SUBIECTE EXAMEN
BIBU ALEXANDRU-SERAFIM
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① $x^2+2x-1=0$

a)

$$(a_k, b_k, x_k) = \begin{cases} a_k = a_{k-1},\ b_k = b_{k-1},\ x_k = x_{k-1}, \text{ dacă } f(x_{k-1}) = 0 \\ a_k = a_{k-1},\ b_k = x_{k-1},\ x_k = \frac{a_{k-1}+b_{k-1}}{2}, \text{ dacă } f(a_{k-1}) f(x_{k-1}) < 0 \\ a_k = x_{k-1},\ b_k = b_{k-1},\ x_k = \frac{a_k + b_k}{2}, \text{ dacă } f(a_{k-1}) \cdot f(x_{k-1}) > 0 \end{cases}$$

$a_0 = 0 \quad b_0 = 1 \quad x_0 = 0{,}5$

b) $x^2+2x-1=0$

$\Delta = b^2 - 4ac = 4+4 = 8$

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm 2\sqrt{2}}{2} \Rightarrow \left.\begin{array}{l} x_1 = -1+\sqrt{2} \in [0,1] \\ x_2 = -1-\sqrt{2} \notin [0,1] \end{array}\right\} \Rightarrow$$

$\Rightarrow x^2+2x-1=0$ admite o singură soluție pe intervalul $[0,1]$

c) $k = \overline{0,3}$, $x_k = ?$

$$e_r = \frac{|x_k - x_{k-1}|}{|x_{k-1}|} = ?$$

$f(x) = x^2+2x-1 \quad f:[0,1] \to \mathbb{R}.$

f este polinomială $\Rightarrow f$ este continuă (1)

$$\left.\begin{array}{l} f(0) \cdot f(1) = -1 \cdot 2 = -2 < 0 \ (2) \\ (1) \end{array}\right\} \Rightarrow (\exists) x \in (a,b) \text{ a.î. } f(x) = 0 \Rightarrow \text{Putem aplica met. Bisecției}$$

x^2+2x-1

$(a_0, b_0, x_0) = (0, 1, 0{,}5)$

$f(0)\cdot f(0{,}5) = (-1)\left(\frac{1}{4}+1-1\right) = -\frac{1}{4} < 0 \Rightarrow (a_1, b_1, x_1) = \left(0, \frac{1}{2}, \frac{1}{4}\right)$

$f(0)\cdot f\left(\frac{1}{4}\right) = (-1)\underbrace{\left(\frac{1}{16}+\frac{1}{2}-1\right)}_{<0} > 0 \Rightarrow (a_2, b_2, x_2) = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{8}\right)$

$f\left(\frac{1}{4}\right)\cdot f\left(\frac{3}{8}\right) = \underbrace{\left(\frac{1}{16}+\frac{1}{2}-1\right)}_{<0}\underbrace{\left(\frac{9}{64}+\frac{3}{4}-1\right)}_{<0} > 0 \Rightarrow (a_3, b_3, x_3) = \left(\frac{3}{8}, \frac{1}{2}, \frac{7}{16}\right)$

(b) $e_{r_1} = \frac{|x_1 - x_0|}{|x_0|}$ $\quad e_{r_2} = \frac{|x_2 - x_1|}{|x_1|}$ $\quad e_{r_3} = \frac{|x_3 - x_2|}{|x_2|}$

$e_{r_1} = \frac{\left|\frac{1}{4}-\frac{1}{2}\right|}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$e_{r_2} = \frac{\left|\frac{3}{8}-\frac{1}{4}\right|}{\frac{1}{4}} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$

$e_{r_3} = \frac{\left|\frac{7}{16}-\frac{3}{8}\right|}{\frac{3}{8}} = \frac{\frac{1}{16}}{\frac{3}{8}} = \frac{1}{6}$

(2) $x^2+2x-1 = 0 \quad x \in [0{,}1, 2]$

a) $f(x) = x^2+2x-1$; $f : [0{,}1, 2] \to \mathbb{R}$

f derivabilă $\underbrace{f(0{,}1)}_{<0}\cdot\underbrace{f(2)}_{>0} < 0 \Rightarrow x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$

$f'(x) = 2x+2 \Rightarrow x_k = x_{k-1} - \frac{x_{k-1}^2 + 2\cdot x_{k-1} - 1}{2\cdot x_{k-1}+2}$

Presupunem că $f \in C^2([a,b])$, f', f'' nu se anulează pe $[a,b]$ și $f(a)\cdot f(b) < 0$. Fie $x_0 \in [a,b]$ a.î. să aibă loc condiția: $f(x_0)\cdot f''(x_0) > 0$. Atunci ecuația $f(x)=0$ are soluție unică $x^* \in (a,b)$, iar șirul $(x_k)_{k\geq 0}$ construit prin metoda Newton-Raphson, rămâne în $[a,b]$ și converge la x^*

3b) $f(x) = x^2+2x-1 \qquad f'(x) = 2x+2 \qquad f''(x) = 2$

$f'(x), f''(x) \neq 0 \Rightarrow$ nu se anulează pe $[0.1, 2]$

$f(0.1)\cdot f(2) < 0$

$\Rightarrow$ Sunt satisfăcute condițiile teoremei de convergență N-R.

$f(x_0)\cdot f''(x_0) > 0 \Leftrightarrow (x_0^2+2x_0-1)\cdot 2 > 0 \Leftrightarrow x_0^2+2x_0-1>0$

Aleg $x_0 = 1 \Rightarrow 1^2+2\cdot 1-1>0$ Adevărat

c) $x_1, x_2, \; e_r = \dfrac{|x_k - x_{k-1}|}{|x_{k-1}|} = ?$ $\qquad x_k = x_{k-1} - \dfrac{f(x_{k-1})}{f'(x_{k-1})}$

$x_0 = 1$

$x_1 = 1 - \dfrac{\cancel{1+1}\; 2}{4} = \cancel{\frac{5}{4}}\; \dfrac{1}{2}$

$x_2 = \cancel{\frac{5}{4}} - \cancel{\dfrac{\left(\frac{5}{4}\right)^2 + 2\cdot\frac{5}{4} - 1}{2\cdot\frac{5}{4}+2}} = \dfrac{1}{2} - \dfrac{\frac{1}{4}+1-1}{1+2} = \dfrac{1}{2} - \dfrac{1}{12} = \dfrac{\cancel{1}5}{12}$

$e_{r_1} = \dfrac{\left|\frac{1}{2} - 1\right|}{1} = \dfrac{1}{2} \qquad e_{r_2} = \dfrac{\left|\frac{5}{12} - \frac{1}{2}\right|}{\frac{1}{2}} = \dfrac{+\frac{1}{12}}{\frac{1}{2}} = \dfrac{1}{6}$

③ Gauss fără pivotare a), pivotare parțială b), pivotare totală c)

a) $\begin{cases} x_1 - 2x_3 = -5 \\ 2x_1 - 2x_2 = -6 \\ x_1 + 2x_2 + x_3 = 5 \end{cases}$

$\bar{A} = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 2 & -2 & 0 & -6 \\ 1 & 2 & 1 & 5 \end{array}\right]$

$K=1$:

$a_{11} \neq 0 \quad L_2 \leftarrow L_2 - 2L_1 \, ; L_3 \leftarrow L_3 - L_1$

$$\overline{A}_1 \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & -2 & 4 & 4 \\ 0 & 2 & \cancel{-1}3 & 10 \end{array}\right)$$

$K=2$

$a_{22} \neq 0 \quad L_3 \leftarrow L_3 + L_2$

$$\overline{A}_1 \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & -2 & 4 & 4 \\ 0 & 0 & 7 & 14 \end{array}\right)$$

$\overline{A}_1$ este superior triunghiulară $\Rightarrow x_3 = \frac{14}{7} = 2 \quad x_2 = \frac{1}{-2}(4-8) = 2$

$x_1 = \cancel{\frac{1}{1}} -5 + 2 \cdot 2 = -1$

$$\begin{cases} 2x_1 - 2x_2 + x_3 = -3 \\ x_1 + 3x_2 - 2x_3 = 1 \\ 3x_1 - x_2 - x_3 = 2 \end{cases}$$

$$\overline{A}_2 = \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right)$$

$k=1$

$a_{11} \neq 0 \quad L_2 \leftarrow L_2 - \frac{L_1}{2} \, ; L_3 \leftarrow L_3 - \frac{3}{2}L_1$

$$\overline{A}_2 \sim \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -\frac{5}{2} & \frac{3}{2} \\ 0 & 2 & -\frac{5}{2} & \frac{13}{2} \end{array}\right)$$

$k=2 \quad a_{22} \neq 0 \quad L_3 \leftarrow L_3 - \frac{1}{2} L_2$

$-\frac{5}{2}\left(-\frac{1}{2} \cdot -\frac{5}{2}\right)$

$-\frac{5}{2} + \frac{5}{4} =$

$\frac{13}{2} - \frac{1}{2} \cdot \frac{5}{2} =$

$$\bar{A}_2 \sim \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -\frac{5}{2} & \frac{5}{2} \\ 0 & 0 & \frac{5}{4} & \frac{21}{4} \end{array}\right)$$

$x_3 = -\frac{21}{5} \quad x_2 = \frac{1}{4}\left(\frac{5}{2} - \left(-\frac{5}{2}\right)\left(-\frac{21}{5}\right)\right) = \frac{1}{4} \cdot \left(\frac{5}{2} - \frac{21}{2}\right) = -\frac{16}{8} = -2$

$x_1 = \frac{1}{2}\left(-3 + \frac{21}{5} - 4\right) = \frac{1}{2} \cdot \frac{-14}{5} = -\frac{7}{5}$

b) pivotare parțială

$$\bar{A}_1 = \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 2 & -2 & 0 & -6 \\ 1 & 2 & 1 & 5 \end{array}\right)$$

$k=1 \quad \max a_{j1} = a_{21} \quad L_1 \leftrightarrow L_2$

$$\bar{A}_1 \sim \left(\begin{array}{ccc|c} 2 & -2 & 0 & -6 \\ 1 & 0 & -2 & -5 \\ 1 & 2 & 1 & 5 \end{array}\right) \quad \begin{array}{l} L_2 \leftarrow L_2 - \frac{1}{2} L_1 \\ L_3 \leftarrow L_3 - \frac{1}{2} L_1 \end{array}$$

$$\bar{A}_1 \sim \left(\begin{array}{ccc|c} 2 & -2 & 0 & -6 \\ 0 & 1 & -2 & -2 \\ 0 & 3 & 1 & 8 \end{array}\right)$$

