A General Approach to the Fusion of Imprecise Information

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We consider the problem of fusion of multiple information sources, particularly in environments when the sensor observations are imprecise. The concept of a combinability relationship is introduced to enable the inclusion in the fusion process of information about the appropriateness of fusing different elements from the observation space. This idea allows for the use of an expert knowledge base, containing information about the domain of the particular problem, in the fusion process and leads to a more intelligent aggregation. We show that if we use a combinability relationship that only allows fusion of identical elements then the only idempotent fusion of any collection of fuzzy observations is their intersection. Using the idea of the fuzzy measure we considered situations in which we allow a partial collection of the observations determine the fused value. © 1997 John Wiley & Sons, Inc.

I. INTRODUCTION

A problem of considerable interest in many domains is the aggregation of information from different sources or sensors, this is often called information or sensor fusion.[1–3] In the following we formally describe this problem.

Let V be some variable taking its value in a set X. Assume we have multiple sensor readings for this variable; $a_1, \ldots, a_n$. The problem of information fusion or condensation consists of combining these pieces of information to obtain one representative value compatible with the observations. In order to accomplish this task we must provide some function F, called the *fussion function*, which takes these multiple readings and provides some combined single value. Thus we have

$$F(a_1, \ldots, a_n) = b$$

where b is called the *fused* value. While the choice of the function F is very situation dependent, intuitively one can require certain basic properties as almost always being necessary in F. One key property characterizing the function F in the environment of concern to us is that of *idempotency*. We recall that idempotency imposes the requirement that if all the a_i are the same then our

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fused value should be this value. Formally this condition states that if $a_i = a$ for all i then

$$F(a_1, \ldots, a_n) = a$$

The imposition of this condition on the function F distinguishes it from other aggregation problems such as pattern recognition and multicriteria decision making where the possibility may exist for reinforcement.[4,5]

A second property often associated with fusion operators is that of *communtativity*. This condition essentially assures us that each of the arguments are treated in the same manner and hence that the indexing of the arguments does not effect the final aggregated value.

In situations in which the underlying space X is the real line, an environment in which we shall be concentrating, a *monotonicity* condition is generally required. This condition specifies that if $a_i \geq b_i$ for all i then

$$F(a_1, \ldots, a_n) \geq F(b_1, \ldots, b_n).$$

A fusion operator F having these three properties is called a mean operator.[6,7] In the numeric environment the prototypical mean operator is the simple averaging operator,

$$F(a_1, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i$$

We must point out that while prototypical, the simple average is one among many possible candidate mean operators (see Yager[7] for a full discussion of mean operators). Other examples of mean operators are Max, Min, Median, and Mode.

In some aggregation situations we have associated with each argument a_i in the aggregation, a weight w_i indicating its importance in the fusion process. The prototypical case of this situation is the weighted average where

$$F(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i.$$

We note that the assumption of monotonicity and idempotency require that the weights sum to one and be in the unit interval.

II. FUSION OF IMPRECISE INFORMATION

In many environments in which we are required to fuse information the information provided may be imprecise. For example, a sensor may say the object is located at about 2 miles. Human information often comes in the form of linguistic values, such as close, nearby, and far. A natural framework for representing this kind of information is terms of a fuzzy subset.[8–10] Assume V is some variable taking its value in the space X. Assume our information regarding the value of V is known in terms of some concept $\mathscr{A}$ near ten, for example. We can express this information in terms of a fuzzy subset A of X, such that for any $x \in$

X, $A(x)$ indicates the degree of compatibility of x with the concept $\mathscr{A}$. We note that if we know V is exactly the value x^*, precise information, we can express this in terms of a fuzzy subset A such that

$$A(x^*) = 1$$
$$A(x) = 0 \text{ for } x \neq x^*.$$

Another special case occurs when we have no information, in this case $A = X$, $A(x) = 1$ for all x. Ranges can be also seen as special cases of this structure. For example if we know that V lies between 20 and 30 we can represent this as a set A where

$$A(x) = 1 \qquad \text{for } 20 \leq x \leq 30$$
$$A(x) = 0 \qquad \text{for all other } x$$

We now turn to the problem of sensor fusion when we have to aggregate pieces of fuzzy information. In order to accomplish this task we must make use of the *extension principle* from fuzzy logic.[8–10] In the following we briefly describe the extension principle. Assume $X_1, X_2, \ldots, X_n$ are a collection of sets and let Y be another set. Assume G is a point mapping

$$G: X_1 \times X_2 \ldots \times X_x \rightarrow Y$$

hence

$$G(x_1, x_2 \ldots x_n) = y$$

where $x_i \in X_i$ and $y \in Y$. Thus G maps points from the cartesian product space, $X = X_1 \times X_2 \ldots \times X_n$, into Y. The extension principle provides a methodology for extending the mapping G from operating on points in the cartesian space to operating on fuzzy subsets in this cartesian space. Assume $A_1, A_2, \ldots, A_n$ are a collection of fuzzy subsets such that A_i is a fuzzy subset of X_i. The cartesian product of the A_i, denoted $A = A_1 \times A_2 \ldots \times A_n$, is a fuzzy subset of X such that for each n-tuple $(x_1, x_2, \ldots, x_n) \in X$

$$A(x_1, x_2, \ldots, x_n) = \mathrm{Min}_i[A_i(x_i)].$$

Let I^{X_i} indicate the collection of fuzzy subsets of X_i. The extension principle allows us to extend G to a mapping $\mathscr{G}$

$$\mathscr{G}: I^{X_1} \times I^{X_2} \times \ldots I^{X_n} \rightarrow I^Y$$

i.e.

$$\mathscr{G}(A_1, \ldots, A_n) = B$$

where $A_i \in I^{X_i}$ and $B \in I^Y$. We define

$$\mathscr{G}(A_1, \ldots, A_n) = \left\{\frac{A(x_1 \ldots x_n)}{G(x_1 \ldots x_n)}\right\} = B$$

where B is a fuzzy set of Y and where we recall

$$A(x_1, x_2, \ldots, x_n) = \text{Min}_i[A_i(x_i)].$$

Alternatively we can express the fuzzy subset B via its membership function as

$$B(y) = \text{Max}[A(x_1 \ldots x_n)]$$

where the Max is taken over all $(x_1 \ldots x_n)$ such that $y = G(x_1 \ldots x_n)$.

