Homological Semantics: A Topological Grammar of Relational Meaning and Epistemic Projection

André Hampshire

andrehampshire@gmail.com

No institutional affiliation

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Abstract

This paper proposes a formal topological framework for natural language syntax and semantics, interpreting sentences as oriented chains in a homological space. Each major part of speech is assigned a distinct topological role: nouns are modeled as volumetric 3-dimensional entities with closed internal structure (Betti number $\beta_3 = 1$); verbs act as directed morphisms (1-simplices) that connect noun entities and induce relational structure; adjectives form epistemic loops that project perceptual variants of their referents. A core thesis of the model is that a well-formed sentence corresponds to a topologically closed chain, where each morphism (e.g., verb, modifier) is properly resolved and no boundary defects remain. Grammatical violations correspond to open chains, unresolved morphisms, or class-incompatible projections. We define a homological criterion for sentence closure and show how common constructions—from simple declaratives to clause embedding, recursion, negation, and quantification—can be analyzed as topological structures whose validity depends on closure and compatibility conditions. Epistemic projection is formally distinguished from ontological identity, with adjectival closure requiring transformation into predicate clauses. We also formalize logical operations such as negation as morphism inversion, and show how quantifiers generate indexed families of chains whose collective closure determines truth. This framework

unifies syntax, semantics, and aspects of perception under a single algebraic-topological grammar. By treating meaning as structural closure, it offers a novel and geometrically grounded approach to linguistic well-formedness.

Contents

1 Introduction

Traditional syntactic theories model language using trees, rules, and symbolic categories. While effective in capturing many surface structures, these approaches often fail to provide a unifying account of how grammatical form, semantic coherence, and perceptual cognition interrelate. This paper introduces a novel framework called *homological semantics*, which recasts sentence structure in topological terms: parts of speech are mapped to topological entities (e.g., objects, chains, and loops), and grammatical validity is determined by conditions of structural closure.

The central thesis is this: a sentence is not merely a linear string of tokens, but a compositional complex—an algebraic-topological object—whose well-formedness corresponds to a condition of topological closure. In this model:

- Nouns are treated as closed ontological entities with internal dimensionality (represented as 3-cycles, $\beta_3 = 1$).
- Verbs function as directed morphisms between noun objects (represented as 1-simplices, $\beta_1 = 1$), expressing interaction and relation.
- Adjectives and modifiers are reflexive projections that transform nouns into perceived variants of themselves, forming epistemic loops.

A grammatically complete sentence is one in which these components resolve into a closed topological structure. For instance, in the sentence *The girl is clever*, the copular verb "is" functions as a morphism that connects the noun "girl" to the adjective "clever," projecting a perceived identity back onto the subject and thereby completing a loop. In contrast, a phrase like *clever girl* is structurally open, as it lacks a predicate-level closure.

By interpreting grammatical validity as the topological closure of morphism chains, this framework offers a geometric explanation for why certain word combinations form coherent sentences and others do not. Furthermore, transformations such as turning an adjective-noun pair into a full clause (*"clever girl"* $\rightarrow$ *"the girl is clever"*) correspond to structural operations that "seal" an otherwise open edge.

This topological grammar not only unifies syntax and semantics, but also suggests a deep structural alignment between linguistic formation and cognitive representation. Meaning arises not just from truth conditions or symbol manipulation, but from the closure of morphisms in an abstract topological space of relations. In what follows, we develop the formal machinery of this model, define its axioms and theorems, and demonstrate its application across a variety of sentence structures—including adjectives, clause embeddings, negation, and quantifiers—all within a unified homological semantics.

2 Topological Modeling of Core Parts of Speech

In this framework, grammatical units are mapped to algebraic-topological structures whose compositionality is governed by homological constraints. Each major part of speech—nouns, verbs, adjectives—is modeled as a specific dimensional entity within a simplicial complex, where grammatical validity corresponds to the condition that these entities form a closed morphism structure. We summarize the core assignments as follows:

Part of Speech	**Topological Representation**	**Betti Number**
Noun	3-cycle (closed object)	$\beta_3 = 1$
Verb	1-simplex (directed morphism)	$\beta_1 = 1$
Adjective	Reflexive loop (epistemic projection)	$\beta_1 = 1$ (local)

2.1 Nouns as Ontological Entities

A noun represents a self-contained ontological entity. Topologically, this corresponds to a closed 3-cycle, meaning the noun does not depend on external morphisms for identity. We define a noun $o \in O$ as a volumetric element with internal closure:

$$\partial o = 0, \quad \beta_3(o) = 1$$

Here, ∂ denotes the topological boundary operator, and the nonzero third Betti number β_3 indicates a fully enclosed three-dimensional structure. Nouns thus form the base "objects" over which other morphisms (verbs, adjectives) operate.

