Simple Analytic Expressions for the Magnetic Field of a Circular Current Loop

James Simpson, John Lane, Christopher Immer, and Robert Youngquist

Abstract – Analytic expressions for the magnetic induction and its spatial derivatives for a circular loop carrying a static current are presented in Cartesian, spherical and cylindrical coordinates. The solutions are exact throughout all space outside the conductor.

Index Terms – Circular Current Loop, Magnetic Field, Spatial Derivatives.

I. INTRODUCTION

Analytic expressions for the magnetic induction (magnetic flux density, **B**) of a simple planar circular current loop have been published in Cartesian and cylindrical coordinates [1,2], and are also known implicitly in spherical coordinates [3]. In this paper, we present explicit analytic expressions for **B** and its spatial derivatives in Cartesian, cylindrical and spherical coordinates for a filamentary current loop. These results were obtained with extensive use of *Mathematica*[TM] and are exact throughout all space outside of the conductor. The field expressions reduce to the well-known limiting cases and satisfy $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ outside the conductor.

These results are general and applicable to any model using filamentary circular current loops. Solenoids of arbitrary size may be easily modeled by approximating the total magnetic induction as the sum of those for the individual loops [4]. The inclusion of the spatial derivatives expands their utility to magnetohydrodynamics where the derivatives are required.

The equations can be coded into any high-level programming language. It is necessary to numerically evaluate complete elliptic integrals of the first and second kind, but this capability is now available with most programming packages.

II. SPHERICAL COORDINATES

We start with spherical coordinates because this is the system usually used in the standard texts. The Cartesian and cylindrical results in Sections III and IV were derived from the spherical coordinate results.

The current loop has radius a, is located in the x-y plane, centered at the origin, and carries a current I which is positive

J. C. Simpson is with Dynacs, Inc., KSC, FL 32899 USA (telephone: 321-867-6937, e-mail: James.Simpson-3@ksc.nasa.gov).

J. E. Lane is with Dynacs, Inc., KSC, FL 32899 USA (telephone: 321-867-6939, e-mail: John.Lane-1@ksc.nasa.gov).

C. D. Immer is with Dynacs, Inc., KSC, FL 32899 USA (telephone: 321-867-6752, e-mail: Christopher.Immer-1@ksc.nasa.gov).

R. C. Youngquist is with NASA., KSC, FL 32899 USA (telephone: 321-867-1829, e-mail: Robert.Youngquist-1@ksc.nasa.gov).

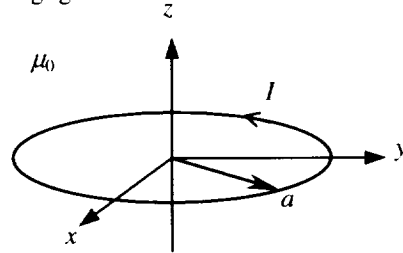as shown (Fig. 1.). It is assumed that the cross section of the conductor is negligible.

Fig. 1. Circular current loop geometry.

The *vector potential*, **A**, of the loop is given by [3]:

$$A_\varphi(r,\theta) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\varphi' \, d\varphi'}{\sqrt{a^2 + r^2 - 2ar\sin\theta\cos\varphi'}}$$
$$= \frac{\mu_0}{4\pi} \frac{4Ia}{\sqrt{a^2 + r^2 + 2ar\sin\theta}} \left[\frac{(2-k^2)K(k^2) - 2E(k^2)}{k^2} \right], \quad (1)$$

where r, θ, and φ are the usual spherical coordinates, and the argument of the elliptic integrals is

$$k^2 = \frac{4ar\sin\theta}{a^2 + r^2 + 2ar\sin\theta} \quad . \quad (2)$$

Note that we use k^2 for the argument of the elliptic integrals. This choice is consistent with the convention of Abramowitz and Stegun [5] where $m = k^2$.

For a static field with constant current, the **B** components in spherical coordinates are [3]:

$$B_r = \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta A_\varphi) \quad (3)$$

$$B_\theta = -\frac{1}{r} \frac{\partial}{\partial r}(r A_\varphi) \quad (4)$$

$$B_\varphi = 0. \quad (5)$$

Analytic expression for the field components and their derivatives in spherical coordinates are given below. For simplicity we use the following substitutions: $\alpha^2 \equiv a^2 + r^2 - 2ar\sin\theta$, $\beta^2 \equiv a^2 + r^2 + 2ar\sin\theta$, $k^2 \equiv 1 - \alpha^2/\beta^2$, and $C \equiv \mu_0 I/\pi$.

