'Pricing under Rough Volatilty Models' Lab Report

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Realized volatility - I

Consider a stochastic volatility model

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{1}$$

where S_t is an asset price process, and σ_t is a stochastic volatility process representing a so-called *spot volatility*.

Definition

The *realized volatility* of a price process S over time interval $[t, t+\delta]$ sampled along the time partition π^n is defined as

$$RV_{t,t+\delta}(\pi^n) = \sqrt{\sum_{\pi^n \cap [t,t+\delta]} \left(\log S_{t^n_{i+1}} - \log S_{t^n_i}\right)^2}. \tag{2}$$

Realized volatility - II

Definition

Let S satisfy (1). Then the integrated variance is defined as

$$IVar_t = \int_0^t \sigma_s^2 ds. \tag{3}$$

Theorem

As time partition scale and δ tend to 0*,*

$$\frac{RV_{t,t+\delta}}{\sqrt{\delta}} \xrightarrow{\mathbb{P}} \sigma_t \tag{4}$$

Fractional stochastic processes - I

Definition

The *fractional Brownian motion* $(W_t^H)_{t\in\mathbb{R}_+}$ with Hurst parameter $H \in (0,1)$ is a Gaussian process with the following properties:

1. $W_0^H = 0$,
2. $\mathbb{E}\left[W_t^H\right] \equiv 0$,
3. $\mathbb{E}\left[W_s^H W_t^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.

Fractional stochastic processes - II

Definition

A stationary fOU process X_t is defined as the stationary solution of the stochastic differential equation

$$dX_t = \nu dW_t^H - \alpha(X_t - m)dt, \tag{5}$$

where $m \in \mathbb{R}$ and ν and α are positive parameters.

Long memory of processes

Definition

A process X_t is said to have a long memory, if

$$\sum_{k=0}^{\infty} \operatorname{cov}\left[X_1, X_k - X_{k-1}\right] = \infty. \tag{6}$$

In particular, the fractional Brownian motion with $H > \frac{1}{2}$ is a long-memory process. Long-memory of the stochastic volatility process in stochastic volatilty models framework used to be a widely-accepted stylized fact.

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RFSV - I

Let there be a riskless asset $B_t \equiv 1$, and a risky asset, whose price S_t is defined by the following equations:

$$dS_t = \alpha S_t dt + \sigma_t S_t dW_t, \tag{7}$$

$$d \log \sigma_t = \alpha(m - \log \sigma_t) dt + \nu dW_t^H. \tag{8}$$

As a stylized fact we shall demand the stationarity of log-increments.

RFSV - II

Let $m(q, \Delta, \pi^n)$ be a sample q-th absolute moment of $\log RV_{t+\Delta} - \log RV_t$:

$$m(q, \Delta, \pi^n) := \frac{1}{n} \sum_t |\log RV_{t+\Delta} - \log RV_t|^q , \tag{9}$$

i.e. $m(q, \Delta, \pi^n)$ is an empirical counterpart of $\mathbb{E}\left[|\log RV_\Delta - \log RV_0|^q\right]$. In this work we shall use the uniform partition of time scale with each step being equal to 15 minutes, so we omit the π^n notation and use $m(q, \Delta)$. Via the explicit formula for the covariance function of the log-volatility in the RFSV model, we can write a closed-form expression for a theoretical $m(2, \Delta)$:

$$m(2, \Delta) = 2\left(\operatorname{var} \log \sigma_t - \operatorname{cov}\left[\log \sigma_t, \log \sigma_{t+\Delta}\right]\right). \tag{10}$$

In the present paper we used high-frequency data for the three types of assets:

1. **Stocks**: Yandex, Sberbank, Gazprom, VTB, Moscow Exchange, Lukoil, and X5 Group;
2. **Depositary reciepts**: Sberbank, Gazprom, VTB, and Lukoil;
3. **Funds**: AEX, AORD, BFX, BVSP, DJI, FCHI, FTMIB, FTSE, GDAXI, GSPTSE, HSI, IBEX, IXIC, KS11, KSE, MXX, N225, OMXC20, OMXHPI, OMXSPI, OSEAX, RUT, SMSI, SPX, SSEC, SSMI.