2.2 Verbs as Directed Morphisms Between Classes

Verbs denote directed relational mappings between ontological classes. Each verb $v \in V$ is defined as a 1-simplex (edge) with source and target entities:

$$v : C_s \to C_o, \quad \text{where } C_s, C_o \subseteq O$$

Here, C_s and C_o are the subject and object noun classes, respectively. The morphism v is grammatically valid only if the subject o_s and object o_o of the sentence satisfy:

$$o_s \in C_s, \quad o_o \in C_o$$

This compatibility condition ensures that verbs impose structural constraints on which nouns may serve as subjects or objects. We formalize this with a type boundary condition:

$$\partial v := o_o - o_s = 0 \quad \text{(under class compatibility)}$$

Although subjects and objects may be distinct entities, the morphism v is said to "close" topologically when both arguments belong to their respective verb classes. This reflects that the verb has resolved its boundary, and the sentence forms a valid 1-chain.

Bidirectional and Reflexive Constructions

Some verbs allow bidirectional or symmetric morphisms (e.g., *meet, love, fight*). These are modeled as symmetric chains:

$$v : C \leftrightarrow C, \quad \text{where } o_s, o_o \in C$$

In such cases, the morphism is valid in either direction, and $\partial v = o_o - o_s$ still yields a closed path if the sentence structure respects the verb's symmetry class. Reflexive verbs form a special case where $o_s = o_o$, and the morphism loops back onto the same noun (e.g., *She prepared herself*).

2.3 Adjectives as Epistemic Loops

Adjectives modify nouns by projecting an internal property or appearance. These are modeled as reflexive morphisms:

$$a : o \to o', \quad \text{where } o' \in \text{EpistemicSpace}(o)$$

Unlike verbs, adjectives do not connect distinct objects but rather induce a perceived variant within an epistemic subspace. The resulting structure has an internal loop:

$$\partial a = o' - o \neq 0$$

This loop does not close the sentence and is considered epistemically open unless transformed into a predicate with a verb (e.g., *"clever girl"* becomes *"The girl is clever"*). Thus, adjectives require resolution through external morphisms to achieve closure.

2.4 Closure Condition for Sentence Validity

We now formalize a key topological criterion:

Theorem 1 (Topological Closure Criterion)**.** *A sentence σ is grammatically and semantically valid if and only if all morphisms v_i within it resolve to boundary-zero chains:*

$$\forall v_i \in \sigma, \quad \partial v_i = 0$$

under the condition that each o_s, o_o satisfies class compatibility.

This generalizes the idea that a sentence must "close" structurally. If any morphism connects incompatible classes (e.g., *cake likes Alice), the chain is broken and the sentence fails the closure condition.

2.5 Summary

This section introduces the topological assignments of parts of speech and formalizes the boundary-resolving structure of verbs. Nouns form closed volumetric bases, verbs are directional morphisms constrained by class compatibility, and adjectives are loops requiring closure. A valid sentence corresponds to a closed topological chain over this grammar space.

