Matrix Calculus

Anton Andreytsev

Let $X \in \mathbb{R}^d, Y \in \mathbb{R}^q$ be some normed linear spaces and let $f : X \to Y$
$f(X + \Delta X) = f(X) + Df(X)[\Delta X] + \bar{o}(\Delta X), \quad \|\Delta X\| \to 0$
$Df(X)[\Delta X]$ – the Fréchet derivative

From the definition above it follows, that

$$Df(X)[\Delta X] = \lim_{t \to +0} \frac{f(X + t \cdot \Delta X) - f(X)}{t}$$

In particular case, when $X \in \mathbb{R}^n, Y \in \mathbb{R}$ (so $f : \mathbb{R}^n \to \mathbb{R}$) $\Rightarrow Df(X)[\Delta X] = \nabla f(X)^T \Delta X$

Generalisations:

$x \in \mathbb{R} \Rightarrow Df(x)[\Delta x] = \nabla f(x) \cdot \Delta x, \quad \nabla f(x) \in \mathbb{R}$
$x \in \mathbb{R}^n \Rightarrow Df(x)[\Delta x] = \nabla f(x)^T \Delta x, \quad \nabla f(x) \in \mathbb{R}^n$
$x \in \mathbb{R}^{mxn} \Rightarrow Df(x)[\Delta x] = Tr\left(\nabla f(x)^T \Delta x\right), \quad \nabla f(x) \in \mathbb{R}^{mxn}$

Example 1: $f : \mathbb{R}^n \to R, \quad f(x) = a^T x$

$$f(x + \Delta x) = a^T(x + \Delta x) = \underbrace{a^T x}_{f(x)} + \underbrace{a^T \Delta x}_{Df(x)[\Delta x]} + \underbrace{0}_{\bar{o}(\|\Delta x\|)}$$

$$Df(x)[\Delta x] = \underbrace{a^T}_{\nabla f(x)^T} \Delta x \Rightarrow \boxed{\nabla(a^T x) = a}$$

Example 2: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = x^T A x$

$$f(x + \Delta x) = (x + \Delta x)^T A(x + \Delta x) = \underbrace{x^T A x}_{f(x)} + \underbrace{\Delta x^T A x + x^T A \Delta x}_{Df(x)[\Delta x]} + \underbrace{\Delta x^T A \Delta x}_{\bar{o}(\|\Delta x\|)}$$

$$Df(x)[\Delta x] = \underbrace{\Delta x^T A x}_{()=()^T, \texttt{const}} + x^T A \Delta x = x^T A^T \Delta x + x^T A \Delta x = x^T(A + A^T)\Delta x = \underbrace{((A + A^T)x)^T}_{\nabla f(x)} \Delta x$$

$$\boxed{\nabla(x^T A x) = (A + A^T)x}$$

Application: The problem of fitting weights of a linear regression can be formalised as follows:

$$\|X\beta - y\|_2^2 \to \min_\beta$$

The necessary condition of extremum is gradient equals zero: $\nabla_\beta \|X\beta - y\|_2^2 = 0$

$\|a\|_2^2 = a^T a$, so $\nabla \left(X\beta - y\right)^T \left(X\beta - y\right) = 0$

$$\nabla \left(\beta^T X^T X\beta \underbrace{-\beta^T X^T y - y^T X\beta}_{-2y^T X\beta} + y^T y\right) = \left(\underbrace{X^T X + X^T X}_{2X^T X}\right)\beta - 2X^T y = 0$$

$$X^T X\beta = X^T y \Rightarrow \boxed{\beta = \left(X^T X\right)^{-1} X^T y}$$

In case of l_2- regularisation:

$$\|X\beta - y\|_2^2 + \lambda\|\beta\|_2^2 \to \min_\beta$$

$\nabla = 2X^T X\beta - 2y^T X\beta + \lambda\left(I + I^T\right)\beta = 0$ where I - is identity matrix

$$\left(X^T X + \lambda I\right)\beta = X^T y \Rightarrow \boxed{\beta = \left(X^T X + \lambda I\right)^{-1} X^T y}$$

Example 3: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = e^{x^T x}$

$$Df(x)[\Delta x] = e^{x^T x} \cdot x^T(I + I^T)\Delta x = \underbrace{2 \cdot x^T \cdot e^{x^T x}}_{\nabla^T} \Delta x$$

$$\nabla\left(e^{x^T x}\right) = 2e^{x^T x} x$$

Example 4: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = x^T e^{xx^T} x, \quad \nabla f(x) = ?$

First consider $e^A = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \ldots$ – this is by definition. So $x^T e^{xx^T} x = x^T \sum\limits_{i=0}^{+\infty} \frac{(xx^T)^i}{i!} x = \sum\limits_{i=0}^{+\infty} \frac{x^T (xx^T)^i x}{i!} = \sum\limits_{i=0}^{+\infty} \frac{\overbrace{x^T(x}^{\text{i+1-times}}\, x^T)(xx^T)\ldots(xx^T)x}{i!} = \sum\limits_{i=0}^{+\infty} \frac{(x^T x)^{i+1}}{i!} = x^T x \sum\limits_{i=0}^{+\infty} \frac{(x^T x)^i}{i!} = x^T x \exp(x^T x)$

Then take into account, that when $f : \mathbb{R}^n \to \mathbb{R}$ the usual differentiation rules hold: $\nabla(u \cdot v) = \nabla u \cdot v + u \cdot \nabla v$, and $\nabla\left(\frac{u}{v}\right) = \frac{\nabla u \cdot v - u \cdot \nabla v}{v^2}$

$\nabla\left(x^T x \exp(x^T x)\right) = \nabla(x^T x) \cdot \exp(x^T x) + x^T x \cdot \nabla(\exp(x^T x)) = 2 \cdot x \cdot \exp(x^T x) + x^T x \cdot 2\exp(x^T x) x = 2\exp(x^T x)\left(1 + x^T x\right) x$

$$\boxed{\nabla\left(x^T \exp(x x^T) x\right) = 2\exp(x^T x)\left(1 + x^T x\right) x}$$

Example 5: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = Det\left(2I + x x^T\right)$

We will need here some properties of matrix differentiation:

$\boxed{1}$ $h(x) = g(f(x)), \quad f : \mathbb{X} \to \mathbb{Y}, \quad g : \mathbb{Y} \to \mathbb{Z} \Rightarrow h : \mathbb{X} \to \mathbb{Z}$

$$Dh(x)[\Delta x] = Dg(\underbrace{f(x)}_{y})[\underbrace{Df(x)[\Delta x]}_{\Delta y}]$$

$\boxed{2}$ $D\left(Tr(X)\right)[\Delta X] = Tr(\Delta X)$

So $D\left(Det(2I + x x^T)\right)[\Delta x] = \begin{cases} D(Det(Y))[\Delta Y] & (1) \\ D(2I + x x^T)[\Delta x] & (2) \end{cases}$

(2) : $(x + \Delta x)(x^T + \Delta x^T) = x x^T + \underbrace{\Delta x x^T + x \Delta x^T}_{D(x x^T[\Delta x])} + \Delta x \Delta x^T \Rightarrow$

$D(2I + x x^T)[\Delta x] = x \Delta x^T + x^T \Delta x$

(1) $D(Det(Y))[\Delta Y] = Det(Y) Tr(Y^{-1} \Delta Y) = Det(2I + x x^T) Tr(\underbrace{\left(2I + x x^T\right)^{-1}}_{A = A^T}[x \Delta x^T + x^T \Delta x]) = Det(2I + x x^T)[Tr\left(A x \Delta x^T\right) + Tr\left(A \Delta x x^T\right)] =$ {`we could rearange the order inside trace and transpose, because` $A = A^T$ $= \underbrace{(2I + x x^T)}_{constant}[2Tr(x^T A \Delta x)] = Tr\left(2Det(2I + x x^T) x^T A \Delta x\right) = Tr(\nabla f(x)^T \Delta x)$

$$\boxed{\nabla f(x) = 2Det(2I + x x^T)(2I + x x^T) x}$$

Example 6: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(x) = \log Det(x), \quad \nabla f(x) = ?$

$$Df(x)[\Delta x] = \lim_{t \to +0} \frac{f(x + t \cdot \Delta x) - f(x)}{t}$$

$$D\left(\log Det(x)\right)[\Delta x] = \begin{cases} D \log Y[\Delta Y] & (1) \\ \Delta Y = D Det(X)[\Delta X] & (2) \end{cases}$$

(1) $D \log Y[\Delta Y] = Y^{-1} \cdot \Delta Y = Det(X)^{-1} \cdot Det(X) \cdot Tr(X^{-1} \Delta X) = Tr(X^{-1} \cdot \Delta X)$

(2) $DDet(X)[\Delta X] = \lim\limits_{t\to+0} \frac{Det(X+t\cdot\Delta X)-Det(X)}{t} = \lim\limits_{t\to+0} \frac{Det(X\cdot[I+t\cdot X^{-1}\cdot\Delta X])-Det(X)}{t} = \lim\limits_{t\to+0} \frac{Det(X)\cdot[Det(I+t\cdot X^{-1}\cdot\Delta X)-1]}{t} = \lim\limits_{t\to+0} \frac{Det(X)\cdot[\exp(\log(Det(I+t\cdot X^{-1}\cdot\Delta X)))-1]}{t} = \lim\limits_{t\to+0} \frac{Det(X)\cdot\log(Det(I+t\cdot X^{-1}\cdot\Delta X))}{t} = \{\texttt{for small } \epsilon \texttt{ holds } : Det(I+\epsilon\cdot A) \approx 1+\epsilon\cdot Tr(A)+\bar{o}(\epsilon)\} = \lim\limits_{t\to+0} \frac{Det(X)\cdot\log(1+t\cdot Tr(X^{-1}\Delta X))}{t} = \lim\limits_{t\to+0} \frac{Det(X)\cdot t\cdot Tr(X^{-1}\Delta X)}{t} = \boxed{Det(X)\cdot Tr(X^{-1}\Delta X)}$

$D(\log Det(X))[\Delta X] = Tr(X^{-1}\Delta X) = Tr(\nabla f(X)^T\Delta X) \Rightarrow \boxed{\nabla(\log Det(X)) = X^{-T}}$

