# Iterative Schemes and Their Convergence For a Symmetric Positive Definite Matrix

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Consider $n \times n$ linear system of equations $Ax = b$. Many iteration schemes are examples of *splitting* of $A$:

$$A = M - N. \tag{1}$$

Assume throughout this note that $M$ is nonsingular. Typically the splitting is chosen in such a way that computing $M^{-1}c$ costs little. An associated iterative scheme is

$$Mx^{(j+1)} = Nx^{(j)} + b, \quad \text{for } j = 0, 1, 2, \ldots. \tag{2}$$

or,

$$x^{(j+1)} = Gx^{(j)} + c, \quad G = M^{-1}N,\ c = M^{-1}b.$$

**Example** (*Jacobi and Gauss-Seidel methods*). Let $A = D - L - U$, where $D$ is $A$'s diagonal part, $L$ is $A$'s strictly lower triangular part, and $U$ is $A$'s strictly upper triangular part. Then

$$\begin{aligned} \text{Jacobi Method:} &\quad M_{\rm J} = D,\ N_{\rm J} = L + U, \\ \text{Gauss-Seidel Method:} &\quad M_{\rm GS} = D - L,\ N_{\rm GS} = U. \end{aligned}$$

We shall write

$$G_{\rm J} = D^{-1}(L + U), \quad G_{\rm GS} = (D - L)^{-1}U.$$

**Theorem** (Householder-John). *Suppose that both $A$ and $M + M^* - A$ are symmetric positive definite. Then the method* (2) *converges.*

PROOF: We have $G = M^{-1}N = M^{-1}(M - A) = I - M^{-1}A$. It suffices to show that $|\lambda| < 1$ for any eigenvalue of $G$. Let $x$ be the corresponding eigenvector. (Note that both $\lambda$ and $x$ could be complex.) We have

$$Gx = \lambda x \quad \Rightarrow \quad (I - M^{-1}A)x = \lambda x \quad \Rightarrow \quad (M - A)x = \lambda Mx \quad \Rightarrow \quad (1 - \lambda)Mx = Ax.$$

First we notice that

$$\lambda \neq 1$$

since otherwise we would have $Ax = 0$, contradicting the fact that $A$ is symmetric positive definite. Pre-multiplying $(1-\lambda)Mx = Ax$ by $x^*$ to get

$$(1-\lambda)x^*Mx = x^*Ax \quad \Rightarrow \quad x^*Mx = \frac{1}{1-\lambda}x^*Ax, \tag{3}$$

taking conjugate transpose of which leads to

$$(1-\bar{\lambda})x^*M^*x = x^*A^*x = x^*Ax \quad \Rightarrow \quad x^*M^*x = \frac{1}{1-\bar{\lambda}}x^*Ax. \tag{4}$$

Adding (3) and (4) and subtracting $x^*Ax$ from the two sides yields

$$x^*(M+M^*-A)x = \left(\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} - 1\right)x^*Ax = \frac{1-|\lambda|^2}{|1-\lambda|^2}x^*Ax.$$

Since $x^*Ax > 0$ (why?) and $x^*(M+M^*-A)x > 0$ (why?), we have

$$1-|\lambda|^2 > 0 \quad \Rightarrow \quad |\lambda| < 1,$$

as was to be shown. $\square$

**Example** (*Symmetric Positive Definite Tridiagonal Matrix*). Let symmetric tridiagonal matrix

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

be positive definite. We claim that both Jacobi and Gauss-Seidel methods converge when applied to it. For Jacobi method, we have

$$M_{\mathrm{J}} = \begin{pmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_{n-1} & \\ & & & & \alpha_n \end{pmatrix}, \quad N_{\mathrm{J}} = -\begin{pmatrix} 0 & \beta_1 & & & \\ \beta_1 & 0 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-2} & 0 & \beta_{n-1} \\ & & & \beta_{n-1} & 0 \end{pmatrix},$$

and thus

$$M_{\mathrm{J}} + M_{\mathrm{J}}^* - A = \begin{pmatrix} \alpha_1 & -\beta_1 & & & \\ -\beta_1 & \alpha_2 & -\beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\beta_{n-2} & \alpha_{n-1} & -\beta_{n-1} \\ & & & -\beta_{n-1} & \alpha_n \end{pmatrix},$$

which is symmetric positive definite (why?). For Gauss-Seidel method, we have

$$M_{\text{GS}} = \begin{pmatrix} \alpha_1 & & & & \\ \beta_1 & \alpha_2 & & & \\ & \ddots & \ddots & & \\ & & \beta_{n-2} & \alpha_{n-1} & \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}, \quad N_{\text{GS}} = -\begin{pmatrix} 0 & \beta_1 & & & \\ & 0 & \beta_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \beta_{n-1} \\ & & & & 0 \end{pmatrix},$$

and thus

$$M_{\text{GS}} + M_{\text{GS}}^* - A = \begin{pmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_{n-1} & \\ & & & & \alpha_n \end{pmatrix},$$

which is symmetric positive definite (why?).

