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Successive overrelaxation (SOR) and related methods ☆

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Abstract

Covering the last half of the 20th century, we present some of the basic and well-known results for the SOR theory and related methods as well as some that are not as well known. Most of the earlier results can be found in the excellent books by Varga (Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962) Young (Iterative Solution of Large Linear systems, Academic Press, New York, 1971) and Berman and Plemmons (Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, PA, 1994) while some of the most recent ones are given in the bibliography of this paper. In this survey, both the point and the block SOR methods are considered for the solution of a linear system of the form $Ax = b$, where $A \in \mathbb{C}^{n,n}$ and $b \in \mathbb{C}^n \setminus \{0\}$. Some general results concerning the SOR and related methods are given and also some more specific ones in cases where A happens to possess some further property, e.g., positive definiteness, L-, M-, H-matrix property, p-cyclic consistently ordered property etc. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and Preliminaries

For the numerical solution of a large nonsingular linear system

$$Ax = b, \quad A \in \mathbb{C}^{n,n}, \quad b \in \mathbb{C}^n \setminus \{0\}, \tag{1.1}$$

we consider iterative methods based on a splitting of the matrix A (see, e.g. [83,93] or [3]). Namely, we write

$$A = M - N \tag{1.2}$$

☆ This work is dedicated to Professors David M. Young and Richard S. Varga on their 75th and 70th birthday, respectively.

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where M, the preconditioner, or preconditioning matrix, is taken to be invertible and cheap to invert, meaning that a linear system with matrix coefficient M is much more economical to solve than (1.1). Based on (1.2), (1.1) can be written in the fixed-point form

$$x = Tx + c, \quad T := M^{-1}N, \quad c := M^{-1}b, \tag{1.3}$$

which yields the following iterative scheme for the solution of (1.1):

$$x^{(m+1)} = Tx^{(m)} + c, \quad m = 0, 1, 2, \ldots, \quad \text{and} \quad x^{(0)} \in \mathbb{C}^n \text{ arbitrary}. \tag{1.4}$$

A sufficient and necessary condition for (1.4) to converge, to the solution of (1.1), is $\rho(T) < 1$, where $\rho(.)$ denotes spectral radius, while a sufficient condition for convergence is $\|T\| < 1$, where $\|.\|$ denotes matrix norm induced by a vector norm (see, e.g. [83,93,3]).

To derive the classical iterative methods one writes $A = D - L - U$, with $D = \operatorname{diag}(A)$, assuming $\det(D) \neq 0$, and L strictly lower and U strictly upper triangular matrices, respectively. Thus, the Jacobi iterative method ($M \equiv D$) is defined by

$$x^{(m+1)} = D^{-1}(L + U)x^{(m)} + D^{-1}b, \tag{1.5}$$

the Gauss–Seidel iterative method ($M \equiv D - L$) by

$$x^{(m+1)} = (D - L)^{-1}Ux^{(m)} + (D - L)^{-1}b \tag{1.6}$$

and the Successive Overrelaxation (SOR) iterative method ($M \equiv (1/\omega)(D - \omega L)$) by

$$x^{(m+1)} = \mathscr{L}_\omega x^{(m)} + c_\omega, \quad \mathscr{L}_\omega := (D - \omega L)^{-1}[(1 - \omega)D + \omega U], \quad c_\omega := \omega(D - \omega L)^{-1}b. \tag{1.7}$$

In (1.7), $\omega \in \mathbb{C} \setminus \{0\}$ is the relaxation factor (or overrelaxation parameter). For $\omega = 1$ the SOR becomes the Gauss–Seidel method.

The above three methods are called point methods to distinguish them from the block methods. For the latter, consider a partitioning of A in the following block form:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \tag{1.8}$$

where $A_{ii} \in \mathbb{C}^{n_i, n_i}$, $i = 1(1)p$, and $\sum_{i=1}^{p} n_i = n$. If we define $D = \operatorname{diag}(A_{11}, A_{22}, \ldots, A_{pp})$, assuming $\det(A_{ii}) \neq 0$, $i = 1(1)p$, set $A = D - L - U$ with L and U being strictly lower and strictly upper triangular matrices, respectively, then the block Jacobi, the block Gauss–Seidel and the block SOR methods associated with the partitioning (1.8) of A are the iterative methods defined by precisely the same iterative schemes as their point counterparts in (1.5)–(1.7), respectively.

2. Successive overrelaxation (SOR) method

The SOR method seems to have appeared in the 1930s and is mentioned in [79]. However, formally its theory was established almost simultaneously by Frankel [16] and Young [90].

In the development of the SOR theory one seeks values of $\omega \in \mathbb{C} \setminus \{0\}$ for which the SOR method converges, the set of which defines the region of convergence, and, if possible, the best value of

ω, ω_b, for which the convergence is asymptotically optimal, namely $\rho(\mathscr{L}_{\omega_b}) = \min_{\omega\in\mathbb{C}\setminus\{0\}}\rho(\mathscr{L}_\omega)$. To find regions of convergence is a problem generally much easier than to determine ω_b. In either case, however, one assumes that some information regarding the spectrum of the associated Jacobi iteration matrix J, $\sigma(J)$, is available. This information comes mostly from the properties of the matrix A (and the partitioning considered).

The only property of the SOR method that does not depend on properties of A, except for those needed to define the method, is the one below due to Kahan [46].

Theorem 2.1 (Kahan). *A necessary condition for the SOR method to converge is* $|\omega-1|<1$. (*For* $\omega\in\mathbb{R}$ *this condition becomes* $\omega\in(0,2)$.)

Note: From now on it will be assumed that $\omega\in\mathbb{R}$ unless otherwise specified.

2.1. Hermitian matrices

Definition 2.2. A matrix $A\in\mathbb{C}^{n,n}$ is said to be Hermitian if and only if (*iff*) $A^{\mathrm{H}}=A$, where the superscript H denotes complex conjugate transpose. (A real Hermitian matrix is a real symmetric matrix and there holds $A^{\mathrm{T}}=A$, where T denotes transpose.)

Definition 2.3. An Hermitian matrix $A\in\mathbb{C}^{n,n}$ is said to be positive definite *iff* $x^{\mathrm{H}}Ax>0$, $\forall x\in\mathbb{C}^n\setminus\{0\}$. (For A real symmetric, the condition becomes $x^{\mathrm{T}}Ax>0$, $\forall x\in\mathbb{R}^n\setminus\{0\}$.)

A well-known result due to Ostrowski, who extended a previous one for the Gauss–Seidel method due to Reich, is given in [83]. Varga [84] gave a different proof and found the best value of ω, ω_b.

Theorem 2.4 (Reich–Ostrowski–Varga). *Let* $A=D-E-E^{\mathrm{H}}\in\mathbb{C}^{n,n}$ *be Hermitian*, D *be Hermitian and positive definite*, *and* $\det(D-\omega E)\neq 0$, $\forall\omega\in(0,2)$. *Then*, $\rho(\mathscr{L}_\omega)<1$ *iff* A *is positive definite and* $\omega\in(0,2)$. (*Note*: *Notice that except for the restrictions in the statement the matrices* $D,E\in\mathbb{C}^{n,n}$ *must satisfy*, *they can be* any *matrices*!)

Note: It is worth mentioning that there is a form of the theorem due to Kuznetsov [53] that applies also in singular cases.

2.2. L-, M- and H-matrices

Notation. Let $A,B\in\mathbb{R}^{n,n}$. If $a_{ij}\geqslant b_{ij}$ ($a_{ij}>b_{ij}$), $i,j=1(1)n$, we write $A\geqslant B$ ($A>B$). The same notation applies to vectors $x,y\in\mathbb{R}^n$.

Definition 2.5. If $A\in\mathbb{R}^{n,n}$ satisfies $A\geqslant 0$ (>0) then it is said to be *nonnegative* (*positive*). The same terminology applies to vectors $x\in\mathbb{R}^n$.

