II. 3. a.) Distributia JPD

X =	-2	-1	1	2
P(X) =	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$

$Y = X^2$

$X = -2 \Rightarrow Y = (-2)^2 = 4$
$X = -1 \Rightarrow Y = (-1)^2 = 1$
$X = 1 \Rightarrow Y = 1^2 = 1$
$X = 2 \Rightarrow Y = 2^2 = 4$

$\rightarrow$ $Y = 4,1,1,4 \Rightarrow Y = 1, 4$

$P(X = -2, Y = 1) = 0 \quad ((-2)^2 \mathrel{!}= 1)$
$P(X = -2, Y = 4) = \frac{1}{4} \quad ((-2)^2 = 4)$
$P(X = -1, Y = 1) = \frac{1}{8} \quad ((-1)^2 = 1)$
$P(X = -1, Y = 4) = 0 \quad ((-1)^2 \mathrel{!}= 4)$

$P(X = -1, Y = 1) = \frac{1}{4} \quad ((-1)^2 = 1)$
$P(X = -1, Y = 4) = 0 \quad ((-1)^2 \mathrel{!}= 4)$

$P(X = -2, Y = 1) = 0 \quad ((-2)^2 \mathrel{!}= 1)$
$P(X = -2, Y = 4) = \frac{3}{8} \quad ((-2)^2 = 4)$

Y \ X	-2	-1	1	2	
1	0	$\frac{1}{8}$	$\frac{1}{4}$	0	$\frac{3}{8}$
4	$\frac{1}{4}$	0	0	$\frac{3}{8}$	$\frac{5}{8}$
	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	Adunare pe linii/coloane

b.) Covariatia & Corelatia

$Cov(X, Y) = E(X * Y) - (E X) * (E Y)$

$E X = (-2) * \frac{1}{4} + (-1) * \frac{1}{8} + 1 * \frac{1}{4} + 2 * \frac{3}{8} = \frac{3}{8}$

$E Y = 1 * \frac{3}{8} + 4 * \frac{5}{8} = \frac{23}{8}$

$X * Y = X * X^2 = X^3 \Rightarrow E(X * Y) = E(X^3) = (-8) * \frac{1}{4} + (-1) * \frac{1}{8} + 1 * \frac{1}{4} + 8 * \frac{3}{8}$

$Cov(X, Y) = \frac{9}{8} - \frac{3}{8} * \frac{23}{8} = \frac{3}{64}$

Corelatia = 0 => X si Y sunt independente

$\rho(X, Y) = \frac{Cov(X,Y)}{StDev(X) * StDev(Y)} = \frac{Cov(X,Y)}{\sqrt{Var(X)} * \sqrt{Var(Y)}}$

$Var(X) = E(X^2) - (E X)^2 = (-4) * \frac{1}{4} + (-1) * \frac{1}{8} + 1 * \frac{1}{4} + 4 * \frac{3}{8} - \frac{9}{64} = \frac{175}{64}$

$Var(Y) = E(Y^2) - (E Y)^2 = 1 * \frac{3}{8} + 16 * \frac{5}{8} - \frac{529}{64} = \frac{135}{64}$

$\rho(X, Y) = \frac{1}{5\sqrt{105}}$

Var (X) = Dispersia lui X

II. 6. a.) Repartitia comuna (JPD)

X = nr de bile albe

Y = nr de bile cu numarul 2

1 by 1, 2 balls (MAXIMUM)

U : $2w_1$, $2w_2$, $2b_1$, $1b_2$

Y \ X	0	1	2	
0	$\frac{1}{21}$	$\frac{4}{21}$	$\frac{1}{21}$	$\frac{6}{21}$
1	$\frac{2}{21}$	$\frac{6}{21}$	$\frac{4}{21}$	$\frac{12}{21}$
2	0	$\frac{2}{21}$	$\frac{1}{21}$	$\frac{3}{21}$
	$\frac{3}{21}$	$\frac{12}{21}$	$\frac{6}{21}$	1

