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Stan Z. Li

# Markov Random Field Modeling in Image Analysis

With 99 Figures



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*In Memory of My Mother*

“An excellent book — very thorough and very clearly written.”

— Stuart Geman

“I have found the book to be a very valuable reference. I am very impressed by both the breadth and depth of the coverage. This must have been a truly monumental undertaking.”

— Charles A. Bouman

# Series Preface

Computer Science Workbench is a monograph series which will provide you with an in-depth working knowledge of current developments in computer technology. Every volume in this series will deal with a topic of importance in computer science and elaborate on how you yourself can build systems related to the main theme. You will be able to develop a variety of systems, including computer software tools, computer graphics, computer animation, database management systems, and computer-aided design and manufacturing systems. Computer Science Workbench represents an important new contribution in the field of practical computer technology.

Tosiyasu L. Kunii

# Foreword by Anil K. Jain

The objective of mathematical modeling in image processing and computer vision is to capture the intrinsic character of the image in a few parameters so as to understand the nature of the phenomena generating the image. Models are also useful to specify natural constraints and general assumptions about the physical world; such constraints and assumptions are necessary to solve the “inverse” problem of three-dimensional scene interpretation, given two-dimensional image(s) of the scene. The introduction of stochastic or random field models has led to the development of algorithms for image restoration, segmentation, texture modeling, classification, and sensor fusion. In particular, Gibbs and Markov random fields for modeling spatial context and stochastic interaction among observable quantities have been quite useful in many practical problems, including medical image analysis and interpretation of remotely sensed images. As a result, Markov random field models have generated a substantial amount of excitement in image processing, computer vision, applied statistics, and neural network research communities.

This monograph presents an exposition of Markov random fields (MRFs) that is likely to be extensively used by researchers in many scientific disciplines. In particular, those investigating the applicability of MRFs to process their data or images are bound to find its contents very useful. The main focus of the monograph, however, is on the application of Markov random fields to computer vision problems such as image restoration and edge detection in the low-level domain, and object matching and recognition in the high-level domain. Using a variety of examples, the author illustrates how to convert a specific vision problem involving uncertainties and constraints into essentially an optimization problem under the MRF setting. In doing so, the author introduces the reader to the various special classes of MRFs, including MRFs on the regular lattice (e.g., auto models and multi-level logistic models) that are used for low-level modeling, and MRFs on relational graphs that are used for high-level modeling.

The author devotes considerable attention to the problems of parameter estimation and function optimization, both of which are crucial in the MRF paradigm. Specific attention is given to the estimation of MRF parameters in the context of object recognition, and to the issue of algorithm selection

for MRF-based function optimization. Another contribution of the book is a study on discontinuities, an important issue in the application of MRFs to image analysis. The extensive list of references, high-level descriptions of algorithms and computational issues associated with various optimization algorithms, are some of the attractive features of this book.

On the whole, the contents of this monograph nicely complement the material in Kindermann and Snell's book *Markov Random Fields and Their Applications*, and Chellappa and Jain's edited volume entitled *Markov Random Fields: Theory and Applications*. In my opinion, the main contribution of this book is the manner in which significant MRF-related concepts are lucidly illustrated via examples from computer vision.

Anil K. Jain

East Lansing, Michigan

# Preface

Since its beginning, computer vision research has been evolving from heuristic design of algorithms to systematic investigation of approaches. In this process, researchers have realized: (1) The solution to a vision problem should be sought based on *optimization* principles, albeit explicitly or implicitly, and (2) *contextual constraints* are ultimately necessary for the understanding of visual information in images. Two questions follow: how to define an optimality criterion under contextual constraints and how to find its optimal solution.

*Markov random field* (MRF) theory, a branch of probability theory, provides a foundation for the characterization of contextual constraints and the derivation of the probability distribution of interacting features. In conjunction with methods from decision and estimation theory, the MRF theory provides a systematic approach for deriving optimality criteria such as those based on the *maximum a posteriori* (MAP) concept. This MAP-MRF framework enables us to systematically develop algorithms for a variety of vision problems using rational principles rather than *ad hoc* heuristics. For these reasons, there has been increasing interest in modeling computer vision problems using MRFs in recent years.

This book provides a coherent reference to theories, methodologies, and recent developments in solving computer vision problems based on MRFs, statistics, and optimization. It treats various problems in low- and high-level computational vision in a systematic and unified way within the MAP-MRF framework. The main issues of concern are: how to use MRFs to encode contextual constraints that are indispensable to image understanding; how to derive the objective function, typically the posterior distribution, for the optimal solution to a problem; and how to design computational algorithms for finding the optimal solution.

As the first thorough reference to the subject, the book has four essential parts for solving computational vision problems using MRFs: (1) introduction to fundamental theories, (2) formulations of various vision models in the MAP-MRF framework, (3) parameter estimation, and (4) optimization methods.

Chapter 1 introduces the notion of visual labeling and describes important results in MRF theory pertinent to applications in vision modeling. A vision

problem is posed in terms of Bayes labeling of an MRF. Its optimal solution is then defined as the MAP configuration of the MRF. The role of optimization in computer vision is discussed. These form the basis on which MAP-MRF models are formulated.

Chapter 2 formulates MRF models for low-level vision problems, such as image restoration, reconstruction, edge detection, texture, and optical flow. The systematic MAP-MRF approach for deriving the posterior distribution is illustrated step by step.

Chapter 3 addresses the issue of discontinuities in low-level vision. An important necessary condition is derived for any MRF prior potential function to be adaptive to discontinuities to avoid oversmoothing. This gives rise to the definition of a class of *adaptive interaction functions* and thereby a class of MRF models capable of dealing with discontinuities.

Chapter 4 provides a comparative study on discontinuity adaptive MRF priors and robust M-estimators based on the results obtained in Chapter 3. To tackle the problems associated with M-estimators, a method is presented to stabilize M-estimators with respect to its initialization and convergence.

Chapter 5 presents MRF models for object recognition and pose determination in high-level vision. Relational measurements are incorporated into the energy function as high-level constraints. The concept of line process is extended to the separation of overlapping objects and the elimination of outlier features.

Chapter 6 describes various methods for both supervised and unsupervised parameter estimation, including coding method, pseudo-likelihood, least squares method, and expectation maximization. A simultaneous image labeling and parameter estimation paradigm is also presented, which enhances the low-level models in Chapter 2.

Chapter 7 presents a theory of parameter estimation for optimization-based object recognition. Two levels of criteria are proposed for the estimation: correctness and optimality. Optimal parameters are learned from examples using supervised learning methods. The theory is applied to parameter learning for the MRF recognition.

Chapters 8 and 9 present local and global methods, respectively, for energy optimization in finding MAP-MRF solutions. These include various algorithms for continuous, discrete, unconstrained and constrained minimization, and strategies for approximating global solutions.

