union.
Abelian group A and subgroups X and Y
we have inclusions from each into A
$X \times Y = X \oplus Y \xrightarrow{h} A$, $(x, y) \mapsto x + y$
h is injective if every $a \in A$ is of the form $x + y$
h is one-to-one $\leftrightarrow$ you can't write $x + y = x' + y'$ unless $x = x'$, $y = y'$
If true, say A is the direct sum of its submodules X and Y.
Suppose A, $X \subset A$, A/X is free (f.g. free): then X has a complement Y in A, $A \cong X \oplus A/X$
$A \xrightarrow{\pi} A/X$
$Y \subset A$, $\pi|_Y$ is an isom $Y \to A/X$.
$\pi|_Y$ inj $\leftrightarrow Y \cap X = (0)$.
$\pi|_Y$ surjective: given $a + X \in A/X$ we can find $y \in Y$ s.t. $y + X = a + X$
$x = y \cdot a \in X$
$a = y \cdot x, x \in X, y \in Y$
A/X free, say $\cong \mathbb{Z}^r$
To map A/X to A is to choose images in A of the generators of A/X corresponding to the unit vectors of $\mathbb{Z}^r$.
There is a unique homomorphism $s : A/X \to A$ so that $s(q_i) = a_i$ for $i = 1, \cdots, r$
$(\pi \cdot s)(q_i) = \pi(a_i) = q_i$
$\pi \circ s = id_{A/X}$
$Y =$ image of $S \subset A$.
$\pi|_Y$ surjective. $\pi(s(q)) = q$ for all $q \in A/X$
$\pi|_Y$ is 1-1. $/pi(s(q_0)) = 0$ but $s(q_0) = q_0$ so equals 0.
A a finitely generated abelian group
$X = A_{tors} = \{a \in A | na = 0 \text{ for some } n \geq 1\}$.
X f.g., tors $\to$ X finite abelian group.
A/X torsion free, f.g. $\to A$ free $\approx \mathbb{Z}^r$
$A \approx \mathbb{Z}^r \oplus A_{tors}$. $A_{tors} =$???
it is a finite abelian group, let $B = A_{tors}$
p prime, $B_p = \{b \in B | p^t \cdot b = 0 \text{ for some } t \geq 0\}$.
$B_P \subset B$.
$\bigoplus_p B_p \xrightarrow{\iota} B$
Proposition: ι is an isomorphism. (formal proof in Lang's book)
Proof essence:
suppose $60 \cdot b = 0$, $60 = 4 \cdot 3 \cdot 5 = 12 \cdot 5$
$(12,5) = 1$
$1 = r5 + s12 = 25 - 24$
$b = r \cdot 5 \cdot b + s \cdot 12 \cdot b$
$12x = 0$, $5y = 0$

Every element can be written as a sum of terms killed by a power of a prime
$A = \mathbb{Z}^r \oplus (\bigoplus_p B_p)$
$\mathbb{Z}^n \approx F \xrightarrow{\varphi} A$ A finitely generated (by n elements)
$Ker(\varphi) = X \subset F$.
? understand A ! understand X inside F.
Elementary division theorem
There exists a basis of $F \approx \mathbb{Z}^n$ s.t. ... $X = \bigoplus_{i \leq r} 0 \oplus a_1\mathbb{Z} \oplus a_2\mathbb{Z} \oplus \cdots \oplus a_{n-r}\mathbb{Z}$, $a_i \geq 1$
$X \subset \mathbb{Z}^n$
$a_1|a_2|a_3|\cdots|a_{n-r}$, increasing multiplicatively
$A = F/X = \mathbb{Z}^r \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \mathbb{Z}/a_2\mathbb{Z} \oplus \cdots$, $a_i|a_{i+1}$
A a finite abelian group $\rightarrow$ A is a direct sum of cyclic groups
p prime, $\#A = p^4 = a_1a_2a_3\cdots$
A is direct sum of cyclic groups of p-power order.
$A \approx \mathbb{Z}|p^i \oplus \mathbb{Z}|p^j \oplus \mathbb{Z}|p^k \oplus \mathbb{Z}|p^l$ at most
$i \leq j \leq k \leq l$, $i+j+k+l = 4$, $i, j, k, l, \geq 1$

9/17

A arbitrary finitely generated group that we want to understand
Pick some generators $g_1, \cdots, g_n$
Get a map from $Y = \mathbb{Z}^n$ to A, has some kernel
Considering $A = Y/X$, and how X lies in Y gives indication of structure of A
Can think of X, Y, as lattices
Theorem: $Y \cong \mathbb{Z}^n$ exists $v_1, \cdots, v_n$ basis of Y
such that in that basis $X = a_1\mathbb{Z} \oplus a_2\mathbb{Z} \oplus \cdots \oplus a_m\mathbb{Z} \oplus 0 \oplus 0 \oplus \cdots \oplus 0$.
$a_i \geq 1$, $a_1|a_2, a_2|a_3, \cdots a_{m-1}|a_m$.
Example: $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
$Y = \mathbb{Z} \oplus \mathbb{Z}$
$Y \supset X = 2\mathbb{Z} \oplus 3\mathbb{Z}$
Not at all true that the integers divide each other.
Puzzle. Not the case as in the theorem.
Need to prove to self that in some new basis, $Y = \mathbb{Z} \oplus \mathbb{Z}$,
and $X = \mathbb{Z} \oplus 6/\mathbb{Z}$, $Y/X = \mathbb{Z}/6\mathbb{Z}$.
$a_1 = 1$, and $a_2 = 6$.

$X \subset \mathbb{Z}^n$. Ask whether $X = (0)$ the zero submodule. If so, simple. So can assume nonzero.
Consider linear forms, homomorphisms $\mathbb{Z}^n \rightarrow \mathbb{Z}$.
For each λ have $\lambda(X) \subset \mathbb{Z}$. e.g., $\lambda(X) = 3\mathbb{Z}$. Some λs are nonzero since X is nonzero.
Choose λ so that $\lambda(X)$ is maximal.
Example: $X = 2\mathbb{Z} \oplus 3\mathbb{Z}$. The first coordinate fn yields $2\mathbb{Z}$,
the second coordinate fn yields $3\mathbb{Z}$.
But with $\lambda(u, v) = v - u$ we can get all of $\mathbb{Z}$.
possible to get λs yielding images $2\mathbb{Z}$, $3\mathbb{Z}$, but not to get λ, $\lambda(X)$ containing both?
In any case, take a maximal λ, fix that λ.