Estimation

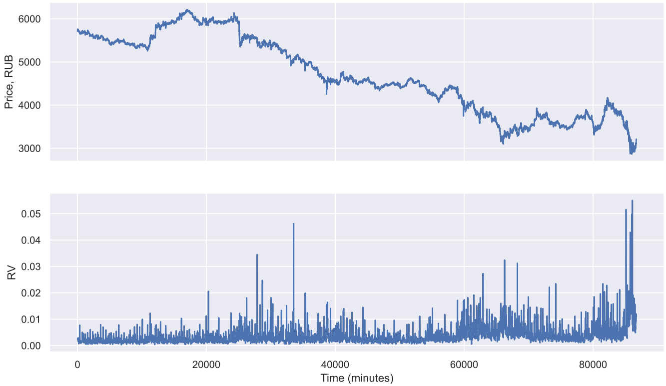

Figure: YNDX RX Equity. Price and Realized Volatility

Main assumption

Main assumption: for some $s_q > 0$, $b_q > 0$ and $N = \left[\frac{T}{\Delta}\right]$ (number of RV estimations via disjoint windows)

$$N^{qs_q} m(q, \Delta) \xrightarrow[\Delta \to 0+]{} b_q. \tag{11}$$

Under additional technical conditions, equation (11) is equivalent to that the volatility process belongs to the Besov smoothness space $B^{s_q}_{q,\infty}$ and for all $\tilde{s}_q > s_q$ does not belong to $B^{\tilde{s}_q}_{q,\infty}$ (see [Ros08]).

Estimation

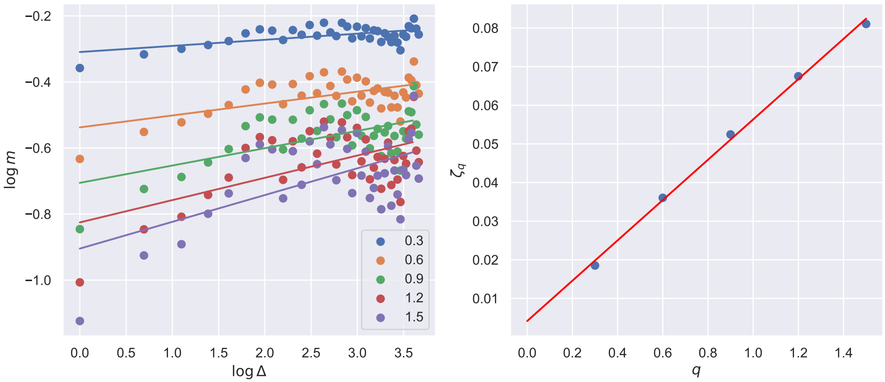

Figure: YNDX RX Equity. Regression-based roughness estimation

It has been shown that under stationarity assumptions and linearity of Figure 2 (left)

$$\mathbb{E}\left[|\log \sigma_{t+\Delta} - \log \sigma_t|^q\right] = K_q \Delta^{\zeta_q}, \tag{12}$$

and the s_q does not depend on q. We note that the graphs for ζ_q are slightly concave, which correlates with [GJR14] results. They conclude that this effect takes place due to the finite statistical population size. It has been proven in [GJR14] that $\log \mathbb{E}[\sigma_t \sigma_{t+\Delta}]$ and $\log \operatorname{cov}[\log \sigma_t, \log \sigma_{t+\Delta}]$ are linear in Δ^{2H}. And we indeed observe this behaviour in the majority of plots (especially for $\Delta < 20$, where we have enough data to work with). Numerical instability occurs when Δ is too large due to the lack of HF data.

