Continuous Mathematics

Partial Differential Equations

Continuous Mathematics HT16

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PDEs

- Ordinary differential equations are differential equations that only depend on total derivatives, e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$
- Partial differential equations are differential equations that depend on partial derivatives, for example

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= x + u \\ \frac{\partial}{\partial x}\left(\frac{u}{1+u}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{3u}{1+u}\frac{\partial u}{\partial y}\right) &= \mathrm{e}^u \end{aligned}$$

Leibniz integral rule

- We need this for the next example:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x = \int_a^b f_t(x,t)\,\mathrm{d}x$$

  provided that f and f_t are continuous
- Outline proof; we have the general limits

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \lim_{h_1\to 0}\frac{g(t+h_1)-g(t)}{h_1} \qquad \int_a^b k(x)\,\mathrm{d}x = \lim_{h_2\to 0}\sum_{a\le x_i\le b} h_2\,k(x_i)$$

  and so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x &= \lim_{h_1\to 0}\frac{1}{h_1}\lim_{h_2\to 0}\sum_{a\le x_i\le b} h_2\,\left(f(x_i,t+h_1)-f(x_i,t)\right) \\ &= \lim_{h_2\to 0}\lim_{h_1\to 0}\sum_{a\le x_i\le b} h_2\left(\frac{f(x_i,t+h_1)-f(x_i,t)}{h_1}\right) \end{aligned}$$

Leibniz integral rule, continued

$$\begin{aligned} &= \lim_{h_2\to 0}\sum_{a\le x_i\le b} h_2\lim_{h_1\to 0}\left(\frac{f(x_i,t+h_1)-f(x_i,t)}{h_1}\right) \\ &= \int_a^b f_t(x,t)\,\mathrm{d}x \end{aligned}$$

- There is a generalisation of this rule that we can use if a and b are functions of t which we quote without further proof:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &= \int_{a(t)}^{b(t)} f_t(x,t)\,\mathrm{d}x \\ &\quad + f(x,b(t))\times b'(t) \\ &\quad - f(x,a(t))\times a'(t) \end{aligned}$$

Example

- We have
$$\int_0^\infty e^{-tx}\,dx = \frac{1}{t}$$
- Differentiate both sides a few times wrt t:
$$\begin{aligned} -\int_0^\infty x e^{-tx}\,dx &= -\frac{1}{t^2} \\ \int_0^\infty x^2 e^{-tx}\,dx &= \frac{2}{t^3} \\ &\dots \\ (-1)^n \int_0^\infty x^n e^{-tx}\,dx &= (-1)^n \frac{n!}{t^n} \\ \Rightarrow \quad \int_0^\infty x^n e^{-x}\,dx &= n! \end{aligned}$$

The heat equation

- Suppose a metal bar occupies the region $0 < x < L$. Assuming the temperature $\Theta(x,t)$ is uniform across the bar's cross section, the temperature in the bar is given by
$$\frac{\partial \Theta}{\partial t} = D\frac{\partial^2 \Theta}{\partial x^2} + f(x,t)$$
where $D > 0$ is the *thermal diffusivity* and $f(x,t)$ is an internal heat source (if one exists)
- We assume that we know the initial temperature
$$\Theta(x,0) = \Theta_0(x)$$
for some given function $\Theta_0(x)$
- We also require boundary conditions for Θ or $\frac{\partial \Theta}{\partial x}$ at $x = 0$ and $x = L$ for all times $t > 0$

Separable solutions to the heat equation

- Example: Find $\Theta(x,t)$ that satisfies
$$\frac{\partial \Theta}{\partial t} = D\frac{\partial^2 \Theta}{\partial x^2} \qquad 0 < x < L$$
with initial conditions
$$\Theta(x,0) = 3\sin\frac{\pi x}{L}$$
with boundary conditions $\Theta = 0$ at $x = 0, L$
- We will assume throughout these examples that $D > 0$
- We will first show that if we can find a solution then it is unique
- Suppose there are two solutions, Θ_1 and Θ_2; Let $\Delta = \Theta_1 - \Theta_2$
- It then follows by substituting Δ into the equation and boundary conditions that
$$\frac{\partial \Delta}{\partial t} = D\frac{\partial^2 \Delta}{\partial x^2} \qquad 0 < x < L$$
with initial conditions and boundary conditions
$$\Delta(x,0) = 0 \qquad \Delta = 0 \text{ at } x = 0, L$$