$\lambda(X) = a\mathbb{Z}$ maximal

Pick $x \in X$ so that $\lambda(x) = a$.

Claim: $\mu(x) = b$ is divisible by a for all $\mu \in Hom(\mathbb{Z}^n, \mathbb{Z})$

$gcd(a,b) = g = ra + sb$

$\tau := r\lambda + s\mu$, $\tau(x) = g$

Now $\tau(X) \supset \mathbb{Z}g \supset \mathbb{Z}a$

So $\tau(x) = \lambda(x)$, $\mathbb{Z}g = \mathbb{Z}a$

$a|b$ for this reason of maximality

"Executive session"

R a commutative ring

R-module: M

1) abelian group

2) endowed with a scalar multiplication $r \in R$, $m \in M$, $rm \in M$

same as a vector space definition except R is not assumed to be a field

The context in which this elementary divisor theorem works.

A a finitely generated abelian group replaced by a finitely generated R-module

And there are 2 conditions on R.

R is an integral domain: $rs = 0 \to r = 0$ or $s = 0$

Ideals of R are principal $M \subset R \to M = R \cdot a$

Digression: motivation. Killer example.

K a field, and R = K[t]. (very much like $\mathbb{Z}$, can do Euclidean division by remainders)

Have V and action of $K[t]$: (action of K and action of t)

V + action of K $\to$ K-vector space

Action of t: $T : V \to V$ multiplication by t, $v \mapsto t \cdot v$, $T(v) = t \cdot v$

Conversely, can form the corresponding polynomial in the linear transformation

Principal Ideal domain. Element of smallest degree, Euclidean algorithm.

Suppose we have an R-module V. This is a K-vector space V with action of t

Multiplication by t gives a linear operator $T : V \to V$ (t commutes with K)

Remark: if V is of finite dimension over K, then it is finitely generated as a K-module

In particular, it's finitely generated over the ring $R = K[t]$

A an abelian group. If A is torsion, we are especially interested.

Suppose we start with a linear operator on a finite-dimension vector space.

There is a characteristic polynomial h such that $h(T) = 0$.

Cayley-Hamilton theorem.

$h(t) \in R = K[t]$. So $h(t) \cdot v = 0$.

V is a torsion module because $h(t)$ annihilates V.

Summary of what we have so far:

$0 \neq X \subset Y = \mathbb{Z}^n$, $\lambda : Y \to \mathbb{Z}$, $\lambda(X)$ is maximal among $\mu(X)$s, $\lambda(X) = a\mathbb{Z}$.

Have shown that $a = \lambda(x)$, then $\mu(x)$ is divisible by a for all μ.

Take μ to be the i^{th} coordinate function, $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$, $a|x_i$ for all $i = 1, \cdots, n$,

$x = a \cdot y$, $y \in \mathbb{Z}^n$, $\lambda(y) = \lambda(x)/a = 1$

Think of Y: contains two submodules (subgroups)

$Y \supset ker(\lambda)$, $Y \supset \mathbb{Z} \cdot y$.

Claim: $Y = ker(\lambda) \oplus \mathbb{Z}y$

1) each $z \in Y$ is: e.g. $(z - \lambda(z) \cdot y) + \lambda(z)y$

2) if my is in $ker(\lambda)$ then $0 = \lambda(my) = m\lambda(y) = m$ so $m = 0$, $my = 0$, intersection is 0

The corresponding statement for X is that $X = (\ker(\lambda|_X)) \oplus \mathbb{Z}_X$

Kind of obvious that the intersection is 0.

Each component is a submodule of the corresp. one in Y.

$z \in X, \lambda(z) = m\lambda(x) = ma\lambda(y)$.

$z = z - \lambda(z)y + \lambda(z)y$

$\lambda(z)y = m \cdot a \cdot y = mx$

$(z - \lambda(z)y) \in ker(\lambda) \cap X = ker(\lambda|_X)$

$\mathbb{Z}^n = Y = ker(\lambda) \oplus \mathbb{Z}y$

$Y \supset X = ker(\lambda|_X) \oplus \mathbb{Z}ay$

Apply inductively to portion of lower rank, having pulled off $\mathbb{Z}a$

$X = a_1\mathbb{Z} \oplus a_2\mathbb{Z} \oplus \cdots \oplus a_m\mathbb{Z} \oplus 0 \cdots 0 \subset Y = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$

need to have some kind of divisibility among these a, need to be explained

$a_1|a_2, \cdots$

$Y = \mathbb{Z} \oplus Y'$ and $X = a\mathbb{Z} + X'$, working rightward

start thinking of various linear maps $\lambda' : Y' \to \mathbb{Z}$, and how they restrict to X

taking a maximal one, etc., etc.

need to understand somehow that if we take this $\lambda'(X') = a'\mathbb{Z}$

we want $a|a'$, meaning $a'\mathbb{Z} \subset a\mathbb{Z}$, do this with some greatest common divisor argument

Introduce $g = gcd(a, a')$ which we want to be a, write in form $ra + sa'$

Need to find some interesting linear map from Y to Z

Have a map $Y' \xrightarrow{\lambda'} \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z}$ the identity

Both of these are linear maps that give linear maps $Y \to \mathbb{Z}$.

Choose $x' \in X'$ so that $\lambda'(x') = a'$

Have $(a, 0)$ in X so that the second linear map (just taking the first coordinate)...

...applied to $(a, 0)$ gives a

Take $Y = \mathbb{Z} \oplus Y'$

$\mathbb{Z} \oplus Y' \xrightarrow{f} \mathbb{Z}$

$\mathbb{Z} \oplus Y' \to Y' \to Y' \xrightarrow{\lambda'} \mathbb{Z}$, the composition of which call g

$Y = \mathbb{Z} \oplus Y' \ni (a, x') \in X$

$f(a, x') = a$

$g(a, x') = \lambda(x') = a'$

$(rf + sg)(a, x') = G$, $rf + sg = \mu$

$\mu(X) \supset \mathbb{Z} \cdot G \supset \mathbb{Z}a$

Maximality $\to G = a$.

