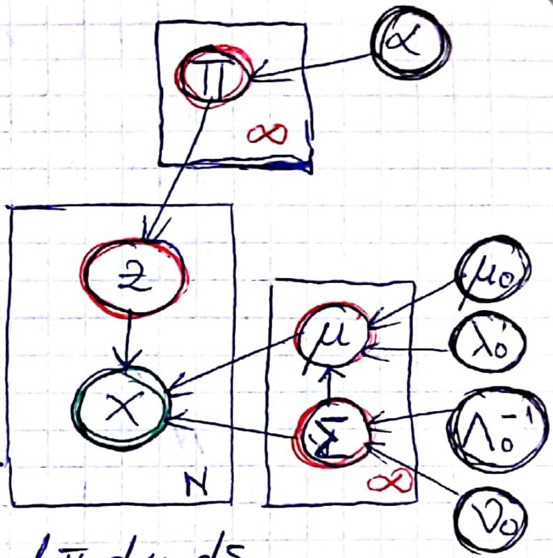
⊛ COLLAPSED GIBBS SAMPLING

→ we integrate out π, μ_k, Σ_k and sample directly the component assignment, z.



$$\iiint_{\pi\,\mu\,\Sigma} p(z_i = k \mid z_{\neg i}, x, \pi, \mu, \Sigma, \alpha) = \iiint_{\pi\,\mu\,\Sigma} \frac{p(z_i = k, z_{\neg i}, x, \mu, \Sigma, \alpha, \pi)}{p(z_{\neg i}, x, \pi, \mu, \Sigma, \alpha)}\, d\pi\, d\mu\, d\Sigma$$

$$\propto \iiint_{\pi\,\mu\,\Sigma} p(z_i = k, z_{\neg i}, x, \pi, \mu, \Sigma, \alpha)\, d\pi\, d\mu\, d\Sigma$$

$$\propto \iiint_{\pi\,\mu\,\Sigma} \underbrace{p(z_i = k \mid z_{\neg i}, \alpha)}_{A} \cdot \underbrace{p(x \mid z_i = k, z_{\neg i}, \alpha, \mu, \Sigma)}_{B}\, d\pi\, d\mu\, d\Sigma$$

(A)

$$p(z_i = k \mid z_{\neg i}, \alpha) = \frac{p(z_i = k, z_{\neg i}; \alpha)}{p(z_{\neg i}; \alpha)} = \frac{p(z \mid \alpha)}{p(z_{\neg i} \mid \alpha)}$$

$$p(z \mid \alpha) = \int_\pi p(z \mid \pi) \cdot p(\pi \mid \alpha)\, d\pi$$

Dirichelet(α)

$$= \int_\pi \prod_{i=1}^{N} p(z_i \mid \pi) \cdot p(\pi \mid \alpha)\, d\pi$$

usual choice $\alpha_k = \frac{\alpha}{K}$

Categorical(π)

$$= \int_\pi \prod_{i=1}^{N} \prod_{k=1}^{K} p(z_i = k \mid \pi) \cdot \frac{1}{B(\alpha)} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1}\, d\pi$$

$$\frac{\prod_{k=1}^{K} \Gamma(\alpha_k)}{\Gamma\left(\sum_{k=1}^{K} \alpha_k\right)}$$

$$= \frac{1}{B(\alpha)} \int_\pi \prod_{i=1}^{N} \prod_{k=1}^{K} \pi_k \cdot \prod_{k=1}^{K} \pi_k^{\frac{\alpha}{K} - 1}\, d\pi$$

$$\sum_{k=1}^{K} \frac{\alpha}{K} = K \cdot \frac{\alpha}{K}$$

$$= \frac{\Gamma\left(\sum_k \alpha_k\right)}{\prod_{k=1}^{K} \Gamma(\alpha_k)} \cdot \int_\pi \prod_{k=1}^{K} (\pi_k)^{N_k} \cdot \prod_{k=1}^{K} \pi_k^{\frac{\alpha}{K} - 1}\, d\pi$$

$$\frac{\prod_{k=1}^{K} \Gamma\left(N_k + \frac{\alpha}{K}\right)}{\Gamma(N + \alpha)}$$

$$= \frac{\Gamma(\alpha)}{\prod_{k=1}^{K} \Gamma\left(\frac{\alpha}{K}\right)} \cdot \int_\pi \prod_{k=1}^{K} \pi_k^{N_k + \frac{\alpha}{K} - 1}\, d\pi$$