Notation. Let $A\in\mathbb{C}^{n,n}$. Then $|A|$ denotes the matrix whose elements are the moduli of the elements of A. The same notation applies to vectors $x\in\mathbb{C}^n$.

From the Perron–Frobenius theory for nonnegative matrices (see [83,93] or [3]) the following statement holds.

Theorem 2.6. *Let* $A \in \mathbb{C}^{n,n}$ *and* $B \in \mathbb{R}^{n,n}$ *satisfy* $0 \leqslant |A| \leqslant B$, *then* $0 \leqslant \rho(A) \leqslant \rho(|A|) \leqslant \rho(B)$.

Definition 2.7. A matrix $A \in \mathbb{R}^{n,n}$ is said to be an L-matrix *iff* $a_{ii} > 0$, $i = 1(1)n$, and $a_{ij} \leqslant 0$, $i \neq j = 1(1)n$.

Definition 2.8. A matrix $A \in \mathbb{R}^{n,n}$ is said to be an M-matrix *iff* $a_{ij} \leqslant 0$, $i \neq j = 1(1)n$, A is nonsingular and $A^{-1} \geqslant 0$.

Remark. It is pointed out that in [3] 50 equivalent conditions for a matrix $A \in \mathbb{R}^{n,n}$, with $a_{ij} \leqslant 0$, $i \neq j = 1(1)n$, to be an M-matrix are given!

Definition 2.9. A matrix $A \in \mathbb{C}^{n,n}$ is said to be an H-matrix *iff* its *companion matrix*, that is the matrix $\mathscr{M}(A)$ with elements $m_{ii} = |a_{ii}|$, $i = 1(1)n$, and $m_{ij} = -|a_{ij}|$, $i \neq j = 1(1)n$, is an M-matrix.

Definition 2.10. A splitting (1.2) of a nonsingular matrix $A \in \mathbb{R}^{n,n}$ is said to be *regular* if $M^{-1} \geqslant 0$ and $N \geqslant 0$. (Varga proved among others that the iterative scheme (1.4) based on a regular splitting is convergent; he also made comparisons of the spectral radii corresponding to two different regular splittings of the same matrix A (see [83]).)

Definition 2.11. A splitting (1.2) of a nonsingular matrix $A \in \mathbb{R}^{n,n}$ is said to be *weak regular* if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$. (As Neumann and Plemmons proved, see, e.g. [3], this definition leads to some results very similar to those of the regular splittings.)

A theorem connecting spectral radii of the Jacobi and the Gauss–Seidel iteration matrices associated with an L-matrix A was given originally by Stein and Rosenberg. In Young [93] a form of it that includes the spectral radius of the SOR iteration matrix is given below. Its proof is mainly based on the Perron–Frobenius theory.

Theorem 2.12. *If* $A \in \mathbb{R}^{n,n}$ *is an L-matrix and* $\omega \in (0, 1]$, *then*:
(a) $\rho(J) < 1$ *iff* $\rho(\mathscr{L}_\omega) < 1$.
(b) $\rho(J) < 1$ *iff* A *is an* M*-matrix*; *if* $\rho(J) < 1$ *then* $\rho(\mathscr{L}_\omega) \leqslant 1 - \omega + \omega\rho(J)$.
(c) *If* $\rho(J) \geqslant 1$ *then* $\rho(\mathscr{L}_\omega) \geqslant 1 - \omega + \omega\rho(J) \geqslant 1$.

Notes: (i) The original form of Stein–Rosenberg theorem restricts to $\omega = 1$ and gives four mutually exclusive relations:

$$\begin{aligned} &\text{(a) } 0 = \rho(J) = \rho(\mathscr{L}_1), \quad \text{(b) } 0 < \rho(\mathscr{L}_1) < \rho(J) < 1, \\ &\text{(c) } 1 = \rho(J) = \rho(\mathscr{L}_1), \quad \text{(d) } 1 < \rho(J) < \rho(\mathscr{L}_1). \end{aligned} \tag{2.1}$$

(ii) Buoni and Varga [5,6] and also Buoni et al. [4] generalized the original Stein–Rosenberg theorem in another direction than that of Theorem 2.12 by assuming that $A \in \mathbb{C}^{n,n}$, D, L and U are any matrices with $D^{-1}L$ and $D^{-1}U$ strictly lower and strictly upper triangular matrices, respectively, and $\mathbb{R}^{n,n} \ni J = D^{-1}(L + U) \geqslant 0$.

In [93] a theorem that gives an interval of ω for which the SOR method converges for M-matrices A is based on the previous statement and on the theory of regular splittings is stated.

Theorem 2.13. *If $A \in \mathbb{R}^{n,n}$ is an M-matrix and if $\omega \in (0, 2/(1+\rho(J)))$ then $\rho(\mathscr{L}_\omega) < 1$.*

The following is a statement extending the previous one to H-matrices.

Theorem 2.14. *If $A \in \mathbb{C}^{n,n}$ is an H-matrix and if $\omega \in (0, 2/(1+\rho(|J|)))$ then $\rho(\mathscr{L}_\omega) < 1$.*

2.3. 2- and p-cyclic consistently ordered matrices

There is a class of matrices for which the investigation for the optimal value of ω leads to the most beautiful theory that has been developed for the last 50 years and which is still going on. It is associated with the class of *p-cyclic consistently ordered* matrices. Such matrices naturally arise, e.g., for $p=2$ in the discretization of second-order elliptic or parabolic PDEs by finite differences, finite element or collocation methods, for $p=3$ in the case of large-scale least-squares problems, and for any $p \geqslant 2$ in the case of *Markov chain analysis.*

Definition 2.15. A matrix $A \in \mathbb{C}^{n,n}$ possesses Young's "*property A*" if there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} D_1 & B \\ C & D_2 \end{bmatrix}, \tag{2.2}$$

where D_1, D_2 are nonsingular diagonal matrices not necessarily of the same order.

A special case of Young's "property A" is what Varga calls *two-cyclic consistently ordered property* [83].

Definition 2.16. A matrix $A \in \mathbb{C}^{n,n}$ is said to be *two-cyclic consistently ordered* if $\sigma(D^{-1}(\alpha L + (1/a)U))$ is independent of $\alpha \in \mathbb{C} \setminus \{0\}$.

Among others, matrices that possess both Young's "property A" and Varga's "two-cyclic consistently ordered property" are the tridiagonal matrices, with nonzero diagonal elements, and the matrices that have already form (2.2).

For two-cyclic consistently ordered matrices A, Young discovered [90,91] that the eigenvalues μ and λ of the Jacobi and the SOR iteration matrices, respectively, associated with A satisfy the functional relationship

$$(\lambda + \omega - 1)^2 = \omega^2 \mu^2 \lambda. \tag{2.3}$$

He also found that if $J = D^{-1}(L+U)$, the eigenvalues of J^2 are nonnegative and $\rho(J) < 1$, then there exists an optimal value of ω, ω_b, such that

$$\omega_b = \frac{2}{1 + (1 - \rho^2(J))^{1/2}}, \quad \rho(\mathscr{L}_{\omega_b}) = \omega_b - 1 \ (< \rho(\mathscr{L}_\omega) \text{ for all } \omega \neq \omega_b). \tag{2.4}$$

(*Note:* For more details see [93].)

Varga generalized the concept of two-cyclic consistently ordered matrices to what he called (*block*) p-cyclic consistently ordered.

Definition 2.17. A matrix $A \in \mathbb{C}^{n,n}$ in the block form (1.8) is said to be *(block) p-cyclic consistently ordered* if $\sigma(D^{-1}(\alpha L + (1/\alpha^{p-1})U))$ is independent of $\alpha \in \mathbb{C} \setminus \{0\}$.

The best representative of such a block partitioned matrix will be the following:

$$A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & A_{1p} \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ 0 & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{p,p-1} & A_{pp} \end{bmatrix}. \tag{2.5}$$

Remark. The spectrum $\sigma(J)$, of the eigenvalues of the (block) Jacobi iteration matrix associated with a p-cyclic consistently ordered matrix A (2.5), which Varga calls *weakly cyclic of index p* [83], presents a p-cyclic symmetry about the origin. That is, with each eigenvalue $\mu \in \sigma(J) \setminus \{0\}$ there are another $p-1$ eigenvalues of J, of the same multiplicity as that of μ, given by the expressions $\mu \exp(\mathrm{i}(2\pi k)/p)$, $k = 1(1)p-1$.

