Math 206

Fall 2015

8/26

Definitions

A *norm* on a vector space X (over $\mathbb{F}$) is a function $\|\| : X \to \mathbb{R}^+$ such that
$\|x\| = 0$ iff $x = 0$
$\|\alpha x\| = |\alpha|\|x\|$ (for $\alpha \in F$)
$\|x + y\| \leq \|x\| + \|y\|$
An *algebra* $\mathscr{A}$ over $\mathbb{F}$ is a vector space with distributive $\cdot$ satisfying
$cx \cdot y = c(x \cdot y)$
$x \cdot cy = c(x \cdot y)$ for all $c \in F$
A *normed algebra* over $\mathbb{R}$ or $\mathbb{C}$ is an algebra $\mathscr{A}$ equipped with (vector space) norm satisfying
$\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathscr{A}$
A norm on $\mathscr{A}$ induces a metric
$d(a, b) = \|a - b\|$ on $\mathscr{A}$ and therefore a topology
if $\mathscr{A}$ is complete for this norm, it is a *Banach algebra*

To figure out (use https://www.math.ksu.edu/ nagy/real-an/2-05-b-alg.pdf)

Supposing $\mathscr{A}$ is not necessarily complete
$\|ab\| \leq \|a\|\|b\|$ gives uniform continuity on the product
hence the norm can be extended to the completion $\bar{\mathscr{A}}$ to form a Banach algebra
A metric space M is complete if all Cauchy sequences converge to an element of M
The completion M is all equivalence classes of Cauchy sequences where
$\{a_n\} \sim \{b_n\}$ iff $\lim\limits_{x\to\infty} d(a_n - b_n) = 0$

Examples

For M a compact space, $C(M)$
the set of continuous $\mathbb{R}/\mathbb{C}$-valued functions on M
pointwise operations
$\|f\|_\infty = sup\{|f(x)| : x \in M\}$
For M locally compact, $C_\infty(M)$

the set of continuous $\mathbb{R}/\mathbb{C}$-valued functions on M vanishing at ∞
vanishing at ∞: $\forall \epsilon \exists$ a compact subset of M, outside of which $f < \epsilon$
note that this is non-unital (lacks an identity)
For $\mathscr{O} \subset \mathbb{C}^n$ open
$H^\infty(\mathscr{O})$ the set of all bounded holomorphic functions on $\mathscr{O}$
(M, d) metric space and $f \in C(M)$
Lipschitz constant (which can be $+\infty$) $L_d(f) = sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in M, x \neq y\}$
The *Lipschitz functions* $\mathscr{L}_d(M,d) = \{f : L(f) < \infty\}$
These form a dense subalgebra of C(M) and are in fact a *-subalgebra
$\|f\|_d := \|f\|_\infty + L_d(f)$, can be shown as a normed-algebra norm
$L_d(M,d)$ is complete for this norm
so $L_d(M,d)$ is a Banach algebra
L_d is a seminorm on $\mathscr{L}_d(M,d)$ since it takes value 0 on the constant functions
can recover d from L_d
M a differentiable manifold (e.g. $T = \mathbb{R}/\mathbb{Z}$ the circle)
C(M) $\supseteq C^{(1)}(M)$ the singly-differentiable functions
$f \in C^{(2)}(T) \to Df : T_x M \to \mathbb{R}, \mathbb{C}$
with Df the derivative and T_x the tangent space
If we put on a Riemmannian metric, define $\|f\|^{(1)} = \|f\|_\infty + \|Df\|_\infty$
If $f \in C^{(1)}(T) : \|f\|^{(1)} = \|f\|_\infty + \|f'\|_\infty$
Banach algebra norm, for which this space of functions is complete
For the circle, $C^{(2)}(T) \to \|f\|^{(2)} = \|f\|_\infty + \|f'\|_\infty + \frac{1}{2}\|f''\|_\infty$
the factor $\frac{1}{2}$ ensures that this satisfies the normed algebra condition
$C^{(n)}(T) = \sum\limits_{k=0}^{n} \frac{1}{k!}\|f^{(k)}\|_\infty$
For $C^\infty(T)$ using the collection of norms $\{\|\|^{(n)}\}_{n=1}^\infty$ yields a Fréchet algebra
A Fréchet algebra has a topology defined by a countable family of seminorms
that respect the algebra structure and is complete **(clarify)**
non-commutative algebras
X a Banach space
$\mathscr{B}(X)$ the algebra of bounded operators on X
$\|\|$ operator norm $\to$ Banach algebra
Any closed subalgebra of $\mathscr{B}(X)$ is a Banach algebra

8/28

Sketch of the course
X a Banach space, $B(X)$ bounded functions on the space
$\mathscr{H}$ a Hilbert space, $\mathscr{B}(\mathscr{H})$ bounded operators on the space
for $T \in \mathscr{B}(\mathscr{H}) \exists$ adjoint operator $T^* \in \mathscr{B}(\mathscr{H})$
$< T\xi, \eta > = < \eta, T^*\xi >$ for $\xi, \eta \in \mathscr{H}$
adjoint is additive, conjugate linear, $T^{**} = T$, $(ST)^* = T^*S^*$
An algebra A over $\mathbb{R}$ or $\mathbb{C}$ is a **-algebra* if it has a $* : A \to A$ satisfying

certain properties (look up)

A **-normal algebra* is a normal *-algebra such that

$(\forall a \in A)\|a^*\| = \|a\|$

A *Banach *-algebra* is a *-normal algebra that is a Banach algebra.

