

Math 206

Fall 2015

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Definitions

A *norm* on a vector space X (over F) is a function $\|\cdot\| : X \rightarrow \mathbb{R}^+$ such that

$$\|x\| = 0 \text{ iff } x = 0$$

$$\|\alpha x\| = |\alpha| \|x\| \text{ (for } \alpha \in F)$$

$$\|x + y\| \leq \|x\| + \|y\|$$

An *algebra* \mathcal{A} over F is a vector space with distributive \cdot satisfying

$$cx \cdot y = c(x \cdot y)$$

$$x \cdot cy = c(x \cdot y) \text{ for all } c \in F$$

A *normed algebra* over \mathbb{R} or \mathbb{C} is an algebra \mathcal{A} equipped with (vector space) norm satisfying

$$\|ab\| \leq \|a\| \|b\| \text{ for all } a, b \in \mathcal{A}$$

A norm on \mathcal{A} induces a metric

$$d(a, b) = \|a - b\| \text{ on } \mathcal{A} \text{ and therefore a topology}$$

if \mathcal{A} is complete for this norm, it is a *Banach algebra*

To figure out (use <https://www.math.ksu.edu/nagy/real-an/2-05-b-alg.pdf>)

Supposing \mathcal{A} is not necessarily complete

$$\|ab\| \leq \|a\| \|b\| \text{ gives uniform continuity on the product}$$

hence the norm can be extended to the completion $\tilde{\mathcal{A}}$ to form a Banach algebra

A metric space M is complete if all Cauchy sequences converge to an element of M

The completion \tilde{M} is all equivalence classes of Cauchy sequences where

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim_{n \rightarrow \infty} d(a_n - b_n) = 0$$

Examples

For M a compact space, $C(M)$

the set of continuous \mathbb{R}/\mathbb{C} -valued functions on M

pointwise operations

$$\|f\|_\infty = \sup\{|f(x)| : x \in M\}$$

For M locally compact, $C_\infty(M)$

the set of continuous \mathbb{R}/\mathbb{C} -valued functions on M vanishing at ∞
 vanishing at ∞ : $\forall \epsilon \exists$ a compact subset of M , outside of which $f < \epsilon$
 note that this is non-unital (lacks an identity)

For $\mathcal{O} \subset \mathbb{C}^n$ open

$H^\infty(\mathcal{O})$ the set of all bounded holomorphic functions on \mathcal{O}

(M, d) metric space and $f \in C(M)$

Lipschitz constant (which can be $+\infty$) $L_d(f) = \sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in M, x \neq y\}$

The Lipschitz functions $\mathcal{L}_d(M, d) = \{f : L(f) < \infty\}$

These form a dense subalgebra of $C(M)$ and are in fact a $*$ -subalgebra

$\|f\|_d := \|f\|_\infty + L_d(f)$, can be shown as a normed-algebra norm

$L_d(M, d)$ is complete for this norm

so $L_d(M, d)$ is a Banach algebra

L_d is a seminorm on $\mathcal{L}_d(M, d)$ since it takes value 0 on the constant functions
 can recover d from L_d

M a differentiable manifold (e.g. $T = \mathbb{R}/\mathbb{Z}$ the circle)

$C(M) \supseteq C^{(1)}(M)$ the singly-differentiable functions

$f \in C^{(2)}(T) \rightarrow Df : T_x M \rightarrow \mathbb{R}, \mathbb{C}$

with Df the derivative and T_x the tangent space

If we put on a Riemannian metric, define $\|f\|^{(1)} = \|f\|_\infty + \|Df\|_\infty$

