OVM Homework Exercises Set 2

Submission deadline : 17th January 2024 (on-line via Brightspace)

Please include all your answers clearly in the main body of your report (PDF), and attach the programming code in the appendix for TA's review. The use of LaTeX is highly recommended. Additionally, submit your code as a separate file (preferably as a Jupyter notebook) for implementation checking. Note that if your solutions are unclear or if we are unable to run your code (in Python or Matlab) correctly, you may not receive full credit. Example code for the programming exercises can be found in the textbook and on the author's website.

1(2P). Implied volatility :

a(0.5). Employ a root-finding algorithm to reproduce Figure 14.2 in the book, by adding 10 units of currency to each option price. For example, 475 becomes 485, 405 becomes 415, etc.

b(1.0). Acquire real-world option data for a company whose name starts with either your initial or the first letter of your surname. This data should have **different strike prices** and **expiry times**. Create an implied volatility figure similar to Figure 14.2 using this data. Be sure to indicate the source of the option data and provide your dataset in the submission.
Hint : For options on stocks from the US market, the risk-free rate can be derived from the daily US Treasury yield curve rates using cubic spline interpolation. For stocks from other markets, you can use a similar approach or select a reasonable estimate, such as $r = 0.01$. To calculate the time to maturity, you may use the Act-365 day count convention, where the year fraction between two dates is computed as the number of days between them divided by 365.

c(0.5). Implied volatility may vary along time to maturity $\tau = T - t$ given a specific strike price. This is so-called term structure of implied volatility. Please make plots and check if the data in Question (b) contains term structure.

2(3P). Binomial method : In the Black-Scholes model, we have the stock price $S_0 = 1.0$, the expiry time $T = 3.0$, the risk-free interest rate $r = 0.05$, volatility $\sigma = 0.3$, strike price $E = 1.5$.

a. Value an American put option using the Binomial method, and find the exercise boundary to produce a figure like Fig. 18.4 in the book.

b. Compute the time-zero value of a corresponding Bermudan put option, respectively, with 3, 6, 12 and 36 equally spaced early-exercise dates. Compared with American options in Question (2.a), what are your findings ?

c. Determine the time-zero price of a shout put option, as follows,

$$\text{Payoff} = \begin{cases} \max(E - S(T), E - S(\tau)), & \text{if holder shouted at time } \tau \text{ (random).} \\ \max(E - S(T), 0), & \text{otherwise.} \end{cases}$$

which allows the holder to shout at most once to the writer before expiration and lock in a payoff of at least $E - S(\tau)$, when $S(\tau) < E$ and $0 \leq \tau < T$.

3(2P). Monte Carlo methods : Suppose there is a financial option contract, with expiry time $T = 1.0$, stock price $S_0 = 1.0$, exercise price $E = 1.2$, risk-free interest rate $r = 0.05$, volatility

$\sigma = 0.3$,

a. Solve programming exercise P21.1 on page 227 in the book for two tasks,
   — value an arithmetic average Asian put option with payoff $\max(E - \frac{1}{n}\sum_{i=1}^{n} S(t_i), 0)$ (here $n = 52$).
   — value a fixed strike lookback put option with payoff $\max(E - S^{min}, 0)$.

b. Employ a variance reduction technique to approximate their Delta at time $t = 0$ and investigate how the numerical error varies over the number of asset paths (refer to Fig 15.3 in the book).

4(3P). Finite difference methods : the Black-Scholes PDE reads,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

with a payoff function $\Lambda(S(T))$ at T, $S_0 = 10$, $E = 10$, $r = 0.06$, $\sigma = 0.3$, $T = 1$ year.

a. Solve Exercise 23.2. Please note half central difference δ Taylor series in Table 23.1 is incorrect.

b. — Show that the transformations $S = e^y$, $\tau = T - t$, and $v(y, \tau) = e^{r\tau}V(y, \tau)$, followed by $x = y + (r - \frac{1}{2}\sigma^2)\tau$ result in the following heat equation for unknown $u(x, \tau)$,

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} \tag{2}$$

   — What is the von Neumann stability condition in FTCS for Equation (2) ? Next explain why the binomial method does not converge uniformly (refer to Section 24.4 in the book).
   — Implement FTCS to solve Equation (2) and value a European put option $\max(E-S(T), 0)$.

c. Considering the stock pays a continuous dividend yield of $q = 0.01$, we have the corresponding Black-Scholes PDE to price European options,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0 \tag{3}$$

The task is to implement the Crank-Nicolson scheme to value a down-and-out European call option using Equation (3) and produce a figure like Fig. 11.3 in the book. Here you can set the barrier $B = 6.0$.

Appendix : (the following problem description appeared in the midterm exam) We can modify the assumptions of the Black-Scholes model to incorporate dividend payments, often seen in index options. In this scenario, dividends are paid out at a constant rate $q > 0$, compounded continuously, causing the stock price to drop by the value of dividend $qS(t_i)\delta t$,

$$\delta S(t_i) = S(t_{i+1}) - S(t_i) = \mu S(t_i)\delta t - \underbrace{qS(t_i)\delta t}_{\text{dividend}} + \sigma S(t_i)\sqrt{\delta t}Y_i.$$

Dividends received from holding one share of stock over Δt are the sum $\sum_i qS(t_i)\delta t = qS\Delta t$. The portfolio Π changes over time interval Δt,

$$\Delta\Pi = \Delta(AS + D) = \Delta(AS) + \Delta D = (A\Delta S + AqS\Delta t) + \Delta D$$

where $\Delta S = S(t + \Delta t) - S(t)$. By Delta-Hedging, the option pricing PDE reads,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0.$$