Example, continued

- We define $E(t)$ by
$$E(t) = \int_0^L [\Delta(x,t)]^2\,dx \quad \Rightarrow \quad E(0) = 0, \qquad E(t) \geq 0$$
$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\int_0^L [\Delta(x,t)]^2\,dx = \int_0^L \frac{\partial}{\partial t}[\Delta(x,t)]^2\,dx \\ &= \int_0^L 2\Delta\frac{\partial \Delta}{\partial t}\,dx = \int_0^L 2\Delta D\frac{\partial^2 \Delta}{\partial x^2}\,dx \\ &= \left[2\Delta D\frac{\partial \Delta}{\partial x}\right]_0^L - \int_0^L 2D\left(\frac{\partial \Delta}{\partial x}\right)^2 dx \\ &= -\int_0^L 2D\left(\frac{\partial \Delta}{\partial x}\right)^2 dx \leq 0 \end{aligned}$$
- Hence $E(t)$ is a decreasing function; the only way all the conditions on $E(t)$ can be met is if $E(t) = 0$ at all times t
$$\Rightarrow \quad \Theta_1 = \Theta_2$$

Example, continued

- We look for a separable solution $\Theta(x,t) = X(x)S(t)$. If we find a solution then we know by uniqueness that it will be the only solution
- We have

$$\frac{\partial \Theta}{\partial t} = X(x)S'(t) \qquad \frac{\partial^2 \Theta}{\partial x^2} = X''(x)S(t)$$

- We may write the governing equation as

$$\begin{aligned} X(x)S'(t) &= DX''(x)S(t) \\ \text{equivalently } \frac{S'(t)}{D\,S(t)} &= \frac{X''(x)}{X(x)} \end{aligned}$$

- The right-hand side of the last equation is a function only of x and not of t
- The left–hand–side is a function only of t and not of x
- The only way this can simultaneously be true is if both sides are equal to a constant, i.e.

$$\frac{S'(t)}{D\,S(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Example, continued

- We now think about boundary conditions
- We have $\Theta(0,t) = 0$ and $\Theta(L,t) = 0$
- As $\Theta(x,t) = X(x)S(t)$ we must have

$$X(0)S(t) = 0, \qquad X(L)S(t) = 0$$

- We don't want $S(t) = 0$ for all times t — this would give $\Theta(x,t) = 0$
- We therefore have boundary conditions on X given by $X(0) = 0$ and $X(L) = 0$

Example, contined

- Suppose $\lambda > 0$. Then we can write $\lambda = k^2$, and so

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - k^2 X = 0$$

- This has general solution $X = A\mathrm{e}^{kx} + B\mathrm{e}^{-kx}$
- The boundary conditions then give $A = B = 0$, and so $X(x) = 0$, and then $\Theta(x,t) = 0$. This isn't what we want — the assumption $\lambda > 0$ must be false
- Suppose now that $\lambda = 0$
- We then have

$$\frac{X''(x)}{X(x)} = 0$$

  together with boundary conditions $X(0) = 0$ and $X(L) = 0$
- Again, this only has solution $X(x) = 0$
- We must have $\lambda < 0$

Example, continued

- We therefore write

$$\frac{X''(x)}{X(x)} = -k^2$$

- This equation has general solution

$$X(x) = A\sin kx + B\cos kx$$

- We now fit the boundary conditions

$$X(0) \Rightarrow B = 0 \qquad X(L) = 0 \Rightarrow A\sin kL = 0$$

- If $A = 0$ we would have the trivial solution $X(x) = 0$, and so $\Theta(x,t) = 0$ which violates the initial conditions
- Instead, $\sin kL = 0$ and so $kL = n\pi$ where $n = 1, 2, 3, \ldots$

Example, continued

- When $kL = n\pi$ the equation for $S(t)$ is

$$S'(t) = -\frac{Dn^2\pi^2}{L^2}S(t) \quad \Rightarrow \quad S(t) = C\mathrm{e}^{-Dn^2\pi^2 t/L^2}$$

- Combining the solutions for $X(x)$ and $S(t)$, we see that

$$E_n \sin\frac{n\pi x}{L}\mathrm{e}^{-Dn^2\pi^2 t/L^2} \qquad \text{where } E_n = AC$$

is a solution that satisfies the boundary conditions for any $n = 1, 2, 3$

- The general solution is the sum of these solutions:

$$\Theta(x,t) = \sum_{i=1}^{N} E_n \sin\frac{n\pi x}{L}\mathrm{e}^{-Dn^2\pi^2 t/L^2}$$

where E_n, $n = 1, 2, 3, \ldots$ are constants that are fitted from the initial conditions