$P(X=0, Y=0) = P(2b_1) = \frac{C_2^2}{C_7^2} = \frac{1}{21}$

$P(X=0, Y=1) = P(1b_2) = \frac{C_2^1 * C_2^1}{C_7^2} = \frac{2}{21} = \frac{2}{21}$

$P(X=0, Y=2) = P(2b_2) = 0$ (Sunt doar $1b_2$)

$P(X=1, Y=0) = P(1w_1, 1b_1) = \frac{C_2^1 * C_2^1}{C_7^2} = \frac{4}{21}$

$P(X=1, Y=1) = P(1w_2, 1b_1) + P(1w_1, 1b_2) = \frac{C_2^1 * C_2^1}{C_7^2} + \frac{C_2^1 * C_2^1}{C_7^2} = \frac{6}{21}$

$P(X=1, Y=2) = P(1w_2, 1b_2) = \frac{C_2^1 * C_2^1}{C_7^2} = \frac{2}{21}$

$P(X=2, Y=0) = P(2w_1) = \frac{C_2^2}{C_7^2} = \frac{1}{21}$

$P(X=2, Y=1) = P(1w_1+1w_2) = \frac{C_2^1 * C_2^1}{C_7^2} = \frac{4}{21}$

$P(X=2, Y=2) = P(2w_2) = \frac{C_2^2}{C_7^2} = \frac{1}{21}$

b.) Verificarea daca variabilele sunt independente

$P(X = X_i, Y = Y_i) = P(X = X_i) * P(Y = Y_i)$, pentru orice i si j => X si Y independent

$$\left.\begin{aligned} &P(X = 0, Y = 0) = \frac{1}{21} = \frac{21}{441} \\ &P(X=0) * P(Y=0) = \frac{3}{21} * \frac{6}{21} = \frac{18}{441} \end{aligned}\right\} \Rightarrow P(X = 0, Y = 0) \mathrel{!=} P(X = 0) * P(Y = 0) \Rightarrow$$

X si Y sunt independente

II. 9.

A and B, three trows

B = 0.4t 0.6 h

X = nr of tails from A

Y = nr of tails from B

a.) Verify if X and Y are independent

Yes, they are physically independent due to the fact that X depends only on A and Y depends only on Y.

b.) Repartitia comuna (JPD)

$P(X=0) = P(h,h,h) = \frac{1}{8}$

$P(X=1) = P(t,h,h;\ h,t,h;\ h,h,t) = \frac{3}{8}$

$P(X=2) = P(t,t,h;\ t,h,t;\ h,t,t) = \frac{3}{8}$

$P(X=3) = P(t,t,t) = \frac{1}{8}$

Y \ X	0	1	2	3	
0	$\frac{27}{1000}$	$\frac{81}{1000}$	$\frac{81}{1000}$	$\frac{27}{1000}$	$\frac{27}{125}$
1	$\frac{54}{1000}$	$\frac{162}{1000}$	$\frac{162}{1000}$	$\frac{54}{1000}$	$\frac{54}{125}$
2	$\frac{36}{1000}$	$\frac{108}{1000}$	$\frac{108}{1000}$	$\frac{36}{1000}$	$\frac{36}{125}$
3	$\frac{8}{1000}$	$\frac{24}{1000}$	$\frac{24}{1000}$	$\frac{8}{1000}$	$\frac{8}{125}$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	

$(X \sim B(3, \frac{1}{2})$ -- 3 with a 50% chance for each

$Y \sim B(3, \frac{2}{3}) \Rightarrow P(Y=0) = (\frac{3}{5})^3$ (3t)

$\Rightarrow P(Y=1) = \frac{2}{5} * (\frac{3}{5})^2 * C_3^1$ (2t 1h)

$\Rightarrow P(Y=2) = (\frac{2}{5})^2 * (\frac{3}{5})^2 * C_3^2$ (1t 2h)

$\Rightarrow P(Y=3) = (\frac{2}{5})^3$ (3h)

c.) $P(X = Y) = \sum_{k=0}^{3} P(X = k, Y = k) = \frac{27+162+108+8}{1000}$

$P(X > Y) = \frac{81+81+162+27+54+36}{1000}$

$P(X + Y >= 4) = \frac{54+108+36+24+24+}{1000}$

III 2.)

