Math 245A

Fall 2015

Chapter 2

2.2 Groups

A group G is a 4-tuple $G = (|G|, \mu, \iota, e)$ with
underlying set $|G|$
law of composition μ
inverse function ι
neutral element e
(Exercise 2.2:1) A homomorphism from a group G to a group H is a function $\phi : G \to H$ satisfying the following for $a, b \in G$:
$\phi(e_G) = e_H$
$\phi(\iota_G(a)) = \iota_H(\phi(a))$
$\phi(\mu_G(a,b)) = \mu_H(\phi(a), \phi(b))$
A more common representation of a group uses symbols $G = (|G|, \cdot, ^{-1}, e)$
(2.2.1) The conditions for a 4-tuple to be a group are as follows
$(\forall x, y, z \in |G|)\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
$(\forall x \in |G|)\ e \cdot x = x = x \cdot e$
$(\forall x \in |G|)\ x^{-1} \cdot x = e = x \cdot x^{-1}$
(2.2.2) We may also say that a set $|G|$ with a map $|G| \times |G| \to |G|$ constitudes a group if
$(\forall x, y, z \in |G|)\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
there exists $e \in |G|$ such that $(\forall x) e \cdot x = x = x \cdot e$ and $(\forall x \in |G|)(\exists y \in |G|) y \cdot x = e = x \cdot y$
(2.2.1) consists of identities (universally quantified equations) and (2.2.2) does not
note: universal quantification is a ”for all” quantification
(Exercise 2.2:2)
(i)
(ii)
(Exercise 2.2:3)

2.3 Indexed Sets

An *I-tuple* of elements of X, $(x_i)_{i \in I}$ is formally defined as an $f : I \to X$
The set of all functions from I to X is denoted X^I

2.4 Arity

The *arity* of an operation is, e.g., 1 if unary, 2 if binary, etc.

An I-ary operation on S is a map $S^I \to S$

Group: a set, a binary operation, a unary operation, and a distinguished element

Can think of the identity as a 0-ary/zeroary operation of the structure

S^0 has exactly one map, $\emptyset \to S$, so a map $S^0 \to S$ is determined by one element

Note these are not strictly identical since one is a map and the other the element itself

But they are in 1-to-1 correspondence and give equivalent information

2.5 Group-theoretic terms

A *group-theoretic relation* in $(\eta_i)_I$ is an equation $p(\eta_i) = q(\eta_i)$ holding in G

p and q are are *group-theoretic terms* which we formally define

The terms in the elements of X under the formal group operations μ, ι, e form a set T:

given with functions $symb_T : X \to T$, $\mu_T : T^2 \to T$, $\iota_T : T \to T$, and $e_T : T^0 \to T$

such that each map is one-to-one, its images disjoint, and T is the union of those images

and T is generated by $symb_T(X)$ under the aforementioned operations

that is, T has no proper subset containing $symb_T(X)$ and closed under those operations

We can represent these terms, for groups, using strings of symbols

We need full parentheses notating order of operations to ensure disjoint images

A set-theoretic approach dispenses with strings and allows for infinite arities

For the example of a group, we would have (using ordered pair, 3-tuple, etc.):

for $x \in X$, $symb_T(x) := (*, x)$

for $s, t \in T$, $\mu_T(s, t) := (\cdot, s, t)$

for $s \in T$, $\iota_T(s) := (^{-1}, s)$

and $e_T = (e)$

and by set theory, no element can be written as such an n-tuple in more than one way

2.6 Evaluation

Given a set map $f : X \to |G|$ for a group G

Recursive evaluation of $s_f \in |G|$ given an X-tuple of symbols $s \in T = T_{X,\cdot,-1,e}$

if $s = symb_T(x)$ for some $x \in X$, then $s_f := f(x)$

$s = \mu_T(t, u) \to s_f = \mu_G(t_f, u_f)$, assuming that given $t, u \in T$ we know $t_f, u_f \in |G|$

similarly, $s = \iota_T(t) \to s_f = \iota_G(t_f)$, assuming we know t_f given t

finally $s = e_T \to s_f = e_G$

Varying f in addition to T gives an evaluation map $(T_{X,\cdot,-1,e}) \times |G|^X \to |G|$

Alternatively, fixing $s \in T$ gives a map $s_G : |G|^X \to |G|$

these represent substitution into s

these s_G are the *derived n-ary operations* (aka *term operations*) of G

distinct terms can induce the same derived operation

e.g. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in general or others for certain groups

Examples of derived operations on groups

conjugation $\xi^\eta = \eta^{-1}\xi\eta$ (binary)

commutator $[\xi,\eta] = \xi^{-1}\eta^{-1}\xi\eta$ (binary)
squaring (unary)
δ (Exercise 2.2:2)
σ (Exercise 2.2:3)

First Class Question

The last example of a derived operation on groups cited the trivial "second component" function, $p_{3,2}(\xi,\eta,\zeta) = \eta$ induced by $y \in T_{\{x,y,z\},^{-1},\cdot,e}$. I wasn't entirely sure how this derived operation would be represented as an element of $T_{\{x,y,z\},^{-1},\cdot,e}$. Would $p_{3,2}$ be the element $(*,y)$ (in the set-theoretic notation)?

Terms in other families of operations

An Ω-*algebra* is a system $A = (|A|,(\alpha_A)_{\alpha\in|\Omega|})$
here $|A|$ is some set, and for each $\alpha \in |\Omega|$, $\alpha_A : |A|^{ari(\alpha)} \to |A|$
note that often people will use $n(\alpha)$ (rather than $ari(\alpha)$) for the arity of an operation α
e.g. for a group, $|\Omega| = \{\mu,\iota,e\}$, $ari(\mu) = 2$, $ari(\iota) = 1$, and $ari(e) = 0$

Lecture 8/28

Operations, terms, algebra

revisit the difficulty with distinguished elements as a zeroary operation when dealing with the empty set
the idea behind distinguishing between terms is e.g. to have distinct objects that we can compare
$(x \cdot y) \cdot z \neq x \cdot (y \cdot z)$ as terms, allowing
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ to be a useful statement about groups
set-theoretic approach, infinite arity
(μ,s,t)
$(\mu,(s,t))$
$\alpha_T : T^X \to T$ using $(\alpha,(S_X)_{x\in X})$
X here shall be some cardinal

Next reading: free groups

$x,y,z \in G$ and $\xi,\eta,\zeta \in H$
when can we have a homomorphism $G \to H$
if and only if the relations that hold in G hold in H for the corresponding elements

Exercises in today's reading

2.7:3

can't have $s(,,,,,,...) = s'(,,,,...) = s''(,,,,,...)$ where the s'' term is the same as the s term

2.2:2 and 2.2:3

$\delta_G(x,y) = xy^{-1}$ and $\sigma_G(x,y) = xy^{-1}x$

$G = \mathbb{Z}$ knowledge of the identity

$x * +y = (x-1) + (y-1) + 1$