

# A Fast Solver For Differential Equations



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# Abstract

Several methods are presented to solve a class of differential equations which are discretized to a linear system and kept in memory using a sparse structure. I have implemented, tested and compared both classic and more recent methods for solving the resulting linear equation. The main results are about the convergence rates of the Multigrid method and of the Preconditioned Conjugate Gradients method. An emphasis is put on the setting of the parameters and of how they affect the solver overall. Some interesting results are presented towards the end regarding the comparison of different parameters selections, including an adaptive method.

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# Chapter 1

## Motivation

### 1.1 A class of differential equations

Einstein's paper: [?]

# Chapter 2

## Discretization of the equations

### 2.1 Deducing the approximations

#### 2.1.1 Finite differences method

#### 2.1.2 Creating a linear system

#### 2.1.3 Backward Euler

### 2.2 Sparse data structures

#### 2.2.1 Requirements

#### 2.2.2 CSR format

#### 2.2.3 Comparison with other formats



# Chapter 3

## Classic iterative algorithms

### 3.1 Successive Over-Relaxation Methods

#### 3.1.1 General Formulation

#### 3.1.2 Jacobi

##### 3.1.2.1 Theoretical aspects

##### 3.1.2.2 Experimental results

#### 3.1.3 Gauss-Seidel

##### 3.1.3.1 Theoretical aspects

##### 3.1.3.2 Experimental results

#### 3.1.4 Symmetric Successive Over-Relaxation

##### 3.1.4.1 Theoretical aspects

##### 3.1.4.2 Experimental results

#### 3.1.5 Comparison

### 3.2 Gradient Methods

#### 3.2.1 Gradient Descent

##### 3.2.1.1 Theoretical aspects

##### 3.2.1.2 Experimental results

#### 3.2.2 Conjugate Gradient Method

##### 3.2.2.1 Theoretical aspects

##### 3.2.2.2 Preconditioned version

##### 3.2.2.3 Experimental results

#### 3.2.3 Comparison

### 3.3 General comparison and comments





# Chapter 4

## The Multigrid Method

### 4.1 Motivation

#### 4.1.1 Capturing different frequencies

#### 4.1.2 The 2-grid algorithm

### 4.2 Formulation

#### 4.2.1 The V-Cycle

#### 4.2.2 Smoothers

#### 4.2.3 Prolongation and restriction operators

#### 4.2.4 Implementation details

### 4.3 Practical aspects

#### 4.3.1 Impact of the smoothing parameter

#### 4.3.2 Impact of prolongation and restriction operators

#### 4.3.3 General experimental results

### 4.4 Multigrid as a preconditioner for Conjugate Gradient Method

#### 4.4.1 Theoretical aspects

#### 4.4.2 Implementation details

#### 4.4.3 Experimental Results

### 4.5 Comparison between solvers

# Chapter 5

## Applications

# Chapter 6

## Conclusions

# Appendix A

## Sample Title

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# Appendix B

## Sample Title

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