Math 245A Fall 2015

Ch 2

A group G is a 4-tuple $G = (|G|, \mu, \iota, e)$ with
  underlying set $|G|$
  law of composition μ
  inverse function ι
  neutral element e
A more common representation of a group uses symbols $G = (|G|, \cdot, ^{-1}, e)$
We may also say that a set $|G|$ with a map $|G| \times |G| \to |G|$ constitutes a group if
  $(\forall x, y, z \in |G|)\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
  there exists $e \in |G|$ such that $(\forall x) e \cdot x = x = x \cdot e$ and $(\forall x \in |G|)(\exists y \in |G|) y \cdot x = e = x \cdot y$
However, unlike the first, does not consist of identities (universally quantified equations)
  note: universal quantification $\leftrightarrow$ "for all"
An I-tuple of elements of X, $(x_i)_{i \in I}$ is an $f : I \to X$
  The set of all such f is denoted X^I

The arity of an operation (e.g. 1 if unary, 2 if binary, etc.)
  An I-ary operation on S is a map $S^I \to S$
Can think of the identity as a 0-ary/zeroary operation of the structure
  S^0 has exactly one map, $\emptyset \to S$, so a map $S^0 \to S$ is determined by one element
  Note that when $S = \emptyset$ there is a unique n-ary operation for $n > 0$ but *no* 0-ary operation

A group-theoretic relation in $(\eta_i)_I$ is an equation $p(\eta_i) = q(\eta_i)$ holding in G
The terms in the elements of X under the formal group operations μ, ι, e form a set T:
  given with functions $symb_T : X \to T$, $\mu_T : T^2 \to T$, $\iota_T : T \to T$, and $e_T : T^0 \to T$
  such that each map is one-to-one, its images disjoint, and T is the union of those images
  and T is generated by $symb_T(X)$ under the aforementioned operations
  that is, T has no proper subset containing $symb_T(X)$ and closed under those operations
We can represent these terms, for groups, using strings of symbols
  We need full parentheses notating order of operations to ensure disjoint images

A set-theoretic approach dispenses with strings and allows for infinite arities
For the example of a group, we would have (using ordered pair, 3-tuple, etc.):
  for $x \in X$, $symb_T(x) := (*, x)$
  for $s, t \in T$, $\mu_T(s, t) := (\cdot, s, t)$
  for $s \in T$, $\iota_T(s) := (^{-1}, s)$
  and $e_T = (e)$
  and by set theory, no element can be written as such an n-tuple in more than one way

Given a set map $f : X \to |G|$ for a group G
Recursive evaluation of $s_f \in |G|$ given an X-tuple of symbols $s \in T = T_{X, \cdot, -1, e}$
  if $s = symb_T(x)$ for some $x \in X$, then $s_f := f(x)$

$s = \mu_T(t,u) \to s_f = \mu_G(t_f,u_f)$, assuming that given $t,u \in T$ we know $t_f,u_f \in |G|$
similarly, $s = \iota_T(t) \to s_f = \iota_G(t_f)$, assuming we know t_f given t
finally $s = e_T \to s_f = e_G$

Varying f in addition to T gives an evaluation map $(T_{X,\cdot,^{-1},e}) \times |G|^X \to |G|$

Alternatively, fixing $s \in T$ gives a map $s_G : |G|^X \to |G|$
these represent substitution into s
these s_G are the derived n-ary operations (aka term operations) of G
distinct terms can induce the same derived operation
e.g. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in general or others for certain groups

Examples of derived operations on groups
conjugation $\xi^\eta = \eta^{-1}\xi\eta$ (binary)
commutator $[\xi,\eta] = \xi^{-1}\eta^{-1}\xi\eta$ (binary)
squaring (unary)

An Ω-algebra is a system $A = (|A|,(\alpha_A)_{\alpha\in|\Omega|})$
here $|A|$ is some set, and for each $\alpha \in |\Omega|$, $\alpha_A : |A|^{ari(\alpha)} \to |A|$
note that often people will use $n(\alpha)$ (rather than $ari(\alpha)$) for the arity of an operation α
e.g. for a group, $|\Omega| = \{\mu,\iota,e\}$, $ari(\mu) = 2$, $ari(\iota) = 1$, and $ari(e) = 0$

