Math 206

Fall 2015

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Definitions

A *norm* on a vector space X (over $\mathbb{F}$) is a function $\|\|: X \to \mathbb{R}^+$ such that
$\|x\| = 0$ iff $x = 0$
$\|\alpha x\| = |\alpha|\|x\|$ (for $\alpha \in F$)
$\|x + y\| \leq \|x\| + \|y\|$
An *algebra* $\mathscr{A}$ over $\mathbb{F}$ is a vector space with distributive $\cdot$ satisfying
$cx \cdot y = c(x \cdot y)$
$x \cdot cy = c(x \cdot y)$ for all $c \in F$
A *normed algebra* over $\mathbb{R}$ or $\mathbb{C}$ is an algebra $\mathscr{A}$ equipped with (vector space) norm satisfying
$\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathscr{A}$
A norm on $\mathscr{A}$ induces a metric
$d(a, b) = \|a - b\|$ on $\mathscr{A}$ and therefore a topology
if $\mathscr{A}$ is complete for this norm, it is a *Banach algebra*

To figure out (use https://www.math.ksu.edu/ nagy/real-an/2-05-b-alg.pdf)

Supposing $\mathscr{A}$ is not necessarily complete
$\|ab\| \leq \|a\|\|b\|$ gives uniform continuity on the product
hence the norm can be extended to the completion $\bar{\mathscr{A}}$ to form a Banach algebra
A metric space M is complete if all Cauchy sequences converge to an element of M
The completion M is all equivalence classes of Cauchy sequences where
$\{a_n\} \sim \{b_n\}$ iff $\lim\limits_{x\to\infty} d(a_n - b_n) = 0$

Examples

For M a compact space, $C(M)$
the set of continuous $\mathbb{R}/\mathbb{C}$-valued functions on M
pointwise operations
$\|f\|_\infty = sup\{|f(x)| : x \in M\}$
For M locally compact, $C_\infty(M)$

the set of continuous $\mathbb{R}/\mathbb{C}$-valued functions on M vanishing at ∞
vanishing at ∞: $\forall\epsilon\exists$ a compact subset of M, outside of which $f < \epsilon$
note that this is non-unital (lacks an identity)
For $\mathscr{O} \subset \mathbb{C}^n$ open
$H^\infty(\mathscr{O})$ the set of all bounded holomorphic functions on $\mathscr{O}$
(M, d) metric space and $f \in C(M)$
Lipschitz constant (which can be $+\infty$) $L_d(f) = sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in M, x \neq y\}$
The *Lipschitz functions* $\mathscr{L}_d(M, d) = \{f : L(f) < \infty\}$
These form a dense subalgebra of C(M) and are in fact a *-subalgebra
$\|f\|_d := \|f\|_\infty + L_d(f)$, can be shown as a normed-algebra norm
$L_d(M, d)$ is complete for this norm
so $L_d(M, d)$ is a Banach algebra
L_d is a seminorm on $\mathscr{L}_d(M, d)$ since it takes value 0 on the constant functions
can recover d from L_d
M a differentiable manifold (e.g. $T = \mathbb{R}/\mathbb{Z}$ the circle)
C(M) $\supseteq C^{(1)}(M)$ the singly-differentiable functions
$f \in C^{(2)}(T) \to Df : T_xM \to \mathbb{R}, \mathbb{C}$
with Df the derivative and T_x the tangent space
If we put on a Riemmannian metric, define $\|f\|^{(1)} = \|f\|_\infty + \|Df\|_\infty$
If $f \in C^{(1)}(T) : \|f\|^{(1)} = \|f\|_\infty + \|f'\|_\infty$
Banach algebra norm, for which this space of functions is complete
For the circle, $C^{(2)}(T) \to \|f\|^{(2)} = \|f\|_\infty + \|f'\|_\infty + \frac{1}{2}\|f''\|_\infty$
the factor $\frac{1}{2}$ ensures that this satisfies the normed algebra condition
$C^{(n)}(T) = \sum_{k=0}^{n} \frac{1}{k!}\|f^{(k)}\|_\infty$
For $C^\infty(T)$ using the collection of norms $\{\|\|^{(n)}\}_{n=1}^\infty$ yields a Fréchet algebra
A Fréchet algebra has a topology defined by a countable family of seminorms
that respect the algebra structure and is complete **(clarify)**
non-commutative algebras
X a Banach space
$\mathscr{B}(X)$ the algebra of bounded operators on X
$\|\|$ operator norm $\to$ Banach algebra
Any closed subalgebra of $\mathscr{B}(X)$ is a Banach algebra