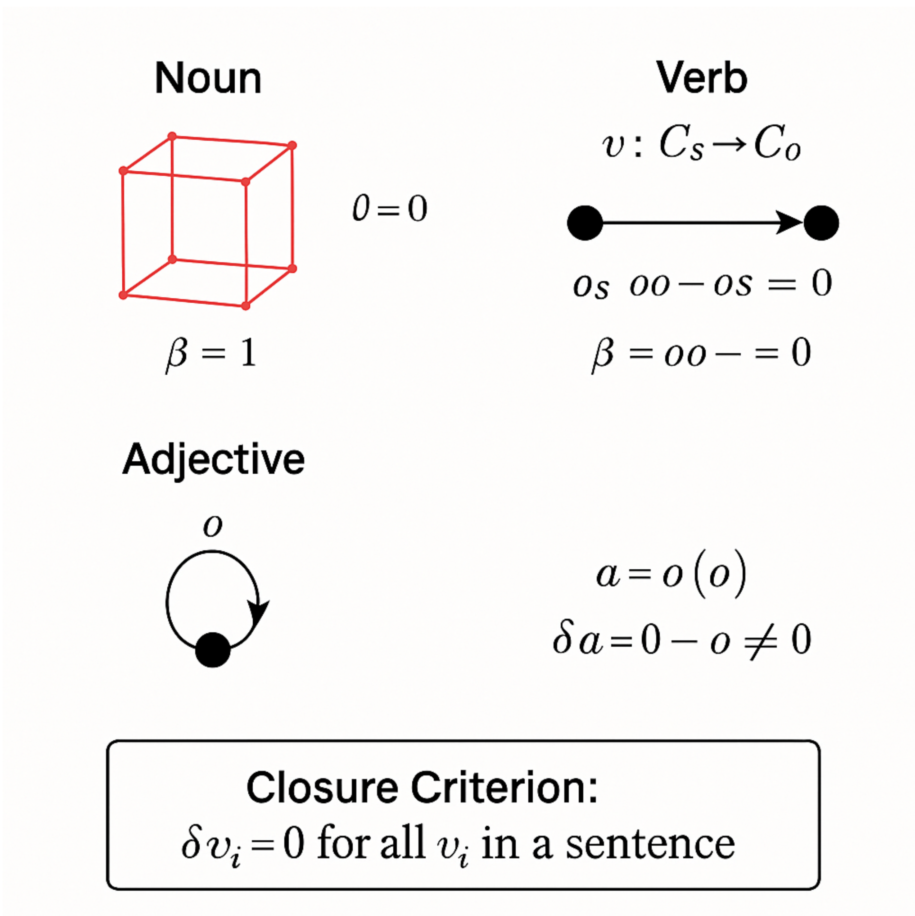

Figure 1: A topological representation of sentence structure. **Left:** An open modifier chain—"clever girl"—where the adjective projects a property onto the noun but no verb resolves the clause. **Right:** A closed sentence loop—"The girl is clever"—where the copular verb resolves the property morphism into a predicate, completing the sentence cycle.

3 Sentence Geometry and Word Order

In homological semantics, a sentence is modeled as a topologically linear structure: an ordered sequence of morphisms and referents that must satisfy structural closure conditions. We formalize this as a one-dimensional geometric path, where the order of tokens determines the orientation and validity of the chain.

3.1 Sentence as Oriented 1-Chain

Let $\mathcal{S}$ be the space of all sentence structures. We define a sentence $\sigma \in \mathcal{S}$ as an oriented 1-chain over a finite ordered set of elements:

$$\sigma = (o_1 \xrightarrow{v_1} o_2 \xrightarrow{v_2} \dots \xrightarrow{v_{n-1}} o_n),$$

where $o_i \in O$ are ontological nouns, and $v_j \in V$ are verb morphisms connecting consecutive terms. The linear geometry is encoded in the word order; reversing the order reverses the semantic meaning or invalidates the structure entirely.

Example 1. *The sentence "Alice likes cake" corresponds to:*

$$A \xrightarrow{\textit{likes}} C$$

Reversing the order yields "Cake likes Alice":

$$C \xrightarrow{\textit{likes}} A,$$

which introduces a different or potentially invalid morphism, depending on verb class compatibility. Thus, sentence orientation determines morphism validity.

3.2 Permutation as Geometric Deformation

Word order permutation acts as a deformation on the sentence path. If a sentence σ corresponds to a valid chain, a permutation $\pi : \sigma \mapsto \sigma'$ may break or alter the topological integrity of the structure. The deformation is interpreted as a morphism rotation, potentially violating boundary conditions or class constraints.

$$\partial(\sigma) = 0 \quad \not\Rightarrow \quad \partial(\pi(\sigma)) = 0$$

This aligns with the intuition that even with the same lexical elements, meaning and grammaticality can vary drastically based on ordering. The sentence manifold is not commutative.

3.3 Magnitude and Clause Depth

We define a sentence's internal complexity via a function $\mu : \mathcal{S} \to \mathbb{N}$, called the *magnitude* or clause depth. It measures the number of properly embedded morphism cycles within the sentence.

Definition 1. *Let $\mu(\sigma)$ be the depth of nested closed clauses within σ. Then:*

$$\mu(\textit{Alice likes cake}) = 1$$

$$\mu(\textit{Alice thinks that Bob likes cake}) = 2$$

$$\mu(\textit{Alice thinks that Bob knows that Carol likes cake}) = 3$$

Each level of embedding introduces a new topological loop into the sentence complex. Magnitude provides a measure of grammatical nesting and structural recursion, which becomes important in clause embedding (Section 6).

3.4 Summary

Sentences are modeled as one-dimensional, oriented chains whose topological shape encodes grammatical validity. Word order determines morphism orientation, and permutations deform sentence topology. Magnitude captures structural depth as a proxy for recursive complexity, paving the way for analyzing nested and hierarchical constructions.

4 Predicate Closure and Copular Resolution

Not all linguistic constructions form topologically closed structures. Adjectival modifiers, in particular, generate epistemic projections that do not inherently yield clause-level closure. In this section, we formalize the transformation required to convert such open structures into closed predicate forms via copular verbs.

4.1 Adjectival Modifiers as Open Projections

Let $a \in A$ be an adjective and $o \in O$ a noun. The composition (a, o) denotes a modified noun phrase. This phrase lacks a verb-induced morphism and thus fails to form a complete 1-cycle. It is represented topologically as a projection:

$$a : o \to o_{perceived},$$

where $o_{perceived} \in \mathcal{O}$ is an epistemically altered representation of o. However, this epistemic projection does not connect to an external entity or resolve into a relational morphism. Thus, it constitutes an open chain with no closure.

Example 2. *The phrase "clever girl" produces a partial structure:*

$$girl \xrightarrow{clever} girl_{perceived},$$

but no clause-level morphism is completed. It does not, on its own, constitute a valid sentence.