P.S: $\boxed{\nabla_X \log Det(X^{-1}) = -\nabla_X \log Det(X) = -X^{-T}}$

Example 7: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = a^T X a$

$$f(X+\Delta X) = \underbrace{a^T X a}_{f(X)} + \underbrace{a^T \Delta X a}_{Df(X)[\Delta X]}$$

$$Df(X)[\Delta X] = Tr(a^T\Delta X a) = Tr(aa^T\Delta X) = Tr(\nabla f(X)^T\Delta X)$$

$$\boxed{\nabla_X(a^T X a) = aa^T}$$

Application: Let's consider the problem of finding the maximum likelyhood estimators for the multidimensional Normal distribution: $x_i \sim \mathcal{N}(\mu, \Sigma)$

$$p(x|\,\mu,\Sigma) = \frac{1}{\sqrt{Det(2\pi\Sigma)}} \exp(-\tfrac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu))$$

The Likelyhood function will look tike that: $L(\mu,\Sigma) = \prod\limits_{i=1}^{n} \frac{1}{\sqrt{Det(2\pi\Sigma)}} \exp(-\frac{1}{2}(x_i-\mu)^T\Sigma^{-1}(x_i-\mu)) = Det(2\pi\Sigma)^{-\frac{n}{2}} \exp(-\frac{1}{2}\sum\limits_{i=1}^{n}(x_i-\mu)^T\Sigma^{-1}(x_i-\mu)) \to \max\limits_{\mu,\Sigma}$

$$\log L = -\frac{n^2}{2}\log Det(2\pi) - \frac{n}{2}\log Det(\Sigma) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^T\Sigma^{-1}(x_i-\mu) \to \max_{\mu,\Sigma}$$

$$\nabla_\mu \log L = -\frac{1}{2}\sum_{i=1}^{n} 2\Sigma^{-1}(x_i-\mu) = 0 \Rightarrow \boxed{\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

$\nabla_\Sigma \log L = \{\texttt{for convenience let } \Lambda = \Sigma^{-1}\} = \nabla_\Lambda \left(-\frac{n}{2}\log Det\Lambda^{-1} - \frac{1}{2}\sum\limits_{i=1}^{n}(x_i-\mu)^T\Lambda(x_i-\mu)\right) = \frac{n}{2}\underbrace{\Lambda^{-T}}_{\Lambda^{-1}} - \frac{1}{2}\sum\limits_{i=1}^{n}(x_i-\mu)(x_i -$

$\mu)^T = 0 \Rightarrow \hat{\Lambda}^{-1} = \boxed{\hat{\Sigma} = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \mu)(x_i - \mu)^T}$

Application: Find the maximum of a function $f(X) = Det(X)^{-1}\exp(-\frac{1}{2}Tr\left(X^{-1} \cdot A\right))$

Necessary condition – $\nabla_X f(X) = 0$:

First let's consider, that the $argmax_X f(X) = argmax_X \log f(X)$

$\log f(X) = -\log Det(X) - \frac{1}{2}Tr(X^{-1}A)$

$DTr(-\frac{1}{2}X^{-1}A) = Tr(\frac{1}{2}X^{-1}\Delta X X^{-1}A) = Tr(\frac{1}{2}X^{-1}AX^{-1}\Delta X) \Rightarrow \nabla_X = \frac{1}{2}X^{-T}A^{-T}X^{-T}$

$\nabla_X f(X) = -X^{-T} + \frac{1}{2}X^{-T}A^{-T}X^{-T} = -X^{-T}\left(I - \frac{1}{2}A^{-T}X^{-T}\right) = 0$

$A^{-T}X^{-T} = 2I \Rightarrow X^TA^T = \frac{1}{2}I \Rightarrow \boxed{X^\star = \dfrac{1}{2}A^{-1}}$

Example 8: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \|x\|_2^3, \quad \nabla f(x) = ?$

$f(x) = (x^Tx)^{3/2} \Rightarrow \nabla f(x) = \frac{3}{2}(x^Tx)^{1/2}2x = 3(x^Tx)^{1/2}x$

Example 9: $f : \mathbb{R}^{nxn} \to \mathbb{R}^{nxn}, \quad f(X) = X^{-1}, \quad Df(X)[\Delta X] = ?$

$D(X^{-1})[\Delta X] = \lim\limits_{t\to+0}\frac{(X+t\cdot\Delta X)^{-1}-X^{-1}}{t} = \lim\limits_{t\to+0}\frac{(X\cdot[I+X^{-1}\cdot t\cdot\Delta X])^{-1}-X^{-1}}{t} = \{(AB)^{-1} = B^{-1}A^{-1}\} = \lim\limits_{t\to+0}\frac{(I+X^{-1}\cdot t\Delta X)^{-1}X^{-1}-X^{-1}}{t} = \lim\limits_{t\to+0}\frac{[(I+X^{-1}\cdot t\cdot\Delta X)^{-1}-I]\cdot X^{-1}}{t} = \{(I + \epsilon \cdot A)^b - I \approx \epsilon \cdot b \cdot A\} = \lim\limits_{t\to+0}\frac{-X^{-1}t\Delta X X^{-1}}{t} = -X^{-1}\Delta X X^{-1}$

$\boxed{D(X^{-1})[\Delta X] = -X^{-1}\Delta X X^{-1}}$

Example 10: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = Tr(X), \quad \nabla f(X) = ?$

$D(Tr(X))[\Delta X] = \lim\limits_{t\to+0}\frac{Tr(X+t\cdot\Delta X)-Tr(X)}{t} = \lim\limits_{t\to+0}\frac{Tr(X+t\cdot\Delta X-X)}{t} = \lim\limits_{t\to+0}\frac{t\cdot Tr(\Delta X)}{t} = Tr(I \cdot \Delta X) = Tr(\nabla f(X)^T\Delta X) \Rightarrow \boxed{\nabla_X Tr(X) = I}$

Example 11: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = Tr(AX^{-1}B), \quad \nabla f(X) = ?$

$$DTr(AX^{-1}B)[\Delta X] = \begin{cases} DTr(Y)[\Delta Y] = Tr(\Delta Y) \\ D(AX^{-1}B)[\Delta X], \quad (1) \end{cases}$$

$$(1) \quad D(AX^{-1}B)[\Delta X] = \begin{cases} D(AZB)[\Delta Z], \quad (2) \\ \Delta Z = D(X^{-1})[\Delta X] = -X^{-1}\Delta X X^{-1} \end{cases}$$

$$f(Z+\Delta Z) = \underbrace{AZB}_{f(Z)} + \underbrace{A\Delta Z B}_{Df(Z)[\Delta Z]}$$

So $DTr(AX^{-1}B)[\Delta X] = Tr(-AX^{-1}\Delta X X^{-1}B) = Tr(\underbrace{-X^{-1}BAX^{-1}}_{\nabla f(X)^T}\Delta X)$

$$\nabla_X f(X) = (-X^{-1}BAX^{-1})^T = -X^{-T}A^TB^TX^{-T}$$

Example 12 : $f : \mathbb{R}^n \to \mathbb{R}^{nxn}, \quad f(x) = xx^T, \quad Df(x)[\Delta x] = ?$

$$f(x+\Delta x) = (x+\Delta x)(x+\Delta x)^T = \underbrace{xx^T}_{f(x)} + \underbrace{x\Delta x^T + \Delta x x^T}_{Df(x)[\Delta x]} + \underbrace{\Delta x \Delta x^T}_{\bar{o}(\|\Delta x\|)}$$

$$Df(x)[\Delta x] = x\Delta x^T + (\underbrace{x\Delta x^T}_{\texttt{symmetric}})^T = 2x\Delta x^T$$

Example 13: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \frac{x^TAx}{x^Tx}, \quad \nabla f(x) = ?$

$$\nabla f(x) = \nabla\left(\frac{g(x)}{h(x)}\right) = \frac{\nabla g(x)\cdot h(x) - g(x)\cdot \nabla h(x)}{h^2(x)} = \frac{(A+A^T)x\cdot x^Tx - 2x^TAx\cdot x}{(x^Tx)^2} = \frac{(A+A^T)x^Tx - 2x^TAx\cdot I}{(x^Tx)^2}\cdot x$$

Example 14: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = Det(AXB), \quad \nabla f(X) = ?$

$$D\left(Det(AXB)\right)[\Delta X] = \begin{cases} DDet(Y)[\Delta Y] = Det(Y)Tr(Y^{-1}\Delta Y) \\ \Delta Y = D(AXB)[\Delta X] = A\Delta X B \end{cases}$$

$$Df(X)[\Delta X] = \underbrace{Det(AXB)}_{const} Tr\left((AXB)^{-1}A\Delta X B\right) = Det(AXB)Tr(B^{-1}X^{-1}A^{-1}A\Delta XB) = Tr(Det(AXB)X^{-1}\Delta X) = Tr(\nabla f(X)^T\Delta X)$$

$$\nabla_X Det(AXB) = Det(AXB)X^{-T}$$

Example 15: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = Tr(AX^{-1})Tr(XB), \quad \nabla f(X) = ?$

Let's apply the property of matrix differentiation:

$h(x) = g(x) \cdot f(x), \quad f : \mathbb{X} \to \mathbb{Y}, \quad g : \mathbb{X} \to \mathbb{R}$, then:

$Dh(x)[\Delta x] = Dg(x)[\Delta x] \cdot f(x) + g(x) \cdot Df(x)[\Delta x]$

$D\left(Tr(AX^{-1})Tr(XB)\right)[\Delta X] = Tr(XB) \cdot \underbrace{DTr(AX^{-1})[\Delta X]}_{(1)} + Tr(AX^{-1}) \cdot \underbrace{DTr(XB)[\Delta X]}_{(2)}$

$$(1) \quad DTr(AX^{-1})[\Delta X] = \begin{cases} DTr(Z)[\Delta Z] = Tr(\Delta Z) \\ \Delta Z = D(AY)[\Delta Y] = A\Delta Y \\ \Delta Y = D(X^{-1})[\Delta X] = -X^{-1}\Delta X X^{-1} \end{cases}$$