For any $T \in \mathscr{B}(\mathscr{H})$, have $\|T^*T\| = \|T\|^2$ **(check: parse through defns)**

For M a locally compact space, $A = C_\infty(M, \mathbb{C})$, $f^* := \bar{f}$ is a Banach *-algebra

Also have $\|f^*f\| = \|f\|^2$ **(verify: should be easier than the other)**

Little Gelfand-Naimark theorem:

Let A be a commutative Banach *-algebra satisfying $\|a^*a\| = \|a\|^2$.

Then $A \cong C_\infty(M)$ for some locally compact M.

One view of the "spectral theorem"

Let $T \in \mathscr{B}(\mathscr{H})$ with $T^* = T$

Let A be the closed subalgebra of $\mathscr{B}(\mathscr{H})$ generated by T and I (i.e. $p(T) := \Sigma\alpha_k T^K$)

Polynomials closed or stable under *

If $S \in A$ then $S^* \in A$ (i.e. A is a *-subalgebra of $\mathscr{B}(\mathscr{H})$

So A is a Banach *-suubalgebra satisfying $\|S^*S\| = \|S\|^2$

Moreover, A is commutative. (unital, since generated by I)

Then by the Little Gelfand-Naimark theorem, $A \cong C(M)$

Indeed $M \subset \mathbb{R}$, the spectrum of T

If $\mathscr{H}$ is finite dimensional, then M is the set of eigenvalues of T

T is normal if $TT^* = T^*T$

A C*-algebra is a Banach *-algebra over $\mathbb{C}$ satisfying

$\|a^*a\| = \|a\|^2$

Theorem: A commutative C*-algebra is $\cong C_\infty(M)$.

Big Gelfand-Naimark Theorem: (*Math 208, C*-algebras*)

Any C*-algebra is $\cong$ to a closed *-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.

Tangent

algebraic topology, differential geometry, Riemann manifolds, "non-commutative geometry" (Connes)

A *von-Neumann algebra* is a *-subalgebra of $\mathscr{B}(\mathscr{H})$

which is closed under the strong operator topology.

Every commutative von-Neumann algebra is $\cong L^\infty(X, S, \mu)$ (measure spaces) acting on $L^2(X, S, \mu)$ by positive sldkjfalksdjf

For group G, $\alpha : G \to Auto(X) \subseteq \mathscr{B}(X)$

$Auto(X)$ a Banach space

Look at subalgebra of $\mathscr{B}(X)$ generated by $\alpha(G)$.

Leads to considering $l'(G)$ with product $(f \star g)(x) = \Sigma f(y)g(y^{-1}x)$ convolution

$f^*(x) = \overline{f(x^{-1})}$

Banach *-algebra, G commuatative $\to$ Fourier transform

8/31

K a field, X a set, $\mathscr{F}(X, K)$ the set of all K-valued functions on X with pointwise operations

Given $f \in \mathscr{F}(X, K)$.

Let $\lambda \in K$. Then $\lambda \in$ range(f) exactly if $(f - \lambda 1)$ is not invertible.

For any $a \in A$, the *spectrum* of a is $\{\lambda \in K : a - \lambda 1_A$ is not invertible in A$\}$.

The spectrum depends on the containing algebra

Assuming that this algebra A (over the field K) has an identity 1_A

Example: Let A = C([0, 1]), and B = polynomials, viewed as a dense subalgebra of A.

Let p be a polynomial of degree ≥ 2. Then

$\sigma_a(p) = p([0,1])$.

$\sigma_B(p) = \mathbb{R}, \mathbb{C}$

Let A be a Banach algebra with 1 ($\|1\| = 1$), and $a \in A$.

If $\|a\| < 1$, then 1 - a is invertible, and $\|(1-a)^{-1}\| \leq \frac{1}{1-\|a\|}$

Proof:

$\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$ ($a^0 := 1_A$)

For any $n > 0$, let $s_n = \sum_{k=0}^{n} a^k$.

Show that $\{s_n\}$ is a Cauchy sequence.

If $n > m$, $\|s_n - s_m\| = \|\sum_{k=m+1}^{n} a^k\| \leq \sum_{k=m+1}^{n} \|a\|^k$.

Given $\epsilon > 0 \exists N$ such that if $m, n \geq N$ then $\sum_{k=m+1}^{n} \|a\|^k \leq \epsilon$

So $\{s_n\}$ is a Cauchy sequence.

By completeness there is a $b \in A$ with $s_n \to b$ as $n \to \infty$.

Want to show $b = (1-a)^{-1}$.

$b(1-a) = lim_{n\to\infty}(s_n(1-a))$

$= lim_{n\to\infty}(1 + a + a^2 + a^3 + \cdots + a^n - (a + a^2 + a^3 + \cdots + a^{n+1}))$

$= lim_{n\to\infty}(1 - a^{n+1}) = 1$

Then $1 - (1-a)$ is invertible, i.e. a is invertible.

$\|(1-a)^{-1}\| = \lim \|s_n\| \leq \lim \sum_{k=0}^{n} \|a^k\| = \frac{1}{1-\|a\|}$ ($\|1\| = 1$)

$\|ab\| \leq \|a\|\|b\|$: can very easily check that multiplication is cts (do this?)

Corollary: If $a \in A$ and $\|1 - a\| < 1$ then a is invertible, and $\|a^{-1}\| \leq \frac{1}{1-\|1-a\|}$

I.e. the open unit ball about 1 consists of invertible elements.

Let $a \in A$. Let L_a, R_a be the operators of left and right multiplication by a on A.

$a \to L_a$ is an algebra homomorphism of A into $\mathscr{L}(A)$ (linear operators on A)

$L_a L_b = L_{ab}$, $R_a R_b = R_{ba}$ (R is an antihomomorphism)

$1 \in A$

If a is invertible, then so is L_a, $L_a L_{a^{-1}} = I_a$

Then if A is a normed algebra, $\|L_a\| = \|a\|$

$\|L_{ab}\| = \|ab\| \leq \|a\|\|b\|$

$\|L_a 1_a\| = \|a\|$

so if $a \in A$ is invertible, then L_a is a homeomorphism of A onto itself.

