

Math 250A

Fall 2015

8/27

Group Action

A group G acts on a set S :

$$G \times S \rightarrow S$$

$$(g, s) \mapsto g \cdot s$$

$$e \cdot s = s$$

$$(gg') \cdot s = g \cdot (g' \cdot s)$$

Alternatively,

$$\phi : G \rightarrow \text{Perm}(S)$$

ϕ is a homomorphism (gives the corresponding properties)

$$(\phi(g))(s) = g \cdot s$$

Examples of Group Actions

The trivial action:

$$G \rightarrow \text{Perm}(S) \text{ where } g \mapsto e_{\text{Perm}(S)}$$

G acting on self by left/right translation, conjugation

G acting on the set of subgroups of G by conjugation:

$$g \cdot H = gHg^{-1} = \{ghg^{-1} | h \in H\}$$

Normal subgroup $N \trianglelefteq G$

$$G \text{ acting on } N, g \cdot n := gng^{-1} \in N$$

$G = S_3$ where S is the set of subgroups of G of order 2.

$$S = \{\{1, (1\ 2)\}, \{1, (1\ 3)\}, \{1, (2\ 3)\}\}$$

recall $\sigma(a_1, a_2, a_3, \dots, a_k)\sigma^{-1} = (\sigma a_1, \sigma a_2, \sigma a_3, \dots, \sigma a_k)$

V vector space over a field K

$$G = \text{GL}(V) = \text{group of invertible linear maps } V \rightarrow V$$

e.g. if $V = K^n$ then $G = \text{GL}(n, K)$

G acts on V (rather simply) by $L \cdot v = L(v)$

Orbits and Stabilizers

Given G acting on S by $G \times S \rightarrow S$ there is an obvious relation on S :

$$s, s': s \sim s' \leftrightarrow \exists g \in G, s' = gs$$

the orbit of s is just the equivalence class of s under this relation

$$\text{i.e., } G \cdot s = \{g \cdot s \mid g \in G\}$$

The conjugacy classes of s are the orbits of S under the group action of G by conjugation

$$\text{the orbit of } s, O(s) = \{s\} \leftrightarrow s = gsg^{-1} \forall g$$

$$\leftrightarrow (\forall g)gs = sg$$

$$\leftrightarrow s \in Z(G) \text{ the center of the group}$$

Example, for $G = S_3$

the orbit of 1 is $\{1\}$

the orbit of $(1\ 2) = \{(1\ 2), (1\ 3), (2\ 3)\}$

the orbit of $(1\ 2\ 3) = \{(1\ 2\ 3), (1\ 3\ 2)\}$

Stabilizer (isotropy group) of a given element $s \in S := G_s$

$$G_s = \{g \in G \mid g \cdot s = s\}$$

stabilizer is closed under inverses: $g \in G_s \rightarrow g \cdot s = s \rightarrow g^{-1}gs = g^{-1}s \rightarrow s = g^{-1}s$

large stabilizer \leftrightarrow small orbit

there exists a natural bijection $\alpha : G/G_s \rightarrow O(s)$ defined $gG_s \mapsto g \cdot s$

well-definition:

$$\text{if } g_1G_s = g_2G_s \text{ then } \exists g \in G_s, g_1 = g_2g \text{ and } \alpha(g_1G_s) = g_1 \cdot s = g_2gs = g_2s = \alpha(g_2G_s)$$

injectivity:

$$\text{if } \alpha(g_1G_s) = g_1 \cdot s = g_2 \cdot s = \alpha(g_2G_s) \text{ then } g_2^{-1}g_1 \cdot s = s, g_2^{-1}g_1 \in G_s \text{ and } g_1G_s = g_2G_s$$

Lang 1.1-1.5

1.1: Monoids

A *monoid* is a set with associative binary operation and unit element.

Abelian \leftrightarrow commutative

A *submonoid* is a subset of a monoid with identity and closure under the operation

Such a submonoid is, itself, a monoid

1.2: Groups

A *group* is a monoid with inverses for each element

The *permutation group* of S is the set of all bijections $S \rightarrow S$ (with composition as product)

A direct product of groups has product defined componentwise

A *subgroup* of a group is a subset closed under composition and inverse

$S \subset G$ generates G if $\forall g \in G, g = \prod s_i$, where $s_i \in S$ or $s_i^{-1} \in S$

$$G = \langle S \rangle$$

The **group of symmetries of the square** is a non-abelian group of order 8

generated by σ, τ such that $\sigma^4 = \tau^2 = e$ and $\tau\sigma\tau^{-1} = \sigma^3$

The **quaternions** are a non-abelian group of order 8

generated by i, j where defining $k = ij, m = i^2$

$i^4 = j^4 = k^4 = e, i^2 = j^2 = k^2 = m, \text{ and } ij = mji$

A *monoid-homomorphism* $f : G \rightarrow G'$ satisfies $f(xy) = f(x)f(y)$ and $f(e_G) = e_{G'}$

If G and G' are groups, f is a group homomorphism ($f(x^{-1}) = f(x)^{-1}$ is implied)

An *isomorphism* is a bijective homomorphism.

