

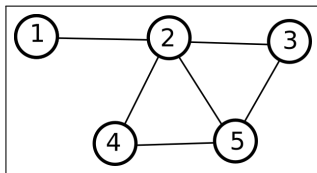
# Graph Signal Processing - Graph Laplacian

Prof. Luis Gustavo Nonato

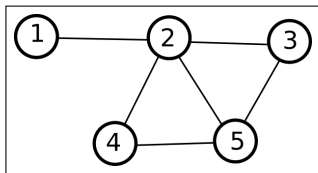
August 23, 2017

# Adjacency Matrix

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$$\mathbf{A} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

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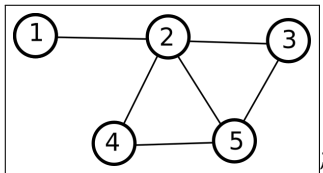
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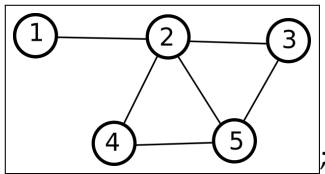
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$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}}_{\mathbf{L}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}}_{\mathbf{D}} - \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}}_{\mathbf{A}}$$

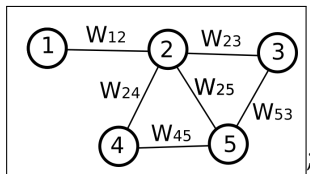


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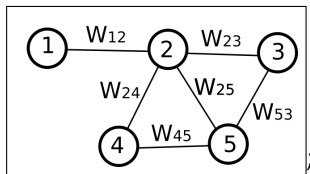
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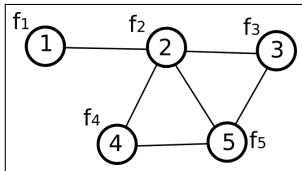
$$\begin{bmatrix} w_{12} & -w_{12} & 0 & 0 & 0 \\ -w_{21} & \sum_{j \neq 2} w_{2j} & -w_{23} & -w_{24} & -w_{25} \\ 0 & -w_{32} & \sum_{j \neq 3} w_{3j} & 0 & -w_{35} \\ 0 & -w_{42} & 0 & \sum_{j \neq 4} w_{4j} & -w_{45} \\ 0 & -w_{52} & -w_{53} & -w_{54} & \sum_{j \neq 5} w_{5j} \end{bmatrix}$$

Let  $G = (V, E)$  be a graph with node set  $V$  and edge set  $E$ .

We can define a scalar function  $f : V \rightarrow \mathbb{R}$  that associates a scalar to each node of the graph.

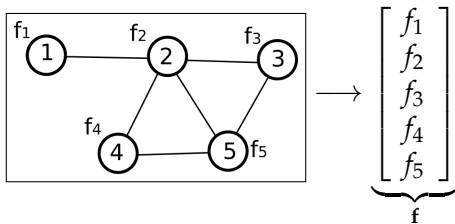
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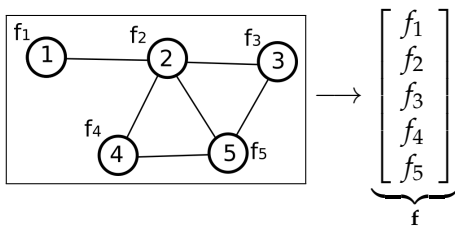
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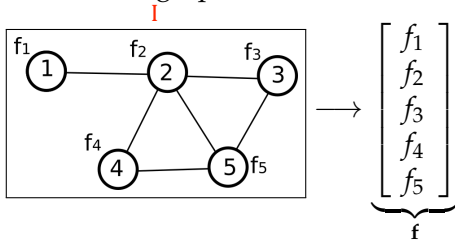


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or equivalently

$$f_i = \frac{1}{l_{ii}} \sum_{j \neq i} f_j$$

(the value in each node is the average of values in neighbor nodes)



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- The eigenvectors are "nice" functions defined on the graph.

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$$\mathbf{L}\mathbf{u}_0 = \begin{bmatrix} \vdots \\ -w_{i1} & 0 & -w_{i3} & \cdots & \sum_j w_{ij} & \cdots & 0 & -w_{3(n-1)} & 0 \\ \vdots \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0\mathbf{u}_0$$

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Imposing the additional constraint  $\|\mathbf{f}\|^2 = 1$ , the Courant-Fiecher theorem ensures that the minimum is reached when  $\mathbf{f}$  is the eigenvector associated to the second smallest eigenvalue.

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where  $\mathbf{u}_i$  are the eigenvectors of  $\mathbf{L}$ .



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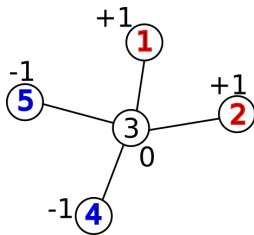
- A positive (or negative) *strong nodal domain* of a function  $f$  defined on  $V$  is a maximal connected subgraph of  $G$  where  $f(v) > 0$  (or  $f(v) < 0$ ). The number of strong nodal domains of  $f$  is  $\mathcal{S}(f)$ .

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$$\begin{aligned}\mathcal{S}(f) &= \{\{1\}, \{2\}, \{4\}, \{5\}\} \\ \mathcal{W}(f) &= \{\{1, 2, 3\}, \{3, 4, 5\}\}\end{aligned}$$

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## Property 4: Discrete Courant's Nodal Theorem

Let  $G$  be a connected graph with  $n$  vertices. Any Graph Laplacian eigenvector  $\mathbf{u}_k$  with corresponding eigenvalue  $\lambda_k$  with multiplicity  $r$  has at most  $k + 1$  weak nodal domains and  $k + r$  strong nodal domains, i.e.,

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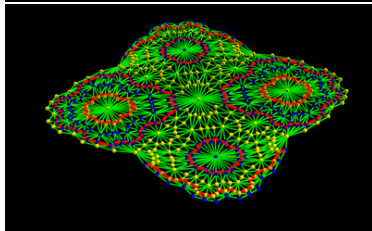
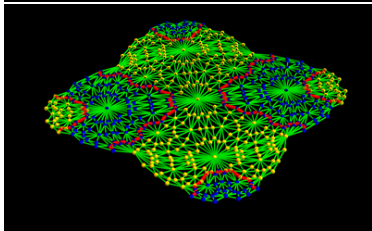
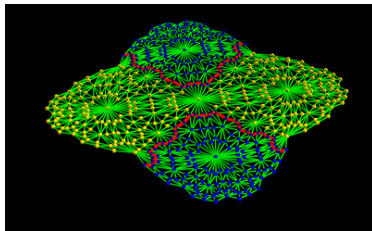
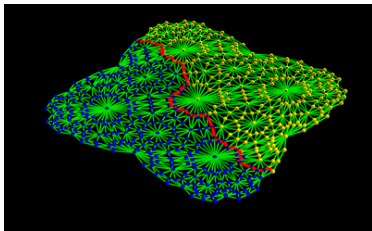
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This theorem was proved by Davies, Gladwell, Leydold, Stadler in 2001 and it is the discrete version of the Courant's Nodal Theorem for the Laplace operator on manifolds.

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- In the case of normalized graph Laplacian, there are a multitude of results giving upper and lower bounds for the eigenvalues, specially related to the diameter of  $G$ .
- Normalized Graph Laplacian is also closely related with random walks on graphs.

$$\mathbf{P} = \mathbf{D}^{-1/2} (\mathbf{I} - \mathcal{L}) \mathbf{D}^{1/2}$$