Graph Signal Processing - Basics on Fourier Analysis

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Inner Product

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The purpose of the conjugate is to ensure that the length of $\mathbf{z} \in \mathbb{C}^d$ is real and nonnegative ($\mathbf{z} = x + iy$, $\mathbf{z}\overline{\mathbf{z}} = x^2 + y^2$).

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Suppose $\mathcal{E} = \{e_1, \ldots, e_n\}$ a set of orthonormal functions basis. Then $\mathcal{E}$ generates a subspace of $L^2[a,b]$ and any function h in this subspace can be written as (h lies in the span of $\mathcal{E}$)

$$h = \sum_{i=1}^{n} < h, e_i > e_i$$

Consider the Taylor expansion of the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

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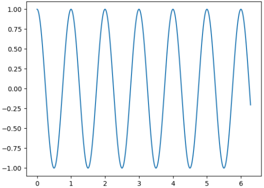
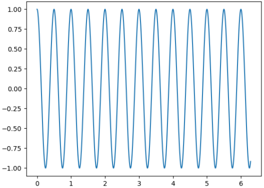
$$e^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)$$

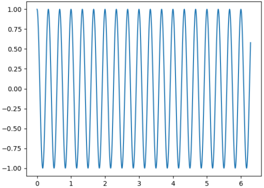
Fourier Basis



$real(e^{-\mathrm{i}x})$



$real(e^{-2(\mathrm{i}x)})$



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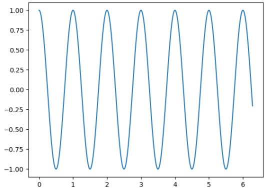

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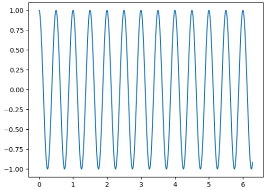

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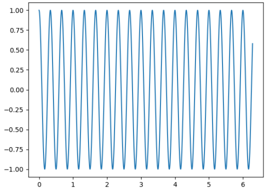

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One-parameter family of (Fourier) basis:

$$e^{-\mathrm{i}\lambda x}$$

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This notation makes clear that $\mathcal{F} : L^2 \to L^2$.

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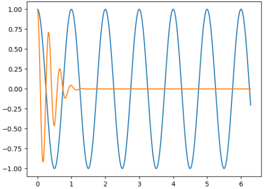

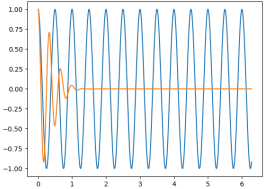

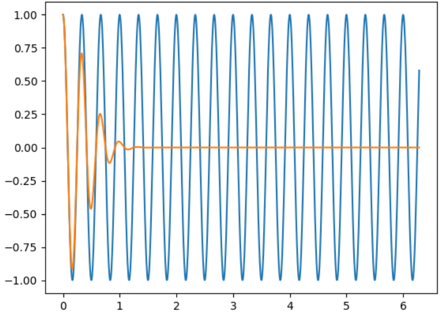

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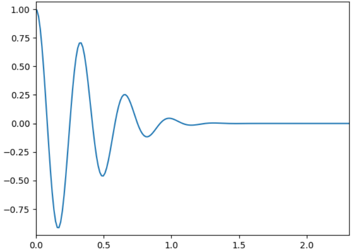

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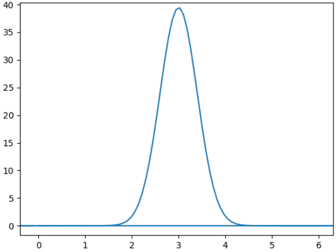

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$$\mathcal{F}[f(cx)](\lambda) = \frac{1}{c}e^{-\mathrm{i}\lambda c}\mathcal{F}[f](\frac{\lambda}{c})$$

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- Perseval's equation

$$\|\mathcal{F}[f]\| = \|f\|$$

Impossibility of Perfect Band-Limiting

A consequence of the Paley–Wiener theorem is that a signal $f(t)$ has perfectly band-limited Fourier transform $\mathcal{F}[f]$, that is,

$$|\mathcal{F}[f](\lambda)| = 0 \text{ for } \lambda > B$$

if only if for any $T \in (-\infty, \infty)$, there exists a $t_0 < T$ sucht that $f(t_0) \neq 0$.

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In other words, a signal has perfectly bandlimited spectrum if and only if the signal persists for all time.

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A filter can easily be understood in the frequency domain.
High-, low-, and band-pass filters are easily designed.