

# Math 206

Fall 2015

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## Definitions

A *norm* on a vector space  $X$  (over  $F$ ) is a function  $\|\cdot\| : X \rightarrow \mathbb{R}^+$  such that

$$\|x\| = 0 \text{ iff } x = 0$$

$$\|\alpha x\| = |\alpha| \|x\| \text{ (for } \alpha \in F)$$

$$\|x + y\| \leq \|x\| + \|y\|$$

An *algebra*  $\mathcal{A}$  over  $F$  is a vector space with distributive  $\cdot$  satisfying

$$cx \cdot y = c(x \cdot y)$$

$$x \cdot cy = c(x \cdot y) \text{ for all } c \in F$$

A *normed algebra* over  $\mathbb{R}$  or  $\mathbb{C}$  is an algebra  $\mathcal{A}$  equipped with (vector space) norm satisfying

$$\|ab\| \leq \|a\| \|b\| \text{ for all } a, b \in \mathcal{A}$$

A norm on  $\mathcal{A}$  induces a metric

$$d(a, b) = \|a - b\| \text{ on } \mathcal{A} \text{ and therefore a topology}$$

if  $\mathcal{A}$  is complete for this norm, it is a *Banach algebra*

**To figure out (use <https://www.math.ksu.edu/nagy/real-an/2-05-b-alg.pdf>)**

Supposing  $\mathcal{A}$  is not necessarily complete

$$\|ab\| \leq \|a\| \|b\| \text{ gives uniform continuity on the product}$$

hence the norm can be extended to the completion  $\tilde{\mathcal{A}}$  to form a Banach algebra

A metric space  $M$  is complete if all Cauchy sequences converge to an element of  $M$

The completion  $\tilde{M}$  is all equivalence classes of Cauchy sequences where

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim_{n \rightarrow \infty} d(a_n - b_n) = 0$$

## Examples

For  $M$  a compact space,  $C(M)$

the set of continuous  $\mathbb{R}/\mathbb{C}$ -valued functions on  $M$

pointwise operations

$$\|f\|_\infty = \sup\{|f(x)| : x \in M\}$$

For  $M$  locally compact,  $C_\infty(M)$

the set of continuous  $\mathbb{R}/\mathbb{C}$ -valued functions on  $M$  vanishing at  $\infty$   
 vanishing at  $\infty$ :  $\forall \epsilon \exists$  a compact subset of  $M$ , outside of which  $f < \epsilon$   
 note that this is non-unital (lacks an identity)

For  $\mathcal{O} \subset \mathbb{C}^n$  open

$H^\infty(\mathcal{O})$  the set of all bounded holomorphic functions on  $\mathcal{O}$

$(M, d)$  metric space and  $f \in C(M)$

Lipschitz constant (which can be  $+\infty$ )  $L_d(f) = \sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in M, x \neq y\}$

The Lipschitz functions  $\mathcal{L}_d(M, d) = \{f : L(f) < \infty\}$

These form a dense subalgebra of  $C(M)$  and are in fact a  $*$ -subalgebra

$\|f\|_d := \|f\|_\infty + L_d(f)$ , can be shown as a normed-algebra norm

$L_d(M, d)$  is complete for this norm

so  $L_d(M, d)$  is a Banach algebra

$L_d$  is a seminorm on  $\mathcal{L}_d(M, d)$  since it takes value 0 on the constant functions  
 can recover  $d$  from  $L_d$

$M$  a differentiable manifold (e.g.  $T = \mathbb{R}/\mathbb{Z}$  the circle)

$C(M) \supseteq C^{(1)}(M)$  the singly-differentiable functions

$f \in C^{(2)}(T) \rightarrow Df : T_x M \rightarrow \mathbb{R}, \mathbb{C}$

with  $Df$  the derivative and  $T_x$  the tangent space

If we put on a Riemannian metric, define  $\|f\|^{(1)} = \|f\|_\infty + \|Df\|_\infty$

If  $f \in C^{(1)}(T) : \|f\|^{(1)} = \|f\|_\infty + \|f'\|_\infty$

Banach algebra norm, for which this space of functions is complete

For the circle,  $C^{(2)}(T) \rightarrow \|f\|^{(2)} = \|f\|_\infty + \|f'\|_\infty + \frac{1}{2}\|f''\|_\infty$

the factor  $\frac{1}{2}$  ensures that this satisfies the normed algebra condition

$$C^{(n)}(T) = \sum_{k=0}^n \frac{1}{k!} \|f^{(k)}\|_\infty$$

For  $C^\infty(T)$  using the collection of norms  $\{\|\cdot\|^{(n)}\}_{n=1}^\infty$  yields a Fréchet algebra