Tells us that a really divides a' by maximality.

The Y and the X really divide off into two separate worlds.

$Y = \mathbb{Z} \oplus Y'$ and $X = a\mathbb{Z} \oplus X'$

The world which we have already considered, and the trailing-off world of Y' and X'

New map μ defined on all of Y and X, by leaving the first coordinate alone.

Go back to the original example of the 2 and the 3. $Y = \mathbb{Z} \oplus \mathbb{Z} \supset X = 2\mathbb{Z} \oplus 3\mathbb{Z}$

$\lambda(u, v) = v - u$

$x = (2, 3), \lambda(x) = 1$

$a = 1, \lambda(X) = \mathbb{Z}$, need to see how that line splits off in $\mathbb{Z}$ and in X.

$Y = \mathbb{Z} \cdot y \oplus ker(\lambda)$
$y = x/a = x$, $ker(\lambda) = \{(u,v) : u = v\} = \mathbb{Z} \cdot (1,1)$
$Y = \mathbb{Z} \cdot (2,3) \oplus \mathbb{Z} \cdot (1,1) = \mathbb{Z}^2$
$X = \mathbb{Z} \cdot (2,3) \oplus \mathbb{Z} \cdot (1,1) \cap (2\mathbb{Z} \oplus 3\mathbb{Z})$
so $X = \mathbb{Z} \cdot (2,3) \oplus 6 \cdot \mathbb{Z}(1,1)$
$Y/X = \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/6\mathbb{Z}$.

9/22

Rings R, A (= 'anneau')
definition: whether or not $1 \in R$ can vary
Lang: $1 \in R$, Hungerford: $1 \notin R$
In the former, $2\mathbb{Z}$ is not a ring, in the latter, it is
Ring:
under +, an abelian group with distinguished element 0
under $\cdot$, associative (not necessarily commutative) with distinguished element 1
distributive laws $(x + y)z = \cdots$ and $z(x + y) = zx + zy$
Integral domain:
Field: under $\cdot$, commutative, and non-zero elements have inverses

Examples
For A an abelian group, the ring of endomorphisms.
$R = End(A) = Hom(A, A)$, $(f + g)(a) = f(a) + g(a)$, $fg = f \circ g$
If $A = \mathbb{Z}^n$ End(A) can be viewed as a ring of matrices
$\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/p\mathbb{Z}, \mathbb{Q}(i) = \{a + bi | a, b \in \mathbb{Q}\}$ are fields.
The "skew field" of Hamilton quaternions over $\mathbb{R}, \mathbb{Q}$, $a + bi + cj + dk$ $(= (\frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2})^{-1})$
Group G (written multiplicatively), $\mathbb{Z}[G] = \mathbb{Z}\langle G\rangle$ the free abelian group on G
elements $\sum n_g \cdot g, n_g \in \mathbb{Z}$ the sum finite
can multiply

$$(\sum n_g \cdot g)(\sum m_h \cdot h) = \sum_{x \in G} (\sum_{g,h,gh=x} n_g m_h) x$$

The term $c_x = \sum_g n_g m_{g^{-1}x}$ represents a convolution product

Ring Homomorphism
a homomorphism of abelian groups respecting multiplication
$\varphi(xy) = \varphi(x)\varphi(y)$
$\varphi(1) \neq 1$ is possible
$ker(\varphi) = \{r \in R | \varphi(r) = 0\}$ is an ideal: $x \in R, r \in ker(\varphi) \to xr, rx \in ker(\varphi)$

Ideals
additive subgroup with ideal property:
$xI \subset I$ left-sided , $Ix \subset I$ right-sided, both $\to$ 2-sided (bilateral)
exact analogues of normal subgroups

two-sided ideal: well-defined quotient multiplication

$(r+I)\cdot(s+I) := rs+I$

$(r+I)(s+I) = r(s+i)+I = rs+ri+I$ and similarly

$(r+I)(s+I) = (r+i)s+I = rs+is+I$

therefore ideals are kernels of ring homomorphisms

Principal Ideal (a) is the minimal ideal containing a

is all multiples of a in R: $I = Ra$, said to be generated by a

Ideal generated by a subset X is the intersection of all ideals containing X

if $X = \{a_1, \cdots, a_t\}$, is written $(a_1, \cdots, a_t)$

Prime ideal $P \subset R$

proper

if $rs \in P$ then $r \in P$ or $s \in P$

Examples

$\mathbb{Z}$ has as ideals the additive subgroups $a\mathbb{Z}$, $a \geq 0 = (a)$; if $a = 0$ or a is prime, (a) is prime

$R = K[x]$ where K is a field: by Euclidean division, all ideals are principal

$R = K[x,y]$

$R \xrightarrow{\varphi} K$, $f(x,y) \mapsto f(0,0) \in K$, $ker(\varphi) = \{$polynomials with constant term of 0$\}$

this is *not* principal

elements $0 + ax + by + cx^2 + \cdots$

$\varphi : R \to S$ a ring homomorphism and P a prime ideal of S

$\varphi^{-1}(P)$ is a prime ideal of R

Proof:

Let $x, y \in R$ and suppose $xy \in \varphi^{-1}(P) = P'$

then $\varphi(x)\varphi(y) = \varphi(xy) \in P \to \varphi(x) \in P$ or $\varphi(y) \in P$

Corollary: $\varphi : R \to S$ a non-trivial homomorphism of rings and (0) prime in S

the kernel of φ is prime

S is called an integral domain if

$(0) \neq S$

if $xy = 0$ then $x = 0$ or $y = 0$

Proposition: $P \subset R$ is a prime ideal $\leftrightarrow R/P$ is an integral domain

Maximal ideal $M \subset R$

$M \neq R$

$M \subset M' \to M = M'$ or $M' = R$

Proposition: M is maximal $\leftrightarrow R/M$ is a field

Example: $\mathbb{Z} \supset a\mathbb{Z}$ maximal $\leftrightarrow a$ is prime

Corollary: Maximal ideals are prime

Pf: Fields are integral domains.