$DTr(AX^{-1})[\Delta X] = Tr(-AX^{-1}\Delta X X^{-1})$

(2) $\quad DTr(XB) = Tr(\Delta XB)$, so

$Df(X)[\Delta X] = \underbrace{Tr(XB)}_{const} \cdot Tr(-AX^{-1}\Delta X X^{-1}) + \underbrace{Tr(AX^{-1})}_{const} \cdot Tr(\Delta XB) = Tr(-Tr(XB)X^{-1}AX^{-1}\Delta X + Tr(AX^{-1}B\Delta X)) = Tr([-Tr(XB)X^{-1}AX^{-1} + Tr(AX^{-1})B] \cdot \Delta X) = Tr(\nabla f(X)^T \Delta X)$

$\nabla f(X) = Tr(AX^{-1})B - Tr(XB)X^{-1}AX^{-1}$

Example 16: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = Det(\exp(X)), \quad \nabla f(X) = ?$

$$DDet(\exp(X))[\Delta X] = \begin{cases} DDet(Y)[\Delta Y] = Det(Y)Tr(Y^{-1}\Delta Y) \\ \Delta Y = D\exp(X)[\Delta X] = \exp(X)\Delta X - \texttt{prove it !} \end{cases}$$

$DDet(\exp(X))[\Delta X] = Det(\exp(X))Tr(exp(X)^{-1}\exp(X)\Delta X) = Tr(Det(\exp(X))\Delta X) = Tr(\nabla f(X)^T \Delta X)$

$$\boxed{\nabla_X \left(Det(\exp(X))\right) = Det(\exp(X)) \cdot I}$$

Example 17: $f : \mathbb{R}^{mxn} \to \mathbb{R}, \quad f(X) = \frac{1}{2}\|X - A\|_F^2, \quad \nabla f(X) = ?$

$\boxed{\|A\|_F^2 = \sum\limits_{i,j} a_{ij}^2 = Tr(A^T A)}$, so $f(X) = \frac{1}{2}Tr\left((X - A)^T(X - A)\right) = \frac{1}{2}Tr(X^T X - X^T A - A^T X + A^T A)$

$Df(X)[\Delta X] = \frac{1}{2}Tr(X^T \Delta X + \Delta X^T X - \Delta X^T A - A^T \Delta X) = Tr(X^T \Delta X) + Tr(-A^T \Delta X) = Tr(\left(X^T - A^T\right) \cdot \Delta X) = Tr(\nabla f(X)^T \Delta X)$

$\nabla f(X) = X - A$

Example 18: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = Tr(Axx^T), \quad \nabla f(x) = ?$

$Df(x)[\Delta x] = Tr(D(Axx^T)[\Delta x]) = Tr(Ax\Delta x^T + A\Delta x x^T) = Tr([x\Delta x^T]^T A^T) + Tr(x^T A\Delta x) = Tr(x^T A^T \Delta x + x^T A \Delta x) = Tr(\nabla f(x)^T \Delta x)$

$\nabla f(x) = x^T \left(A + A^T\right)$

Example 19: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \frac{1}{2}\|xx^T - A\|_F^2, \quad \nabla f(x) = ?$

$f(x) = \frac{1}{2}Tr((xx^T - A)^T(xx^T - A)) = \frac{1}{2}Tr(xx^T xx^T - xx^T A - Axx^T - A^T A)$

$Df(x)[\Delta x] = \frac{1}{2}Tr(4x^T xx^T \Delta x - x^T A\Delta x - x^T A^T \Delta x)$

$\nabla f(x) = [2x^T xx^T - \frac{1}{2}x^T(A + A^T)]^T = [x^T \left(2xx^T - \frac{1}{2}A - \frac{1}{2}A^T\right)]^T = (2xx^T - \frac{1}{2}(A + A^T))x$

Example 20: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \frac{1}{2}\left(x^T x + (a^T x)^2\right), \quad \nabla f(x) = ?$

$\nabla f(x) = x + a^T xa$

Example 21: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = x^T \vec{\log}(x), \nabla f(x) = ?,$ `where` $\vec{\log}(x) = (\log x_1, \dots, \log x_n)$

$f(x) = x^T \log(x) = \sum\limits_{i=1}^{n} x_i \cdot \log(x_i), \quad \nabla f(x) = \left(\frac{\partial f(x_1)}{\partial x_1}, \dots, \frac{\partial f(x_n)}{\partial x_n}\right)$
$\frac{\partial f(x_i)}{\partial x_i} = \log(x_i) + 1 \Rightarrow \nabla f(x) = \vec{\log}(x) + \vec{\mathbf{1}}$

Example 22: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = Det(A + X^{-1}), \quad \nabla f(X) = ?$

$$DDet(A + X^{-1})[\Delta X] = \begin{cases} DDet(Y)[\Delta Y] = Det(Y)Tr(Y^{-1}\Delta Y) \\ \Delta Y = D(A + X^{-1})[\Delta X] = -X^{-1}\Delta X X^{-1} \end{cases}$$

$Df(X)[\Delta X] = Det(A + X^{-1})Tr(-(A + X^{-1})^{-1}X^{-1}\Delta X X^{-1}) = Tr\left(-Det(A + X^{-1})X^{-1}(A + X^{-1})^{-1}X^{-1}\Delta X\right) = Tr(\nabla f(X)^T \Delta X)$

$\nabla f(X) = -Det(A + X^{-1})X^{-T}(A + X^{-1})^{-T}X^{-T}$

Example 23: (The problem of constructing surrogat eigenvector)

The problem comes from the fact, that the linear system $Ax = b$ can equalently be rewritten as $f(x) = \frac{1}{2}x^T Ax - b^T x \to \min\limits_x$, where equalence means that the solution of both problems would be the same.

$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \frac{1}{2}x^T (A - \lambda \cdot I)\, x \to \min\limits_x$, where λ is an eigenvalue of A, $A = A^T > 0$

$f(x) \to \min\limits_x \Rightarrow \nabla_x f(x) = 0$

$\nabla f(x) = (A - \lambda \cdot I)x = 0$, from this follows, that a minimiser of this function is an eigenvector of a matrix A, corresponding to an eigenvalue λ, but if we can't find an eigenvector, we can use this gradient to find it iteratively and numerically:

Initialisation: x_0 – some random vector,

$x_{k+1} = x_k - \alpha_k \cdot \nabla f(x_k) = ((1 + \alpha_k \cdot \lambda) \cdot I - \alpha_k \cdot A)\, x_k = \{\texttt{if } \alpha_k \texttt{ is constant, then }\} = ((1 + \alpha \cdot \lambda) \cdot I - \alpha \cdot A)^{k+1} \cdot x_0$

Application: (Logistic Regression fitting)

$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = -\sum\limits_{i=1}^{n} \log(\sigma(w^T x_i y_i)) \to \min\limits_w, \texttt{where } \sigma(z) = \frac{1}{1+\exp(-z)}, \quad x_i, w \in \mathbb{R}^n, \quad y_i \in \mathbb{R}$

$\sigma(z)'_z = \frac{\exp(-z)}{(1+\exp(-z))^2} = \frac{1}{(1+\exp(z))^2} = \sigma^2(-z)$

$\nabla_x f(x) = -\sum\limits_{i=1}^{n} \frac{\sigma^2(-w^T x_i y_i)}{\sigma(w^T x_i y_i)} y_i x_i$

Example 24 Find minimum of a function $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \frac{x^T Ax}{x^T Bx}$, where $A \geq 0, B \geq 0$

Necessary condition $\nabla f(x) = 0$

$\nabla f(x) = \frac{(A+A^T)x \cdot x^T Bx - x^T Ax \cdot (B+B^T)x}{(x^T Bx)^2}$

Example 25: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = a^T X^{-1} a, \quad \nabla f(X) =?$

$D\left(a^T X^{-1} a\right)[\Delta X] = Tr(aa^T DX^{-1}[\Delta X]) = Tr(-aa^T X^{-1} \Delta X X^{-1}) = Tr(-X^{-1} aa^T X^{-1} \Delta X) = Tr(\nabla f(X)^T \Delta X) \Rightarrow \nabla f(X) = -X^{-T} aa^T X^{-T}$

Example 26: $f:\mathbb{R}^{nxn}\to\mathbb{R},\quad f(X)=Det(X^2),\quad \nabla f(X)=?$

There are 2 ways of dealing with it, let's compare these ways:

1) $Det(X^2) = Det(X)^2 \Rightarrow DDet(X)^2[\Delta X] = \begin{cases} DY^2[\Delta Y]=Y\Delta Y+\Delta Y Y \quad (1) \\ \Delta Y = DDet(X)[\Delta X]=Det(X)Tr(X^{-1}\Delta X)\end{cases}$

(1) $Df(X)[\Delta X] = 2Det(X)\cdot Det(X)Tr(X^{-1}\Delta X) \Rightarrow \nabla f(X) = 2Det(X)^2X^{-T}$

2)$Det(X^2),\quad DDet(X^2)[\Delta X]=\begin{cases} DDet(Y)[\Delta Y]=Det(Y)Tr(Y^{-1}\Delta Y) \\ \Delta Y = D\left(X^2\right)[\Delta X]=X\Delta X+\Delta X X\end{cases}$

$Df(X)[\Delta X]=Det(X^2)Tr(2X^{-2}X\Delta X)=2Det(X)^2Tr(X^{-1}\Delta X)$, which is indeed the same result as in the first way

Example 27: $f:\mathbb{R}^{nxn}\to\mathbb{R},\quad f(X)=Tr(AX^{-1}),\quad \nabla f(X)=?$

$Df(X)[\Delta X]=\begin{cases} DTr(AY)[\Delta Y]=Tr(A\Delta Y) \\ \Delta Y = DX^{-1}[\Delta X]=-X^{-1}\Delta X X^{-1}\end{cases}$

$DTr(AX)[\Delta X]=Tr(-AX^{-1}\Delta X X^{-1})=Tr(-X^{-1}AX^{-1}\Delta X)$

$\nabla f(X)=-X^{-T}A^{-T}X^{-T}$

Application: Kernel Linear Regression Problem

$Q(a)=\frac{1}{2}\|\Phi\Phi^Ta-y\|_2^2+\frac{\lambda}{2}a^T\Phi\Phi^Ta\to\min\limits_a,\quad K=\Phi\Phi^T=K^T>0$

$\nabla_a Q=\left(\Phi\Phi^T\right)^2a-\Phi\Phi^Ty+\lambda\Phi\Phi^Ta=0\Big|\cdot K^{-1}\Rightarrow Ka-y+\lambda a=0$