4.2 Copular Transformation and Predicate Formation

To induce closure, we introduce a copular verb v_{cop}, typically "is" or "was", forming the following transformation:

Definition 2 (Copular Transformation)**.** *Let T be a function that maps an adjective–noun pair to a predicate clause:*

$$T : (a, o) \mapsto (o \xrightarrow{v_{cop}} a)$$

This transformation yields a closed morphism:

$$girl \overset{is}{\to} clever,$$

which now completes a clause-level structure by reinterpreting the adjective as the morphism target.

4.3 Predicate Closure Theorem

Theorem 2 (Predicate Closure)**.** *Let $o \in O$, $a \in A$, and $v_{cop} \in V$ be a copular verb such that $v_{cop} : o \to a$. Then the structure $(o \xrightarrow{v_{cop}} a)$ is a topologically closed clause if and only if a denotes a valid epistemic state compatible with o.*

Proof. A predicate clause is closed if the morphism $v_{cop} : o \to a$ maps o into a semantically valid projection space. That is, the target a must correspond to a perceptual state admissible under the referent class of o. If this holds, then the epistemic projection becomes grounded via relational resolution, closing the structure. □

4.4 Diagrammatic Example

A visual depiction of this transformation is shown in Figure 2:

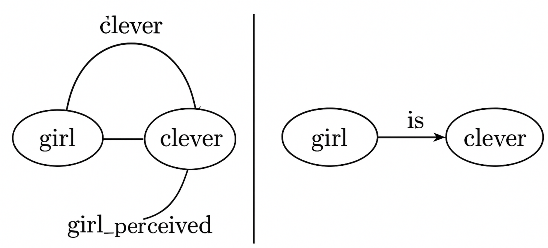

Figure 1: Copular closure resolves an open adjectival projection by introducing a relational morphism to a perceptual state.

Figure 2: Copular closure resolves an open adjectival projection by introducing a relational morphism to a perceptual state.

4.5 Summary

Adjectival modifiers generate open structures by projecting internal variants of nouns. These open chains are transformed into closed predicate structures by inserting a copular verb, which binds the noun to its epistemic projection. The result is a morphism that satisfies the clause-level topological closure criterion.

5 Epistemic Modifiers and Projective Identity

Adjectives and other modifiers do not function as externally directed morphisms; rather, they produce reflexive epistemic projections. These projections alter the perceptual presentation of a noun without changing its ontological structure. In this section, we formalize the role of modifiers as topological transformations that preserve ontological identity but shift referential appearance. We then characterize the conditions under which modifier chains contribute to epistemic or grammatical closure.

5.1 Modifiers as Reflexive Morphisms

Let $a \in A$ be an adjective and $o \in O$ a noun. The adjective acts as a morphism:

$$a : o \to o^*,$$

where o^* is the perceived or epistemically projected form of o. Crucially, $o^* \notin O$ in general: it lies within a perceptual or epistemic subspace $\mathcal{O}_{perceived}$:

$$o^* \in \mathcal{O}_{perceived} \subset \mathcal{O}_{semantic}.$$

These projections are not externally relational; they are intrinsically reflective and maintain a loop-like structure:

$$\partial a = o^* - o \neq 0.$$

Such loops are incomplete at the clause level. That is, while they may alter the internal presentation of an object, they do not yield a complete sentence structure. The morphism remains suspended unless a verb (e.g., a copula) introduces predicate closure.

5.2 Failure of Closure in Modifier Chains

Consider the modifier chain:

$$\text{“clever girl”} \Rightarrow (girl \xrightarrow{a_{clever}} girl^*)$$

Here, the loop is epistemically grounded but grammatically incomplete. No external morphism resolves the structure at the clause level. Such chains fail the sentence closure condition:

$$\partial(\text{“clever girl”}) \neq 0$$

A valid sentence requires that such a projection be transformed via a closure operation, as described in Section 4.

5.3 Epistemic Sufficiency and Modifier Sets

A collection of modifiers can jointly distinguish a noun referent from all others in its class. This gives rise to the notion of epistemic sufficiency.

Definition 3 (Epistemically Sufficient Modifier Set)**.** *Let $M = \{a_1, a_2, ..., a_k\}$ be a set of adjectives. We say M is* epistemically sufficient *for a noun o if:*

$$M(o) = o^* \quad \textit{and} \quad \forall o' \in \mathcal{O},\ M(o') = o^* \Rightarrow o' = o.$$

That is, M uniquely identifies o via its epistemic projections.

This condition reflects informational completeness under perceptual constraints. Even without predicate closure, modifier sets can function as referentially unique identifiers within discourse.