If $f \in C^{(1)}(T) : \|f\|^{(1)} = \|f\|_\infty + \|f'\|_\infty$

Banach algebra norm, for which this space of functions is complete

For the circle, $C^{(2)}(T) \rightarrow \|f\|^{(2)} = \|f\|_\infty + \|f'\|_\infty + \frac{1}{2}\|f''\|_\infty$

the factor $\frac{1}{2}$ ensures that this satisfies the normed algebra condition

$$C^{(n)}(T) = \sum_{k=0}^n \frac{1}{k!} \|f^{(k)}\|_\infty$$

For $C^\infty(T)$ using the collection of norms $\{\|\cdot\|^{(n)}\}_{n=1}^\infty$ yields a Fréchet algebra

A Fréchet algebra has a topology defined by a countable family of seminorms

that respect the algebra structure and is complete (**clarify**)

non-commutative algebras

X a Banach space

$\mathcal{B}(X)$ the algebra of bounded operators on X

$\|\cdot\|$ operator norm \rightarrow Banach algebra

Any closed subalgebra of $\mathcal{B}(X)$ is a Banach algebra

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Sketch of the course

X a Banach space, $B(X)$ bounded functions on the space

\mathcal{H} a Hilbert space, $\mathcal{B}(\mathcal{H})$ bounded operators on the space

for $T \in \mathcal{B}(\mathcal{H}) \exists$ adjoint operator $T^* \in \mathcal{B}(\mathcal{H})$

$\langle T\xi, \eta \rangle = \langle \eta, T^*\xi \rangle$ for $\xi, \eta \in \mathcal{H}$

adjoint is additive, conjugate linear, $T^{**} = T$, $(ST)^* = T^*S^*$

An algebra A over \mathbb{R} or \mathbb{C} is a $*$ -algebra if it has a $*$: $A \rightarrow A$ satisfying

certain properties (look up)

A **-normal algebra* is a normal *-algebra such that

$$(\forall a \in A) \|a^*\| = \|a\|$$

A *Banach *-algebra* is a *-normal algebra that is a Banach algebra.

For any $T \in \mathcal{B}(\mathcal{H})$, have $\|T^*T\| = \|T\|^2$ (**check: parse through defns**)

For M a locally compact space, $A = C_\infty(M, \mathbb{C})$, $f^* := \bar{f}$ is a Banach *-algebra

Also have $\|f^*f\| = \|f\|^2$ (**verify: should be easier than the other**)

Little Gelfand-Naimark theorem:

Let A be a commutative Banach *-algebra satisfying $\|a^*a\| = \|a\|^2$.

Then $A \cong C_\infty(M)$ for some locally compact M.

One view of the “spectral theorem”

Let $T \in \mathcal{B}(\mathcal{H})$ with $T^* = T$

Let A be the closed subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T and I (i.e. $p(T) := \sum \alpha_k T^k$)

Polynomials closed or stable under *

If $S \in A$ then $S^* \in A$ (i.e. A is a *-subalgebra of $\mathcal{B}(\mathcal{H})$)

So A is a Banach *-subalgebra satisfying $\|S^*S\| = \|S\|^2$

Moreover, A is commutative. (unital, since generated by I)

Then by the Little Gelfand-Naimark theorem, $A \cong C(M)$

Indeed $M \subset \mathbb{R}$, the spectrum of T

If \mathcal{H} is finite dimensional, then M is the set of eigenvalues of T

T is normal if $TT^* = T^*T$

A C*-algebra is a Banach *-algebra over \mathbb{C} satisfying

$$\|a^*a\| = \|a\|^2$$

Theorem: A commutative C*-algebra is $\cong C_\infty(M)$.

Big Gelfand-Naimark Theorem: (Math 208, C*-algebras)

Any C*-algebra is \cong to a closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} .

Tangent

algebraic topology, differential geometry, Riemann manifolds, “non-commutative geometry” (Connes)

A *von-Neumann algebra* is a *-subalgebra of $\mathcal{B}(\mathcal{H})$

which is closed under the strong operator topology.