The final version of this manuscript benefited from comments on earlier versions by a number of people. I am very grateful to Anil K. Jain and Kanti V. Mardia for their valuable suggestions. I would like to thank Kap Luk Chan, Lihui Chen, Yi-Ping Hung, Eric Sung, Han Wang, Ming Xie, and Dekun Yang. Their corrections have had a very positive effect on the book. I am particularly indebted to Yunjun Zhang, Weiyun Yau, and Yihong Huang for their proofreading of the whole manuscript. Finally, I owe a deep debt of gratitude to my wife for her understanding, patience, and support.

# Preface to the Second Edition

Progress has been made since the first edition of this book was published five years ago. The second edition has included most important progress in Markov modeling in image analysis in recent years, such as Markov modeling of images with “macro” patterns (e.g., the FRAME model), Markov chain Monte Carlo (MCMC) methods, and reversible jump MCMC. Work done by the author in this area after publication of the first edition is also included. The author would like to thank SongChun Zhu for valuable discussions and suggestions.

Stan Z. Li

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# Chapter 1

## Introduction

Modeling problems in this book are addressed mainly from the computational viewpoint. The primary concerns are how to define an objective function for the optimal solution to a vision problem and how to find the optimal solution. The reason for defining the solution in an *optimization* sense is due to various uncertainties in vision processes. It may be difficult to find the perfect solution, so we usually look for an optimal one in the sense that an objective in which constraints are encoded is optimized.

*Contextual constraints* are ultimately necessary in the interpretation of visual information. A scene is understood in the spatial and visual context of the objects in it; the objects are recognized in the context of object features at a lower level representation; the object features are identified based on the context of primitives at an even lower level; and the primitives are extracted in the context of image pixels at the lowest level of abstraction. The use of contextual constraints is indispensable for a capable vision system.

Markov random field (MRF) theory provides a convenient and consistent way for modeling context dependent entities such as image pixels and correlated features. This is achieved through characterizing mutual influences among such entities using conditional MRF distributions. The practical use of MRF models is largely ascribed to a theorem stating the equivalence between MRFs and Gibbs distributions which was established by Hammersley and Clifford (1971) and further developed by Besag (1974). This is because the joint distribution is required in most applications but deriving the joint distribution from conditional distributions turns out to be very difficult for MRFs. The MRFs-Gibbs equivalence theorem points out that the joint distribution of an MRF is a Gibbs distribution, the latter taking a simple form. This gives us a not only mathematically sound but also mathematically tractable means for statistical image analysis (Grenander 1983; Geman and Geman 1984). From the computational perspective, the local property of MRFs leads to algorithms which can be implemented in a local and mas-

sively parallel manner. Furthermore, MRF theory provides a foundation for multi-resolution computation (Gidas 1989).

For the above reasons, MRFs have been widely employed to solve vision problems at all levels. Most of the MRF models are for low level processing. These include image restoration and segmentation, surface reconstruction, edge detection, texture analysis, optical flow, shape from X, active contours, deformable templates, data fusion, visual integration, and perceptual organization. The use of MRFs in high level vision, such as for object matching and recognition, has also emerged in recent years. The interest in MRF modeling in computer vision is still increasing, as reflected by books as well as journal/conference papers published in recent years<sup>1</sup>.

MRF theory tells us how to model the *a priori* probability of contextual dependent patterns, such as textures and object features. A particular MRF model favors the class of patterns encoded by itself by associating them with larger probabilities than other pattern classes. MRF theory is often used in conjunction with statistical decision and estimation theories, so as to formulate objective functions in terms of established optimality principles. *Maximum a posteriori* (MAP) probability is one of the most popular statistical criteria for optimality and in fact, has been the most popular choice in MRF vision modeling. MRFs and the MAP criterion together give rise to the MAP-MRF framework adopted in this book as well as in most other MRF works. This framework, advocated by Geman and Geman (1984) and others, enables us to develop algorithms for a variety of vision problems systematically using rational principles rather than relying on *ad hoc* heuristics. See also introductory statements in (Mardia 1989; Chellappa and Jain 1993; Mardia and Kanji 1994).

An objective function is completely specified by its *form*, *i.e.* the parametric family, and the involved *parameters*. In the MAP-MRF framework, the objective is the joint posterior probability of the MRF labels. Its form and parameters are determined, in turn, according to the Bayes formula, by those of the joint prior distribution of the labels and the conditional probability of the observed data. “A particular MRF model” referred in the previous paragraph means a particular probability function (of patterns) specified by the functional form and the parameters. Two major parts of the MAP-MRF modeling is to derive the form of the posterior distribution and to determine the parameters in it, so as to completely define the posterior probability. Another important part is to design optimization algorithms for finding the maximum of the posterior distribution.

This book is organized in four parts in accordance with the motivations and issues brought out above. The first part (Chapter 1) introduces basic notions, fundamentals and background materials, including labeling problems, relevant results from MRF theory, optimization-based vision and the

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<sup>1</sup>There are numerous recent publications in this area. They are not cited here to keep the introductory statements neat. They will be given subsequently.

systematic MAP-MRF approach. The second part (Chapters 2 – 5) formulates various MRF models in low and high level vision in the MAP-MRF framework and studies the related issues. The third part (Chapters 6 – 7) addresses the problem of MRF parameter estimation. Part four (Chapters 8 – 9) presents search algorithms for computing optimal solutions and strategies for global optimization.

In the rest of this chapter, basic definitions, notations and important theoretical results for the MAP-MRF vision modeling will be introduced. These background materials will be used throughout the book.

## 1.1 Visual Labeling

Many vision problems can be posed as labeling problems in which the solution to a problem is a set of labels assigned to image pixels or features. Labeling is also a natural representation for the study of MRFs (Besag 1974).

### 1.1.1 Sites and Labels

A *labeling problem* is specified in terms of a set of *sites* and a set of *labels*. Let  $\mathcal{S}$  index a discrete set of  $m$  sites

$$\mathcal{S} = \{1, \dots, m\} \quad (1.1)$$

in which  $1, \dots, m$  are indices. A site often represents a point or a region in the Euclidean space such as an image pixel or an image feature such as a corner point, a line segment or a surface patch. A set of sites may be categorized in terms of their “regularity”. Sites on a lattice are considered as spatially *regular*. A rectangular lattice for a 2D image of size  $n \times n$  can be denoted by

$$\mathcal{S} = \{(i, j) \mid 1 \leq i, j \leq n\} \quad (1.2)$$

Its elements correspond to the locations at which an image is sampled. Sites which do not present spatial regularity are considered as *irregular*. This is the usual case corresponding to features extracted from images at a more abstract level, such as the detected corners and lines.

We normally treat the sites in MRF models as un-ordered. For an  $n \times n$  image, pixel  $(i, j)$  can be conveniently re-indexed by a single number  $k$  where  $k$  takes on values in  $\{1, 2, \dots, m\}$  with  $m = n \times n$ . This notation of single-number site index will be used in this book even for images unless an elaboration is necessary. The inter-relationship between sites is maintained by a so-called *neighborhood system* (to be introduced later).

