Graph Signal Processing - Wavelets and Graph Wavelets

Prof. Luis Gustavo Nonato

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Wavelet Transform

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The admissibility conditition means that $\psi(x)$ wiggle up and down like a wave and that $\psi(x)$ should decay when x goes to $\pm\infty$.

Therefore, the admissibility condition allows for and effective localization in both time and frequency for the basis functions, contrary to the Fourier basis that are of infinite duration waves.

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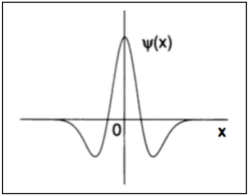

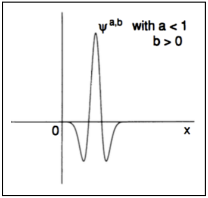

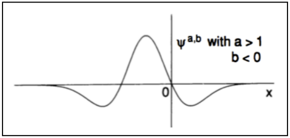

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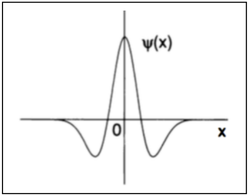
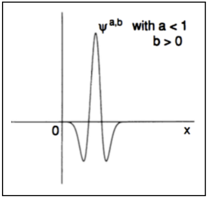
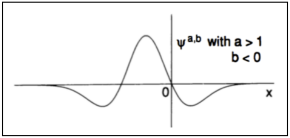
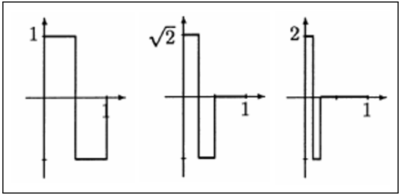






Haar:



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A common choice is to fix a_0 and b_0, defining the discretization as:

$$\psi_{m,n}(x) = \frac{1}{\sqrt{a_0^m}} \psi\left(\frac{x - nb_0 a_0^m}{a_0^m}\right), \qquad \text{where } m, n \in \mathbb{Z}.$$

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What about the discrete case?

- Do the discrete coefficients $< f, \psi_{m,n} >$ completely characterize f?
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The answers for those questions come from the concept of frames.

DWT: Frames

Frames

A family of functions φ_j (in a Hilbert space) is called a *frame* if there exist $A > 0$ and $B < \infty$ such that for all f

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If φ_j is a frame then there exist a *dual frame* $\tilde{\varphi}_j$ such that

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In other words, f can be reconstructed from a frame !!

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Proposition

Assuming $a_0 > 1$, if

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for some C, $\alpha > 0$, and $\gamma > \alpha + 1$, then there exist $\tilde{b}_0$ such that $\psi_{m,n}$ is a frame for all $b_0 < \tilde{b}_0$.

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Mexican Hat ($a_0 = 2, b_0 \leq .75$), Daubechies family, and Haar basis ($a_0 = 2, b_0 = 1$) give rise to tight frames.

Scaling functions

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Translations are limited by the duration of the signal under analysis, so there is an upper boundary for the number of translations.

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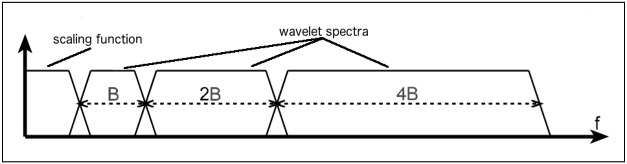

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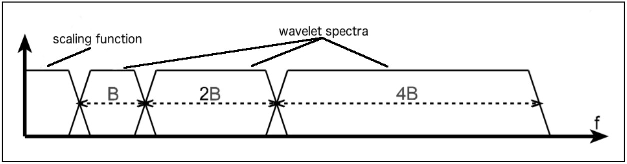

PS: Notice that wavelets in different scales correspond to band-pass filters. The larger the scale the higher the frequencies that are filtered.

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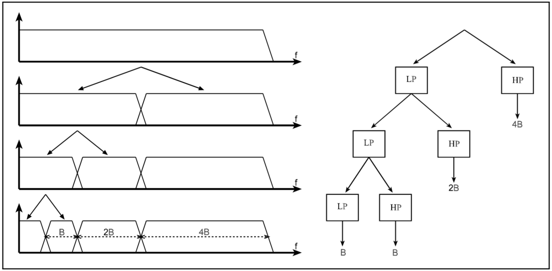

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Other approaches are:

- Crovella and Kolaczyk (Second Generation Wavelets)
- Coifman and Maggioni (Diffusion Wavelets)
- Lee (Treelets)

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The kernel g should behave as a band-pass filter (similarly to scaled wavelets) satisfying $g(0) = 0$ and $\lim_{\lambda \to \infty} g(\lambda) = 0$.

Graph Wavelet Transform

Given the band-pass filter g defined in the spectral domain, we define the mother wavelet ψ as the inverse graph Fourier transform of g.

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PS. Definition above assumes the scale parameter s is continuous.

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Is it possible to reconstruct f given $\{\mathcal{W}_f(s,n)\}$?

Lemma

If the SGWT kernel g satisfies $g(0) = 0$ and the admissibility condition

$$\int_0^\infty \frac{g^2(x)}{x} dx = C_g < \infty$$

then

$$\frac{1}{C_g} \sum_n \int_0^\infty \mathcal{W}_f(s,n) \psi_{s,n}(i) \frac{ds}{s} = \tilde{f}(i)$$

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Notice that the integral is assuming the scale is continuous.

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The scaling function in each vertex n can then be defined as:

$$\phi_n(i) = \sum_l h(\lambda_l)\mathbf{u}_l(n)\mathbf{u}_l(i)$$

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Theorem

Given scales $\{s_j\}$, $j = 1, \ldots, J$ the set $B = \{\phi_n\} \cup \{\psi_{s_j,n}\}$ forms a frame (but not a tight frame).

However, in the context fo *SGWT* the guarantee of forming a frame is not enough to ensure reconstruction.

Function Reconstruction

In fact, there are more wavelets $\psi_{s_j,n}$ than vertices in the graph ($N\,J$ wavelets, where N is the number of vertices and J the number of scales).

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In other words, given the whole set of coefficients $f_{s_j,h}$, the system

$$Wf = f_{s_j,h},$$

where W is the matrix with rows given by the basis function in B, is over determined.

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Therefore, $B = \{\phi_n\} \cup \{\psi_{s_j,n}\}$ is an overcomplete basis.

In other words, given the whole set of coefficients $f_{s_j,h}$, the system

$$Wf = f_{s_j,h},$$

where W is the matrix with rows given by the basis function in B, is over determined.

A possible solution is given by the pseudoinverse, that is, find f that minimizes

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The solution via pseudoinverse can be obtained by solving

$$W^\top Wf = W^\top f_{s_j,h} \qquad \text{(least square solution)}$$

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