

Ch 2

A group G is a 4-tuple $G = (|G|, \mu, \iota, e)$ with
 underlying set $|G|$
 law of composition μ
 inverse function ι
 neutral element e

A more common representation of a group uses symbols $G = (|G|, \cdot, {}^{-1}, e)$

We may also say that a set $|G|$ with a map $|G| \times |G| \rightarrow |G|$ constitutes a group if

$$(\forall x, y, z \in |G|) (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

there exists $e \in |G|$ such that $(\forall x) e \cdot x = x = x \cdot e$ and $(\forall x \in |G|)(\exists y \in |G|) y \cdot x = e = x \cdot y$

However, unlike the first, does not consist of identities (universally quantified equations)

note: universal quantification \leftrightarrow "for all"

An I -tuple of elements of X , $(x_i)_{i \in I}$ is an $f : I \rightarrow X$

The set of all such f is denoted X^I

The arity of an operation (e.g. 1 if unary, 2 if binary, etc.)

An I -ary operation on S is a map $S^I \rightarrow S$

Can think of the identity as a 0-ary/zeroary operation of the structure

S^0 has exactly one map, $\emptyset \rightarrow S$, so a map $S^0 \rightarrow S$ is determined by one element

Note that when $S = \emptyset$ there is a unique n -ary operation for $n > 0$ but *no* 0-ary operation

A group-theoretic relation in $(\eta_i)_I$ is an equation $p(\eta_i) = q(\eta_i)$ holding in G

The terms in the elements of X under the formal group operations μ, ι, e form a set T :

given with functions $\text{symp}_T : X \rightarrow T$, $\mu_T : T^2 \rightarrow T$, $\iota_T : T \rightarrow T$, and $e_T : T^0 \rightarrow T$

such that each map is one-to-one, its images disjoint, and T is the union of those images

and T is generated by $\text{symp}_T(X)$ under the aforementioned operations

that is, T has no proper subset containing $\text{symp}_T(X)$ and closed under those operations

We can represent these terms, for groups, using strings of symbols

We need full parentheses notating order of operations to ensure disjoint images

A set-theoretic approach dispenses with strings and allows for infinite arities

For the example of a group, we would have (using ordered pair, 3-tuple, etc.):

for $x \in X$, $\text{symp}_T(x) := (*, x)$

for $s, t \in T$, $\mu_T(s, t) := (\cdot, s, t)$

for $s \in T$, $\iota_T(s) := ({}^{-1}, s)$

and $e_T = (e)$

and by set theory, no element can be written as such an n -tuple in more than one way

Given a set map $f : X \rightarrow |G|$ for a group G

Recursive evaluation of $s_f \in |G|$ given an X -tuple of symbols $s \in T = T_{X, \cdot, {}^{-1}, e}$

if $s = \text{symp}_T(x)$ for some $x \in X$, then $s_f := f(x)$

$s = \mu_T(t, u) \rightarrow s_f = \mu_G(t_f, u_f)$, assuming that given $t, u \in T$ we know $t_f, u_f \in |G|$
 similarly, $s = \iota_T(t) \rightarrow s_f = \iota_G(t_f)$, assuming we know t_f given t
 finally $s = e_T \rightarrow s_f = e_G$

Varying f in addition to T gives an evaluation map $(T_{X, -, e}) \times |G|^X \rightarrow |G|$

Alternatively, fixing $s \in T$ gives a map $s_G : |G|^X \rightarrow |G|$

these represent substitution into s

these s_G are the derived n -ary operations (aka term operations) of G

distinct terms can induce the same derived operation

e.g. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in general or others for certain groups

Examples of derived operations on groups

conjugation $\xi^\eta = \eta^{-1} \xi \eta$ (binary)

commutator $[\xi, \eta] = \xi^{-1} \eta^{-1} \xi \eta$ (binary)

squaring (unary)