Estimation

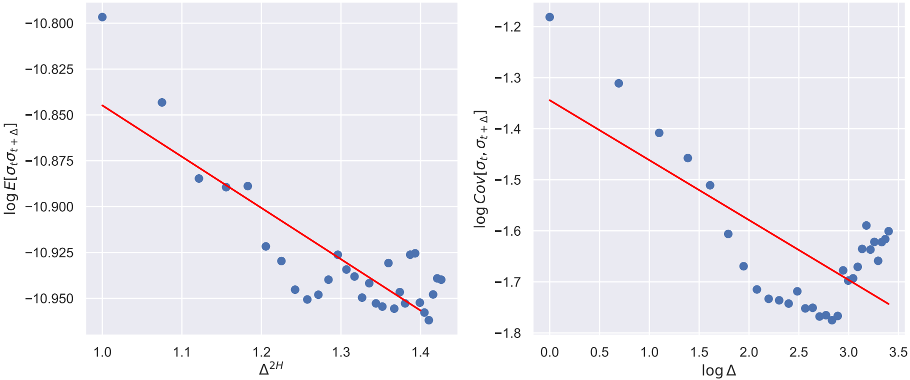

Figure: YNDX RX Equity. Empirical counterpart of $\log \mathbb{E}\left[\sigma_t \sigma_{t+\Delta}\right]$ as a function of Δ^{2H} (left) and Empirical counterpart of $\log \operatorname{cov}[\log \sigma_t, \log \sigma_{t+\Delta}]$ as a function of $\log \Delta$ (right)

Asset Type	Ticker	$\hat{H}$
Stock	YNDX	0.0521766
Stock	SBER	0.1551646
Stock	VTBR	0.0917236
Stock	MOEX	0.0853878
Stock	LKOH	0.0730521
Stock	GAZP	0.1309705
Stock	FIVE	0.0630289
Depositary reciept	OGZD	0.0523981
Depositary reciept	VTBR	0.0370185
Depositary reciept	SBER	0.0578053
Depositary reciept	LKOD	0.0352792

Table: Hurst parameter estimations

In order to test the normality of the log-increments of the realized volatility, we used the following tests:

1. Visual analysis of histograms: KDE vs normal fit vs empirical fit
2. Visual analysis of excessed kurtosis plot
3. D'Agostino's K Squared normality test
4. Shapiro-Wilk normality test

Visual analysis of histograms and excessed kurtosis plot - I

1. KDE is the *kernel density estimator* of the data.
2. *Normal fit* $NF(\Delta)$ is the normal distribution fitted to the data with the same mean and variance.
3. *Empirical fit* $EF(\Delta)$ is the scaled normal distribution:
   - $EF(1)$ is said to be same as the $NF(1)$
   - $EF(\Delta)$ for $\Delta > 1$ is said to be a scaled $NF(1)$ by the factor of $\Delta^{\hat{H}}$ (by this we test the monofractal scaling property of normal distribution)

Visual analysis of histograms and excessed kurtosis plot - II

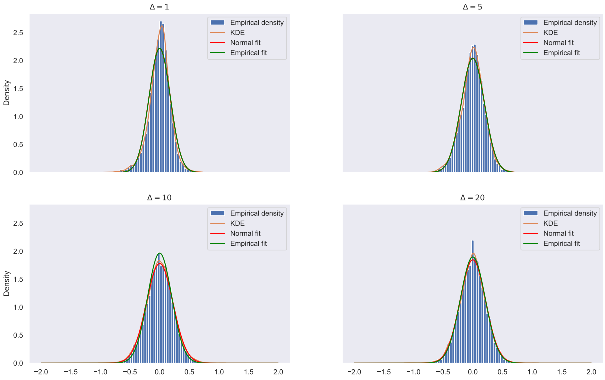

Figure: YNDX RX Equity. Empirical density of $\log \sigma_{t+\Delta} - \log \sigma_t$ for $\Delta = 1, 5, 10, 20$ days.