- In this case, $E_1 = 3$ and $E_n = 0$ for $n = 2, 3, 4, \ldots$ and so

$$\Theta(x,t) = 3\sin\frac{n\pi x}{L}\mathrm{e}^{-Dn^2\pi^2 t/L^2}$$

Example, continued

- Note that the *only* Fourier solution to $f(x) = 0\ \forall x$ is $a_m = b_n = 0$

- We see that initially the bar has a positive temperature inside the bar
- The ends of the bar are maintained at a temperature $\Theta = 0$, and so we would expect that this would cool the bar down
- This is evident from the solution; as $t \to \infty$ we see that $\Theta(x,t) \to 0$ for all values of x

- This use of 'common sense' is standard in solving equations that deal with physical systems
- The equations are in fact reversible wrt time; but we know that 'hot things get cooler' (in the absence of a heat source)

Another Example

- Consider the PDE

$$\frac{\partial\Theta}{\partial t} = D\frac{\partial^2\Theta}{\partial x^2}, \qquad 0 < x < L$$

with boundary conditions $\frac{\partial\Theta}{\partial x} = 0$ at $x = 0$, and $\Theta = 0$ at $x = L$, and initial conditions

$$\Theta(x,0) = L^2 - x^2$$

- We again proceed by seeking a separable solution $\Theta(x,t) = X(x)S(t)$
- Boundary conditions give $X'(0) = 0$ and $X(L) = 0$
- As with the previous example we may write

$$\frac{X''(x)}{X(x)} = \frac{S'(t)}{D\,S(t)} = -k^2$$

from which we may deduce that

$$X(x) = A\sin kx + B\cos kx$$

Another Example, continued

- The boundary condition $X'(0) = 0$ implies $A = 0$
- The boundary condition $X(L) = 0$ gives $B\cos kL = 0$
- For a non–trivial solution we require $kL = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$
- We therefore have, for $n = 0, 1, 2, \ldots$

$$X(x) = B\cos\frac{(2n+1)\pi x}{2L}, \qquad k = \frac{(2n+1)\pi}{2L}$$

- Associated with this $X(x)$ is the equation for $S(t)$:

$$S'(t) = -\frac{(2n+1)^2\pi^2 D}{4L^2}S(t)$$

with solution

$$S(t) = C\mathrm{e}^{-(2n+1)^2\pi^2 Dt/(4L^2)}$$

Another Example, continued

- For $n = 1, 2, 3, \ldots$ the following is a solution of the PDE:

$$P_n \cos\frac{(2n+1)\pi x}{2L} e^{-(2n+1)^2\pi^2 Dt/(4L^2)}$$

where P_n is a constant

- A general solution is a linear sum of these solutions:

$$\Theta(x,t) = \sum_{n=1}^{\infty} P_n \cos\frac{(2n+1)\pi x}{2L} e^{-(2n+1)^2\pi^2 Dt/(4L^2)}$$

- The constants P_n are determined by the initial conditions $\Theta(x,0) = L^2 - x^2$
- Setting $t = 0$ in the infinite sum and equating to the initial conditions gives

$$L^2 - x^2 = \sum_{n=1}^{\infty} P_n \cos\frac{(2n+1)\pi x}{2L}$$

Another Example, continued

- We now use an approach similar to that used for Fourier series to determine the constants P_n
- Assuming we may interchange the order of the infinite summation and integration gives

$$\int_0^L (L^2 - x^2) \cos\tfrac{(2m+1)\pi x}{2L}\, dx = \sum_{n=1}^{\infty} P_n \int_0^L \cos\tfrac{(2m+1)\pi x}{2L} \cos\tfrac{(2n+1)\pi x}{2L}\, dx$$

- Remembering that $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ we have, for integers $m \neq n$:

$$\begin{aligned}
&\int_0^L \cos\tfrac{(2m+1)\pi x}{2L} \cos\tfrac{(2n+1)\pi x}{2L}\, dx \\
&\quad = \tfrac{1}{2} \int_0^L \cos\tfrac{(n+m+1)\pi x}{L} + \cos\tfrac{(n-m)\pi x}{L}\, dx \\
&= \tfrac{1}{2}\left[\tfrac{L}{(n+m+1)\pi} \sin\tfrac{(n+m+1)\pi x}{L} + \tfrac{L}{(n-m)\pi} \sin\tfrac{(n-m)\pi x}{L}\right]_0^L = 0
\end{aligned}$$