$k=2 \quad \max_{j=\overline{2,3}} a_{j2} = a_{32} \quad L_3 \leftrightarrow L_2$

$$\bar{A}_1 \sim \left(\begin{array}{ccc|c} 2 & -2 & 0 & -6 \\ 0 & 3 & 1 & 8 \\ 0 & 1 & -2 & -2 \end{array}\right) \quad L_3 \leftarrow L_3 - \frac{1}{3} L_2$$

$$\bar{A}_1 \sim \left(\begin{array}{ccc|c} 2 & -2 & 0 & -6 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & -\frac{7}{3} & -\frac{14}{3} \end{array}\right) \Rightarrow \begin{array}{l} x_3 = -\frac{14}{3} \cdot \frac{3}{7}(-1) = 2 \\ x_2 = \frac{1}{3}(8-2) = 2 \\ x_1 = \frac{1}{2}(-6+4) = -1 \end{array}$$

$$\overline{A_2} = \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right)$$

$K = 1 \Rightarrow \max_{j} a_{j1} = a_{33} \Rightarrow L_3 \leftrightarrow L_1$

$$\overline{A_2} \sim \left(\begin{array}{ccc|c} 3 & -1 & -1 & 2 \\ 1 & 3 & -2 & 1 \\ 2 & -2 & 1 & -3 \end{array}\right) \qquad \begin{array}{l} L_2 \leftarrow L_2 - \frac{1}{3} L_1 \\ L_3 \leftarrow L_3 - \frac{2}{3} L_1 \end{array}$$

$$\overline{A_2} \sim \left(\begin{array}{ccc|c} 3 & -1 & -1 & 2 \\ 0 & \frac{10}{3} & -\frac{5}{3} & \frac{1}{3} \\ 0 & -\frac{4}{3} & \frac{5}{3} & -\frac{13}{3} \end{array}\right)$$

$\frac{5}{3} + \frac{2}{5} \cdot -\frac{5}{3}$

$\frac{5}{3} - \frac{2}{3} = 1$

$K = 2 \Rightarrow \max(a_{22}, a_{32}) = a_{22} \qquad L_3 \leftarrow L_3 + \frac{2}{5} \cdot L_2$

$-\frac{13}{3} + \frac{1}{3} \cdot \frac{2}{5}$

$\frac{-13 \cdot 5 + 2}{15} = -\frac{63}{15}$

$$\overline{A_2} \sim \left(\begin{array}{ccc|c} 3 & -1 & -1 & 2 \\ 0 & \frac{10}{3} & -\frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{21}{5} \end{array}\right)$$

$x_3 = -\frac{21}{5}$; $x_2 = \frac{3}{10} \cdot \left(\frac{1}{3} - \frac{5}{3} \cdot \frac{21}{5}\right) = \frac{3}{10} \cdot \frac{-20}{3} = -2$

$x_1 = \frac{1}{3}\left(\cancel{2} - \frac{21}{5} - \cancel{2}\right) = -\frac{7}{5}$

e) Cu pivotare totală

$$\overline{A_1} = \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 2 & -2 & 0 & -6 \\ 1 & 2 & 1 & 5 \end{array}\right)$$

$5 - \frac{5}{2} = \frac{5}{2}$

$K = 1 \quad \max|a_{ij}| = a_{13} \Rightarrow p_1 = 1, m = 3$

$m \neq K \Rightarrow C_1 \leftrightarrow C_3 \qquad x = (3, 2, 1)$

$$\overline{A_1} \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & -5 \\ 0 & -2 & 2 & -6 \\ 1 & 2 & 1 & 5 \end{array}\right) \Rightarrow \overline{A_1} \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & -5 \\ 0 & -2 & 2 & -6 \\ 0 & 2 & \frac{3}{2} & \frac{5}{2} \end{array}\right)$$

$L_3 \leftarrow L_3 + \frac{1}{2} L_1$

$k=2 \quad \max_{i,j=\overline{2,3}} |a_{ij}| = |a_{22}| \Rightarrow p = m = 2 \quad L_3 \leftarrow L_3 + L_2$ $\quad \frac{5}{2} \cdot \frac{12}{2} =$

$$\overline{A_1} \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & -5 \\ 0 & -2 & 2 & -6 \\ 0 & 0 & \frac{7}{2} & -\frac{7}{2} \end{array}\right) \quad \left.\begin{array}{l} x_3 = -1 \\ x_2 = \frac{1}{-2}(-6+2) = 2 \\ x_1 = \frac{1}{-2}(-5+1) = 2 \\ x = (3, 2, 1) \end{array}\right\} \Rightarrow \text{Soluție}: (-1, 2, 2)$$

$$\overline{A_2} = \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right)$$

$k=1 \quad \max |a_{ij}| = |a_{22}| \Rightarrow L_2 \leftrightarrow L_1 ; C_2 \leftrightarrow C_1$

$$\overline{A_2} \sim \left(\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ -2 & 2 & 1 & -3 \\ -1 & 3 & -1 & 2 \end{array}\right) \quad \begin{array}{l} L_2 \leftarrow L_2 + \frac{2}{3} L_1 \\ L_3 \leftarrow L_2 + \frac{1}{3} L_1 \end{array}$$

$33 + 30 = 63$

$\frac{21}{5}$

$-\frac{1}{3} + \frac{4}{3} \cdot + \frac{5}{3}$

$5)$

$-\frac{7}{3} - \frac{4}{3} \cdot + \frac{7}{3}$

$\frac{35+28}{15} = \frac{21}{5}$

$$\overline{A_2} \sim \left(\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ 0 & \frac{8}{3} & -\frac{1}{3} & -\frac{7}{3} \\ 0 & \frac{10}{3} & -\frac{5}{3} & \frac{7}{3} \end{array}\right)$$

$k=2 \quad \max_{i,j=\overline{2,3}} |a_{ij}| = \left|\frac{10}{3}\right| \Rightarrow L_3 \leftrightarrow L_2$

$$\overline{A_2} \sim \left(\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ 0 & \frac{10}{3} & -\frac{5}{3} & +\frac{7}{3} \\ 0 & \frac{8}{3} & -\frac{1}{3} & -\frac{7}{3} \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 - \frac{4}{5} L_2} \overline{A} \sim \left(\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ 0 & \frac{10}{3} & -\frac{5}{3} & +\frac{7}{3} \\ 0 & 0 & 1 & \frac{-21}{5} \end{array}\right)$$

$x_3 = \frac{-21}{5} \quad x_2 = \cancel{\frac{3}{10}\left(\frac{7}{3} + \frac{5}{3} \cdot \frac{21}{5} \cdot \frac{7}{7}\right)}$

$$x_2 = \frac{3}{10}\left(+\frac{7}{3} - \frac{21}{5} \cdot \frac{5}{3}\right) = \frac{3}{10} \cdot \frac{-14}{3} = \frac{-7}{5}$$

$$x_1 = \frac{1}{3}\left(1 - -2 \cdot -\frac{21}{5} - 1 \cdot -\frac{7}{5}\right) = \frac{1}{3} \cdot \frac{5-42+7}{5} = \frac{30}{15} = 2$$

$\text{Soluție} = \left(-\frac{7}{5}, 2, -\frac{21}{5}\right)$

(4)

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 8 & 5 \\ -4 & 0 & 10 \end{pmatrix}$$

L - inf Δ
U - sup Δ

a) i) Gauss fără pivotare.

$2)\ 1 - \frac{5}{2} = \frac{-3}{2}$

$W = (1, 2, 3)$

$K = 1 \quad |a_{11}| \neq 0$

$m_{21} = \frac{a_{21}}{a_{11}} = \frac{4}{2} = 2; \quad m_{31} = \frac{-4}{2} = -2$

$L_2 \leftarrow L_2 - 2L_1; \quad L_3 \leftarrow L_3 + 2L_1$

$$A \sim \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 6 & 12 \end{pmatrix}$$

$K = 2 \quad |a_{22}| \neq 0 \quad m_{32} = \frac{6}{2} = 3 \quad L_3 \leftarrow L_3 - 3L_2$

$$A \sim \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix} = U \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ +2 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}$$

ii) Gauss cu piv. parțială

$w = (1, 2, 3)$

$K = 1 \quad \max a_{j1} = a_{j2}. \quad L_2 \leftrightarrow L_1 \quad w = (2, 1, 3)$

$$A \sim \begin{pmatrix} 4 & 8 & 5 \\ 2 & 3 & 1 \\ -4 & 0 & 10 \end{pmatrix} \quad \begin{matrix} L_2 \leftarrow L_2 - \frac{1}{2} L_1 \\ L_3 \leftarrow L_3 + L_1 \end{matrix} \qquad A \sim \begin{pmatrix} 4 & 8 & 5 \\ 0 & -1 & -\frac{3}{2} \\ 0 & 8 & 15 \end{pmatrix}$$

~~K = 2 max~~ $m_{21} = \frac{2}{4} = \frac{1}{2} \quad m_{31} = \frac{-4}{4} = -1$

$K = 2 \quad \max_{j=2,3} a_{j2} = a_{32} \quad L_3 \leftrightarrow L_2 \quad w = (2, 3, 1)$

$A \sim$

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$$A \sim \begin{pmatrix} 4 & 8 & 5 \\ 0 & 8 & 15 \\ 0 & -1 & -\frac{3}{2} \end{pmatrix} \quad L_3 \leftarrow L_3 + \frac{1}{8}L_2 \quad m_{32} = -\frac{1}{8} \qquad A \sim \begin{pmatrix} 4 & 8 & 5 \\ 0 & 8 & 15 \\ 0 & 0 & \frac{3}{8} \end{pmatrix} = U$$

$\frac{15}{8} + \frac{4/3}{-\frac{3}{2}} = \frac{3}{8}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -1 & -\frac{1}{8} & 1 \end{pmatrix}$$

$\frac{12}{5} - \frac{3}{8} \cdot 6$

$48 - 45$

iii) Gauss cu piv. totală $\quad A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 8 & 5 \\ -4 & 0 & 10 \end{pmatrix}$