A Fréchet algebra has a topology defined by a countable family of seminorms

that respect the algebra structure and is complete (**clarify**)

non-commutative algebras

$X$  a Banach space

$\mathcal{B}(X)$  the algebra of bounded operators on  $X$

$\|\cdot\|$  operator norm  $\rightarrow$  Banach algebra

Any closed subalgebra of  $\mathcal{B}(X)$  is a Banach algebra

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Sketch of the course

$X$  a Banach space,  $B(X)$  bounded functions on the space

$\mathcal{H}$  a Hilbert space,  $\mathcal{B}(\mathcal{H})$  bounded operators on the space

for  $T \in \mathcal{B}(\mathcal{H}) \exists$  adjoint operator  $T^* \in \mathcal{B}(\mathcal{H})$

$\langle T\xi, \eta \rangle = \langle \eta, T^*\xi \rangle$  for  $\xi, \eta \in \mathcal{H}$

adjoint is additive, conjugate linear,  $T^{**} = T$ ,  $(ST)^* = T^*S^*$

An algebra  $A$  over  $\mathbb{R}$  or  $\mathbb{C}$  is a  $*$ -algebra if it has a  $*$  :  $A \rightarrow A$  satisfying

certain properties (look up)

A *\*-normal algebra* is a normal \*-algebra such that

$$(\forall a \in A) \|a^*\| = \|a\|$$

A *Banach \*-algebra* is a \*-normal algebra that is a Banach algebra.

For any  $T \in \mathcal{B}(\mathcal{H})$ , have  $\|T^*T\| = \|T\|^2$  (**check: parse through defs**)

For M a locally compact space,  $A = C_\infty(M, \mathbb{C})$ ,  $f^* := \bar{f}$  is a Banach \*-algebra

Also have  $\|f^*f\| = \|f\|^2$  (**verify: should be easier than the other**)

Little Gelfand-Naimark theorem:

Let A be a commutative Banach \*-algebra satisfying  $\|a^*a\| = \|a\|^2$ .

Then  $A \cong C_\infty(M)$  for some locally compact M.

One view of the “spectral theorem”

Let  $T \in \mathcal{B}(\mathcal{H})$  with  $T^* = T$

Let A be the closed subalgebra of  $\mathcal{B}(\mathcal{H})$  generated by T and I (i.e.  $p(T) := \sum \alpha_k T^k$ )

Polynomials closed or stable under \*

If  $S \in A$  then  $S^* \in A$  (i.e. A is a \*-subalgebra of  $\mathcal{B}(\mathcal{H})$ )

So A is a Banach \*-subalgebra satisfying  $\|S^*S\| = \|S\|^2$

Moreover, A is commutative. (unital, since generated by I)

Then by the Little Gelfand-Naimark theorem,  $A \cong C(M)$

Indeed  $M \subset \mathbb{R}$ , the spectrum of T

If  $\mathcal{H}$  is finite dimensional, then M is the set of eigenvalues of T

T is normal if  $TT^* = T^*T$

A C\*-algebra is a Banach \*-algebra over  $\mathbb{C}$  satisfying

$$\|a^*a\| = \|a\|^2$$

Theorem: A commutative C\*-algebra is  $\cong C_\infty(M)$ .

Big Gelfand-Naimark Theorem: (Math 208, C\*-algebras)

Any C\*-algebra is  $\cong$  to a closed \*-subalgebra of  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ .

Tangent

algebraic topology, differential geometry, Riemann manifolds, “non-commutative geometry” (Connes)

A *von-Neumann algebra* is a \*-subalgebra of  $\mathcal{B}(\mathcal{H})$

which is closed under the strong operator topology.

Every commutative von-Neumann algebra is  $\cong L^\infty(X, S, \mu)$  (measure spaces) acting on  $L^2(X, S, \mu)$  by positive sldkjfalksdjf

For group G,  $\alpha : G \rightarrow \text{Auto}(X) \subseteq \mathcal{B}(X)$

$\text{Auto}(X)$  a Banach space

Look at subalgebra of  $\mathcal{B}(X)$  generated by  $\alpha(G)$ .

Leads to considering  $l'(G)$  with product  $(f \star g)(x) = \sum f(y)g(y^{-1}x)$  convolution

$$f^*(x) = \overline{f(x^{-1})}$$

Banach \*-algebra, G commutative  $\rightarrow$  Fourier transform

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K a field, X a set,  $\mathcal{F}(X, K)$  the set of all K-valued functions on X with pointwise operations

Given  $f \in \mathcal{F}(X, K)$ .

Let  $\lambda \in K$ . Then  $\lambda \in \text{range}(f)$  exactly if  $(f - \lambda 1)$  is not invertible.