We note that the function $\mathcal{G}$ is a true extension of G in that for the case when the A_i's are point sets we get the same result as we would get using G. Consider $G(x_1, \ldots, x_n) = y$. Assume we represent the value x_i as a fuzzy subset. In this case we have

$$A_i = \left\{\frac{1}{x_i}\right\}.$$

Then we must find

$$\mathcal{G}(A_1, \ldots, A_n) = B = \left\{\frac{A(x_1, \ldots, x_n)}{G(x_1, \ldots, x_n)}\right\}.$$

However we note that in this special case

$$A = \left\{\frac{1}{A(x_1, \ldots, x_n)}\right\}$$

and thus

$$B = \left\{\frac{1}{G(x_1, \ldots, x_n)}\right\} = \{G(x_1, \ldots, x_n)\} = y.$$

It should be noted that via the extension principle we can introduce fuzzy arithmetic. Consider any arithmetic operation $\perp$ $(+, -, *, /)$ where $x_1 \perp x_2 = y$, we can express this as a mapping

$$G(x_1, x_2) = x_1 \perp x_2 = y.$$

We now extend this to operate on fuzzy number A_1 and A_2, fuzzy subsets of the real line,

$$\mathcal{G}(A_1, A_2) = A_1 \boxed{\perp} A_2 = B$$

where B is also fuzzy subset of R. In particular

$$\mathrm{A}_1 \boxed{\perp} A_2 = \left\{\frac{A_1(x_1) \wedge A_2(x_2)}{x_1 \perp x_2}\right\}.$$

A special case of the above is that of multiplying a fuzzy number by a constant. Assume k is some constant and A is some fuzzy number. We express kA as the fuzzy number

$$B = \left\{\frac{A(x)}{kx}\right\}.$$

Having introduced the extension principle we can now easily provide for the fusion of fuzzy information. Let F ordinary fusion function defined as,

$$F: X_1 \times X_2 \times \ . \ . \ . \times X_n \rightarrow Y,$$

that is, F takes points in the cartesian space $X_1 \times X_2 \times \ . \ . \ . \times X_n$ and delivers a value in the space Y. In the purely numeric environment all X_i and Y are the real line $\mathbb{R}$.* Assume $A_1, \ . \ . \ . \ , A_n$ are fuzzy observations then we obtain the fusion of these observations, $\mathbf{F}(A_1, \ . \ . \ . \ , A_n)$, as

$$\mathbf{F}(A_1, \ . \ . \ . \ , A_n) = B = \left\{\frac{\operatorname{Min}_i[A_i(a_i)]}{F(a_1, \ . \ . \ . \ , a_n)}\right\}.$$

Consider the special case when our fusion function F is the simple average,

$$F(a_1, \ . \ . \ . \ , a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

To obtain the average of fuzzy n observations we get

$$F(A_1, \ . \ . \ . \ , A_n) = \frac{A_1 \oplus A_2 + \oplus A_n}{n} = B$$

and thus

$$B = \left\{\frac{\operatorname{Min}_i A_i(a_i)}{\frac{1}{n}\sum_{i=1}^{n} a_i}\right\}$$

For the case of the weighted average we obtain

$$F(A_1, \ . \ . \ . \ , A_n) = w_1 A, \oplus w_2 A_2 \oplus \ . \ . \ . \oplus w_n A_m = B$$

and thus from the extension principle we get

$$B = \left\{\frac{\operatorname{Min} A_i(a_i)}{\sum_{i=1}^{n} w_i a_i}\right\}.$$

We now look at the special case when the A_i's are fuzzy numbers.[11,12] We recall that a fuzzy number is a unimodal fuzzy subset on the real line $\mathbb{R}$.

*When working in infinite domains, such as the real line, a number of technical mathematical issues such as the use of the supremum instead of the Max and the issue of noncontinuous fuzzy sets arise. As this work is at an engineering level we shall not formally address these issues and assume all indicated operations have results that are meaningful.

That is if A is a fuzzy number there exists two values on the real line a and b such that

$$\begin{array}{lll} A(x) \geq A(y) & \text{for} & y \leq x \leq a \\ A(x) = 1 & \text{for} & a \leq x \leq b \\ A(x) \leq A(y) & \text{for} & x \geq y \geq b \end{array}$$

We next recall the concept of a level set associated with a fuzzy subset. Assume A is a fuzzy subset of R, the α-level set of A, denoted A_α, is a crisp subset of $\mathbb{R}$ defined such that

$$A_\alpha = \{x/A(x) \geq \alpha\}.$$

It can be easily shown that for any $\alpha_1 \geq \alpha_2$

$$A_{\alpha_1} \subseteq A_{\alpha_2}.$$

Furthermore the convexity of A leads to a situation in which each α-level set is an interval thus we can express

$$A_\alpha = [a_\alpha, b_\alpha].$$

We recall[13,14] that given the level sets associated with a fuzzy subset we can express the associated fuzzy subset as

$$A = \bigcup_\alpha \tilde{A}_\alpha$$

where $\tilde{A}_\alpha$ is the fuzzy subset defined such that

$$\tilde{A}_\alpha(x) = \alpha \wedge A_\alpha(x).$$

We note that since A_α is a crisp set then

$$\begin{array}{ll} \tilde{A}_\alpha(x) = \alpha & \text{if } A_\alpha(x) = 1 \\ \tilde{A}_\alpha(x) = 0 & \text{if } A_\alpha(x) = 0. \end{array}$$

One reason for introducing the level sets is that many of the operations on fuzzy subsets can be performed in terms of level sets. Assume $A_1, \ldots, A_n$ are a collection of fuzzy numbers, let A_{i_α} indicate the α level set of the fuzzy number A_i. Furthermore let us denote

$$A_{i_\alpha} = [a_{i_\alpha}, b_{i_\alpha}]$$

as the interval associated with this level set. Consider the addition of these fuzzy numbers

$$B = A_1 \oplus A_2 \oplus \ldots \oplus A_n.$$

We can express this in terms of operations on the level sets. In particular

$$B_\alpha = \left[\sum_{i=1}^{n} a_{i_\alpha}, \sum_{i=1}^{n} b_{i_\alpha}\right]$$

More generally if

$$B = w_1 A_1 \oplus w_2 A_2 \oplus \ldots \oplus w_n A_n$$

where w_i are a collection of regular weights then

$$B_\alpha = \left[\sum_{i=1}^{n} w_i a_{i_\alpha}, \sum_{i=1}^{n} w_i b_{i_\alpha}\right].$$

In the special case where $w_i = 1/n$ then

$$B_\alpha = \left[\frac{1}{n}\sum_{i=1}^{n} a_{i_\alpha}, \frac{1}{n}\sum_{i=1}^{n} b_{i_\alpha}\right]$$

In the preceding we have provided the basic methodology for fusing fuzzy information.

III. COMPATIBILITY CONSTRAINED FUSION OPERATORS

Let us return to the simple problem of the fusion of ordinary numbers,

$$F(a_1, \ldots, a_n) = b.$$

As we noted a prototypical operation used to perform this task is $b = \frac{1}{n}\sum a_i$. By its very nature these fusion operators provide an answer that lies in the middle; it tries to balance extremes. One problem that arises in using these operators is that we may get fused values that are not really satisfactory to any of the components in the aggregation. For example if we have $a_1 = 100$ and $a_2 = 0$ the average is $b = 50$. This is a value that is not very compatible with either of the components being aggregated. The reason for this lack of compatibility is the attempt to aggregate disparate values. In constructing intelligent fusion systems we may desire to avoid the aggregation of items that are very dissimilar or at the very least provide information that the aggregated value is based upon an aggregation of values that are essentially incompatible. In the following we shall introduce the idea of a combinability relationship which shall convey information about the allowability of fusing elements. The actual form of this relationship will be situationally dependent and as such can be seen as a knowledge base which contains meta-information about the problem in which we are using the fusion operation.

