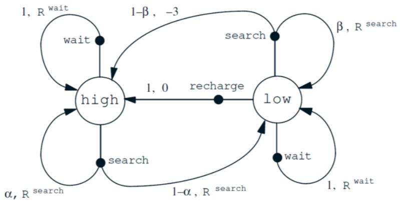


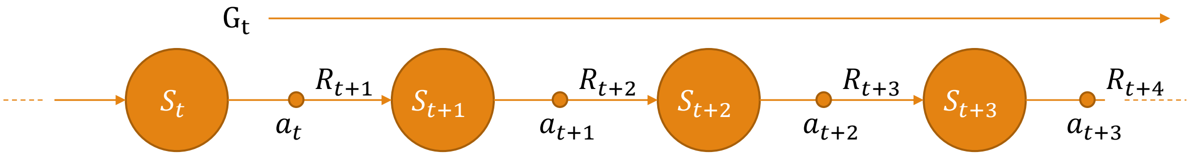
ECE 414/517 Reinforcement Learning

LECTURE 6: BELLMAN EQUATIONS

SEP. 12 2022

(ADAPTED FROM SLIDES BY DR. ITAMAR AREL)

Returns and state value function



$$\mathrm{G_t} = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{\{i=0\}}^{\infty} \gamma^i R_{t+i+1}$$

$$\mathrm{G_t} = R_{t+1} + \gamma \mathrm{G_{t+1}}$$

$$V^{\pi}(s) = E_{\pi}[\mathrm{G_t} | \mathrm{S_t} = \mathrm{s}]$$

Bellman Equations

Given the definitions for the value function and the return:

$$V^{\pi}(s) = E_{\pi}[\mathrm{G_t}|\mathrm{S_t} = \mathrm{s}]$$
$$\mathrm{G_t} = R_{t+1} + \gamma \mathrm{G_{t+1}}$$

We can derive the following:

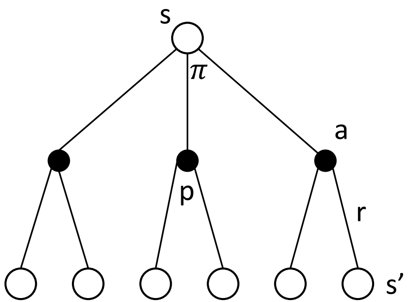
$$V^{\pi}(s) = E_{\pi}[R_{t+1} + \gamma G_{t+1}|\mathrm{S_t} = \mathrm{s}]$$
$$V^{\pi}(s) = \sum_{a} \pi(a|s) \sum_{s',r} p(s', r|s, a)[r + \gamma V^{\pi}(s')]$$

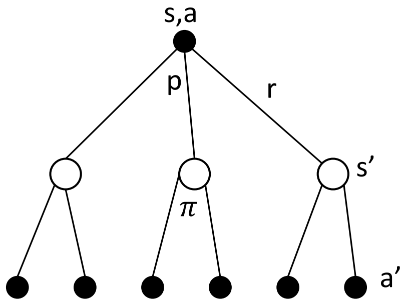
Can be solved as linear equations ($|s|$ equations with $|s|$ unknowns)

Backup Diagram

$$V^{\pi}(s) = \textstyle\sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma V^{\pi}(s')] = \displaystyle\sum_a \pi(a|s)\, Q^{\pi}(s,a)$$

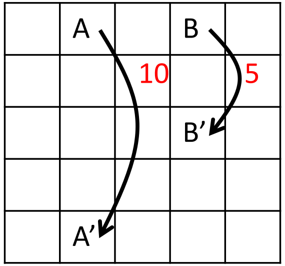
$$Q^{\pi}(s,a) = \sum_{s',r} p(s',r|s,a)\left[r + \gamma \sum_{a'} \pi(a'|s') Q^{\pi}(s',a')\right] = \sum_{s',r} p(s',r|s,a)[r + \gamma V^{\pi}(s')]$$





Grid World

1. An agent starts at a certain cell on a grid and at each time step can make one of 4 moves: **north, south, east, west.**
2. If a move is made which takes the agent off the grid, it does not move but is given a reward of -1
3. Two special grid cells which produce reward:
   - Grid cell A give a reward of 10 for any move and you always end up at A'
   - Grid cell B give a reward of 10 for any move and you always end up at B
4. All other moves give a zero reward
5. Calculate $V_\pi(s)$ for all states where $\pi = \{0.25, 0.25, 0.25, 0.25\}$ for all states
6. Set $\gamma = 0.9$