Markov : X >= 0 , P (X >= t) <= $\frac{EX}{t}$, t > 0

M : P (X >= 2) <= $\frac{EX}{2} = \frac{1}{2}$ (t=2)

|X – 1| >= 2 → $\begin{cases} X-1 \geq 2 \\ OR \\ X-1 \leq 2 \end{cases}$ => $\begin{cases} X \geq 3 \\ OR \\ X \leq -1 \end{cases}$ => X >= 3 (X >= 0) → P (|X-1| >= 2) = P (X>=3) <= $\frac{EX}{3} = \frac{1}{3}$

P (X <= -3) = 0 , because X >= 0

III 2.)

Chebyshe : P (|X – E X | >= t) <= $\frac{\text{var (X)}}{t}$, t > 0

2-1 (t-EX)

C :

P (X >= 2) = P (X-1 >= 1) <= P (|X-1| >= 1)

X-1 >= 1 => |X-1| >= 1 → {X-1 >= 1} apartine {|X-1| >= 1}

P (|X-1| >= 2) <= $\frac{\text{Var (X)}}{2^2}$ = 1

III 6.) 300 coin tosses, with the prob of head is 3/10, what is the prob to get head at least 100 times

X = nr of heads X ~ B (300, $\frac{3}{10}$)

M : P (X >= 100) <= $\frac{EX}{100} = \frac{300*\frac{3}{10}}{100} = \frac{9}{10}$ EX = 90 (300*3/10) ---(n*p) Var X : (n*p*(1-p))

C : P (X >= 100) = P (X - 90 >= 10) <= P (|X-90| >= 10) <= $\frac{\text{Var (X)}}{10^2} = \frac{300*\frac{3}{10}*(1-\frac{3}{10})}{100} = \frac{63}{100}$

III 7.) n coin tosses, with the prob of 2/10, what is the prob to get head (at least) 50% of times

X = nr of heads $\quad X \sim B\left(n, \frac{1}{5}\right) \quad EX = n*1/5 \quad Var(X) = n*1/5*(1-1/5)$

$$M : P\left(X >= \frac{n}{2}\right) <= \frac{EX}{\frac{n}{2}} = \frac{\frac{n}{5}}{\frac{n}{2}} = \frac{2}{5}$$

$$C : P\left(X >= \frac{n}{2}\right) = P\left(X - \frac{n}{5} >= \frac{n}{2} - \frac{n}{5}\right) <= P\left(\left|X - \frac{n}{5}\right| >= \frac{3n}{10}\right) <= \frac{Var(X)}{\left(\frac{3n}{10}\right)^2} = \frac{n*\frac{1}{5}*\frac{4}{5}}{\frac{9n}{100}^2} = \frac{16n}{9n^2} = \frac{16}{9n}$$

III 8.) A = prob 1/4 for t, B = prob of 4/5, A and B are flipped 25 times, prob of 2t at least 10

X = success = 2t $\quad$ P (success) = 0.25 * 0.8 = 0.2

$$M : P(X >= 10) <= \frac{EX}{10} = \frac{25*\frac{1}{5}}{10} = \frac{1}{2}$$

$$C : P(X >= 10) = P(X - 5 >= 5) <= P(|X - 5| >= 5) <= \frac{Var(X)}{5^2} = \frac{25*\frac{1}{5}*\frac{4}{5}}{25} = \frac{4}{25}$$