Notation. From now on the Jacobi iteration matrix associated with a block p-cyclic consistently ordered matrix will be denoted by J_p.

For such matrices Varga [82] extended Young's results (2.3)–(2.4) to any $p \geqslant 3$, namely

$$(\lambda + \omega - 1)^p = \omega^p \mu^p \lambda^{p-1}. \tag{2.6}$$

He also proved that if the pth powers of the eigenvalues $\mu \in \sigma(J_p)$ are real nonnegative and $\rho(J_p) < 1$, then there exists an optimal value of ω, ω_b, which is the unique positive real root in $(1, p/(p-1))$ of the equation

$$\left(\rho(J_p)\omega_b\right)^p = \frac{p^p}{(p-1)^{p-1}}(\omega_b - 1), \tag{2.7}$$

which is such that

$$\rho(\mathscr{L}_{\omega_b}) = (p-1)(\omega_b - 1) \ (< \rho(\mathscr{L}_\omega) \text{ for all } \omega \neq \omega_b). \tag{2.8}$$

Similar optimal results for $\sigma(J_p^p)$ nonpositive have been obtained for $p=2$ by Kredell [52] and Niethammer [66], for $p=3$ by Niethammer et al. [67] and for any $p \geqslant 3$ by Wild and Niethammer [88] and also by Galanis et al. [18].

In the analyses given in [83,93,52,66,67,88], the regions of convergence, in all the previous cases where optimal ω's were obtained, are also determined. In the following statement [3] the optimal values and the regions of convergence are given.

Theorem 2.18. *Let the matrix $A \in \mathbb{C}^{n,n}$ be p-cyclic consistently ordered and suppose that all the eigenvalues of J_p^p are nonnegative* (*nonpositive*). *Let $s = 1$ (-1) if the signs of the eigenvalues of*

J_p^p *are nonnegative* (*nonpositive*). *If*

$$\rho(J_p) < \frac{p-1-s}{p-2}, \tag{2.9}$$

then the regions of convergence of the SOR method ($\rho(\mathscr{L}_\omega) < 1$) are

$$\text{For } s=1, \quad \omega \in \left(0, \frac{p}{p-1}\right) \text{ and for } s=-1,\ \omega \in \left(\frac{p-2}{p-1}, \frac{2}{1+\rho(J_p)}\right). \tag{2.10}$$

The optimal relaxation factor ω_b is the unique real positive root $\omega_b \in ((2p-3+s)/(2(p-1)), (2p-1+s)/(2(p-1)))$ of the equation

$$(\rho(J_p)\omega_b)^p = s p^p (p-1)^{1-p}(\omega_b - 1) \tag{2.11}$$

and the optimal SOR spectral radius is given by

$$\rho(\mathscr{L}_{\omega_b}) = s(p-1)(\omega_b - 1)\ (< \rho(\mathscr{L}_\omega) \text{ for all } \omega \neq \omega_b). \tag{2.12}$$

Note: For $p=2$, $(p-2)/(p-2)$ and $p/(p-2)$ should be interpreted as 1 and ∞, respectively.

In passing we mention that the only case in which a complex optimal ω_b has been determined [52] is the case of a two-cyclic consistently ordered matrix with $\sigma(J_2)$ on a straight line segment, namely $\sigma(J_2) \subset [-\rho(J_2)\exp(\mathrm{i}\theta), \rho(J_2)\exp(\mathrm{i}\theta)]$, with any $\rho(J_2)$ and any $\theta \in (0, \pi)$. The corresponding optimal values are given by

$$\omega_b = \frac{2}{1+(1-\rho^2(J_2)\exp(2\mathrm{i}\theta))^{1/2}}, \quad \rho(\mathscr{L}_{\omega_b}) = |\omega_b - 1|\ (< \rho(\mathscr{L}_\omega) \text{ for all } \omega \neq \omega_b), \tag{2.13}$$

where of the two square roots the one with the nonnegative real part is taken. It is noted that for $\theta = 0$ and $\rho(J_2) < 1$, and also for $\theta = \pi/2$ and any $\rho(J_2)$, the optimal formulas by Young [90,91,93], and by Kredell [52] and Niethammer [66], respectively, are recovered.

As Varga first noticed [82], the transformation (2.6) that maps μ to $\lambda^{1/p}$ is a conformal mapping transformation. The study of this transformation, to find regions of convergence for ω and its optimal value, ω_b, involves ellipses for $p=2$ and p-cyclic hypocycloids (cusped, shortened and stretched) for $p \geqslant 3$. The latter curves for $p=5$ are depicted in Fig. 1. (In [88] not only ω_b and $\rho(\mathscr{L}_{\omega_b})$ are determined but also an excellent analysis with hypocycloids is done which allows the authors to obtain regions of convergence for ω.)

So, for matrices $A \in \mathbb{C}^{n,n}$ p-cyclic consistently ordered, because of $\omega \in \mathbb{R}$, the p-cyclic symmetry of the spectrum $\sigma(J_p)$ and of the p-cyclic hypocycloids about the origin and the symmetry of the latter with respect to (*wrt*) the real axis, the optimal problems that have been considered so far can be called *one-point problems*. This is justified from the fact that the coordinates of only one *critical* point suffice to determine the optimal parameters. E.g., for any $p \geqslant 2$ and $0 \leqslant \mu^p < 1$, the point $(\rho(J_p), 0)$ is the only information needed, for $p=2$ and $\mu^2 \leqslant 0$ we only need the point $(0, \mathrm{i}\rho(J_2))$ while for $p \geqslant 3$ and $-(p/(p-2))^p < \mu^p \leqslant 0$, only the point $(\rho(J_p)\cos(\pi/2p), \mathrm{i}\rho(J_p)\sin(\pi/2p))$ suffices.

One may notice that in the case of the *one-point problem* because we are dealing with complex matrices A, in general, one has to consider not only the spectrum $\sigma(J_p)$ but also its symmetric *wrt* the real axis. E.g., for $p=2$, if there is a rectangle symmetric *wrt* both axes that contains $\sigma(J_2)$ in the closure of its interior and which lies within the infinite unit strip $S := \{z \in \mathbb{C} \mid |Re\, z| < 1\}$, then

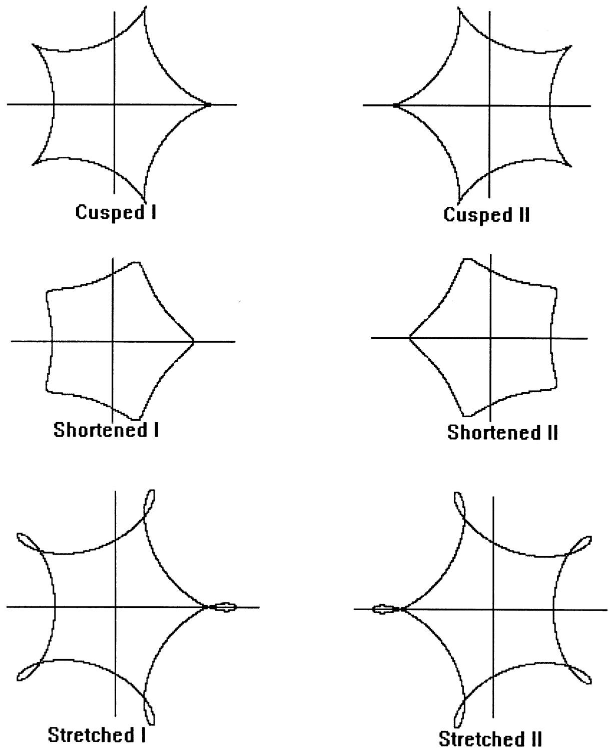

Fig. 1. Hypocycloids of all kinds and types for $p = 5$.

the only information needed to find ω_b and $\rho(\mathscr{L}_{\omega_b})$ is the pair of coordinates of its vertex in the first quadrant ($(2p) - ant$ with $p = 2$). This problem was solved by Kjellberg [50] and Russell [73] and the optimal values are given by the elements of the unique *best* ellipse that passes through the vertex in question (see also [93]).