A label is an event that may happen to a site. Let  $\mathcal{L}$  be a set of *labels*. A label set may be categorized as being continuous or discrete. In the continuous case, a label set may correspond to the real line  $\mathbb{R}$  or a compact interval of it

$$\mathcal{L}_c = [X_l, X_h] \subset \mathbb{R} \quad (1.3)$$

An example is the dynamic range for an analog pixel intensity. It is also possible that a continuous label takes a vector or matrix value, for example,  $\mathcal{L}_c = \mathbb{R}^{a \times b}$  where  $a$  and  $b$  are dimensions.

In the discrete case, a label assumes a discrete value in a set of  $M$  labels

$$\mathcal{L}_d = \{\ell_1, \dots, \ell_M\} \quad (1.4)$$

or simply

$$\mathcal{L}_d = \{1, \dots, M\} \quad (1.5)$$

In edge detection, for example, the label set is  $\mathcal{L} = \{\text{edge}, \text{non-edge}\}$ .

Besides the continuity, another essential property of a label set is the ordering of the labels. For example, elements in the continuous label set  $\mathbb{R}$  (the real space) can be ordered by the relation “smaller than”. When a discrete set, say  $\{0, \dots, 255\}$ , represents the quantized values of intensities, it is an ordered set because for intensity values we have  $0 < 1 < 2 < \dots < 255$ . When it denotes 256 different symbols such as texture types, it is considered to be un-ordered unless an artificial ordering is imposed.

For an ordered label set, a numerical (quantitative) measure of similarity between any two labels can usually be defined. For an unordered label set, a similarity measure is symbolic (qualitative), typically taking a value on “equal” or “non-equal”. Label ordering and similarity not only categorize labeling problems but more importantly, affect our choices of labeling algorithms and hence the computational complexity.

### 1.1.2 The Labeling Problem

The labeling problem is to assign a label from the label set  $\mathcal{L}$  to each of the sites in  $\mathcal{S}$ . Edge detection in an image, for example, is to assign a label  $f_i$  from the set  $\mathcal{L} = \{\text{edge}, \text{non-edge}\}$  to site  $i \in \mathcal{S}$  where elements in  $\mathcal{S}$  index the image pixels. The set

$$f = \{f_1, \dots, f_m\} \quad (1.6)$$

is called a *labeling* of the sites in  $\mathcal{S}$  in terms of the labels in  $\mathcal{L}$ . When each site is assigned a unique label,  $f_i = f(i)$  can be regarded as a function with domain  $\mathcal{S}$  and image  $\mathcal{L}$ . Because the support of the function is the whole domain  $\mathcal{S}$ , it is a *mapping* from  $\mathcal{S}$  to  $\mathcal{L}$ , that is,

$$f : \mathcal{S} \longrightarrow \mathcal{L} \quad (1.7)$$

Mappings with continuous and discrete label sets are demonstrated in Figure 1.1. A labeling is also called a *coloring* in mathematical programming.

In the terminology of random fields (*cf.* Section 1.2.2), a labeling is called a *configuration*. In vision, a configuration or labeling can correspond to an image, an edge map, an interpretation of image features in terms of object features, or a pose transformation, and so on.

When all the sites have the same label set  $\mathcal{L}$ , the set of all possible labelings, that is, the configuration space, is the following Cartesian product

$$\mathbb{F} = \underbrace{\mathcal{L} \times \mathcal{L} \cdots \times \mathcal{L}}_{m \text{ times}} = \mathcal{L}^m \quad (1.8)$$

where  $m$  is the size of  $\mathcal{S}$ . In image restoration, for example,  $\mathcal{L}$  contains admissible pixel values which are common to all pixel sites in  $\mathcal{S}$  and  $\mathbb{F}$  defines all admissible images. When  $\mathcal{L} = \mathbb{R}$  is the real line,  $\mathbb{F} = \mathbb{R}^m$  is the  $m$  dimensional real space. When  $\mathcal{L}$  is a discrete set, the size of  $\mathbb{F}$  is combinatorial. For a problem with  $m$  sites and  $M$  labels, for example, there exist a total number of  $M^m$  possible configurations in  $\mathbb{F}$ .

In certain circumstances, admissible labels may not be common to all the sites. Consider, for example, feature based object matching. Supposing there are three types of features: points, lines and regions, then a constraint is that a certain type of image features can be labeled or interpreted in terms of the same type of model features. Therefore, the admissible label for any site is restricted to one of the three types. In an extreme case, every site  $i$  may have its own admissible set,  $\mathcal{L}_i$ , of labels and this gives the following configuration space

$$\mathbb{F} = \mathcal{L}_1 \times \mathcal{L}_2 \cdots \times \mathcal{L}_m \quad (1.9)$$

This imposes constraints on the search for wanted configurations.

### 1.1.3 Labeling Problems in Vision

In terms of the regularity and the continuity, we may classify a vision labeling problem into one of the following four categories:

*LP1:* Regular sites with continuous labels.

*LP2:* Regular sites with discrete labels.

*LP3:* Irregular sites with discrete labels.

*LP4:* Irregular sites with continuous labels.

The first two categories characterize low level processing performed on observed images and the other two do high level processing on extracted token features. The following describes some vision problems in terms of the four categories.

Restoration or smoothing of images having continuous pixel values is an LP1. The set  $\mathcal{S}$  of sites corresponds to image pixels and the set  $\mathcal{L}$  of labels is a real interval. The restoration is to estimate the true image signal from a degraded or noise-corrupted image.

Restoration of binary or multi-level images is an LP2. Similar to the continuous restoration, the aim is also to estimate the true image signal from

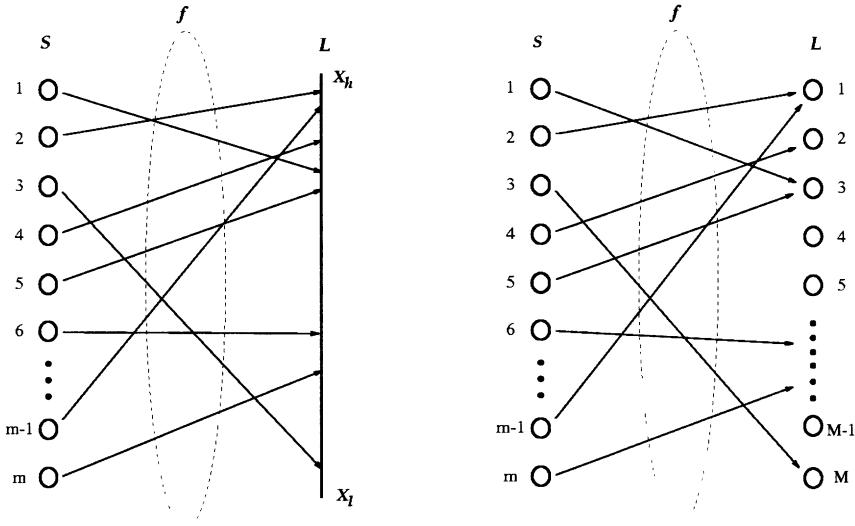


Figure 1.1: A labeling of sites can be considered as a mapping from the set of sites  $S$  to the set of labels  $\mathcal{L}$ . The above shows mappings with continuous label set (left) and discrete label set (right).

the input image. The difference is that each pixel in the resulting image here assumes a discrete value and thus  $\mathcal{L}$  in this case is a set of discrete labels.