9/29

(Charlie)
A a ring, I an ideal in A
  have a correspondence between ideals J of A containing I and the ideals of A/I
  $\pi : A \to A/I$ and $\pi(J) = J/I$ an ideal of A/I
  for K ideal of A/I, $\pi^{-1}(K)$ is an ideal of A

A a ring, its group of units $A^* = \{u \in A | \exists v \in A, uv = 1\}$
  $(\mathbb{Z}[i])^* = \{1, -1, i, -i\} \cong \mathbb{Z}/4\mathbb{Z}$
  $(\mathbb{R}[x])^* = \mathbb{R}^*$
  $(\mathbb{Z}[\sqrt{5}])^* \ni 1, -1, 2+\sqrt{5}, 2-\sqrt{5}$

A a field $\leftrightarrow A^* = A - \{0\}$ and $A \neq \{0\}$
  $\{0\}$ is a maximal ideal

Every proper ideal of A is contained in a maximal ideal.
  Proof by Zorn's Lemma.

Chinese Remainder Theorem
  a ring A with ideals $I_1, \cdots, I_k$, $k \geq 2$
  the ideals coprime: that is, $I_i + I_j = A$.
  then there exists a surjective map $A \to A/I_1 \times \cdots \times A/I_k$
example
  $r\mathbb{Z} + s\mathbb{Z} = gcd(r,s)\mathbb{Z}$
  $(r\mathbb{Z})(s\mathbb{Z}) = r \cdot s\mathbb{Z}$
  $r\mathbb{Z} \cap s\mathbb{Z} = lcm(r,s)\mathbb{Z}$
  $(lcm)(gcd) = rs$
for two: IJ, $A/(IJ) \leftrightarrow (A/I) \times (A/J)$
  $(IJ = I \cap J)$
Proof:
  Assume $I, J \subset A$, $I + J = A$
  $A \to A/I \times A/J$
  let $x + y = 1$
  $x \to (0,1)$ and $y \to (1,0)$, $cx + dy \to (c,d)$

Quotient Fields
  e.g. $\mathbb{Z} \to \mathbb{Q}$
  A an integral domain and S a "multiplicative subset" of A
  $1 \in S$, $x, y \in S \to xy \in S$
  $S^{-1}A =$ equivalence class

10/1

Principal Ideal Domain: $\forall$ ideals I, $I = (a)$

Noetherian ring

  every ideal is finitely generated $I = (a_1, \cdots, a_m) = \{\sum_{i=1}^m r_i a_i | r_i \in A\}$

$I_1 \subset I_2 \subset I_3 \subset \cdots$ increasing chain of ideals in A

  becomes stable: $\exists N \geq 1$ so that $I_n = I_N$ for all $n \geq N$

  e.g. in $\mathbb{Z}$, have $(2^{100}) \subset (2^{99}) \subset \cdots$ (arbitrarily long chains exist, but all terminate)

the following are equivalent

  (1) each ideal is finitely generated

  (2) chains become stable

  (3) every non-empty set of ideals of A contains a maximal element.

(1) implies (2)

  given, $I_1 \subset I_2 \subset \cdots$ take $I = \bigcup_{i=1}^\infty I_i$

  I finitely generated, each a_i needs to be in some I

  eventually all of them are in some I_N, so $I \subset I_N$ and we are done

(2) implies (3)

  S some set of ideals, $I_1 \in S$. If I_1 not maximal, $I_1 \subset I_2$, $I_2 \in S$, iterate to construct a chain

  by (2), becomes stable; I_N is maximal

(3) implies (1)

  I an ideal, $a_0 \in I$, $I \neq (a_0)$, $\exists a_1$, $a_0 \subsetneq a_1$

  iterate $\to$ ascending sequence: has a maximal element $(a_0, \cdots, a_r) = I$

irreducible elements of A cannot be factored

  $a \in A$, not a unit and $\neq 0$

  if $a = bc$ then b is a unit or c is a unit

$(0) \subset (a) \subset A$; if A is a principal ideal domain, (a) is maximal

  $I \supset (a) = (b) \subset A \to a \in (b)$, $a = bc$

  b a unit $\to I = A$, c a unit $\to I = (a)$

Principal ideal domain A, $t \in A$, $t \neq 0$, t not a unit

  Proposition: t can be written as a product of irreducible elements

Proof:

  Let $S =$ the set of (principal) ideals (t) for which the proposition is false

  If nonempty, has maximal element (m); if $(m) \subsetneq (m')$, (m') can be factored

  m irreducible else $m = m'm''$ where m', m'' not units

  $(m) \subsetneq (m')$, $(m) \subsetneq (m'')$, hence neither are in S

Proof works for Noetherian rings generally

prime elements of A

  $a \neq 0$, not a unit, a prime $\leftrightarrow (a)$ is prime

  if $a|bc$ then $a|b$ or $a|c$

Primes are irreducible:

  if a is prime and $a = bc$ then $a|b$ or $a|c$

  if $a|b$ then b is a multiple of a and a is a multiple of b

  so $a \sim b$: $b = u \cdot a$ and $a = u^{-1} \cdot b$, differ by a unit irreducible elements might not be prime

$A = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} | a, b \in \mathbb{Z}\}$

$2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$

2 is irreducible and not prime, 2|4 but doesn't divide either on the right side

exists norm $N : z \mapsto z\bar{z}$

$a + b\sqrt{-3} \mapsto a^2 + 3b^2$

2 is irreducible

$2 = \alpha\beta$, $N(2) = N(\alpha)N(\beta)$, $4 = N(\alpha)N(\beta)$

but norms can never be 2 so one of these must be a unit ($N = 1$ implies ± 1)

In a principal ideal domain, irreducible elements are prime

(a) irreducible $\to (a)$ maximal $\to (a)$ is prime $\to a$ is prime

Unique factorization domain: every $a \neq 0$, unit has a factorization as a prod of irreducibles

this is unique up to reordering and transformation by units

$a \sim b$, a and b are associated, if $a = b \cdot u$ and $b = a \cdot u^{-1}$ for some unit u