Field Components:

$$B_r = \frac{Ca^2\cos\theta}{\alpha^2\beta}E(k^2)$$ (6)

$$B_\theta = \frac{C}{2\alpha^2\beta\sin\theta}\left[(r^2 + a^2\cos2\theta)E(k^2) - a^2K(k^2)\right]$$ (7)

Spatial Derivatives of the Field Components:

$$\frac{\partial B_r}{\partial r} = \frac{Ca^2\cos\theta}{2r\alpha^4\beta^3}\left\{\left[a^4 - 7r^4 - 6a^2r^2\cos2\theta\right]E(k^2) + \left[a^2(r^2-a^2)\right]K(k^2)\right\}$$ (8)

$$\frac{\partial B_r}{\partial\theta} = \frac{-Ca^2}{4\sin\theta\,\alpha^4\beta^3}\left\{\left[a^4 - 7a^2r^2 + r^4 + \cos2\theta(-3a^4 + 2a^2r^2 - 3r^4) + a^2r^2\cos4\theta\right]E(k^2) + \left[2\alpha^2(a^2+r^2)\cos^2\theta\right]K(k^2)\right\}$$ (9)

$$\frac{\partial B_\theta}{\partial r} = \frac{-C}{4\alpha^4\beta^3 r\sin\theta}\left\{\left[a^6 - 3a^4r^2 + a^2r^4 + 2r^6 + a^2(3r^2-a^2)(a^2+r^2)\cos2\theta + 3a^4r^2\cos4\theta\right]E(k^2) + \left[a^2(-a^4 + a^2r^2 - 2r^4 + a^2(a^2-3r^2)\cos^2 2\theta)\right]K(k^2)\right\}$$ (10)

$$\frac{\partial B_\theta}{\partial\theta} = \frac{-C\cos\theta}{4\alpha^4\beta^3\sin^2\theta}\left\{\left[5a^6 + 3a^4r^2 - 3a^2r^4 + 2r^6 + (-3a^6 + 2a^4r^2 + 9a^2r^4)\cos2\theta + a^4r^2\cos4\theta\right]E(k^2) - \left[3a^6 + 2a^4r^2 + a^2r^4 + 2r^6 + a^2(5r^2-a^2)(a^2+r^2)\cos2\theta + (-7a^5r + 7a^3r^3 - 4ar^5)\sin\theta + a^3r(a^2-5r^2)\sin3\theta\right]K(k^2)\right\}$$ (11)

III. Cartesian Coordinates

The field components and their derivatives in Cartesian coordinates are given below. These are easier to use when rotations or translations are needed and obviate the need to transform the basis vectors. The following substitutions are used for simplicity: $\rho^2 \equiv x^2 + y^2$, $r^2 \equiv x^2 + y^2 + z^2$, $\alpha^2 \equiv a^2 + r^2 - 2a\rho$, $\beta^2 \equiv a^2 + r^2 + 2a\rho$, $k^2 \equiv 1 - \alpha^2/\beta^2$, $\gamma \equiv x^2 - y^2$, and $C \equiv \mu_0 I/\pi$. Note that ρ and r are non-negative.

Field Components:

$$B_x = \frac{Cxz}{2\alpha^2\beta\rho^2}\left[(a^2+r^2)E(k^2) - a^2K(k^2)\right]$$ (12)

$$B_y = \frac{Cyz}{2\alpha^2\beta\rho^2}\left[(a^2+r^2)E(k^2) - a^2K(k^2)\right] = \frac{y}{x}B_x$$ (13)

$$B_z = \frac{C}{2\alpha^2\beta}\left[(a^2-r^2)E(k^2) + a^2K(k^2)\right]$$ (14)

Spatial Derivatives of the Field Components:

$$\frac{\partial B_x}{\partial x} = \frac{Cz}{2\alpha^4\beta^3\rho^4}\left\{\left[a^4(-\gamma(3z^2+a^2)+\rho^2(8x^2-y^2)) - a^2(\rho^4(5x^2+y^2) - 2\rho^2z^2(2x^2+y^2)+3z^4\gamma) - r^4(2x^4 + \gamma(y^2+z^2))\right]E(k^2) + \left[a^2(\gamma(a^2+2z^2) - \rho^2(3x^2-2y^2)) + r^2(2x^4 + \gamma(y^2+z^2))\right]a^2K(k^2)\right\}$$ (15)

$$\frac{\partial B_x}{\partial y} = \frac{Cxyz}{2\alpha^4\beta^3\rho^4}\left\{\left[3a^4(3\rho^2-2z^2) - r^4(2r^2+\rho^2) - 2a^6 - 2a^2(2\rho^4 - \rho^2z^2 + 3z^4)\right]E(k^2) + \left[r^2(2r^2+\rho^2) - a^2(5\rho^2-4z^2)+2a^4\right]a^2K(k^2)\right\}$$ (16)

$$\frac{\partial B_x}{\partial z} = \frac{Cx}{2\alpha^4\beta^3\rho^2}\left\{\left[(\rho^2-a^2)^2(\rho^2+a^2) + 2z^2(a^4 - 6a^2\rho^2 + \rho^4)+z^4(a^2+\rho^2)\right]E(k^2) - \left[(\rho^2-a^2)^2 + z^2(\rho^2+a^2)\right]a^2K(k^2)\right\}$$ (17)