Region segmentation is an LP2. It partitions an observation image into mutually exclusive regions, each of which has some uniform and homogeneous properties whose values are significantly different from those of the neighboring regions. The property can be, for example, grey tone, color or texture. Pixels within each region are assigned a unique label.

The prior assumption in the above problems is that the signal is smooth or piecewise smooth. This is complementary to the assumption of abrupt changes made for edge detection.

Edge detection is also an LP2. Each edge site, located between two neighboring pixels, is assigned a label in  $\{\text{edge, non-edge}\}$  if there is a significant difference between the two pixels. Continuous restoration with discontinuities can be viewed as a combination of LP1 and LP2.

Perceptual grouping is an LP3. The sites usually correspond to initially segmented features (points, lines and regions) which are irregularly arranged. The fragmentary features are to be organized into perceptually more significant features. Between each pair of the features is assigned a label in  $\{\text{connected, disconnected}\}$ , indicating whether the two features should be linked.

Feature-based object matching and recognition is an LP3. Each site indexes an image feature such as a point, a line segment or a region. Labels are discrete in nature and each of them indexes a model feature. The resulting configuration is a mapping from the image features to those of a model object.

Pose estimation from a set of point correspondences might be formulated as an LP4. A site is a given correspondence. A label represents an admissible (orthogonal, affine or perspective) transformation. A prior (unary) constraint is that the label of transformation itself must be orthogonal, affine or perspective. A mutual constraint is that the labels  $f_1, \dots, f_m$  should be close to each other to form a consistent transformation.

For a discrete labeling problem of  $m$  sites and  $M$  labels, there exist a total number of  $M^m$  possible labelings. For a continuous labeling problem, there are an infinite number of them. However, among all labelings, there are only a small number of them which are good solutions and may be just a few are optimal in terms of a criterion. How to define the optimal solution for a problem and how to find it are two important topics in the optimization approach to visual labeling.

#### 1.1.4 Labeling with Contextual Constraints

The use of contextual information is ultimately indispensable in image understanding (Pavlidis 1986). The use of contextual information in image analysis and pattern recognition dates back to (Chow 1962; Abend et al. 1965). In (Chow 1962) character recognition is considered as a statistical decision problem. A nearest neighborhood dependence of pixels on an image lattice is obtained by going beyond the assumption of statistical independence. Information on the nearest neighborhood is used to calculate conditional probabilities. That system also includes parameter estimation from sample characters; recognition is done by using the estimated parameters. The work by (Abend et al. 1965) is probably the earliest work using the Markov assumption for pattern recognition. There, a Markov mesh model is used to reduce the number of parameters required for the processing using contextual constraints. Fu and Yu (1980) use MRFs defined on an image lattice to develop a class of pattern classifiers for remote sensing image classification. Another development of context-based models is relaxation labeling (RL) (Rosenfeld et al. 1976). RL is a class of iterative procedures which use contextual constraints to reduce ambiguities in image analysis. A theory is given in (Haralick 1983) to explain RL from a Bayes point of view.

In probability terms, contextual constraints may be expressed locally in terms of conditional probabilities  $P(f_i | \{f_{i'}\})$ , where  $\{f_{i'}\}$  denotes the set of labels at the other sites  $i' \neq i$ , or globally as the joint probability  $P(f)$ . Because local information is more directly observed, it is normal that a global inference is made based on local properties.

In situations where labels are independent of one another (no context), the joint probability is the product of the local ones

$$P(f) = \prod_{i \in \mathcal{S}} P(f_i) \quad (1.10)$$

The above implies conditional independence

$$P(f_i \mid \{f_{i'}\}) = P(f_i) \quad i' \neq i \quad (1.11)$$

Therefore, a global labeling  $f$  can be computed by considering each label  $f_i$  locally. This is advantageous for problem solving.

In the presence of context, labels are mutually dependent. The simple relationships expressed in (1.10) and (1.11) do not hold any more. How to make a global inference using local information becomes a non-trivial task. Markov random field theory provides a mathematical foundation for solving this problem.

## 1.2 Markov Random Fields and Gibbs Distributions

Markov random field theory is a branch of probability theory for analyzing the spatial or contextual dependencies of physical phenomena. It is used in visual labeling to establish probabilistic distributions of interacting labels. This section introduces notations and results relevant to vision.

### 1.2.1 Neighborhood System and Cliques

The sites in  $\mathcal{S}$  are related to one another via a neighborhood system. A neighborhood system for  $\mathcal{S}$  is defined as

$$\mathcal{N} = \{\mathcal{N}_i \mid \forall i \in \mathcal{S}\} \quad (1.12)$$

where  $\mathcal{N}_i$  is the set of sites neighboring  $i$ . The neighboring relationship has the following properties:

- (1) a site is not neighboring to itself:  $i \notin \mathcal{N}_i$ ;
- (2) the neighboring relationship is mutual:  $i \in \mathcal{N}_{i'} \iff i' \in \mathcal{N}_i$ .

For a regular lattice  $\mathcal{S}$ , the set of neighbors of  $i$  is defined as the set of sites within a radius of  $\sqrt{r}$  from  $i$

$$\mathcal{N}_i = \{i' \in \mathcal{S} \mid [\text{dist}(\text{pixel}_{i'}, \text{pixel}_i)]^2 \leq r, i' \neq i\} \quad (1.13)$$

where  $\text{dist}(A, B)$  denotes the Euclidean distance between  $A$  and  $B$  and  $r$  takes an integer value. Note that sites at or near the boundaries have fewer neighbors.

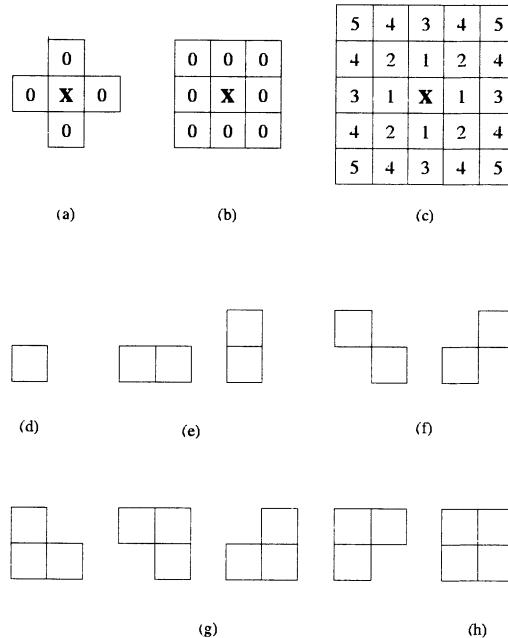


Figure 1.2: Neighborhood and cliques on a lattice of regular sites.