$(K+\lambda I)\,a=y\Rightarrow \boxed{a^\star=(K+\lambda I)^{-1}y}$

Example 28: $f:\mathbb{R}^{n}\to\mathbb{R},\quad f(X)=\log Det(X)+Tr(X^{-1}A)\to\min\limits_X$

$D\log Det(X)[\Delta X]=Tr(X^{-1}\Delta X)\Rightarrow\nabla\log Det(X)=X^{-T}$

$DTr(X^{-1}A)[\Delta X] = Tr(-X^{-1}\Delta X X^{-1}A) = Tr(-X^{-1}AX^{-1}\Delta X) \Rightarrow \nabla Tr(X^{-1}A)=-X^{-T}A^TX^{-T}$

$\nabla f(X) = X^{-T} - X^{-T}A^TX^{-T} = 0 \Rightarrow X^{-T}\left(I - A^TX^{-T}\right) = 0$

$X^{-1}A = I \Rightarrow X = A$

Application: Robust linear regression

$\sum\limits_{i=1}^{n} w_i(< x_i, \beta > -y_i)^2 = (X\beta - y)^T W (X\beta - y)$

$Q(\beta) = (X\beta - y)^T W (X\beta - y) \to \min\limits_{\beta}$

$Q(\beta) = \beta^T X^T W X\beta - 2y^T W X\beta + y^T W y$

$\nabla_\beta Q = 2X^T W X\beta - 2X^T W y = 0$

$\boxed{\beta^* = \left(X^T W X\right)^{-1} X^T W y}$ `where W is a weight matrix`

Application: Available GLS estimator

Problem: $y = X\beta + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \Omega) \to$ `Efficient` $\beta^{GLS} =?$

$\epsilon = y - X\beta \sim \mathcal{N}(0, \Omega)$

$p(\epsilon) = \frac{1}{\sqrt{Det(2\pi\Omega)}} \exp\{-\frac{1}{2}(y - X\beta)^T\Omega^{-1}(y - X\beta)\}$

Let's find the ML estimator of Ω:

$L(\Omega) = \prod\limits_{i=1}^{l} \frac{1}{\sqrt{Det(2\pi\Omega)}} \exp\{-\frac{1}{2}(y - X\beta)_i^T\Omega^{-1}(y - X\beta)_i\} = \left(\frac{1}{\sqrt{Det(2\pi\Omega)}}\right)^l \cdot$
$\exp\{-\frac{1}{2}\sum\limits_{i=1}^{l}(y - X\beta)_i^T\Omega^{-1}(y - X\beta)_i\} \to \max\limits_{\Omega}$

$\log L(\Omega) = -\frac{l \cdot n}{2}\log 2\pi - \frac{l}{2}\log Det(\Omega) - \frac{1}{2}\sum\limits_{i=1}^{l}(y - X\beta)_i^T\Omega^{-1}(y - X\beta)_i \to \max\limits_{i=1}^{l}$

$\{\Lambda = \Omega^{-1}\} \to \nabla_\Lambda \log L(\Lambda) = -\frac{l}{2}\underbrace{\Lambda^{-T}}_{\Lambda^{-1}} - \frac{1}{2}\sum\limits_{i=1}^{l}(y - X\beta)_i(y - X\beta)_i^T = 0$

$\Lambda^{-1} = \frac{1}{l}\sum\limits_{i=1}^{l}(y - X\beta)_i(y - X\beta)_i^T \Rightarrow \boxed{\hat{\Omega} = \frac{1}{l}\sum_{i=1}^{l}(y - X\beta)_i(y - X\beta)_i^T}$

$\beta^{GLS} = \left(X^T\hat{\Omega}^{-1}X\right)^{-1} X^T\hat{\Omega}^{-1}y$

Application: Consider the linear model $y = X\beta + \epsilon$, $\epsilon \sim \mathcal{N}(0, \sigma^2 \cdot I)$. Let's call the $MSE(\hat{\beta}) = \mathbb{E}\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)^T$. Compute the $MSE(\hat{\beta})$, where $\hat{\beta} = Ay$

$$\hat{\beta} - \beta = AX\beta + A\epsilon - \beta = (AX - I)\beta + A\epsilon$$

$$\mathbb{E}\left(A\epsilon\epsilon^T A^T\right) = A\underbrace{\mathbb{E}(\epsilon\epsilon^T)}_{\sigma^2 \cdot I} A^T = \sigma^2 AA^T$$

$$MSE(\hat{\beta}) = \mathbb{E}\left[((AX - I)\beta + A\epsilon)\left(\beta^T(AX - I)^T + \epsilon^T A^T\right)\right] = (AX - I)\beta\beta^T(AX - I)^T + \mathbb{E}\left(A\epsilon\epsilon^T A^T\right) = \boxed{(AX - I)\beta\beta^T(AX - I)^T + \sigma^2 AA^T}$$

Consider the problem of minimizing $tr(MSE(\hat{\beta}))$, find the optimal matrix A:

$$f(A) = tr((AX - I)\beta\beta^T(AX - I)^T + \sigma^2 AA^T) \to \min_A$$

$$f(A) = tr\left(AX\beta\beta^T X^T A^T - 2AX\beta\beta^T + \beta\beta^T + \sigma^2 AA^T\right)$$

$$trD\left(ABA^T\right)[\Delta A] = tr\left(\Delta ABA^T + AB\Delta A^T\right) = tr\left(BA^T\Delta A + \Delta AB^T A^T\right) = tr\left(BA^T\Delta A + B^T A^T \Delta A\right) = tr\left((B + B^T)A^T\Delta A\right) \Rightarrow \boxed{\nabla_A tr(ABA^T) = A(B + B^T)}$$

$$trD(AB)[\Delta A] = tr(\Delta AB) \Rightarrow \boxed{\nabla_A tr(AB) = B^T}$$

$$trD(AA^T)[\Delta A] = tr(A\Delta A^T + \Delta AA^T) = tr(2A^T\Delta A) \Rightarrow \boxed{\nabla_A tr(AA^T) = 2A}$$

$$\nabla_A f(A) = 2AX\beta\beta^T X^T - 2\beta\beta^T X^T + 2\sigma^2 A = 0$$

$$A\left(X\beta\beta^T X^T + \sigma^2 I\right) = \beta\beta^T X^T$$

$$\boxed{A^* = \beta\beta^T X^T\left(X\beta\beta^T X^T + \sigma^2 I\right)^{-1}}$$

Example 29: Prove, that $f(x) = a^T x + x^T Ax \geq -\frac{1}{4}a^T Aa$, where $A = A^T > 0$

$$\nabla_x\left(a^T x + x^T Ax\right) = a + 2Ax = 0 \Rightarrow x^* = -\frac{1}{2}A^{-1}a$$

So $f(x^*) = -\frac{1}{2}a^T A^{-1}a = \frac{1}{4}a^T A^{-1}AA^{-1}a = -\frac{1}{4}a^T A^{-1}a$

x^* is a minimum, because: $\nabla^2_{x^2} f(x) = 2A > 0$

Example 30: $f(X) = tr(AX^{-1}), \quad \nabla_X f(X) =?$

$Df(X)[\Delta X] = \lim\limits_{t\to+0} \frac{tr(A[X+t\Delta X]^{-1}) - tr(AX^{-1})}{t} = \lim\limits_{t\to+0} \frac{tr\left(A[(X+t\Delta X)^{-1} - X^{-1}]\right)}{t} = \lim\limits_{t\to+0} \frac{tr(A[(I+tX^{-1}\Delta X)^{-1}X^{-1} - X^{-1}])}{t} = \lim\limits_{t\to+0} \frac{tr(A[(I+tX^{-1}\Delta X)^{-1} - I]X^{-1})}{t} = \lim\limits_{t\to+0} \frac{tr(A(-tX^{-1}\Delta X)X^{-1})}{t} = -tr(AX^{-1}\Delta X X^{-1}) = -tr(X^{-1}AX^{-1}\Delta X) \Rightarrow$ $\boxed{\nabla_X f(X) = -X^T A^T X^{-T}}$

Example 31: $f(X) = a^T X X^T a, \quad \nabla_X f(X) =?$

$f(X + \Delta X) = a^T(X + \Delta X) \cdot (X^T + \Delta X^T)a = a^T X X^T a + \underbrace{a^T \Delta X X^T a + a^T X \Delta X^T a}_{Df(X)[\Delta X]} + o(\|\Delta X\|)$

So $Df(X)[\Delta X] = a^T(X\Delta X^T + \Delta X X^T)a = tr(a^T \Delta X X^T a) + tr(aa^T X \Delta X^T) = tr(a^T \Delta X X^T a) + tr(\Delta X X^T aa^T) = 2tr(a^T \Delta X X^T a) = 2tr(X^T aa^T \Delta X) \Rightarrow \boxed{\nabla_X f(X) = 2aa^T X}$

Higher order derivatives

When $f : \mathbb{R}^n \to \mathbb{R} \Rightarrow \boxed{D^2 f(x)[\Delta x_1, \Delta x_2] = \Delta x_1^T H(x) \Delta x_2}$, where $H(x)$ – *Hessian*

$D^k f(x)[\Delta x_1, \ldots, \Delta x_k] = \frac{\partial^k}{\partial t_1 \cdot \partial t_2 \cdot \cdots \cdot \partial t_k}\Big|_{t_1 = \cdots = t_k = 0} f(x + t_1 \cdot \Delta x_1 + \cdots + t_k \cdot \Delta x_k)$

Example 1: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = x^T A x, \quad D^2 f(x)[\Delta x_1, \Delta x_2] = ?, \quad H(x) = ?$

$Df(x)[\Delta x_1] = 2x^T A \Delta x_1$

$D^2 f(x)[\Delta x_1, \Delta x_2] = \lim\limits_{t \to +0} \frac{Df(x + t \cdot \Delta x_2)[\Delta x_1] - Df(x)[\Delta x_1]}{t} = \lim\limits_{t \to +0} \frac{2(x^T + t \cdot \Delta x_2^T) A \Delta x_1 - 2x^T A \Delta x_1}{t} = \boxed{2\Delta x_2^T A \Delta x_1} \Rightarrow \boxed{H(x) = 2A}$

Example 2: $f : \mathbb{R}^{nxn} \to \mathbb{R}, \quad f(X) = \log Det(X), \quad D^2 f(X[\Delta X_1, \Delta X_2] = ?$