Assume we have selected some fusion function F, the average for example. Let X be the space from which our observations are drawn. We shall further assume the existence of a relationship.

$$R: X \times X \rightarrow I,$$

called the *combinability relationship*. For any two values a and b drawn from X,

$R(a, b)$ is used to indicate the degree to which it is acceptable to fuse the values a and b under F. Alternatively we can view $R(a, b)$ as indicating our confidence in fusing these two values. It is natural to require that $R(a, a) = 1$ for all $a \in X$. Furthermore, as the distance between a and b, on the metric associated with the space X, increases then $R(a, b)$ should not get larger, that is if Distance(a, b) $\geq$ Distance(a, b') then $R(a, b) \leq R(a, b')$. We also assume that R is symmetric, $R(a, b) = R(b, a)$. It should be noted that this combinability relationship is in spirit close to the idea of a similarity relationship introduced by Zadeh[13]; however, transitivity is not required. While we have defined R as a binary relationship on X we can naturally extend it to act on m-tuples form X as follows

$$R(a_1, \ldots, a_n) = \underset{\substack{\text{overall pairs} \\ a_i, a_j}}{\mathrm{Min}} [R(a_i, a_j)].$$

If we are now faced with the problem of fusing the collection $(a_1, \ldots, a_n)$ we can then calculate $F(a_1, \ldots, a_n)$ as our fused value and also provide the value $R(a_1, \ldots, a_n)$ as our degree of confidence in this fused value. In spirit, this combinability value is related to the inverse of the variance. However we should note it allows for a more sophisticated representation of the knowledge about the combinability.

Let us now turn to the case when the values to be fused are fuzzy subsets, it is in this case that we shall see the full impact of the use of the combinability relationship. Assume $A_1, A_2, \ldots, A_n$ are a collection of fuzzy subsets, over the space X, which are to be fused. We recall that if F is our fusion function then

$$F(A_1, \ldots, A_n) = \left\{ \frac{A(x_1 \ldots x_n)}{F(x_1 \ldots x_n)} \right\} = B$$

where

$$A(x_1 \ldots x_n) = \mathrm{Min}_i[A_i(x_i)].$$

Thus B is a fuzzy subset of X defined as such that for each x,

$$B(x) = \underset{\substack{\text{all} \\ (x_1, \ldots, x_n) \\ \text{such that} \\ F(x_1, \ldots, x_n) = x}}{\mathrm{Max}} [A(x_1, \ldots, x_n)]$$

In the above for each $xB(x)$ indicates the possibility of that x being the fused value under F in the face of the observations. In situations in which we have a combinability function R by considering the allowability of fusing the elements in each of the tuples. In particular we now require that the resulting fusion be one that only allows the aggregation of points that are combinable. This additional requirement leads to a fused value that must satisfy

$$F(A_1, \ldots, A_n) \textbf{ and } R(A_1, \ldots, A_n).$$

Hence in the situation in which we have a combinability function R we must modify our process and calculate the fused value F_R as

$$F_R(A_1, . . . , A_n) = \left\{\frac{R(x_1, . . . , x_n) \wedge A(x_1, . . . , x_n)}{F(x_1, . . . , x_n)}\right\} = B$$

where again B denotes the fuzzy subset of X corresponding to the fused value.

In the above the effect of the combinability relationship R is to essentially reduce the membership grade associated with tuples that are incompatible. In the above we let $\text{Max}_x B(x)$ be an indication of the overall compatibility of this fusion. In particular, if we try to fuse two fuzzy subsets which are far away from each other then we shall attain a situation in which $\text{Max}_x B(x)$ is low.

In the special case when the A_i's are singletons, $A_i = \left\{\frac{1}{a_i}\right\}$, we get

$$A(a_1, . . . , a_n) = 1$$
$$A(x_1, . . . , x_n) = 0 \qquad \text{for all other tuples}$$

hence B is a fuzzy set with one element having nonzero membership grade, $F(a_1, . . . , a_n)$, and its membership grade is $R(a_1, . . . , a_n)$.

With the aid of the combinability function R we are able to introduce meta-knowledge about the underlying space X regarding the acceptability of fusing objects. Let us now look at the performance of his knowledge-based approach to fusion for some notable combinability relationships.

We first consider the case when

$$R(x, y) = 1 \qquad \text{for all } x \text{ and } y.$$

Here we are assuming that it is acceptable to fuse any values from the space X. In this case

$$F_R(A_1, . . . , A_n) = \left\{\frac{R(x_1 . . . x_n) \wedge A(x_1 . . . x_n)}{F(x_1 . . . x_n)}\right\}$$

and since $R(x_1, . . . , x_n) = 1$ for all tuples, we get

$$F_R(A_1, . . . , A_n) = F(A_1, . . . , A_n)$$

the ordinary fusion of fuzzy values. Thus the ordinary fusion function is a special case of the constrained fusion when all fusions are completely acceptable.

We now consider the opposite extreme

$$R(x, x) = 1$$
$$R(x, y) = 0 \qquad \text{for } x \neq y.$$

In this case we only allow the fusion of elements that are the same. In this case we first note that the extended combinability relationship is

$$R(x_1, . . . , x_n) = 1 \qquad \text{if all } x_i \text{ are the same}$$
$$R(x_1, . . . , x_n) = 0 \qquad \text{otherwise.}$$

Let us first consider the situation where F is the simple average. In this case

$$\text{Average}_R(A_1, \ldots, A_n) = \left\{\frac{A(x, \ldots, x)}{F(x, \ldots, x)}\right\}.$$

Since

$$\frac{1}{n}\sum_{i=1}^{n} x = x$$

we have

$$\text{Average}_R(A_1, \ldots, A_n) = \left\{\frac{A(x, x, \ldots, x)}{x}\right\}$$

where

$$A(x, \ldots, x) = \text{Min}_i[A_i(x)]$$

Thus we get

$$B = \text{Average}_R(A_1, \ldots, A_n) = \left\{\frac{\text{Min}_i A_i(x)}{x}\right\}$$

However, we recall that if

$$B(x) = \text{Min}_i[A_i(x)]$$

then

$$B = A_1 \cap A_2 \qquad \cap A_n$$

Thus we see that with choice of combinability relationship the fusion based upon the average operation reduces to the intersection of the fuzzy values to be fused.

The following generalization of the above produces a very important and useful theorem.

THEOREM. *Assume X is any space. Let F be any* idempotent *fusion operator, $F(x, \ldots, x) = x$, and let the combinability relationship be $R(x, x) = 1$ and $R(x, y) = 0$ for $x \neq y$ then for any fuzzy values $A_1, \ldots, A_n$.*

$$B = F_R(A_1, \ldots, A_n) = A_1 \cap A_2 \cap \qquad \cap A_n$$

which is simply the intersection of the fused values.