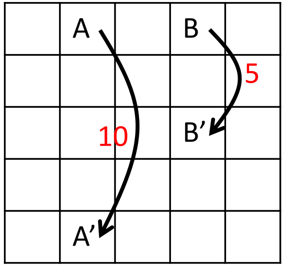
$$v^{\pi}(s) = \sum_{a} \pi(a,s) \sum_{s',r} p(s',r|s,a)\left[r + \gamma V^{\pi}(s')\right]$$

$$v^{\pi}(0,0) = \overset{W}{0.25} \cdot \left[-1 + 0.9\, V^{\pi}(0,0)\right] + \overset{N}{0.25}\left[-1 + 0.9\, V^{\pi}(0,0)\right]$$
$$+ \underset{}{\overset{S}{0.25}} \cdot \left[0 + 0.9\, V^{\pi}(1,0)\right] + \overset{E}{0.25}\left[0 + 0.9\, V^{\pi}(0,1)\right]$$

$$v^{\pi}(0,1) = \overset{W}{0.25} \cdot \left[10 + 0.9\, V^{\pi}(4,1)\right] \times 4$$

Grid World

$$V^{\pi}(s) = \mathrm{E}_{\pi}\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \mid S_t = s\right] = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma V^{\pi}(s')]$$



1. Show that the bellman equations hold for the middle square.

$$\begin{aligned} V(2,2) = &\ 0.\overset{w}{2}5[0 + 0.9 \cdot 0.74] \\ &+ 0.\overset{n}{2}5[0 + 0.9 \cdot 2.25] \\ &+ 0.\overset{e}{2}5[0 + 0.9 \cdot 0.30] \\ &+ 0.\overset{s}{2}5[0 + 0.9 \cdot -0.35] = 0.67 \end{aligned}$$

3.31	8.79	4.43	5.32	1.49
1.52	2.99	2.25	1.91	0.55
0.05	0.74	0.67	0.36	-0.40
-0.97	-0.44	-0.35	-0.59	-1.18
-1.86	-1.34	-1.23	-1.42	-1.97

Optimal Value Functions:

1. A policy π^* is defined to be better than or equal to a policy π, if its expected return is greater than or equal to that of π **for all states:**
$$\pi^* \geq \pi \quad \Leftrightarrow \quad V^{\pi^*}(s) \geq V^{\pi}(s) \quad \forall s \in S$$

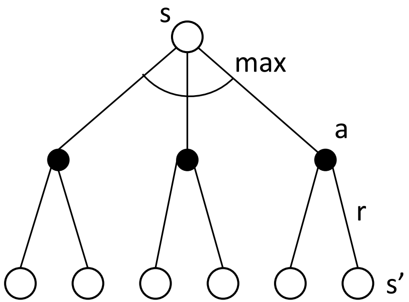
2. There is always **at least** one policy (a.k.a optimal policy) that is better than or equal to all other policies:
$$V^*(s) = \max_{\pi} V^{\pi}(s) \quad \forall s \in S$$

3. Optimal policies also share the same optimal action-value function, defined as:
$$Q^*(s,a) = \max_{\pi} Q^{\pi}(s,a) \quad \forall s \in S, a \in A(s)$$

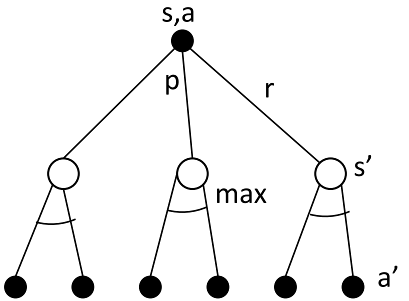
4. Since the action value is simply the return for taking action a in state s and thereafter following an optimal policy, we can write:
$$Q^*(s,a) = \mathrm{E}[\mathrm{r}_{t+1} + \gamma V^*(s_{t+1}) | s_t = s, a_t = a]$$

Bellman Optimality Equation

1. Since $V^*(s)$ is the value function for a policy, it must satisfy the Bellman Equation
2. When using the optimal policy we call the equation the **Bellman Optimality Equation**.
3. Intuitively, the Bellman Optimality Equation expresses the fact that the value of a state under an optimal policy must equal the expected return for the best action from that state:



$$V^*(s) = \max_a E[r_{t+1} + \gamma V^*(s_{t+1}) | S_t = s, A_t = a]$$
$$= \max_a \sum_{s',r} p(s', r | s, a)[r + \gamma V^*(s')]$$