In the first order neighborhood system, also called the 4-neighborhood system, every (interior) site has four neighbors, as shown in Fig.1.2(a) where  $x$  denotes the considered site and 0's its neighbors. In the second order neighborhood system, also called the 8-neighborhood system, there are eight neighbors for every (interior) site, as shown in Fig.1.2(b). The numbers  $n = 1, \dots, 5$  shown in Fig.1.2(c) indicate the outermost neighboring sites in the  $n$ -th order neighborhood system. The shape of a neighbor set may be described as the hull enclosing all the sites in the set.

When the ordering of the elements in  $\mathcal{S}$  is specified, the neighbor set can be determined more explicitly. For example, when  $\mathcal{S} = \{1, \dots, m\}$  is an ordered set of sites and its elements index the pixels of a 1D image, an interior site  $i \in \{2, \dots, m-1\}$  has two nearest neighbors,  $\mathcal{N}_i = \{i-1, i+1\}$ , and a site at the boundaries (the two ends) has one neighbor each,  $\mathcal{N}_1 = \{2\}$  and  $\mathcal{N}_m = \{m-1\}$ . When the sites in a regular rectangular lattice  $\mathcal{S} = \{(i, j) \mid 1 \leq i, j \leq n\}$  correspond to the pixels of an  $n \times n$  image in the 2D plane, an internal site  $(i, j)$  has four nearest neighbors as  $\mathcal{N}_{i,j} = \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$ , a site at a boundary has three and a site at the corners has two.

For an irregular  $\mathcal{S}$ , the neighbor set  $\mathcal{N}_i$  of  $i$  is defined in the same way as (1.13) to comprise nearby sites within the radius of  $\sqrt{r}$

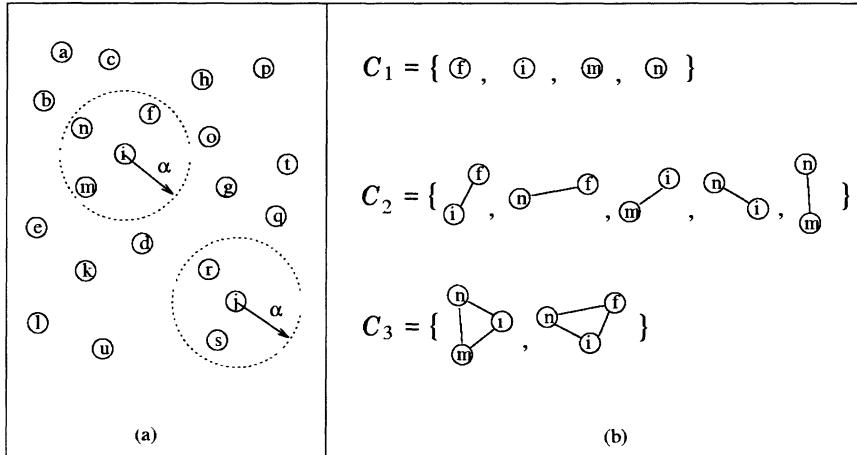


Figure 1.3: Neighborhood and cliques on a set of irregular sites.

$$\mathcal{N}_i = \{i' \in \mathcal{S} \mid [\text{dist}(\text{feature}_{i'}, \text{feature}_i)]^2 \leq r, i' \neq i\} \quad (1.14)$$

The  $\text{dist}(A, B)$  function needs to be defined appropriately for non-point features. Alternatively, the neighborhood may be defined by the Delaunay triangulation,<sup>2</sup> or its dual, the Voronoi polygons, of the sites (Besag 1975). In general, the neighbor sets  $\mathcal{N}_i$  for an irregular  $\mathcal{S}$  have varying shapes and sizes. Irregular sites and their neighborhoods are illustrated in Fig.1.3(a). The neighborhood areas for sites  $i$  and  $j$  are marked by the dotted circles. The sizes of the two neighbor sets are  $\#\mathcal{N}_i = 3$  and  $\#\mathcal{N}_j = 2$ .

The pair  $(\mathcal{S}, \mathcal{N}) \triangleq \mathcal{G}$  constitutes a graph in the usual sense;  $\mathcal{S}$  contains the nodes and  $\mathcal{N}$  determines the links between the nodes according to the neighboring relationship. A *clique*  $c$  for  $(\mathcal{S}, \mathcal{N})$  is defined as a subset of sites in  $\mathcal{S}$ . It consists either of a single site  $c = \{i\}$ , or of a pair of neighboring sites  $c = \{i, i'\}$ , or of a triple of neighboring sites  $c = \{i, i', i''\}$ , and so on. The collections of single-site, pair-site and triple-site cliques will be denoted by  $\mathcal{C}_1$ ,  $\mathcal{C}_2$  and  $\mathcal{C}_3$ , respectively, where

$$\mathcal{C}_1 = \{i \mid i \in \mathcal{S}\} \quad (1.15)$$

$$\mathcal{C}_2 = \{\{i, i'\} \mid i' \in \mathcal{N}_i, i \in \mathcal{S}\} \quad (1.16)$$

and

$$\mathcal{C}_3 = \{\{i, i', i''\} \mid i, i', i'' \in \mathcal{S} \text{ are neighbors to one another}\} \quad (1.17)$$

Note that the sites in a clique are *ordered*, and  $\{i, i'\}$  is not the same clique as  $\{i', i\}$ , and so on. The collection of all cliques for  $(\mathcal{S}, \mathcal{N})$  is

<sup>2</sup>Algorithms for constructing a Delaunay triangulation in  $k \geq 2$  dimensional space can be found in (Bowyer 1981; Watson 1981).

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \dots \quad (1.18)$$

where “...” denotes possible sets of larger cliques.

The type of a clique for  $(\mathcal{S}, \mathcal{N})$  of a regular lattice is determined by its size, shape and orientation. Fig.1.2(d)-(h) show clique types for the first and second order neighborhood systems for a lattice. The single-site and horizontal and vertical pair-site cliques in (d) and (e) are all those for the first order neighborhood system (a). The clique types for the second order neighborhood system (b) include not only those in (d) and (e) but also diagonal pair-site cliques (f) and triple-site (g) and quadruple-site (h) cliques. As the order of the neighborhood system increases, the number of cliques grow rapidly and so the involved computational expenses.

Cliques for irregular sites do not have fixed shapes as those for a regular lattice. Therefore, their types are essentially depicted by the number of involved sites. Consider the four sites  $f, i, m$  and  $n$  within the circle in Fig.1.3(a) in which  $m$  and  $n$  are supposed to be neighbors to each other and so are  $n$  and  $f$ . Then the single-site, pair-site and triple-site cliques associated with this set of sites are shown in Fig.1.3(b). The set  $\{m, i, f\}$  does not form a clique because  $f$  and  $m$  are not neighbors.

