Problem 2b. Calculate $L[x_0, \ldots, x_n;\ X^{n+2}]$.

Answer. With $L[f] := L[x_0, \ldots, x_n; f]$, we know that $L\left[X^{n+1}\right] = X^{n+1} - \prod_{i=0}^{n}(X - x_i)$. Let

$$Q_n := X^{n+2} - X\prod_{i=0}^{n}(X - x_i) - X^{n+1}\sum_{i=0}^{n} x_i.$$

Q_n is a polynomial of degree at most n. It follows that

$$\begin{aligned} Q_n = L\left[Q_n\right] &= L\left[X^{n+2} - X\prod_{i=0}^{n}(X - x_i) - X^{n+1}\sum_{i=0}^{n} x_i\right] \\ &= L\left[X^{n+2}\right] - 0 - L\left[X^{n+1}\right]\sum_{i=0}^{n} x_i \\ &= L\left[X^{n+2}\right] - 0 - \left(X^{n+1} - \prod_{i=0}^{n}(X - x_i)\right)\sum_{i=0}^{n} x_i, \quad \text{hence} \end{aligned}$$

$$L\left[X^{n+2}\right] := X^{n+2} - X\prod_{i=0}^{n}(X - x_i) - \left(\prod_{i=0}^{n}(X - x_i)\right)\sum_{i=0}^{n} x_i \quad ✓$$

Problem 3.

Let $P_n : \mathbb{R} \longrightarrow \mathbb{R}$ be a polynomial of degree at most n, $n \in \mathbb{N}$.
Prove the partial fraction decomposition,

$$\frac{P_n(x)}{(x-x_0)\dots(x-x_n)} = \sum_{i=0}^{n} \frac{A_i}{x-x_i}.$$

Answer.

With $\ell(x) = (x-x_0)\dots(x-x_n)$, we have

$$\frac{P_n(x)}{\ell(x)} = \frac{L[x_0,\dots,x_n;\, P_n](x)}{\ell(x)} = \sum_{i=0}^{n} \frac{1}{x-x_i} \cdot \frac{P(x_i)}{\ell'(x_i)}.$$

Problem 4. Calculate the sum

$$\sum_{i=0}^{n} \frac{x_i^k}{(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)},$$

$k = 0, \dots, n.$

Answer.

$$\begin{aligned}
&\sum_{i=0}^{n} \frac{x_i^k}{(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)} \\
= \ & [x_0, \dots, x_n;\ x^k] \\
= \ & \begin{cases} 0, & \text{if } 0 \le k \le n-1; \\ 1, & \text{if } k = n. \end{cases}
\end{aligned}$$

Problem 5. Calculate

$$[x_0, \dots, x_n; x^{n+1}].$$

Answer. We have:

$$\begin{aligned}
& 0 \\
= \ & [x_0, \dots, x_n; (X - x_0) \dots (X - x_n)] \\
= \ & [x_0, \dots, x_n; X^{n+1} - (x_0 + \dots + x_n)X^n + \cdots] \\
= \ & [x_0, \dots, x_n; X^{n+1}] - (x_0 + \dots + x_n)[x_0, \dots, x_n; X^n] + 0 \\
= \ & [x_0, \dots, x_n; X^{n+1}] - (x_0 + \dots + x_n) \cdot 1.
\end{aligned}$$

Consequently

$$[x_0, \dots, x_n; X^{n+1}] = x_0 + \dots + x_n.$$

Problem 6. Calculate

$$[x_0, \dots, x_n; (X - x_0) \dots (X - x_{k-1})], \quad k \in \mathbb{N}.$$

Answer. For $k < n$, the degree of the polynomial $(X - x_0) \dots (X - x_{k-1})$ is at most $n - 1$, hence

$$[x_0, \dots, x_n; (X - x_0) \dots (X - x_{k-1})] = 0.$$

For $k = n$,

$$\begin{aligned} & [x_0, \dots, x_n; (X - x_0) \dots (X - x_{n-1})] \\ = \; & [x_0, \dots, x_n; X^n - \cdots] \\ = \; & [x_0, \dots, x_n; X^n] - 0 \\ = \; & 1. \end{aligned}$$

For $k > n$, the polynomial $(X - x_0) \dots (X - x_{k-1})$ takes zero value for $X = x_0, \dots, x_n$, hence

Problem 7. Prove that

$$L[x_0,\dots,x_n; L[x_1,\dots,x_n;f]] = L[x_1,\dots,x_n; L[x_0,\dots,x_n;f]]$$

Answer. Since the degree of the polynomial $L[x_1,\dots,x_n;f]$ is at most $n-1$, we have

$$L[x_0,\dots,x_n; L[x_1,\dots,x_n;f]] = L[x_1,\dots,x_n;f].$$

Next, consider the polynomial

$$P = L[x_1,\dots,x_n; L[x_0,\dots,x_n;f]].$$

We have:

$$\begin{aligned} P(x_i) &= L[x_1,\dots,x_n; L[x_0,\dots,x_n;f]](x_i) \\ &= L[x_0,\dots,x_n;f](x_i), \qquad i=1,\dots,n \\ &= f(x_i), \qquad i=1,\dots,n, \end{aligned}$$

hence

$$P = L[x_1,\dots,x_n;f].$$

Problem 8. Prove that, for $x \notin \{x_0, \dots, x_n\}$,

$$\left[x_0, \dots, x_n;\ \frac{f(t)}{x-t}\right]_t = \frac{L[x_0, \dots, x_n;\ f](x)}{(x-x_0)\dots(x-x_n)}.$$

Answer. With $\ell(x) = (x-x_0)\dots(x-x_n)$, we have

$$L[x_0, \dots, x_n;\ f](x) = \sum_{k=0}^{n} \frac{\ell(x)}{x-x_i} \cdot \frac{f(x_i)}{\ell'(x_i)},$$

hence

$$\frac{L[x_0, \dots, x_n;\ f](x)}{\ell(x)} = \sum_{k=0}^{n} \frac{\frac{f(x_i)}{x-x_i}}{\ell'(x_i)} = \left[x_0, \dots, x_n;\ \frac{f(t)}{x-t}\right]_t.$$

Problem 9. If $f(x) = \frac{1}{x}$ and $0 < x_0 < x_1 < \cdots < x_n$, then

$$[x_0, \ldots, x_n;\ f] = \frac{(-1)^n}{x_0 \ldots x_n}.$$

Solution 1. By using Problem 8 we have

$$\left[x_0, \ldots, x_n;\ \frac{1}{t-0}\right]_t = -\frac{L[x_0, \ldots, x_n;\ 1](0)}{(0-x_0)\ldots(0-x_n)} = \frac{(-1)^n}{x_0 \ldots x_n}.$$

Solution 2. We write the obvious equality

$$\left[x_0, \ldots, x_n; \frac{(t-x_0)\ldots(t-x_n)}{t}\right] = 0$$

in the form

$$\left[x_0, \ldots, x_n; t^n - (x_0 + \cdots + x_n)t^{n-1} + \cdots + (-1)^{n+1}\frac{x_0 \ldots x_n}{t}\right] = 0,$$

i.e.,

$$1 - (x_0 + \cdots + x_n)0 + \cdots + 0 + (-1)^{n+1}\, x_0 \ldots x_n \left[x_0, \ldots, x_n; \frac{1}{t}\right] = 0.$$

✔ ✔ ✔