$Df(X)[\Delta X_1] = Tr(X^{-1} \Delta X_1)$

$D^2 f(X)[\Delta X_1, \Delta X_2] = \lim\limits_{t \to +0} \frac{Df(X + t \cdot \Delta X_2)[\Delta X_1] - Df(X)[\Delta X_1]}{t} = \lim\limits_{t \to +0} \frac{Tr((X + t \cdot \Delta X_2)^{-1}[\Delta X_1]) - Tr(X^{-1}[\Delta X_1])}{t} = \lim\limits_{t \to +0} \frac{Tr\Big([(X + t \cdot \Delta X_2)^{-1} - X^{-1}][\Delta X_1]\Big)}{t} = \lim\limits_{t \to +0} \frac{Tr\Big(([X(I + t \cdot X^{-1} \Delta X_2)]^{-1} - X^{-1})[\Delta X_1]\Big)}{t} = \lim\limits_{t \to +0} \frac{Tr\Big([(I + t \cdot X^{-1} \Delta X_2)^{-1} - I] X^{-1} \Delta X_1\Big)}{t} = \lim\limits_{t \to +0} \frac{Tr(-t \cdot X^{-1} \Delta X_2 X^{-1} \Delta X_1)}{t} = Tr\left(-\Delta X_1 X^{-1} \Delta X_2 X^{-1}\right)$

Example 3: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(\beta) = \|X\beta - y\|_2^2, \quad H(\beta) = ?$

$Df(\beta)[\Delta\beta_1] = \left(2X^T X \beta - 2X^T y\right)^T [\Delta\beta_1]$

$D^2 f(\beta)[\Delta\beta_1, \Delta\beta_2] = \lim\limits_{t \to +0} \frac{[2(\beta^T + t \cdot \Delta\beta_2^T) X^T X - 2y^T X - 2\beta^T X^T X + 2y^T X][\Delta\beta_1]}{t} = \Delta\beta_2^T 2X^T X \Delta\beta_1 = \beta_1^T 2X^T X \Delta\beta_2 \Rightarrow \boxed{H(\beta) = 2X^T X}$

Which we could have attained easier: $\nabla_\beta \nabla_\beta f(\beta) = \nabla_\beta \left(2X^T X \beta\right) = 2X^T X$

Constraint optimisation

Example 1: $\begin{cases} \|x\|_2^2 \to \min\limits_x \\ \texttt{s.t.} Ax \leq b \end{cases}$

$L = x^T x + \lambda^T(Ax - b)$

K.K.T conditions:

$$\begin{cases} \nabla_x L = 2x + A^T\lambda = 0 \\ Ax - b \leq 0 \\ \lambda^T(Ax - b) = 0 \\ \lambda \geq 0 \end{cases} \Rightarrow \begin{cases} x^\star = -\frac{1}{2}A^T\lambda \\ \lambda^T(-\frac{1}{2}AA^T\lambda - b) = 0 \Rightarrow \lambda^\star = -2(AA^T)^{-1}b \end{cases}$$

$\boxed{x^\star = A^T(AA^T)^{-1}b}$

Application: OLS under constraints:

$$\begin{cases} \|X\beta - y\|_2^2 \to \min\limits_\beta \\ \texttt{s.t.} C\beta = d \end{cases}$$

$L = \beta^T X^T X\beta - 2y^T X\beta + y^T y + \mu^T(C\beta - d)$

K.K.T. conditions:

$$\begin{cases} \nabla_\beta L = 2X^T X\beta - 2X^T y + C^T\mu = 0 \\ C\beta - d = 0 \end{cases} \Rightarrow \begin{cases} \beta = (X^T X)^{-1}(X^T y - \frac{1}{2}C^T\mu) \\ C\beta = d \end{cases}$$

$C\beta = d \Rightarrow C(X^T X)^{-1}X^T y - \frac{1}{2}C(X^T X)^{-1}C^T\mu = d$

$\frac{1}{2}C(X^T X)^{-1}C^T\mu = C(X^T X)^{-1}X^T y - d$, $\mu = 2 \cdot \left(C(X^T X)^{-1}C^T\right)^{-1}\left(C(X^T X)^{-1}X^T y - d\right)$

$$\boxed{\beta^* = \underbrace{(X^T X)^{-1}X^T y}_{\beta^{\texttt{ols}}} - (X^T X)^{-1}C^T\left(C(X^T X)^{-1}C^T\right)^{-1}\left(C\underbrace{(X^T X)^{-1}X^T y}_{\beta^{\texttt{ols}}} - d\right)}$$

Let's derive $V(\beta^*)$:

$V(\beta^*) = \sigma^2\left(X^T X\right)^{-1} - \sigma^2 \cdot (X^T X)^{-1}C^T\left(C(X^T X)^{-1}C^T\right)^{-1}C(X^T X)^{-1} \cdot$
$\cdot((X^T X)^{-1}C^T\left(C(X^T X)^{-1}C^T\right)^{-1}C)^T = \sigma^2\left(X^T X\right)^{-1} - \sigma^2 \cdot$

$(X^TX)^{-1}C^T\left(C(X^TX)^{-1}C^T\right)^{-1}C(X^TX)^{-1}C^T\left(C(X^TX)^{-1}C^T\right)^{-1}C(X^TX)^{-1} =$
$\sigma^2(X^TX)^{-1} - \sigma^2\left(X^TX\right)^{-1}\left(C(X^TX)^{-1}C^T\right)^{-1}C(X^TX)^{-1} =$
$\boxed{V(\beta^*) = \sigma^2(X^TX)^{-1}\left(I - \left(C(X^TX)^{-1}C^T\right)^{-1}C(X^TX)^{-1}\right)}$

When C is invertable this could be reduced to:

$\sigma^2(X^TX)^{-1}\left(I - \left(C(X^TX)^{-1}C^T\right)^{-1}C(X^TX)^{-1}C^TC^{-T}\right) =$
$\boxed{V(\beta^*) = \sigma^2(X^TX)^{-1}\left(I - C^{-T}\right)}$

Example 2: $\begin{cases} a^Tx \to \min\limits_x \\ \texttt{s.t.}\, x^TAx \le 1 \end{cases}$, `where` $A = A^T > 0$

$L = a^Tx + \lambda(x^TAx - 1)$

K.K.T. conditions:

$$\begin{cases} \nabla_x L = a + 2\lambda Ax = 0 \\ x^TAx - 1 \le 0 \\ \lambda(x^TAx - 1) = 0 \\ \lambda \ge 0 \end{cases} \Rightarrow \begin{cases} x^\star = -\frac{1}{2\lambda}A^{-1}a \\ \lambda((-\frac{1}{2\lambda}A^{-1}a)^TA(-\frac{1}{2\lambda}A^{-1}a) - 1) = 0 \quad (1) \end{cases}$$

(1) $\frac{1}{4\lambda}a^TA^{-1}a - \lambda = 0\Big| \cdot \lambda \Rightarrow \lambda^\star = \pm\frac{1}{2}\sqrt{a^TA^{-1}a}, \quad \lambda \ge 0 \Rightarrow \lambda^\star = \frac{1}{2}\sqrt{a^TA^{-1}a}$

$$\boxed{x^\star = -\frac{A^{-1}a}{\sqrt{a^TA^{-1}a}}}$$

Example 3: $\begin{cases} x^TQx \to \min\limits_x \\ \texttt{s.t.}\, \|Ax - b\|_2^2 \le 1 \end{cases}$, `where` $Q = Q^T > 0, A = A^T > 0$

$L = x^TQx + \lambda\left((Ax-b)^T(Ax-b) - 1\right) = x^TQx + \lambda\left(x^TA^TAx - 2b^TAx + b^Tb - 1\right)$

K.K.T. conditions:

$$\begin{cases} \nabla_x L = 2Qx + 2\lambda A^TAx - 2\lambda A^Tb = 0 \\ \lambda \ge 0 \\ \lambda\left(x^TA^TAx - 2b^TAx + b^Tb - 1\right) = 0 \\ \|Ax - b\|_2^2 \le 1 \end{cases} \Rightarrow$$

$$\begin{cases} \boxed{x^\star = \lambda\left(Q + \lambda A^TA\right)^{-1}A^Tb} \\ \lambda \ge 0 \\ \lambda\left(x^TA^TAx - 2b^TAx + b^Tb - 1\right) = 0 \\ \|Ax - b\|_2^2 \le 1 \end{cases}$$

$\lambda^\star$ one can find from the dual function, which will look a bit complex here

Application: (2-rank update)

$$\begin{cases} \|B - B_k\|_F^2 \to \min\limits_B \\ \quad \texttt{s.t.} Bs_k = y_k \end{cases}$$

$$L = Tr(B^T B) + \mu^T (Bs_k - y_k)$$

K.K.T. conditions:

$$\begin{cases} \nabla_B L = 2B + \mu s_k^T = 0 \\ \quad Bs_k = y_k \end{cases} \Rightarrow B = -\frac{1}{2}\mu s_k^T$$

Dual function $q(\mu) = Tr(\frac{1}{4}s_k \mu^T \mu s_k^T) - \frac{1}{2}\mu^T \mu s_k^T s_k - \mu^T y_k \to \min\limits_\mu$

$\nabla_\mu q(\mu) = \frac{1}{2}s_k^T s_k \mu - s_k^T s_k \mu - y_k = 0, \quad \mu^\star = \frac{-2y_k}{s_k^T s_k}$

So $\boxed{B^\star = \dfrac{y_k s_k^T}{s_k^T s_k}}$

Example 4:

$$\begin{cases} \frac{1}{2}x^T A x + b^T x + c \to \min\limits_x \\ \quad \texttt{s.t.} x^T x \leq 1 \end{cases} \texttt{where } A = A^T > 0$$

$$L = \frac{1}{2}x^T A x + b^T x + c + \lambda(x^T x - 1)$$

K.K.T. conditions:

$$\begin{cases} \nabla_x L = Ax + b + \lambda x = 0 \\ x^T x - 1 \leq 0 \\ \lambda(x^T x - 1) = 0 \\ \lambda \geq 0 \end{cases} \Rightarrow \begin{cases} \boxed{x^\star = -\left(A + I \cdot \lambda\right)^{-1} b} \\ \lambda b^T (A + \lambda \cdot I)^{-T} (A + \lambda \cdot I)^{-1} b - \lambda = 0, \quad (\star) \end{cases}$$