### 1.2.2 Markov Random Fields

Let  $F = \{F_1, \dots, F_m\}$  be a family of random variables defined on the set  $\mathcal{S}$ , in which each random variable  $F_i$  takes a value  $f_i$  in  $\mathcal{L}$ . The family  $F$  is called a random field. We use the notation  $F_i = f_i$  to denote the event that  $F_i$  takes the value  $f_i$  and the notation  $(F_1 = f_1, \dots, F_m = f_m)$  to denote the joint event. For simplicity, a joint event is abbreviated as  $F = f$  where  $f = \{f_1, \dots, f_m\}$  is a *configuration* of  $F$ , corresponding to a realization of the field. For a discrete label set  $\mathcal{L}$ , the probability that random variable  $F_i$  takes the value  $f_i$  is denoted  $P(F_i = f_i)$ , abbreviated  $P(f_i)$  unless there is a need to elaborate the expressions, and the joint probability is denoted  $P(F = f) = P(F_1 = f_1, \dots, F_m = f_m)$  and abbreviated  $P(f)$ . For a continuous  $\mathcal{L}$ , we have probability density functions (p.d.f.'s),  $p(F_i = f_i)$  and  $p(F = f)$ .

$F$  is said to be a Markov random field on  $\mathcal{S}$  with respect to a neighborhood system  $\mathcal{N}$  if and only if the following two conditions are satisfied:

$$P(f) > 0, \quad \forall f \in \mathbb{F} \quad (\text{positivity}) \quad (1.19)$$

$$P(f_i \mid f_{\mathcal{S}-\{i\}}) = P(f_i \mid f_{\mathcal{N}_i}) \quad (\text{Markovianity}) \quad (1.20)$$

where  $\mathcal{S} - \{i\}$  is the set difference,  $f_{\mathcal{S}-\{i\}}$  denotes the set of labels at the sites in  $\mathcal{S} - \{i\}$  and

$$f_{\mathcal{N}_i} = \{f_{i'} \mid i' \in \mathcal{N}_i\} \quad (1.21)$$

stands for the set of labels at the sites neighboring  $i$ . The positivity is assumed for some technical reasons and can usually be satisfied in practice. For example, when the positivity condition is satisfied, the joint probability

$P(f)$  of any random field is uniquely determined by its local conditional probabilities (Besag 1974). The Markovianity depicts the local characteristics of  $F$ . In MRFs, only neighboring labels have direct interactions with each other. If we choose the largest neighborhood in which the neighbors of any sites include all other sites, then any  $F$  is an MRF with respect to such a neighborhood system.

An MRF can have other properties such as homogeneity and isotropy. It is said to be homogeneous if  $P(f_i | f_{\mathcal{N}_i})$  is independent of the relative location of the site  $i$  in  $\mathcal{S}$ . So, for a homogeneous MRF, if  $f_i = f_j$  and  $f_{\mathcal{N}_i} = f_{\mathcal{N}_j}$ , there will be  $P(f_i | f_{\mathcal{N}_i}) = P(f_j | f_{\mathcal{N}_j})$  even if  $i \neq j$ . The isotropy will be illustrated in the next subsection with clique potentials.

In modeling some problems, we may need to use several *coupled* MRFs; each of the MRFs is defined on one set of sites, and the sites due to different MRFs are spatially interwoven. For example, in the related tasks of image restoration and edge detection, two MRFs, one for pixel values ( $\{f_i\}$ ) and the other for edge values ( $\{l_{i,i'}\}$ ), can be defined on the image lattice and its dual lattice, respectively. They are coupled to each other *e.g.* via conditional probability  $P(f_i | f_{i'}, l_{i,i'})$  (see Section 2.3.1).

The concept of MRFs is a generalization of that of Markov processes (MPs) which are widely used in sequence analysis. An MP is defined on a domain of time rather than space. It is a sequence (chain) of random variables  $\dots, F_1, \dots, F_m, \dots$  defined on the time indices  $\{\dots, 1, \dots, m, \dots\}$ . An  $n$ -th order unilateral MP satisfies

$$P(f_i | \dots, f_{i-2}, f_{i-1}) = P(f_i | f_{i-1}, \dots, f_{i-n}) \quad (1.22)$$

A bilateral or non-causal MP depends not only on the past but also on the future. An  $n$ -th order bilateral MP satisfies

$$P(f_i | \dots, f_{i-2}, f_{i-1}, f_{i+1}, f_{i+2}, \dots) = P(f_i | f_{i+n}, \dots, f_{i+1}, f_{i-1}, \dots, f_{i-n}) \quad (1.23)$$

It is generalized into MRFs when the time indices are considered as spatial indices.

There are two approaches for specifying an MRF, that in terms of the conditional probabilities  $P(f_i | f_{\mathcal{N}_i})$  and that in terms of the joint probability  $P(f)$ . Besag (1974) argues for the joint probability approach in view of the disadvantages of the conditional probability approach: Firstly, no obvious method is available for deducing the joint probability from the associated conditional probabilities. Secondly, the conditional probabilities themselves are subject to some non-obvious and highly restrictive consistency conditions. Thirdly, the natural specification of an equilibrium of statistical process is in terms of the joint probability rather than the conditional distribution of the variables. Fortunately, a theoretical result about the equivalence between Markov random fields and Gibbs distributions (Hammersley and Clifford 1971; Besag 1974) provides a mathematically tractable means of specifying the joint probability of an MRF.

### 1.2.3 Gibbs Random Fields

A set of random variables  $F$  is said to be a *Gibbs random field* (GRF) on  $\mathcal{S}$  with respect to  $\mathcal{N}$  if and only if its configurations obey a *Gibbs distribution*. A Gibbs distribution takes the following form

$$P(f) = Z^{-1} \times e^{-\frac{1}{T}U(f)} \quad (1.24)$$

where

$$Z = \sum_{f \in \mathbb{F}} e^{-\frac{1}{T}U(f)} \quad (1.25)$$

is a normalizing constant called the *partition function*,  $T$  is a constant called the *temperature* which shall be assumed to be 1 unless otherwise stated, and  $U(f)$  is the *energy function*. The energy

$$U(f) = \sum_{c \in \mathcal{C}} V_c(f) \quad (1.26)$$

is a sum of *clique potentials*  $V_c(f)$  over all possible cliques  $\mathcal{C}$ . The value of  $V_c(f)$  depends on the local configuration on the clique  $c$ . Obviously, the Gaussian distribution is a special member of this Gibbs distribution family.

A GRF is said to be homogeneous if  $V_c(f)$  is independent of the relative position of the clique  $c$  in  $\mathcal{S}$ . It is said to be isotropic if  $V_c$  is independent of the orientation of  $c$ . It is considerably simpler to specify a GRF distribution if it is homogeneous or isotropic than one without such properties. The homogeneity is assumed in most MRF vision models for mathematical and computational convenience. The isotropy is a property of direction-independent blob-like regions.

To calculate a Gibbs distribution, it is necessary to evaluate the partition function  $Z$  which is the sum over all possible configurations in  $\mathbb{F}$ . Since there are a combinatorial number of elements in  $\mathbb{F}$  for a discrete  $\mathcal{L}$ , as illustrated in Section 1.1.2, the evaluation is prohibitive even for problems of moderate sizes. Several Approximation methods exist for solving this problem (*cf.* Chapter 7).