Another Example, continued

- We also have

$$\begin{aligned}
\int_0^L \cos^2\frac{(2n+1)\pi x}{2L}\, dx &= \frac{1}{2}\int_0^L 1 + \cos\frac{(2n+1)\pi x}{L}\, dx \\
&= \frac{L}{2}
\end{aligned}$$

- The constants P_n are therefore given by

$$P_n = \frac{2}{L}\int_0^L (L^2 - x^2)\cos\frac{(2n+1)\pi x}{2L}\, dx$$

which can be evaluated by integration by parts; I get

$$P_n = (-1)^n \frac{32L^2}{(2n+1)^3\pi^3}$$

Yet Another Example

- Solve

$$\frac{\partial\Theta}{\partial t} = \frac{\partial^2\Theta}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions

$$\Theta = 2 \text{ at } x = 0, \qquad \text{and} \qquad \Theta = 4 \text{ at } x = 1$$

and initial conditions

$$\Theta(x,0) = 2 + 2x + 3\sin\pi x$$

- All the boundary conditions we have considered before are of the form $\Theta = 0$ or $\frac{\partial\Theta}{\partial x} = 0$, which are known as *homogeneous boundary conditions*
- This allows us to find non–trivial sine and cosine solutions in the x variable
- At first sight, we can't do this for the non homogeneous boundary conditions for this problem.
- But there is a way around it — write $U = \Theta - (2 + 2x)$

Yet Another Example, continued

- Noting that

$$\frac{\partial U}{\partial t} = \frac{\partial \Theta}{\partial t}, \qquad \frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 \Theta}{\partial t^2}$$

we see that U satisfies the PDE

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

- The boundary conditions become

$$U = 0 \text{ at } x = 0, \qquad \text{and} \qquad U = 0 \text{ at } x = 1$$

and the initial conditions become

$$U(x, 0) = 3 \sin \pi x$$

- The equation for U has homogeneous boundary conditions, so can be solved in the same way as we have solved earlier equations
- The solution for Θ can then be recovered by writing

$$\Theta(x, t) = U(x, t) + 2 + 2x$$

Similarity solutions to the heat equation

- Suppose we want to solve

$$\frac{\partial \Theta}{\partial t} = D\frac{\partial^2 \Theta}{\partial x^2}, \qquad x, t > 0$$

with initial and boundary conditions

$$\Theta(x, 0) = 0 \ (x > 0) \qquad \Theta(0, t) = U, \Theta(\infty, t) = 0 \ (t > 0)$$

- Physically this corresponds to a semi-infinite bar occupying the region $0 < x < \infty$, that is initially at zero temperature. At time $t = 0$ the end at $x = 0$ is raised to temperature U
- Let $\eta = x/\sqrt{Dt}$, and let $\Theta = f(\eta)$
- η is known as a *similarity variable*

Yet Another Example, continued

- On Worksheet 1 you showed that for the heat equation on the previous slide this implies that

$$f''(\eta) + \tfrac{1}{2}\eta f'(\eta) = 0$$

- Integrating once gives, for arbitrary constant B,

$$f'(\eta) = B\mathrm{e}^{-\eta^2/4}$$

- Integrating once more gives, for arbitrary constant A,

$$f(\eta) = A + B\int_{s=0}^{\eta} \mathrm{e}^{-s^2/4}\,\mathrm{d}s$$

- Noting that $x > 0, t = 0$ corresponds to $\eta = x/\sqrt{Dt} = \infty$, the initial condition corresponds to $f(\infty) = 0$
- Similarly, $x = 0, t > 0$ corresponds to $\eta = 0$ and so the first boundary condition corresponds to $f(0) = U$.