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Sketch of the course
X a Banach space, $B(X)$ bounded functions on the space
$\mathscr{H}$ a Hilbert space, $\mathscr{B}(\mathscr{H})$ bounded operators on the space
for $T \in \mathscr{B}(\mathscr{H})\exists$ adjoint operator $T^* \in \mathscr{B}(\mathscr{H})$
$< T\xi, \eta > = < \eta, T^*\xi >$ for $\xi, \eta \in \mathscr{H}$
adjoint is additive, conjugate linear, $T^{**} = T$, $(ST)^* = T^*S^*$
An algebra A over $\mathbb{R}$ or $\mathbb{C}$ is a **-algebra* if it has a $* : A \to A$ satisfying

certain properties (look up)

A **-normal algebra* is a normal *-algebra such that

$(\forall a \in A) \|a^*\| = \|a\|$

A *Banach *-algebra* is a *-normal algebra that is a Banach algebra.

For any $T \in \mathscr{B}(\mathscr{H})$, have $\|T^*T\| = \|T\|^2$ **(check: parse through defns)**

For M a locally compact space, $A = C_\infty(M, \mathbb{C})$, $f^* := \bar{f}$ is a Banach *-algebra

Also have $\|f^* f\| = \|f\|^2$ **(verify: should be easier than the other)**

Little Gelfand-Naimark theorem:

Let A be a commutative Banach *-algebra satisfying $\|a^*a\| = \|a\|^2$.

Then $A \cong C_\infty(M)$ for some locally compact M.

One view of the "spectral theorem"

Let $T \in \mathscr{B}(\mathscr{H})$ with $T^* = T$

Let A be the closed subalgebra of $\mathscr{B}(\mathscr{H})$ generated by T and I (i.e. $p(T) := \Sigma \alpha_k T^K$)

Polynomials closed or stable under *

If $S \in A$ then $S^* \in A$ (i.e. A is a *-subalgebra of $\mathscr{B}(\mathscr{H})$

So A is a Banach *-suubalgebra satisfying $\|S^*S\| = \|S\|^2$

Moreover, A is commutative. (unital, since generated by I)

Then by the Little Gelfand-Naimark theorem, $A \cong C(M)$

Indeed $M \subset \mathbb{R}$, the spectrum of T

If $\mathscr{H}$ is finite dimensional, then M is the set of eigenvalues of T

T is normal if $TT^* = T^*T$

A C*-algebra is a Banach *-algebra over $\mathbb{C}$ satisfying

$\|a^*a\| = \|a\|^2$

Theorem: A commutative C*-algebra is $\cong C_\infty(M)$.

Big Gelfand-Naimark Theorem: (*Math 208, C*-algebras*)

Any C*-algebra is $\cong$ to a closed *-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.

Tangent

algebraic topology, differential geometry, Riemann manifolds, "non-commutative geometry" (Connes)

A *von-Neumann algebra* is a *-subalgebra of $\mathscr{B}(\mathscr{H})$

which is closed under the strong operator topology.

Every commutative von-Neumann algebra is $\cong L^\infty(X, S, \mu)$ (measure spaces) acting on $L^2(X, S, \mu)$ by positive sldkjfalksdjf

For group G, $\alpha : G \to Auto(X) \subseteq \mathscr{B}(X)$

$Auto(X)$ a Banach space

Look at subalgebra of $\mathscr{B}(X)$ generated by $\alpha(G)$.

Leads to considering $l'(G)$ with product $(f \star g)(x) = \Sigma f(y) g(y^{-1}x)$ convolution

$f^*(x) = \overline{f(x^{-1})}$

Banach *-algebra, G commuatative $\to$ Fourier transform

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K a field, X a set, $\mathscr{F}(X, K)$ the set of all K-valued functions on X with pointwise operations

Given $f \in \mathscr{F}(X, K)$.