$k = 1 \quad \max |a_{ij}| = a_{33} \quad C_3 \leftrightarrow C_1; L_3 \leftrightarrow L_1$

$$A \sim \begin{pmatrix} 1 & 3 & 2 \\ 5 & 8 & 4 \\ 10 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 10 & 0 & -4 \\ 5 & 8 & 4 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{matrix} L_2 \leftarrow L_2 - \frac{1}{2}L_1 & m_{21} = \frac{5}{10} = \frac{1}{2} \\ L_3 \leftarrow L_2 - \frac{1}{10}L_1 & m_{31} = \frac{1}{10} \end{matrix}$$

$$A \sim \begin{pmatrix} 10 & 0 & -4 \\ 0 & 8 & 6 \\ 0 & 3 & \frac{12}{5} \end{pmatrix} \quad m_{32} = \frac{3}{8} \quad L_3 \leftarrow L_3 - \frac{3}{8}L_2 \qquad A \sim \begin{pmatrix} 10 & 0 & -4 \\ 0 & 8 & 6 \\ 0 & 0 & \frac{3}{20} \end{pmatrix} = U$$

$k = 2 \quad \max\limits_{i,j=\overline{2,3}} |a_{ij}| = a_{22}$ nu interschimb nimic

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{10} & \frac{3}{8} & 1 \end{pmatrix}$$

$\frac{1}{2}(\cancel{18-5})$

b) $Ax = b \qquad b = \begin{pmatrix} 5 \\ 18 \\ 20 \end{pmatrix}$

$\frac{1}{2}(8-6)$

a) i) $\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$

$\frac{1}{2}(5-2-3) = 0$

$Ax = b \Leftrightarrow L \cdot U \cdot x = b \Leftrightarrow L \cdot y = b \Rightarrow \begin{cases} y_1 = 5 \\ y_2 = 8 \\ y_3 = 6 \end{cases}$

$Ux = y \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$

⑤ $A = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 14 \end{pmatrix}$

a) A e simetrică și pozitiv definită ??

A e simetrică $\Leftrightarrow A^T = A$

$A^T = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 14 \end{pmatrix} = A \Rightarrow$ A este simetrică

A e poz. def $\Leftrightarrow \langle Av, v \rangle > 0, (\forall) v \in \mathbb{R}^n \setminus \{0^n\}$

Crt lui Sylvester: Mat ~~A~~ simetrică $A \in M_n(\mathbb{R})$ este poz. def. ddacă

$\det A_k > 0, A_k = (a_{ij})_{i,j=\overline{1,k}}$

$$\left.\begin{array}{l} |A_1| = |4| = 4 > 0 \\ |A_2| = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4 > 0 \\ |A_3| = \begin{vmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 14 \end{vmatrix} = \overset{4}{\cancel{8}} \cdot 14 + \cancel{6 \cdot 4} + \cancel{8 \cdot 3} - \overset{-8}{\cancel{16 \cdot 2}} - \overset{-12}{\cancel{9 \cdot 4}} - \cancel{13 \cdot 4} \\ = \underbrace{14 - 12}_{2} + \underbrace{14 - 8}_{6} + 28 = 36 \end{array}\right\} \Rightarrow \text{A este poz. def}$$

b) fact. Cholesky.

a) $\Rightarrow$ ~~A~~ $(\exists)$ desc. Cholesky. LL^T

L mat inf Δ

$A = LL^T \Leftrightarrow \begin{pmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 14 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$

$a_{11} = l_{11}^2 \Rightarrow l_{11} = 2$

$a_{22} = l_{22}^2 + l_{21}^2$

$$l_{kk} = \sqrt{a_{kk} - \sum_{s=1}^{k-1} l_{ks}^2} \qquad \underline{l_{11}} = \sqrt{4} = \underline{2} \quad \underline{l_{22}} = \sqrt{2 - l_{21}^2} = \underline{1}$$

$$\underline{l_{33}} = \sqrt{14 - \underbrace{l_{31}^2}_{4} - \underbrace{l_{32}^2}_{1}} = \cancel{\sqrt{9}}\ \underline{3}$$

$$l_{ik} = \frac{1}{l_{kk}}\left(a_{ik} - \sum_{s=1}^{k-1} l_{is} l_{ks}\right)$$

$$\underline{l_{21}} = \frac{1}{2}(2-0) = \underline{1} \qquad \cancel{l_{32} = \frac{1}{1}(3 - l_{31} - l_{21}) = 0}$$

$$\underline{l_{31}} = \frac{1}{2}(4-0) = \underline{2} \qquad \underline{\underset{i\,k}{l_{32}}} = \frac{1}{1}(3 - 2\cdot 1) = \underline{1}$$

$\cancel{l_{33}}$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \qquad L^T = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \qquad \frac{1}{3}\underbrace{(11 - 10 - 1)}_{0}$$

c) $Ax = b$, $b = \begin{pmatrix} 10 \\ 6 \\ 11 \end{pmatrix}$

$LL^T x = b \quad Ly = b \Rightarrow \begin{cases} y_1 = 5 \\ y_2 = 1 \\ y_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$

⑥ $A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ λ - val. proprii

a) $\det(A - \lambda I_3) = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)^2(3-\lambda) + 4 - (3-\lambda) - 4(2-\lambda) = 0$$

$\cancel{(\lambda^2 - 4\lambda + 4)(3-\lambda) - 7 + 3\lambda = \lambda^3 + 7\lambda^2 - 14\lambda + 5 = 0}$

$-12\lambda \quad -4\lambda$

1

$(\lambda^2-4\lambda+4)(3-\lambda)+4-3+\lambda-8+4\lambda=0$

$-\lambda^3+7\lambda^2-11\lambda+5=0 \Leftrightarrow (\lambda-1)^2(\lambda-5)=0$

$\lambda\in\{1,5\}$ val. proprii ale lui A

b) $\rho(A)=\max|\lambda_i|=5$

c) $\|A\|_1$, $\|A\|_\infty$

$\cancel{\|A\|_1=\sum_{i=1}^{n} \sup_{v\in\mathbb{R}^n\setminus\{0\}}\frac{\|Av\|}{\|v\|}}$

$\|A\|_1=\max_{j=\overline{1,n}}\sum_{i=1}^{n}|a_{ij}|=\max\{5,5,5\}=5$

$\|A\|_\infty=\max_{i=\overline{1,n}}\sum_{j=1}^{n}|a_{ij}|=\max\{5,7,4\}=7$

(7) A e simetrică $\Rightarrow \|A\|_2=\rho(A)=\max_{i=\overline{1,n}}|\lambda_i|$, λ_i val. pr. ale lui A.

$\|A\|_2=\max_{i=\overline{1,n}}\sqrt{\lambda_i^B}$, unde $B=A\cdot A^T$, λ_i^B sunt val. proprii mat B

A simetrică $\Rightarrow A\cdot A^T=A^2=B$

Trebuie să arătăm că $\max_{i=\overline{1,n}}\sqrt{\lambda_i^B}=\max_{i=\overline{1,n}}|\lambda_i|$. Este suficient să arătăm că $\lambda_i^2=\lambda_i^B$

~~det(A² − λ verif~~ verificăm dacă $\det(A^2-\lambda^2\cdot I_3)=0 \Leftrightarrow$

$\Leftrightarrow \det(A-\lambda I_3)\det(A+\lambda I_3)=0 \Leftrightarrow \det(A-\lambda I_3)=0$ sau $\det(A+\lambda I_3)=0$

știm λ_i val. proprii de lui A, dacă $\det(A-\lambda I_3)=0 \Rightarrow$
$\det(A^2-\lambda^2 I_3)=0 \Rightarrow \lambda_i^2$ sunt val. proprii de lui $A^2 \Rightarrow$

$\Rightarrow \max\limits_{i=\overline{1,n}} |\lambda_i|^2 = \max\limits_{i=\overline{1,n}} \lambda_i^B \Rightarrow \|A\|_2 = \rho(A)$.

③ $Ax = a \quad A = \begin{pmatrix} 0,2 & 0,01 & 0 \\ 0 & 1 & 0,04 \\ 0 & 0,02 & 1 \end{pmatrix} \quad a = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

a) $Ax = a \Leftrightarrow -Ax = -a \Leftrightarrow x - Ax = x - a \Leftrightarrow x = (I_3 - A)x + a$

Considerând $B = I_3 - A$, $b = a$, obținem $x = Bx + b$

b) Metoda Jacobi este convergentă d.dacă $\rho(B) < 1$.

Dacă $\|I_3 - A\| = q \in (0,1)$, atunci metoda Jacobi este convergentă.

c)

$B = I_3 - A = \begin{pmatrix} 0,8 & -0,01 & 0 \\ 0 & 0 & -0,04 \\ 0 & -0,02 & 0 \end{pmatrix}$

$\|B\| = \max\limits_{j=\overline{1,3}} \sum\limits_{i=1}^{3} |b_{ij}| = \max\{0,8; -0,03; -0,04\} = 0,8 \in (0,1) \Rightarrow$

$\Rightarrow$ metoda Jacobi este convergentă

d) $x^{(1)}, x^{(2)} = ?$ $\|x^{(1)} - x^*\|, \|x^{(2)} - x^*\|$, $x^* = \begin{pmatrix} 5,00 \\ -0,08 \\ 2,00 \end{pmatrix}$ este sol. exactă a sist. cu precizie 2

$x^0 = 0$

$x^{(1)} = B \cdot 0 + b = b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$x^{(2)} = B \cdot x^{(1)} + b = \begin{pmatrix} 0,8 & -0,01 & 0 \\ 0 & 0 & -0,04 \\ 0 & -0,02 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0,8 \\ -0,08 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1,8 \\ -0,08 \\ 2 \end{pmatrix}$

$$\|x^{(1)}-x^*\| = \left\|\begin{pmatrix}1\\0\\2\end{pmatrix}-\begin{pmatrix}5\\-0{,}08\\2\end{pmatrix}\right\| = \left\|\begin{pmatrix}-4\\0{,}08\\0\end{pmatrix}\right\| = 4$$

$$\|x^{(1)}-x^*\| = 8{,}5-1{,}8 = 3{,}2.$$

③ $Ax=a$, $A=\begin{pmatrix}4&1&2\\0&3&1\\2&4&8\end{pmatrix}$ $\quad a=\begin{pmatrix}1\\0\\1\end{pmatrix}$

a) $A\in M_n(\mathbb{R})$ este diag pe linii dacă $|a_{ii}| > \sum_{\substack{j=\overline{1,n}\\ j\neq i}} |a_{ij}|,\ i=\overline{1,n}$

b) Fie $\Delta = diag(A)$. $(a_{ii})>0 \Rightarrow \Delta$ este inversabilă

$Ax=a \Leftrightarrow \Delta^{-1}Ax = \Delta^{-1}a \Leftrightarrow x-\Delta^{-1}Ax = x-\Delta^{-1}a \Leftrightarrow x=(I-\Delta^{-1}A)x+\Delta^{-1}a$