Yet Another Example, continued

- $x = \infty, t > 0$ corresponds to $\eta = \infty$ — the second boundary condition corresponds to $f(\infty) = 0$. Note this is consistent with other conditions on f
- Using these conditions on $f(0)$ and $f(\infty)$ we may determine A and B to give

$$u(x, t) = f(\eta) = U\left(1 - \frac{\int_{s=0}^{\eta} \mathrm{e}^{-s^2/4}\,\mathrm{d}s}{\int_{s=0}^{\infty} \mathrm{e}^{-s^2/4}\,\mathrm{d}s}\right)$$

Vibrating String

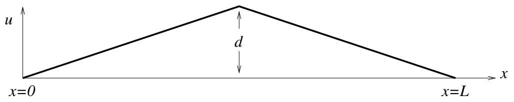
- Another standard example. Consider a tensioned string of length L attached its ends. The equation connecting its (small) displacement u as a function of distance along the string x and time t is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = u(L,t) = 0$$

- We justify this later. We also want initial conditions

$$u_t(x,0) = 0, \text{ and given } u(x,0) = \begin{cases} \frac{2xd}{L} & 0 < x \le \frac{L}{2} \\ \frac{2(L-x)d}{L} & \frac{L}{2} < x \le L \end{cases}$$

which is equivalent to the string being plucked in its centre



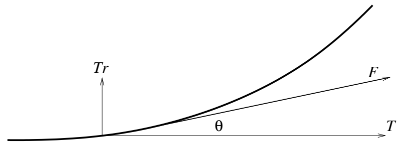
Model Justification

- Can model the string as many discrete masses connected by short springs
- The longitudual tension in the string T is assumed large compared with the transversal forces, so if $F(x,t)$ is the force to the right in the string and $Tr(x,t)$ the transversal component the string

$$T = F(x,t)\cos\theta \quad Tr(x,t) = F(x,t)\sin\theta = T\tan\theta = T\frac{\partial u}{\partial x}(x,t)$$

- The overall transversal force on a small piece of string is proportional to the rate of change of $Tr(x,t)$:

$$\frac{\partial^2 u}{\partial t^2}(x,t) \propto \frac{\partial}{\partial x}\frac{\partial u}{\partial x}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$



Solution

- Try for a separated solution

$$u(x,t) = X(x)T(t) \quad \Rightarrow \quad XT'' = c^2 X'' T$$

- Re-organising, and avoiding exponential solutions

$$\frac{T''}{T} = c^2\frac{X''}{X} = -\lambda^2 \quad \Rightarrow \quad X(x) = A_\lambda \sin\tfrac{\lambda}{c}x + B_\lambda \cos\tfrac{\lambda}{c}x$$

- $u(0,t) = 0 \Rightarrow B_\lambda = 0$
- $u(L,t) = 0 \Rightarrow \lambda = \frac{cn\pi}{L}$
- Now include $T(t)$; we must have

$$X(x) = A_n \sin\tfrac{n\pi}{L}x \quad T(t) = C_n \cos\tfrac{n\pi}{L}ct + D_n \sin\tfrac{n\pi}{L}ct$$

- $u_t(x,0) = 0 \Rightarrow D_n = 0$
- WLOG, $C_n = 1$ and

$$u(x,t) = \sum_n A_n \sin(\tfrac{n\pi}{L}x)\cos(\tfrac{n\pi}{L}ct)$$

Solution, continued

- Only at this point do we need to match the initial shape of the string. We have the Fourier series for this shape from an example on Worksheet 3;

$$\frac{8d}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{(2k+1)\pi x}{L}\right)$$

and so

$$A_n = \begin{cases} \left(\frac{8d}{\pi^2 n^2}\right) \times (-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

- So there is a fundamental frequency, followed by odd harmonics, with harmonic n of magnitude $\frac{1}{n^2}$ compared to the fundamental.
- The *spectrum* of the relative size of the harmonics is what makes up the tone of a musical note

Standing Waves

- We had the general solution (before imposing initial shape):

$$u(x,t) \;=\; \sum_n A_n \sin(\tfrac{n\pi}{L}x)\cos(\tfrac{n\pi}{L}ct)$$

- Using the identity $2\sin X\cos Y = \sin(X+Y) + \sin(X-Y)$ we get

$$\begin{aligned} u(x,t) \;&=\; W(x+ct) + W(x-ct) \\ &\quad \text{where } W(\theta) = \tfrac{1}{2}\sum_n A_n \sin(\tfrac{n\pi}{L}\theta) \end{aligned}$$

- This provides another interpretation of the solution, namely as two *standing waves* travelling in opposite directions with velocity c, and reflecting at the end-points.