$(\star)\lambda(b^T(A + \lambda \cdot I)^{-2} b - 1) = 0 \Rightarrow \lambda^\star = \max\{0, \texttt{solution of } (\star)\}$

Example 5: (Projection on the radius 1 Ball)

$$\begin{cases} \frac{1}{2}\|x - v\|_2^2 \to \min\limits_x \\ \quad \texttt{s.t.} x^T x \leq 1 \end{cases}$$

$$L = \frac{1}{2}\left(x^T x - 2v^T x + v^T v\right) + \lambda(x^T x - 1)$$

K.K.T. conditions:

$$\begin{cases} \nabla_x L = x - v + 2\lambda x = 0 \\ \lambda \geq 0 \\ \lambda(x^T x - 1) = 0 \\ x^T x - 1 \leq 0 \end{cases} \Rightarrow \begin{cases} x^\star = \frac{v}{1+2\lambda} \\ \lambda \left(\frac{v^T v}{(1+2\lambda)^2} - 1 \right) = 0 \quad (\star) \end{cases}$$

$(\star) 1 + 2\lambda = \sqrt{v^T v} = \|v\|_2$

So $\boxed{x^\star = \dfrac{v}{\|v\|_2}}$, which is quite intuitive

Application: (Hard margin SVM)

$$\begin{cases} \frac{1}{2}\|w\|_2^2 \to \min\limits_w \\ y_i \left(x_i^T w - b \right) \geq 1 \end{cases}$$

$L = \frac{1}{2} w^T w - \sum\limits_{i=1}^n \lambda_i \left(y_i \left(x_i^T w - b \right) - 1 \right)$

K.K.T. conditions:

$$\begin{cases} \nabla_w L = w - \sum\limits_{i=1}^n \lambda_i y_i x_i = 0 \\ \nabla_b L = \sum\limits_{i=1}^n \lambda_i y_i = 0 \\ \lambda_i \geq 0, \forall i = 1, \ldots, n \\ \lambda_i \left(y_i \left(x_i^T w - b \right) - 1 \right) = 0, \forall i = 1, \ldots, n \end{cases}$$

$$\boxed{w^\star = \sum_{i=1}^n \lambda_i y_i x_i}$$

Application: Consider the multiple regression model $y = X \cdot \beta + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$, lets propose a linear estimator of the form $\hat{\beta} = L \cdot y$. Find the unbiased estimator $\hat{\beta}$, for which $Tr(Var(\hat{\beta})) \to \min\limits_\beta$

$$\begin{cases} Tr(Var(Ly)) \to \min\limits_L \\ \mathbb{E}\left(\hat{\beta}\right) = \beta \end{cases}$$

$\mathcal{L} = Tr(Var\left(L\left(X\beta + \epsilon\right)\right)) + \mu^T \left(\mathbb{E}\left(\hat{\beta}\right) - \beta \right) = Tr\left(Var(L\epsilon)\right) + \mu^T \left(LX\beta - \beta \right) = Tr\left(\sigma^2 L L^T\right) + \mu^T \left(LX\beta - \beta \right)$

$$DTr\left(LL^T\right)[\Delta L] = Tr\left(L \Delta L^T + L^T \Delta L\right) = Tr(\nabla f(L)^T \Delta L) \Rightarrow \nabla_L = 2L$$

$$D\mu^T L X\beta[\Delta L] = \mu^T \Delta L X \beta = Tr\left(X\beta\mu^T \Delta L\right) \Rightarrow \nabla_L = \mu \left(X\beta\right)^T = \mu\beta^T X^T$$

K.K.T. conditions: $\begin{cases} \nabla_L \mathcal{L} = 2\sigma^2 L + \mu\beta^T X^T = 0 \\ LX\beta = \beta \end{cases} \Rightarrow$

$$\begin{cases} L^\star = -\frac{\mu}{2\sigma^2}\beta^T X^T \\ -\frac{\mu}{2\sigma^2}\beta^T X^T X\beta = \beta \quad (\star) \end{cases}$$

$$(\star) \quad -2\sigma^2\mu\beta^T X^T X = I \Rightarrow \mu\beta^T = -2\sigma^2\left(X^T X\right)^{-1} \Rightarrow L^\star = \left(X^T X\right)^{-1} X^T$$

$$\boxed{\hat{\beta} = \left(X^T X\right)^{-1} X^T y}$$

P.S: The same problem, but for $\epsilon \sim \mathcal{N}(0, \Sigma)$

$$\begin{cases} Tr\left(Var(Ly)\right) \to \min\limits_L \\ \mathbb{E}\left(Ly\right) = \beta \end{cases}$$

$$\mathcal{L} = Tr(Var(LX\beta + L\epsilon)) + \mu^T\left(LX\beta - \beta\right) = Tr(L\Sigma L^T) + \mu^T\left(LX\beta - \beta\right)$$

$$\nabla_L\left(Tr(L\Sigma L^T)\right) = 2L\Sigma^T = 2L\Sigma$$

$$DTr(L\Sigma L^T)[\Delta L] = Tr(L\Sigma\Delta L^T + \Delta L\Sigma L^T) = Tr(2\Sigma L^T\Delta L) = Tr(\nabla f(L)^T\Delta L)$$

K.K.T. conditions:

$$\begin{cases} \nabla_L\mathcal{L} = 2L\Sigma + \mu\beta^T X^T = 0 \\ LX\beta = \beta \end{cases} \Rightarrow \begin{cases} L^\star = -2\mu\beta^T X^T\Sigma^{-1} \\ -2\mu\beta^T X^T\Sigma^{-1}X\beta = \beta \quad (\star) \end{cases}$$

$$(\star) \quad -2\mu\beta^T X^T\Sigma^{-1}X = I \Rightarrow \mu\beta^T = -\tfrac{1}{2}\left(X^T\Sigma^{-1}X\right)^{-1}$$

$$L^\star = \left(X^T\Sigma^{-1}X\right)^{-1} X^T\Sigma^{-1}$$

$$\boxed{\hat{\beta} = \left(X^T\Sigma^{-1}X\right)^{-1} X^T\Sigma^{-1}y}$$

Application (Principle Component Analysis)

Finding the 1st component:

$$\begin{cases} \|Xa\|_2^2 \to \max\limits_a \\ \texttt{s.t.}\, \|a\|_2^2 = 1 \end{cases}$$

$$L = a^T X^T Xa + \mu\left(a^T a - 1\right)$$

K.K.T. conditions: $\begin{cases} \nabla_a L = 2X^T Xa + 2\mu a = 0 \quad (1) \\ a^T a = 1 \end{cases}$

From the (1) it follows that, $X^T Xa = -\mu a \Rightarrow a$ is an eigenvector of $X^T X$. Let's call the corresponding eigenvalue λ, then the problem $\|Xa\|_2^2 \to \max\limits_a$ would take the form: $a^T X^T X a = a^T \lambda a = \lambda \cdot a^T a = \lambda \underbrace{\|a\|_2^2}_{1} \to \max \Rightarrow \lambda =$ maximum eigenvalue of $X^T X$, so the answer would be $a =$ the eigenvector, corresponding to the maximum eigenvalue of matrix $X^T X$

Finding the k^{th} principal component:

$$\begin{cases} \|Xa_k\|_2^2 \to \max\limits_{a_k} \\ < a_k, a_i >= 0, \forall i \neq k \\ \|a_k\|_2^2 = 1 \end{cases}$$

$$L = a^T X^T X a + \mu \left(a^T a - 1\right) + \sum_{i=1}^{k-1} \gamma_i a_k^T a_i$$

$\nabla_{a_k} L = 2X^T X a_k + 2\mu a_k + \sum\limits_{i=1}^{k-1} \gamma_i a_i = 0 \Big| \cdot a_k^T \Rightarrow 2a_k^T X^T X a_k + 2\mu a_k^T + \underbrace{\sum\limits_{i=1}^{k-1} \gamma_i a_k^T a_i}_{0} = 0 \Rightarrow a_k$ is an eigenvector of $X^T X$, corresponding to the next biggest eigenvalue.

Application: Consider the model $y = X\beta + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Find efficient unbiased quadratic estimator of σ^2

Quadratic estimator takes the form: $\hat{\sigma}^2 = y^T A y$, so the problem can be formalised as follows:

$$\begin{cases} Var(y^T A y) \to \min\limits_{A} \\ \mathbb{E}(y^T A y) = \sigma^2 \end{cases}$$

$\mathbb{E}(y^T A y) = \mathbb{E}((X\beta + \epsilon)^T A (X\beta + \epsilon)) = \beta^T X^T A X \beta + \mathbb{E}\left(\epsilon^T A \epsilon\right) = \beta^T X^T A X \beta + \sigma^2 tr(A) = \sigma^2 \Rightarrow \begin{cases} X^T A X = (0) \\ tr(A) = 1 \end{cases}$

$$\left\{ \mathbb{E}\left(\epsilon^T A \epsilon\right) = \mathbb{E}(tr(\epsilon^T A \epsilon)) = \mathbb{E}(tr(A\epsilon\epsilon^T)) = tr(A \underbrace{\mathbb{E}(\epsilon\epsilon^T)}_{\sigma^2 I}) = \sigma^2 tr(A) \right\}$$

$Var(y^T A y) = Var(\beta^T X^T A X \beta + \underbrace{2\beta^T X^T A \epsilon}_{\texttt{A would be symm}} + \epsilon^T A \epsilon) = 4\sigma^2 \beta^T X^T A^2 X \beta + 2\sigma^4 tr(A^2)$

$L = 4\sigma^2\beta^T X^T A^2 X\beta + 2\sigma^4 tr(A^2) + tr(\Lambda \cdot X^T AX) + \mu \cdot (tr(A) - 1)$

$DL[\Delta A] = 4\sigma^2\beta^T X^T (\Delta AA + A\Delta A) X\beta + 2\sigma^4 tr(\Delta AA + A\Delta A) + tr(\Lambda X^T \Delta AX) + \mu tr(\Delta A) = tr\left([8\sigma^2 X\beta\beta^T X^T A + 4\sigma^4 A + X\Lambda X^T + \mu I]\Delta A\right) = tr(\nabla_A L^T \Delta A)$