Theorem 3 (Epistemic Sufficiency Theorem)**.** *Let $M = \{a_1, ..., a_k\}$ be a set of modifiers acting on a noun o. Then M is sufficient if and only if the composite projection:*

$$o^* = a_k \circ ... \circ a_1(o)$$

satisfies:

$$\forall o' \in \mathcal{O},\ a_k \circ ... \circ a_1(o') = o^* \Rightarrow o' = o.$$

Proof. If $M(o) = o^*$ and some $o' \neq o$ yields the same projection, o is not uniquely identified, contradicting sufficiency. Conversely, if o^* is reached only by o, M uniquely determines the referent. □

5.4 Clausehood and the Necessity of Verbal Resolution

Although modifier sets can produce epistemic closure (referential uniqueness), they do not produce grammatical closure. As shown in Section 4, transformation into a clause requires the insertion of a verb morphism that binds the noun to its projected identity.

Only through this resolution can the modifier complex function as a sentence:

$$(girl \xrightarrow{a_{clever}} girl^{*}) \Rightarrow (girl \xrightarrow{is} clever).$$

5.5 Summary

Modifiers act as epistemic loops, projecting an object onto its perceived variant. They can identify referents uniquely but do not yield clause-level closure. Such closure requires resolution via a verb morphism. This distinction separates perceptual identity from grammatical validity and introduces a layered interpretation of structural closure in language.

6 Clause Embedding and Nested Loop Structures

Natural language allows clauses to be embedded within larger clauses, recursively increasing syntactic and semantic complexity. In our topological framework, clause embedding corresponds to the nesting of closed morphism cycles within higher-order chains. This section formalizes the embedding process, defines substitution operations on clause-level morphisms, and characterizes nested structure as topological composition.

6.1 Closed Clauses as Substitutable Units

Let $\sigma \in \mathcal{S}$ be a sentence, modeled as a closed morphism complex. A clause is said to be closed if all of its internal morphisms satisfy the boundary condition $\partial = 0$:

$$\partial(\sigma) = 0.$$

Such a clause is structurally valid and may function grammatically as a subject, object, or modifier in a higher-order sentence. This process is analogous to treating

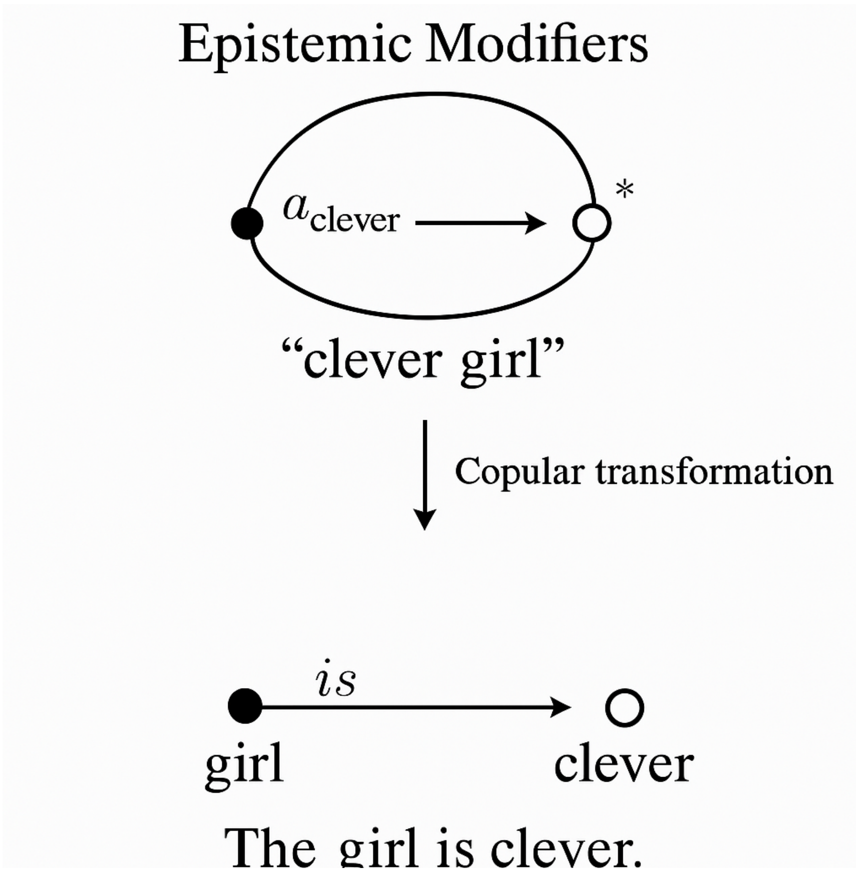

Figure 3: Epistemic modifiers as reflexive morphisms. **Left:** An adjectival phrase (e.g., "clever girl") projects a perceived variant o^* from the base noun o, forming an open epistemic loop without predicate closure. **Right:** Copular resolution introduces a verb morphism ("is") that binds the noun to its projected property, completing a clause-level morphism and achieving topological closure.

the closed clause as a higher-order noun:

$$\sigma \in \mathcal{S}_{closed} \Rightarrow \sigma \in \mathcal{O}^{(1)},$$

where $\mathcal{O}^{(1)}$ denotes the space of embedded nominal clauses—objects that can themselves contain morphism structure.