Theorem: PIDs are UFDs

PID: $a = \pi_1 \cdots \pi_n = \sigma_1 \cdots \sigma_m$

σ_m prime so σ_m divides some π_i

can assume $\sigma_m | \pi_n$, $\phi_n = \sigma_m \cdot c$, c unit

proceed by induction on indices, end

A PID $a, b \in A$, $(a, b) = \{ax + by | x, y \in A\} = (g)$ since principal

$g = gcd(a, b)$: $(g) = (a, b) \ni a, b$

a and b are multiples of g, g divides a, b

t can't be factored as a product of irreducibles, (t) is maximal in this property

if t irreducible $t = t$; impossible

if t not irreducible, $t = r \cdot s$, r, s non-units

$(t) \subsetneq (r)$ $(t) \subsetneq (s)$

$A = \mathbb{Z}[\cdots (7)^{\frac{1}{2^N}} \cdots]$

7 is not a unit in A

Lemma: every element of A is "integral"

it satisfies an equation (monic polynomial) $x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0$

monic: first coefficient = 1

$c_i \in \mathbb{Z}$

integral ring

$1/7$ satisfies no such polynomial

7 can be factored on and on $n(7^{1/n})$; not a Noetherian ring

10/6

A-Modules (left modules)

M = abelian group with an action of scalar multiplication of A (= ring)

(same axioms as for an A-vector space except that $A \neq$field)

$End(M) = Hom(M, M)$

$M = \mathbb{Z}^n$, $End(M) = M(n, \mathbb{Z})$
action of A on M: a homomorphism of rings $A \xrightarrow{\varphi} End(M)$
$\varphi(a) \in End(M)$, $\varphi(a) : M \to M)$, $(\varphi(a))(m) := a \cdot m$
$f, g \in End(M)$: $fg = f \circ g$
Diversion: Fresh water (Chicago) algebra: $a \in A, m \in M, m^a, (m^{ab}) = (m^a)^b$
instead of $a \cdot m)$ or $a(m)$
Module properties
$\varphi(ab) = \varphi(a)\varphi(b)$
$(ab) \cdot m = a \cdot (b \cdot m)$
$a \cdot (m + m') = a \cdot m + a \cdot m'$
$\varphi(a) \in End(M)$
$(a + b) \cdot m = a \cdot m + b \cdot m$
$\varphi(a + b) = \varphi(a) + \varphi(b)$

Examples:
$A =$ field: an A-module is an A-vector space
Th: (uses choice) every A-vector space has a basis $\leftrightarrow$ all A-modules are free
M free on the set of generators $\{x_i\}_{i \in I}$
if every $m \in M$ is uniquely a finite A-linear combination of the x_i
For I, the free A-module on the set I
$\{\sum_{i \in I} a_i x_i | a_i \in A$ all but finitely many are $0\}$
could also notate $\{\sum_{i \in I} a_i i | a_i \in A$ all but finitely many are $0\}$, just indexed by I
Direct sums $\{M_i\}_{i \in I}$, $\oplus_{i \in I} M_i$
set of tuples indexed by I, with the i^{th} entry in M_i, all but finitely many entries are 0
$a \cdot (\cdots m_i \cdots)_{i \in I} = (\cdots a m_i \cdots)_{i \in I}$
Homomorphisms of A-modules M, N
$M \xrightarrow{h} N$, conditions of linearity $h(x + y) = h(x) + h(y)$, $h(a \cdot x) = ah(x)$
$A =$ field: linear map
$Hom_A(M, N)$ is an A-module
A map from a direct sum to a module uniquely determined by action on the summands

$$M \hookrightarrow \oplus_{j \in I} M_j \xrightarrow{h} N$$

$$M_i \xrightarrow{h_i} N$$

$$Hom_A(\oplus M_i, N) \xrightarrow{\alpha} \prod_{i \in I} Hom_A(M_i, N), h \mapsto (\cdots, h_i, \cdots)$$

α is a bijection
To map a free module to N is to choose the images of each of the generators
Unconstrained: can choose arbitrarily the images of the generators

Examples
$A = \mathbb{Z}$, $M =$ ab grp, $\mathbb{Z} \to End(M)$, $1 \mapsto \varphi(1) = id$, $2 \mapsto id + id$, $-1 \mapsto -id$
$A = A, I \subset A$ left ideal, $I =$A-module, $a \cdot i = ai \in I$

ring hom $A \to A'$, $M = A'$-module, $A \to A' \xrightarrow{\varphi} End(M)$, A'-modules $\mapsto$ A-modules
M = $\mathbb{Z}$-module, $n \geq 1$, $M^n = \oplus_{i=1}^{n} M$

$A = M(n, \mathbb{Z})$ acts on M^n by left matrix multiplication
could replace $\mathbb{Z}$ by some ring R, new construction
An exercise: A-modules $\leftrightarrow$ abelian groups, leftwards, $M \mapsto M^n$, rightwards, ?
Morita equivalence

Exact sequence $X \xrightarrow{h} Y \xrightarrow{g} Z$; $Im(h) = Ker(g)$ (implies $g \circ h = 0$, but even stronger)

can make these as long as we like $\cdots X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$
exact if exact at each place X_i, i.e. $Ker(f_{i+1}) = Im(f_i)$ for all i

Examples
$Y \xrightarrow{g} Z \xrightarrow{0} 0$, exact. g is surjective (epimorphism)
$0 \to X \xrightarrow{h} Y$, exact. h is injective (monomorphism)
$0 \to X \xrightarrow{h} Y \xrightarrow{g} Z \to 0$ is called a short exact sequence. $Y/h(X) \cong Z$
$X \xrightarrow{h} Y$, $0 \to Ker(h) \to X \xrightarrow{h} Im(h) \to 0$, exact, $X/Ker(h) \cong Im(h)$
$0 \to Im(h) \to Y \to Coker(h) \to 0$
$0 \hookrightarrow Ker(h) \hookrightarrow X \xrightarrow{h} Y \to Y/Im(h) = Coker(h) \to 0$

$N \to X \to Y$ $N \to Y$, $0 \to X \to Y \to Z \to 0$ exact. $Hom_A(N, X) \to Hom_A(N, Y)$

use a functor, get a $0 \to Hom(N, X) \to Hom(N, Y) \to Hom(N, Z) \to 0$
have exactness at $Hom(N, X)$, $Hom(N, Y)$
what about exactness at $Hom(N, Z)$?
equivalent statement: every homomorphism $N \to Z$ lifts to a homomorphism $N \to Y$
the entering map not necessarily surjective
e.g. $A = \mathbb{Z}, X = 2\mathbb{Z}.Y = \mathbb{Z}$ and $Z = Y/X = \mathbb{Z}/2\mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, lift does not exist
go from left to right using functor/construction $Hom_A(N, \cdot)$
this functor/construction is "left exact" but not "right exact/fully exact"
the class of modules with full exactness are the projective modules

10/8

(Tal)
A a ring and M, N modules

$Hom_A(M, N)$ is an abelian (group?/ring?)
if A is commutative, then it is an $A - module$

$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ (exact?)