Proof.

$$F_R(A_1, \ldots, A_n) = \left\{\frac{R(x_1, \ldots, x_n) \wedge A(x_1, \ldots, x_n)}{F(x_1, \ldots, x_n}\right\}$$

where $A(x_1, \ldots, x_n) = \mathrm{Min}_i[A_i(x_i)]$. Since $R(x_1, \ldots, x_n) = 0$ if there exists at least one pair $x_i \neq x_j$ otherwise $R(x_1, \ldots, x_n) = 1$ we get

$$F_R(A_1, \ldots, A_n) = \left\{\frac{A(x, \ldots, x)}{F(x, \ldots, x)}\right\}.$$

From the idempotency property of F, we get

$$F(x, x, \ldots, x) = x$$

thus

$$F_R(A_1, \ldots, A_n) = \left\{\frac{A(x, \ldots, x)}{x}\right\} = B$$

however, since

$$A(x, x, \ldots, x) = \mathrm{Min}_i[A_i(x)]$$

we see that

$$F_R(A_1, \ldots, A_n) = A_1 \cap A_2 \cap \ldots \cap A_n.$$

The result of this theorem shows us that the only idempotent fusion operation in the case in which only identical elements are combinable is the intersection.

Assume R_1 and R_2 are two combinability functions and let B_1 and B_2 be the fused value of the collection $A_1, \ldots, A_n$ with respect to the fusion operator F using these relationships, respectively. It is easily seen that if $R_1 \subset R_2$, $R_1(x, y) \leq R_2(x, y)$ for all x and y, then $B_1 \subseteq B_2$. Let us denote R^* as the combinability function with $R(x, y) = 1$ for all x and y and R_* as the combinability function with

$$R_*(x, x) = 1$$
$$R_*(x, y) = 0 \qquad \text{for all } x \neq y.$$

We easily see that for any combinability function R

$$R \subset R^*$$
$$R_* \subset R.$$

From this it follows that if B is the fused value using R then

$$B_* \subseteq B \subseteq B^*.$$

Thus the unconstrained fusion always leads to the largest fuzzy set while the totally constrained fusion leads to the smallest fuzzy subset.

It is significant to point out that in introducing this compatibility function we have provided a means for introducing information about the space underlying the observation domain. While in the preceding we have considered some unique examples of R, more generally, R is highly dependent upon the situation in which we are in. Specifically R can be seen as a carrier of our expert knowledge of the situation. Thus we can envision the structure of R being generated by some knowledge base technology such as an expert system or fuzzy system model.[6] Thus R provides a means for inserting expert meta-knowledge about the observation space and help in making more intelligent type aggregations.

IV. AGGREGATION OF FUZZY NUMBERS USING THE COMPATIBILITY

Let us now consider the special case of combinability constrained fusion where the arguments are fuzzy numbers. Assume $A_1, \ldots, A_n$ are a collection of fuzzy numbers and our fusion function is the simple average. Furthermore, assume we have a combinability function R defined on the space of real numbers. The following describes an approach, using the idea of level sets for performing the average of these numbers. We let

$$\frac{A_1 \oplus A_2 \oplus \ldots \oplus A_n}{n} = B.$$

For any level α we denote

$$A_{i_\alpha} = [a_{i_\alpha}, b_{i_\alpha}]$$

as the α-level set associated with the fuzzy number A_i. Furthermore we let

$$a^*_\alpha = \text{Max}_i[a_{i_\alpha}]$$

$$b^*_\alpha = \text{Min}_i[b_{i_\alpha}].$$

Thus a^*_α is the largest lower bound on any α-level set and b^*_α is the smallest upper bound on any α-level set. Next we let U^*_α be the minimum x such that $R(a^*_\alpha, x) \geq \alpha$. Thus we see that U^*_α is the smallest value that is within α combinability of a^*_α. We also let V^*_α be the maximum x such that $R(b^*_\alpha, x) \geq \alpha$. Thus V^*_α is the largest value that is within α combinability of b^*_α.

Using this information we are able to calculate the α-level sets of the aggregated value B,

$$B_\alpha = [d_\alpha, e_\alpha].$$

In particular

$$d_\alpha = \frac{1}{n}\sum_{i=1}^{n} g_{i\alpha}$$

where

$$g_{i\alpha} = a_{i\alpha} \qquad \text{if} \qquad a_{i\alpha} \geq U^*_\alpha$$

$$g_{i\alpha} = U_{i\alpha} \qquad \text{if} \qquad a_{i\alpha} < U^*_\alpha$$

and

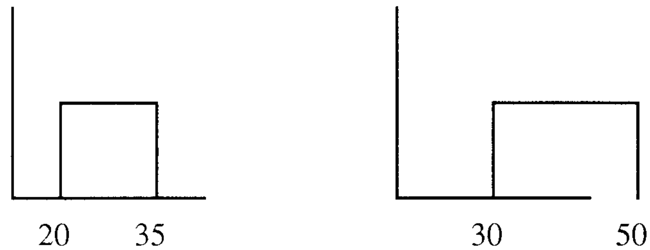

Figure 1. Values to be aggregated.

$$e_\alpha = \frac{1}{n} \sum_{i=1}^{n} h_{i\alpha}$$

where

$$h_{i\alpha} = b_{i\alpha} \quad \text{if} \quad b_{i\alpha} \le V^*_\alpha$$

$$h_{i\alpha} = V^*_\alpha \quad \text{if} \quad b_{i\alpha} > V^*_\alpha.$$

We further note that if we are using a weighted aggregation,

$$B = w_1 A_1 \oplus w_2 A_2 \oplus \ldots \oplus W_n A_n,$$

then the only change in the above is that

$$d_\alpha = \sum_{i=1}^{n} w_i g_{i\alpha}$$

$$e_\alpha = \sum_{i=1}^{n} w_i h_{i\alpha}.$$

The following simple example illustrates the procedure.

Example. Assume we have two sources of information and our fusion function is the simple average,

$$B = \frac{A_1 \oplus A_2}{2}.$$

For the example at hand we consider that

A_1 = "between 20 and 35"

A_2 = "between 30 and 50"

Figure 1 illustrates these sets. We note that in this special case for each argument the level sets are the same at each level, thus we shall suppress the indication of the degree of level,

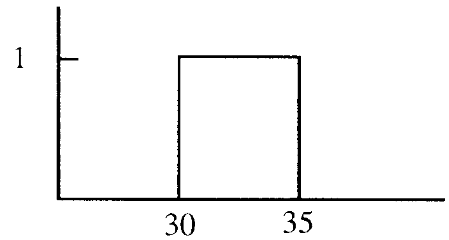

Figure 2. Fused value under R_*.

$$A_{1\alpha} = A_1 = [20, 35]$$
$$A_{2\alpha} = A_2 = [30, 50].$$

In the following we shall investigate the aggregation of these values for different combinability functions

(I) First we assume that R is the very strict combinability function R_*:

$$R(x, y) = 1$$
$$R(x, y) = 0 \qquad x \neq 1.$$

As we have previously indicated in this case the aggregated value B is simply the intersection of the arguments hence

$$B = A_1 \cap A_2$$

and thus we get the result shown in Figure 2.