Thus if A is a Banach algebra, with 1, and a is invertible:

$\{L_a b : \|1 - b\| < 1\}$ is an open neighborhood of a consisting of invertible elements

Let GL(A) be the set of invertible elements of A. (general linear group)

Then (for A a unital Banach algebra) GL(A) is an open subset of A.

(Fails for Poly $\subseteq$ C([0, 1]))

Two Fréchet algebras, for one, GL(A) is an open subset, for another it isn't.

ask about this?: not sure what he was talking about

$C^\infty(T)$, $\|f^{(n)}\|$

C(R) cont fns on $\mathbb{R}$ (or $\mathbb{C}$) maybe unbounded
For each n let $\|f\|_n = \sup\{|f(t)| : |t| \leq n\}$
Corollary: For A a Banach algebra with 1 and $a \in A$, $\sigma(a)$ is a closed subset of $\mathbb{C}$

9/2

Proposition: Let A be a unital Banach algebra and $a \in A$.
Then $\sigma(a)$ is a closed subset of $\mathbb{C}$ or $\mathbb{R}$. If $\lambda \in \sigma(a)$ then $\|\lambda\| \leq \|a\|$.
Proof: $\sigma(a) = \{\lambda : (a - \lambda) \text{ is not invertible}\}$
Its complement, the *resolvant set*, of a is $\{\lambda : (a - \lambda) \in GL(A)\}$, is open.
If $|\lambda| > \|a\|$ then $(\lambda - a) = \lambda(1 - \frac{a}{\lambda})$, $\|a/\lambda\| < 1$
so $(\lambda - a)$ is invertible, ie $\lambda \in \sigma(a)$.
Over $\mathbb{R}$, can have $\sigma(a) = \emptyset$, e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
"If $a \in GL(A)$ and b is close to a then b^{-1} is not much bigger than a^{-1}".
Let $\mathscr{O} = \{c : \|1 - c\| < 1/2\}$
So c is invertible, and $\|c^{-1}\| \leq \frac{1}{1 - \|1-c\|} \leq 2\}$
Let $b \in a\mathscr{O}$, so $b = ac$ for $c \in \mathscr{O}$, then $\|b^{-1}\| = \|c^{-1}a^{-1}\| \leq 2\|a^{-1}\|$.
For $a, b \in GL(A)$.
$b^{-1} - a^{-1} = b^{-1}(a - b)a^{-1}$
Thus $\|b^{-1} - a^{-1}\| \leq \|b^{-1}\|\|a - b\|\|a^{-1}\|$.
So $b \to b^{-1}$ is continuous for the norm.
So $GL(A)$ is a topological group for topology from norm.
$b^{-1} = (1 + b^{-1}(a - b))a^{-1}$
On $\rho(a)$ (the resolvant set, complement of the spectrum) define the resolvant of a
This is the function $R(a, \lambda) = (\lambda - a)^{-1}$
$R(a, \lambda)$ is an analytic function on $\rho(a)$.
Proof: Let $f(z) = R(a, z)$.
$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{(z+h-a)^{-1} - (z-a)^{-1}}{h}$
$= \lim_{h \to 0} \frac{1}{h}(z + h - a)^{-1}((z - a) - (z + h - a))(z - a)^{-1}$
$= \lim_{h \to 0} -(z + h - a)^{-1}(z - a)^{-1} = -(z - a)^{-2}$
$f'' = +z(z - a)^{-3}$
Given $z_0 \in \rho(a)$
Will use $b^{-1} = (1 + b^{-1}(a - b))a^{-1}$ and $f(z) = (z - a)^{-1} = \sum c_n(z - z_0)^n$
$f(z) = (z - a)^{-1}$
$b \to z - a$
$f(z) = (1 + (z - a)^{-1}((z_0 - a) - (z - a))(z_0 - a)^{-1}$
$= (1 + (z - a)^{-1}(z_0 - z))(z_0 - a)^{-1}$
where $(z - a)^{-1}(z_0 - z) \leq 1$ then the above
$= \sum(-1)^n(z - a)^{-n}(z - z_0)^n = \sum(-1)^n(z - a)^{-n-1}(z - z_0)^n$
a proper power series expansion.
Examine $R(a, z)$ at ∞.
$R(a, z^{-1}) = (z^{-1} - a)^{-1} = \frac{1}{z^{-1} - a}$

$= z(1 - za)^{-1}$ (for small z, ie $\|za\| < 1$)
$R(a,z^{-1})$ approaches 0 as $z \to 0$.
defn $R(a,0^{-1}) = 0$, see $R(a,z)$ is analytic at ∞.
Theorem: For a Banach algebra over $\mathbb{C}$ with 1, and for any $a \in A$,
$\sigma(a) \neq \emptyset$, that is, the spectrum is non-empty.
Proof: Suppose that $\sigma(a) = \emptyset$.
Then $R(a,z)$ is defined on all of $\mathbb{C}$ and is bounded.
By Liouville's, $R(a,z)$ is constant, $= 0$, $(a-z)^{-1} = 0 \ \forall z$
Why can we use Liouville's in this Banach space case?
Let A' be the dual Banach space to A.
For $\varphi \in A'$, $z \mapsto \varphi(R(a,z))$ is a $\mathbb{C}$-valued analytic function.
So set $\varphi(R(a,z)) = 0 \ \forall z, \forall \varphi$
so $R(a,z) = 0$.
Knowing that there is anything in here is the Hahn-Banach Theorem, depending on the axiom of choice.
Theorem (Gelfand-Mazer)
Let A be a unital Banach algebra over $\mathbb{C}$.
If every nonzero element is invertible, then $z \to z1_A$ is an isomorphism from $\mathbb{C}$ onto A.
Proof:
Given $a \in A$ let $z \in \sigma(a) \neq \emptyset$.
So $(z - a)$ is not invertible so $z - a = 0$.
Fails over $\mathbb{R}$ since have $\mathbb{R}$, $\mathbb{C}$, quaternions