$0 \to Hom_A(N, X) \to Hom_A(N, Y) \to Hom_A(N, Z) \to 0$
this sequence is left exact and exact at the center

surjectivity of the map $Hom_A(N, Y) \to Hom_A(N, Z)$

$Y \twoheadrightarrow Z$ via G, $h : N \to Z$
does H exists such that $g \circ H = h$?
the same question, rephrased:

suppose we map $Hom_A(N,Y) \to Hom_A(N,Z)$, taking H to h
is this map surjective?
an example of a case where it does not lift
take $h > 1$
$\mathbb{Z} \to \mathbb{Z}/h\mathbb{Z}$ surjective
identity $\mathbb{Z}/h\mathbb{Z}$, look for map $\mathbb{Z}/h\mathbb{Z} \to \mathbb{Z}$
map doesn't exist, no lifting

(1) Suppose $y \xrightarrow{g} Z$ is surjective.
If $Hom_A(N,Y) \xrightarrow{g_*} Hom_A(N,Z)$ is also surjective, we say N is projective
an equivalent statement: the functor $Hom_A(N,\cdot)$ is right exact
another equivalent statement: for all g,h there is a lifting H
$g : y \twoheadrightarrow Z$, $h : N \to Z$, $H : N \to y$
(2) given a sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} N \to 0$
if $\exists s$ such that $g \circ s = id_N$ we say that the sequence splits
all exact sequences split (misreading notes?)
given $y \xrightarrow{g} N \to 0$, we can find s such that $g \circ s = id$
(1) implies (2)
if N is projective, then the exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} N \to 0$ splits
take $h = id \to Z = N$
$y \xrightarrow{g} N$, $id : N \to N$, $N \to Y$
(3) the module N is a direct summand of a free module:
$\exists M$ such that $N \oplus M \cong F$ where $F = A\langle S\rangle$
(2) implies (3)
choose a set of generators of N, call it S. You can have $N \subset S$
induces $A\langle S\rangle \to N$ surjective
we have $f : N \to A\langle S\rangle$ by hypothesis
so $A\langle S\rangle = Ker(g) \oplus f(N)$
(3) implies (1)
$F = M \oplus N$ where F is free
want to show that if $g : Y \twoheadrightarrow Z$ then $Hom_A(N,Y) \xrightarrow{g_*} Hom_A(N,Z)$ is surjective
$F = A\langle S\rangle$ same as $Hom_A(F,X) = Maps(S,X)$
S generates F, so a map $F \to X$ is determined by S
$Hom_A(F,Y) = Maps(S,Y)$ and $Hom_A(F,Z) = Maps(S,Z)$
$Maps(S,Y) \to Maps(S,Z)$ obviously surjects, $Hom_A(F,y) \to Hom_A(F,Z)$ surjects
$s \to z \in Z$ and $Y \ni y \to z \in Z$
$Hom_A(M \oplus N, y) = Hom_A(M,Y) \times Hom_A(N,Y)$
have surjective $Hom_A(M \oplus N, y) \xrightarrow{\sigma} Hom_A(M \oplus N, Z)$
$Hom_A(M,Y) \times Hom_A(N,Y) \xrightarrow{(g_*,g_*)} Hom_A(M,Z) \times Hom_Z(N,Z)$
since σ surjective, (g_*,g_*) surjective and g_* surjective
Thus $Hom_A(N,Y) \xrightarrow{g_*} Hom_A(N,Z)$ surjective.

Diagram

$Hom(M \oplus N, y) \to Hom(N, Y)$ by $h \mapsto h \circ i$

$Hom(M \oplus N, y) \xrightarrow{g_*} Hom(M \oplus N, Z)$

$Hom(N, Y) \to Hom(N, Z)$

$Hom(M \oplus N, Z) \to Hom(N, Z)$

$g \circ h \mapsto (g \circ h) \circ i = g \circ (h \circ i)$

Diagram

$N \xrightarrow{h \circ i} Y$

$N \hookrightarrow M \oplus N$

$M \oplus N \xrightarrow{h} Y$

Examples: free modules are projective.

Any free module is a summand of another module that generates (word unsure) a free module

If A is a field, all A-modules are free.

$A = K \oplus K = \{(a, b)|a, b \in K\}$, K a field

$F = A = N \oplus M$ where $N = \{(a, 0)|a \in K\}$ and $M = \{(0, b)|b \in K\}$

Projective, but not free over A.

Suppose $N \cong A\langle S \rangle$, basis over $A \leftrightarrow S$

$N \cong A^n$, $dim_k N = 2h = 1$ (not sure about these figures)

$n > 1$, $A = M(n, k)$, $F = A$

$M \in A$, $x \in F$

$M_1 x$ matrix (not sure if right), $x = (c_1 \cdots c_n)$ n columns

$M \circ X = (M_{c_1}, \cdots, M_{c_n})$

$F = K^n \oplus \cdots \oplus K^n$

K^n projective, not free.

example (justification left for homework)

k a number field; that is, contains Q ($\mathbb{Q}$?) and $dim_Q k < \infty$

let $\alpha \in \mathbb{C}$ and α algebraic

$k = span(1, \alpha, \alpha^2, \cdots, \alpha^{n-1})$

k field

Diagram

$A \subset k$, $\mathbb{Z} \subset \mathbb{Q}$, A assoc with $\mathbb{Z}$, k with $\mathbb{Q}$

$A = \{\beta \in k | \beta$ satisfies a monic polynomial with integer coefficients$\}$

Theorem: the ring A is a Dedekind domain.