Considerând $B=I-\Delta^{-1}A$, $b=\Delta^{-1}a$ se obține sist: $x=Bx+b$

c) $q := \|B\|_\infty < 1$

$$B = I_3-\Delta^{-1}A = I_3-\begin{pmatrix}\frac14&0&0\\0&\frac13&0\\0&0&\frac18\end{pmatrix}\cdot\begin{pmatrix}4&1&2\\0&3&1\\2&4&8\end{pmatrix} = I_3-\begin{pmatrix}1&\frac14&\frac12\\0&1&\frac13\\\frac14&\frac12&1\end{pmatrix} = \begin{pmatrix}0&-\frac14&-\frac12\\0&0&-\frac13\\-\frac14&-\frac12&0\end{pmatrix}$$

$\Rightarrow \|B\|_\infty = \frac34 < 1 \Rightarrow$ asigură convergența prin metoda Jacobi.

d) $\left.\begin{array}{l}a_{11}=4>3\\a_{22}=3>1\\a_{33}=8>6\end{array}\right\}\Rightarrow A$ este diag dominantă pe linii

$q=\frac34$

e) $x^{(1)}, x^{(2)}, \|x^{(1)}-x^*\|, \|x^{(2)}-x^*\|$ $\cancel{\angle}$? $\quad x^*=\begin{pmatrix}\frac{3}{14}\\-\frac{1}{35}\\\frac{3}{35}\end{pmatrix}$

$x^{0} = 0 \qquad b = D^{-1}\cancel{a}$

$x^{(k)} = Bx^{(k-1)} + b$

$$x^{(1)} = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -1/3 \\ -1/4 & -1/2 & 0 \end{pmatrix} \cdot 0 + b = \cancel{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{8} \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & -1/3 \\ -1/4 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 1/4 \\ 0 \\ 1/8 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \\ 1/8 \end{pmatrix} = \begin{pmatrix} -1/16 \\ -1/24 \\ -1/16 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \\ 1/8 \end{pmatrix} = \begin{pmatrix} 3/16 \\ -1/24 \\ 1/16 \end{pmatrix}$$

$$\|x^{(1)} - x^{*}\| = \left\| \begin{pmatrix} 1/4 \\ 0 \\ 1/8 \end{pmatrix} - \begin{pmatrix} 3/14 \\ -1/35 \\ 3/35 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0{,}0357 \\ 0{,}02857 \\ 0{,}0392 \end{pmatrix} \right\| \cong 0{,}08 \quad \cancel{0{,}0392}$$

$$\|x^{(2)} - x^{*}\| = \left\| \begin{pmatrix} 3/16 \\ -1/24 \\ 1/16 \end{pmatrix} - \begin{pmatrix} 3/14 \\ 1/35 \\ 3/35 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -0{,}0267 \\ -0{,}07023 \\ -0{,}02321 \end{pmatrix} \right\| \cong 0{,}11 \quad \cancel{0{,}07023}$$

(10) $Ax = a$, $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 2 & 4 & 6 \end{pmatrix}$ $\quad a = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

a) A este simetrică ~~și poz. definită~~

$$\left.\begin{aligned} \det(A_1) &= 4 > 0 \\ \det(A_2) &= 40 - 4 > 0 \\ \det(A_3) &= 240 + 16 + 16 - 40 - 64 - 24 > 0 \end{aligned}\right\} \Rightarrow A \text{ este poz. definită.}$$

$Ax = a \Leftrightarrow \tau Ax = \tau a$ sau $(\Leftrightarrow)$ $-\tau Ax = -\tau a \Leftrightarrow (I_3 - \tau A)x = x - \tau a$

$\Leftrightarrow x = B_\tau x + b_\tau$, $\tau > 0$ cu $B_\tau = I_3 - \tau A$ $b_\tau = \tau a$

b) τ_0 optim

$\tau_0 = \dfrac{2}{\lambda_m + \lambda_1}$, $\lambda_{1,m}$ val. proprii ale matricei $A \in M_m(\mathbb{R})$

c) $\sigma_0 = 0{,}125 = \frac{1}{8}$

$x^{(1)}, x^{(2)}, \underbrace{\|x^{(1)}-x^*\|}_{X}, \underbrace{\|x^{(2)}-x^*\|}_{Y} = ?$ $\quad x^* = \begin{pmatrix} 2/9 \\ -1/9 \\ 1/6 \end{pmatrix}$ sol. sist.

$x^{(0)} = 0$

$1 - \frac{5}{4}$

$x^{(k)} = B_\sigma x^{(k-1)} + b_\sigma$

$1 - \frac{3}{4}$

$$B_\sigma = I_3 - \sigma A = I_3 - \frac{1}{8} \cdot \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$b_\sigma = \sigma a = \frac{1}{8}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/8 \\ 0 \\ 1/8 \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} + \begin{pmatrix} 1/8 \\ 0 \\ 1/8 \end{pmatrix} = \begin{pmatrix} 1/8 \\ 0 \\ 1/8 \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{8} \\ 0 \\ \frac{1}{8} \end{pmatrix} + \begin{pmatrix} \frac{1}{8} \\ 0 \\ \frac{1}{8} \end{pmatrix} = \begin{pmatrix} \frac{1}{16} - \frac{1}{32} + \frac{1}{8} \\ -\frac{1}{32} - \frac{1}{16} + 0 \\ -\frac{1}{32} + \frac{1}{16} + \frac{1}{8} \end{pmatrix} = \begin{pmatrix} \frac{5}{32} \\ \frac{-3}{32} \\ \frac{5}{32} \end{pmatrix}$$

$$X = \left\| \begin{pmatrix} \frac{1}{8} \\ 0 \\ \frac{1}{8} \end{pmatrix} - \begin{pmatrix} 2/9 \\ -1/9 \\ 1/6 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -0{,}0972 \\ +0{,}11111 \\ -0{,}04166 \end{pmatrix} \right\| \cong 0{,}23$$

$\neq$ ~~0,1111~~

$$Y = \left\| \begin{pmatrix} \frac{5}{32} \\ \frac{-3}{32} \\ \frac{5}{32} \end{pmatrix} - \begin{pmatrix} \frac{2}{9} \\ -\frac{1}{9} \\ \frac{1}{6} \end{pmatrix} \right\| = \left\| \begin{pmatrix} -0{,}06 \\ 0{,}01 \\ -0{,}01 \end{pmatrix} \right\| \cong 0{,}08$$

11) a) Metoda directă

Pol. de interpolare LaGrange, $P_2(x)$ asociat funcției f și nodurilor de interpolare x_1, x_2, x_3.

$$P_2(x_1)=y_1;\ P_2(x_2)=y_2;\ P_2(x_3)=y_3$$

$$P_2(x)=a_1+a_2x+a_3x^2$$

$$\begin{cases} a_1+a_2x_1+a_3x_1^2=y_1 \\ a_1+a_2x_2+a_3x_2^2=y_2 \\ a_1+a_2x_3+a_3x_3^2=y_3 \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}}_{A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

b) Metoda LaGrange

$$P_m(x)=\sum_{k=1}^{m+1} L_{m,k}(x)\,y_k,\ x\in\mathbb{R},$$ unde $L_{m,k}$ sunt pol. ce urmează să fie determinate. Deoarece P_m interpolează funcția f în nodurile $\{x_i\},\ i=\overline{1,3}$ atunci au loc relațiile $P_m(x_i)=y_i$, de unde rezultă $L_{m,k}(x_i)=\delta_{ik}$. Deoarece $L_{m,k}$ sunt pol. de gradul ~~m~~ 2 și $L^2_{~~m~~,k}(x_i)=0$, $i\neq k$ rezultă că ~~$x_{1,2,3}$ sunt $x_1 \ldots x_{k-1}$~~ sunt ~~soluții~~ rădăcini $x_i,\ i=\overline{1,3},\ i\neq k$

$L_{2,k}$ se reprezintă:

$\cancel{L_{2,k}(x)=C_k(x-x_1)}$ $\qquad \cancel{C_1-(x_1-x_1)(x_i}$

$$L_{2,1}(x)=C_1(x-x_2)(x-x_3) \qquad C_1=\frac{1}{(x_1-x_2)(x_1-x_3)}$$

$$L_{2,2}(x)=C_2(x-x_1)(x-x_3) \qquad C_2=\left[(x_2-x_1)(x_2-x_3)\right]^{-1}$$

$$L_{2,3}(x)=C_3(x-x_1)(x-x_2) \qquad C_3=\left[(x_3-x_1)(x_3-x_2)\right]^{-1}$$

$$P_m(x)=\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}+\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}+\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

c) Metoda Newton

$$\cancel{P_m(x) = c_1 + c_2(x-x_1) + c_3(x-x_2)(x-x_1)}$$

$$P_2(x) = c_1 + c_2(x-x_1) + c_3(x-x_1)(x-x_2)$$

$$\begin{cases} c_1 = y_1 \\ c_1 + c_2(x_2 - x_1) = y_2 \\ c_1 + c_2(x_3 - x_1) + c_3(x_3 - x_2) = y_3 \end{cases} \Leftrightarrow \qquad P_2(x_i) = y_i,\ i = \overline{1,3}$$

$$\Leftrightarrow \begin{cases} c_1 = y_1 \\ c_2 = \dfrac{y_2 - y_1}{x_2 - x_1} \\ c_3 = \dfrac{y_3 - y_1 - \frac{y_2 - y_1}{x_2 - x_1}(x_3 - x_1)}{x_3 - x_2} \end{cases}$$

$$P_2(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}\cdot(x - x_1) + \frac{y_3 - y_1 - \frac{(y_2 - y_1)(x_3 - x_1)}{x_2 - x_1}}{x_3 - x_2}(x - x_1)(x - x_2)$$

d) Met Newton cu DD

$$P_m(x) = f[x_1] + \sum_{i=2}^{m+1} f[x_1, \dots, x_i] \prod_{j=1}^{i-1}(x - x_j)$$

$$P_2(x) = f[x_1] + f[x_1, x_2]\cdot(x - x_1) + f[x_1, x_2, x_3](x - x_1)(x - x_2)$$

$$f[x_1] = f(x_1)$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{f(x_3) - f(x_2)}{(x_3 - x_1)(x_3 - x_2)} - \frac{f(x_2) - f(x_1)}{(x_2 - x_1)(x_3 - x_1)}$$

$$P_2(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + \frac{f(x_3) - f(x_2)}{(x_3 - x_1)(x_3 - x_2)}(x - x_1)(x - x_2) - \frac{f(x_2) - f(x_1)}{(x_2 - x_1)(x_3 - x_1)}\cdot(x - x_1)(x - x_2)$$