9/4

Let A be an algebra or ring. Ideals I, left, right, 2-sided.
A/I, for a left ideal get a left A-module, right ideal get a right A-module
Two-sided get an algebra or ring.
Let A be a normed algebra, and if I is an ideal in A.
Then $\bar{I}$ (the closure) is again an ideal in A.
$\{a_n\} \subset I$, $a_n \to c \in A$ then $ba_n \to bc$
Proposition: If A is a unital Banach algebra and if I is an proper ideal in A.
Then $\bar{I}$ is proper.
Proof:
Use GL(A) is open. The original ideal cannot contain any invertible elements.
The complement of the invertible elements is going to be closed.
The ideal is in its closure; the closure is in the complement of the invertible elements.
So the closure will not contain any invertible elements.
Counter-example: C locally compact, $C_c(\mathbb{R}) \subset C_\infty(\mathbb{R})$
In fact $C_c(\mathbb{R})$ is the minimal dense ideal in $C_\infty(\mathbb{R})$
Lack of identity element.
Counter-example: Look at A the polynomials viewed as a subset of $C([0,1])$
Using the sup-norm.
Lack of completeness.

Let $I = \{p : p(2) = 0\}$ is an ideal, in fact a maximal proper ideal.
This ideal of polynomials will be dense inside A and dense inside all polynomials.
Counter-example: A = $C(\mathbb{R})$, including unbounded, $I = C_C(\mathbb{R})$, compact open topology
Compact in here for the Fréchet open topology.
Corollary: Let A be a unital Banach algebra.
Then every maximal ideal is closed.
Taking ideals to form quotients.
Recall that if X is a normed vector space and if Y is a vector subspace of X:
Then we can form the quotient vector space X/Y.
Have the evident $\pi : X \to X/Y$ by $\pi(x) = x + Y$.
Set $\|\pi(x)\| = inf\{\|x - y\| : y \in Y\}$ i.e. the distance from x to the subspace Y.
It is easily seen that this is a seminorm.
Problem: if Y is not closed, then it is not a norm, since $\|\pi(y)\| = 0$ for $y \in \overline{Y}$.
Therefore, if Y is closed, then $\|\|_{X/Y}$ is a norm.
(Should know from $202B$) If X is a Banach space and Y is a closed subspace.
Then X/Y with the norm defined above is a Banach space.
Trickier to prove, but a true statement.
Proposition: Let A be a normed algebra and let I be a closed ideal in A.
(can be left or right or two-sided)
Then if I is a two-sided ideal, then A/I with $\|\|_{A/I}$ is a normed algebra.
That is $\|\pi(a)\pi(b)\| \leq \|\pi(a)\|\|\pi(b)\|$.
Then if I is a left ideal, so that A/I with $\|\|_{A/I}$ is a left A-module.
$\|a\pi(b)\|_{A/I} \leq \|a\|_A\|\pi(b)\|_{A/I}$
And if I is a right ideal, so that A/I with $\|\|_{A/I}$ is a right A-module, similar.
Proof:
For I a two-sided ideal.
If $c, d \in I$, then $\|\pi(a)\pi(b)\|_{A/I} \leq \|(a - c)(b - d)\| = \|(ab - (cb + ad - cd)\|$
where $cb + ad - cd \in I$.
Take inf over $c, d \in I$.
$\leq \|a - c\|\|b - d\|$
Proposition: If a is a Banach algebra and I is a closed 2-sided ideal, then A/I is a Banach algebra.
An algebra or ring is *simple* if it contains no proper 2-sided ideals.
e.g. $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(K)$ for K a field.
Proposition: Let A be an algebra or ring and let I be a maximal 2-sided ideal in A.
The A/I is a simple algebra/ring.
Corollary: If A is a Banach algebra and if I is a maximal closed 2-sided ideal.
Then A/I is a simple Banach algebra.
Let A be a commutative algebra/ring with 1.
If $a \in A$ and a is not invertible, then Aa is a two-sided ideal.
And $1 \notin Aa$ (else it would be invertible).
Thus Aa is a proper ideal.
Corollary: If A is a commutative algebra/ring with 1 and if A is simple, then A is a field.
A is simple: every nonzero element is invertible.
Since if one weren't we would have a proper ideal.

Theorem: Let A be a commutative Banach algebra with 1.
Let I be a maximal ideal in A (which of course is necessarily closed).
Then $A/I \cong$ either $\mathbb{C}$ (isomorphic in every sense).
Proof:
Banach-Mazer theorem applied to the previous discussion.
$c \in A/I$, $z \in \sigma(c)$, $c - z1_A$ is noninvertible by def'n spectrum.
Since it is a field, $c - z1_A = 0$.
From a maximal ideal, I, get a homomorphism $\phi : A \to \mathbb{C}$, unital
($\phi(1) = 1$) such that $I = ker(\phi)$
Let A be a Banach algebra with 1 and let $\phi : A \to \mathbb{C}$ be a homomorphism.
Then ϕ is continuous and $ker(\phi)$ is a maximal 2-sided ideal in A.
$\|\phi\| = 1$
Lemma: For any given $a \in A$, $\phi(a) \in \sigma(a)$
Proof: $\phi(a - \phi(a)1_A) = 0$ so $a - \phi(a)1$ is not invertible.
so $\phi(a) \in \sigma(a)$
Then $\|\phi(a)\| \leq \|a\|$, so $\|phi\| \leq 1$ but $\phi(1) = 1$.