The most general *one-point problem* has been solved recently in [19] where, among others, use of most of the previous results and also of those in [69] was made. In [19] it is assumed that $A \in \mathbb{C}^{n,n}$ is p-cyclic consistently ordered and there exists one element of $\sigma(J_p)$ or of its mirror image $\sigma'(J_p)$ *wrt* the real axis in the first $(2p) - ant$ with polar coordinates (r, θ) such that the cusped hypocycloid of type II that passes through (r, θ) crosses the real axis at a point with abscissa strictly less that 1 and on the other hand, the hypocycloid just mentioned and the cusped hypocycloid of type I through (r, θ) contain both $\sigma(J_p)$ and $\sigma'(J_p)$ in the closure of the intersection of their interiors. In such a case ω_b and $\rho(\mathscr{L}_{\omega_b})$ can be found through analytical expressions in terms of the semiaxes of the unique *best* shortened hypocycloid that passes through (r, θ). It is worth pointing out that all the previous cases mentioned so far are particular subcases of the one just described.

The case of the *two-point problem* in its simplest form is when $p = 2$ and its spectrum $\sigma(J_2)$ is real and such that $-a^2 \leqslant \mu^2 \leqslant b^2$ with $a, b > 0$ and $b < 1$. This problem was solved by Wrigley [89] (see also [93]) and the optimal parameters are given by

$$\omega_b = \frac{2}{1 + (1 - b^2 + a^2)^{1/2}}, \quad \rho(\mathscr{L}_{\omega_b}) = \left(\frac{b + a}{1 + (1 - b^2 + a^2)^{1/2}}\right)^2 \ (< \rho(\mathscr{L}_{\omega}) \text{ for all } \omega \neq \omega_b). \tag{2.14}$$

(*Note:* The solution just given solves a more general problem, namely the one where $\sigma(J_2)$ lies in the closed interior of the ellipse with semiaxes b and a. We note that the cases of nonnegative and nonpositive μ^2 presented previously are particular subcases of the present one.)

The solution to the *two-point problem* for any $p \geqslant 3$, provided that $-a^p \leqslant \mu^p \leqslant b^p$, with $a, b > 0$ and $a < p/(p-2)$, $b < 1$, was given by Eiermann et al. [11] by means of the unique p-cyclic shortened hypocycloid through both points $(b, 0)$ and $(a\cos(\pi/2p), \mathrm{i}a\sin(\pi/2p))$ *iff* $(p-2)/p < a/b < p/(p-2)$ which becomes a cusped I through $(b, 0)$ *iff* $a/b \leqslant (p-2)/p$ and a cusped II through $(a\cos(\pi/2p),\ \mathrm{i}a\sin(\pi/2p))$ *iff* $p/(p-2) \leqslant a/b$. More specifically:

Theorem 2.19. *Under the notation and the assumptions so far, for $(p-2)/p < a/b < p/(p-2)$, ω_b is given as the unique positive real root in $((p-2)/(p-1), p/(p-1))$ of the equation*

$$\left(\frac{b+a}{2}\omega_b\right)^p = \frac{b+a}{b-a}(\omega_b - 1), \tag{2.15}$$

which is such that

$$\rho(\mathscr{L}_{\omega_b}) = \frac{b+a}{b-a}(\omega_b - 1)\ (< \rho(\mathscr{L}_\omega) \textit{ for all } \omega \neq \omega_b). \tag{2.16}$$

(*Note*: When $a/b \leqslant (p-2)/p$ and $a/b \geqslant p/(p-2)$ the above equations and expressions reduce to the ones of the *one-point problem* for the nonnegative and nonpositive case, respectively.)

For $p = 2$ and for a *two-point problem* where the vertices of two rectangles, both lying in the open infinite unit strip S and are symmetric *wrt* both axes, in the first quadrant are given and the closure of the intersection of their interiors contains $\sigma(J_2)$, the solution was given by Young and Eidson [94] (see also [93]) by a simple algorithm that uses the two *best* ellipses through each of the vertices and also the ellipse through the two points. An ingenious extension of this algorithm [94] gives the solution to the corresponding *many-point problem*.

The analogs to the *two-* and *many-point problems* for any $p \geqslant 3$ has been solved very recently by Galanis et al. [20]. The solutions are given by means of algorithms analogous to the ones by Young and Eidson where instead of ellipses shortened hypocycloids are used.

2.3.1. *Generalized $(q, p-q)$-cyclic consistently ordered matrices*

The block form of these matrices is the following:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & A_{1,p-q+1} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 & 0 & A_{2,p-q+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A_{qp} \\ A_{q+1,1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{p,p-q} & 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \tag{2.17}$$

where the diagonal blocks satisfy the same restrictions as in (1.8) and p and q are relatively prime. Obviously, for $q = 1$ the *generalized* $(1, p-1)$-*cyclic consistently ordered* matrices reduce to the block p-cyclic consistently ordered ones of the previous section.

This time the functional relationship that connects the spectra of the block Jacobi iteration matrix $J_{q,p-q}$ and of the block SOR matrix associated with A in (2.17) is

$$(\lambda + \omega - 1)^p = \omega^p \mu^p \lambda^{p-q}. \tag{2.18}$$

(2.18) is attributed to Verner and Bernal [87]. However, it seems that Varga implies the corresponding class of matrices and in some cases the optimal parameters of the associated SOR method (see [83, pp. 108–109, Exs 1,2]). For the basic theory concerning matrices of the present class as well as of their point counterparts the reader is referred to Young [93].

Remarks. (i) In Young [93], a more general than (2.17) form of matrices, called *generalized* $(q, p-q)$*-consistently ordered* (GCO$(q, p-q)$), are analyzed and studied extensively. (ii) Varga brought to the attention of the present author [85] that it appears that GCO$(q, p-q)$ matrices are, from a graph-theoretic point of view, essentially reorderings of the (block) p-cyclic consistently ordered matrices. This *new* result seems to make the theory of GCO$(q, p-q)$ matrices redundant! (iii) Two more points: (a) Optimal SOR results to cover all possible cases for the two classes of matrices (p-cyclic consistently ordered and GCO$(q, p-q)$ ones) have not been found, and (b) As was shown in [36] there are cases where for certain values of $\omega \in (0,2)$, the SOR method applied to a GCO$(q, p-q)$ matrix A converges while when it is applied to the corresponding reordered p-cyclic consistently ordered matrix diverges. (iv) In view of the two points in (iii) in the following we shall keep on considering the GCO$(q, p-q)$ matrices mostly in the form (2.17).

For the optimal parameters little has been done because it seems that the corresponding problems are not only difficult to attack but also there are no obvious practical applications associated with them. The only optimal results known to us are those by Nichols and Fox [64] who found that for $\sigma(J^p_{q,p-q})$ nonnegative and $\rho(J_{q,p-q}) < 1$, it is $\omega_b = 1$ and $\rho(\mathscr{L}_{\omega_b}) = \rho^{p/q}(J_{q,p-q})$ and also the one by Galanis et al. [21] who treated the nonpositive case for $q = p-1$ and $p = 3$ and 4 and obtained analytical expressions for ω_b and $\rho(\mathscr{L}_{\omega_b})$.