(II) Next we consider the most lax combinability function R^*,

$$R(x, y) = 1 \qquad \text{for all } x \text{ and } y.$$

This situation reduces to the simple case of fuzzy averaging

$$B = [e, d]$$

where

$$e = \frac{1}{2}(20 + 30) = 25$$

$$d = \frac{1}{2}(35 + 50) = 42.5.$$

Thus the fused value is as in Figure 3.

(III) Here we consider the situation where

$$R(x, y) = 1 \qquad \text{if } |x - \mathrm{y}| \leq 5$$
$$R(x, y) = 0 \qquad \text{if } |x - \mathrm{y}| > 5,$$

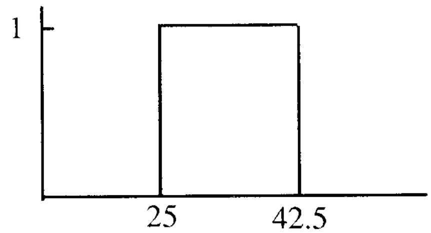

Figure 3. Fused under R^*.

thus aggregation is allowed only if the values are within five units of each other. With

$$A_1 = [20, 35]$$
$$A_2 = [30, 50]$$

we get

$$a^* = \mathrm{Max}[20, 30] = 30$$
$$b^* = \mathrm{Min}[35, 50] = 35.$$

Furthermore

$$U^* = \mathrm{Min}[x \text{ s.t.R}(30, x) \geq \alpha] = 25$$
$$V^* = \mathrm{Max}[x \text{ s.t.R}(35, x) \geq \alpha] = 40.$$

In this situation $g_1 = 25$ and $g_2 = 30$ and thus $d = 27.5$. In addition $h_1 = 35$ and $h_2 = 40$ and thus $e = 37.5$. Figure 4 illustrates this aggregated value.

(IV) Here we consider a more fuzzy combinability relation. Denoting $z = |x - y|$ we let

$$R(x, y) = 0 \qquad \text{if } z > 10$$
$$R(x, y) = 1 - \frac{1}{10}x \quad \text{if } z \leq 10$$

In the case

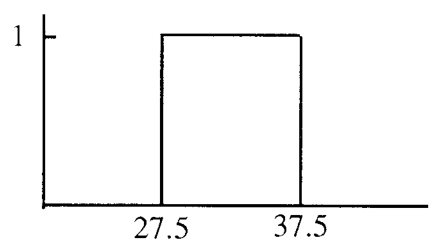

Figure 4.

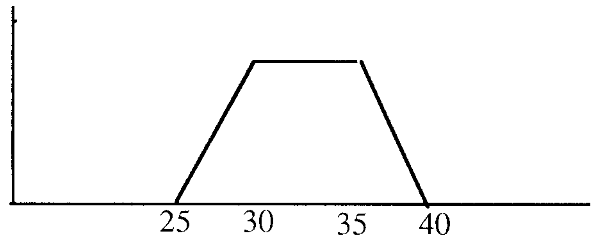

Figure 5. Fusion with Fuzzy R.

$$a^*_\alpha = 30$$

$$b^*_\alpha = 35$$

Furthermore $U^*_\alpha = 20 + 10\alpha$ and $V^*_\alpha = 45 - 10\alpha$. In this situation

$$g_{1\alpha} = 20 + 10\alpha$$

$$g_{2\alpha} = 30$$

and hence

$$d_\alpha = 25 + 5\alpha.$$

Furthermore

$$h_{1\alpha} = 35$$

$$h_{2\alpha} = 45 - 10\alpha.$$

This gives us

$$e_\alpha = 40 - 5\alpha$$

Thus we get

$$B_\alpha = [25 + 5\alpha, 40 - 5x]$$

Figure 5 illustrates the resulting aggregate value.

V. FUSING SUBSETS OF AGGREGATES

In the process of finding the fused value of n observations we have implicitly assumed that we must consider all the arguments in the aggregation. An often used approach when handling collections of data in which a few of the observations are in contradiction with the majority of observations is to discard these outliers in the aggregation process. Thus it appears natural to consider a general approach in which we can consider only some subset of arguments for inclusion in the aggregation operation. Furthermore, in aggregations in which we use only some subset of the observations it would be appropriate to penalize in some way those aggregations that use less than all of the arguments. In this section we look at a general model for aggregation in which we allow less than all the arguments determine our aggregated value.

In order to accomplish this task we shall use the idea of a fuzzy measure which was introduced by Sugeno.[15,16] Assume Y is a finite set, a *fuzzy measure* m is a mapping

$$m: 2^Y \to [0, 1]$$

such that

(1) $m(\emptyset) = 0$
(2) $m(Y) = 1$
(3) For any subsets A and B of Y if $A \subset B$ then $m(A) \leq m(B)$.

Thus a fuzzy measure is a set mapping that exhibits a monotonicity property as seen by property three, a larger subset cannot have a smaller measure.

In the framework of information aggregation we provide the following semantics for the fuzzy measure. Let $Y = \{A_1, \ldots, A_n\}$ be our collection of fuzzy observations. For any subset E of Y we let $m(E)$ be the degree to which we are satisfied using the set of observations in E to formulate our aggregated value. $m(E)$ can be considered as the reliability of any result obtained from a fusion of the sensors whose observations constitute the elements in the set E. Because of the structure of the fuzzy measure we see that when we use all the arguments in Y our satisfaction is complete, one. When we use none of the arguments our satisfaction is zero. In between these two extremes we have reliability values lying in the interval [0, 1] where the monotonicity implies that increasing the number of arguments cannot cause a decrease in our reliability.

Assume F is our basic fusion function, simple average for example, let R be our combinability function based on the underlying space from which we are drawing our sensor observations and let E be any subset of the collection of arguments. We shall let $F(E)$ indicate the fused value of these arguments using the aggregation procedure described in the preceding section.† Furthermore, if we fuse the subset of E elements then we note that $m(E)$ indicates our degree of satisfaction in using only these particular observations in our fusion process. Since we know the degree of reliability obtained in using any subset of Y in calculating our fused value we can consider a subset E_1 of Y get its fused value $F(E_1)$ and have $m(E_1)$ satisfaction *or* we can consider another subset E_2 gets its fused value $F(E_2)$ and have $m(E_2)$ satisfaction *or* we can consider another subset E_3 gets its fused value $F(E_3)$ and have $m(E_3)$ satisfaction, etc. Essentially the above generates a weighted union[17–20] of the fuzzy subsets corresponding to the use of different collections of arguments in the aggregation process. In this weighted union the weights associated with each fuzzy subset are determined from the fuzzy measure adjudicating the allowable aggregations.

†$F(E)$ is what we previously indicated as $F_R(E)$, however, for notational simplicity we have chosen to suppress the subscript.