12) $P_2(x)$ Met Lagrange și Met Newton DD $x=(x_1,x_2,x_3)$

a) $f(x)=\ln x$, $x=(1,e,e^2)$

Met Lagrange

$$P_2(x)=L_{2,1}(x)y_1+L_{2,2}(x)y_2+L_{2,3}(x)y_3$$

$(e-1)(e+1)=$
$(e^2-\cancel{e+e}-1)$
$\cancel{(x+e)(x}$

$$L_{2,1}(x)=\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}=\frac{(x-e)(x-e^2)}{(1-e)(1-e^2)}$$

$$L_{2,2}(x)=\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}=\frac{(x-1)(x-e^2)}{(e-1)(e-e^2)}$$

$$L_{2,3}(x)=\frac{(x-x_2)(x-x_1)}{(x_3-x_1)(x_3-x_2)}=\frac{(x-1)(x-e)}{(e^2-1)(e^2-e)}$$

$$P_2(x)=\frac{(x-e)(x-e^2)}{(1-e)(1-e^2)}\cdot\underbrace{\ln 1}_{0}+\frac{(x-1)(x-e^2)}{(e-1)(e-e^2)}\cdot\underbrace{\ln e}_{1}+\frac{(x-1)(x-e)}{(e^2-1)(e^2-e)}\cdot\underbrace{\ln e^2}_{2}$$

$$P_2(x)=\frac{^{e+1}(x-1)(x-e^2)}{(e-1)(e)(1-e)}+\frac{2(x-1)(x-e)}{(e^2-1)(e)(1-e)}=\cancel{(e+1)(x-1}$$

$$P_2(x)=\frac{(x-1)\left[(x-e^2)(e+1)+2(x-e)\right]}{e(e^2-1)(1-e)}$$

Met Newton DD

$$P_2(x)=\underbrace{f(x_1)}_{0}+\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+\left[\frac{f(x_3)-f(x_2)}{(x_3-x_1)(x_3-x_2)}-\frac{f(x_2)-f(x_1)}{(x_2-x_1)(x_3-x_1)}\right]\cdot\frac{(x-x_1)}{\frac{1}{(x-x_2)}}$$

$$P_2(x)=\frac{2-1}{e^2-1}(x-1)+\left[\frac{2-1}{(e^2-1)(e^2-e)}-\frac{e-1}{(e-1)(e^2-1)}\right]\cdot(x-1)(x-e)$$

$$P_2(x)=(x-1)\left[\frac{1}{e-1}+\frac{(1-e^2+e)(x-e)}{e(e^2-1)(e-1)}\right]=\frac{(x-1)\left[e(e^2-1)+(1-e^2+e)(x-e)\right]}{e(e^2-1)(e-1)}$$

$$P_2(x) = 0 + \frac{1-0}{e-1}(x-1) + \frac{2-1}{(e^2-1)(e^2-e)}(x-1)(x-e) - $$
$$- \frac{1-0}{(e-1)(e^2-1)} \cdot (x-1)(x-e)$$

$$P_2(x) = (x-1)\left[\frac{\overset{(e-1)e}{1}}{e-1} + (x-e)\left[\frac{1}{(e^2-1)e(e-1)} - \frac{\overset{e/}{1}}{(e-1)(e^2-1)}\right]\right]$$

$$P_2(x) = (x-1)\frac{e(e-1)+(x-e)(1-e)}{e(e-1)(e^2-1)}$$

b) $f(x) = \sin\left(\frac{\pi x}{2}\right)$, $x = (-1, 0, 1)$

$f(x_1) = \sin\left(-\frac{\pi}{2}\right) = -1 = y_1$
$f(x_2) = \sin 0 = 0 = y_2$
$f(x_3) = \sin\frac{\pi}{2} = 1 = y_3$

MetLaGrange

$$P_2(x) = L_{2,1}(x) \cdot \underbrace{y_1}_{-1} + L_{2,2}(x)\underbrace{y_2}_{0} + L_{2,3}(x) \cdot \underbrace{y_3}_{1}$$

$$L_{2,1}(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{x(x-1)}{-1 \cdot -2} = \frac{x(x-1)}{2}$$

$$L_{2,3}(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{x(x+1)}{2 \cdot 1} = \frac{x(x+1)}{2}$$

$$P_2(x) = -\frac{x(x-1)}{2} + \frac{x(x+1)}{2} = \frac{-x^2+x+x^2+x}{2} = x$$

MetNewton

$$P_2(x) = f(x_1) + f[x_1, x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2) = -1+x+1 = x$$

$$f[x_1,x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{1}{1} = 1$$

$$f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} = \frac{1-1}{x_3-x_1} = 0$$

$$f[x_2,x_3] = \frac{f(x_3)-f(x_2)}{(x_3-x_2)} = \frac{1}{1}$$

(13) $\overset{x_1}{(-2,1)}, \overset{x_2}{(-1,4)}, \overset{x_3}{(0,11)}, \overset{x_4}{(1,16)}, \overset{x_5}{(2,13)}, \overset{x_6}{(3,-4)}$. Să se arate că pol care interpolează datele este de gradul 3 folosind Newton DD

	1	2	3	4	5	6
1	$f[x_1]=1$					
2	$f[x_2]=4$	$f[x_1,x_2]=\frac{3}{1}$	$\cancel{f[x_1,x_2,x_3]=}$			
3	$f[x_3]=11$	$f[x_2,x_3]=\frac{7}{1}$	$f[x_1,x_2,x_3]=\frac{4}{+2}=+2$			
4	$f[x_4]=16$	$f[x_3,x_4]=\frac{5}{1}$	$f[x_2,x_3,x_4]=\frac{-2}{2}=-1$	$f[x_1,x_2,x_3,x_4]=\frac{-3}{3}=-1$		
5	$f[x_5]=13$	$f[x_4,x_5]=\frac{-3}{1}$	$f[x_3,x_4,x_5]=\frac{-8}{2}=-4$	$f[x_2,x_3,x_4,x_5]=\frac{-3}{3}=-1$	$f[x_1\ldots x_5]=0$	
6	$f[x_6]=-4$	$f[x_5,x_6]=\frac{-17}{1}$	$f[x_4,x_5,x_6]=\frac{-14}{2}=-7$	$f[x_3,x_4,x_5,x_6]=\frac{-3}{3}=-1$	$f[x_2\ldots x_6]=0$	$f[x_1\ldots x_6]=0$

$$Q_{i,j}=\frac{Q_{i,j-1}-Q_{i-1,j-1}}{x_i-x_{i-j+1}},\ j=\overline{2,m+1},\ i=\overline{j,m+1}$$

$$P_m(x)=f[x_1]+\sum_{i=2}^{m+1} f[x_1\ldots x_i]\prod_{j=1}^{i-1}(x-x_j)$$

$$P_m(x)=1+3(x+2)+2(x+2)(x+1)-1(x+2)(x+1)x \Rightarrow \text{Pol. de grad } 3$$

(14) $P_3(x)$ pol. de interpolare LaGrange asociat setului $\underset{x_1}{(0,0)}, \underset{x_2}{(0.5,y)}, \underset{x_3}{(1,3)}, \underset{x_4}{(2,2)}$.
$y=?$ coef lui $x^3=6$

$$L_{3,1}(x)=\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}=\frac{(x-1)(x-2)(x-\frac{1}{2})}{-\frac{1}{2}\cdot+1\cdot+2}=(x-1)(x-2)(x-\frac{1}{2})$$

$$L_{3,2}(x)=\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}=\frac{x(x-1)(x-2)}{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}=\frac{8}{3}x(x-1)(x-2)$$

$$L_{3,3}(x)=\frac{x(x-\frac{1}{2})(x-2)}{1\cdot\frac{1}{2}\cdot-1}=-2\cdot(x)(x-\frac{1}{2})(x-2)$$

$$L_{3,4}(x)=\frac{x(x-\frac{1}{2})(x-1)}{2\cdot\frac{3}{2}\cdot1}=\frac{1}{3}x(x-1)(x-\frac{1}{2})$$

$P_3(x) = L_{3,1}y_1 + L_{3,2}y_2 + L_{3,3}y_3 + L_{3,4}y_4 = 0 + y\cdot\frac{8}{3}x(x-1)(x-2) + 3\cdot(-2)\,x(x-2)(x-\frac{1}{2}) + 2\cdot\frac{1}{3}x(x-1)(x-\frac{1}{2})$

Coef lui $x^3 = 6 \Rightarrow \frac{8}{3}y + (-6) + \frac{2}{3} = 6 \Leftrightarrow \cancel{8}^{4}y - \cancel{18}^{5} + \cancel{2}^{1} = \cancel{18}^{9} \Leftrightarrow y = \frac{17}{4}$

(15) Să se det. coef lui x^2 dacă toate dd. de ord 3 sunt 1

x	0	1	2
$P_m(x)$	2	-1	4

$P_m(x_i) = f(x_i)$ (def interpolării LaGrange)

	x	DD0	DD1	DD2	DD3	DD4
1	0	2	$f_1[x_1,x_2]$			
2	1	-1	-3			
3	2	4	5	$4 = f[x_1,x_2,x_3]$		

$f[x_1,\dots,x_4] = 1$ cf. enunțului

$P_m(x) = 2 + (-3)(x - \cancel{x_0}) + 4x(x-1) + x(x-1)(x-2)$

$= 2 + (-3)(x) + 4x^2 - 4x + x^3 + \cancel{\dots}\,(x^2 - \cancel{x})(x-2)$

coef lui $x^2 = 4 + (-2) \cancel{+1} + (-1) = 1$

(16) $P_2(x) = f[x_1] + f[x_1,x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2)$

$f[x_1] = f(x_1) = f(0)$

$f[x_1,x_2] = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

$f(x_1,x_2,x_3) = \dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} = \dfrac{10 - f[x_1,x_2]}{0,7} = \dfrac{5\cancel{0}}{0,7} \Rightarrow f[x_1,x_2] = 5$