An Ω -algebra is a system $A = (|A|, (\alpha_A)_{\alpha \in |\Omega|})$

here $|A|$ is some set, and for each $\alpha \in |\Omega|$, $\alpha_A : |A|^{ari(\alpha)} \rightarrow |A|$

note that often people will use $n(\alpha)$ (rather than $ari(\alpha)$) for the arity of an operation α

e.g. for a group, $|\Omega| = \{\mu, \iota, e\}$, $ari(\mu) = 2$, $ari(\iota) = 1$, and $ari(e) = 0$

Ch 3

Ch 4

the subgroup and normal subgroup of G generated by $S \subset |G|$

$\langle S \rangle$ contains S and is contained in every subgroup which contains S

$\forall x \in \langle S \rangle$ $x = e$ or $x = \prod s_i$, $s_i \in S$ or $s_i^{-1} \in S$

is the image of the map into G of the free group F on S induced by the inclusion $S \rightarrow |G|$

there is additionally a least normal subgroup of G containing S .

relations on a group/quotient groups

Quotient groups: homomorphisms causing certain elements to fall together.

Satisfies e.g. $(\forall i \in I) f(x_i) = f(y_i) \leftrightarrow (\forall i \in I) f(x_i y_i^{-1}) = e$

A set of elements annihilated by a group homomorphism form a normal subgroup.

Leads to $q : G \rightarrow G/N$, where N is this normal subgroup.

We have a quotient map and a quotient group.

This map has the universal property desired:

For every homomorphism $h : G \rightarrow K$ satisfying the above, $\exists! g : N \rightarrow K$, s.t. $h = g \circ q$.

This construction *imposes the relations* $x_i = y_i$ ($i \in I$) on G , forming $G/(x_i = y_i | i \in I)$.

For G a group, a G -set is a pair $S = (|S|, m)$, $|S|$ a set and $m : |G| \times |S| \rightarrow |S|$, satisfying

$(\forall s \in |S|, g, g' \in |G|) g(g's) = (gg')s$

$(\forall s \in |S|) es = s$

That is, a set on which G acts by permutations.

A homomorphism $S \rightarrow S'$ of G -sets (for G fixed) is a map $a : |S| \rightarrow |S'|$ satisfying

$(\forall s \in |S|, g \in |G|) a(gs) = ga(s)$

The set of left cosets of H in G is $|G/H|$ and a typical coset $[g] = gH$.

Then $|G/H|$ is the underlying set of a left G -set G/H by $g[g'] = [gg']$

groups presented by generators and relations

Let X be a set, T the set of all group-theoretic terms in X , and $R \subset T \times T$.

\exists a universal example of a group with X -tuples of elements satisfying the relations R .

That is, there is a pair (G, u) with G a group and $u : X \rightarrow |G|$ satisfying:

$(\forall (s, t) \in R) s_u = t_u$

such that for any group H and X -tuple v of elements in H satisfying

$$(\forall (s, t) \in R) s_u = t_u$$

$\exists ! f : G \rightarrow H$ a homomorphism, such that $v = fu$

The pair (G, u) is determined up to canonical isomorphism by these properties

The group G is generated by $u(X)$

Proving the existence of such a universal construction.

Two ways: construction from terms and subgroups of direct products.

In these approaches, apply the additional conditions to the group axioms.

A proof that builds upon prior constructions:

Let (F, u_F) be the free group on X and N be a normal subgroup.

Define N such that it is generated by $\{s_{u_F} t_{u_F}^{-1} \mid (s, t) \in R\}$

Take a canonical map $q : F \rightarrow F/N$.

Then the pair $(F/N, q \circ u_F)$ has the desired universal properties.

As a consequence of the universal properties of free and quotient groups.

Class Question #9

Definition 5.5.13 Unimportant

Just to clarify my understanding of the definition of a cardinal, is it correct that each of the natural numbers is itself a cardinal, but ordinals such as $\omega + 1$ and ω^2 are not, themselves, cardinals?