Definition of a Dedekind domain

I an ideal in A, then there exists $J \subset A$ (is it an ideal?) such that

$$IJ = \{\sum_{r=1}^{t} x_r y_r | x_r \in I, y_r \in J, t \geq 0\}$$

is principal

What is J?

Define $I^{-1} = \{y \in k | yI \subset A\} \supset A$
I^{-1} is an A-submodule of k.
$II^{-1} \subset A$, in fact $II^{-1} = A$
Theorem: If I is a nonzero ideal in a Dedekind Domain A, I is a projective.

10/13

Category
notation e.g. $\mathcal{A}$, (sets)
objects, and $Mor(A,B)$ a set of morphisms from the object A to the object B
axioms: every object has an identity morphism
composition of which preserves morphisms, etc.
define isomorphisms in terms of the existence of inverses
in some categories, bijections are not isomorphisms
Examples
for A a ring, the category of A-modules
morphisms of which are the A-linear homomorphisms $X \to Y$, $Hom_A(X,Y)$
pointed sets (X,x) in which a morphism is an $f : (X,x) \to (Y,y)$, $f(x) = y$
A-modules X,Y,Z,W and fixed X take as objects pairs (Y,f) where $f : Y \to X$
we have created a new category relative to X
category of partially ordered sets, whose morphisms are isotone maps

Functor
takes objects to objects, and also morphisms to morphisms
Diagram(F is the functor, f is a morphism, A, A' are objects)
$A \xrightarrow{F} F(A)$
$A \xrightarrow{f} A'$
$A \xrightarrow{F} F(A')$
$F(A) \xrightarrow{Ff} F(A')$
since the arrows go in the same direction, this desribes a covariant functor
if, say, $F(A') \xrightarrow{Ff} F(A)$, this would be a contravariant functor
Examples
forgetful functors from for instance (groups) $\to$ (sets) or (A-modules $\to$ (abelian groups)
Fix X. Functor from $A \in Ob(A) \to Mor(X,A)$ (morphisms in the category of sets)
or, contravariantly $A \in Ob(A) \to Mor(A,X)$
in A-modules, fix X and take N to $N \oplus X$
or from (sets) to (abelian groups) using the free group construction

Representable Functors
covariant $\mathcal{A} \xrightarrow{F}$ (sets)
Fix X. By the hom-functor h_X, $A \mapsto Mor(X,A)$.
Given an F, can it be written as a hom-functor?
That is, for some X, is $F \cong h_X$?

Those which can be are said to be represented by X (not a complete definition)
Fully defining a representable functor F
we need an $X \in \mathcal{A}$ and a $u \in F(X)$ such that for all A have a bijection

$$Mor(X, A) \to F(A)$$

if we have an $h : X \to A$, it induces a morphism $h_* : F(X) \to F(A)$
the lower-star signifies a covariant (push-forward)
a contravariant (pull-back) would be represented by an upper-star
can associate $h \in Mor(X, A)$ with $h(u)$ and this $h \mapsto h(u)$ is a bijection
epithet: to give an element of $F(A)$ is to give a map $X \to A$

Example of a Representable Functors
Fix a set S, let $\mathcal{A}$ be the category of abelian groups
Take $G \mapsto F(G) = Maps(S, G)$
Want an abelian group X such that $Maps(S, G) \cong Hom(X, G)$
Take X to be the free abelian group on S
The universal element u is the set map taking s to $1 \cdot s$
Diagram:
$X = \mathbb{Z}\langle S \rangle \xrightarrow{h} G$
$S \xrightarrow{u} \mathbb{Z}\langle S \rangle$
$S \xrightarrow{h_*(u)} G$
a set map in the category of sets is given by a group map from the free group

Another example
From $\mathcal{A}$ the category of abelian groups to sets
Fix $M, N \in \mathcal{A}$. Define the functor $A \mapsto Hom(M, A) \times Hom(N, A)$
to give a pair of maps $M \to A$, $N \to A$ is to give a map from the direct sum to A
Take $X = M \oplus N$ and $u \in F(X) = Hom(M, X) \times Hom(N, X)$
u is a universal pair of inclusions
to give a map of the direct sum is to give a map of the first and a map of the second

The uniqueness of (X, u)
if they represent the same functor, they are isomorphic in a canonical sense
no choice involved in the formulation of isomorphism
say (X, u) and (X', u') represent the functor F
then $Mor(X, A) \ni h \mapsto h_*(u) \in F(A)$ and $Mor(X', A) \ni h' \mapsto h'_*(u') \in F(A)$
taking the particular cases when $A = X'$, $A = X$
not totally sure if the next two lines are totally right
I remember he said in class that these are "the same" h as those in the above line
there is a bijection $Mor(X, X') \to F(X')$; so for some $h \in Mor(X, X')$, $h(u) = u'$
there is a bijection $Mor(X', X) \to F(X)$ so for some $h' \in Mor(X', X)$, $h'(u') = u$
the representing property of X and X' gives two morphisms
their compositions are the identity on X and the identity on X' (why?)

Tensor Products
    can be defined on noncommutative rings
    one must be a left-module and the other a right-module
    will be defined on a commutative ring $\mathcal{A}$ for simplicity
    A-modules X, Y, Z, M, N, T
    bilinear maps $Bil(X \times Y, Z)$: linear in each variable
    i.e. $Bil(X \times Y, Z) = Hom_A(X, Hom_A(Y, Z))$
two examples of bilinear maps
    $X, Y = k^n$ for k a field, $f(x, y) = det(x|y|c_1|\cdots|c_{n-2}) \in k$
    $x \in X$, $Y = Hom_A(X, A) \ni \varphi$ a linear form, $(x, \varphi) \mapsto \varphi(x)$

Define a functor whose representing element is T a tensor product.
    Fix X, Y in the category of A-modules and have $F : Z \mapsto Bil(X \times Y, Z)$
    $F(Z) = Mor(T, Z) = Hom_A(T, Z)$
    $u \in F(T)$ gives the universal bilinear map $u : X \times Y \to T$
    A homomorphism $T \to Z$ gives a bilinear map $X \times Y \to Z$
    i.e. there is a set bijection $Bil(X \times Y, Z) \leftrightarrow Mor(T, Z)$
    How one constructs such a T: next lecuture.
T has uniqueness property by canonical isomorphism.
    We do some amount of work to show this construction is possible
    Then we can abstract away this work because of the universality of T
    (last line unsure; maybe ask about it again)

