Exercícios Atiyah-Macdonald

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Exercício 1. *Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.*

Demonstração. Seja $x \in A$ um nilpotente qualquer, e $y = -x$, que também é nilpotente. Sendo $n \in \mathbb{N}$ o menor natural tal que $y^n = 0$, temos

$$(1 - y)(1 + y + \cdots + y^{n-1}) = 1 - y^n = 1\,,$$

de onde $1 - y = 1 + x$ é invertível.
Se $\lambda \in A$ é também invertível, então $\lambda^{-1}x$ é nilpotente, de onde $1 + \lambda^{-1}x$ é inversível, e portanto $\lambda + x = \lambda(1 + \lambda^{-1}x)$ é inversível. □

Exercício 2. *Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that*

- *i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]*
- *ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.*
- *iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leqslant r \leqslant n)$.]*
- *iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.*

Demonstração. too long
i) ($\Rightarrow$) Suponha que $g(x) = b_0 + b_1x + \cdots b_mx^m$ é o inverso de f. Temos que

$$1 = gf = \sum_{i=1}^{m+n} c_ix^i\,, \quad \text{com } c_i = \sum_{j=0}^{i} a_jb_{i-j}\,,$$

e note que $c_0 = 1$ e $c_1 = 0$ para $i > 0$. Assim vemos que a_0 e b_0 são inversíveis. Vemos também em particular, de $c_{m+n} = 0$, que $a_n b_m = 0$. Ainda, multiplicando $c_{m+n-1} = a_n b_{m-1} + a_{n-1} b_m$ por a_n e usando que $a_n b_m = 0$, vemos que $a_n^2 b_{m-1} = 0$. Prosseguindo indutivamente, obtemos □

Exercício 3. *Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.*

Exercício 4. *In the ring $A[x]$, the Jacobson radical is equal to the nilradical.*

Exercício 5. *Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that i) f is a unit in $A[[x]] \Leftrightarrow a_0$ is a unit in A. ii) If f is nilpotent, then a_n is nilpotent for all $n \geqslant 0$. Is the converse true? (See Chapter 7 , Exercise 2.) iii) f belongs to the Jacobson radical of $A[[x]] \Leftrightarrow a_0$ belongs to the Jacobson radical of A. iv) The contraction of a maximal ideal m of $A[[x]]$ is a maximal ideal of A, and m is generated by m^{ε} and x. v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$*

Exercício 6. *A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.*

Demonstração. Suponha que $\operatorname{rad}(A) \nsubseteq N(A)$, e tome $a \in \operatorname{rad}(A) \backslash N(A)$. Como $(a) \nsubseteq N(A)$, existe $b \in A$ tal que $0 \neq ab = abab$. Como $a \in \operatorname{rad}(A)$, tem-se que $1 - ab \in U(A)$. Mas

$$(1 - ab)ab = 0\,,$$

o que é uma contradição, pois um elemento não pode ser divisor de zero e inversível simultaneamente. Assim, $\operatorname{rad}(A) \subseteq N(A)$, o que mostra o resultado. □

Exercício 7. *Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.*

Exercício 8. *Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.*

Exercício 9. *Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.*

Exercício 10. *Let A be a ring, $\mathfrak{R}$ its nilradical. Show that the following are equivalent: i) A has exactly one prime ideal; ii) every element of A is either a unit or nilpotent; iii) $A/\mathfrak{R}$ is a field.*

Exercício 11. *A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that i) $2x = 0$ for all $x \in A$; ii) every prime ideal p is maximal, and $A/\mathfrak{p}$ is a field with two elements; iii) every finitely generated ideal in A is principal.*

Exercício 12. *A local ring contains no idempotent $\neq 0, 1$.*

Demonstração. Seja $e \in A$ um idempotente. Como $A = \mathrm{rad}(A) \cup U(A)$ é local, $e \in \mathrm{rad}(A)$ ou $e \in U(A)$. Suponha $e \neq 0, 1$. Temos de $e(e-1) = 0$ que e é divisor de zero, e assim, $e \in \mathrm{rad}(A)$. Temos que $1 - e \in U(A)$. Mas $1 - e$ também é divisor de zero, absurdo. $\square$