Preconditioned conjugate gradient algorithm

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In order to accelerate the iterative method, we usually introduce the so-called preconditioner B to the linear system $Ax = b$. The idea is that matrix A itself maybe ill-conditioned (condition number of A is quite large), however, hopefully, by choose some good preconditioner B, the condition number of BA will be much better, i.e. $\text{cond}(BA) << \text{cond}(A)$. Then we can solve $BAx = Bb$ by iterative method instead of $Ax = b$. Preconditioned Conjugate Gradient (PCG) method is one of this kind methods. Roughly speaking, it is CG method applied to $BAx = Bb$. Alogorithm 1 is the detailed PCG algorithm. Implement the PCG method, and apply it to solve the above linear system with $n = 2^l$, $l = 5, 6, 7, 8$ (stopping criterion is $||b - Ax^k||/||b|| < 10^{-6}$). Choose the preconditioner $B = I$, $B = D^{-1}$ and $B = (D + U)^{-1}D(D + L)^{-1}$ (decompose $A = D + L + U$). Make a table to report the number of iterations for these three different preconditioners.

Algorithm 1 Preconditioned Conjugate Gradient Method

Given initial guess x^0
Set residual $r^0 = b - Ax^0$
$z^0 = Br^0$
$v^0 = z^0$
$k \leftarrow 0$
while $\frac{||b-Ax^k||}{||b||} > 10^{-6}$ **do**
    $\alpha_k = \frac{(r^k, z^k)}{(Av^k, v^k)}$
    $x^{k+1} = x^k + \alpha_k v^k$
    $r^{k+1} = r^k - \alpha_k A v^k$
    $z^{k+1} = Br^{k+1}$
    $\beta_{k+1} = \frac{(r^{k+1}, z^{k+1})}{(r^k, z^k)}$
    $v^{k+1} = z^{k+1} + \beta_{k+1} v^k$
    $k = k + 1$
end while