Săceleanu Andrei-Iulian
IA-1B

Homework 5 - CV

1. Let ℓ_1 be an arbitrary line in the 2D plane and $A(u_0, v_0)$ a point such that $A \in \ell_1$

$\ell_1 : ax + by + c = 0$
$A \in \ell_1$
$\Rightarrow au_0 + bv_0 + c = 0$

For every line, there is a normal to that line ($\vec{n}$), such that $\vec{\ell_1} \cdot \vec{n} = 0$

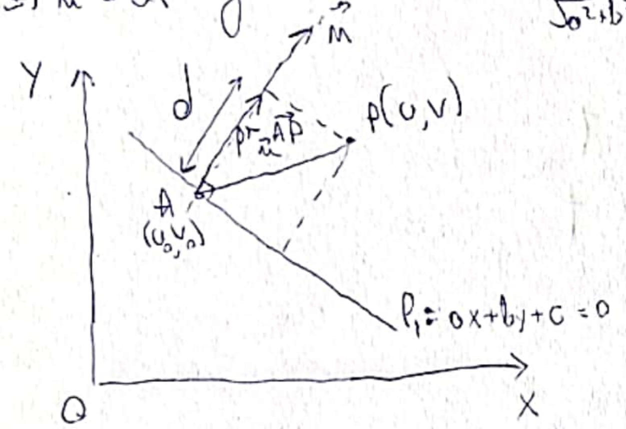
We will prove that $\vec{n} = a\vec{i} + b\vec{j}$

Let 2 points $B(x_1, y_1)$, $C(x_2, y_2)$ such that $B \in \ell_1 \wedge C \in \ell_1 \Rightarrow \begin{cases} ax_1 + by_1 + c = 0 \\ ax_2 + by_2 + c = 0 \end{cases} \xRightarrow{\text{subtracting}}$

$a(x_2 - x_1) + b(y_2 - y_1) = 0 \iff \vec{n} \cdot \vec{\ell_1} = 0$

The direction vector of ℓ_1 can be derived as $(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$ $\Rightarrow$

$\Rightarrow \vec{n} = a\vec{i} + b\vec{j}$ or $\vec{n} = \dfrac{a\vec{i} + b\vec{j}}{\sqrt{a^2 + b^2}}$ in order to have $\|\vec{n}\| = 1$



The distance from a point P to a line $\ell_1(a, b, c)$ is d, where

$\vec{n}$ - the normal starting from A

$d = \| pr_{\vec{n}} \vec{AP} \|$

$\vec{AP} = (u - u_0)\vec{i} + (v - v_0)\vec{j}$

$\Rightarrow d = \left\| \dfrac{\vec{AP} \cdot \vec{n}}{\|\vec{n}\|} \vec{n} \right\| =$

$= \left| \dfrac{\vec{AP} \cdot \vec{n}}{\underbrace{\|\vec{n}\|}_{=1}} \right| \underbrace{\|\vec{n}\|}_{=1} = |\vec{AP} \cdot \vec{n}| = \left| \dfrac{a(u - u_0) + b(v - v_0)}{\sqrt{a^2 + b^2}} \right| = \dfrac{|au + bv \overbrace{- au_0 - bv_0 - c}^{=0} + c|}{\sqrt{a^2 + b^2}} =$

$= \dfrac{|au + bv + c|}{\sqrt{a^2 + b^2}}$

$a^2 + b^2 = 1$

$\Rightarrow d = |au + bv + c|$

A and ℓ_1 were arbitrarily chosen $\Rightarrow$ the result is generally true for any line and point in the plane

2. $x(t)^2+y(t)^2 = \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2} = \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = \frac{t^4+2t^2+1}{(1+t^2)^2} = \frac{(t^2+1)^2}{(t^2+1)^2} = 1$

A circular arc can be expressed as $\begin{cases} x = x_0 + r\cos\theta \\ y = y_0 + r\sin\theta \end{cases}$, $\theta \in [\theta_{min}, \theta_{max}]$

$\left.\begin{array}{l}(x-x_0)^2+(y-y_0)^2 = r^2 \\ x(t)^2+y(t)^2 = 1\end{array}\right| \Rightarrow$ the points of the curve lie on the circle of radius 1, centered at origin, a circular arc if theta or respectively t lies on a single interval or several ones that overlap (considering $2k\pi$ additional revolutions, $k \in \mathbb{Z}$)

$\left.\begin{array}{l}x(t) = \frac{1-t^2}{1+t^2} = \cos\theta \\ y(t) = \frac{2t}{1+t^2} = \sin\theta\end{array}\right| \Rightarrow$ from trigonometry, $\left.\begin{array}{l}\cos\theta = \frac{1-tg^2\frac{\theta}{2}}{1+tg^2\frac{\theta}{2}} \\ \sin\theta = \frac{2tg\frac{\theta}{2}}{1+tg^2\frac{\theta}{2}}\end{array}\right| \Rightarrow t = tg\frac{\theta}{2} \Rightarrow \theta = 2\,arctg\,t$