2.3.2. Regions of convergence

Besides optimal results in the case of p-cyclic and GCO$(q, p-q)$ matrices researchers in the area are also interested in the regions of convergence of the SOR method in the $(\rho(J_p), \omega)$-plane especially in case the spectrum $\sigma(J_p^p)$ is nonnegative or nonpositive. The 2-cyclic consistently ordered cases are trivial but the cases of p-cyclic consistently ordered matrices for any $p \geqslant 3$ are not. For $p=3$, Niethammer et al. [67] determined the exact regions in the nonnegative and nonpositive cases. For any $p \geqslant 3$ the solution was given in [28] where use of the famous Schur–Cohn algorithm was made (see [45]). The only other case where the Schur–Cohn algorithm was successfully applied was in the case of nonnegative and nonpositive spectra $\sigma(J_p^p)$ for $p \geqslant 3$ in the case of the GCO$(p-1, 1)$ matrices (see [36]). By using *asteroidal* hypocycloids, regions of convergence for the SOR are found in [34] for GCO$(q, p-q)$ matrices when $\sigma(J_p^p)$ is nonnegative or nonpositive. Finally, as in the previous case, but dropping the assumption on nonnegativeness and nonpositiveness, using the Rouché's Theorem [80], as in [29], one can find that sufficient conditions for the SOR method to converge for all $p \geqslant 3$ are $\rho(J_{q,p-q}) < 1$ and $0 < \omega < 2/(1 + \rho(J_{q,p-q}))$, that is the same basic conditions as in Theorem 2.13.

2.3.3. Singular linear systems and p-cyclic SOR

For singular linear systems the associated Jacobi iteration matrix has in its spectrum the eigenvalue 1. The very first theoretical results in this case for the SOR method were given by Buoni, Neumann and Varga [4]. If, however, the matrix coefficient A happens to be p-cyclic consistently ordered and in the Jordan form of the Jacobi iteration matrix the eigenvalue 1 is associated with 1×1 blocks only then the theory regarding convergence and optimal results seems to be precisely that of the nonsingular case where simply the eigenvalue 1 is discarded (see, e.g., [26,51,38]). Since this case is of much practical importance in the Markov Chain Analysis the reader is specifically referred to [51] for details on this and also on the concept of what is called *Extended SOR*.

2.3.4. Block p-cyclic repartitioning

In a case arising in the solution of large linear systems for least-squares problems Markham, Neumann and Plemmons [60] observed that if a block 3-cyclic consistently ordered matrix as in (2.5), with $\sigma(J_3^3)$ nonpositive and $\rho(J_3) < 3$, was repartitioned and considered as a block 2-cyclic consistently ordered matrix as

$$A = \left[\begin{array}{cc|c} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & 0 \\ \hline 0 & A_{23} & A_{33} \end{array}\right], \tag{2.19}$$

then the SOR method associated with the latter had much better convergence properties than the SOR associated with the former and also it was convergent for any $\rho(J_3)$. This was mainly based on the observation that $\sigma(J_2^2) \setminus \{0\} \equiv \sigma(J_3^3) \setminus \{0\}$.

The previous work was the starting point for an investigation that followed. So, Pierce, Hadjidimos and Plemmons [72] proved that for block p-cyclic consistently ordered matrices when the spectrum $\sigma(J_p^p)$ was either nonnegative, with $\rho(J_p) < 1$, or nonpositive, with any $\rho(J_p)$, the 2-cyclic repartitioning was not only always the *best* among all possible repartitionings but was also giving convergent SOR methods in the nonpositive case even when the corresponding to the original partitioning SOR method failed to converge!

Later Eiermann et al. [11], using theoretical and numerical examples, showed that the result obtained in [72] was *not* always true for *real* spectra $\sigma(J_p^p)$.

Finally, Galanis and Hadjidimos [17] considered the general case of the real spectra $\sigma(J_p^p)$ and all q-cyclic repartitionings for $2 \leqslant q \leqslant p$ of the original block p-cyclic matrix A and found the *best* q-cyclic repartitioning out of all possible repartitionings.

3. Modified SOR method

The idea of Modified SOR (or MSOR) method is to associate a different ω with each (block) row of the original linear system. The idea goes back to Russell [73] but it was mainly McDowell [61] and Taylor [81], who analyzed its convergence properties (see also [49]). It is best applied when the matrix A is 2-cyclic consistently ordered of the form (2.2). In such a case the MSOR method will be defined by the following iterative scheme:

$$x^{(m+1)} = \mathscr{L}_{\omega_1,\omega_2} x^{(m)} + c_{\omega_1,\omega_2}, \tag{3.1}$$

where

$$\begin{aligned} \mathscr{L}_{\omega_1,\omega_2} &:= (D-\omega_2 L)^{-1}[\operatorname{diag}((1-\omega_1)D_1,(1-\omega_2)D_2)+\omega_1 U], \\ c_{\omega_1,\omega_2} &:= (D-\omega_2 L)^{-1}\operatorname{diag}(\omega_1 I_{n_1},\omega_2 I_{n_2})b \end{aligned} \tag{3.2}$$

with I_{n_1}, I_{n_2} the unit matrices of the orders of D_1, D_2, respectively.

In such a case the basic relationship that connects the eigenvalues μ and λ of the spectra $\sigma(J_2)$ and $\sigma(\mathscr{L}_{\omega_1,\omega_2})$ is

$$(\lambda+\omega_1-1)(\lambda+\omega_2-1)=\omega_1\omega_2\mu^2\lambda, \tag{3.3}$$

which reduces to the classical one by Young for the SOR method for $\omega_1=\omega_2$. Optimal results for spectra $\sigma(J_2)$ of various configurations have been successfully obtained in some cases. For example: (i) For $\sigma(J_2)$ lying on a cross-shaped region optimal results can be found in [43] from which several other ones previously obtained by Taylor and other researchers can be easily recovered. (ii) For spectra $\sigma(J_2)$ lying on the unit circle and at the origin, except at the points $(\pm 1, 0)$, which is the case of the Jacobi iteration matrices arising in the discretization of second order elliptic boundary value problems by the finite-element collocation method with Hermite elements [40] and (iii) For $\sigma(J_2)$ lying in a "bow-tie" region which is the case arising in the discretization of the convection-diffusion equation by finite differences [2]. It is proved that the optimal MSOR method converges much faster than the optimal SOR and it also converges even in cases where the optimal SOR diverges. (For more details see [7,13] and especially Section 6 of [2].)

For extensions of the theory to GCO$(q, p-q)$-matrices the reader is referred to [43].

A problem which seemed to have been dealt with by Young and his colleagues (see [93]) in the 1960s, (see also [47]), was recast rather recently by Golub and de Pillis [22] in a more general form. More specifically, because the spectral radius is only an asymptotic rate of convergence of a linear iterative method the question raised was to determine, for each $k \geqslant 1$, a relaxation parameter $\omega \in (0,2)$ and a pair of relaxation parameters ω_1, ω_2 which minimize the Euclidean norm of the kth power of the SOR and MSOR iteration matrices associated with a real symmetric positive-definite matrix with *property A*. In [31] these problems were solved completely for $k=1$. Here are the corresponding results:

Theorem 3.1. *Let $A \in \mathbb{R}^{n,n}$ be a symmetric positive-definite matrix having property A and the block form*

$$A = \begin{bmatrix} I_{n_1} & -M \\ -M^T & I_{n_2} \end{bmatrix} =: I - J_2, \quad n_1+n_2=n. \tag{3.4}$$

Then for any fixed $t := \rho^2(J_2) \in [0,1)$ the value of ω, call it $\hat{\omega}$, which yields the minimum in $\min_{\omega\in(0,2)}\|\mathscr{L}_\omega\|_2$ is the unique real positive root in $(0,1)$ of the quartic equation

$$(t^2+t^3)\omega^4+(1-4t^2)\omega^3+(-5+4t+4t^2)\omega^2+(8-8t)\omega+(-4+4t)=0. \tag{3.5}$$