Let $\lambda \in K$. Then $\lambda \in$ range(f) exactly if $(f - \lambda 1)$ is not invertible.
For any $a \in A$, the *spectrum* of a is $\{\lambda \in K : a - \lambda 1_A$ is not invertible in A$\}$.
The spectrum depends on the containing algebra
Assuming that this algebra A (over the field K) has an identity 1_A
Example: Let A = C([0, 1]), and B = polynomials, viewed as a dense subalgebra of A.
Let p be a polynomial of degree ≥ 2. Then
$\sigma_a(p) = p([0,1])$.
$\sigma_B(p) = \mathbb{R}, \mathbb{C}$
Let A be a Banach algebra with 1 ($\|1\| = 1$), and $a \in A$.
If $\|a\| < 1$, then 1 - a is invertible, and $\|(1-a)^{-1}\| \leq \frac{1}{1-\|a\|}$
Proof:
$\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$ ($a^0 := 1_A$)
For any $n > 0$, let $s_n = \sum_{k=0}^{n} a^k$.
Show that $\{s_n\}$ is a Cauchy sequence.
If $n > m$, $\|s_n - s_m\| = \|\sum_{k=m+1}^{n} a^k\| \leq \sum_{k=m+1}^{n} \|a\|^k$.
Given $\epsilon > 0 \exists N$ such that if $m, n \geq N$ then $\sum_{k=m+1}^{n} \|a\|^k \leq \epsilon$
So $\{s_n\}$ is a Cauchy sequence.
By completeness there is a $b \in A$ with $s_n \to b$ as $n \to \infty$.
Want to show $b = (1-a)^{-1}$.
$b(1-a) = lim_{n\to\infty}(s_n(1-a))$
$= lim_{n\to\infty}(1 + a + a^2 + a^3 + \cdots + a^n - (a + a^2 + a^3 + \cdots + a^{n+1}))$
$= lim_{n\to\infty}(1 - a^{n+1}) = 1$
Then $1 - (1-a)$ is invertible, i.e. a is invertible.
$\|(1-a)^{-1}\| = \lim \|s_n\| \leq \lim \sum_{k=0}^{n} \|a^k\| = \frac{1}{1-\|a\|}$ ($\|1\| = 1$)
$\|ab\| \leq \|a\|\|b\|$: can very easily check that multiplication is cts (do this?)
Corollary: If $a \in A$ and $\|1 - a\| < 1$ then a is invertible, and $\|a^{-1}\| \leq \frac{1}{1-\|1-a\|}$
I.e. the open unit ball about 1 consists of invertible elements.
Let $a \in A$. Let L_a, R_a be the operators of left and right multiplication by a on A.
$a \to L_a$ is an algebra homomorphism of A into $\mathscr{L}(A)$ (linear operators on A)
$L_a L_b = L_{ab}$, $R_a R_b = R_{ba}$ (R is an antihomomorphism)
$1 \in A$
If a is invertible, then so is L_a, $L_a L_{a^{-1}} = I_a$
Then if A is a normed algebra, $\|L_a\| = \|a\|$
$\|L_{ab}\| = \|ab\| \leq \|a\|\|b\|$
$\|L_a 1_a\| = \|a\|$
so if $a \in A$ is invertible, then L_a is a homeomorphism of A onto itself.
Thus if A is a Banach algebra, with 1, and a is invertible:
$\{L_a b : \|1 - b\| < 1\}$ is an open neighborhood of a consisting of invertible elements
Let GL(A) be the set of invertible elements of A. (general linear group)
Then (for A a unital Banach algebra) GL(A) is an open subset of A.
(Fails for Poly $\subseteq$ C([0, 1]))
Two Fréchet algebras, for one, GL(A) is an open subset, for another it isn't.
ask about this?: not sure what he was talking about
$C^{\infty}(T)$, $\|f^{(n)}\|$

C(R) cont fns on $\mathbb{R}$ (or $\mathbb{C}$) maybe unbounded

For each n let $\|f\|_n = \sup\{|f(t)| : |t| \leq n\}$

Corollary: For A a Banach algebra with 1 and $a \in A$, $\sigma(a)$ is a closed subset of $\mathbb{C}$

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Proposition: Let A be a unital Banach algebra and $a \in A$.