In the following we shall let $F(m)$ indicate the fusion operation guided by the fuzzy measure m, that is $F(m)$ is the overall fused value obtained by considering the fused value obtained using E_1 or E_2 or E_3 or . . . and considering their appropriate reliabilities, that is, $F(m)$ is the weighted union of the $F(E_i)$. We recall from Ref. 18 if $H_1, \ldots, H_n$ are a collection of fuzzy subsets with associated weights w_i then their weighted union is

$$\bigcup_{i=1}^{n} \hat{H}_i$$

where the $\hat{H}_i$ are fuzzy subsets over the same space as the H_i and defined such that

$$\hat{H}_i(z) = w_i \wedge H_i(z).$$

In the following if H is a fuzzy subset and w is a number in the unit interval we shall use the notation $w \boxed{\wedge} H$ to indicate a fuzzy subset H where $H(x) = w \boxed{\wedge} H(x)$.

Using these ideas we can formally express $F(m)$ as

$$F(m) = \bigcup_{E \subset Y} \hat{F}(E)$$

where

$$\hat{F}(E) = m(E) \boxed{\wedge} F(E).$$

That is for each x

$$\hat{F}(E)(x) = m(E) \wedge F(E)(x).$$

In particular, we see that for any x its membership grade in $\hat{F}(E)$ is obtained by the conjunction of the degree of it is membership in the fusion of the elements in E, $F(E)$ and the degree to which the arguments in E are an acceptable fusion, $m(E)$. Generally speaking we see that $F(E)$ can contribute to the overall aggregated value $F(m)$ if not all the elements in $F(E)$ have low membership grades. There are two ways in which $F(E)$ is precluded from making a significant contribution. The first way is if E is not an acceptable aggregation based upon it having to low an $m(E)$ value. Essentially this corresponds to situations in which we are considering too few of the observations in E. The second way in $F(E)$ can be precluded from making a significant contribution to $F(m)$ is if $F(E)$ has low membership value for all its elements. Since the observations are all assumed to be normal the only way for this to happen is for the combinability function, which is used in the construction of $F(E)$, to drive the membership grades down. This situation occurs when the constituent observations in E are not combinable, that is, some of the observations making up E are conflicting. Thus in summary we see that $F(m)$ is constructed from subsets of observations that are nonconflicting and contain enough of the observations to constitute a meaningful selection of the observations.

As noted above the fuzzy measure provides some indication of the reliability of the result of different subsets of observations by providing weights associated with each of the possible subset fusions. In the context of our problem we shall refer to m as our sensor reliability measure. In the following we shall consider some typical sensor reliability measure.

First we consider the aggregation agenda m_* defined such that

$$m_*(Y) = 1$$
$$m_*(E) = 0 \qquad E \neq Y.$$

In using this measure of satisfaction to guide our fusion process we see that we are only satisfied when we include all observations in the fusion operation. In this case we are saying the only reliable situation is one which all sensors have considered. Specifically, no partial satisfaction is obtained by considering some subset of arguments. We note that this situation is effectively the same as the case we studied before introducing the idea of considering subsets of sensors. Formally in this case we have

$$F(m_*) = \bigcup_{E \subset Y} \hat{F}(E),$$

where from the definition of m_* we get

$$\hat{F}(E) = \Phi \qquad \text{for } E \neq Y$$
$$\hat{F}(Y) = F(Y)$$

thus

$$F(m_*) = F(Y).$$

Thus in this case we obtain as our aggregated value the same value of the previous section.

At the other extreme is the situation in which each sensor is considered to be completely reliable by itself. In this case, which we shall denote as m^*, we have

$$m^*(\{A_i\}) = 1 \qquad \text{for all } A_i \in Y.$$

In this case because of the monotonicity of m we have

$$m^*(E) = 1 \qquad \text{for all } E \subset Y \text{ and } E \neq \varnothing$$
$$m^*(\varnothing) = 0.$$

In this case we are completely satisfied by considering any nonnull subset of the arguments as determining our aggregated value. In particular the value obtained from every subset of sensors must be considered as completely reliable. In this case since $m^*(E) = 1$ for any nonnull E we have

$$\hat{F}(E) = F(E)$$

and hence

$$F(m^*) = \bigcup_{E \subset Y} F(E).$$

In this case since $F(\{A_i\}) = A_i$ we have that

$$F(m^*) \supseteq \bigcup_i A_i.$$

We note that for any other measure m we have for any $E \subset Y$

$$m^*(E) \leq m(E) \leq m^*(E).$$

THEOREM. *Let m be any fuzzy measure then for any aggregation function F and any combinability function R*,

$$F(m_*) \subseteq F(m) \subseteq F(m^*).$$

Proof. The result follows simply from the fact for any x

$$m_*(E) \wedge F(E)(x) \leq m(E) \wedge F(E)(x) \leq m^*(E) \wedge F(E)(x).$$

Another interesting class of fuzzy measures useful for representing the satisfaction of using subsets of arguments for our aggregation are the fixed cardinality measures. In this class the measure, weight, of a subset E of Y just depends upon the number of elements in E,

$$m(E) = g(\text{Card}(E)).$$

In the above g of course must be some monotonic mapping from the nonnegative integers into the unit interval which attains a value of one for Card(Y). In this case reliability of a result is directly related to the number of sensors considered. Thus using this measure all subsets with i arguments have the same degree of reliability. Furthermore, this degree must be monotonic with respect to the cardinality. At times we shall find it convenient to denote these type measures as m_c. We should note that the reliability functions m_* and m^* can be seen as special cases of this more general formulation. In particular m^* has $g(1) = 1$ and m_* has $g(n) = 1$ and $g(i) = 0$ for all other i.

A generalization of this type of structure can be had if we provide a partitioning of the n sensors, the set Y, into subsets, for example, Y_1, Y_2, Y_3, and Y_4. Now consider any subset E of Y. We associate with E a measure m based on the following, where $a_1 \geq a_2 \geq a_3$

$g(E) = a_1$ if E has elements from only one of the Y_i

$g(E) = a_2$ if E has elements from only two of the Y_i

$g(E) = a_3$ if E has elements from only three of the Y_i

$g(E) = 1$ if E has elements from all of the Y_i.

One situation motivating such a structure would be one in which the sensors in class Y_i are essentially the same types of sensors, for example, using the heat given off by an object to determine its location, while the sensors in another class are using some other feature of the object to determine its location.

Thus far we have considered situations in which the reliability of a collection of sensors just depends upon the number of sensors considered. In particular, no distinction was considered regarding the reliability of the different sensors. Another dimension can be added to the formulation of m if we consider differing reliabilities associated with each of the sensors. Specifically if we assign to each sensor and its associated observation A_i a degree of reliability $\alpha_i \in [0, 1]$ we can construct m from these individual reliability values. One notable case of this situation occurs when we consider the reliability of a collection of sensors is determined by the reliability of the most reliable sensor in the collection. In this case

$$m(E) = \underset{\substack{\text{over all } i \\ \text{s. t. } A_i \in E}}{\text{Max}} \alpha_i.$$

In using this formulation it is required in order to have $m(Y) = 1$ that at least one of the sensors be assigned reliability of one. Thus in using this approach we must assign the most reliable sensor an α value of one.