Poisson's equation

- Poisson's equations in two dimensions is the partial differential equation

$$D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y) = 0$$

  where D is constant
- This models many time independent diffusion processes — e.g. chemical, heat — where $f(x,y)$ is a source term
- An exercise on Worksheet 1 was to show that, for cylindrical polar coordinates $x = r\cos\theta$, $y = r\sin\theta$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

- We may therefore write Poisson's equation as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + f(r,\theta) = 0$$

Example

- Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - x^2 - y^2 = 0$$

  for $x^2 + y^2 < 4$, with boundary condition $u = 5$ on $x^2 + y^2 = 4$
- Noting that $x^2 + y^2 = r^2$ we may write this as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^2$$

  for $r < 2$, with boundary condition $u = 5$ on $r = 2$
- As there is no dependence on θ in the boundary conditions or source term we seek a solution $u = u(r)$, i.e. we neglect the dependence on θ
- The partial derivatives with respect to r are now total derivatives, and the partial derivatives with respect to θ are zero
- The equation therefore becomes

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = r^2$$

Example, continued

- This has general solution

$$u = \frac{1}{16}r^4 + A\log r + B$$

- We first note that u must be finite at $r = 0$ — as $\lim_{r\to 0}\log r = -\infty$ this requires $A = 0$
- The other boundary condition is $u = 5$ on $r = 2$ — this yields $B = 4$
- The solution is therefore

$$\begin{aligned} u \;&=\; \frac{1}{16}r^4 + 4 \\ &=\; \frac{1}{16}\left(x^2 + y^2\right)^2 + 4 \end{aligned}$$

Separable solutions to Poisson's equation

- Suppose we want to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x, y < 1$$

with boundary conditions

$$\begin{aligned} u &= 0, & x &= 0, x = 1, y = 1 \\ u &= x(1-x), & y &= 0 \end{aligned}$$

- We seek a separable solution $u(x, y) = X(x)Y(y)$, with boundary conditions

$$\begin{aligned} X(0) &= 0, & X(1) &= 0 \\ &, & Y(1) &= 0 \end{aligned}$$

- Substitution into the given PDE gives $X''Y + XY'' = 0$ which may be written

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$$

Separable solutions, continued

- We have two boundary conditions on X, so start with this equation first:

$$X'' + k^2 X = 0$$

- Boundary conditions give, for arbitrary constant B

$$X(x) = B \sin n\pi x, \qquad n = 1, 2, 3, \ldots$$

and $k = n\pi$.
- ODE for $Y(y)$ becomes

$$Y'' - n^2\pi^2 Y = 0$$

- This has general solution

$$Y(y) = C\mathrm{e}^{n\pi y} + D\mathrm{e}^{-n\pi y}$$

- Applying the boundary condition $Y(1) = 0$ gives $D = -C\mathrm{e}^{2n\pi}$
- General solution becomes $Y(y) = C\left(\mathrm{e}^{n\pi y} - \mathrm{e}^{n\pi(2-y)}\right)$

Separable solutions, continued

- Solution may be written

$$u(x, y, t) = \sum_{n=1}^{\infty} P_n \left(\mathrm{e}^{n\pi y} - \mathrm{e}^{n\pi(2-y)}\right) \sin n\pi x$$

- To set $u = x(1-x)$ on $y = 0$ we note that

$$\begin{aligned} x(1-x) &= u(x, 0) \\ &= \sum_{n=1}^{\infty} P_n \left(1 - \mathrm{e}^{2n\pi}\right) \sin n\pi x \end{aligned}$$

- We will now follow the Fourier type approach to determine the constants P_n. Noting that for integers m, n we have

$$\int_0^1 \sin m\pi x \sin n\pi x \, \mathrm{d}x = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$$

we may write

$$P_n = \frac{2}{1 - \mathrm{e}^{2n\pi}} \int_0^1 x(1-x) \sin n\pi x \, \mathrm{d}x = \begin{cases} \frac{8}{n^3\pi^3(\mathrm{e}^{2n\pi} - 1)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Summary

- We can normally assume that PDEs have a unique solution
- Separation of variable used to construct separate (but linked) ODEs for each variable
- Boundary conditions on one variables can be used to constrain the common constant
- Common sense is also useful. For physical systems quantities can't be infinite, entropy tends to increase with time, etc.
- Tricks are also useful in solving the equations
- Fourier techniques can be used to combine solutions, with the constants established from the boundary conditions