$x(t) = \cos(2\,arctg\,t + 2k\pi)$
$y(t) = \sin(2\,arctg\,t + 2k\pi)$ $\quad k \in \mathbb{Z}$

3. If the point lies on the arc $\Rightarrow x(t) = \cos(2\,arctg\,t + 2k\pi)$
$y(t) = \sin(2\,arctg\,t + 2k\pi)$

$$\begin{cases} t = tg\left(\frac{\arccos x - 2k\pi}{2}\right) = tg\left(\frac{\arccos x}{2}\right) \\ t = tg\left(\frac{\arcsin y - 2k\pi}{2}\right) = tg\left(\frac{\arcsin y}{2}\right) \end{cases}$$

If the point doesn't lie on the arc, let p' the closest point on the arc

$p(x,y);\ p'(x(t), y(t)) \Rightarrow \|pp'\|^2 = (x(t)-x)^2 + (y(t)-y)^2 = x(t)^2 - 2x(t)x + x^2 + y(t)^2 - 2y(t)y + y^2 =$

$= 1+x^2+y^2 - 2x(t)x - 2y(t)y = 1+x^2+y^2 - 2x\frac{1-t^2}{1+t^2} - 2y\frac{2t}{1+t^2} = 1+x^2+y^2 + \frac{-2x+2xt^2-4yt}{1+t^2}$

$f(t) = d^2(t) = \|PP'\|^2$.

$$f'(t) = 0 \Rightarrow \frac{(4xt-4y)(1+t^2) - (2xt^2-4yt-2x)2t}{(1+t^2)^2} = 0 \Rightarrow$$

$$\Rightarrow 4xt + \cancel{4xt^3} - 4y - 4yt^2 - \cancel{4xt^3} + 8yt^2 + 4xt = 0 \Rightarrow 4(yt^2 + 2xt - y) = 0 \Rightarrow$$

$$\Rightarrow yt^2 + 2xt - y = 0$$

$\underline{I}$ $y = 0$

$2xt = 0$

a) $x = 0$

all the points on the arc are equally distant / close

b) $x \neq 0$

$t = 0 \Rightarrow$ at most one solution, depending if $t=0$ lies in $[t_{min}, t_{max}]$ that characterises the arc

$\underline{II}$ $y \neq 0$

$$t_{1,2} = \frac{-2x \pm \sqrt{4x^2+4y^2}}{2y} = \frac{-x \pm \sqrt{x^2+y^2}}{y}$$

$y \neq 0 \Rightarrow x^2 + y^2 > 0$ $\Rightarrow$ the equation has 2 distinct solutions, it would be needed to check $t_1 \in [t_{min}, t_{max}]$ and/or $t_2 \in [t_{min}, t_{max}]$

5. 2D

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \det R = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta) = 1$$

$$R^{-1} = \frac{1}{\det R} \begin{pmatrix} \cos\theta & -(-\sin\theta) \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R^T$$

3D

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_x^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\det R_x = 1\cdot(-1)^{1+1} \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$R_x^{-1} = \frac{1}{\det R_x} R_x^* = R_x^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -(-\sin\theta) \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Analogous for $R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$ and $R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

For a general 3D rotation (consider Tait-Bryan convention for Euler Angles) $\Rightarrow R = R_z R_y R_x$

$\det R = \det R_z \det R_y \det R_x = 1$,

$$R^{-1} = R_x^{-1} R_y^{-1} R_z^{-1} = R_x^T R_y^T R_z^T = (R_z R_y R_x)^T = R^T$$

nD

$$R_{n,a,b}(\theta) = [r_{ij}] = \begin{cases} r_{aa} = r_{bb} = \cos\theta \\ r_{ab} = -\sin\theta \\ r_{ba} = \sin\theta \\ r_{jj} = 1, & j \neq a,\ j \neq b \\ r_{ij} = 0, & \text{otherwise} \end{cases}$$

By matrix multiplication, it can be observed that $R_{n,a,b}$ can become $R_{n,1,2}$

$R_{n,a,b}\, P_{1a} P_{2b}$, where P_{ij} is the permutation matrix that swaps rows i and j

$R_{n,1,2} = P_{2b} P_{1a}$

$$R_{n,1,2} = \left(\begin{array}{c|c} \begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix} & O_{2\times n-2} \\ \hline O_{n-2\times 2} & I_{n-2} \end{array}\right)$$

In a permutation matrix, the rows/columns are ~~orthogonal~~ orthonormal (the one values always have distinct indices) $\Rightarrow$

$\Rightarrow v^t v = 1,\ v^t w = 0,\ v, w$ - ~~diferent~~ different columns/rows $\Rightarrow P^T P = P P^T = I_n$

$P^T = P^{-1}$

* follow-up after exercise 8

Inputs:

6. $A(x_A, y_A, z_A)$

$B(x_B, y_B, z_B)$

$C(x_C, y_C, z_C)$

$D(x_D, y_D, z_D)$

Outputs:

is_inside : ~~boolean~~ int $\left(\begin{array}{l} 1\text{ - inside} \\ 0\text{ - outside} \\ 2\text{ - other errors} \end{array}\right)$