Every commutative von-Neumann algebra is $\cong L^\infty(X, S, \mu)$ (measure spaces) acting on $L^2(X, S, \mu)$ by positive sldkjfalksdjf

For group G, $\alpha : G \rightarrow \text{Auto}(X) \subseteq \mathcal{B}(X)$

$\text{Auto}(X)$ a Banach space

Look at subalgebra of $\mathcal{B}(X)$ generated by $\alpha(G)$.

Leads to considering $l'(G)$ with product $(f \star g)(x) = \sum f(y)g(y^{-1}x)$ convolution

$$f^*(x) = \overline{f(x^{-1})}$$

Banach *-algebra, G commutative \rightarrow Fourier transform

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K a field, X a set, $\mathcal{F}(X, K)$ the set of all K-valued functions on X with pointwise operations

Given $f \in \mathcal{F}(X, K)$.

Let $\lambda \in K$. Then $\lambda \in \text{range}(f)$ exactly if $(f - \lambda 1)$ is not invertible.

For any $a \in A$, the *spectrum* of a is $\{\lambda \in K : a - \lambda 1_A \text{ is not invertible in } A\}$.

The spectrum depends on the containing algebra

Assuming that this algebra A (over the field K) has an identity 1_A

Example: Let $A = C([0, 1])$, and $B = \text{polynomials}$, viewed as a dense subalgebra of A .

Let p be a polynomial of degree ≥ 2 . Then

$$\sigma_a(p) = p([0, 1]).$$

$$\sigma_B(p) = \mathbb{R}, \mathbb{C}$$

Let A be a Banach algebra with 1 ($\|1\| = 1$), and $a \in A$.

If $\|a\| < 1$, then $1 - a$ is invertible, and $\|(1 - a)^{-1}\| \leq \frac{1}{1 - \|a\|}$

Proof:

$$\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n \quad (a^0 := 1_A)$$

For any $n > 0$, let $s_n = \sum_{k=0}^n a^k$.

Show that $\{s_n\}$ is a Cauchy sequence.

If $n > m$, $\|s_n - s_m\| = \|\sum_{k=m+1}^n a^k\| \leq \sum_{k=m+1}^n \|a\|^k$.

Given $\epsilon > 0 \exists N$ such that if $m, n \geq N$ then $\sum_{k=m+1}^n \|a\|^k \leq \epsilon$

So $\{s_n\}$ is a Cauchy sequence.

By completeness there is a $b \in A$ with $s_n \rightarrow b$ as $n \rightarrow \infty$.

Want to show $b = (1 - a)^{-1}$.

$$b(1 - a) = \lim_{n \rightarrow \infty} (s_n(1 - a))$$

$$= \lim_{n \rightarrow \infty} (1 + a + a^2 + a^3 + \dots + a^n - (a + a^2 + a^3 + \dots + a^{n+1}))$$

$$= \lim_{n \rightarrow \infty} (1 - a^{n+1}) = 1$$

Then $1 - (1 - a)$ is invertible, i.e. a is invertible.

$$\|(1 - a)^{-1}\| = \lim \|s_n\| \leq \lim \sum_{k=0}^n \|a\|^k = \frac{1}{1 - \|a\|} \quad (\|1\| = 1)$$

$\|ab\| \leq \|a\|\|b\|$: can very easily check that multiplication is cts (do this?)

Corollary: If $a \in A$ and $\|1 - a\| < 1$ then a is invertible, and $\|a^{-1}\| \leq \frac{1}{1 - \|1 - a\|}$

I.e. the open unit ball about 1 consists of invertible elements.

Let $a \in A$. Let L_a, R_a be the operators of left and right multiplication by a on A .

$a \rightarrow L_a$ is an algebra homomorphism of A into $\mathcal{L}(A)$ (linear operators on A)

$L_a L_b = L_{ab}, R_a R_b = R_{ba}$ (R is an antihomomorphism)

$1 \in A$

If a is invertible, then so is $L_a, L_a L_{a^{-1}} = I_a$

Then if A is a normed algebra, $\|L_a\| = \|a\|$

$$\|L_{ab}\| = \|ab\| \leq \|a\|\|b\|$$

$$\|L_a 1_a\| = \|a\|$$

so if $a \in A$ is invertible, then L_a is a homeomorphism of A onto itself.