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T a universal object: is $u \in F(T)$
    $\forall z \in F(A)$ exists unique $h \in Mor(T, A)$ such that $z = h_* u$
Representable Functor: example
    $\mathcal{A} =$ (rings), $F : \mathcal{A} \to$ (sets) 'forgetful', $F(A) = A$
    want a ring T and $u \in T$ such that $h \in Mor(T, A)$ corresponds to $h(u) \in A$
    Take $T = \mathbb{Z}[x]$, $u = x$
    $Hom(\mathbb{Z}[x], A)$ determined by image of x
    so mapping $\mathbb{Z}[x]$ to $A \leftrightarrow$ choosing elt of A

A non-representable functor: example
    $\mathcal{A} =$ (rings) and $F(A) = \{a \in A | a = b^2, b \in A\}$
    representability corresponds to automorphisms!
non-representability
    $A = \mathbb{Z}[x]$, $A \xrightarrow{\alpha} A$ an involution ($\alpha^2 = Id$) defined $f \mapsto (x \mapsto f(-x))$
    Assume F representable by (T, u); say $u = v^2$.
    In definition, take $A = \mathbb{Z}[x]$, $z \in F(A)$, $z = x^2$.
    Exists a unique homomorphism $h : T \to \mathbb{Z}[x]$ such that $h(u) = x^2$
    Here we're not using the notation h_* since h_* is the restriction of h to the squares
    $(\alpha h)(u) = (-x)^2 = x^2$

$h(u) = h(v^2) = x^2 \to (h(v))^2 = x^2 \to h(v) = x$ or $h(v) = -x$
$((\alpha h)(v))^2 = x^2 \to (\alpha h)(v) = -x$ when $h(v) = x$ and $(\alpha h)(v) = x$ when $h(v) = -x$
αh and h differ on v so they are distinct, violating uniqueness of h

Tensors again
Commutative ring A; $\mathcal{A} = (A\text{-modules})$; M, N fixed A-modules
$F(X) = Bil(M \times N, X)$
Theorem (construction): F representable by $T = M \otimes_A N$ and $u : M \times N \to M \otimes_A N$
all bilinear maps $M \times N \xrightarrow{b} X$:
unique homomorphism of A-modules $T \xrightarrow{h} X$, $h \circ u = b$
$(m, n) \mapsto m \otimes n$ pure tensors
Fact: all elements of T are sums of pure tensors

Example from linear algebra
$A = K$ a field, $V = M$, $W = N$
If V has dimension m we have a basis $v_1, \cdots, v_m$, similarly $w_1, \cdots, w_n$
$V \otimes_K W$ has dimension mn, basis $v_i \otimes w_j$
A similar expression can be found if we have rings and v_i, w_j finitely generate them

Tensor identities
$A \otimes_A N$: consider $Bil(A \times N, X) \leftrightarrow Hom_A(A, Hom_A(N, X)) = Hom_A(N, X)$
$T = N$, $u \in Bil(A \times N, N) : (a, n) \mapsto an$
Conclusion: $A \otimes_A N = N$ with universal bilinear map u
$(M_1 \oplus M_2) \otimes_A N \cong (M_1 \otimes_A N) \oplus (M_2 \otimes_A N)$
$Bil((M_1 \oplus M_2) \otimes N, X) = Hom_A(M_1 \oplus M_2, Hom_A(N, X))$
$= Hom_A(M_1, Hom_A(N, X)) \times Hom_A(M_2, Hom_A(N, X))$
$= Bil(M_1 \times N, X) \times Bil(M_2 \times N, X) = Hom_A(M_1 \otimes_A N, X) \times Hom_A(M_2 \otimes_A N, X)$
$= Hom_A((M_1 \otimes_A N) \oplus (M_2 \otimes_A N), X)$
tensor products commute with direct sums

Tensor Product: the construction
$Bil(M \times N, X) \subset Maps(M \times N, X) = Hom_A(A\langle M \times N \rangle, X)$
pass to a quotient of $A\langle M \times N \rangle$ satisfying conditions
conditions include $h((am, n)) = h(a(m, n))$
take T to be the largest quotient of $A\langle M \times N \rangle$ in which $(am, n) = a(m, n)$
this is the quotient by the submodule generated by $(am, n) - a(m, n)$, etc.
$u : M \times N \to T$ taking (m, n) to the image of (m, n) in the free module $:= m \otimes n$

$A = K$, $V = K \otimes \cdots \otimes K$ (m times), $W = K^n$
$V \otimes_K W = (K \otimes \cdots \otimes K) \otimes (K \oplus \cdots \oplus K)$ = big sum of $K \otimes K$s, $K \otimes_K K = K$
$M \otimes_A N \cong N \otimes_A M$, $m \otimes n \mapsto n \otimes n$
Consider bilinear maps $M \times N \to N \otimes_A M$, $(m, n) \mapsto n \otimes m$
universality: bilinear maps f factor through unique map $h : m \otimes n \mapsto n \otimes m$
$b : (m, n) \mapsto m \otimes n$

Tensors and sequences

$X \xrightarrow{f} Y$, $N \to X \otimes N \to Y \otimes N$ defined $f \otimes id$

Proposition: If f is onto, then $f \otimes id$ is surjective.

Need it to make all pure tensors in $Y \otimes N$, since these are generators

This is simple

A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, ($f$ injective, g onto, $Im(f) = Ker(g)$)

$X \otimes N \xrightarrow{f \otimes id} Y \otimes N \xrightarrow{g \otimes id} Z \otimes N \to 0$

call $F = f \otimes id$ and $G = g \otimes id$

$G \circ F = 0$, $Im(F) \subset Ker(G)$

proposition: this new sequence is exact; i.e. $Ker(G) \subset Im(F)$

thus the tensor product is right-exact

$\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z} = \mathbb{Z}/g\mathbb{Z}$ where $g = gcd(a, b)$

Example

$A = \mathbb{Z}$. $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, first map is multiplication by two

$N = \mathbb{Z}/2\mathbb{Z}$.

Get an exact sequence of $\mathbb{Z}/2\mathbb{Z}$s.