$\nabla_A L = 8\sigma^2 AX\beta\beta^T X^T + 4\sigma^4 A + X\Lambda^T X^T + \mu I$

K.K.T conditions:

$$\begin{cases} \nabla_A L = 0 \\ X^T AX = (0) \\ tr(A) = 1 \end{cases} \Rightarrow \left\{ 4\sigma^2 A\left(2X\beta\beta^T X^T + \sigma^2 I\right) = \left(X\Lambda^T X^T + \mu I\right) \right.$$

Example 6: Find the extremum of a function $f(x) = \frac{x^T Ax}{x^T Bx}$, where A and B>0 are symmetric matrices

We can reduce this problem to the constrained optimisation problem:

$$\begin{cases} x^T Ax \to extr_x \\ \texttt{s.t.}\, x^T Bx = 1 \end{cases}$$

$L = x^T Ax + \mu\left(x^T Bx - 1\right)$

As $B > 0 \quad B = B^{1/2} \cdot B^{1/2}$, let's denote $B^{1/2} \cdot x = y$, so the problem will transform to:

$$\begin{cases} y^T B^{1/2} AB^{1/2} y \to extr_y \\ y^T y = 1 \end{cases}$$

$L = y^T B^{1/2} AB^{1/2} y + \mu(y^T y - 1)$

K.K.T conditions:

$$\begin{cases} \nabla_y L = 2B^{1/2} AB^{1/2} y + 2\mu y = 0 \\ y^T y = 1 \end{cases} \Rightarrow y^* \texttt{ eigenvec of } B^{1/2} AB^{1/2} \texttt{ with } \lambda$$

So the our function takes the form: $\lambda y^T y \to extr_y \Rightarrow \lambda \to extr$

The minimum of a function is attained in $y_1 : B^{1/2} AB^{1/2} y = \lambda_{min} y$, and the maximum is attained in $y : B^{1/2} AB^{1/2} y = \lambda_{max} y$

Example 7:

$$\begin{cases} tr(A^TA) & -2tr(A) \to \min\limits_A \\ \texttt{s.t.}\, AX & = 0 \end{cases}$$

$L = tr(A^TA) - 2tr(A) + tr(\Lambda^T AX) = tr(A^TA - 2A + \Lambda^T AX)$

K.K.T. conditions:
$\begin{cases} \nabla_A L & = 2A - 2I + \Lambda X^T = 0 \\ AX & = 0 \end{cases} \Rightarrow A^* = I - \frac{1}{2}\Lambda X^T$

Dual problem: $q(\Lambda) = tr[(I - \frac{1}{2}\Lambda X^T)^T(I - \frac{1}{2}\Lambda X^T)] - 2tr(I - \frac{1}{2}\Lambda X^T) + tr(\Lambda^T X - \frac{1}{2}\Lambda^T\Lambda X^T X) = tr[-I + X^T\Lambda - \frac{1}{4}\Lambda X\Lambda^T\Lambda X^T] \to \max\limits_\Lambda$

$\nabla_\Lambda q = X - \frac{1}{2}\Lambda X^T X = 0, \Rightarrow \Lambda^* = 2X(X^TX)^{-1}$

$\boxed{A^* = I - X(X^TX)^{-1}X^T}$

Example 8: $f(x) = \frac{(a^Tx)^2}{x^TBx} \to \max\limits_x, B > 0$

This could be represented as a constrained optimisation problem:

$$\begin{cases} a^Tx \to \max\limits_x \\ \texttt{s.t.} \quad x^TBx = 1 \end{cases}$$

$L = a^Tx + \lambda\left(x^TBx - 1\right)$

K.K.T. conditions:
$\begin{cases} \nabla_x L = a + 2\lambda \cdot Bx = 0 \\ x^TBx = 1 \end{cases} \Rightarrow \begin{cases} x^* = -\frac{1}{2\lambda}B^{-1}a \\ \frac{1}{4\lambda^2}a^TB^{-1}BB^{-1}a = 1 \end{cases}$

Dual function: $q(\lambda) = -\frac{1}{2\lambda}a^TB^{-1}a + \lambda\left(\frac{1}{4\lambda^2}a^TB^{-1}a - 1\right) = -\frac{1}{4\lambda}a^TB^{-1}a - \lambda \to \max\limits_\lambda$

$q'_\lambda = \frac{1}{4\lambda^2}a^TB^{-1}a - 1 = 0 \Rightarrow \lambda^* = \frac{\sqrt{a^TB^{-1}a}}{2}$

$x^* = \frac{B^{-1}a}{\sqrt{a^TB^{-1}a}} \Rightarrow f(x^*) = \frac{a^TB^{-1}a}{\frac{a^TB^{-1}BB^{-1}a^T}{a^TB^{-1}a}} = \boxed{a^TB^{-1}a}$

Example 9: Proove that $f(y) = \frac{y^TAy}{(y^Tx)^2} \geq (x^TA^{-1}x)^{-1}$

$f(y) \to \min\limits_y$ is equal to:
$$\begin{cases} y^TAy \to \min\limits_y \\ \texttt{s.t.}\, y^Tx = 1 \end{cases}$$

$L = y^T A y + \lambda(x^T y - 1)$

K.K.T. conditions:

$$\begin{cases} \nabla_y L = 2Ay + \lambda x = 0 \\ x^T y = 1 \end{cases} \Rightarrow \begin{cases} y^* = -\frac{\lambda}{2} A^{-1} x \\ -\frac{\lambda}{2} x^T A^{-1} x = 1 \end{cases}$$

$\lambda^* = -\frac{2}{x^T A^{-1} x} \Rightarrow \boxed{y^* = \dfrac{A^{-1}x}{x^T A^{-1} x}}$

So $f(y^*) = \frac{\frac{x^T A^{-1} x}{(x^T A^{-1} x)(x^T A^{-1} x)}}{\left(\frac{x^T A^{-1} x}{x^T A^{-1} x}\right)^2} = \left(x^T A^{-1} x\right)^{-1}$

Example 10: Let A be an m x n matrix of rank r. Let $\delta_1, \dots, \delta_r$ be the singular values of A (that is the square roots of the non-zero eigenvalues of AA^T) and let $\delta = \delta_1 + \cdots + \delta_r$. Proove, that $-\delta \le tr(AX) \le \delta$ for every n x m matrix X satisfying $X^T X = I_m$

Constrained extremum problem:

$$\begin{cases} tr(AX) \to extr_X \\ \texttt{s.t.} X^T X = I_m \end{cases}$$

$L = tr(AX) + tr(\Lambda^T X^T X - \Lambda^T)$

Since the matrix $X^T X$ is symmetric $\Rightarrow \Lambda$ should also be symmetric, so:

K.K.T. conditions:

$$\begin{cases} \nabla_X L = A^T + 2X\Lambda = 0 \\ X^T X = I_m \end{cases} \Rightarrow \begin{cases} X^* = -\frac{1}{2} A^T \Lambda^{-1} \\ \frac{1}{4} \Lambda^{-1} A A^T \Lambda^{-1} = I_m (*) \end{cases}$$

$(*) \Lambda^2 = \frac{1}{4} A A^T \Rightarrow \boxed{\Lambda^* = \pm \dfrac{1}{2} \sqrt{AA^T}}$

$\boxed{X^* = \pm A^T (AA^T)^{-\frac{1}{2}}}$

To sum up:

$tr(AX^*) = tr(AA^T (AA^T)^{-\frac{1}{2}}) = tr(\sqrt{AA^T})$ which results the proof

Example 11:

$$\begin{cases} \|y-\theta\|_2^2 \to \min\limits_{\theta} \\ \texttt{s.t.} A\theta = 0 \end{cases}, \quad A = A^T \geq 0$$

$L = y^T y - 2y^T\theta + \theta^T\theta - \lambda^T A\theta$

K.K.T. conditions:

$$\begin{cases} \nabla_\theta L = -2y + 2\theta - A^T\lambda = 0 \\ A\theta = 0 \end{cases} \Rightarrow \begin{cases} \theta^* = y + \frac{1}{2}A^T\lambda \\ Ay + \frac{1}{2}AA^T\lambda = 0(*) \end{cases}$$

$(*)\lambda^* = -2(AA^T)^{-1}Ay \Rightarrow \theta^* = y - A^T(AA^T)^{-1}Ay = \left(I - A^T(AA^T)^{-1}A\right)y$

$$\boxed{\theta^* = \left(I - A^T(AA^T)^{-1}A\right)y}$$

Example : $\begin{cases} x^T Ax \to \min\limits_{x} \\ \texttt{s.t.} \|x\|_2^2 \leq 1 \end{cases}$, `where` $A = A^T > 0$

$L = x^T Ax + \lambda(x^T x - 1)$

K.K.T conditions:

$\begin{cases} \nabla_x L = 2Ax + 2\lambda x = 0 \\ x^T x \leq 1 \\ \lambda(x^T x - 1) = 0 \\ \lambda \geq 0 \end{cases} \Rightarrow Ax = -\lambda x$, so x should be the eigenvector of matrix A for example with eigenvalue γ, then the dual function:

Other

Example 1: Find $\mathbb{E}\left(x^T x\right)$, where $x \sim \mathcal{N}(\mu, \Sigma)$

$\mathbb{E}x = \mu, \quad \mathbb{E}\left(x-\mu\right)\left(x-\mu\right)^T = \Sigma$

$\mathbb{E}\left(x-\mu\right)\left(x-\mu\right)^T = \mathbb{E}xx^T - 2\mu^T\mathbb{E}x + \mathbb{E}\mu^T\mu = \mathbb{E}xx^T - 2\mu\mu^T + \mu\mu^T = \mathbb{E}xx^T - \mu\mu^T = \Sigma \Rightarrow \mathbb{E}xx^T = \Sigma + \mu\mu^T$

$\mathbb{E}x^T x = tr\left(\mathbb{E}x^T x\right) = \mathbb{E}\left(tr(x^T x)\right) = \mathbb{E}\left(tr(xx^T)\right) = tr\left(\mathbb{E}xx^T\right) = tr\left(\Sigma + \mu\mu^T\right) = tr\Sigma + \mu^T\mu$

$\boxed{\mathbb{E}x^T x = tr\Sigma + \mu^T\mu}$

Here we used the fact, that $tr(aa^T) = a^T a$

Example 2 Find the ML estimator for the Σ parameter of Wishart distribution.