An *automorphism* or *endomorphism* of G is an isomorphism $\varphi : G \rightarrow G$

The group **Aut(G)** is the set of all automorphisms of G

The *kernel* of a homomorphism $f : G \rightarrow G'$ is $\{g \in G : f(g) = e_{G'}\}$

the kernel and the image f are subgroups of their respective groups

An *embedding* is a homomorphism $f : G \rightarrow G'$ where $G \cong \text{Im}(f)$.

Fact: A homomorphism with trivial kernel is injective.

Forward is obvious.

Supposing trivial kernel: $f(x) = f(y) \leftrightarrow f(x)[f(y)]^{-1} = e \leftrightarrow f(xy^{-1}) = e \leftrightarrow xy^{-1} = e$

For G a group, and $H, K \leq G$ such that $H \cap K = e, HK = G$, and $xy = yx \forall x \in H \forall y \in K$

The map $H \times K \rightarrow G$ defined $(x, y) \mapsto xy$ is an isomorphism

This generalizes to finitely many such subgroups by induction

A *left coset* of H in G ($H \leq G$) is $aH = \{ax : x \in H\} \leq G$

$x \mapsto ax$ gives bijection between cosets of H , are all of equal cardinality

The *index* of H in G ($G : H$) is the number of cosets of H in G (right or left)

The *order* of G is the index ($G : 1$) of its trivial subgroup

For any subgroup H of G , G is the disjoint union of its cosets in H

For $H \leq G$, $(G : H)(H : 1) = (G : 1)$, holding if at least two are finite

If $(G : 1)$ is finite, the order of H divides the order of G .

Given:

$H, K \leq G, K \subset H$

$\{x_i\}$ a set of coset representatives of K in H

$\{y_i\}$ a set of coset representatives of H in G

Then:

$\{y_j x_i\}$ is a set of coset representatives of K in G .

Therefore the above can be generalized to $(G : K) = (G : H)(H : K)$

Conclusion: groups of prime order are cyclic.

$J_n = \{1, \dots, n\}, S_n = \text{Perm}(J_n)$

$\tau \in S_n$ is a **transposition** if $\exists r \neq s \in J_n, \tau(r) = s, \tau(s) = r, \tau(k) = k \forall k \neq r, s$

The set of transpositions generate S_n

Consider $H \leq S_n$ those which leave n fixed. Then $H \cong S_{n-1}$.

Now if $\sigma_i \in S_n$ for $1 \leq i \leq n$ are defined with $\sigma_i(n) = i, \{\sigma_i\}$ are coset reps for H

Hence $(S_n : 1) = n(H : 1) = n!$.

1.3: Normal subgroups

For H the kernel of $f : G \rightarrow G'$ a group-homomorphism, $xH = f^{-1}(f(x)) = Hx$

Such a relation is equivalent to e.g. $xH \subset Hx$ and $H \subset xHx^{-1}$

A subgroup $H \trianglelefteq G$ (satisfying $xHx^{-1} = H \forall x \in G$) is termed *normal*
 H is normal $\leftrightarrow H$ is the kernel of some homomorphism
 The *factor group* of G by $H \trianglelefteq G$ is the group of cosets, denoted G/H
 $f : G \rightarrow G/H$ defined $x \mapsto xH$ is the canonical map for H
 The *normalizer* N_S of $S \subset G$ is $\{x \in G | xSx^{-1} = S\}$
 The normalizer of H is the largest subgroup of G in which H is normal
 The *centralizer* Z_S of S is $\{x \in G | xyx^{-1} = y \forall y \in S\}$
 The centralizer of G is called its *center*; its elements commute with all others in G
 The **special linear group** is the kernel of the determinant (a homomorphism)
 G is the *semidirect product* of N and H if $G = NH$ and $H \cap N = \{e\}$
 An *exact sequence* $G' \xrightarrow{f} G \xrightarrow{g} G''$ satisfies $\text{Im}(f) = \text{Ker}(g)$.
 Can extend to larger sequences as long as each triple satisfies the above
 Some canonical homomorphisms, given $f : G \rightarrow G'$
 $H = \text{ker}(f) \rightarrow \exists! f' : G/H \rightarrow G'$ injective $\rightarrow \exists \lambda : G/H \rightarrow \text{Im}(f)$ an isomorphism
 $H \leq G, N$ the minimal $N \trianglelefteq G$ s.t. $H \leq N, H \subset \text{ker}(f)$, then $N \subset \text{ker}(f)$, $\exists! f' : G/N \rightarrow G'$
 $H, K \trianglelefteq G, K \subset H$, then $K \trianglelefteq H \rightarrow (G/K)/(H/K) \cong G/H$
 $H, K \leq G, H \subset N_K \rightarrow H \cap K \trianglelefteq H, HK = KH \leq G, \rightarrow H/(H \cap K) \cong HK/K$
 $H' \trianglelefteq G', H = f^{-1}(H') \rightarrow H \trianglelefteq G \rightarrow \bar{f} : G/H \rightarrow G'/H'$ injective

1.4: Cyclic groups

1.5: Operations of a group on a set