$P(f)$  measures the probability of the occurrence of a particular configuration, or “pattern”,  $f$ . The more probable configurations are those with lower energies. The temperature  $T$  controls the sharpness of the distribution. When the temperature is high, all configurations tend to be equally distributed. Near the zero temperature, the distribution concentrates around the global energy minima. Given  $T$  and  $U(f)$ , we can generate a class of “patterns” by sampling the configuration space  $\mathbb{F}$  according to  $P(f)$ ; see Section 2.4.1.

For discrete labeling problems, a clique potential  $V_c(f)$  can be specified by a number of *parameters*. For example, letting  $f_c = (f_i, f_{i'}, f_{i''})$  be the local configuration on a triple-clique  $c = \{i, i', i''\}$ ,  $f_c$  takes a finite number of states and therefore  $V_c(f)$  takes a finite number of values. For continuous

labeling problems,  $f_c$  can vary continuously. In this case,  $V_c(f)$  is a (possibly piecewise) continuous function of  $f_c$ .

Sometimes, it may be convenient to express the energy of a Gibbs distribution as the sum of several terms, each ascribed to cliques of a certain size, that is,

$$U(f) = \sum_{\{i\} \in \mathcal{C}_1} V_1(f_i) + \sum_{\{i, i'\} \in \mathcal{C}_2} V_2(f_i, f_{i'}) + \sum_{\{i, i', i''\} \in \mathcal{C}_3} V_3(f_i, f_{i'}, f_{i''}) + \dots \quad (1.27)$$

The above implies a homogeneous Gibbs distribution because  $V_1$ ,  $V_2$  and  $V_3$  are independent of the locations of  $i$ ,  $i'$  and  $i''$ . For non-homogeneous Gibbs distributions, the clique functions should be written as  $V_1(i, f_i)$ ,  $V_2(i, i', f_i, f_{i'})$ , and so on.

An important special case is when only cliques of size up to two are considered. In this case, the energy can also be written as

$$U(f) = \sum_{i \in \mathcal{S}} V_1(f_i) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}) \quad (1.28)$$

Note that in the second term on the RHS,  $\{i, i'\}$  and  $\{i', i\}$  are two distinct cliques in  $\mathcal{C}_2$  because the sites in a clique are *ordered*. The conditional probability can be written as

$$P(f_i \mid f_{\mathcal{N}_i}) = \frac{e^{-[V_1(f_i) + \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'})]}}{\sum_{f_i' \in \mathcal{L}} e^{-[V_1(f_i) + \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'})]}} \quad (1.29)$$

### 1.2.4 Markov-Gibbs Equivalence

An MRF is characterized by its local property (the Markovianity) whereas a GRF is characterized by its global property (the Gibbs distribution). The Hammersley-Clifford theorem (Hammersley and Clifford 1971) establishes the equivalence of these two types of properties. The theorem states that *F is an MRF on S with respect to N if and only if F is a GRF on S with respect to N*. Many proofs of the theorem exist, e.g. in (Besag 1974; Moussouris 1974; Kindermann and Snell 1980).

A proof that a GRF is an MRF is given as follows. Let  $P(f)$  be a Gibbs distribution on  $\mathcal{S}$  with respect to the neighborhood system  $\mathcal{N}$ . Consider the conditional probability

$$P(f_i \mid f_{\mathcal{S}-\{i\}}) = \frac{P(f_i, f_{\mathcal{S}-\{i\}})}{P(f_{\mathcal{S}-\{i\}})} = \frac{P(f)}{\sum_{f_i' \in \mathcal{L}} P(f')} \quad (1.30)$$

where  $f' = \{f_1, \dots, f_{i-1}, f_i', \dots, f_m\}$  is any configuration which agrees with  $f$  at all sites except possibly  $i$ . Writing  $P(f) = Z^{-1} \times e^{-\sum_{c \in \mathcal{C}} V_c(f)}$  out gives<sup>3</sup>

<sup>3</sup>This also provides a formula for calculating the conditional probability  $P(f_i \mid f_{\mathcal{N}_i}) = P(f_i \mid f_{\mathcal{S}-\{i\}})$  from potential functions.

$$P(f_i | f_{\mathcal{S}-\{i\}}) = \frac{e^{-\sum_{c \in \mathcal{C}} V_c(f)}}{\sum_{f'_i} e^{-\sum_{c \in \mathcal{C}} V_c(f')}} \quad (1.31)$$

Divide  $\mathcal{C}$  into two set  $\mathcal{A}$  and  $\mathcal{B}$  with  $\mathcal{A}$  consisting of cliques containing  $i$  and  $\mathcal{B}$  cliques not containing  $i$ . Then the above can be written as

$$P(f_i | f_{\mathcal{S}-\{i\}}) = \frac{[e^{-\sum_{c \in \mathcal{A}} V_c(f)}] [e^{-\sum_{c \in \mathcal{B}} V_c(f)}]}{\sum_{f'_i} \{ [e^{-\sum_{c \in \mathcal{A}} V_c(f')}] [e^{-\sum_{c \in \mathcal{B}} V_c(f')}] \}} \quad (1.32)$$

Because  $V_c(f) = V_c(f')$  for any clique  $c$  that does not contain  $i$ ,  $e^{-\sum_{c \in \mathcal{B}} V_c(f)}$  cancels from both the numerator and denominator. Therefore, this probability depends only on the potentials of the cliques containing  $i$ ,

$$P(f_i | f_{\mathcal{S}-\{i\}}) = \frac{e^{-\sum_{c \in \mathcal{A}} V_c(f)}}{\sum_{f'_i} e^{-\sum_{c \in \mathcal{A}} V_c(f')}} \quad (1.33)$$

that is, it depends on labels at  $i$ 's neighbors. This proves that a Gibbs random field is a Markov random field. The proof that an MRF is a GRF is much more involved; a result to be described in the next subsection, which is about the unique GRF representation (Griffeath 1976), provides such a proof.

The practical value of the theorem is that it provides a simple way of specifying the joint probability. One can specify the joint probability  $P(F = f)$  by specifying the clique potential functions  $V_c(f)$  and choosing appropriate potential functions for desired system behavior. In this way, he encodes the *a priori* knowledge or preference about interactions between labels.

How to choose the forms and parameters of the potential functions for a proper encoding of constraints is a major topic in MRF modeling. The forms of the potential functions determine the form of the Gibbs distribution. When all the parameters involved in the potential functions are specified, the Gibbs distribution is completely defined. Defining the functional forms is the theme in Chapters 2–3 while estimating parameters is the subject in Chapters 6 and 7.

To calculate the joint probability of an MRF, which is a Gibbs distribution, it is necessary to evaluate the partition function (1.25). Because it is the sum over a combinatorial number of configurations in  $\mathbb{F}$ , the computation is usually intractable. The explicit evaluation can be avoided in maximum-probability based MRF vision models when  $U(f)$  contains no unknown parameters, as we will see subsequently. However, this is not true when the parameter estimation is also a part of the problem. In the latter case, the energy function  $U(f) = U(f | \theta)$  is also a function of parameters  $\theta$  and so is the partition function  $Z = Z(\theta)$ . The evaluation of  $Z(\theta)$  is required. To circumvent the formidable difficulty therein, the joint probability is often approximated in practice. Several approximate formulae will be introduced in Chapter 6 where the problem of MRF parameter estimation is the subject.