Visual analysis of histograms and excessed kurtosis plot - III

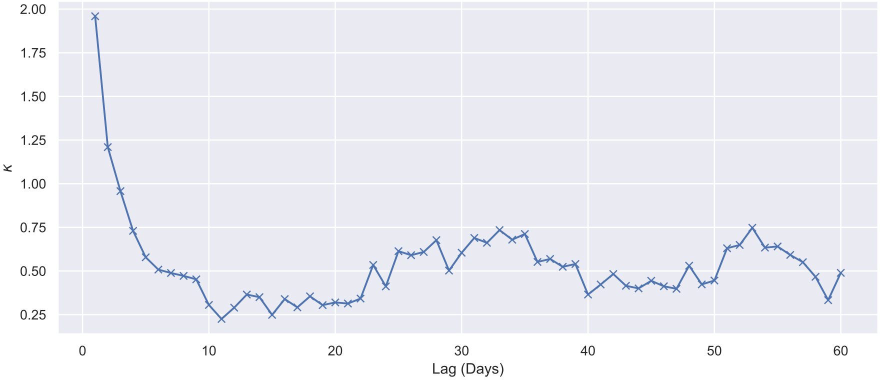

Figure: YNDX RX Equity. Excessed kurtosis κ as a function of Δ

Statistical tests for normality

The three possible explanations are:

1. The tests are correct and the data is not normally distributed or is correlated strongly.
2. The visual analysis of the histograms show that for many lags the KDE plot, the normal fit and the empirical fit are very similar, therefore, the distribution is normal, but the data is correlated strongly. The excessed kurtosis plot shows that the data is distributed very close to the normal distribution for $\Delta > 5$, and at its closest distance for $\Delta \in [10, 22]$.
3. We get a population sampling error (not enough data).

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Let us consider a sequence of partitions π^n of $[0, T]$ with $|\pi^n| := \max_{t_i^n \in \pi^n}(t_{i+1}^n - t_i^n) \to 0$.

Definition

A function $x \in C[0, T]$ is said to have the finite p-th variation along the sequence of partitions π^n if there exists a continious increasing function $[x]_\pi^{(p)}$ such that for all subpartitions $\tilde{\pi}^n(t) = \pi^n \cap [0, t]$

$$\sum_{t_i^n \in \tilde{\pi}^n(t)} \left|x(t_{i+1}^n) - x(t_i^n)\right|^p \to [x]_\pi^{(p)}(t), \quad n \to \infty, \tag{13}$$

and the set of all functions having finite p-th variation along π we denote V_π^p.

Definition

The *variation index* of a path x is defined as $p^\pi(x) := \inf\left\{p \geq 1\colon \quad x \in V_\pi^p\right\}$, and the *roughness index* is defined as $H^\pi(x) := \frac{1}{p^\pi(x)}$.

It has been proven that for fBm with Hurst parameter H
$p^\pi(W^H) = \frac{1}{H}$ and $H^\pi(W^H) = H$.

Definition

$x \in V_\pi^p$ is said to have p-th normalized variation if there exists such continious function $w(x,p,\pi)\colon [0,T] \to \mathbb{R}$ that

$$\sum_{\tilde{\pi}^n(t)} \frac{\left|x(t_{i+1}^n) - x(t_i^n)\right|^p}{[x]_\pi^{(p)}(t_{i+1}^n) - [x]_\pi^{(p)}(t_i^n)} (t_{i+1}^n - t_i^n) \to w(x,p,\pi). \tag{14}$$

Let us consider π^L, π^K – partitions with sampling frequencies $L \gg K$ $(\pi^K \subset \pi^L)$.

Definition

Sample normalized p-th variation is defined as

$$W(L,K,p,t,X) = \sum_{\tilde{\pi}^K(t)} \frac{\left|x(t^K_{i+1}) - x(t^K_i)\right|^p}{\sum_{t^n_i \in \tilde{\pi}^n(t)} \left|x(t^n_{i+1}) - x(t^n_i)\right|^p} (t^n_{i+1} - t^n_i) \tag{15}$$

It has been proven that

- W converges to w as sampling frequencies tend to ∞;
- W is strictly monotonic in p.

This method was introduced in [CD22].