Another class of measures useful for expressing the satisfaction we get in using a subset of sensors is based upon the classic probability measure. We associate with each $A_i \in Y$ a value $\alpha_i \in [0, 1]$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Then for any $E \subset Y$ we have

$$m(E) = \sum_{i \in E} \alpha_i$$

We call this the additive measure and denote it m_A. In a sense the α_i play the role of relative reliability associated with each of the sensor.

We now illustrate the approach described previously with an example. For simplicity we shall assume that the readings are crisp singletons.

Example. Assume we have four readings

$$A_1 = 6, A_2 = 9, A_3 = 6, A_4 = 3.$$

Since we have four arguments we have 16 subsets in the power set of A, we denote these as

$$E_0 = \Phi, E_1 = \{A_1\}, E_2 = \{A_2\}, E_3 = \{A_3\}, E_4 = \{A_4\}, E_5 = \{A_1, A_2\},$$

$$E_6 = \{A_1, A_3\}, E_7 = \{A_1, A_4\}, E_8 = \{A_2, A_3\}, E_9 = \{A_2, A_4\},$$

$$E_{10} = \{A_3, A_4\}, E_{11} = \{A_1, A_2, A_3\}, E_{12} = \{A_1, A_2, A_4\}, E_{13} = \{A_1, A_3, A_4\},$$

$$E_{14} = \{A_2, A_3, A_4\}, E_{15} = \{A_1, A_2, A_3, A_4\}.$$

We shall assume our fusion function F is the simple average and let b_i indicate the average value of the observations in the set E_i, hence we get

$$b_1 = 6, b_2 = 9, b_3 = 6, b_4 = 3, b_5 = 7.5, b_6 = 6, b_7 = 4.5, b_8 = 7.5, b_9 = 6,$$
$$b_{10} = 4.5, b_{11} = 7, b_{12} = 6, b_{13} = 5, b_{14} = 6, b_{15} = 6.$$

We shall assume that the measure m of the reliability of any collection of sensor observations is simply related to the cardinality of the number elements in the subset, hence $m(E_i) = g(\text{Card}(E_i))$. Furthermore we shall assume this measure is defined as

$$\begin{aligned} g(\text{Card}(0)) = 0, g(\text{Card}(1)) = 0.2, g(\text{Card}(2))& \\ = 0.5, g(\text{Card}(3)) = 0.8, g(\text{Card}(4)) = 1.& \end{aligned}$$

Furthermore, we shall let R be the combinability function and use $R(E_i)$ to denote the degree of combinability of the elements in E_i using this function.

Since each of the arguments are singletons, all membership grades in the arguments are one, we get that for each collection of sensor observations

$$F(E_i) = \left\{\frac{R(E_i)}{b_i}\right\}$$

and therefore

$$\hat{F}(E_i) = \left\{\frac{R(E_i) \wedge m(E_i)}{b_i}\right\}.$$

For simplicity we shall denote $\hat{F}(E_i)$ as B_i. In this situation our aggregated value is

$$B = \bigcup_{i=1}^{15} B_i.$$

We shall now investigate the calculation of B for different formulations of R.

(i) We shall first assume that our combinability function R is $R(x, y) = 1$ for all x and y. In this case $R(E_i) = 1$ for all E_i. Thus we get $B_i(b_i) = m(E_i)$ and therefore

$$B = \bigcup_{i=1}^{15} \left\{\frac{m(E_i)}{b_i}\right\}$$

where $m(E_i) = g(\text{Card}(E_i))$.

Furthermore we note that for any x

$$B(x) = \operatorname*{Max}_{\substack{i \text{ such that} \\ b_i = x}} [m(E_i)].$$

From this we get that the aggregated value is the fuzzy subset B where

$$B = \left\{\frac{0.2}{3}, \frac{0.5}{4.5}, \frac{0.8}{5}, \frac{1}{6}, \frac{0.8}{7}, \frac{0.5}{7.5}, \frac{0.2}{9}\right\}.$$

(ii) We now assume R is the strictest combinability relation, $R(x, x) = 1$ and $R(x, y) = 0, x \neq y$. In this situation we note that $R(E_i) = 1$ if all the elements in A are the same otherwise it is zero. Following are the subsets which have nonzero combinability:

$$E_1,\ E_2,\ E_3,\ E_4,\ E_6.$$

We let $G = \{1, 2, 3, 4, 6\}$. We further note that

$$B_i(b_i) = 0 \qquad \text{if } i \notin G$$

$$B_i(b_i) = m(E_i) \qquad \text{if } i \in G.$$

From this we see that

$$B = \bigcup_{i \in G} B_i$$

$$B = \left\{\frac{0.2}{3}, \frac{0.5}{6}, \frac{0.2}{9}\right\}.$$

We note that in this case there exists no element in B with membership grade one which indicates that there exists some conflict between the observations under this combinability relationship.

(iii) Here we consider a combinability function of the form

$$R(x, y) = -\frac{1}{10}|x - y| + 1 \qquad |x - y| \leq 10$$

$$R(x, y) = 0 \qquad |x - y| > 10.$$

Here we are assuming that values ten or more units apart are incombinable, completely conflicting, while within ten units the combinability is linear.

We now calculate $R(E_i)$ for each E_i. First we note that if E_i just has one element then $R(E_i) = 1$ and if E_i has more then two elements $R(E_i)$ is the minimal of all the pairs of elements in E_i. Using this formulation for R we get

$$R(E_1) = R(E_2) = R(E_3) = R(E_4) = R(E_6) = 1$$

$$R(E_i) = 0.7 \qquad \text{for all other subsets.}$$

Thus for $i = 1, 2, 3, 4$, and 6

$$B_i(b_i) = m(E_i)$$

and for the other i

$$B_i(b_i) = 0.7 \wedge m(E_i).$$

Using this we get as our aggregated value

$$B = \left\{\frac{0.2}{3}, \frac{0.5}{4.5}, \frac{0.7}{5}, \frac{0.7}{6}, \frac{0.7}{7}, \frac{0.5}{7.5}, \frac{0.2}{9}\right\}.$$

Again we note that the fuzzy subset is subnormal indicating that their is no value that is completely satisfactory.

As we have shown in the earlier section if we use the strict combinability function, R_*, where

$$R_*(x, y) = 1$$
$$R_*(x, y) = 0 \qquad \text{for all } x \neq y$$

along with the requirement of aggregating all elements, implicitly m_*, and if F has the property of idempotency then we end up essentially taking the intersection of arguments. We now show that a dual type result occurs when we used the function m^*.

THEOREM. *Assume X is any space. Let F be an idempotent fusion operator and let our combinability relation be R_*. If our aggregation is guided by the measure m^* then*

$$F(A_1, \ldots, A_n) = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$$

Proof. We recall that

$$B = F(A_1, \ldots, A_n) = \bigcup_{ECA} \hat{F}(E)$$

Let partition the power set of Y into three mutual disjoint classes; Φ, G, and H. G consists of the subsets of Y containing just one observation, that is $E_i = \{A_i\} \in G$. We let H be the remaining collection of subsets of Y. Furthermore since $m(\Phi) = 0$ then $F(\Phi) = \Phi$ and hence

$$B = \bigcup_{E_i \in G} \hat{F}(E_i) \cup \bigcup_{E \in H} \hat{F}(E).$$

Since the E_i are just singleton subsets and $m^*(E_i) = 1$ then

$$\bigcup_{E_i \in G} \hat{F}(E_i) = \bigcup_{i=1}^{n} A_i.$$

Consider now any arbitrary $E \in H$. We first note that since $m^*(E) = 1$ for all $E \neq \Phi$ then $F(E) = F(E)$. Since we have assumed the strict combinability function

$$F(E) = \bigcap_{i \in E} A_i.$$

Since

$$\bigcap_{i \in E} A_i \subseteq \bigcup_{i=1}^{n} A_i$$

our result follows.