Then $\sigma(a)$ is a closed subset of $\mathbb{C}$ or $\mathbb{R}$. If $\lambda \in \sigma(a)$ then $\|\lambda\| \leq \|a\|$.

Proof: $\sigma(a) = \{\lambda : (a - \lambda) \text{ is not invertible}\}$

Its complement, the *resolvant set*, of a is $\{\lambda : (a - \lambda) \in GL(A)\}$, is open.

If $|\lambda| > \|a\|$ then $(\lambda - a) = \lambda(1 - \frac{a}{\lambda})$, $\|a/\lambda\| < 1$

so $(\lambda - a)$ is invertible, ie $\lambda \in \sigma(a)$.

Over $\mathbb{R}$, can have $\sigma(a) = \emptyset$, e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

"If $a \in GL(A)$ and b is close to a then b^{-1} is not much bigger than a^{-1}".

Let $\mathscr{O} = \{c : \|1 - c\| < 1/2\}$

So c is invertible, and $\|c^{-1}\| \leq \frac{1}{1 - \|1-c\|} \leq 2\}$

Let $b \in a\mathscr{O}$, so $b = ac$ for $c \in \mathscr{O}$, then $\|b^{-1}\| = \|c^{-1}a^{-1}\| \leq 2\|a^{-1}\|$.

For $a,b \in GL(A)$.

$b^{-1} - a^{-1} = b^{-1}(a - b)a^{-1}$

Thus $\|b^{-1} - a^{-1}\| \leq \|b^{-1}\|\|a - b\|\|a^{-1}\|$.

So $b \to b^{-1}$ is continuous for the norm.

So $GL(A)$ is a topological group for topology from norm.

$b^{-1} = (1 + b^{-1}(a - b))a^{-1}$

On $\rho(a)$ (the resolvant set, complement of the spectrum) define the resolvant of a

This is the function $R(a,\lambda) = (\lambda - a)^{-1}$

$R(a,\lambda)$ is an analytic function on $\rho(a)$.

Proof: Let $f(z) = R(a,z)$.

$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{(z+h-a)^{-1} - (z-a)^{-1}}{h}$

$= \lim_{h \to 0} \frac{1}{h}(z + h - a)^{-1}((z - a) - (z + h - a))(z - a)^{-1}$

$= \lim_{h \to 0} -(z + h - a)^{-1}(z - a)^{-1} = -(z - a)^{-2}$

$f'' = +z(z - a)^{-3}$

Given $z_0 \in \rho(a)$

Will use $b^{-1} = (1 + b^{-1}(a - b))a^{-1}$ and $f(z) = (z - a)^{-1} = \sum c_n(z - z_0)^n$

$f(z) = (z - a)^{-1}$

$b \to z - a$

$f(z) = (1 + (z - a)^{-1}((z_0 - a) - (z - a))(z_0 - a)^{-1}$

$= (1 + (z - a)^{-1}(z_0 - z))(z_0 - a)^{-1}$

where $(z - a)^{-1}(z_0 - z) \leq 1$ then the above

$= \sum(-1)^n(z - a)^{-n}(z - z_0)^n = \sum(-1)^n(z - a)^{-n-1}(z - z_0)^n$

a proper power series expansion.

Examine $R(a,z)$ at ∞.