9/9

thanks for notes from Roy

Let A be a commutative Banach algebra over $\mathbb{C}$ with 1.
$\exists$ natural bijection b/t the maximal idea of A and the $\not\equiv 0$ homomorphisms $A \to \mathbb{C}$.
Given a homomorphism φ, then $I_\varphi = ker(\varphi)$.
Notation: $\hat{A}$ denotes the set of maximal ideals
equivalently, the set of $\not\equiv 0$ hom $\varphi : A \to \mathbb{C}$, with multiplicative linear functionals
$\hat{A}$ is also the maximal ideal space of A

For $\varphi \in \hat{A}$, $\phi(a) \in \sigma(a)$.
Corollary: $\|\varphi\| = 1$.

$\hat{A} \subset A'$. in fact $\hat{A} \subset$ unit ball of A'.
On A', have the weak-* topology. Alogue's Thm $\to$ unit ball of G' is compact.

Proposition: $\hat{A}$ is closed under the weak-* topology and thus compact. Proof:
Let $\{\varphi_x\}$ be a net of elements of $\hat{A}$ which converges to $\psi \in A'$.
Then for $a, b \in A$, we have:
$\psi(ab) = \lim \phi_\alpha(ab) = \lim \phi_\alpha(a)\phi_\alpha(b) = \phi(a)\phi(b)$
$\psi(1a) = \lim \phi_\alpha(1a) = \lim 1 = 1$.

Gelfand transform
For all $a \in A$, define $\hat{a} \in C(\hat{A})$ by $\hat{A}(\varphi) := \varphi(a)$.
$\hat{A}$ is continuous since if $\{\varphi_\alpha\} \to \varphi$ for weak-* topology,
then $\hat{A}(\varphi) = \varphi(a) = \lim \varphi_\alpha(a) = \lim \hat{A}(\varphi_\alpha)$.

Also $a \mapsto \hat{a}$ is a unital hom of A into $C(\hat{A})$.
Indeed for $a \cdot b \in A$, $(\hat{ab})\phi = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi)\hat{b}(\phi)$.
Therefore $(\hat{ab}) = \hat{a}\hat{b}$.
$\|\hat{A}\|_\infty \leq \|A\|$, since $\forall \varphi$, $\hat{a}(\varphi) = \varphi(a) \in \sigma(a)$.
$|\hat{a}(\phi)| = |\phi(a)| \leq \|a\|$.
Definition: The mapping $A \to \hat{A}$ is called the Gelfand transformation for A.

Note for $A = C_\infty(\mathbb{Z})$. Define $\varphi_n(f) = f(n)$ for $n \in \mathbb{Z}$. $\lim_{n\to\infty} =?$ in weak-*.

Let X be a Banach space. Define a product on X by setting all products = 0. Proposition:
The image of A under Gelfand transform separates the points of $\hat{A}$.
i.e. if $\hat{a}(\varphi) = \hat{a}(\varphi')$ for $a \in A$, $\varphi = \varphi'$, $\varphi(a) = \varphi'(a)$.
Suppose that a is generated by one element a_0 (+1), i.e. poly(a_0) is dense in A.
Then $\hat{A} \cong \sigma(a_0)$ homomorphic.
Proof:
Consider $\hat{a_0} : \hat{a} \to \sigma(a_0)$, $\hat{a_0}(\varphi) = \varphi(a_0) \in \sigma(a_0)$.
If $\varphi, \psi \in \hat{A}$ and $\varphi(a_0) = \psi(a_0)$, then $\varphi(p(a_0)) = \psi(p(a_0))$ for $p \in$poly.
As the poly(a_0) are dense $\varphi = \psi$.
Therefore $\hat{a_0} : \hat{a} \to \sigma(a_0)$ is inj.
Let $Z \in \sigma(a_0)$. Then $a_0 - Z$ is not invertible.
Thus $(a_0 - Z)A$ is a proper ideal. Let I be its closure.
Claim: I is maximal.
Consider A/I. $C_{1a+I} \supset$poly(a_0).
Use $(a_0 - Z)A \supset (a_0 - z)$poly.
C_1 = poly(a_0).
Let φ be the multiplicative linear functional such that $\varphi(a_0) = Z$.
image of $C_1 + I$ in A/I. $C_1 + I$ dense in A.
Therefore $A/I \cong \mathbb{C}$.

9/11

X a compact (hausd) space and $A = C(X)$. What is $\hat{A}$? (the maximal ideal space)
Each $x \in X$, $\phi_x(f) = f(x)$. Multiplicative linear functional.
Get map $X \to \hat{A}$, $x \mapsto \phi_x$. Question: is this onto?
Proposition: Let I be a proper ideal in $A = C(X)$.
There is at least one $x \in X$ such that $f(x) = 0$ for all $f \in I$.
Proof:
Suppose no such x exists. (Goal: show that $I = A$.)
Then for each $x \in X$ is $f_x \in I$ with $f_x(x) \neq 0$.
Multiply it by $\overline{f_x}$, then we have $\overline{f_x} f_x$ non-negative and real.
Let $U_x = \{y : f_x(y) > 0\}$ open, $x \in U_x$.
The U_x's cover X, so there is a finite subcover, corresponding to $\{x_1, \cdots, x_n\}$
Then let $f = \sum_{j=1}^n f_{x_j}$. Then $f(x) > 0$ all $x \in X$.

Certainly $f \in I$. In A, f is invertible. Hence I = A.
We can conclude that $X \to \overline{A}$ is onto, one-to-one, continuous (easy to see).
Compact space to a Hausdorff space this is a homeomorphism.
Proposition: $\overline{A}$ "=" X.