**Example** (*SOR*). The *Successive Over-Relaxation (SOR) method* is an iterative procedure that accelerate the Gauss-Seidel method by the help of a positive parameter:

$$x^{(j+1)} = G_{\text{SOR}(\omega)} x^{(j)} + c_{\text{SOR}(\omega)},$$

where

$$G_{\text{SOR}(\omega)} = (D - \omega L)^{-1}\big[(1-\omega)D + \omega U\big].$$

It can be seen that SOR is the iterative scheme associated with the following splitting of $A$:

$$M_{\text{SOR}(\omega)} = (1/\omega)D - L, \quad N_{\text{SOR}(\omega)} = (1/\omega - 1)D + U.$$

**Theorem** (*Ostrowski-Reich*). *If $A$ is symmetric positive definite, SOR method converges if and only if $0 < \omega < 2$.*

Proof: Kahan's theorem implies that $0 < \omega < 2$ is necessary for SOR to converge. On the other hand, notice that

$$M_{\text{SOR}(\omega)} + M_{\text{SOR}(\omega)}^* - A = (2/\omega - 1)D \qquad (\text{since } L = U^*)$$

is symmetric positive definite if $0 < \omega < 2$; Householder-John theorem implies that SOR method converges. $\square$

**Example** (*SSOR*). A *symmetric version of SOR (SSOR)* is defined as follows

$$\begin{aligned} x^{(j+1/2)} &= (D-\omega L)^{-1}\big[(1-\omega)D + \omega U\big]x^{(j)} + \big[(1/\omega)D - L\big]^{-1}b, \\ x^{(j+1)} &= (D-\omega U)^{-1}\big[(1-\omega)D + \omega L\big]x^{(j+1/2)} + \big[(1/\omega)D - U\big]^{-1}b, \end{aligned}$$

which yields

$$x^{(j+1)} = G_{\text{SSOR}} x^{(j)} + c_{\text{SSOR}},$$

where

$$G_{\text{SSOR}} = (D - \omega U)^{-1}\big[(1-\omega)D + \omega L\big](D - \omega L)^{-1}\big[(1-\omega)D + \omega U\big].$$

It can be proved that SSOR is the iterative scheme associated with the following splitting of $A$:

$$M_{\text{SSOR}} = \frac{1}{\omega(2-\omega)}(D-\omega L)D^{-1}(D-\omega U), \quad N_{\text{SSOR}} = \frac{1}{\omega(2-\omega)}\big[(1-\omega)D+\omega L\big]D^{-1}\big[(1-\omega)D+\omega U\big].$$

(Verify that $A = M_{\text{SSOR}} - N_{\text{SSOR}}$ and $G_{\text{SSOR}} = M_{\text{SSOR}}^{-1}N_{\text{SSOR}}$!)

**Theorem**. *If $A$ is symmetric positive definite, SSOR method converges if and only if $0 < \omega < 2$.*

Proof: It can be shown that $\rho(G_{\text{SSOR}}) \geq (\omega - 1)^2$, which implies that $0 < \omega < 2$ is necessary for SSOR to converge. On the other hand, notice that

$$\begin{aligned} M_{\text{SSOR}(\omega)} + M^*_{\text{SSOR}(\omega)} - A &= \frac{2}{\omega(2-\omega)}(D - \omega L)D^{-1}(D - \omega L^*) - A \\ &= \frac{2-\omega}{2\omega}D + \frac{2\omega}{2-\omega}\left(\frac{1}{2}D^{1/2} - LD^{-1/2}\right)\left(\frac{1}{2}D^{1/2} - LD^{-1/2}\right)^*, \end{aligned}$$

is symmetric positive definite if $0 < \omega < 2$; Householder-John theorem implies that SSOR method converges. $\square$