$\vec{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$

$\vec{AC}\,(x_C - x_A, y_C - y_A, z_C - z_A)$

$\vec{n} = \vec{AB} \times \vec{AC}\ (a, b, c)$ (normal to the potential plane $ax+by+cz+d=0$ defined by A, B and C)

if $\vec{n} = (0,0,0)$
  print "Error, collinear points can't properly define a plane"
  ~~exit with error~~
  return 2

// similar with problem 1, the distance from D to the plane is:

$$t = \frac{|a x_D + b y_D + c z_D + d|}{\sqrt{a^2+b^2+c^2}}$$

D' - projection of D in plane of ΔABC

~~$D' = A + (D - V) = A + D - V$~~

~~$A(x_A, y_A, z_A)$~~

~~$D(x_D, y_D, z_D)$~~

$V = t\,\vec{n}$

sgn = 0

if $\vec{AD} \cdot \vec{n} >= 0$ // if D on same side of plane as $\vec{n}$, we need to subtract

  sgn = -1

else: sgn = 1

$D' = D + sgn \cdot t \cdot \frac{\vec{n}}{\|\vec{n}\|}$

~~cond~~ cond1 = $(AB \times AD') \cdot (AB \times AC)$ >= 0

cond2 = $(AC \times AD') \cdot (AC \times AB)$ >= 0

cond3 = $(BC \times BD') \cdot (BC \times BA)$ >= 0

if cond1 & cond2 & cond3
  return 1
return 0

7. Let a square neighbourhood ($M \times M$) on which a median filter is applied.

The pixel values are uniform, $[\underbrace{c, c, \ldots, c}_{M \times M}]$. Considering the sorted list of pixel values (same as noted above), the median, in this case, is $\begin{cases} \text{middle element} = c, & \text{if } M \text{ is odd} \\ \text{average of middle elems} = c, & \text{if } M \text{ is even} \end{cases}$

I up to half salt & pepper noise $\wedge$ M is odd

a) all pepper / all salt (similar cases, will treat only one)

pixel values in the neighbourhood (sorted) would be, in the maximal noise case,

$$[\underbrace{0, 0, \ldots 0}_{\lceil \frac{M^2}{2} \rceil}, \underbrace{c, c, \ldots, c}_{\lfloor \frac{M^2}{2} \rfloor}] \rightarrow \text{result} = c, \text{ still correct}$$

b) both pepper and salt

$$[\underbrace{0, 0, \ldots 0}_{k}, \underbrace{c, c, \ldots, c}_{\lceil \frac{M^2}{2} \rceil}, \underbrace{255, \ldots, 255}_{\lfloor \frac{M^2}{2} \rfloor - k}]$$, middle point is in the $(c, \ldots, c)$ subarray, result still correct

II up to half — " — $\wedge$ M is even

a) both pepper and salt

similar list as in I b); however, we are interested in the 2 middle elements now, both c, result still correct

b) all pepper / all salt

Considering up to half as strict inequality (i.e. $<$ instead of $\leq$), the result is still correct. If not, one middle element would be 0 and the ~~other~~ other c, so result $= \frac{c}{2}$ for ~~the~~ the maximal noise case

8. $J(w) = (Aw-b)^T(Aw-b) = (w^TA^T - b^T)(Aw-b) =$

$= w^TA^TAw - w^TA^Tb - b^TAw + b^Tb = w^TA^TAw - 2w^TA^Tb + b^Tb \quad (b^T(Aw) = (Aw)^Tb = w^TA^Tb)$

$J'(w) = 2A^TAw - 2A^Tb$
$J'(w) = 0$
$\Rightarrow A^TAw = A^Tb \Rightarrow w = (A^TA)^{-1}A^Tb$ - Moore-Penrose inverse / pseudoinverse of A $* b$

5. $R_{mab} = P_{1a}^{-1} P_{2b}^{-1} R_{m12} P_{2b}^{-1} P_{1a}^{-1} \overset{(1)}{=} P_{1a}^T P_{2b}^T R_{m12} P_{2b}^T P_{1a}^T$ (2)

$R_{mab}^{-1} \overset{(1)}{=} P_{1a} P_{2b} R_{m12}^{-1} P_{2b} P_{1a}$

$R_{mab}^{T} \overset{(2)}{=} P_{1a} P_{2b} R_{m12}^{T} P_{2b} P_{1a}$

$R_{m12}^{-1} = \frac{1}{\det(R_{m12})} R_{m12}^{*} = \frac{1}{1}\left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0_{2\times m-2} \\ -\sin\theta & \cos\theta & \\ \hline \multicolumn{2}{c|}{0_{m-2\times 2}} & I_{m-2} \end{array}\right) = R_{m12}^T$

$\Rightarrow R_{mab}^T = R_{mab}^{-1}$

$\det R_{mab} = \det(R_{m12}) \cdot \underbrace{\det(P_{2b})^2 \cdot \det(P_{1a})^2}_{\text{any permutation matrix has det either 1 or -1 + } \det P = \det P^T} = \det(R_{m12}) = 1$