A commutative Banach algebra with 1.
Have the Gelfand transform: $A \to C(\hat{A})$, where $a \mapsto \hat{a}$.
If a is nilpotent, $a^n = 0$, for any $\phi \in \hat{A}$:
$0 = \phi(a^n) = (\phi(a))^n$, so $\phi(a) = 0$. Thus $\hat{a} \equiv 0$. (0 on all elements)

A comm. Banach, $1 \in A$. ($\mathbb{C}$)
We have ssen that $a \in A$, range($\hat{a}$) $\subset \sigma(a)$.
Is the converse true? Yes, observe:
If $\lambda \in \sigma(a)$, then $\lambda 1_a - a$ is not invertible so $(\lambda - a)A$ is a proper ideal (2-sided).
From algebra: every ideal is contained in a maximal ideal; call it M.
So there is $\phi \in \hat{A}$ having this maximal ideal as its kernel.
Then $\phi(\lambda - a) = 0$ and $\phi(a) = \lambda$.
Proposition: range$(\hat{a}) = \sigma_a(a)$.

Let $A = C_b(\mathbb{Z}^-)$ be the algebra of all bounded $\mathbb{C}$-valued sequences $\|\|_\infty$.
Let $I = C_\infty(\mathbb{Z}^+)$ an ideal. (proper, norm-closed)
Then I is contained in a maximal ideal of A, so there is a $\phi \in \hat{A}$ with $\phi(I) = \{0\}$.
Such maximal ideals are "not constructive". (logician term)
Consequence of the use of Zorn's Lemma to say that a maximal ideal exists.
Let R be any unital ring, e.g. finite field, and $\mathscr{R} = \prod_{n=1}^{\infty} R$, $I = \bigoplus_{n=1}^{\infty} \mathscr{R}$.
Then exists maximal ideal of R containing I, same difficulty/situation.
I.e. does not have anything to do with Banach algebras.
$\hat{A} = \beta\mathbb{Z}^+$ the Stone-Cêch compactification of the positive integers

The *expected radius* of a is $\max\{|\lambda| : \lambda \in \sigma(a)\}$
Because of range$(\hat{a}) = \sigma(a)$ this is the same as $\|\hat{a}\|_\infty$.
This definition works also for a non-commutative algebra A.
But consequence applies only to commutative algebras. (Gelfand transform def'd)

For A commutative Banach algebra with 1 and any $a, b \in A$.
$r(ab) \leq r(a)r(b)$ and $r(a+b) \leq r(a) + r(b)$. ($ab \neq ba \to$ can fail)
Proof:
$r(ab) = \|(\hat{ab})\|_\infty = \|\hat{a}\hat{b}\|_\infty \leq \|\hat{a}\|_\infty \|\hat{b}\|_\infty = r(a)r(b)$.
Note: the spectral radius is independent of the containing algebra.

First form of "holomorphic functional calculus".
Given $a \in A$ Banach algebra with 1.
Let f be a function holomorphic on an open subset of $\mathbb{C}$ containing $\{z : |z| \leq \|a\|\}$.
Thus f has a power series expansion $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ that converges absolutely and uniformly on $\{z : |z| \leq \|a\|\}$.

Thus can define $f(a) := \sum_{n=0}^{\infty} \alpha_n a^n$.
Proposition (proof next time): If $\lambda \in \sigma(a)$, then $f(\lambda) \in \sigma(f(a))$.

9/14

Let A be unital Banach algebra, let $a \in A$.
Proposition: let f be analytic function with power series expansion that converges on some open subset of $\mathbb{C}$ that contains $\{z : |z| \leq \|a\|\}$. Then $f(a) = \sum \alpha_n a^n \in A$ and if $\lambda \in \sigma(a)$ then $f(\lambda) \in \sigma(f(a))$.
Proof:

$f(\lambda) = f(a) = \sum_{n=0}^{\infty} \alpha_n \lambda^n - \sum_{n=0}^{\infty} \alpha_n a^n = \sum_{n=1}^{\infty} \alpha_n (\lambda^n - a^n)$

$= \sum_{n=1}^{\infty} \alpha_n (\lambda - a)(\lambda^{n-1} + \lambda^{n-2} a + \cdots + \lambda a^{n-2} + a^{n-1})$

Call the telescoping sum $P_n(\lambda, a)$.

Then $\|P_n(\lambda, a)\| \leq n\|a\|^{n-1}$
Continue, equals $(\lambda - a) \sum \alpha_n P_n(\lambda, a)$
$f'(z) = \sum_{n=1}^{\infty} \alpha_n n z^{n-1}$, converges absolutely uniformly for $|z| \leq \|a\|$.
$\sum_{n=1}^{\infty} \alpha_n P_n(\lambda, a) = b \in A$.
So $f(\lambda) - f(a) = (\lambda - a)b$
If $(f(\lambda) - f(a))$ has inverse c, then $1 = (\lambda - a)bc$, so $\lambda - a)$ is invertible.
Thus since $\lambda \in \sigma(a)$, $f(\lambda) = f(a)$ is not invertible so $f(\lambda) \in \sigma(f(a))$.

This proof above is the beginnings of the spectral mapping theorem.

Consider $f(z) = z^n$.
Then if $\lambda \in \sigma(a)$ then $\lambda^n \in \sigma(a^n)$.
Thus $|\lambda^n| \leq \|a^n\|$, so $|\lambda| \leq \|a^n\|^{1/n}$.
Thus $|\lambda| \leq inf_n\{\|a^n\|^{1/n}\}$.
Corollary: $r(a) \leq inf_n\{\|a^n\|^{1/n}\}$.
This expression doesn't depend on the containing algebra.