~~$P_2(x)$~~ $\dfrac{f[x_3]-f[x_2]}{x_3-x_2} = f[x_2,x_3] \Leftrightarrow \dfrac{6-f[x_2]}{0,3} = 10 \Rightarrow f[x_2] = 3$

$5 = \dfrac{3-f(x_1)}{0,4} \Rightarrow f(x_1) = 1$

$P_2(x) = 1 + 5(x-0) + \dfrac{50}{7}x\cdot(x-0,4) = \cancel{\dfrac{50}{7}x^2 + \dfrac{5}{7}x + 1}$

$= 1 + 5x + \dfrac{50x^2}{7} - \dfrac{20}{7}x = \dfrac{50x^2}{7} + \dfrac{15x}{7} + 1$

(17) a) Funcția $S:[a,b]\to\mathbb{R}$ s.n. funcție spline liniară pt. funcția $f:[a,b]\to\mathbb{R}$ dacă

i) S este liniară pe porțiuni:

$S(x) = S_j(x), (\forall)\, x\in I_j,\ j=\overline{1,m}$ unde

$S_j: I_j \to \mathbb{R},\ S_j(x) = a_j + b_j(x-x_j),\ j=\overline{1,m}$ cu $a_j, b_j \in \mathbb{R},\ j=\overline{1,m}$, ce trebuie determinați

ii) S interpolează f în $x_j,\ j=\overline{1,m+1}$

$S(x_j) = f(x_j),\ j=\overline{1,m+1}$

iii) S este continuă în nodurile interioare, i.e. $x_{j+1},\ j=\overline{1,m-1}$

$S(x_{j+1}) = S_{j+1}(x_{j+1}),\ j=\overline{1,m-1}$

b) $f(x) = \ln x \quad (1, e, e^2)$

$S(x) = \begin{cases} S_1(x), & x\in[x_1,x_2] \\ S_2(x), & x\in[x_2,x_3] \end{cases}$, unde $\begin{cases} S_1(x) = a_1 + b_1(x-x_1) \\ S_2(x) = a_2 + b_2(x-x_2) \end{cases}$

$S(x) = \begin{cases} a_1 + b_1(x-1), & x\in(1,e] \\ a_2 + b_2(x-e), & x\in[e,e^2] \end{cases}$

S interpolează f în 3 noduri $\Rightarrow S(x_i)=f(x_i), i=\overline{1,3} \Rightarrow$

$\Rightarrow \cancel{S(x)}\ S(1)=0 \qquad S(e)=1 \qquad S(e^2)=2 \Leftrightarrow$ 4

$\Leftrightarrow S_1(1)=0;\ S_2(e)=1;\ S_2(e^2)=2 \Rightarrow$

$\Rightarrow \underline{a_1=0}$

$\underline{a_2=1}$

$\cancel{a_2+b_2=2 \Rightarrow b_2=1}\ a_2+b_2(e^2-e)=2 \Rightarrow b_2=\frac{1}{e(e-1)}$

De asemenea, S e continuă în $x_1=e \Rightarrow S_1(x_2)=S_2(x_2) \Leftrightarrow$

$\Leftrightarrow \cancel{e-1=2\ e+b_1=2 \Rightarrow b_1=2-e} \Leftrightarrow b_1(e-1)=a_2+b_2(e-e) \Rightarrow \underline{b_1=\frac{1}{e-1}}$

$\cancel{S(x)=\begin{cases}(2-e)(x), & x\in[0,1]\\ 1+\end{cases}}$

$$S(x)=\begin{cases}\frac{1}{e-1}(x-1), & x\in[1,e]\\ 1+\frac{1}{e(e-1)}\cdot(x-e), & x\in[e,e^2]\end{cases}$$

(12) Funcția $S:[a,b]\to\mathbb{R}$ s.n. funcție spline pătratică pentru funcția $f:[a,b]\to\mathbb{R}$ dacă:

i) S este pătratică pe porțiuni:

$S(x)=S_j(x), \forall x\in I_j, j=\overline{1,n}$ unde $S_j: I_j\to\mathbb{R}$

$S_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2, j=\overline{1,n}$ cu $a_j,b_j,c_j\in\mathbb{R}, j=\overline{1,n}$ ce trebuie det.

ii) S interpolează f în $x_j, j=\overline{1,n+1}$

$S(x_j)=f(x_j), j=\overline{1,n+1}$

iii) S este continuă în nodurile interioare $x_{j+1}, j=\overline{1,n-1}$

$S_j(x_{j+1})=S_{j+1}(x_{j+1}), j=\overline{1,n-1}$

iv) S' este continuă în nodurile interioare $x_{j+1}, j=\overline{1,n-1}$

$S'_j(x_{j+1})=S'_{j+1}(x_{j+1}), j=\overline{1,n-1}$

v) Una din urm. condiții este satisfăcută:

- $S'(x_1)=f'(x_1)$
- $S'(x_{n+1})=f'(x_{n+1})$

b) $f(x)=\ln x \quad (1, e, e^2)$

$$S(x)=\begin{cases} S_1(x), & x\in[1,e] \\ S_2(x), & x\in[e,e^2]\end{cases} \Leftrightarrow S(x)=\begin{cases} a_1+b_1(x-x_1)+c_1(x-x_1)^2, & x\in[1,e] \\ a_2+b_2(x-x_2)+c_2(x-x_2)^2, & x\in[e,e^2]\end{cases}$$

$(e^2-e)^2$

$e^4-2e^3+e^2$

$e^2(e^2-2e+1)$

~~Dar~~ S interpolează f în ~~e~~ 3 noduri

$S(x_1)=S_1(x_1)=f(x_1)$; $S(x_2)=S_2(x_2)=f(x_2)$

$S_{\cancel{3}}(x_3)=S_2(x_3)=f(x_3)$

$S_1(1)=0 \Rightarrow a_1=0$

$S_2(e)=1 \Rightarrow a_2=1$

$S_{\cancel{3}}(e^2)=2 \Rightarrow 1+b_2e(e-1)+c_2(e^4-2e^3+e^2)=2$

$$1+b_2e(e-1)+c_2e^2(e^2-2e+1)=2 \quad (*)(1)$$

S continuă în $x_2 \Rightarrow S_1(x_2)=S_2(x_2)$

$b_1(e-1)+c_1(e^2-2e+1)=1 \quad (*)(2)$

Derivata $S'(x)$ este continuă $x_2=e \Rightarrow S_1'(e)=S_2'(e)$

$S_i'(x)=b_i+2c_i(x-1)$, $i=\overline{1,2} \Rightarrow b_1+2c_1(e-1)=b_2+2c_2(e-1) \quad (*)(3)$

$$1,2,3 \Rightarrow \begin{cases} b_2e(e-1)+c_2e^2(e^2-2e+1)-1=0 \\ b_1(e-1)+c_1(e^2-2e+1)-1=0 \\ b_1+2c_1(e-1)-b_2+2c_2(e-1)=0 \quad (4)\end{cases}$$

Alegem $S'(x_1)=f'(x_1) \Rightarrow b_1+2c_1\underbrace{(1-1)}_{0}=\frac{1}{1}=1 \Rightarrow b_1=1$

$$\begin{cases} b_2e(e-1)+c_2(e^2-2e+1)=1 \\ e-1+c_1(e^2-2e+1)=1 \Rightarrow c_1=\dfrac{2-e}{(e-1)^2} \quad (5) \\ \cancel{b_1+2c_1} \end{cases}$$

$e-1+4-2e$

$3-e$

$4,5 \Rightarrow 1+2\dfrac{2-e}{e-1}-b_2-2c_2(e-1)=0$

$b_2+c_2\cdot 2(e-1)=\dfrac{e-3}{e-1}$

$$\begin{cases} b_2 + c_2 \cdot 2(e-1) = \dfrac{e-3}{e-1} \\ b_2 \cdot e(e-1) + c_2(e^2)(e^2-2e+1) = 1 \end{cases}$$

4

$$b_2 = \frac{e-3}{e-1} - c_2\, 2(e-1)$$

$$\left[\frac{e-3}{e-1} - c_2\, 2(e-1)\right] \cdot e\,(e-1) + c_2\,(e^2)(e^2-2e+1) = 1$$

$$c_2\left[e^2(e^2-2e+1) - 2\,e(e-1)^2\right] = 1 - (e-3)(e)$$

$$c_2 = \frac{-e^2 + 3e + 1}{e^4 - 2e^3 + e^2 - 2e^3 + 4e^2 - 2e} = \frac{-e^2+3e+1}{e^4-4e^3+5e^2-2e} = \frac{-e^2+e\cdot 3+1}{e(e-1)^2(e-2)}$$

$$b_2 = \frac{e-3}{e-1} - \frac{-e^2+e\cdot 3+1}{e(e-1)\cancel{(e-1)}(e-2)} \cdot 2\cancel{(e-1)}$$

$$b_2 = \frac{(e-3)\,e\,(e-2) - e^2 + 3e + 1}{e\,(e-1)\,(e-2)} = \frac{e^3 - 2e^2 - 3e^2 + 6e - e^2 + 3e + 1}{e\,(e-1)(e-2)}$$

$$b_2 = \frac{e^3 - 6e^2 + 9e + 1}{e(e-1)(e-2)}$$

$$S = \begin{cases} x - 1 + \dfrac{2-e}{(e-1)^2}(x-1)^2, & x \in [1,e] \\[2ex] 1 + \dfrac{e^3-6e^2+9e+1}{e(e-1)(e-2)}(x-e) + \dfrac{-e^2+3e+1}{e(e-1)^2(e-2)}(x-e)^2, & x \in [e,e^2] \end{cases}$$