Thus if A is a Banach algebra, with 1 , and a is invertible:

$\{L_a b : \|1 - b\| < 1\}$ is an open neighborhood of a consisting of invertible elements

Let $GL(A)$ be the set of invertible elements of A . (general linear group)

Then (for A a unital Banach algebra) $GL(A)$ is an open subset of A .

(Fails for $\text{Poly} \subseteq C([0, 1])$)

Two Fréchet algebras, for one, $GL(A)$ is an open subset, for another it isn't.

ask about this?: not sure what he was talking about

$$C^\infty(T), \|f^{(n)}\|$$

$\mathbb{C}(\mathbb{R})$ cont fns on \mathbb{R} (or \mathbb{C}) maybe unbounded

For each n let $\|f\|_n = \sup\{|f(t)| : |t| \leq n\}$

Corollary: For A a Banach algebra with 1 and $a \in A$, $\sigma(a)$ is a closed subset of \mathbb{C}

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Proposition: Let A be a unital Banach algebra and $a \in A$.

Then $\sigma(a)$ is a closed subset of \mathbb{C} or \mathbb{R} . If $\lambda \in \sigma(a)$ then $\|\lambda\| \leq \|a\|$.

Proof: $\sigma(a) = \{\lambda : (a - \lambda) \text{ is not invertible}\}$

Its complement, the *resolvent set*, of a is $\{\lambda : (a - \lambda) \in GL(A)\}$, is open.

If $|\lambda| > \|a\|$ then $(\lambda - a) = \lambda(1 - \frac{a}{\lambda})$, $\|a/\lambda\| < 1$

so $(\lambda - a)$ is invertible, ie $\lambda \in \sigma(a)$.

Over \mathbb{R} , can have $\sigma(a) = \emptyset$, e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

"If $a \in GL(A)$ and b is close to a then b^{-1} is not much bigger than a^{-1} ".

Let $\mathcal{O} = \{c : \|1 - c\| < 1/2\}$

So c is invertible, and $\|c^{-1}\| \leq \frac{1}{1 - \|1 - c\|} \leq 2\}$

Let $b \in a\mathcal{O}$, so $b = ac$ for $c \in \mathcal{O}$, then $\|b^{-1}\| = \|c^{-1}a^{-1}\| \leq 2\|a^{-1}\|$.

For $a, b \in GL(A)$.

$$b^{-1} - a^{-1} = b^{-1}(a - b)a^{-1}$$

$$\text{Thus } \|b^{-1} - a^{-1}\| \leq \|b^{-1}\| \|a - b\| \|a^{-1}\|.$$

So $b \rightarrow b^{-1}$ is continuous for the norm.

So $GL(A)$ is a topological group for topology from norm.

$$b^{-1} = (1 + b^{-1}(a - b))a^{-1}$$

On $\rho(a)$ (the resolvent set, complement of the spectrum) define the resolvent of a

This is the function $R(a, \lambda) = (\lambda - a)^{-1}$

$R(a, \lambda)$ is an analytic function on $\rho(a)$.

Proof: Let $f(z) = R(a, z)$.

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \frac{(z+h-a)^{-1} - (z-a)^{-1}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} (z+h-a)^{-1} ((z-a) - (z+h-a)) (z-a)^{-1}$$

$$= \lim_{h \rightarrow 0} -(z+h-a)^{-1} (z-a)^{-1} = -(z-a)^{-2}$$

$$f'' = +z(z-a)^{-3}$$

Given $z_0 \in \rho(a)$

Will use $b^{-1} = (1 + b^{-1}(a - b))a^{-1}$ and $f(z) = (z - a)^{-1} = \sum c_n (z - z_0)^n$

$$f(z) = (z - a)^{-1}$$

$$b \rightarrow z - a$$

$$f(z) = (1 + (z - a)^{-1}((z_0 - a) - (z - a)))(z_0 - a)^{-1}$$

$$= (1 + (z - a)^{-1}(z_0 - z))(z_0 - a)^{-1}$$

where $(z - a)^{-1}(z_0 - z) \leq 1$ then the above

$$= \sum (-1)^n (z - a)^{-n} (z - z_0)^n = \sum (-1)^n (z - a)^{-n-1} (z - z_0)^n$$

a proper power series expansion.

