Math 245A

Fall 2015

Chapter 2

2.2 Groups

A group G is a 4-tuple $G = (|G|, \mu, \iota, e)$ with
  underlying set $|G|$
  law of composition μ
  inverse function ι
  neutral element e
(Exercise 2.2:1) A homomorphism from a group G to a group H is a function $\phi : G \to H$ satisfying the following for $a, b \in G$:
  $\phi(e_G) = e_H$
  $\phi(\iota_G(a)) = \iota_H(\phi(a))$
  $\phi(\mu_G(a,b)) = \mu_H(\phi(a), \phi(b))$
A more common representation of a group uses symbols $G = (|G|, \cdot, ^{-1}, e)$
(2.2.1) The conditions for a 4-tuple to be a group are as follows
  $(\forall x, y, z \in |G|)\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
  $(\forall x \in |G|)\ e \cdot x = x = x \cdot e$
  $(\forall x \in |G|)\ x^{-1} \cdot x = e = x \cdot x^{-1}$
(2.2.2) We may also say that a set $|G|$ with a map $|G| \times |G| \to |G|$ constitudes a group if
  $(\forall x, y, z \in |G|)\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
  there exists $e \in |G|$ such that $(\forall x) e \cdot x = x = x \cdot e$ and $(\forall x \in |G|)(\exists y \in |G|) y \cdot x = e = x \cdot y$
(2.2.1) consists of identities (universally quantified equations) and (2.2.2) does not
  note: universal quantification is a "for all" quantification
(Exercise 2.2:2)
  (i)
  (ii)
(Exercise 2.2:3)

2.3 Indexed Sets

An *I-tuple* of elements of X, $(x_i)_{i \in I}$ is formally defined as an $f : I \to X$
  The set of all functions from I to X is denoted X^I

2.4 Arity

The *arity* of an operation is, e.g., 1 if unary, 2 if binary, etc.
An I-ary operation on S is a map $S^I \to S$
Group: a set, a binary operation, a unary operation, and a distinguished element
Can think of the identity as a 0-ary/zeroary operation of the structure
S^0 has exactly one map, $\varnothing \to S$, so a map $S^0 \to S$ is determined by one element
Note these are not strictly identical since one is a map and the other the element itself
But they are in 1-to-1 correspondence and give equivalent information

2.5 Group-theoretic terms

A *group-theoretic relation* in $(\eta_i)_I$ is an equation $p(\eta_i) = q(\eta_i)$ holding in G
p and q are are *group-theoretic terms* which we formally define
The terms in the elements of X under the formal group operations μ, ι, e form a set T:
given with functions $symb_T : X \to T$, $\mu_T : T^2 \to T$, $\iota_T : T \to T$, and $e_T : T^0 \to T$
such that each map is one-to-one, its images disjoint, and T is the union of those images
and T is generated by $symb_T(X)$ under the aforementioned operations
that is, T has no proper subset containing $symb_T(X)$ and closed under those operations
We can represent these terms, for groups, using strings of symbols
We need full parentheses notating order of operations to ensure disjoint images
A set-theoretic approach dispenses with strings and allows for infinite arities
For the example of a group, we would have (using ordered pair, 3-tuple, etc.):
for $x \in X$, $symb_T(x) := (*, x)$
for $s, t \in T$, $\mu_T(s,t) := (\cdot, s, t)$
for $s \in T$, $\iota_T(s) := (^{-1}, s)$
and $e_T = (e)$
and by set theory, no element can be written as such an n-tuple in more than one way