(13) $f:[a,b]\to\mathbb{R}$, $a = x_1 < x_2 < x_3 = b$

$$S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2, x\in[x_1,x_2)$$

$$S_2(x) = a_2 + b_2(x-x_2) + c_2(x-x_2)^2, x\in[x_2,x_3]$$

$a_{1,2}, b_{1,2}, c_{1,2}$ $\cancel{=}$? Dacă $s'(x_1) = f'(x_1)$

$S_1(x_1) = f(x_1)$; $S_2(x_2) = f(x_2)$; $S_2(x_3) = f(x_3)$ $\Rightarrow$

$\Rightarrow \underline{a_1} = \cancel{f(a)} = f(x_1)$

$\underline{a_2} = f(x_2)$

$a_2 + b_2(x_3-x_2) + c_2(x_3-x_2)^2 = f(x_3)$

$\bar{S}$ - continuă $\Rightarrow S_1(x_2) = S_2(x_2) \Rightarrow f(x_1) + b_1(x_2-x_1) + c_1(x_2-x_1)^2 = \cancel{f(x_3)}$ $f(x_2)$

$$s'(x) = \begin{cases} b_1 + 2c_1(x-x_1), & x\in[x_1,x_2) \\ b_2 + 2c_2(x-x_2), & x\in[x_2,x_3] \end{cases}, \text{cont în } x_2 \Rightarrow$$

$\Rightarrow s'(x_2) = S_2'(x_2) \Leftrightarrow b_1 + 2c_1(x_2-x_1) = b_2$

~~Als~~ $s'(x_1) = f'(x_1) \Leftrightarrow \underline{b_1} = f'(x_1)$ $\Bigg\}\Rightarrow$

$$\frac{f(x_1) \mp f'(x_1)(x_2-x_1) - f(x_2)}{(x_2-x_1)^2} = \underline{c_1}$$

~~$s'(x_1) = f(x_1)$~~

$$\Rightarrow b_2 = f'(x_1) + 2\cdot\frac{f(x_1) \mp f'(x_1)(x_2-x_1) f(x_2)}{(x_2-x_1)^{\cancel{2}}}\cdot\cancel{(x_2-x_1)} \Rightarrow$$

(x_2-x_1)

$$\Rightarrow \underline{b_2} = f'(x_1) + 2\cdot\frac{f(x_1) \mp f'(x_1)(x_2-x_1) - f(x_2)}{x_2-x_1} = \frac{2(f(x_1)-f(x_2)) - f'(x_1)(x_2-x_1)}{x_2-x_1}$$

$$c_2 = \frac{f(x_3) - a_2 - b_2(x_3 - x_2)}{(x_3 - x_2)^2}$$

$$c_2 = \frac{1}{(x_3 - x_2)^2} \cdot \left[f(x_3) - f(x_2) - \frac{2[f(x_1) - f(x_2)] - f'(x_1)(x_2 - x_1)}{x_2 - x_1} \right]$$

4

(20) Def: $S:[a,b] \to \mathbb{R}$ s.n. f. spline cubică a f. $f:[a,b] \to \mathbb{R}$ dacă:

i) S este cubică pe porțiuni:

$S(x) = S_j(x), \forall x \in I_j, j = \overline{1,n}$ unde

$S_j : I_j \to \mathbb{R}, S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, j = \overline{1,n}$

cu $a_j, b_j, c_j, d_j \in \mathbb{R}, j = \overline{1,n}$ ce trebuie det.

ii) S interpolează f în $x_j, j = \overline{1, n+1}$

$S(x_j) = f(x_j), j = \overline{1, n+1}$

iii) S e continuă în $x_{j+1}, j = \overline{1, n-1}$: $S_j(x_{j+1}) = S_{j+1}(x_{j+1}), j = \overline{1, n-1}$

iv) S' ——— // — : $S_j'(x_{j+1}) = S_{j+1}'(x_{j+1})$, — // —

v) S'' ——— // ——— : $S_j''(x_{j+1}) = S_{j+1}''(x_{j+1})$, — // —

vi) Unul din seturile de condiții este adevărat

1) $S'(x_1) = f'(x_1), S'(x_{n+1}) = f'(x_{n+1})$ ← constrânsă

2) $S''(x_1) = 0, S''(x_{n+1}) = 0$ ← naturală

a) ~~(1,2), (2,3), (3,5), S'(1)=2, S'(3)=1~~

a) $f(0)=0,\ f(1)=1,\ f(2)=2$

$$S(x)=\begin{cases} a_1+b_1(x-x_1)+c_1(x-x_1)^2+d_1(x-x_1)^3, & x\in[0,1)=S_1(x)\\ a_2+b_2(x-x_2)+c_2(x-x_2)^2+d_2(x-x_2)^3, & x\in[1,2]=S_2(x)\end{cases}$$

S interpolează f în $0,1,2 \Rightarrow S_1(0)=f(0),\ S_2(1)=f(1),\ S_2(2)=f(2)$

$a_1=S_1(0)=f(0)\Rightarrow \underline{a_1=0}$

$S_2(1)=f(1)\Rightarrow \underline{a_2=1}$

$a_2+b_2+c_2+d_2=2 \Leftrightarrow b_2+c_2+d_2=1$

S e cont. în $1 \Rightarrow S_1(1)=S_2(1)\Leftrightarrow a_1+b_1+c_1+d_1=1 \Leftrightarrow b_1+c_1+d_1=1$

$$S'(x)=\begin{cases} b_1+2c_1x+3d_1x^2, & x\in[0,1)\\ b_2+2c_2(x-1)+3d_2(x-1)^2 & x\in[1,2]\end{cases}$$

$S'(x)$ cont în $1 \Rightarrow b_1+2c_1+3d_1=b_2+2c_2\underbrace{(x-1)}_{0}+0$

$b_1+2c_1+3d_1=b_2$

$$S''(x)=\begin{cases} 2c_1+6d_1x, & x\in[0,1)\\ 2c_2+6d_2(x-1), & x\in[1,2]\end{cases}$$

$S''(x)$ cont în $1 \Rightarrow 2c_1+6d_1=2c_2 \overset{(1)}{\Rightarrow} 3d_1=c_2$

Aleg $S''(x_1)=0 \quad S''(x_3)=0 \Rightarrow \begin{cases} 2\cdot c_1=0\\ 2c_2+6d_2=0\end{cases} \Leftrightarrow \begin{cases} \underline{c_1=0}\ (1)\\ 2c_2+3d_2=0\end{cases}$

$$\begin{cases} \underline{a_1=0}\\ \underline{a_2=1}\\ \underline{b_2+c_2+d_2=1}\\ b_1+0+d_1=1\\ b_1+0+3d_1-b_2=0\\ \left.\begin{array}{l}3d_1-c_2=0\\ c_2+3d_2=0\end{array}\right\}(2)\\ \underline{c_1=0}\end{cases} \Leftrightarrow \cancel{(2)\Leftrightarrow} \begin{cases} b_2+c_2+d_2=1\\ b_1+d_1=1\\ b_1+3d_1-b_2=0\\ \left.\begin{array}{l}3d_1-c_2=0\\ c_2+3d_2=0\end{array}\right\}(3)\\ \overset{(3)}{\Rightarrow} d_1+d_2=0\end{cases}$$

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$a_1 = 0 \mid a_2 = 1 \mid c_1 = 0$

$$\begin{cases} b_2 + c_2 + d_2 = 1 \\ \left.\begin{array}{l} b_1 + d_1 = 1 \\ b_1 + 3d_1 = b_2 \end{array}\right| \Rightarrow 2d_1 = b_2 - 1 \\ 3d_1 = c_2 \Rightarrow d_1 = \frac{c_2}{3} \\ -3d_2 = c_2 \Rightarrow d_2 = \frac{c_2}{-3} \\ b_2 = 1 + \frac{2}{3}c_2 \end{cases}$$

$\Rightarrow 1 + \frac{2}{3}c_2 + c_2 + \frac{c_2}{-3} = 1 \Leftrightarrow \cancel{3} + 2c_2 + \cancel{3c_2} - \cancel{3c_2} = \cancel{3} \Rightarrow c_2 = 0 \Rightarrow \begin{array}{l} d_1 = 0 \\ d_2 = 0 \end{array}$

$b_1 = 1$
$b_2 = 1$

$$S(x) = \begin{cases} x, & x \in [0,1) \\ \cancel{1} + x - \cancel{1}, & x \in [1, 2] \end{cases} \qquad S(x) = x,\ x \in [0, 2]$$

b) $f(1) = 2$, $f(2) = 3$, $f(3) = 5$

$$S(x) = \begin{cases} a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3, & x \in [1,2) \\ a_2 + b_2(x-x_2) + c_2(x-x_2)^2 + d_2(x-x_2)^3, & x \in [2,3] \end{cases}$$ S interpolează f

$\Rightarrow$ $a_1 = 2$

$a_2 = 3$

$a_2 + b_2 + c_2 + d_2 = 5 \Leftrightarrow b_2 + c_2 + d_2 = 2$

S continuă în 2 $\Rightarrow a_1 + b_1 + c_1 + d_1 = a_2 \Leftrightarrow b_1 + c_1 + d_1 = 1 \Rightarrow b_1 + d_1 = 1$

$$S'(x) = \begin{cases} b_1 + 2c_1(x-1) + 3d_1(x-1)^2, & x \in [1,2) \\ b_2 + 2c_2(x-2) + 3d_2(x-2)^2, & x \in [2,3] \end{cases}$$ cont. în 2 $\Rightarrow b_1 + 2c_1 + 3d_1 = b_2$

$$S''(x) = \begin{cases} 2c_1 + 6d_1(x-1), & x \in [1,2) \\ 2c_2 + 6d_2(x-2), & x \in [2,3] \end{cases}$$ cont în 2 $\Rightarrow 2c_1 + 6d_1 = 2c_2 \Rightarrow 3d_1 = c_2 = -3d_2$

Aleg $S''(1) = 0$ și $S''(3) = 0$ $\Rightarrow$ $2c_1 = 0 \Rightarrow c_1 = 0$; $2c_2 + 6d_2 = 0 \Rightarrow c_2 = -3d_2$

$$\begin{cases} c_1 = 0 \\ a_1 = 2 \\ a_2 = 3 \\ b_2 + c_2 + d_2 = 2 \\ b_1 + d_1 = 1 \\ b_1 + 3d_1 = b_2 \\ 3d_1 = c_2 = -3d_2 \end{cases}$$

$\Rightarrow \cancel{2}d_1 + \cancel{1} + 3d_1 + (-d_1) = \cancel{2}^{1} \Rightarrow d_1 = \frac{1}{4} \Rightarrow$ $d_2 = -\frac{1}{4}$, $c_2 = \frac{3}{4}$, $b_2 = \frac{3}{2}$, $b_1 = \frac{3}{4}$

$\Rightarrow 2d_1 = b_2 - 1 \Leftrightarrow b_2 = 2d_1 + 1$

$d_1 = -d_2 \Leftrightarrow d_2 = -d_1$

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, & x \in [1,2) \\ \cancel{2} + 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 + \left(-\frac{1}{4}\right)(x-2)^3 & x \in [2,3] \end{cases}$$