Wishart distribution: $M \in \mathbb{R}^{PxP}$ $p(M|\Sigma, m) = \underbrace{\dfrac{1}{2^{\frac{mP}{2}}\pi^{\frac{P(P-1)}{4}}\prod\limits_{p=1}^{P}\Gamma[\frac{1}{2}(m+1-p)]}}_{const \quad of \quad \Sigma} \cdot det(\Sigma)^{-\frac{m}{2}} \cdot det(M)^{\frac{m-P-1}{2}} \cdot \exp[-\frac{1}{2}tr(\Sigma^{-1}M)]$

$L(\Sigma) = \prod\limits_{i=1}^{n} Const \cdot det(\Sigma)^{-\frac{m}{2}} \cdot det(M_i)^{\frac{m-P-1}{2}} \cdot \exp[-\frac{1}{2}tr(\Sigma^{-1}M_i)] \to \max\limits_{\Sigma}$

$\log L(\Sigma) = -\frac{mn}{2}\log det(\Sigma) + \frac{m-P-1}{2}\sum\limits_{i=1}^{n}\log det(M_i) - \frac{1}{2}\sum\limits_{i=1}^{n} tr\left(\Sigma^{-1}M_i\right) \to \max\limits_{\Sigma}$

$\Lambda = \Sigma^{-1} \to \log L(\Lambda) = \frac{mn}{2}\log det(\Lambda) + \dots - \frac{1}{2}\sum\limits_{i=1}^{n} tr(\Lambda M_i)$

$\nabla_\Lambda \log L = \frac{mn}{2}\Lambda^{-1} - \frac{1}{2}\sum\limits_{i=1}^{n} M_i^T = 0$

$mn\Sigma = \sum\limits_{i=1}^{n} M_i^T \Rightarrow \boxed{\hat{\Sigma} = \dfrac{1}{mn}\sum_{i=1}^{n} M_i^T}$

Application: R^2 representation

$R^2 = \frac{\sum\limits_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum\limits_{i=1}^{n}(y_i - \bar{y})^2} = 1 - \frac{\sum\limits_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum\limits_{i=1}^{n}(y_i - \bar{y})^2}$, let's consider, that we're working with the standartised data: $\bar{y} = 0, \sigma_y = 1 \Rightarrow \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 = \sigma_y^2 = 1$, so

$R^2 = 1 - e^T e = 1 - (X\beta - y)^T (X\beta - y) = 1 - \beta^T X^T X\beta + 2y^T X\beta - \underbrace{y^T y}_{\sigma_y^2=1} =$ $2y^T X\beta - \beta^T X^T X\beta = \{\beta^{\texttt{ols}} = (X^T X)^{-1} X^T y\} = 2y^T X(X^T X)^{-1} X^T y - y^T X(X^T X)^{-1} X^T X(X^T X)^{-1} X^T y = y^T X(X^T X)^{-1} X^T y = \{V(\beta^{\texttt{ols}}) = \sigma_\epsilon^2 (X^T X)^{-1} = \Sigma_\beta\} = \frac{1}{\sigma_\epsilon^2} y^T X \underbrace{\Sigma_\beta}_{\texttt{Grammian matrix}} X^T y$

Let's recall, that in the ortonormal basis $\{e\}$ the scale product of 2 vectors $\vec{a}$ and $\vec{b}$, could be repsresented as: $\boxed{< a, b >_e = a^T b}$.

If the basis $\{f\}$ is not an ortonormal one, it would be represented as: $\boxed{< a, b >_f = a^T \Gamma b}$, where $\Gamma = \begin{pmatrix} < f_1, f_1 > & \dots & < f_1, f_n > \\ \vdots & \ddots & \vdots \\ < f_n, f_1 > & \dots & < f_n, f_n > \end{pmatrix}$ – Grammian matrix of the basis vectors.

So $R^2 = \frac{1}{\sigma_\epsilon^2} \left(X^T y\right)^T \Sigma_\beta \left(X^T y\right) = \frac{1}{\sigma_\epsilon^2} \|X^T y\|_\beta^2$ could be interpreted as a squared norm of a $X^T y$ vector in the space of parameters β: $\mathcal{L} = L\{\beta_1, \dots, \beta_k\}$

Example R^2 representation

$R^2 = 1 - \frac{ESS}{TSS} = 1 - \frac{e^T e}{(y-\bar{y})^T (y-\bar{y})}$

$e^T e = (y - X\beta)^T (y - X\beta) = (y - X(X^T X)^{-1} X^T y)^T (y - X(X^T X)^{-1} X^T y) =$ $y^T \underbrace{\left(I - X(X^T X)^{-1} X^T\right)^2}_{idempotent} y = y^T (I - X(X^T X)^{-1} X^T) y$

$R^2 = 1 - \frac{y^T (I - X(X^T X)^{-1} X^T) y}{y^T y - 2\bar{y}^T y + n \cdot (\bar{y})^2} = \frac{y^T (X(X^T X)^{-1} X^T) y - 2\bar{y}^T y + n \cdot (\bar{y})^2}{y^T y - 2\bar{y}^T y + n \cdot (\bar{y})^2} =$ $1 - \frac{(y + \bar{y} \cdot \vec{1})^T (I - X(X^T X)^{-1} X^T)(y + \bar{y} \cdot \vec{1})}{y^T y}$

Example: $x \sim \mathcal{N}(\mu, \Omega)$, find $\mathbb{V}(x^T A x), \quad A = A^T$

Theorem: if $y \sim \mathcal{N}(\mu, \Sigma) \Rightarrow M_y(t) = \exp\{t^T \mu + \frac{1}{2} t^T \Sigma t\}$, where $M_y(t)$ – is a moment generating function.

Proof: $M_y(t) = \frac{1}{\sqrt{Det(2\pi\Sigma)}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\{t^T y - \frac{1}{2}(y - \mu)^T \Sigma^{-1} (y - \mu)\} dy = \frac{1}{\sqrt{Det(2\pi\Sigma)}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\{t^T \mu + \frac{1}{2} t^T \Sigma t -$

$\frac{1}{2}(y - \mu - \Sigma t)^T\Sigma^{-1}(y - \mu - \Sigma t)\}dy = \exp\{t^T\mu + \frac{1}{2}t^T\Sigma\} \cdot \underbrace{\frac{1}{\sqrt{Det(2\pi\Sigma)}}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\exp\{-\frac{1}{2}\left(y-(\mu+\Sigma t)\right)^T\Sigma^{-1}\left(y-(\mu+\Sigma t)\right)\}dy}_{1} =$

$$\boxed{\exp\{t^T\mu + \frac{1}{2}t^T\Sigma t\}}$$

Equialent actions will lead to:

$$\boxed{M_{x^TAx}(t) = |I - 2tA\Sigma|^{-1/2}\exp\{-\frac{1}{2}\mu^T[I-(I-2tA\Sigma)^{-1}]\Sigma^{-1}\mu\}}$$

Other way:

$Var(x^TAx) = cov(x^TAx, x^TBx)$, where $B = A$

$cov(x^TAx, x^TBx) = \mathbb{E}\left(x^TAxx^TBx\right) - \mathbb{E}\left(x^TAx\right)\cdot\mathbb{E}\left(x^TBx\right)$

$\mathbb{E}(x^TAx) = tr(A\mathbb{E}xx^T) = tr(A\cdot(\Sigma+\mu\mu^T))$

.

Example: $\log Det(A) = tr\left(\log A\right)$

Proof: $tr(A) = \sum\limits_{j=1}^{d}\lambda_j, \quad Det(A) = \prod\limits_{j=1}^{d}\lambda_j, \quad \lambda_j$ – j-th eigenvalue of matrix A

$\log Det(A) = \log\left(\prod\limits_{j=1}^{d}\lambda_j\right) = \sum\limits_{j=1}^{d}\log\lambda_j = \{\log x = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots\} = \sum\limits_{j=1}^{d}\sum\limits_{k=1}^{+\infty}(-1)^{k-1}\cdot\frac{1}{k}\cdot\lambda_j^k$

$tr\left(\log A\right) = tr\left(\sum\limits_{k=1}^{+\infty}(-1)^{k-1}\cdot\frac{1}{k}\cdot A^k\right) = \sum\limits_{k=1}^{+\infty}(-1)^{k-1}\cdot\frac{1}{k}\cdot tr(A^k) = \sum\limits_{k=1}^{+\infty}(-1)^{k-1}\cdot\frac{1}{k}\cdot\sum\limits_{j=1}^{d}\lambda_j^k = \sum\limits_{j=1}^{d}\sum\limits_{k=1}^{+\infty}(-1)^{k-1}\cdot\frac{1}{k}\cdot\lambda_j^k$

Example: $Det(e^A) = e^{tr(A)}, \quad A = A^T$

Proof: $Det(e^A) = Det\left(T\cdot\begin{pmatrix}e^{\lambda_1} & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdot & e^{\lambda_d}\end{pmatrix}\cdot T^{-1}\right) = Det(T)\cdot Det(T^{-1})\cdot$

$\prod_{j=1}^{d} e^{\lambda_j} = e^{\sum_{j=1}^{d} \lambda_j} = e^{tr(A)}$

Example: Find Eigvals(xx^T), where $x = (x_1, \ldots, x_n)$

Let's look at the 4x4 case and find some patterns:

$$xx^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ x_1x_2 & x_2^2 & x_2x_3 & x_2x_4 \\ x_1x_3 & x_2x_3 & x_3^2 & x_3x_4 \\ x_1x_4 & x_2x_4 & x_3x_4 & x_4^2 \end{pmatrix} \rightarrow$$

`Gauss transformations` $\rightarrow \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$ this shows us, that xx^T has $\lambda = 0$, whose geometry multyplicity is n-1. The left eigenvalue can be guessed from the 2x2 case and it is $\lambda = \sum_{i=1}^{n} x_i^2$

So Eigvals(xx^T) $= \begin{cases} 0, & AM = n-1 \\ \sum_{i=1}^{n} x_i^2, & AM = 1 \end{cases}$

Theorem

if $P = P^T$ and $P^2 = P \Rightarrow P \geq 0$

Proof: $x^T P x = x^T P^2 x = x^T \underbrace{P}_{P^T} P x = (Px)^T (Px) = \|Px\|_2^2 \geq \Rightarrow$ $x^T P x \geq 0$