For any  $a \in A$ , the *spectrum* of  $a$  is  $\{\lambda \in K : a - \lambda 1_A \text{ is not invertible in } A\}$ .

The spectrum depends on the containing algebra

Assuming that this algebra  $A$  (over the field  $K$ ) has an identity  $1_A$

Example: Let  $A = C([0, 1])$ , and  $B = \text{polynomials}$ , viewed as a dense subalgebra of  $A$ .

Let  $p$  be a polynomial of degree  $\geq 2$ . Then

$$\sigma_a(p) = p([0, 1]).$$

$$\sigma_B(p) = \mathbb{R}, \mathbb{C}$$

Let  $A$  be a Banach algebra with  $1$  ( $\|1\| = 1$ ), and  $a \in A$ .

If  $\|a\| < 1$ , then  $1 - a$  is invertible, and  $\|(1 - a)^{-1}\| \leq \frac{1}{1 - \|a\|}$

Proof:

$$\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n \quad (a^0 := 1_A)$$

For any  $n > 0$ , let  $s_n = \sum_{k=0}^n a^k$ .

Show that  $\{s_n\}$  is a Cauchy sequence.

If  $n > m$ ,  $\|s_n - s_m\| = \|\sum_{k=m+1}^n a^k\| \leq \sum_{k=m+1}^n \|a\|^k$ .

Given  $\epsilon > 0 \exists N$  such that if  $m, n \geq N$  then  $\sum_{k=m+1}^n \|a\|^k \leq \epsilon$

So  $\{s_n\}$  is a Cauchy sequence.

By completeness there is a  $b \in A$  with  $s_n \rightarrow b$  as  $n \rightarrow \infty$ .

Want to show  $b = (1 - a)^{-1}$ .

$$b(1 - a) = \lim_{n \rightarrow \infty} (s_n(1 - a))$$

$$= \lim_{n \rightarrow \infty} (1 + a + a^2 + a^3 + \dots + a^n - (a + a^2 + a^3 + \dots + a^{n+1}))$$

$$= \lim_{n \rightarrow \infty} (1 - a^{n+1}) = 1$$

Then  $1 - (1 - a)$  is invertible, i.e.  $a$  is invertible.

$$\|(1 - a)^{-1}\| = \lim \|s_n\| \leq \lim \sum_{k=0}^n \|a\|^k = \frac{1}{1 - \|a\|} \quad (\|1\| = 1)$$

$\|ab\| \leq \|a\|\|b\|$ : can very easily check that multiplication is cts (do this?)

Corollary: If  $a \in A$  and  $\|1 - a\| < 1$  then  $a$  is invertible, and  $\|a^{-1}\| \leq \frac{1}{1 - \|1 - a\|}$

I.e. the open unit ball about  $1$  consists of invertible elements.

Let  $a \in A$ . Let  $L_a, R_a$  be the operators of left and right multiplication by  $a$  on  $A$ .

$a \rightarrow L_a$  is an algebra homomorphism of  $A$  into  $\mathcal{L}(A)$  (linear operators on  $A$ )

$L_a L_b = L_{ab}, R_a R_b = R_{ba}$  ( $R$  is an antihomomorphism)

$1 \in A$

If  $a$  is invertible, then so is  $L_a, L_a L_{a^{-1}} = I_a$

Then if  $A$  is a normed algebra,  $\|L_a\| = \|a\|$

$$\|L_{ab}\| = \|ab\| \leq \|a\|\|b\|$$

$$\|L_a 1_a\| = \|a\|$$

so if  $a \in A$  is invertible, then  $L_a$  is a homeomorphism of  $A$  onto itself.

Thus if  $A$  is a Banach algebra, with  $1$ , and  $a$  is invertible:

$\{L_a b : \|1 - b\| < 1\}$  is an open neighborhood of  $a$  consisting of invertible elements

Let  $GL(A)$  be the set of invertible elements of  $A$ . (general linear group)

Then (for  $A$  a unital Banach algebra)  $GL(A)$  is an open subset of  $A$ .

(Fails for  $\text{Poly} \subseteq C([0, 1])$ )

Two Fréchet algebras, for one,  $GL(A)$  is an open subset, for another it isn't.

**ask about this?:** not sure what he was talking about

$$C^\infty(T), \|f^{(n)}\|$$

$C(\mathbb{R})$  cont fns on  $\mathbb{R}$  (or  $\mathbb{C}$ ) maybe unbounded

For each  $n$  let  $\|f\|_n = \sup\{|f(t)| : |t| \leq n\}$

Corollary: For  $A$  a Banach algebra with  $1$  and  $a \in A$ ,  $\sigma(a)$  is a closed subset of  $\mathbb{C}$