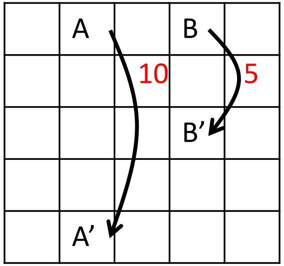
$$Q^*(s,a) = E[r_{t+1} + \gamma \max_{a'} Q^*(s_{t+1}, a') | S_t = s, A_t = a]$$
$$= \sum_{s',r} p(s', r | s, a)[r + \gamma \max_{a'} Q^*(s', a')]$$

Bellman Optimality Equation

$$V^*(s) = \max_a E[r_{t+1} + \gamma V^*(s_{t+1}) | S_t = s, A_t = a]$$

$$= \max_a \sum_{s',r} p(s', r | s, a)[r + \gamma V^*(s')]$$

$$V^*(0,0) = \max \begin{array}{c} w \\ n \\ e \\ s \end{array} \begin{bmatrix} -1 + 0.9 \cdot 21.98 \\ -1 + 0.9 \cdot 21.98 \\ 0 + 0.9 \cdot 24.42 \\ 0 + 0.9 \cdot 19.78 \end{bmatrix} = 21.98$$



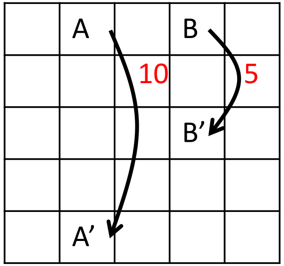
21.98	24.42	21.98	19.42	17.48
19.78	21.98	19.78	17.80	16.02
17.80	19.78	17.80	16.02	14.42
16.02	17.80	16.02	14.42	12.98
14.42	16.02	14.42	12.98	11.68

Bellman Optimality Equation

1. Why optimal state value functions are useful?
   - Any policy which is greedy with respect to V^*, is an optimal policy.
     - Why?
2. Therefore, given V^*, one-step-ahead search produces the long-term optimal actions. (This is a deterministic policy)

$$\pi^*(s) = \underset{a \in A}{argmax} \left[\sum_{s',r} p(s', r | s, a)[r + \gamma V^*(s')] \right]$$

3. E.g. back to the grid world:



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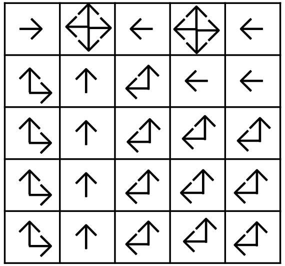
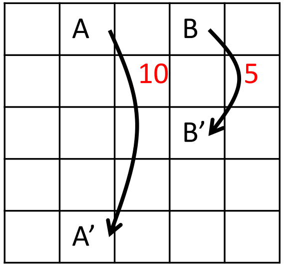


Bellman Optimality Equation

1. Given $Q^*(s,a)$, the agent does not even need to do a one-step-ahead search:
$$\pi^*(s) = \underset{a \in A}{\operatorname{argmax}}\, Q^*(s,a)$$
2. If the function is known, agent does not need to know anything about the dynamics of the environment
3. What are the implementation tradeoffs?



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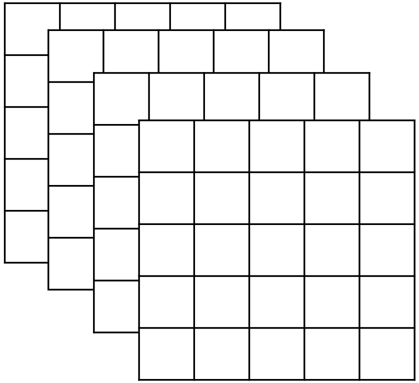
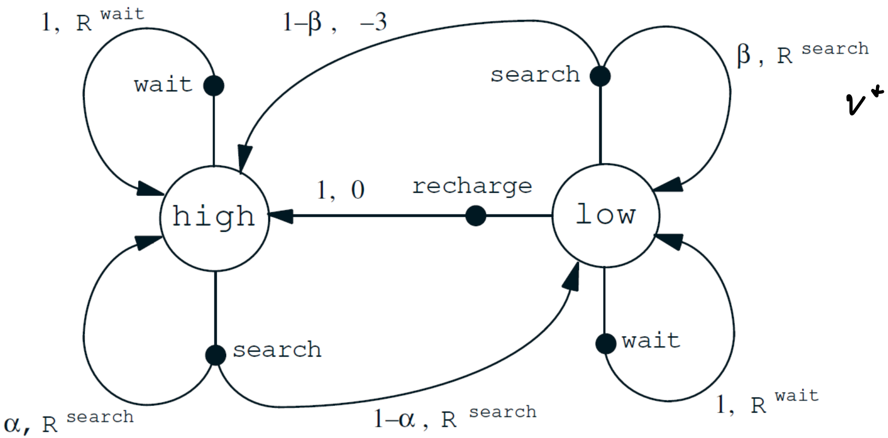