2.6 Evaluation

Given a set map $f : X \to |G|$ for a group G
Recursive evaluation of $s_f \in |G|$ given an X-tuple of symbols $s \in T = T_{X,\cdot,-1,e}$
if $s = symb_T(x)$ for some $x \in X$, then $s_f := f(x)$
$s = \mu_T(t,u) \to s_f = \mu_G(t_f, u_f)$, assuming that given $t, u \in T$ we know $t_f, u_f \in |G|$
similarly, $s = \iota_T(t) \to s_f = \iota_G(t_f)$, assuming we know t_f given t
finally $s = e_T \to s_f = e_G$
Varying f in addition to T gives an evaluation map $(T_{X,\cdot,-1,e}) \times |G|^X \to |G|$
Alternatively, fixing $s \in T$ gives a map $s_G : |G|^X \to |G|$
these represent substitution into s
these s_G are the *derived n-ary operations* (aka *term operations*) of G
distinct terms can induce the same derived operation
e.g. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in general or others for certain groups
Examples of derived operations on groups
conjugation $\xi^\eta = \eta^{-1}\xi\eta$ (binary)

commutator $[\xi,\eta] = \xi^{-1}\eta^{-1}\xi\eta$ (binary)
squaring (unary)
δ (Exercise 2.2:2)
σ (Exercise 2.2:3)

Class Question #1

end of Section 2.6: Unimportant

The last example of a derived operation on groups cited the trivial "second component" function, $p_{3,2}(\xi,\eta,\zeta) = \eta$ induced by $y \in T_{\{x,y,z\},^{-1},\cdot,e}$. I wasn't entirely sure how this derived operation would be represented as an element of $T_{\{x,y,z\},^{-1},\cdot,e}$. Would $p_{3,2}$ be the element $(*,y)$ (in the set-theoretic notation)?

Terms in other families of operations

An Ω-*algebra* is a system $A = (|A|,(\alpha_A)_{\alpha\in|\Omega|})$
here $|A|$ is some set, and for each $\alpha \in |\Omega|$, $\alpha_A : |A|^{ari(\alpha)} \to |A|$
note that often people will use $n(\alpha)$ (rather than $ari(\alpha)$) for the arity of an operation α
e.g. for a group, $|\Omega| = \{\mu,\iota,e\}$, $ari(\mu) = 2$, $ari(\iota) = 1$, and $ari(e) = 0$

Lecture 8/28

Operations, terms, algebra

revisit the difficulty with distinguished elements as a zeroary operation when dealing with the empty set

the idea behind distinguishing between terms is e.g. to have distinct objects that we can compare
$(x\cdot y)\cdot z \neq x\cdot(y\cdot z)$ as terms, allowing
$(x\cdot y)\cdot z = x\cdot(y\cdot z)$ to be a useful statement about groups

set-theoretic approach, infinite arity
(μ,s,t)
$(\mu,(s,t))$
$\alpha_T : T^X \to T$ using $(\alpha,(S_X)_{x\in X})$
X here shall be some cardinal

Next reading: free groups

$x,y,z \in G$ and $\xi,\eta,\zeta \in H$
when can we have a homomorphism $G \to H$
if and only if the relations that hold in G hold in H for the corresponding elements

Exercises in today's reading

2.7:3

can't have $s(,,,,,...) = s'(,,,,...) = s''(,,,,...)$ where the s" term is the same as the s term

2.2:2 and 2.2:3

$\delta_G(x,y) = xy^{-1}$ and $\sigma_G(x,y) = xy^{-1}x$

$G = \mathbb{Z}$ knowledge of the identity

$x * +y = (x-1) + (y-1) + 1$

Chapter 3

3.1 Motivation

3.2 The logician's approach: construction from terms

3.3 Free groups as subgroups of big enough direct products

3.4 The classical construction: groups of words

Class Question #2

near 3.3.1 Important

The question concerns the set of all groups G (I'll call it X) whose underlying sets $|G|$ are subsets of S, some countably infinite set. I wanted to clarify for myself why for any countable group H we can find an isomorphism from one of these groups to H. Is it sufficient to justify the statement by declaring that X contains all countable groups up to isomorphism and hence for some $G' \in X$, G' is isomorphic to H? For some reason this feels like incomplete justification to me, and there may be some set-theoretic considerations that may need to be explicated more clearly.