In fact $\hat{\omega} \in (0,\omega^*)$, where ω^* is the unique real positive root in $(0,1)$ of the cubic

$$(t+t^2)\omega^3-3t\omega^2+(1+2t)\omega-1=0. \tag{3.6}$$

Theorem 3.2. *Under the assumptions and notation of the previous theorem and for any fixed* $t \in [0,1)$ *the pair* (ω_1, ω_2), *call it* $(\hat{\omega}_1, \hat{\omega}_2)$, *which yields the minimum in* $\hat{\delta} := \min_{\omega_1,\omega_2 \in (0,2)} \|\mathscr{L}_{\omega_1,\omega_2}\|_2$ *is as follows*: *For* $t \in [0, \frac{1}{3}]$

$$(\hat{\omega}_1, \hat{\omega}_2) = \left(\frac{1}{1+t}, \frac{1}{1-t}\right) \text{ when } \hat{\delta} = \left(\frac{t}{1+t}\right)^{1/2} \tag{3.7}$$

while for $t \in [\frac{1}{3}, 1)$

$$(\hat{\omega}_1, \hat{\omega}_2) = \left(\frac{4}{5+t}, \frac{4}{3-t}\right) \text{ when } \hat{\delta} = \frac{1+t}{3-t}. \tag{3.8}$$

Remark. (i) It is worth pointing out that in [93] the values of $\hat{\omega}$ and the corresponding ones for $\|\mathscr{L}_{\hat{\omega}}\|_2$ are given for $t^{1/2} = \rho(J_2) = 0(0.1)1$. (ii) Part of Theorem 3.2 is also given in [93] where its proof at some points is based on strong numerical evidence.

We conclude this section by giving the functional eigenvalue relationship connecting the spectra of the Jacobi iteration matrix of a GCO($q, p-q$) matrix A of the class (2.17) and of the corresponding MSOR operator when each block is associated with a different relaxation factor ω_j, $j = 1(1)p$. The formula below is an extension of the one given by Taylor [81]

$$\prod_{j=1}^{p}(\lambda + \omega_j - 1) = \prod_{j=1}^{p} \omega_j \mu^p \lambda^{p-q}. \tag{3.9}$$

4. Symmetric SOR method

Each iteration step of the Symmetric SOR (SSOR) method consists of two semi-iterations the first of which is a usual (forward) SOR iteration followed by a backward SOR iteration, namely an SOR where the roles of L and U have been interchanged. More specifically

$$\begin{aligned} x^{(m+(1/2))} &= (D-\omega L)^{-1}[(1-\omega)D + \omega U]x^{(m)} + \omega(D-\omega L)^{-1}b, \\ x^{(m+1)} &= (D-\omega U)^{-1}[(1-\omega)D + \omega L]x^{(m+(1/2))} + \omega(D-\omega U)^{-1}b. \end{aligned} \tag{4.1}$$

An elimination of $x^{(m+(1/2))}$ from the above equations yields

$$x^{(m+1)} = \mathscr{S}_\omega x^{(m)} + c_\omega, \quad k = 0,1,2,\ldots, \; x^{(0)} \in \mathbb{C}^n \text{ arbitrary} \tag{4.2}$$

with

$$\begin{aligned} \mathscr{S}_\omega &:= (D-\omega U)^{-1}[(1-\omega)D + \omega L](D-\omega L)^{-1}[(1-\omega)D + \omega U], \\ c_\omega &:= \omega(2-\omega)(D-\omega U)^{-1}D(D-\omega L)^{-1}b. \end{aligned} \tag{4.3}$$

The SSOR method was introduced by Sheldon and constitutes a generalization of the method introduced previously by Aitken for $\omega = 1$ (see [93]).

Statements analogous to Kahan's theorem and also to Reich–Ostrowski–Varga's theorem of the SOR method can be proved. Specifically we have:

Theorem 4.1. *A necessary condition for the SSOR method defined in* (4.2)–(4.3) *to converge is* $|\omega - 1| < 1$. *For* $\omega \in \mathbb{R}$ *the condition becomes* $\omega \in (0,2)$.

Theorem 4.2. *Let* $A \in \mathbb{C}^{n,n}$ *be Hermitian with positive diagonal elements. Then for any* $\omega \in (0,2)$ *the SSOR iteration matrix* $\mathscr{S}_\omega$ *has real nonnegative eigenvalues. In addition, if A is positive definite then the SSOR method converges. Conversely, if the SSOR method converges and* $\omega \in \mathbb{R}$ *then* $\omega \in (0,2)$ *and A is positive definite.*

Note: Compared to SOR, SSOR requires more work per iteration and in general converges slower. Due to its symmetry, however, it can be combined with the semi-iterative method to produce other methods with nice convergence properties (see, e.g. [93]).

For 2-cyclic consistently ordered matrices the first functional relationship between the eigenvalues μ and λ of the associated Jacobi and SSOR iteration matrices was given by D'Sylva and Miles [10] and Lynn [55] and is the following:

$$(\lambda - (\omega - 1)^2)^2 = \omega^2 (2 - \omega)^2 \mu^2 \lambda. \tag{4.4}$$

It can be found that for A as in (2.2) the optimal ω, $\omega_b = 1$. Then $\rho(\mathscr{S}_1) = \rho(\mathscr{L}_1) = \rho^2(J_2)$.

In case A is block two-cyclic consistently ordered and $\sigma(J_2)$ lies in the open infinite unit strip S one can develop a Young-Eidson's-type algorithm for the determination of the optimal parameter ω_b and subsequently of $\rho(\mathscr{L}_{\omega_b})$ (see [32]).

The functional eigenvalue relationship in the case of block p-cyclic consistently ordered matrices was discovered by Varga, Niethammer and Cai [86], who obtained the relationship

$$(\lambda - (\omega - 1)^2)^p = \omega^p (2 - \omega)^2 \mu^p \lambda (\lambda - (\omega - 1))^{p-2}. \tag{4.5}$$

The relationship above was then extended by Chong and Cai [8] to cover the class of $GCO(q, p-q)$ matrices in (2.17) to

$$(\lambda - (\omega - 1)^2)^p = \omega^p (2 - \omega)^{2q} \mu^p \lambda^q (\lambda - (\omega - 1))^{p-2q}. \tag{4.6}$$

Optimal values of the SSOR method for spectra $\sigma(J_p^p)$ nonnegative or nonpositive for any $p \geqslant 3$ cannot be found anywhere in the literature except in a very recent article [37], where a number of cases are covered analytically and experimentally and a number of conjectures based on strong numerical evidence are made.

As for the SOR method also for the SSOR method researchers have tried to find regions of convergence for various classes of matrices. Thus Neumaier and Varga [63] determined for the class of H-matrices the region in the $(\rho(|D^{-1}(L+U)|), \omega)$-plane for which the SSOR method converges. Motivated by their work, Hadjidimos and Neumann [29], using Rouché's theorem, studied and determined the region of convergence of the SSOR method in the $(\rho(J_p), \omega)$-plane for each value of $p \geqslant 3$ for the class of the p-cyclic consistently ordered matrices. It is noted that the intersection of all the domains obtained for all $p \geqslant 3$ is the same domain as the one obtained by Neumaier and Varga for the whole class of the H-matrices with the only difference being that in the latter case the domain is obtained in the $(\rho(|D^{-1}(L+U)|), \omega)$-plane. An extension of the work in [29] is given in [30], where $GCO(q, p-q)$ matrices for each possible value of $l = q/p < \frac{1}{2}$ and each $p \geqslant 3$ are considered. Finally, in [35] the domains of convergence of the SSOR method for the class of p-cyclic consistently ordered matrices for each $p \geqslant 3$ in the $(\rho(J_p), \omega)$-plane is determined in the two cases of the nonnegative and nonpositive spectra $\sigma(J_p^p)$.

