QF620 Project Part I

Black-Scholes Model

$$dS_t = rS_t dt + \sigma S_t dW_t$$

$$S_t = S_0 \cdot e^{(r-\frac{\sigma^2}{2})t + \sigma W_t}$$

$$S_T > K \Rightarrow (r - \tfrac{1}{2}\sigma^2)T + \sigma W_T > \ln\frac{K}{S_0}$$

$$\sigma\sqrt{T}\,x > \ln\frac{K}{S_0} - (r-\tfrac{1}{2}\sigma^2)T, \quad x \sim N(0,1)$$

$$x > \frac{\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -\frac{\ln\frac{S_0}{K} + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -d_2 = x^*$$

$$d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln\frac{S_0}{K} + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$C_{van} = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

$$P_{van} = K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$$

$$C_{DCN} = \text{cash} \times e^{-rT} \Phi(d_2)$$

$$P_{DCN} = \text{cash} \times e^{-rT} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} \mathbb{1} \cdot e^{-\frac{x^2}{2}} dx$$

$$= \text{cash} \times e^{-rT} \Phi(-d_2)$$

$$C_{DAN} = S_0 \cdot \Phi(d_1)$$

$$P_{DAN} = e^{-rT} \cdot E[S_T \mathbb{1}_{S_T < K}]$$

$$= S_0 \cdot \int_{-\infty}^{y^*} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy, \quad y^* = x^* - \sigma\sqrt{T}$$

$$= S_0 \cdot \Phi(y^*) = S_0 \cdot \Phi(x^* - \sigma\sqrt{T})$$

$$= S_0 \cdot \Phi(-d_2 - \sigma\sqrt{T}) = S_0 \Phi(-d_1)$$

$$\text{Black 76}(F, K, r, \sigma, T) = \text{Black-Scholes}(S, K, r, \sigma, T)$$

$$\text{Displaced-Diffusion}(F, K, r, \sigma, T, \beta) = \text{Black-Scholes}\left(\frac{F}{\beta}, K + \frac{1-\beta}{\beta}F, r, \sigma\beta, T\right)$$

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Bachelier Model

$$dS_t = \sigma dW_t$$

$$S_T = S_0 + \sigma W_T = S_0 + \sigma \cdot \sqrt{T} \cdot x \qquad x \sim N(0,1)$$

$$S_T > K \Rightarrow x > \frac{K - S_0}{\sigma\sqrt{T}} = x^*$$

$$C_{van} = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 + \sigma\sqrt{T}x - K) e^{-\frac{x^2}{2}} dx$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \cdot \int_{x^*}^{\infty} (S_0 - K) e^{-\frac{x^2}{2}} dx + \frac{e^{-rT}}{\sqrt{2\pi}} \cdot \int_{x^*}^{\infty} \cdot \sigma\sqrt{T}\, x \cdot e^{-\frac{x^2}{2}} dx$$

let $u = \frac{x^2}{2} \Rightarrow \frac{du}{dx} = \frac{2x}{2} = x$

$\Rightarrow du = x \cdot dx$

$$= e^{-rT}(S_0 - K)[\Phi(\infty) - \Phi(x^*)] + \frac{e^{-rT}}{\sqrt{2\pi}} \cdot \sigma\sqrt{T} \cdot e^{-\frac{(x^*)^2}{2}}$$

$$C_{van} = e^{-rT}[(S_0 - K)\Phi(-x^*) + \sigma\sqrt{T} \cdot \phi(-x^*)]$$

$$\int_{x^*}^{\infty} x \cdot e^{-\frac{x^2}{2}} dx = \int_{u^*}^{\infty} e^{-u} du = -[e^{-u}]_{x^*}^{\infty}$$

$$= -\left[0 - e^{-\frac{(x^*)^2}{2}}\right] = e^{-\frac{(x^*)^2}{2}}$$

$$P_{van} = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{x^*} (K - S_0 - \sigma\sqrt{T}x) e^{-\frac{x^2}{2}} dx$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{x^*} (K - S_0) \cdot e^{-\frac{x^2}{2}} dx - \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{x^*} \cdot \sigma\sqrt{T} \cdot x e^{-\frac{x^2}{2}} dx$$

$$= e^{-rT} \cdot (K - S_0) \cdot \Phi(x^*) + \frac{e^{-rT}}{\sqrt{2\pi}} \cdot \sigma\sqrt{T} \cdot [e^{-u}]_{-\infty}^{x^*} \qquad u = \frac{x^2}{2}$$

$$= e^{-rT} \cdot (K - S_0) \cdot \Phi(x^*) + e^{-rT} \cdot \sigma\sqrt{T} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x^*)^2}{2}}$$

$$= e^{-rT}[(K - S_0)\Phi(x^*) + \sigma\sqrt{T} \cdot \phi(x^*)]$$

$$C_{DCN} = \left(e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx\right) \times cash = cash \cdot e^{-rT}(\Phi(\infty) - \Phi(x^*)) = cash \cdot e^{-rT}\Phi(-x^*)$$

$$P_{DCN} = cash \cdot e^{-rT} \cdot \Phi(x^*)$$

$$C_{DAN} = e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 + \sigma\sqrt{T}x) e^{-\frac{x^2}{2}} dx$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \cdot \left(\int_{x^*}^{\infty} \cdot S_0 \cdot e^{-\frac{x^2}{2}} dx + \sigma\sqrt{T} \cdot \int_{x^*}^{\infty} \cdot x \cdot e^{-\frac{x^2}{2}} dx\right)$$

$$= e^{-rT}(S_0 \Phi(-x^*) + \sigma\sqrt{T}\phi(-x^*))$$

$$P_{DAN} = e^{-rT} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x^*} (S_0 + \sigma\sqrt{T}x) e^{-\frac{x^2}{2}} dx = e^{-rT}(S_0 \cdot \Phi(x^*) - \sigma\sqrt{T} \cdot \phi(x^*))$$