Thus we see that the strictest combinability relation with the most optimistic measure of reliability leads to an aggregation which is simply the union of the observations.

Actually we can provide a more general result. Assume F is any idempotent aggregation operator, assume R is the strict combinability relation, $R_*(x, x) = 1$ and $R_*(x, y) = 0$ for $x \neq y$ and let m be the measure of satisfaction with respect to the arguments considered.

$$F(m) = \bigcup_{ECA} \hat{F}(E)$$

where

$$\hat{F}(E) = M(E) \boxed{\triangle} F(E)$$

with

$$F(E) = \left\{ \frac{R(z) \wedge \operatorname*{Min}_{i \in E} [A_i(x_i)]}{F(z)} \right\}$$

where $z = (x_1, x_2, \ldots, x_{n_E})$ is an element in the cartesian product space of X^{n_E} (n_E is the cardinality of E). Since $R(z) = 0$ if all the elements of z are not the same and one if they are the same and since F is idempotent we get

$$F(E) = \left\{ \frac{\operatorname*{Min}_{i \in E} [A_i(x)]}{x} \right\}$$

for all $x \in X$, thus

$$F(E) = \bigcap_{i \in E} A_i$$

and hence

$$\hat{F}(E) = m(E) \boxed{\triangle} \bigcap_{i \in E} A_i.$$

Let us denote $\bigcap_{i \in E} A_i$ as the fuzzy subset G_E of X. Using this notation we have

$$F(m) = \bigcup_{E \subset Y} m(E) \boxtimes G_E$$

$$F(m)(x) = \operatorname*{Max}_{E \subset Y} [m(E) \wedge G_E(x)].$$

As we have already shown if $m(E) = 1$ for all $E \neq \Phi$, m^*, then

$$F(m) = \bigcup_{E \subset Y} G_E.$$

If m is such that $m(Y) = 1$ and $m(E) = 0$ for other E, m_*, then we have shown

$$F(m) = \bigcap_{i=1,n} A_i.$$

Next assume m is m_c, $m_c(E) = g(\text{card}E)$ in this case

$$F(m)(x) = \operatorname*{Max}_{E \subset Y} [m(E) \wedge G_E(x)].$$

In this case let H_j be the collection of subsets of Y then have cardinality j and let $m(j)$ indicate the measure of reliability associated with subsets of Y with j elements. Furthermore, let $B_j(x)$ be the jth largest of the $A_i(x)$ then

$$F(m)(x) = \operatorname*{Max}_{E \subset \cup_j H_j} [m(j) \wedge G_E(x)]$$

and, therefore,

$$F(m)(x) = \operatorname*{Max}_{j=1 \text{ to } n} [m(j) \wedge B_j(x)].$$

VI. CONDENSATION OF THE FUSION SET

In the preceding sections we have investigated the problem of fusing imprecise pieces of data. The key concept we introduced was the combinability relation between elements in the output space as well as considering the possibility of using subsets of the observations. The end result of this process is a fuzzy subset B over the output space. In some applications we may need to have one crisp single value as our representative fused value. This process can seen as one of *condensation* or defuzzification. In the following we shall describe a technique introduced in Refs. 21 and 22 to accomplish this task. We note that this technique also makes use of the combinability function. In Refs. 21 and 22 we called this knowledge-based defuzzification and discussed its application to fuzzy logic control.

The first step in this process in to find a collection of tentative defuzzified values. We obtain these values by using the following algorithm which is described in more detail in Refs. 21 and 22. We assume that B is a fuzzy subset over the space X, R is the combinability relation, and $\delta \in [0, 1]$ is a parameter

(1) Calculate

$$\mathbf{Max} = \mathrm{Max}_x B(x).$$

(2) Initialize $i = 1$ and set

$$F_1(x) = F(x)$$

(3) Calculate x_i such that

$$F_i(x_i) = \mathrm{Max}_x F_i(x)$$

(4) If $F_i(x_i) < \delta * \mathbf{Max}$ then stop
(5) Place x_i in the pile of tentative potential defuzzified values
(6) Calculate the new fuzzy subset F_{i+1} such that

$$F_{i+1}(x) = (F_i(x) - (F_i(x_i)R(x, x_i))) \vee 0$$

(7) Go to step 3.

As a result of this algorithm we have identified a set of q distanct tentative potential defuzzified values, $x_1, \ldots, x_q$. The next step in the process is to find a modified potential defuzzified value, $\hat{x}_i$, associated with each of the x_i. In particular for each i we calculate

$$\hat{x}_i = \frac{\sum_{x \in X} B(x)R(x_i, x)x}{\sum_{x \in X} B(x)R(x_i, x)}.$$

The next step is to calculate the power T_i associated with each of these modified potential defuzzified values,

$$T_i = \sum_{x \in X} B(x)R(x, \hat{x}_i).$$

We now have a collection of potential defuzzified values, $\hat{x}_i$ and associated with each we have a power value T_i. At this point we have two possible ways of selecting the defuzzified value. One way is to the select the $\hat{x}_i$ with the largest power. The second way is to convert the power into a probability distribution where

$$P_i = \frac{T_i}{\sum_{i=1}^{q} T_i}$$

is the probability associated with $\hat{x}_i$. We then use these probabilities to perform a single random experiment. In particular

(1) Set

$$a_0 = 0$$
$$a_i = a_i - 1 + P_i \qquad \text{for } i = 1, q$$

(2) Set

$$S_i = [a_{i-1}, a_i] \qquad \text{for } i = 1, \quad q$$

(3) Generate a random number $r \in [0, 1]$.
(4) If $r \in S_i$ then the defuzzified value is x_i.

In Ref. 21 Yager showed that if R is such that $R(x, y) = 1$ for x and y this method reduces to the COA method of defuzzification.[23]

VII. CONCLUSION

We have considered the problem of aggregation of multiple source information and introduced a general approach to this problem. This approach is based upon the use of two knowledge structures. The first is the combinability relationship which is used to include information about the appropriateness of aggregating different values from the observation space. The second is a fuzzy measure which carries information about the confidence of using various subsets of the available sensors. By appropriately selecting these knowledge structures we can model many different classes of fusion processes. In this spirit we showed that if we assume a idempotent fusion rule and use a combinability relation that only allows fusion of identical elements then we found that the fusion of any fuzzy subsets is their intersection. In future research we are looking at the problem of developing fast and efficient algorithms to implement the methodology described here.

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