Definitions

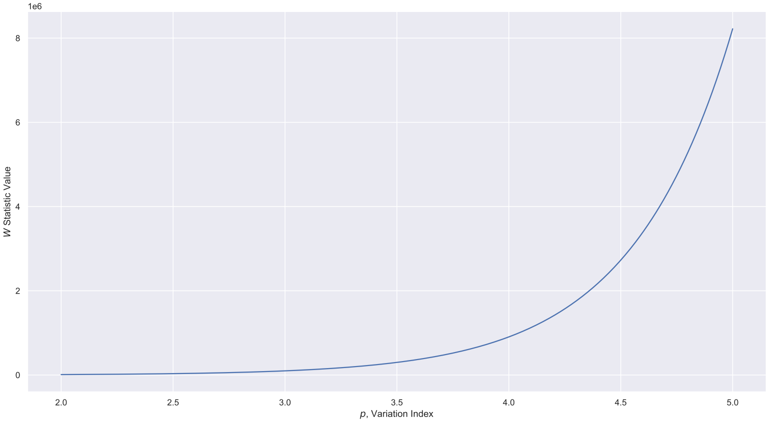

Figure: The W statistic as a function of p

Roughness estimation of Monte-Carlo simulations of fBm

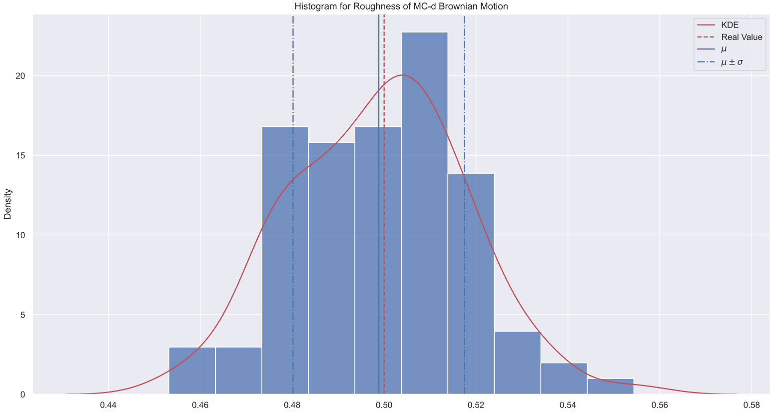

Figure: Histogram for roughness of Brownian motion, $\hat{H} = 0.4988$

Roughness estimation of Monte-Carlo simulations of fBm

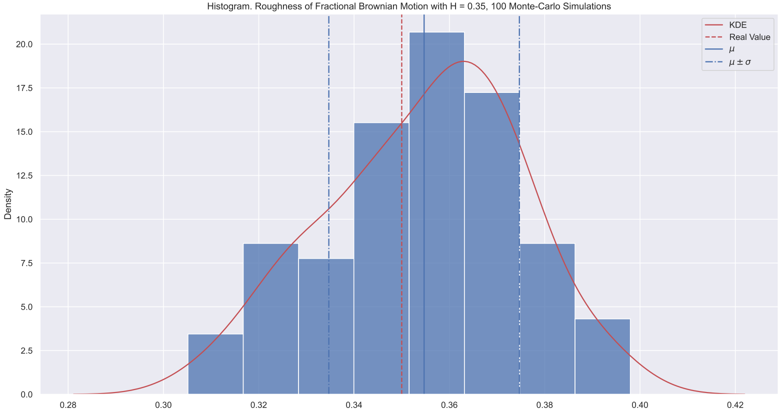

Figure: Histogram for roughness of fractional Brownian motion, $H = 0.35$, $\hat{H} = 0.3547$

Roughness estimation of Monte-Carlo simulations of fBm

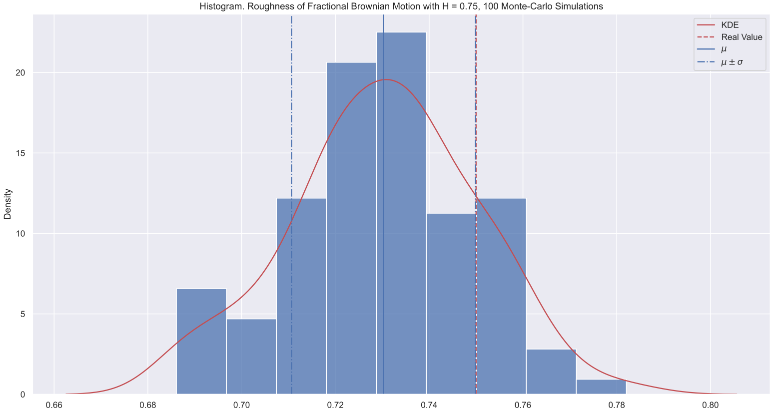

Figure: Histogram for roughness of fractional Brownian motion, $H = 0.35$, $\hat{H} = 0.7302$