Consider the resolvant of a, at ∞.
$R(a, z^{-1}) \frac{1}{z-a} = z(1 - az)^{-1} = z \sum_{n=0}^{\infty} a^n z^n$ converges for $\|az\| < 1$ i.e. $|z| < \|a\|^{-1}$

However since $R(a, z)$ is analytic for $|z| > r(a)$, $R(a, z^{-1})$ is analytic for $|z| < r(a)^{-1}$.
Power series converges in the largest circle in which it's analytic.
Should converge in the larger circle. Note we are in Banach-algebra-valued functions.
Right now we assume we don't know that, so use method from before.
Composing with linear operators on the algebra.
For any $\varphi \in A'$ the dual space of the algebra, $f_\varphi(z) = \varphi(z(1 - az)^{-1})$.
Then f_φ is an ordinary holomorphic function, holomorphic in $\{z : |z| < r(a)^{-1}\}$.
So the power series expansion of f_φ about 0 converges for $|z| < r(a)^{-1}$.
But the power series for f_φ is $z \sum_{n=0}^{\infty} \varphi(a^n) z^n$.

This will converge for $|z| < r(a)^{-1}$.

Thus for any $r > r(a)$, $z\sum_{n=0}^{\infty} \varphi(a^n)z^n$ will converge absolutely and uniformly for $|z| \leq r^{-1}$.

So there is M_φ such that $|\phi(a^n)||z^n| \leq M_\varphi \forall n$, $\forall z$ with $|z| \leq r^{-1}$.

So $|\varphi(a^n)r^{-n}| \leq M_\varphi$ for all n.

For each n define $F_n \in A''$ by $F_n(\varphi) = \varphi(a^n)r^{-n}$.

Thus $|F_n(\varphi)| \leq M_\varphi$ for all φ.

Recall the uniform boundedness theorem (consequence of the Baire category theorem).

Says here that $\exists M$ such that $\|F_n\| \leq M \; \forall n$.

But it is clear that $\|F_n\| = \|a^n\|r^{-n}$.

Thus $\|a^n\|r^{-n} \leq M$ i.e. $\|a^n\| \leq Mr^n \to \|a^n\|^{1/n} \leq M^{1/n}r$

As $n \to \infty$, $M^{1/n} \to 1$.

So the $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r \; \forall r > r(a)$ (Box it!)

Thus $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$.

And we have $r(a) \leq \inf_n\{\|a^n\|^{1/n}\}$.

Thus, Theorem: $r(a) = \lim \|a^n\|^{1/n}$. In particular, this limit exists.

(Gelfand's spectral radius formula).

Just needed a unital Banach algebra over $\mathbb{C}$.

Corollary: $r(a)$ does not depend on the containing algebra.

Corollary: Let A be a commutative Banach algebra with 1 over $\mathbb{C}$.

Then the Gelfand transform $a \mapsto \hat{a}$ from A to $C(\hat{A})$ is isometric exactly if $\|a^2\| = \|a\|^2$ for all $a \in A$.

9/16

Proposition: Let A be a commutative unital Banach algebra.

The Gelfand transform is isometric iff $\|a^2\| = \|a\|^2$ for all $a \in A$.

Proof:

Forward: If equality holds then $\|a^4\| = \|a^2\|^2 = \|a\|^4$.

$\|a^{2^n}\| \cdots = \|a\|^{2^n}$. So $\|a^{2^n}\|^{1/2^n} = \|a\|$.

So $r(a) = \|a\|$, $r(a) \to \|\hat{a}\|_\infty$.

Conversely: If for some a we have $\|a^2\| = s^2 \leq \|a\|^2$, then

$\|a^4\| \leq \|a^2\|^2 \leq s^4$.

$\|a^{2^n}\| \cdots \leq s^{2^n}$ so $\|a^{2^n}\| \leq s < \|a\|$ and $r(a) < \|a\|$

Let $\mathcal{O}$ be an open bounded region in $\mathbb{C}$ and A be the bounded holomorphic fns on $\mathcal{O}$

That's a unital Banach algebra. Pretty clear that $\|f^2\|_\infty = \|f\|_\infty^2$.

The Gelfand transform will be isometric.

Will take functions to themselves... well there will be some things on the boundary.

Let's add the requirement that the function is continuous on the boundary.

i.e. consider continuous functions on boundary $\overline{\mathcal{O}}$ holorphic on the interior

Gelfand transform isometric but not onto $C(\mathcal{O})$.

Let A be an algebra over $\mathbb{C}/\mathbb{R}$.
An involution on A is a map $a \to a^*$.
$(a+b)^* = a^* + b^*$
$(\alpha a)^* = \overline{\alpha} a^*$
$(ab)^* = b^* a^*$
$(a^*)^* = a$
Given a, set $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$ with real and imaginary part, which are self-adjoint
A *-algebra is going to be an algebra equipped with an involution
A normed *-algebra is a *-algebra that is a normed algebra, with $\|a^*\| = \|a\|$.
And similarly Banach *-algebra

A *-algebra is symmetric if for any $a \in A$ with $a^* = a$ (self-adjoint), $\sigma(a) \subset \mathbb{R}$.
A commutative algebra is semi-simple if the intersection of all its maximal ideals is $\{0\}$
Proposition: Let A be a unital Banach algebra, commutative.
Then the Gelfand transform for A is one-to-one, exactly if A is semi-simple. No real content in the proposition, just relating the terminology.
Let A be a unital Banach *-algebra, commutative.
If it is symmetric, then the image of A under the Gelfand transform is a dense *-subalgebra of $C(\hat{A})$.
Proof
Always the image of A separates the points of $\hat{A}$. Image contains 1.
By separating the the image is closed under complex conjugation, because if
$a = b + ic$, b, c self-adjoint
$\hat{a} = \hat{b} + i\hat{c}$, $\hat{b}, \hat{c}$ are real-valued
$a^* = b - ic$, $\hat{a^*} = \hat{b} - i\hat{c} = (\hat{a})^-$.
Apply Stone-Weierstrass.
If, in addition, $\|a^2\| = \|a\|^2$ for all $a \in A$:
then the Gelfand transform is an isometric *-isomorphism of A onto $C(\hat{A}$.

