

Math 245A

Fall 2015

Chapter 2

2.2 Groups

A group G is a 4-tuple $G = (|G|, \mu, \iota, e)$ with
underlying set $|G|$
law of composition μ
inverse function ι
neutral element e

(Exercise 2.2:1) A homomorphism from a group G to a group H is a function $\phi : G \rightarrow H$ satisfying the following for $a, b \in G$:

$$\begin{aligned}\phi(e_G) &= e_H \\ \phi(\iota_G(a)) &= \iota_H(\phi(a)) \\ \phi(\mu_G(a, b)) &= \mu_H(\phi(a), \phi(b))\end{aligned}$$

A more common representation of a group uses symbols $G = (|G|, \cdot, {}^{-1}, e)$

(2.2.1) The conditions for a 4-tuple to be a group are as follows

$$\begin{aligned}(\forall x, y, z \in |G|) \quad (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ (\forall x \in |G|) \quad e \cdot x &= x = x \cdot e \\ (\forall x \in |G|) \quad x^{-1} \cdot x &= e = x \cdot x^{-1}\end{aligned}$$

(2.2.2) We may also say that a set $|G|$ with a map $|G| \times |G| \rightarrow |G|$ constitutes a group if

$$\begin{aligned}(\forall x, y, z \in |G|) \quad (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ \text{there exists } e \in |G| \text{ such that } (\forall x) e \cdot x &= x = x \cdot e \text{ and } (\forall x \in |G|) (\exists y \in |G|) y \cdot x = e = x \cdot y\end{aligned}$$

(2.2.1) consists of identities (universally quantified equations) and (2.2.2) does not

note: universal quantification is a "for all" quantification

(Exercise 2.2:2)

- (i)
- (ii)

(Exercise 2.2:3)

2.3 Indexed Sets

An I -tuple of elements of X , $(x_i)_{i \in I}$ is formally defined as an $f : I \rightarrow X$

The set of all functions from I to X is denoted X^I

2.4 Arity

The *arity* of an operation is, e.g., 1 if unary, 2 if binary, etc.

An I -ary operation on S is a map $S^I \rightarrow S$

Group: a set, a binary operation, a unary operation, and a distinguished element

Can think of the identity as a 0-ary/zeroary operation of the structure

S^0 has exactly one map, $\emptyset \rightarrow S$, so a map $S^0 \rightarrow S$ is determined by one element

Note these are not strictly identical since one is a map and the other the element itself

But they are in 1-to-1 correspondence and give equivalent information

2.5 Group-theoretic terms

A *group-theoretic relation* in $(\eta_i)_I$ is an equation $p(\eta_i) = q(\eta_i)$ holding in G

p and q are *group-theoretic terms* which we formally define

The terms in the elements of X under the formal group operations μ, ι, e form a set T :

given with functions $\text{symb}_T : X \rightarrow T$, $\mu_T : T^2 \rightarrow T$, $\iota_T : T \rightarrow T$, and $e_T : T^0 \rightarrow T$

such that each map is one-to-one, its images disjoint, and T is the union of those images

and T is generated by $\text{symb}_T(X)$ under the aforementioned operations

that is, T has no proper subset containing $\text{symb}_T(X)$ and closed under those operations

We can represent these terms, for groups, using strings of symbols

We need full parentheses notating order of operations to ensure disjoint images

A set-theoretic approach dispenses with strings and allows for infinite arities

For the example of a group, we would have (using ordered pair, 3-tuple, etc.):

for $x \in X$, $\text{symb}_T(x) := (*, x)$

for $s, t \in T$, $\mu_T(s, t) := (\cdot, s, t)$

for $s \in T$, $\iota_T(s) := (-^1, s)$

and $e_T = (e)$

and by set theory, no element can be written as such an n -tuple in more than one way

2.6 Evaluation

Given a set map $f : X \rightarrow |G|$ for a group G

Recursive evaluation of $s_f \in |G|$ given an X -tuple of symbols $s \in T = T_{X, \cdot, -^1, e}$

if $s = \text{symb}_T(x)$ for some $x \in X$, then $s_f := f(x)$

$s = \mu_T(t, u) \rightarrow s_f = \mu_G(t_f, u_f)$, assuming that given $t, u \in T$ we know $t_f, u_f \in |G|$

similarly, $s = \iota_T(t) \rightarrow s_f = \iota_G(t_f)$, assuming we know t_f given t

finally $s = e_T \rightarrow s_f = e_G$

Varying f in addition to T gives an evaluation map $(T_{X, \cdot, -^1, e}) \times |G|^X \rightarrow |G|$

Alternatively, fixing $s \in T$ gives a map $s_G : |G|^X \rightarrow |G|$

these represent substitution into s

these s_G are the *derived n -ary operations* (aka *term operations*) of G

distinct terms can induce the same derived operation

e.g. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in general or others for certain groups

Examples of derived operations on groups

conjugation $\zeta^\eta = \eta^{-1} \zeta \eta$ (binary)

commutator $[\zeta, \eta] = \zeta^{-1}\eta^{-1}\zeta\eta$ (binary)
 squaring (unary)
 δ (Exercise 2.2:2)
 σ (Exercise 2.2:3)

First Class Question

The last example of a derived operation on groups cited the trivial “second component” function, $p_{3,2}(\zeta, \eta, \zeta) = \eta$ induced by $y \in T_{\{x,y,z\}, -1, \cdot, e}$. I wasn’t entirely sure how this derived operation would be represented as an element of $T_{\{x,y,z\}, -1, \cdot, e}$. Would $p_{3,2}$ be the element $(*, y)$ (in the set-theoretic notation)?

Terms in other families of operations

An Ω -algebra is a system $A = (|A|, (\alpha_A)_{\alpha \in |\Omega|})$

here $|A|$ is some set, and for each $\alpha \in |\Omega|$, $\alpha_A : |A|^{\text{ari}(\alpha)} \rightarrow |A|$

note that often people will use $n(\alpha)$ (rather than $\text{ari}(\alpha)$) for the arity of an operation α
 e.g. for a group, $|\Omega| = \{\mu, \iota, e\}$, $\text{ari}(\mu) = 2$, $\text{ari}(\iota) = 1$, and $\text{ari}(e) = 0$