Phase Retrieval Homework Report for "Algorithms for Big-Data Analysis"

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1 Problem a

1.1 Formulation

As pointed out in the assignment, one popular formulation of the phase retrieval problem is solving a system of quadratic equations in the form

$$b_r = | < a_r, x > |^2, r = 1, 2, ..., m, \tag{1}$$

where $x \in \mathcal{C}^n$ is the decision variable, $a_r \in \mathcal{C}^n$ are known sampling vectors, and $y_r \in \mathcal{R}$ are the observed measurements.

Our goal is to recover the true signal using the finite measurements a_r we observed. Our recovered signal is the solution x.

As we know, quadratic measurements can be lifted up and interpreted as linear measurements about the rank-one matrix $X = xx\top$. That is,

$$| < a_r, x > |^2 = Tr(x^T a_r a_r^T x) = Tr(a_r a_r^T x x^T) := Tr(A_r X), \tag{2}$$

where $A_r := a_r a_r^T$.

Hence the phase retrieval problem above is equivalent to

$$\begin{array}{rl} \text{find} & X \\ \text{subject to} & Tr(A_r X) = b_r, r = 1, ..., m \\ & X \succeq 0, \\ & \text{rank}(X) = 1 \end{array}$$

Since by definition $b_r = Tr(A_r x_0 x_0^T)$ [1]. There exists a rank-one solution to the above problem. And therefore the problem is equivalent to

$$\begin{array}{rl} \text{minimize} & \text{rank}(X) \\ \text{subject to} & Tr(A_r X) = b_r, r = 1, ..., m \\ & X \succeq 0. \end{array}$$

[1] here x_0 is the true signal

This is a rank-minimization problem over an affine slice of the positive semi-definite cone. And it is NP-hard. The paper [1] proposed using the trace norm to replace the rank functional, giving us the SDP problem,

$$\begin{aligned} \text{minimize} \quad & \mathrm{Tr}(X) \\ \text{subject to} \quad & Tr(A_r X) = b_r, r = 1, ..., m \\ & X \succeq 0. \end{aligned}$$

This SDP problem can be solved using existing solvers or algorithms like ADMM. In this homework, we use mosek to solve it. The results are in the next subsection.

Note that by solving this problem we can get X, and our wanted recovered signal is x and $X = xx^T$. Since X is a low-rank (ideally rank-1), we choose x to be the (scaled) leading eigenvector.

1.2 Numerical Results

We try the data in the link. In the following, n will be the length of the true signal and our recovered signal and m will be the number of measurements we use. We use the data generation files in the link provided in the assignment, and our main file is phase_main.m. We plot our results in Table 1.2.

Error in 2-norm	n=20	n=40	n=60
m=100	4.762e-10	4.640e-01	6.767e-01
m=200	7.123e-11	7.720e-09	4.470e-01
m=300	4.196e-14	4.057e-13	1.897e-09

2 Problem b

2.1 Algorithm

In this problem, the formulation of phase retrieval is the same as (2). We are going to solve for x in the following problem,

$$\text{minimize} \quad f(x) = \frac{1}{2m} \sum_{r=1}^{m} (b_r - |a_r^T x|^2)^2, x \in \mathcal{C}^n \tag{3}$$

And the algorithm we will use is Wirtinger Flow in the paper [2].

The first step is to carefully initialize x_0 as the initial guess via a spectral method, see Algorithm 1.

Then from the initial guess x_0, for $t = 0, 1, 2...$, we update x as the following:

$$x_{t+1} = x_t - \frac{\mu_{t+1}}{||x_0||^2} \left(\frac{1}{m} \sum_{r=1}^{m} (|a_r^T x|^2 - b_r)(a_r a_r^T) x \right) := x_t - \frac{\mu_{t+1}}{||x_0||^2} \nabla f(x_t).$$

This gradient-descent-like algorithm can iteratively refine our guess x and finally we can get our recovered signal.

Algorithm 1 Wirtinger Flow: Initilization

Input: Observations $\{b_r\} \in \mathcal{R}^m$.

1: Set $\lambda^2 = n\frac{\sum_r b_r}{\sum_r ||a_r||^2}$

2: Set x_0 to be the eigenvector corresponding to the largest eigenvalue of

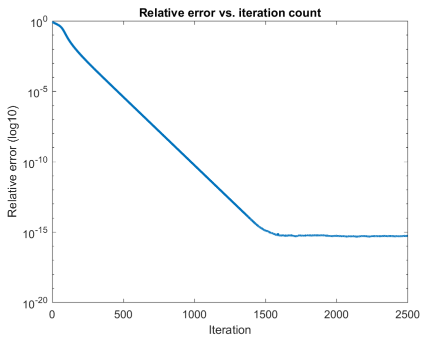
$$B = \frac{1}{m}\sum_{r=1}^{m} b_r a_r a_r^T,$$

and normalize to $||x_0|| = \lambda$.

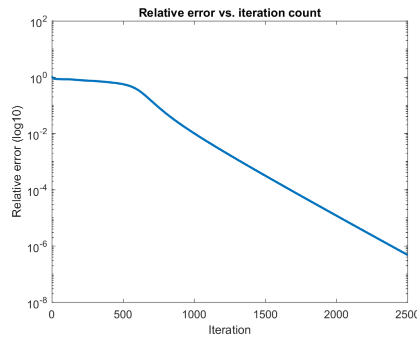
Output: Initial guess x_0

2.2 Numerical Results

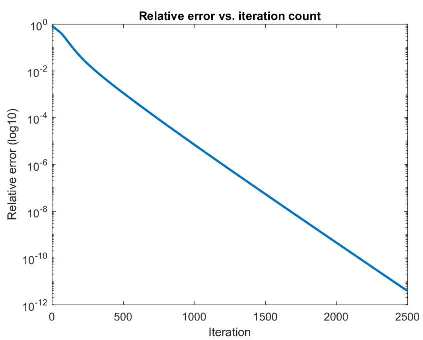
To implement this algorithm, we choose the input data as phaseless measurements about a Gaussian complex valued 1D signal. The results can be seen in 1. During experiments, we found that the algorithm would work well when m is larger than approximately $4n$, and would crash if m is relatively small.



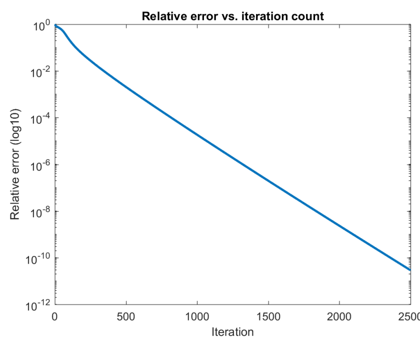
(a) n=20, m=100



(b) n=60, m=200



(c) n=100, m=400



(d) n=200, m=800

Figure 1: Wirtinger Flow phase retrieval

References

[1] Emmanuel J. Cands, Yonina C. Eldar, Thomas Strohmer, and Vladislav Voroninski. Phase retrieval via matrix completion. *SIAM Journal on Imaging Sciences*, 6(1):199–225, 2013.

[2] Emmanuel J. Cands, Xiaodong Li, and Mahdi Soltanolkotabi. Phase retrieval via wirtinger flow: Theory and algorithms. *CoRR*, abs/1407.1065, 2014.