Examine $R(a, z)$ at ∞ .

$$R(a, z^{-1}) = (z^{-1} - a)^{-1} = \frac{1}{z^{-1} - a}$$

$= z(1 - za)^{-1}$ (for small z , ie $\|za\| < 1$)

$R(a, z^{-1})$ approaches 0 as $z \rightarrow 0$.

defn $R(a, 0^{-1}) = 0$, see $R(a, z)$ is analytic at ∞ .

Theorem: For a Banach algebra over \mathbb{C} with 1, and for any $a \in A$,

$\sigma(a) \neq \emptyset$, that is, the spectrum is non-empty.

Proof: Suppose that $\sigma(a) = \emptyset$.

Then $R(a, z)$ is defined on all of \mathbb{C} and is bounded.

By Liouville's, $R(a, z)$ is constant, $= 0$, $(a - z)^{-1} = 0 \forall z$

Why can we use Liouville's in this Banach space case?

Let A' be the dual Banach space to A .

For $\varphi \in A'$, $z \mapsto \varphi(R(a, z))$ is a \mathbb{C} -valued analytic function.

So set $\varphi(R(a, z)) = 0 \forall z, \forall \varphi$

so $R(a, z) = 0$.

Knowing that there is anything in here is the Hahn-Banach Theorem, depending on the axiom of choice.

Theorem (Gelfand-Mazur)

Let A be a unital Banach algebra over \mathbb{C} .

If every nonzero element is invertible, then $z \rightarrow z1_A$ is an isomorphism from \mathbb{C} onto A .

Proof:

Given $a \in A$ let $z \in \sigma(a) \neq \emptyset$.

So $(z - a)$ is not invertible so $z - a = 0$.

Fails over \mathbb{R} since have \mathbb{R} , \mathbb{C} , quaternions

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Let A be an algebra or ring. Ideals I , left, right, 2-sided.

A/I , for a left ideal get a left A -module, right ideal get a right A -module

Two-sided get an algebra or ring.

Let A be a normed algebra, and if I is an ideal in A .

Then \bar{I} (the closure) is again an ideal in A .

$\{a_n\} \subset I, a_n \rightarrow c \in A$ then $ba_n \rightarrow bc$

Proposition: If A is a unital Banach algebra and if I is a proper ideal in A .

Then \bar{I} is proper.

Proof:

Use $GL(A)$ is open. The original ideal cannot contain any invertible elements.

The complement of the invertible elements is going to be closed.

The ideal is in its closure; the closure is in the complement of the invertible elements.

So the closure will not contain any invertible elements.

Counter-example: \mathbb{C} locally compact, $C_c(\mathbb{R}) \subset C_\infty(\mathbb{R})$

In fact $C_c(\mathbb{R})$ is the minimal dense ideal in $C_\infty(\mathbb{R})$

Lack of identity element.

Counter-example: Look at A the polynomials viewed as a subset of $C([0, 1])$

Using the sup-norm.

Lack of completeness.

Let $I = \{p : p(2) = 0\}$ is an ideal, in fact a maximal proper ideal.

This ideal of polynomials will be dense inside A and dense inside all polynomials.

Counter-example: $A = C(\mathbb{R})$, including unbounded, $I = C_c(\mathbb{R})$, compact open topology

Compact in here for the Fréchet open topology.

Corollary: Let A be a unital Banach algebra.

Then every maximal ideal is closed.

Taking ideals to form quotients.