Example 3. *In the sentence:*

Alice knows that Bob likes cake,

the embedded clause "Bob likes cake" forms a closed morphism cycle and serves as the object of the higher-order verb "knows."

6.2 Functorial Substitution Principle

We define a substitution operation that maps closed clauses into grammatical positions normally occupied by nouns. Let F be a functorial mapping from clause space to noun space:

$$F : \mathcal{S}_{closed} \to \mathcal{O}^{(1)},$$

Then substitution corresponds to:

$$v : o_s \to F(\sigma'),$$

where σ' is a closed embedded clause and v is a higher-order verb whose object class includes embedded propositions.

6.3 Loop Nesting and Clause Depth

Each embedded clause introduces a new loop in the topological structure of the full sentence. The sentence's structural depth is the number of loop-nestings it contains:

Definition 4 (Clause Depth)**.** *Let σ be a sentence composed of n nested closed clauses. Then the clause depth $\mu(\sigma) = n$.*

This measure aligns with linguistic recursion and maps directly onto the topology of sentence embedding. We visualize each clause as a contractible loop, and recursive sentence structure as a sequence of loops nested within loops.

Example 4.

$$\mu(\textit{Alice believes that Bob knows that Carol likes cake}) = 3.$$

6.4 Nested Closure Conditions

To preserve global grammaticality, each clause must satisfy its own closure condition:

$$\forall i \in \{1, ..., n\}, \quad \partial(\sigma_i) = 0.$$

If even one internal clause fails to resolve (e.g., due to class incompatibility or a missing verb), the entire sentence collapses topologically, regardless of its surface grammatical form.

6.5 Summary

Clause embedding corresponds to topological nesting of closed sentence loops. Closed clauses can be treated as substitutable nominal objects in higher-order sentence morphisms. This recursive compositionality gives rise to depth metrics and complexity hierarchies, which are structurally measured by the number of nested loop cycles required to complete the sentence manifold.

7 Logical Structure: Negation, Contradiction, and Semantic Saturation

Logical operators such as negation and contradiction can be naturally represented in our homological semantics framework through topological inversions, boundary reflections, and epistemic overlays. In this section, we introduce a structural criterion called *semantic saturation* that defines the minimal condition under which a linguistic expression becomes logically evaluable.

7.1 Negation as Boundary Inversion

Negation operates on a verb morphism $v : C_s \to C_o$ by reversing its orientation. If v represents the claim "Alice likes cake", then the negation $\neg v$ is a morphism with flipped orientation:

$$\neg v : C_s \leftarrow C_o$$

This inversion disrupts the directionality of the original relation, severing the flow from source to target. Visually, it can be represented by reversing the arrow in the sentence path:

$$\text{Alice} \xleftarrow{\neg\text{likes}} \text{cake}$$

This inversion constitutes an epistemic obstruction that reflects the claim's opposite truth status.

7.2 Contradiction as Topological Collapse

A contradiction is modeled as the conjunction of a morphism and its negation:

$$\sigma \wedge \neg\sigma$$

This produces a closed loop in which the forward and backward morphisms cancel one another. The result is a structural collapse:

$$\partial(\sigma \wedge \neg\sigma) = 0$$

This represents a null structure—a topological manifestation of logical inconsistency. The sentence space contracts into a trivial complex with no interpretable extension.

7.3 Semantic Saturation

We define *semantic saturation* as the condition under which all required morphisms and referential entities are present in a sentence space such that logical operations (e.g., negation, quantification, inference) can be coherently applied.

Proposition 1 (Semantic Saturation Criterion). *A sentence σ is semantically saturated if and only if all morphisms v_i are clause-complete, and every noun node o_j admits sufficient epistemic closure via modifiers or prior contextual grounding.*

Proof Sketch. Clause-completeness ensures that each verb v_i has an admissible subject and object from the proper classes (C_s, C_o) and forms a closed syntactic loop. Epistemic closure ensures that each noun is structurally resolved to a referent or a determinate class (e.g., "the girl" versus "girl"). Without this, ambiguity persists, and logical operators such as negation would lack a fixed scope. Therefore, both criteria are necessary and jointly sufficient for semantic saturation. □