$R(a,z^{-1}) = (z^{-1} - a)^{-1} = \frac{1}{z^{-1} - a}$

$= z(1-za)^{-1}$ (for small z, ie $\|za\| < 1$)
$R(a,z^{-1})$ approaches 0 as $z \to 0$.
defn $R(a,0^{-1}) = 0$, see $R(a,z)$ is analytic at ∞.
Theorem: For a Banach algebra over $\mathbb{C}$ with 1, and for any $a \in A$,
$\sigma(a) \neq \emptyset$, that is, the spectrum is non-empty.
Proof: Suppose that $\sigma(a) = \emptyset$.
Then $R(a,z)$ is defined on all of $\mathbb{C}$ and is bounded.
By Liouville's, $R(a,z)$ is constant, $= 0$, $(a-z)^{-1} = 0\ \forall z$
Why can we use Liouville's in this Banach space case?
Let A' be the dual Banach space to A.
For $\varphi \in A'$, $z \mapsto \varphi(R(a,z))$ is a $\mathbb{C}$-valued analytic function.
So set $\varphi(R(a,z)) = 0\ \forall z, \forall \varphi$
so $R(a,z) = 0$.
Knowing that there is anything in here is the Hahn-Banach Theorem, depending on the axiom of choice.
Theorem (Gelfand-Mazer)
Let A be a unital Banach algebra over $\mathbb{C}$.
If every nonzero element is invertible, then $z \to z1_A$ is an isomorphism from $\mathbb{C}$ onto A.
Proof:
Given $a \in A$ let $z \in \sigma(a) \neq \emptyset$.
So $(z-a)$ is not invertible so $z - a = 0$.
Fails over $\mathbb{R}$ since have $\mathbb{R}$, $\mathbb{C}$, quaternions

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Let A be an algebra or ring. Ideals I, left, right, 2-sided.
A/I, for a left ideal get a left A-module, right ideal get a right A-module
Two-sided get an algebra or ring.
Let A be a normed algebra, and if I is an ideal in A.
Then $\bar{I}$ (the closure) is again an ideal in A.
$\{a_n\} \subset I$, $a_n \to c \in A$ then $ba_n \to bc$
Proposition: If A is a unital Banach algebra and if I is an proper ideal in A.
Then $\bar{I}$ is proper.
Proof:
Use GL(A) is open. The original ideal cannot contain any invertible elements.
The complement of the invertible elements is going to be closed.
The ideal is in its closure; the closure is in the complement of the invertible elements.
So the closure will not contain any invertible elements.
Counter-example: C locally compact, $C_c(\mathbb{R}) \subset C_\infty(\mathbb{R})$
In fact $C_c(\mathbb{R})$ is the minimal dense ideal in $C_\infty(\mathbb{R})$
Lack of identity element.
Counter-example: Look at A the polynomials viewed as a subset of $C([0,1])$
Using the sup-norm.
Lack of completeness.

Let $I = \{p : p(2) = 0\}$ is an ideal, in fact a maximal proper ideal.
This ideal of polynomials will be dense inside A and dense inside all polynomials.
Counter-example: A = $C(\mathbb{R})$, including unbounded, $I = C_C(\mathbb{R})$, compact open topology
Compact in here for the Fréchet open topology.
Corollary: Let A be a unital Banach algebra.
Then every maximal ideal is closed.
Taking ideals to form quotients.
Recall that if X is a normed vector space and if Y is a vector subspace of X:
Then we can form the quotient vector space X/Y.
Have the evident $\pi : X \to X/Y$ by $\pi(x) = x + Y$.
Set $\|\pi(x)\| = inf\{\|x - y\| : y \in Y\}$ i.e. the distance from x to the subspace Y.
It is easily seen that this is a seminorm.
Problem: if Y is not closed, then it is not a norm, since $\|\pi(y)\| = 0$ for $y \in \overline{Y}$.
Therefore, if Y is closed, then $\|\|_{X/Y}$ is a norm.
(Should know from $202B$) If X is a Banach space and Y is a closed subspace.
Then X/Y with the norm defined above is a Banach space.
Trickier to prove, but a true statement.
Proposition: Let A be a normed algebra and let I be a closed ideal in A.
(can be left or right or two-sided)
Then if I is a two-sided ideal, then A/I with $\|\|_{A/I}$ is a normed algebra.
That is $\|\pi(a)\pi(b)\| \leq \|\pi(a)\|\|\pi(b)\|$.
Then if I is a left ideal, so that A/I with $\|\|_{A/I}$ is a left A-module.
$\|a\pi(b)\|_{A/I} \leq \|a\|_A\|\pi(b)\|_{A/I}$
And if I is a right ideal, so that A/I with $\|\|_{A/I}$ is a right A-module, similar.
Proof:
For I a two-sided ideal.
If $c, d \in I$, then $\|\pi(a)\pi(b)\|_{A/I} \leq \|(a - c)(b - d)\| = \|(ab - (cb + ad - cd)\|$
where $cb + ad - cd \in I$.
Take inf over $c, d \in I$.
$\leq \|a - c\|\|b - d\|$
Proposition: If a is a Banach algebra and I is a closed 2-sided ideal, then A/I is a Banach algebra.
An algebra or ring is *simple* if it contains no proper 2-sided ideals.
e.g. $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(K)$ for K a field.
Proposition: Let A be an algebra or ring and let I be a maximal 2-sided ideal in A.
The A/I is a simple algebra/ring.
Corollary: If A is a Banach algebra and if I is a maximal closed 2-sided ideal.
Then A/I is a simple Banach algebra.
Let A be a commutative algebra/ring with 1.
If $a \in A$ and a is not invertible, then Aa is a two-sided ideal.
And $1 \notin Aa$ (else it would be invertible).
Thus Aa is a proper ideal.
Corollary: If A is a commutative algebra/ring with 1 and if A is simple, then A is a field.
A is simple: every nonzero element is invertible.
Since if one weren't we would have a proper ideal.