Ch 3

Ch 4

the subgroup and normal subgroup of G generated by $S \subset |G|$
$\langle S\rangle$ contains S and is contained in every subgroup which contains S
$\forall x \in \langle S\rangle$ $x = e$ or $x = \prod s_i$, $s_i \in S$ or $s_i^{-1} \in S$
is the image of the map into G of the free group F on S induced by the inclusion $S \to |G|$
there is additionally a least normal subgroup of G containing S.

relations on a group/quotient groups
Quotient groups: homomorphisms causing certain elements to fall together.
Satisfies e.g. $(\forall i \in I) f(x_i) = f(y_i) \leftrightarrow (\forall i \in I) f(x_i y_i^{-1}) = e$
A set of elements annihilated by a group homomorphism form a normal subgroup.
Leads to $q : G \to G/N$, where N is this normal subgroup.
We have a quotient map and a quotient group.

This map has the universal property desired:
For every homomorphism $h : G \to K$ satisfying the above, $\exists! g : N \to K$, s.t. $h = g \circ q$.

This construction *imposes the relations* $x_i = y_i (i \in I)$ on G, forming $G/(x_i = y_i | i \in I)$.

For G a group, a G-set is a pair $S = (|S|,m)$, $|S|$ a set and $m : |G| \times |S| \to |S|$, satisfying
$(\forall s \in |S|, g, g' \in |G|)\ g(g's) = (gg')s$
$(\forall s \in |S|)\ es = s$
That is, a set on which G acts by permutations.

A homomorphism $S \to S'$ of G-sets (for G fixed) is a map $a : |S| \to |S'|$ satisfying
$(\forall s \in |S|, g \in |G|)\ a(gs) = ga(s)$

The set of left cosets of H in G is $|G/H|$ and a typical coset $[g] = gH$.

Then $|G/H|$ is the underlying set of a left G-set G/H by $g[g'] = [gg']$

groups presented by generators and relations

Let X be a set, T the set of all group-theoretic terms in X, and $R \subset T \times T$.

$\exists$ a universal example of a group with X-tuples of elements satisfying the relations R.

That is, there is a pair (G, u) with G a group and $u : X \to |G|$ satisfying:

$(\forall (s,t) \in R)\ s_u = t_u$

such that for any group H and X-tuple v of elements in H satisfying

$$(\forall (s,t) \in R)\ s_u = t_u$$

$\exists! f : G \to H$ a homomorphism, such that $v = fu$

The pair (G, u) is determined up to canonical isomorphism by these properties

The group G is generated by u(X)

Proving the existence of such a universal construction.

Two ways: construction from terms and subgroups of direct products.

In these approaches, apply the additional conditions to the group axioms.

A proof that builds upon prior constructions:

Let (F, u_F) be the free group on X and N be a normal subgroup.

Define N such that it is generated by $\{s_{u_F} t_{u_F}^{-1} | (s,t) \in R\}$

Take a canonical map $q : F \to F/N$.

Then the pair $(F/N, q \circ u_F)$ has the desired universal properties.

As a consequence of the universal properties of free and quotient groups.

Class Question #9

Definition 5.5.13 Unimportant

Just to clarify my understanding of the definition of a cardinal, is it correct that each of the natural numbers is itself a cardinal, but ordinals such as $\omega + 1$ and $\omega 2$ are not, themselves, cardinals?

Class Question #11

Proposition 6.2.3 Unimportant

Sorry about the lateness of the question for today. In the proof, is the first paragraph intended to show that (i) is equivalent to (i*)?

–Andre

Class Question Section 7.5

Example, beneath definition 7.5.7 Pro forma

"You should verify that the behavior of $h_Z(G)$ on morphisms agrees with the underlying set functor."

Given a homomorphism $h: G \to H$, h_z of that homomorphism takes a $\phi \in Group(Z,G)$ to $h\phi \in Group(Z,H)$. Each ϕ in $Group(Z,G)$ is determined by its action on the generator x of Z, so there is a correspondence between the ϕ and the elements of $|G|$. The function $h\phi$ in $Group(Z,H)$ shall be determined by its action on the generator, which, if $\phi(x) = g$ an element of G, equals $h(g)$ so that the image functions $h\phi$ correspond to images of elements $g \in |G|$ under the map of underlying sets $h' : |G| \to |H|$ induced by h. This shows that the behavior of $h_Z(G)$ on morphisms agrees with the underlying set functor.