7.4 Diagrammatic Representation

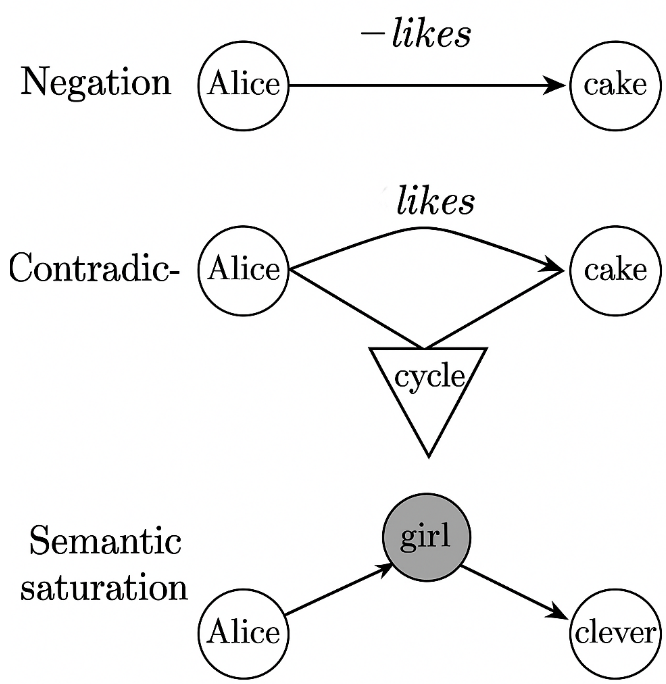

Figure 5. Diagram illustrating negation as a reversed morphism (top), contradiction as a cycle collapse (middle), and semantic saturation as a fully closed and contextually resolved sentence structure (bottom).

7.5 Discussion

This formal treatment emphasizes that logical evaluation depends on structural sufficiency—not merely syntactic completion but referential and epistemic resolution. By modeling negation as a topological inversion and contradiction as structural annihilation, we reveal that logic in language emerges from the geometry of closure and the epistemology of projection. Semantic saturation is thus the minimal ground condition for any logical calculus in linguistic space.

8 Quantifiers and Indexed Closure

Quantifiers introduce abstraction over sets of entities, allowing statements to generalize beyond individual morphisms. In the homological semantics framework, quantification corresponds to generating a family of parallel morphism chains indexed over a domain. The closure or failure of this indexed structure determines the truth of the quantified expression.

8.1 Quantifiers as Set-to-Subset Mappings

Let Q be a quantifier operator (e.g., $\forall$, $\exists$), and let O be a class of referents (e.g., all girls). We define:

$$Q : O \mapsto \mathcal{P}(O),$$

mapping a noun class to a specified subset satisfying a morphic condition.

For example: - $\forall x \in \text{Girl},\ v(x)$ asserts that the verb morphism v holds for all elements of the class. - $\exists x \in \text{Girl},\ v(x)$ asserts that the morphism holds for at least one.

8.2 Universal Quantification and Indexed Morphism Families

A universally quantified sentence is interpreted as a product of morphism chains:

$$\sigma = \prod_{x \in O} (x \xrightarrow{v} o_t),$$

where each x must individually satisfy the morphism condition. The sentence is topologically valid only if every such morphism forms a closed and compatible chain.

Example 5. *"Every girl likes cake" yields:*

$$\forall x \in Girl,\ x \xrightarrow{likes} Cake.$$

If any girl fails to map validly to "cake" via the verb likes, the structure becomes incomplete.

8.3 Existential Quantification and Minimal Closure

In contrast, existential quantification requires closure for only one valid instantiation:

$$\exists x \in O,\ x \xrightarrow{v} o_t.$$

Saturation is achieved upon the first successfully resolved morphism. All others may remain uninstantiated without compromising grammatical or logical validity.

8.4 Quantified Closure Theorem

Theorem 4 (Quantified Closure Theorem)**.** *Let Q be a quantifier over class O, and let v be a morphism such that $v : O \to O'$. Then:*

- $\forall x \in O,\ v(x)$ *is valid iff* $\forall x,\ \partial(v(x)) = 0$*;*
- $\exists x \in O,\ v(x)$ *is valid iff* $\exists x,\ \partial(v(x)) = 0$*.*

Proof Sketch. Each instance $v(x)$ defines a morphism. For $\forall$, all morphisms must close; for $\exists$, a single closure suffices. This follows directly from the closure condition $\partial(v) = 0$ applied over an indexed domain. □

8.5 Failure of Saturation and Truth Collapse

If any required morphism fails to resolve (e.g., due to class incompatibility), the structure remains semantically unsaturated and cannot be evaluated truthfully. For universal statements, a single failure invalidates the whole set:

$$\exists x \in O,\ \partial(v(x)) \neq 0 \Rightarrow \partial\left(\prod v(x)\right) \neq 0.$$

8.6 Summary

Quantifiers generate indexed families of morphisms whose validity depends on the resolution of each constituent path. Universal statements require full saturation and closure of all morphisms; existential statements require minimal saturation. The quantified closure theorem formalizes these conditions as topological constraints on indexed morphism sets.

9 Theoretical Positioning and Discussion

The homological semantics framework introduced in this paper offers a unified, geometrically grounded model of grammatical structure, semantic validity, and logical inference. It interprets linguistic constructions as morphism complexes whose well-formedness is governed by topological closure. In this final section, we position this framework within the landscape of prior linguistic, mathematical, and cognitive theories.