continuous functions on a compact hausdorff space is an example

By a C*-algebra we mean a Banach *-algebra such that $\|a^* a$
$= \|a\|^2$ for all $a \in A$.
e.g. $C(X)$, $C_\infty(X)$ for X locally compact.
Proposition: Let $\mathcal{H}$ be a Hilbert space, $T \in B(\mathcal{H})$, then $\|T^*T\| = \|T\|^2$.
Proof:
$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$.
For any $\xi \in \mathcal{H}$, $\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle \leq$ (Cauchy-Schwartz) $\|T^*T\|\|\xi\|^2$
so $\|T\| \leq \|T^*T\|^{1/2}$.

Any closed *-subalgebra of $B(\mathcal{H})$ is a C*-algebra. (C from closed, * from *).

Proposition: Let A be a unital C*-algebra. Let $a \in A$, with $a^* = a$, then $\|a^{2^n}\| = \|a\|^{2^n}$, $r(a) = \|a\|$.
Corollary: For a unital C*-algebra, its norm is determined by its *-algebra structure.
Proof:
Given $a \in A$, $\|a\|^2 = \|a^*a\| = r(a^*a)$.
The C*-algebras are by far the nicest-behaving Banach algebras.
Tied into this correspondence between *-algebra structure and norm.

Let V be a finite-dimensional vector space over $\mathbb{C}$.
Then $\mathcal{L}(V)$ the algebra of linear operators on V.
For each inner product on V, get a * on A and a norm on V and so a norm on A.
C*-algebra.

9/21

Let A be a C*-algebra with 1.
Then A is symmetric, i.e. if $a \in A$ and $a^* = a$, then $\sigma(a) \subset \mathbb{R}$.
(in C*-subalgebra generated by a and 1_a)
Proof: (Arens' trick)
Let $a \in A$, $a^* = a$, let $(r, s \in mathdsR)$ $r + is \in \sigma(a)$.
Need to show $s = 0$. Given $t \in \mathbb{R}$. Let $b = a + it$.
Then $b^*b = (a - it)(a + it) = a^2 + t^t$
Thus $\|b\|^2 = \|b^*b\| \leq \|a^2\| + t^2$.
$r + i(s + t) \in \sigma(b)$. Thus $|r + i(s + t)|^2 \leq \|b\|^2$
$r^2 + (s + t)^2 \leq \|b\|^2 \leq \|a^2\| + t^2$
$r^2 + s^2 + 2st + t^2$
$\leq \|a^2\|$ $\forall t \in \mathbb{R}$
So $s = 0$.
See the power of this little identity $\|a^*a\| = \|a\|^2$

Theorem (Gelfand-Naimark "little")
Let A be a unital commutative C*-algebara. Then the Gelfand transform is an isometric *-isomorphism of A onto $C(\hat{A})$.
Proof:
Now have all the pieces we need to apply the Stone-Weierstrass theorem.

An element a in a *-algebra is normal if a^* commutes with a.
(e.g. a is unitary, i.e. $a^*a = 1 = aa^*$)
Continuous functional calculus:
Let A be a unital C*-algebra (e.g. C*-subalgebra of some $B(\mathcal{H}) \ni T$)
Let $a \in A$, and let a be normal.
Let $f \in C(\sigma(a))$.
Let B be the C*-subalgebra generated by a and 1 (so $a^* \in B$).
If ϕ is a multiplicative linear functional on a then $\phi(a) \in \sigma(a)$

$\hat{B}'' = ''\sigma(a)$ by $\phi \mapsto \phi(a)$

In terms of the isomorphism $B \xrightarrow{\hat{}} C(\sigma(a))$ there is a unique element $b \in B$ such that $\hat{b} = f$.

Call this element $f(a)$.

Then $f \mapsto f(a)$ is an isometric *-isomorophism of $C(\sigma_B(a))$ with B.

The inverse of the Gelfand transform.

Theorem: "Spectral permanence"

Let A be a unital C*-algebra, and let B be any unital C*-subalgebra of A.

For any $b \in B$, $\sigma_B(b) = \sigma_A(b)$.

Equivalent statement: If b is invertible in A then it is invertible in B.

Proof:

Deal first with the case $b = b^*$, and let $\mathcal{C}$ be the unital C*-algebra generated by b and 1.

It suffices to show that $\sigma_C(b) = \sigma_A(b)$.

Show tht if b is not invertible in the algebra C, then b is not invertible in A.

Suppose $d \in A$ is an inverse for b.

Since b is not invertible in $\mathcal{C} \cong C(\sigma_{\mathcal{C}}(b))$

then $\hat{b}$ must take value 0 at some point of $\sigma_{\mathcal{C}}(b))$.

Then there is $g \in C(\sigma_{\mathcal{C}}(b))$ such that $\|g\|_\infty = 1$, $\|g\hat{b}\|_\infty < \frac{1}{2\|d\|}$

Let $c = g(b)$. So $c \in \mathcal{C}$, $\|c\| = 1$, $\|cb\| \le \frac{1}{2\|d\|}$

$bd = 1 = db$

Then $1 = \|c\| = \|c(bd)\| = \|(cb)d\| \le \|cb\|\|d\| \le \frac{1}{2}$, $\perp$.

For general case, if b is not invertible in $\mathcal{B}$, then it's easy to see b^*b is not invertible in $\mathcal{B}$.

So b^*b is not invertible in A so b is not invertible in A.

9/21