Recall that if X is a normed vector space and if Y is a vector subspace of X :

Then we can form the quotient vector space X/Y .

Have the evident $\pi : X \rightarrow X/Y$ by $\pi(x) = x + Y$.

Set $\|\pi(x)\| = \inf\{\|x - y\| : y \in Y\}$ i.e. the distance from x to the subspace Y .

It is easily seen that this is a seminorm.

Problem: if Y is not closed, then it is not a norm, since $\|\pi(y)\| = 0$ for $y \in \bar{Y}$.

Therefore, if Y is closed, then $\|\cdot\|_{X/Y}$ is a norm.

(Should know from 202B) If X is a Banach space and Y is a closed subspace.

Then X/Y with the norm defined above is a Banach space.

Trickier to prove, but a true statement.

Proposition: Let A be a normed algebra and let I be a closed ideal in A .

(can be left or right or two-sided)

Then if I is a two-sided ideal, then A/I with $\|\cdot\|_{A/I}$ is a normed algebra.

That is $\|\pi(a)\pi(b)\| \leq \|\pi(a)\| \|\pi(b)\|$.

Then if I is a left ideal, so that A/I with $\|\cdot\|_{A/I}$ is a left A -module.

$\|a\pi(b)\|_{A/I} \leq \|a\|_A \|\pi(b)\|_{A/I}$

And if I is a right ideal, so that A/I with $\|\cdot\|_{A/I}$ is a right A -module, similar.

Proof:

For I a two-sided ideal.

If $c, d \in I$, then $\|\pi(a)\pi(b)\|_{A/I} \leq \|(a - c)(b - d)\| = \|ab - (cb + ad - cd)\|$

where $cb + ad - cd \in I$.

Take inf over $c, d \in I$.

$\leq \|a - c\| \|b - d\|$

Proposition: If A is a Banach algebra and I is a closed 2-sided ideal, then A/I is a Banach algebra.

An algebra or ring is *simple* if it contains no proper 2-sided ideals.

e.g. $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(K)$ for K a field.

Proposition: Let A be an algebra or ring and let I be a maximal 2-sided ideal in A .

The A/I is a simple algebra/ring.

Corollary: If A is a Banach algebra and if I is a maximal closed 2-sided ideal.

Then A/I is a simple Banach algebra.

Let A be a commutative algebra/ring with 1.

If $a \in A$ and a is not invertible, then Aa is a two-sided ideal.

And $1 \notin Aa$ (else it would be invertible).

Thus Aa is a proper ideal.

Corollary: If A is a commutative algebra/ring with 1 and if A is simple, then A is a field.

A is simple: every nonzero element is invertible.

Since if one weren't we would have a proper ideal.

Theorem: Let A be a commutative Banach algebra with 1.

Let I be a maximal ideal in A (which of course is necessarily closed).

Then $A/I \cong \mathbb{C}$ (isomorphic in every sense).

Proof:

Banach-Mazur theorem applied to the previous discussion.

$c \in A/I, z \in \sigma(c), c - z1_A$ is noninvertible by def'n spectrum.

Since it is a field, $c - z1_A = 0$.

From a maximal ideal, I , get a homomorphism $\phi : A \rightarrow \mathbb{C}$, unital

($\phi(1) = 1$) such that $I = \ker(\phi)$

Let A be a Banach algebra with 1 and let $\phi : A \rightarrow \mathbb{C}$ be a homomorphism.

Then ϕ is continuous and $\ker(\phi)$ is a maximal 2-sided ideal in A .

$\|\phi\| = 1$

Lemma: For any given $a \in A, \phi(a) \in \sigma(a)$

Proof: $\phi(a - \phi(a)1_A) = 0$ so $a - \phi(a)1$ is not invertible.

so $\phi(a) \in \sigma(a)$

Then $\|\phi(a)\| \leq \|a\|$, so $\|\phi\| \leq 1$ but $\phi(1) = 1$.