4.1. Unsymmetric SOR method

The unsymmetric SOR (USSOR) method differs from the SSOR method in the second (backward) SOR part of each iteration where a different relaxation factor is used (see [10,55,93]). It consists of the following two half steps:

$$\begin{aligned} x^{(m+(1/2))} &= (D-\omega_1 L)^{-1}[(1-\omega_1)D + \omega_1 U]x^{(m)} + \omega_1 (D-\omega_1 L)^{-1} b, \\ x^{(m+1)} &= (D-\omega_2 U)^{-1}[(1-\omega_2)D + \omega_2 L]x^{(m+(1/2))} + \omega_2 (D-\omega_2 U)^{-1} b. \end{aligned} \tag{4.7}$$

On elimination of $x^{(m+(1/2))}$ it is produced

$$x^{(m+1)} = \mathscr{S}_{\omega_1,\omega_2} x^{(m)} + c_{\omega_1,\omega_2}, \quad k = 0, 1, 2, \ldots, \; x^{(0)} \in \mathbb{C}^n \text{ arbitrary} \tag{4.8}$$

with

$$\begin{aligned} \mathscr{S}_{\omega_1,\omega_2} &:= (D-\omega_2 U)^{-1}[(1-\omega_2)D + \omega_2 L](D-\omega_1 L)^{-1}[(1-\omega_1)D + \omega_1 U], \\ c_{\omega_1,\omega_2} &:= (\omega_1 + \omega_2 - \omega_1\omega_2)(D-\omega_2 U)^{-1} D (D-\omega_1 L)^{-1} b. \end{aligned} \tag{4.9}$$

Theory analogous to that of the SSOR method can be developed and the interested reader is referred to [92,93].

The only point we would like to make is that for p-cyclic consistently ordered and for GCO$(q, p-q)$ matrices A there are functional eigenvalue relationships connecting the eigenvalue spectra of the Jacobi and of the USSOR iteration matrices. They were discovered by Saridakis [75] and the most general one below by Li and Varga [54]

$$\begin{aligned} &(\lambda - (1-\omega_1)(1-\omega_2))^p \\ &\quad = (\omega_1 + \omega_2 - \omega_1\omega_2)^{2q} \mu^p \lambda^q (\lambda\omega_1 + \omega_2 - \omega_1\omega_2)^{|\zeta_L|-q} (\lambda\omega_2 + \omega_1 - \omega_1\omega_2)^{|\zeta_U|-q}, \end{aligned} \tag{4.10}$$

where $|\zeta_L|$ and $|\zeta_U|$ are the cardinalities of the sets ζ_L and ζ_U, which are the two disjoint subsets of $P \equiv \{1, 2, \ldots, p\}$ associated with the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ as these are defined in [54].

5. Accelerated overrelaxation (AOR) method

A technique that sometimes "accelerates" the convergence of a convergent iterative scheme or makes it converge if it diverges is the introduction of an "acceleration" or "relaxation" parameter $\omega \in \mathbb{C} \setminus \{0\}$ as follows. Based on (1.2) we consider as a new preconditioner the matrix $M_\omega = \frac{1}{\omega} M$. It is then readily seen that the new iterative scheme is given by

$$x^{(m+1)} = T_\omega x^{(m)} + c_\omega, \quad T_\omega := (1-\omega)I + \omega T, \quad c_\omega := \omega c. \tag{5.1}$$

The parameter ω is called the *extrapolation* parameter and the corresponding scheme is the *extrapolated* of the original one. The most general algorithm to determine the best extrapolation parameter

$\omega \in \mathbb{C} \setminus \{0\}$ under some basic assumptions regarding some information on the spectrum $\sigma(T)$ of T can be found in [25] (see also the references cited therein and also [71] which treats a similar case).

Exploiting the idea of extrapolation a two-parameter SOR-type iterative method was introduced in [24]. It was called Accelerated overrelaxation (AOR) method and can be defined as follows:

$$x^{(m+1)} = \mathscr{L}_{r,\omega} x^{(m)} + c_{r,\omega}, \quad m = 0,1,2,\ldots, \; x^{(0)} \in \mathbb{C}^n \text{ arbitrary}, \tag{5.2}$$

where

$$\mathscr{L}_{r,\omega} := (D - rL)^{-1}[(1-\omega)D + (\omega - r)L + \omega U], \quad c_{r,\omega} := \omega(D - rL)^{-1}b. \tag{5.3}$$

It can be readily proved that the AOR method is the union of the Extrapolated Jacobi method ($r=0$) with extrapolation parameter ω and of the Extrapolated SOR method ($r \neq 0$) with extrapolation parameter $s = \omega/r$ of an SOR method with relaxation factor r. It is obvious that the Jacobi, the Gauss–Seidel, the SOR method and their extrapolated counterparts can be considered as special cases of the AOR method.

Note: Niethammer [65] refers to a similar to the AOR method that was introduced in a series of papers by Sisler [76–78].

For Hermitian matrices $A \in \mathbb{C}^{n,n}$ a statement analogous to the Reich–Ostrowski–Varga theorem holds for the AOR method as well. Here is one version of it given in [42].

Theorem 5.1. *Let $A = D - E - E^H \in \mathbb{C}^{n,n}$ be Hermitian, D be Hermitian and positive definite,* $\det(D - rE) \neq 0$, $\forall \omega \in (0,2)$ *and $r \in (\omega + (2-\omega)/\mu_m, \omega + (2-\omega)/\mu_M)$ with $\mu_m < 0 < \mu_M$ being the smallest and the largest eigenvalues of $D^{-1}(E + E^H)$. Then, $\rho(\mathscr{L}_{r,\omega}) < 1$ iff A is positive definite.* (*Note*: *Except for the restrictions in the statement the matrices $D, E \in \mathbb{C}^{n,n}$ can be any matrices.*)

Many more theoretical results can be proved in case A is p-cyclic consistently ordered. For example, if A is 2-cyclic consistently ordered and $\sigma(J_2^2)$ is either nonnegative or nonpositive then optimal parameters for the AOR method can be derived. They are better than the optimal ones for the corresponding SOR method if some further assumptions are satisfied. These results can be found in [1,62,27].

Theorem 5.2. *Under the notation and the assumptions so far, let $\underline{\mu}$ and $\bar{\mu}$ denote the absolutely smallest and the absolutely largest of the eigenvalues of the Jacobi iteration matrix J_2 of a* 2-*cyclic consistently ordered matrix A. Then: For $\sigma(J_2^2)$ nonnegative and $0 < \underline{\mu} < \bar{\mu} < 1$ if $1 - \underline{\mu}^2 < (1 - \bar{\mu}^2)^{1/2}$ the optimal parameters of the AOR method are given by the expressions*

$$r_b = \frac{2}{1 + (1 - \bar{\mu}^2)^{1/2}}, \quad \omega_b = \frac{1 - \underline{\mu}^2 + (1 - \bar{\mu}^2)^{1/2}}{(1 - \bar{\mu}^2)(1 + (1 - \bar{\mu}^2)^{1/2})}, \tag{5.4}$$

$$\rho(\mathscr{L}_{r_b,\omega_b}) = \frac{\underline{\mu}(\bar{\mu}^2 - \underline{\mu}^2)^{1/2}}{(1 - \underline{\mu}^2)^{1/2}(1 + (1 - \bar{\mu}^2)^{1/2})}.$$

Furthermore, for $0 < \underline{\mu} = \bar{\mu} < 1$ there are two pairs of optimal parameters

$$(r_b, \omega_b) = \left(\frac{2}{1 + \varepsilon(1 - \bar{\mu}^2)^{1/2}}, \frac{\varepsilon}{(1 - \bar{\mu}^2)^{1/2}} \right), \quad \varepsilon = \pm 1, \tag{5.5}$$

both of which give $\rho(\mathscr{L}_{r_b,\omega_b}) = 0$. For $\sigma(J_2^2)$ nonpositive and if $(1 + \bar{\mu}^2)^{1/2} < 1 + \underline{\mu}^2$ the optimal parameters of the AOR method are given by the expressions

$$r_b = \frac{2}{1 + (1 + \bar{\mu}^2)^{1/2}}, \quad \omega_b = \frac{1 + \underline{\mu}^2 + (1 + \bar{\mu}^2)^{1/2}}{(1 + \bar{\mu}^2)(1 + (1 + \bar{\mu}^2)^{1/2})}, \tag{5.6}$$

$$\rho(\mathscr{L}_{r_b,\omega_b}) = \frac{\underline{\mu}(\bar{\mu}^2 - \underline{\mu}^2)^{1/2}}{(1 + \underline{\mu}^2)^{1/2}(1 + (1 + \bar{\mu}^2)^{1/2})}.$$

Again for $0 < \underline{\mu} = \bar{\mu}$ there are two pairs of optimal parameters

$$(r_b, \omega_b) = \left(\frac{2}{1 + \varepsilon(1 + \bar{\mu}^2)^{1/2}}, \frac{\varepsilon}{(1 + \bar{\mu}^2)^{1/2}} \right), \quad \varepsilon = \pm 1, \tag{5.7}$$

both of which give $\rho(\mathscr{L}_{r_b,\omega_b}) = 0$.