$$p(z \mid \alpha) = \frac{\Gamma(\alpha)}{\Gamma(N + \alpha)} \prod_{k=1}^{K} \frac{\Gamma\left(N_k + \frac{\alpha}{K}\right)}{\Gamma\left(\frac{\alpha}{K}\right)}$$

$$p(z_i = k \mid z_{\neg i}, \alpha) = \frac{p(z \mid \alpha)}{p(z_{\neg i} \mid \alpha)} =$$

$$= \frac{\dfrac{\Gamma(\alpha)}{\Gamma(N+\alpha)} \cdot \displaystyle\prod_{k=1}^{K} \frac{\Gamma(N_k + \frac{\alpha}{K})}{\Gamma(\alpha/K)}}{\dfrac{\Gamma(\alpha)}{\Gamma(N-1+\alpha)} \cdot \displaystyle\prod_{k=1}^{K} \frac{\Gamma(N_{k,\neg i} + \frac{\alpha}{K})}{\Gamma(\alpha/K)}}$$

Between the 2 $\prod_{k=1}^{K}\ldots$, all factors are similar except the one containing $\Gamma(N_k + \frac{\alpha}{K}) / \Gamma(N_{k,\neg i} + \frac{\alpha}{K})$

$$= \frac{\Gamma(N+\alpha-1) \cdot \Gamma(N_k + \frac{\alpha}{K})}{\Gamma(N+\alpha) \quad \Gamma(N_{k,\neg i} + \frac{\alpha}{K})}$$

$N_{k,\neg i} = N_k - 1$

$\Gamma(x+1) = x \cdot \Gamma(x)$

$$= \frac{\cancel{\Gamma(N+\alpha-1)}}{(N+\alpha-1) \cdot \cancel{\Gamma(N+\alpha-1)}} \quad \frac{\Gamma(N_k + \frac{\alpha}{K})}{\cancel{\Gamma(N_k - 1 + \frac{\alpha}{K})}} = (N_k + \frac{\alpha}{K} - 1) \cdot \Gamma(\underbrace{\ldots}_{N_k + \frac{\alpha}{K} - 1})$$

$$= \frac{N_k + \frac{\alpha}{K} - 1}{N + \alpha - 1}$$

$$\Rightarrow p(z_i = k \mid z_{\neg i}, \alpha) = \frac{N_k + \frac{\alpha}{K} - 1}{N + \alpha - 1}$$

⊛ When $K \to \infty$

a) if x_i joins a cluster k with $N_k - 1$ members

$$\boxed{p(z_i = k \mid z_{\neg i}, \alpha) = \frac{N_{k,\neg i}}{N + \alpha - 1}}$$

b) if x_i joins a cluster with no members (creates its own cluster) [and this is possible since the total number of cluster can grow up to ∞], $N_{k,\neg i} \to 0$

$$p(z_i = k^* \mid z_{\neg i}, \alpha) = \frac{\frac{\alpha}{\cancel{K}}}{N + \alpha - 1} \cdot \cancel{K} = \frac{\alpha}{N + \alpha - 1}$$

there are K possible clusters "to create"

$$\Rightarrow \boxed{p(z_i = k^* \mid z_{\neg i}, \alpha) = \frac{\alpha}{N + \alpha - 1}}$$

Conclusion: $\boxed{\text{EFFECT OF } \alpha}$

→ an instance X_i joins a cluster with members in it with a probability $\sim$ # of members OR creates a new cluster with a probability $\sim \alpha$.

(B) $\iint_{\mu\Sigma} p(x_i \mid z_i = k, \cancel{z_{\setminus i}}, \mu, \Sigma) = \iint \frac{p(x_i, z_i = k, \mu, \Sigma)}{p(z_i = k, \mu, \Sigma)} d\mu d\Sigma$

$$\propto \iint_{\mu\Sigma} p(x_i, z_i = k, \mu, \Sigma)\, d\mu\, d\Sigma$$

$$= \iint_{\mu\Sigma} p(x_i \mid z_i = k, \mu, \Sigma) \cdot p(\mu \mid \Sigma) \cdot p(\Sigma)\, d\mu\, d\Sigma$$

$$= \iint_{\mu\Sigma} \underbrace{\mathcal{N}(\mu_k, \Sigma_k)(x_i)}_{ⓐ} \cdot \underbrace{\mathcal{N}(\mu_0, \tfrac{1}{\lambda_0} \Sigma)(\mu)}_{ⓑ} \cdot \underbrace{iW(\Sigma_0, \nu_0)}_{ⓒ}\, d\mu d\Sigma$$