Theorem: Let A be a commutative Banach algebra with 1.
Let I be a maximal ideal in A (which of course is necessarily closed).
Then $A/I \cong$ either $\mathbb{C}$ (isomorphic in every sense).
Proof:
Banach-Mazer theorem applied to the previous discussion.
$c \in A/I$, $z \in \sigma(c)$, $c - z1_A$ is noninvertible by def'n spectrum.
Since it is a field, $c - z1_A = 0$.
From a maximal ideal, I, get a homomorphism $\phi : A \to \mathbb{C}$, unital
($\phi(1) = 1$) such that $I = ker(\phi)$
Let A be a Banach algebra with 1 and let $\phi : A \to \mathbb{C}$ be a homomorphism.
Then ϕ is continuous and $ker(\phi)$ is a maximal 2-sided ideal in A.
$\|\phi\| = 1$
Lemma: For any given $a \in A$, $\phi(a) \in \sigma(a)$
Proof: $\phi(a - \phi(a)1_A) = 0$ so $a - \phi(a)1$ is not invertible.
so $\phi(a) \in \sigma(a)$
Then $\|\phi(a)\| \leq \|a\|$, so $\|phi\| \leq 1$ but $\phi(1) = 1$.

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X a compact (hausd) space and $A = C(X)$. What is $\hat{A}$? (the maximal ideal space)
Each $x \in X$, $\phi_x(f) = f(x)$. Multiplicative linear functional.
Get map $X \to \hat{A}$, $x \mapsto \phi_x$. Question: is this onto?
Proposition: Let I be a proper ideal in $A = C(X)$.
There is at least one $x \in X$ such that $f(x) = 0$ for all $f \in I$.
Proof:
Suppose no such x exists. (Goal: show that $I = A$.)
Then for each $x \in X$ is $f_x \in I$ with $f_x(x) \neq 0$.
Multiply it by $\overline{f_x}$, then we have $\overline{f_x} f_x$ non-negative and real.
Let $U_x = \{y : f_x(y) > 0\}$ open, $x \in U_x$.
The U_x's cover X, so there is a finite subcover, corresponding to $\{x_1, \cdots, x_n\}$
Then let $f = \sum_{j=1}^{n} f_{x_j}$. Then $f(x) > 0$ all $x \in X$.
Certainly $f \in I$. In A, f is invertible. Hence I = A.
We can conclude that $X \to \overline{A}$ is onto, one-to-one, continuous (easy to see).
Compact space to a Hausdorff space this is a homeomorphism.
Proposition: $\overline{A}$ "=" X.

A commutative Banach algebra with 1.
Have the Gelfand transform: $A \to C(\hat{A})$, where $a \mapsto \hat{a}$.
If a is nilpotent, $a^n = 0$, for any $\phi \in \hat{A}$:
$0 = \phi(a^n) = (\phi(a))^n$, so $\phi(a) = 0$. Thus $\hat{a} \equiv 0$. (0 on all elements)

A comm. Banach, $1 \in A$. ($\mathbb{C}$)

We have ssen that $a \in A$, range($\hat{a}$) $\subset \sigma(a)$.