Notes: (i) The assumptions on $\underline{\mu}$ and $\bar{\mu}$ of Theorem 5.2 are very demanding. Practically, to have an optimal AOR better than the optimal SOR, $\underline{\mu}$ must be "*different*" from 0 and "*very close*" to $\bar{\mu}$. It is not known whether these assumptions are true for any *real life* problem. (ii) The assumptions $\underline{\mu} = \bar{\mu} \neq 0$ indicate that the Jacobi iteration matrix J_2 has only two distinct, of opposite sign and of the same multiplicity eigenvalues. This leads directly to the fact that all eigenvalues of $\mathscr{L}_{r_b,\omega_b}$ are zero.

Methods analogous to the MSOR, SSOR, etc, have been developed and thus MAOR [39], SAOR [41], etc., can be found in the literature. Here we only give the functional eigenvalue relationship for GCO$(q, p - q)$ matrices that generalizes many other similar equations and especially the one by Saridakis [74] for the AOR method

$$\prod_{j=1}^{p} (\lambda + \omega_j - 1) = \prod_{j=1}^{q} \omega_j \mu^p \prod_{j=q+1}^{p} (\omega_j - r_j + r_j \lambda). \tag{5.8}$$

6. Linear (non-)stationary higher-order(degree), semi-iterative methods and SOR

All the iterative methods studied so far are *linear stationary first-order*(*degree*) ones. The term *stationary* means that any parameters involved in the iterative scheme are kept fixed during the iterations, *first order*(*degree*) means that the new iteration $x^{(m+1)}$ depends only on the previous one $x^{(m)}$ and *linear* that $x^{(m+1)}$ is a linear function of $x^{(m)}$.

Among the linear non-stationary first-order methods the adaptive SOR method is one of the most important and most popular in practical problems and is now incorporated in all the well-known computer packages like, e.g., ITPACK [48]. For an introduction to the adaptive SOR which was first considered for real symmetric positive definite 2-cyclic consistently ordered matrices but now is of a more general application the reader is referred to [44].

A class of linear stationary second-order methods, where each new iteration depends linearly on the two previous ones, that can handle effectively linear systems $Ax = b$, rewritten equivalently as $x = Tx + c$, where $\sigma(T)$ is assumed to be enclosed by an ellipse lying strictly to the left of the line $\operatorname{Re} z < 1$ of the complex plane, are described in [93]. In [93] the reader can find interesting results, when A is 2-cyclic consistently ordered with $\sigma(J_2^2)$ nonnegative and $\rho(J_2) < 1$, as well as some other interesting references.

A similar linear stationary second-order method is also given by Manteuffel in [59]. This method is derived directly from a linear non-stationary second-order one [56–58] which, in turn, is developed by using translated and scaled Chebyshev polynomials in the complex plane. It is worth pointing out that a 2-cyclic MSOR method is equivalent in the *Chebyshev sense* to a linear stationary second-order one and therefore "optimal" values of its parameters can be found by using either Manteuffel's algorithm [40] or a "continuous" analog of it [2].

There is also a class of iterative methods that are called Semi-Iterative and are described in a very nice way in [83] (see also [93]). In [83] it is shown that if one uses Chebyshev polynomials and bases one's analysis on them one can derive a linear non-stationary second-order scheme with very nice properties. The study of semi-iterative methods seems to have begun in [68] followed by a number of papers among which are [12–15]. Especially in the last two (see also [9]) when as the matrix T in $x = Tx + c$, the SOR iteration matrix, associated with a 2-cyclic consistently ordered matrix A with $\sigma(J_2^2)$ nonnegative and $\rho(J_2) < 1$, is considered, it is proved that it converges for all $\omega \in (-\infty, 2/(1-(1-\rho^2(J_2))^{1/2})) \setminus \{0\}$ which constitutes an amazingly wider range than that of the SOR method!

7. Operator relationships for generalized $(q, p-q)$-cyclic consistently ordered matrices

Before we conclude this article we would like to mention one more point. As we have seen so far in case A is a GCO$(q, p-q)$ matrix there is always a functional relationship that connects the eigenvalues of the Jacobi iteration matrix and the eigenvalues of the iteration operator associated with any of the methods considered. E.g., SOR, MSOR, SSOR, USSOR, AOR, MAOR, SAOR, etc. However, it seems that exactly the same functional relationship holds for the iteration operators involved.

The first who observed that such a relationship held was Young and Kincaid [95] (see also [93]), who proved that for a 2-cyclic consistently ordered matrix there holds

$$(\mathscr{L}_\omega + (\omega - 1)I)^2 = \omega^2 J_2^2 \mathscr{L}_\omega. \tag{7.1}$$

Using this equation as a starting point a discussion started whether similar relationships held as well for other functional relationships associated with operators of a p-cyclic consistently ordered matrix. The theory behind the proof of such relationships is basically graph theory and combinatorics. The most general relationships that can be found in the literature are the following two which refer to the MSOR and to the USSOR methods associated with a GCO$(q, p-q)$ matrix, respectively,

$$\prod_{j=1}^{p}(\mathscr{L}_\Omega + (\omega_j - 1)I) = (\Omega J_{q,p-q})^p \mathscr{L}_\Omega^{p-q}, \tag{7.2}$$

where $\Omega = \mathrm{diag}(\omega_1 I_{n_1}, \omega_2 I_{n_2}, \ldots, \omega_p I_{n_p}), \sum_{i=1}^{p} n_i = n$, and

$$\begin{aligned}(\mathscr{S}_{\omega_1,\omega_2} - (1-\omega_1)(1-\omega_2)I)^p \\ = (\omega_1+\omega_2-\omega_1\omega_2)^{2q} J_{q,\,p-q}^{p} \mathscr{S}_{\omega_1,\omega_2}^{q} (\omega_1 \mathscr{S}_{\omega_1,\omega_2} + \omega_2(1-\omega_1)I)^{|\zeta_L|-q} \\ \times(\omega_2 \mathscr{S}_{\omega_1,\omega_2} + \omega_1(1-\omega_2)I)^{|\zeta_U|-q},\end{aligned} \tag{7.3}$$

where for the various notations see previous section and [33,54,70]. From these relationships simpler ones can be obtained, e.g., for the p-cyclic consistently ordered SOR, for the GCO$(q, p-q)$ SSOR, and also the same relationships can be extended to cover the p-cyclic AOR, MAOR, SAOR, etc., cases.

Use of the functional relationships can be made in order to transform a one-step iterative scheme into another equivalent p-step one. For more details the reader is referred to the references of this section and also to [68].

8. Final remarks

In this article an effort was made to present the SOR method and some of its properties together with some other methods closely related to it. For the methods presented the most common classes of matrices A that led to some interesting results were considered. Of course, not all of the well-known classes of matrices A was possible to cover. For example, matrices like strictly diagonally dominant, irreducibly diagonally dominant, etc., were left out.

Finally, we mentioned only very briefly the role of the SOR, SSOR, etc, methods as preconditioners for the class of semi-iterative (see [93]) and we did not examine at all their roles for the class of conjugate gradient methods (see [23]). This was done purposefully for otherwise the basic theory of the other classes of methods involved should have been analyzed to some extent and this would be beyond the scope of the present article. On the other hand, it is the author's opinion that other expert researchers in the corresponding areas will cover these subjects in a much better and more efficient way in their articles in the present volume.

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