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$ (18)

$$\frac{\partial B_y}{\partial y} = \frac{Cz}{2\alpha^4\beta^3\rho^4}\left\{\left[a^4(\gamma(3z^2+a^2)+\rho^2(8y^2-x^2)) - a^2(\rho^4(5y^2+x^2) - 2\rho^2z^2(2y^2+x^2)-3z^4\gamma) - r^4(2y^4 - \gamma(x^2+z^2))\right]E(k^2) + \left[a^2(-\gamma(a^2+2z^2) - \rho^2(3y^2-2x^2)) + r^2(2y^4 - \gamma(x^2+z^2))\right]a^2K(k^2)\right\}$$ (19)

$$\frac{\partial B_y}{\partial z} = \frac{y}{x}\frac{\partial B_x}{\partial z}$$ (20)

$$\frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}$$ (21)

$$\frac{\partial B_z}{\partial y} = \frac{\partial B_y}{\partial z}$$ (22)

$$\frac{\partial B_z}{\partial z} = \frac{Cz}{2\alpha^4\beta^3}\left[6a^2(\rho^2-z^2) - 7a^4 + r^4\right]E(k^2) + a^2\left[a^2-r^2\right]K(k^2)$$ (23)

IV. CYLINDRICAL COORDINATES

The following substitutions are used for simplicity:

$\alpha^2 \equiv a^2 + \rho^2 + z^2 - 2a\rho$, $\beta^2 \equiv a^2 + \rho^2 + z^2 + 2a\rho$, $k^2 \equiv 1 - \alpha^2/\beta^2$, $C \equiv \mu_0 I/\pi$.

Field Components:

$$B_\rho = \frac{C z}{2\alpha^2 \beta \rho}\left[(a^2 + \rho^2 + z^2)E(k^2) - \alpha^2 K(k^2)\right] \tag{24}$$

$$B_z = \frac{C}{2\alpha^2 \beta}\left[(a^2 - \rho^2 - z^2)E(k^2) + \alpha^2 K(k^2)\right] \tag{25}$$

Spatial Derivatives of the Field Components:

$$\frac{\partial B_\rho}{\partial \rho} = \frac{-Cz}{2\rho^2\alpha^4\beta^3}\left\{\left[a^6 + (\rho^2 + z^2)^2(2\rho^2 + z^2) + \right.\right.$$
$$a^4(3z^2 - 8\rho^2) + a^2(5\rho^4 - 4\rho^2 z^2 +$$
$$\left. 3z^4)\right]E(k^2) - \alpha^2\left[a^4 - 3a^2\rho^2 + 2\rho^4 + \right.$$
$$\left.\left. (2a^2 + 3\rho^2)z^2 + z^4\right]K(k^2)\right\} \tag{26}$$

$$\frac{\partial B_\rho}{\partial z} = \frac{C}{2\rho\alpha^4\beta^3}\left\{\left[(a^2 + \rho^2)(z^4 + (a^2 - \rho^2)^2) + \right.\right.$$
$$2z^2(a^4 - 6a^2\rho^2 + \rho^4)\right]E(k^2) -$$
$$\left. \alpha^2\left[(a^2 - \rho^2)^2 + (a^2 + \rho^2)z^2\right]K(k^2)\right\} \tag{27}$$

$$\frac{\partial B_z}{\partial z} = \frac{Cz}{2\alpha^4\beta^3}\left\{\left[(6a^2(\rho^2 - z^2) - 7a^4 + \right.\right.$$
$$\left.\left. (\rho^2 + z^2)^2\right]E(k^2) + \alpha^2\left[a^2 - \rho^2 - z^2\right]K(k^2)\right\} \tag{28}$$

$$\frac{\partial B_z}{\partial \rho} = \frac{\partial B_\rho}{\partial z} \tag{29}$$

V. LIMITING CASES

Several special limiting cases are given for completeness. We have confirmed that our results given above do indeed converge to these formulas. We also give additional expressions for B_x and B_y near the axis that may prove useful.

Along the axis of the loop:

$$B_z = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \tag{30}$$

Near the axis of the loop ($x, y << a$):

$$B_x = \frac{3a^2\mu_0 I xz}{4(a^2 + z^2)^{5/2}} \tag{31}$$

$$B_y = \frac{3a^2\mu_0 I yz}{4(a^2 + z^2)^{5/2}} \tag{32}$$

Far from the loop ($r >> a$):

$$B_r = \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\cos\theta}{r^3} \tag{33}$$

$$B_\theta = \frac{\mu_0}{4\pi}(I\pi a^2)\frac{\sin\theta}{r^3} \tag{34}$$

VI. CONCLUSION

We have presented simple, closed-form algebraic formulas for the magnetic induction and its spatial derivatives of a filamentary current loop that are exact everywhere in space outside the conductor. Although these formulas are exact, they do require the numerical evaluation of elliptic integrals.

Solenoids with circular cross sections of arbitrary size and configuration can be modeled by simply summing the contributions of each individual loop.

There are, of course, other ways to obtain **B** for the basic circular current loop. For example, series expansions are available [3] and numerical integration via a finite element approach can be performed [6]. However, these suffer from limitations such as truncating the series expansions after some tolerance is reached or accepting some graininess when using a discrete grid. Our approach has neither of these limitations and yields results are that exact up to the limitations of the numerical arithmetic and the elliptic integral routines.

The inclusion of the spatial derivatives allows convective derivatives to be found exactly and may prove useful for magnetohydrodynamics applications.

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