Bellman Optimality Equations Example

$$V^*(s) = \max_a E[r_{t+1} + \gamma V^*(s_{t+1}) | S_t = s, A_t = a]$$
$$= \max_a \sum_{s',r} p(s', r | s, a)[r + \gamma V^*(s')]$$

h: high
l: low
s: search
w: wait
re: recharge



$$V^*(h) = \max_a \begin{cases} \alpha(R^{search} + \gamma V^*(h)) + (1-\alpha)(R^{search} + \gamma V^*(l)) \\ R^{wait} + \gamma V^*(h) \end{cases}$$

Bellman Optimality Equations Example

$$Q^*(s,a) = E[r_{t+1} + \gamma \max_{a'} Q^*(s_{t+1},a') | S_t = s, A_t = a]$$

$$= \sum_{s',r} p(s',r|s,a)\left[r + \gamma \max_{a'} Q^*(s',a')\right]$$

h: high
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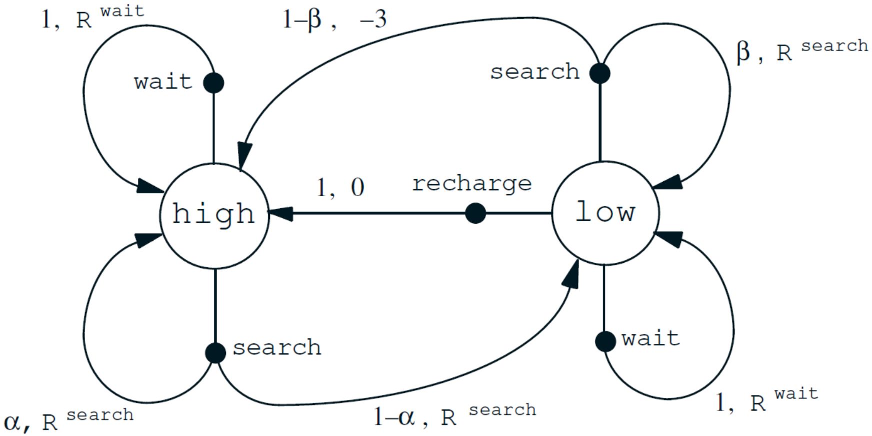

Solving the Bellman Optimality Equation

1. Finding an optimal policy by solving the Bellman Optimality Equation (for example with dynamic programming) requires the following:
    - Accurate knowledge of the environment dynamics (environment model)
    - Enough space and time to do the computation
    - The Markov property
2. How much space and time do we need:
    - Polynomial in the number of states (will discuss in next chapter)
    - However, number of states can be extremely large (e.g. backgammon has about 10^{20} states)
3. We usually have to settle for approximations
4. Many RL methods can be understood as approximately solving the Bellman Optimality Equation.
    - Advantages of approximation?
        - learn more effectively
        - feature extraction can reduce noise
        - can address large scale problems

Solving the Bellman Optimality Equation

1. During the next few weeks we'll talk about techniques for solving the RL problem:
   - Dynamic Programming – well developed mathematically, solves the Bellman equations, but requires an accurate model of the environment.
   - Monte Carlo Methods – Do not require an exact model, but work only on episodic tasks since returns only calculated at the end.
   - Temporal Difference Learning – Do not require an exact model and can work on non episodic tasks.
     - More complex to analyze
     - Launched the revisiting of RL as a pragmatic framework (1988)
2. All methods differ in efficiency and speed of convergence to the optimal solution.

Summary

1. Agent environment interaction
   - states
   - actions
   - rewards
2. Policy:
   - $\pi(a|s)$
   - probability of selecting action in state
3. Return: (weighted) sum of future rewards
   - episodic vs. continuing tasks
4. Markov property
5. Markov Decision Process:
   - Transition probability
   - Expected reward
6. Value Functions
   - State value function for a policy
   - Action value function for a policy
7. Bellman Equations
8. Optimal value functions
9. Optimal Policies
10. The need for approximation