Is the converse true? Yes, observe:

If $\lambda \in \sigma(a)$, then $\lambda 1_a - a$ is not invertible so $(\lambda - a)A$ is a proper ideal (2-sided).

From algebra: every ideal is contained in a maximal ideal; call it M.

So there is $\phi \in \hat{A}$ having this maximal ideal as its kernel.

Then $\phi(\lambda - a) = 0$ and $\phi(a) = \lambda$.

Proposition: range$(\hat{a}) = \sigma_a(a)$.

Let $A = C_b(\mathbb{Z}^-)$ be the algebra of all bounded $\mathbb{C}$-valued sequences $\|\|_\infty$.

Let $I = C_\infty(\mathbb{Z}^+)$ an ideal. (proper, norm-closed)

Then I is contained in a maximal ideal of A, so there is a $\phi \in \hat{A}$ with $\phi(I) = \{0\}$.

Such maximal ideals are "not constructive". (logician term)

Consequence of the use of Zorn's Lemma to say that a maximal ideal exists.

Let R be any unital ring, e.g. finite field, and $\mathscr{R} = \prod_{n=1}^\infty R$, $I = \bigoplus_{n=1}^\infty \mathscr{R}$.

Then exists maximal ideal of R containing I, same difficulty/situation.

I.e. does not have anything to do with Banach algebras.

$\hat{A} = \beta\mathbb{Z}^+$ the Stone-Cêch compactification of the positive integers

The *expected radius* of a is $\max\{|\lambda| : \lambda \in \sigma(a)\}$

Because of range$(\hat{a}) = \sigma(a)$ this is the same as $\|\hat{a}\|_\infty$.

This definition works also for a non-commutative algebra A.

But consequence applies only to commutative algebras. (Gelfand transform def'd)

For A commutative Banach algebra with 1 and any $a, b \in A$.

$r(ab) \le r(a)r(b)$ and $r(a+b) \le r(a) + r(b)$. ($ab \ne ba \to$ can fail)

Proof:

$r(ab) = \|(\hat{ab})\|_\infty = \|\hat{a}\hat{b}\|_\infty \le \|\hat{a}\|_\infty \|\hat{b}\|_\infty = r(a)r(b)$.

Note: the spectral radius is independent of the containing algebra.

First form of "holomorphic functional calculus".

Given $a \in A$ Banach algebra with 1.

Let f be a function holomorphic on an open subset of $\mathbb{C}$ containing $\{z : |z| \le \|a\|\}$.

Thus f has a power series expansion $f(z) = \sum_{n=0}^\infty \alpha_n z^n$ that converges absolutely and uniformly on $\{z : |z| \le \|a\|\}$.

Thus can define $f(a) := \sum_{n=0}^\infty \alpha_n a^n$.

Proposition (proof next time): If $\lambda \in \sigma(a)$, then $f(\lambda) \in \sigma(f(a))$.

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Let A be unital Banach algebra, let $a \in A$.

Proposition: let f be analytic function with power series expansion that converges on some open subset of $\mathbb{C}$ that contains $\{z : |z| \le \|a\|\}$. Then $f(a) = \sum \alpha_n a^n \in A$ and if $\lambda \in \sigma(a)$ then $f(\lambda) \in \sigma(f(a))$.

Proof:

$f(\lambda) = f(a) = \sum_{n=0}^{\infty} \alpha_n \lambda^n - \sum_{n=0}^{\infty} \alpha_n a^n = \sum_{n=1}^{\infty} \alpha_n (\lambda^n - a^n)$

$= \sum_{n=1}^{\infty} \alpha_n (\lambda - a)(\lambda^{n-1} + \lambda^{n-2} a + \cdots + \lambda a^{n-2} + a^{n-1})$

Call the telescoping sum $P_n(\lambda, a)$.