9.1 Comparison to Classical Formal Semantics

In classical model-theoretic semantics (e.g., Montague grammar), meaning is built compositionally via functional types and truth-conditional interpretation. Our framework differs in that it does not require truth assignment in external models; instead, it defines meaning internally as a structural resolution condition: closure of morphism paths between referents.

Rather than relying on symbol-interpretation pairs, our model identifies the compositional logic of language with structural compatibility between topologically typed components. In this way, it offers a structural justification for compositional semantics, showing why particular combinations "work" by virtue of boundary-resolving morphism geometry.

9.2 Connection to Category-Theoretic Grammar

Our formalization aligns naturally with category-theoretic grammar models (e.g., Lambek calculus, categorial grammars), in which grammatical roles are represented as morphisms and composition corresponds to type matching. However, we extend this by assigning **homological properties** (e.g., Betti numbers, closure conditions) to each construct, allowing one to reason not just about composition but about the **global shape** of a sentence structure.

In particular, functorial substitution of embedded clauses and the role of epimorphisms in predicate closure reveal categorical coherence in the sentence space. One might view our model as a topologically enriched category in which morphism families admit loop-completion as a semantic criterion.

9.3 Distinction from Tree-Based Syntax

Traditional generative syntax relies on hierarchical tree structures to represent phrase composition. Our model shifts from a hierarchical to a **cyclical and geometric** view: noun-verb-object triads are interpreted as 1-chains, predicates as loop closures, and recursion as nested homological embeddings. Unlike trees, which enforce strictly acyclic structure, our model embraces loop formation as central to linguistic validity.

This topological lens resolves several ambiguities that trees cannot, such as incomplete closure in modifier phrases or logical collapse in contradictory clauses.

9.4 Cognitive Implications

By treating adjectives as epistemic projections and modifiers as perceptual loops, this framework connects grammar to cognition. It reflects how referents are identified not only by naming but also by perceptual differentiation. The requirement for predicate closure aligns with the cognitive need for **assertive completeness**: statements must resolve to usable meaning.

Semantic saturation, likewise, corresponds to the psychological intuition that vague or structurally incomplete utterances (e.g., "A girl...") are not logically operable until grounded in sufficient referential or structural context.

9.5 Axiomatic Summary

Across the paper, the following core principles have emerged:

- **Parts of speech** correspond to simplices of varying dimensionality, each with assigned Betti numbers.
- **Verbs** are directional morphisms; sentences are 1-chains.
- **Closure** of morphism paths defines grammaticality.
- **Adjectives** are epistemic projections and require copular closure.

- **Quantification** generates indexed morphism families whose closure reflects truth.
- **Negation and contradiction** arise from orientation reversal and loop collapse.
- **Semantic saturation** is the precondition for logical evaluation.

9.6 Positioning and Potential Impact

To our knowledge, no prior framework has unified the structural constraints of grammar, the epistemic nature of reference, and the logical operations of language under one continuous geometric model. The topological grammar developed here offers a new vocabulary for reasoning about linguistic form, and may also be extended to discourse analysis, information structure, and cognitive grounding.

By shifting from symbol manipulation to homological coherence, we introduce a framework in which form and meaning arise not from syntax or logic alone, but from the structure of resolution within the space of linguistic possibility

10 Conclusion

This paper has introduced a homological semantics framework that reinterprets sentence structure, grammatical validity, and semantic inference through the lens of algebraic topology. By modeling linguistic expressions as morphism complexes with associated homological properties, we have shown that:

- Nouns act as ontologically closed entities with internal volumetric structure ($\beta_3 = 1$);
- Verbs function as directed morphisms that impose relational structure and induce 1-chains ($\beta_1 = 1$);
- Adjectives and modifiers operate as reflexive epistemic projections, forming open loops requiring predicate closure;
- Grammatical well-formedness corresponds to topological closure: every morphism must resolve to zero boundary;
- Clause embedding and recursion are modeled as nested homological loops;

- Negation and contradiction manifest as orientation inversions and boundary cancellations;
- Quantification generates indexed morphism families, with truth determined by their saturation and closure;
- Logical evaluation is structurally grounded in semantic saturation: the completeness of morphism instantiations and referent grounding.

This framework unifies syntactic structure, semantic content, logical operations, and epistemic constraints within a single geometric formalism. It extends and complements classical compositional semantics, categorial grammar, and tree-based syntax, while introducing new tools to analyze structural insufficiency, perceptual grounding, and clause-level recursion.

More broadly, this work suggests that the deep regularities of language—its capacity for structured expression, inference, and ambiguity—arise from its topological organization. Meaning is not assigned externally; it emerges from internal structural closure.

Future work will extend this model to discourse structure, anaphora, modal operators, and multi-agent interaction. The integration of diagrammatic reasoning, categorical logic, and homotopical invariants may further deepen our understanding of linguistic structure as a space of perceptual, relational, and inferential possibility.

We conclude that homological semantics offers a principled, extensible foundation for modeling the geometry of language—one that brings meaning, grammar, and cognition into structural alignment.

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