21 a) ~~(1,2),(2,3)~~ $f(1) = 2$ $f(2) = 3$ $f(3) = 5$ $s'(1) = 2$ $s'(3) = 1$

$$\begin{cases} a_j = f(x_j), \ j = \overline{1,m} \\ b_j(x - x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3 = f(x_{j+1}) - f(x_j), \ j = \overline{1,m} \\ b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2 = b_{j+1}, \ j = \overline{1,m} \\ b_1 = f'(x_1) = 2, \ b_{m+1} = f'(x_{m+1}) = 1 \\ c_{j-1} + 3d_{j-1}(x_j - x_{j-1}) = c_j, \ j = \overline{2,m} \end{cases}$$

$$\begin{cases} a_1 = 2, a_2 = 3 \\ b_1(2-1) + c_1 + d_1 = f(3) - 2 = 1 \\ \cancel{b_2 + 2c_2 + 3d_2 =} \\ b_2 + c_2 + d_2 = 2 \\ b_1 + 2c_1 + 3d_1 = b_2 \\ b_2 + 2c_2 + 3d_2 = 1 \end{cases} \Leftrightarrow \begin{cases} a_1 = 2; a_2 = 3 \\ b_1 = 1; \\ \cancel{b_1} + c_1 + d_1 = 0 \\ \cancel{b_1} + 2c_1 + 3d_1 = b_2 \\ b_2 + c_2 + d_2 = 2 \\ b_2 + 2c_2 + 3d_2 = 1 \\ c_1 + 3d_1 = c_2 \end{cases}$$

$c_1 = -d_1 \Rightarrow d_1 = b_2 - 1$

$c_1 + 2d_1 = b_2 - 1$

$\Rightarrow c_2 + 2d_2 = -1 \Rightarrow d_2 = \frac{-1 - c_2}{2}$

$\Rightarrow c_2 = 2d_1$ $d_2 = \frac{-1 - 2d_1}{2}$

$c_2 = 2b_2 - 2$ $d_2 = \frac{+1 - 2b_2}{2}$

$c_1 + 3d_1 \underbrace{(x_2 - x_1)}_{1} = c_2$

$b_1 = 2$

$\cancel{b_2} + 2b_1 - 2 + \frac{1}{2} - \cancel{b_2} = 2 \Rightarrow b_2 = \frac{7}{4}$

$x_1 = 0$

$x_2 = 1$

22 B, C, D a.î.

$$S(x) = \begin{cases} S_1(x) = 1 + 2x - x^3, & x \in [0,1) \\ S_2(x) = 2 + B(x-1) + C(x-1)^2 + D(x-1)^3, & x \in [1,2] \end{cases}$$

$S''(0) = S''(2) = 0$

$$S'(x) = \begin{cases} 2 - 3x^2, & x \in [0,1) \\ B + 2C(x-1) + 3D(x-1)^2 \end{cases} \Rightarrow S''(x) = \begin{cases} -6x, & x \in [0,1) \\ 2C + 6D(x-1), & x \in [1,2] \end{cases}$$

$S''(0) = 0$; $S''(2) = 2C + 6D = 0 \Rightarrow C = -3D$

S'' cont în 1 $\Rightarrow -6 = 2C \Rightarrow \boxed{C = -3} \Rightarrow \boxed{D = 1}$

$$S'(x) = \begin{cases} 2 - 3x^2, & x \in [0,1) \\ B + (-6)(x-1) + 3(x-1)^2, & x \in [1,2] \end{cases}$$ cont în 1 $\Rightarrow \boxed{-1 = B}$

23 $S'(1) = f'(1)$; $S'(3) = f'(3)$

$$S(x) = \begin{cases} S_1(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & x \in [1,2) \\ S_2(x) = A + B(x-2) + C(x-2)^2 + D(x-2)^3, & x \in [2,3] \end{cases} \quad (1)$$

$f'(1) = f'(3) \Rightarrow S'(1) = S'(3)$

$$S'(x) = \begin{cases} 3 + 4(x-1) - 3(x-1)^2, & x \in [1,2) \\ B + 2C(x-2) + 3D(x-2)^2, & x \in [2,3] \end{cases} \quad (2) \Rightarrow 3 = \underset{4}{B} + 2\underset{-1}{C} + 3D$$

$3 = 2 + 3D \Rightarrow \boxed{D = \frac{1}{3}}$

(1) cont. in 2 $\Rightarrow 3 + 2 - 1 = A \Rightarrow \boxed{A = 4}$

(2) cont în 2 $\Rightarrow \boxed{4 = B}$

$$S''(x) = \begin{cases} 4 - 6(x-1), & x \in [1,2) \\ 2C + 6D(x-2), & x \in [2,3] \end{cases}$$ cont în 2 $\Rightarrow -2 = 2C \Rightarrow \boxed{C = -1}$

(24) $S(x)=\begin{cases} S_1(x)=1+Bx+2x^2-2x^3, & x\in[0,1) \\ S_2(x)=1+b(x-1)-4(x-1)^2+7(x-1)^3, & x\in[1,2] \end{cases}$

$f'(0), f'(2)=?$

S funcție spline cu constrângeri $\Rightarrow$ $\cancel{s''(0)}=$
$s'(0)=f'(0)$,
$s'(2)=f'(2)$

$s'(x)=\begin{cases} B+4x-6x^2, & x\in[0,1) \\ b-8(x-1)+21(x-1)^2 & x\in[1,2] \end{cases}$ cont în $1 \Rightarrow \cancel{s'(1)}=B+(-2)=b \Rightarrow \boxed{b=-2}$

S cont în $1 \Rightarrow 1+B+\cancel{2}-\cancel{2}=1 \Rightarrow \boxed{B=0}$

$s'(0)=B=0=f'(0)$

$s'(2)=-2-8+21=11=f'(2)$

(25)
$S(x)=\begin{cases} S_1(x)=1+B(x-1)-D(x-1)^3, & x\in[1,2) \\ S_2(x)=1+b(x-2)-\frac{3}{4}(x-2)^2+d(x-2)^3, & x\in[2,3] \end{cases}$

$B, D, d, b=?$

$f(1)=1; f(2)=1, f(3)=0$ S interpolează f $\Rightarrow$

$\Rightarrow S(1)=1; S(2)=1; S(3)=0 \Leftrightarrow 1+b-\frac{3}{4}+d=0 \Leftrightarrow \underline{b+d=-\frac{1}{4}}$

S cont în $2 \Rightarrow 1+B-D=1 \Leftrightarrow B-D=0 \Leftrightarrow \boxed{B=D}$

$s'(x)=\begin{cases} B-3D(x-1)^2, & x\in[1,2) \\ b-\frac{3}{2}(x-2)+3d(x-2)^2, & x\in[2,3] \end{cases}$ cont în $2 \Rightarrow \underline{B-3D=b}$

$s''(x)=\begin{cases} -6D(x-1), & x\in[1,2) \\ -\frac{3}{2}+6d(x-2), & x\in[2,3] \end{cases}$ cont. în $2 \Rightarrow -6D=-\frac{3}{2} \Rightarrow \boxed{D=\frac{1}{4}=B}$

$\boxed{b=\frac{1}{4}-\frac{3}{4}=-\frac{1}{2}}$

$\boxed{d=\frac{1}{4}}$

(26) $f(x-h), f(x+h), f(x+2h), f(x+3h)$

$A f(x-h) + B f(x+h) + C f(x+2h) + D f(x+3h)$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$\cancel{A f(x-h) = A(f(x) - f'(}$

$a \leftarrow x-h$

$$A f(x+h) = A\left(f(x-h) + f'(x-h)(h) + \frac{f''(x-h)}{2}(h)^2 + \frac{f'''(x-h)}{6} \cdot h^3 + \frac{f''''(x-h)}{24} h^4\right)$$

(29) $\int_0^2 f(x)dx = 5$ prin formula trapezului

$\int_0^2 f(x)dx = 4$ — // — dreptunghiului

$\int_0^2 f(x)\,dx = ?$ — // — Simpson

Prin formula trapezului $\int_0^2 f(x)dx = \frac{2-0}{2}\left[f(0)+f(2)\right] = 5 \Rightarrow$

$\Rightarrow f(0) + f(2) = 5$

Prin formula dreptunghiului $\int_0^2 f(x)dx = (2-0) f\left(\frac{2}{2}\right) = 2f(1) = 4 \Rightarrow f(1) = 2$

Formula Simpson: $\int_a^b f(x)dx = \frac{b-a}{6}\left[\underline{f(a)} + 4f\left(\frac{a+b}{2}\right) + \underline{f(b)}\right]$

$$= \frac{2}{6}(5+8) = \frac{13}{3}$$

(30)

Formula trapezului $\int_0^2 f(x)dx = 4 = I$

— // — Simpson ——— $= 2 \Rightarrow I = 2$

$f(1) = ?$

Formula trapezului $I = f(0) + f(2) = 4$

Formula Simpson $I = \frac{1}{3}(4 + 4f(1)) = 2 \Rightarrow f(1) = \frac{2}{4} = \frac{1}{2}$

(31) $f(0)=1$, $f(0,5)=2,5$, $f(1)=2$, $f(0,25)=f(0,75)=\alpha$

$\alpha=?$

Prin formula trapezului însemnată cu $m=4$ $\underbrace{\int_0^1 f(x)dx}_{I} \approx 1,75$

$$I=\frac{h}{2}\left(f(x_1)+2\sum_{k=2}^{m} f(x_k)+f(x_{m+1})\right), \text{ unde } h=\frac{b-a}{m}=\frac{1}{4}$$

$(x_1, x_2, x_3, x_4, x_5)=(0, 0,25, 0,5, 0,75, 1)$

$$I=\frac{1}{8}\left(1+2(2,5+2\alpha)+2\right)\approx 1,75 \Leftrightarrow$$

$$\underbrace{1+5+2}_{8}+4\alpha \simeq 14 \Leftrightarrow \alpha \simeq \frac{3}{2}$$

(32)
$$\begin{cases} x'=-5x+5t^2+2t, & 0\leq x\leq 0,1 \\ x(0)=\frac{1}{3}, & h=0,1 \end{cases}$$

a) Să se dem. că ec. admite proprietatea de E.U.L.

b) $x(t)=t^2+\frac{1}{3}e^{-5t}$ este soluție?