Roughness estimation of Monte-Carlo simulations of Heston stochastic volatility model

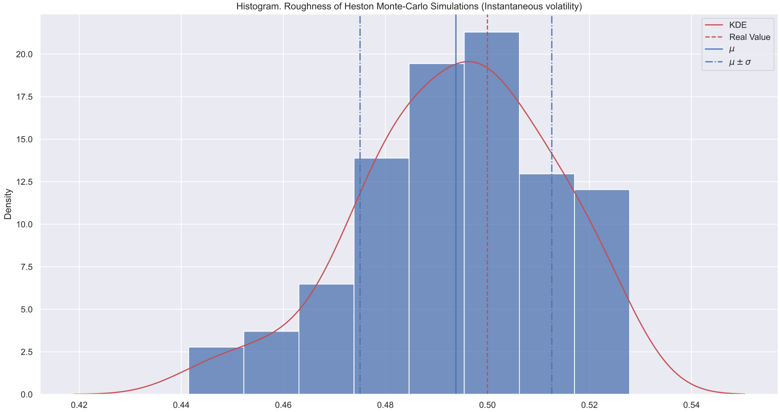

Figure: Histogram for roughness of Heston SVM (Instantaneous volatility)

Roughness estimation of Monte-Carlo simulations of Heston stochastic volatility model

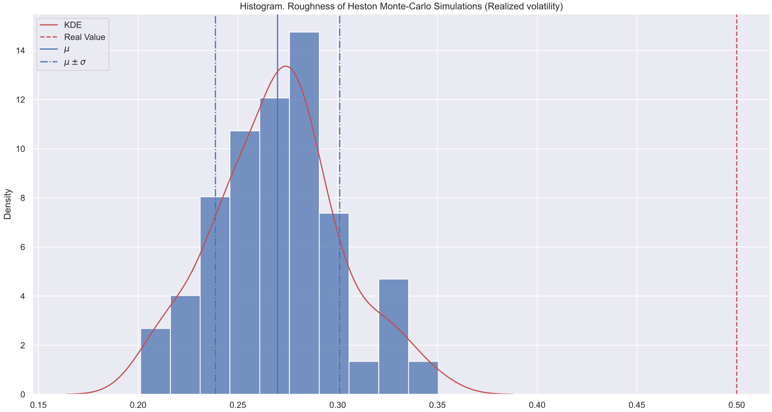

Figure: Histogram for roughness of Heston SVM (Realized volatility)

Roughness estimation of real market data

Ticker	Roughness Index
YNDX RX Equity	0.372691
SBER RX Equity	0.313109
VTBR RX Equity	0.304677
MOEX RX Equity	0.295378
LKOH RX Equity	0.301795
GAZP RX Equity	0.316125
FIVE RX Equity	0.284704
OGZD LI Equity	2.968608
VTBR LI Equity	0.306763
SBER LI Equity	1.176616
LKOD LI Equity	0.306061

Table: Roughness index estimation

Base articles

GJR14 Jim Gatheral, Thibault Jaisson, and Mathieu Rosenbaum. "Volatility is rough". In: *Quantitative Finance* 18.6 (2014), pp. 933–949;

Ros08 Mathieu Rosenbaum. "Estimation of the volatility persistence in a discretely observed diffusion model". In: *Stochastic Processes and their Applications* 118.8 (2008), pp. 1434–1462;

CD22 Rama Cont and Purba Das. "Rough volatility: fact or artefact?" In: (2022).