ⓐ $\mathcal{N}(\mu_k, \Sigma_k)(x_i) = \frac{1}{\sqrt{(2\pi)^D \cdot |\Sigma_k|}} \cdot e^{-\frac{1}{2}(x_i - \mu_k)^T \Sigma_k^{-1} (x_i - \mu_k)}$

$$= (2\pi)^{-\frac{D}{2}} \cdot |\Sigma_k|^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}(x_i - \mu_k)^T \Sigma_k^{-1} (x_i - \mu_k)}$$

ⓑ $\mathcal{N}(\mu_0, \frac{1}{\lambda_0} \cdot \Sigma)(\mu) = \frac{1}{\sqrt{(2\pi)^D \cdot |\frac{1}{\lambda_0}\Sigma|}} \cdot e^{-\frac{1}{2}(\mu - \mu_0)^T (\frac{1}{\lambda_0} \Sigma)^{-1} (\mu - \mu_0)}$

$\left(\frac{1}{\lambda_0}\right)^{-\frac{D}{2}} = \lambda_0^{\frac{D}{2}}$

$$= (2\pi)^{-\frac{D}{2}} \cdot \lambda_0^{\frac{D}{2}} \cdot |\Sigma|^{-\frac{1}{2}} \cdot e^{-\frac{\lambda_0}{2}(\mu - \mu_0)^T \cdot \Sigma^{-1} (\mu - \mu_0)}$$

ⓒ $\text{Inv Wishart}(\Sigma_0, \nu_0)(\Sigma) = \frac{|\Sigma_0|^{\frac{\nu_0}{2}}}{2^{\frac{\nu_0 D}{2}} \Gamma_D(\frac{\nu_0}{2})} |\Sigma|^{-\frac{\nu_0 + D + 1}{2}} \cdot e^{-\frac{1}{2} tr(\Sigma_0 \Sigma^{-1})}$

$$\iint_{\mu\Sigma} ⓐ \cdot ⓑ \cdot ⓒ = \iint_{\mu\Sigma} (2\pi)^{-D} \cdot |\Sigma|^{-\frac{\nu_0 + D + 3}{2}} \cdot \underbrace{\lambda_0^{\frac{D}{2}} \cdot \frac{|\Sigma_0|^{\frac{\nu_0}{2}}}{2^{\frac{\nu_0 D}{2}} \Gamma_D(\frac{\nu_0}{2})}}_{\mathcal{Z}(\lambda_0, \nu_0, \Sigma_0, D)} \cdot e^{-\frac{1}{2}\left[(\mu-\mu_0)^T \Sigma^{-1} (\mu - \mu_0) \lambda_0 + (x_i - \mu_k)^T \Sigma_k^{-1} (x_i - \mu_k) + tr(\Sigma_0 \Sigma)^{-1}\right]}$$

$$= (2\pi)^{-D} \cdot \mathcal{Z}(\lambda_0, \nu_0, \Sigma_0, D) \cdot \iint_{\mu\Sigma} |\Sigma|^{-\frac{\nu_0 + D + 3}{2}} \cdot e^{-\frac{1}{2}\{\dots\}}\, d\mu d\Sigma$$

It turns out that the $\iint_{\mu\Sigma} \dots d\mu d\Sigma$ is:

$\mathcal{Z}(\lambda_N, \nu_N, \Sigma_N, D)$, where

$\lambda_N = \lambda_0 + N$ $\quad \Sigma_N = \Sigma_0 + (x_i - \mu_k)^T (x_i - \mu_k) + \frac{\lambda_0 N}{\lambda_0 + N} (\bar{x} - \mu_0)^T (\bar{x} - \mu_0)$

$\nu_N = \nu_0 + N$

$\bar{x} = \frac{1}{N} \sum_{n=1}^{N} x_n$

⊗ EFFECT of $\mu_0, \lambda_0, \Sigma_0, \nu_0$

- μ_0 is our prior mean for μ.
- λ_0 is how strongly we believe this prior for μ.
- S_0 is proportional to our prior mean for Σ;
- ν_0 is how strongly we believe this prior for Σ $(\nu_0 \geq D)$