Homework part B.

14 points [1p: B1] + [1p: B2] + [1p: B3] + [4p: B4 or B5] + [7p: B6]

B1. (1 point) The volume of the cone

$$Con(r,h) = \left\{(x,y,z)\ :\ x^2+y^2 \leqslant \frac{r^2}{h^2}z^2, 0 \leqslant z \leqslant h\right\} \subseteq [-r,r]\times[-r,r]\times[0,h]$$

is $\frac{\pi r^2 h}{3}$. Estimate this volume using MC method for $r=2$, $h=3$ and compare the result with the exact value. Do this for 10000, 20000 and 50000 trials and compute the relative errors.

B2. (1 point) Let T be the following triangle $\{(x,y)\ :\ y \geqslant 0, 2y \leqslant x, x+y \leqslant 3\}$. Determine a rectangular area $[a,b]\times[c,d]$ that includes all of its interior points and then estimate the (unknown) area covered by T using MC method with sample of size 15000.

B3. (1 point) Estimate the value of the following integrals and compare the result with the exact value:

$$(a)\int_0^1 \frac{2\sqrt{x}}{x+1}dx = 4-\pi,\ (b)\int_0^{+\infty}(1+2x)e^{-x}dx = 3.$$

B4. (4 points) Three web servers deliver the same pages to web clients. The time to process a HTTP request is $\Gamma(6,4)$ distributed on the first server, $\Gamma(6,2)$ on the second, and $\Gamma(5,3)$ on the third server (in milliseconds). To this time we add the latency between the client computer and the serveres on the Internet which has an exponential distribution with $\lambda = 2$ (in milliseconds). It is known that a client is directed to the first server with probability 0.35, o the second server with probability 0.4, and to the third server with probability 0.25. Estimate the average time a client has to wait until it receives a response to its request.

B5. (4 points) Twenty five computers are connected in a network. A virus infects this network in the following way: every day, the virus spreads independently at random from any infected computer to any uninfected computer with probability 0.15. Also, every day (starting from the second day), the system admin choose at random three infected computers (if exist) and independently cleans each of them with probability 0.65. In the first day we have four infected computers. Estimate the expected time it takes to remove the virus from the whole network (if possible).

B6. (7 points) A forest has the shape of a triangle with vertices of coordinates $(0,0)$, $(196,0)$, and $(98,98)$. Row p contains the trees in the positions (p,p), $(p+2,p)$, $(p+4,p),\ldots,(196-p,p)$, where $0 \leqslant p \leqslant 98$.

The forest catches fire from a tree of coordinates (i,j). After each our a tree can catch fire from one of its already burning neighbors. The wind blows from the northeast, therefore, the probability that any tree catches fire from its burning northeastern neighbor is 0.7. The probabilities to catch fire from the remaining neighbors are: 0.3 from west or south, 0.5 from north or east, and 0.4 from northwest or southeast. Furthermore, a tree catches fire if all of its neighbors burn.

(a) (3 points) Estimate the expected number of hours after the tree of coordinates $(98,98)$ catches fire, if the first burning tree has coordinates $(50,100)$.

(b) (3 points) Estimate the probability that at least one quarter of the forest will burn after h hours, if the first burning tree has coordinates $(98,98)$ ($h = 20,30,40,50,60,70$).

Solutions to these exercises (the corresponding R functions and their calls) will be written in an single R script.