Lecture 8/31

Free Groups: the motivation

factor-set: given a set and an equivalence relation, the set of equivalence classes

... as subgroups of big products

If G generated by an X-tuple of elements then has cardinality $\leq max(card(X), \aleph_0)$

Chapter 4.1-4.5

4.1 The subgroup and normal subgroup of G generated by $S \subset |G|$

$\langle S \rangle$ contains S and is contained in every subgroup which contains S
$\forall x \in \langle S \rangle$ $x = e$ or $x = \prod s_i$, $s_i \in S$ or $s_i^{-1} \in S$
$\langle S \rangle$ is also the image of the map into G of the free group F on S induced by the inclusion map $S \to |G|$
There is additionally a least *normal* subgroup of G containing S.

4.2 Imposing relations on a group. Quotient groups

Quotient groups: homomorphisms causing certain elements to fall together.
Satisfies e.g. $(\forall i \in I) f(x_i) = f(y_i) \leftrightarrow (\forall i \in I) f(x_i y_i^{-1}) = e$
A set of elements annihilated by a group homomorphism form a normal subgroup.
Leads to $q : G \to G/N$, where N is this normal subgroup.
We have a quotient map and a quotient group.
This map has the universal property desired:
For every homomorphism $h : G \to K$ satisfying the above, $\exists! g : N \to K$, s.t. $h = g \circ q$.
This construction *imposes the relations* $x_i = y_i (i \in I)$ on G, forming $G/(x_i = y_i | i \in I)$.
For G a group, a G-set is a pair $S = (|S|, m)$, $|S|$ a set and $m : |G| \times |S| \to |S|$, satisfying
$(\forall s \in |S|, g, g' \in |G|)$ $g(g's) = (gg')s$
$(\forall s \in |S|)$ $es = s$
That is, a set on which G acts by permutations.
A homomorphism $S \to S'$ of G-sets (for G fixed) is a map $a : |S| \to |S'|$ satisfying
$(\forall s \in |S|, g \in |G|)$ $a(gs) = ga(s)$
The set of left cosets of H in G is $|G/H|$ and a typical coset $[g] = gH$.
Then $|G/H|$ is the underlying set of a left G-set G/H by $g[g'] = [gg']$

4.3 Groups presented by generators and relations

Let X be a set, T the set of all group-theoretic terms in X, and $R \subset T \times T$.
$\exists$ a universal example of a group with X-tuples of elements satisfying the relations R.
That is, there is a pair (G, u) with G a group and $u : X \to |G|$ satisfying:
$(\forall (s,t) \in R)$ $s_u = t_u$
such that for any group H and X-tuple v of elements in H satisfying

$$(\forall (s,t) \in R)\ s_u = t_u$$

$\exists! f : G \to H$ a homomorphism, such that $v = fu$
The pair (G, u) is determined up to canonical isomorphism by these properties
The group G is generated by u(X)
Proving the existence of such a universal construction.
Two ways: construction from terms and subgroups of direct products.
In these approaches, apply the additional conditions to the group axioms.

A proof that builds upon prior constructions:

Let (F, u_F) be the free group on X and N be a normal subgroup.

Define N such that it is generated by $\{s_{u_F} t_{u_F}^{-1} | (s,t) \in R\}$

Take a canonical map $q : F \to F/N$.

Then the pair $(F/N, q \circ u_F)$ has the desired universal properties.

As a consequence of the universal properties of free and quotient groups.

Class Question #3

3.3 proof of a universal group satisfying relations Important

I am unsure about the maps in the diagram describing the construction of the pair $(F/N, qu_F)$ using the free group on X and an appropriate quotient group. The map $q : F \to G$ is a map between groups as indicated in the right-sided diagram, whereas it is composed to form a set map $u : X \to |G|$ as in the left-hand diagram. What sort of distinctions between q as set-map and q as group-map do I need to consider here? I feel that I might not be clearly expressing the source of my confusion, so I apologize for that, but maybe my question betrays some fundamental misunderstanding about the nature of free groups or quotient groups to be cleared up.

P.S. I do not yet see how to show X contains all countable groups up to isomorphism, but hope to spend some more time thinking about it.

4.4 Abelian groups, free abelian groups, and abelianizations

4.5 The Burnside Problem