Then $\|P_n(\lambda, a)\| \le n\|a\|^{n-1}$
Continue, equals $(\lambda - a)\sum \alpha_n P_n(\lambda, a)$
$f'(z) = \sum_{n=1}^{\infty} \alpha_n n z^{n-1}$, converges absolutely uniformly for $|z| \le \|a\|$.
$\sum_{n=1}^{\infty} \alpha_n P_n(\lambda, a) = b \in A$.
So $f(\lambda) - f(a) = (\lambda - a)b$
If $(f(\lambda) - f(a))$ has inverse c, then $1 = (\lambda - a)bc$, so $\lambda - a)$ is invertible.
Thus since $\lambda \in \sigma(a)$, $f(\lambda) = f(a)$ is not invertible so $f(\lambda) \in \sigma(f(a))$.
This proof above is the beginnings of the spectral mapping theorem.

Consider $f(z) = z^n$.
Then if $\lambda \in \sigma(a)$ then $\lambda^n \in \sigma(a^n)$.
Thus $|\lambda^n| \le \|a^n\|$, so $|\lambda| \le \|a^n\|^{1/n}$.
Thus $|\lambda| \le inf_n\{\|a^n\|^{1/n}\}$.
Corollary: $r(a) \le inf_n\{\|a^n\|^{1/n}\}$.
This expression doesn't depend on the containing algebra.

Consider the resolvant of a, at ∞.
$R(a, z^{-1})\frac{1}{z-a} = z(1 - az)^{-1} = z\sum_{n=0}^{\infty} a^n z^n$ converges for $\|az\| < 1$ i.e. $|z| < \|a\|^{-1}$

However since $R(a, z)$ is analytic for $|z| > r(a)$, $R(a, z^{-1})$ is analytic for $|z| < r(a)^{-1}$.
Power series converges in the largest circle in which it's analytic.
Should converge in the larger circle. Note we are in Banach-algebra-valued functions.
Right now we assume we don't know that, so use method from before.
Composing with linear operators on the algebra.
For any $\varphi \in A'$ the dual space of the algebra, $f_\varphi(z) = \varphi(z(1 - az)^{-1})$.
Then f_φ is an ordinary holomorphic function, holomorphic in $\{z : |z| < r(a)^{-1}\}$.
So the power series expansion of f_φ about 0 converges for $|z| < r(a)^{-1}$.
But the power series for f_φ is $z\sum_{n=0}^{\infty} \varphi(a^n) z^n$.
This will converge for $|z| < r(a)^{-1}$.
Thus for any $r > r(a)$, $z\sum_{n=0}^{\infty} \varphi(a^n) z^n$ will converge absolutely and uniformly for $|z| \le r^{-1}$.
So there is M_φ such that $|\phi(a^n)||z^n| \le M_\varphi \forall n$, $\forall z$ with $|z| \le r^{-1}$.
So $|\varphi(a^n) r^{-n}| \le M_\varphi$ for all n.
For each n define $F_n \in A''$ by $F_n(\varphi) = \varphi(a^n) r^{-n}$.
Thus $|F_n(\varphi)| \le M_\varphi$ for all φ.
Recall the uniform boundedness theorem (consequence of the Baire category theorem).

Says here that $\exists M$ such that $\|F_n\| \leq M \; \forall n$.
But it is clear that $\|F_n\| = \|a^n\| r^{-n}$.
Thus $\|a^n\| r^{-n} \leq M$ i.e. $\|a^n\| \leq M r^n \to \|a^n\|^{1/n} \leq M^{1/n} r$
As $n \to \infty$, $M^{1/n} \to 1$.
So the $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r \; \forall r > r(a)$ (Box it!)
Thus $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$.
And we have $r(a) \leq \inf_n \{\|a^n\|^{1/n}\}$.

Thus, Theorem: $r(a) = \lim \|a^n\|^{1/n}$. In particular, this limit exists.
(Gelfand's spectral radius formula).
Just needed a unital Banach algebra over $\mathbb{C}$.

Corollary: $r(a)$ does not depend on the containing algebra.

Corollary: Let A be a commutative Banach algebra with 1 over $\mathbb{C}$.
Then the Gelfand transform $a \mapsto \hat{a}$ from A to $C(\hat{A})$ is isometric exactly if $\|a^2\| = \|a\|^2$ for all $a \in A$.

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