### 1.2.5 Normalized and Canonical Forms

It is known that the choices of clique potential functions for a specific MRF are not unique; there may exist many equivalent choices which specify the same Gibbs distribution. However, there exists a unique normalized potential, called the *canonical potential*, for every MRF (Griffeath 1976).

Let  $\mathcal{L}$  be a countable label set. A clique potential function  $V_c(f)$  is said to be *normalized* if  $V_c(f) = 0$  whenever for some  $i \in c$ ,  $f_i$  takes a particular value in  $\mathcal{L}$ . The particular value can be any element in  $\mathcal{L}$ , e.g. 0 in  $\mathcal{L} = \{0, 1, \dots, M\}$ . Griffeath (1976) establishes the mathematical relationship between an MRF distribution  $P(f)$  and the unique canonical representation of clique potentials  $V_c$  in the corresponding Gibbs distribution (Griffeath 1976; Kindermann and Snell 1980). The result is described below.

Let  $F$  be a random field on a finite set  $\mathcal{S}$  with local characteristics  $P(f_i \mid f_{\mathcal{S}-\{i\}}) = P(f_i \mid f_{\mathcal{N}_i})$ . Then  $F$  is a Gibbs field with *canonical potential function* defined by the following:

$$V_c(f) = \begin{cases} 0 & c = \phi \\ \sum_{b \subset c} (-1)^{|c-b|} \ln P(f_i^b) & c \neq \phi \end{cases} \quad (1.34)$$

where  $\phi$  denotes the empty set,  $|c-b|$  is the number of elements in the set  $c-b$  and

$$f_i^b = \begin{cases} f_i & \text{if } i \in b \\ 0 & \text{otherwise} \end{cases} \quad (1.35)$$

is the configuration which agrees with  $f$  on set  $b$  but assigns the value 0 to all sites outside of  $b$ . For nonempty  $c$ , the potential can also be obtained as

$$V_c(f) = \sum_{b \subset c} (-1)^{|c-b|} \ln P(f_i^b \mid f_{\mathcal{N}_i}^b) \quad (1.36)$$

where  $i$  is any element in  $b$ . Such canonical potential function is *unique* for the corresponding MRF. Using this result, the canonical  $V_c(f)$  can be computed if  $P(f)$  is known.

However, in MRF modeling using Gibbs distributions,  $P(f)$  is defined after  $V_c(f)$  is determined and therefore, it is difficult to compute the canonical  $V_c(f)$  from  $P(f)$  directly. Nonetheless, there is an indirect way: Use a non-canonical representation to derive  $P(f)$  and then canonicalize it using Griffeath's result to obtain the unique canonical representation.

The normalized potential functions appear to be immediately useful. For instance, for the sake of economy, one would use the minimal number of clique potentials or parameters to represent an MRF for a given neighborhood system. The concept of normalized potential functions can be used to reduce the number of nonzero clique parameters (see Chapter 6).

## 1.3 Useful MRF Models

The following introduces some useful MRF models for modeling image properties such as regions and textures. We are interested in their conditional and joint distributions, and the corresponding energy functions. They will be used in the subsequent chapters for deriving MAP-MRF vision models and for MRF parameter estimation. The interested reader may refer to Derin and Kelly (1989) for a systematic study and categorization of Markov random processes and fields in terms of what is called there strict-sense Markov and wide-sense Markov properties.

### 1.3.1 Auto-Models

Contextual constraints on two labels are the lowest order constraints to convey contextual information. They are widely used because of their simple form and low computational cost. They are encoded in the Gibbs energy as pair-site clique potentials. With clique potentials of up to two sites, the energy takes the form

$$U(f) = \sum_{i \in \mathcal{S}} V_1(f_i) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}) \quad (1.37)$$

where “ $\sum_{i \in \mathcal{S}}$ ” is equivalent to “ $\sum_{\{i\} \in \mathcal{C}_1}$ ” and “ $\sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i}$ ” to “ $\sum_{\{i, i'\} \in \mathcal{C}_2}$ ”. The above is a special case of (1.27), which we call a second-order energy because it involves up to pair-site cliques. It is the most frequently used form owing to the mentioned feature that it is the simplest in form but conveys contextual information. A specific GRF or MRF can be specified by proper selection of  $V_1$ ’s and  $V_2$ ’s. Some important such GRF models are described subsequently. Derin and Kelly (1989) present a systematic study and categorization of Markov random processes and fields in terms of what they call strict-sense Markov and wide-sense Markov properties.

When  $V_1(f_i) = f_i G_i(f_i)$  and  $V_2(f_i, f_{i'}) = \beta_{i,i'} f_i f_{i'}$ , where  $G_i(\cdot)$  are arbitrary functions and  $\beta_{i,i'}$  are constants reflecting the pair-site interaction between  $i$  and  $i'$ , the energy is

$$U(f) = \sum_{\{i\} \in \mathcal{C}_1} f_i G_i(f_i) + \sum_{\{i, i'\} \in \mathcal{C}_2} \beta_{i,i'} f_i f_{i'} \quad (1.38)$$

The above is called *auto-models* (Besag 1974). The auto-models can be further classified according to assumptions made about individual  $f_i$ ’s.

An auto-model is said to be an *auto-logistic* model, if the  $f_i$ ’s take on values in the discrete label set  $\mathcal{L} = \{0, 1\}$  (or  $\mathcal{L} = \{-1, +1\}$ ). The corresponding energy is of the following form

$$U(f) = \sum_{\{i\} \in \mathcal{C}_1} \alpha_i f_i + \sum_{\{i, i'\} \in \mathcal{C}_2} \beta_{i,i'} f_i f_{i'} \quad (1.39)$$

where  $\beta_{i,i'}$  can be viewed as the *interaction coefficients*. When  $\mathcal{N}$  is the nearest neighborhood system on a lattice (4 nearest neighbors on a 2D lattice or 2 nearest neighbors on a 1D lattice), the auto-logistic model is reduced to the *Ising model*. The conditional probability for the auto-logistic model with  $\mathcal{L} = \{0, 1\}$  is

$$P(f_i | f_{\mathcal{N}_i}) = \frac{e^{\alpha_i f_i + \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} f_i f_{i'}}}{\sum_{f_i \in \{0,1\}} e^{\alpha_i f_i + \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} f_i f_{i'}}} = \frac{e^{\alpha_i f_i + \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} f_i f_{i'}}}{1 + e^{\alpha_i + \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} f_{i'}}} \quad (1.40)$$

When the distribution is homogeneous, we have  $\alpha_i = \alpha$  and  $\beta_{i,i'} = \beta$ , regardless of  $i$  and  $i'$ .

An auto-model is said to be an *auto-binomial* model if the  $f_i$ 's take on values in  $\{0, 1, \dots, M-1\}$  and every  $f_i$  has a conditionally binomial distribution of  $M$  trials and probability of success  $q$

$$P(f_i | f_{\mathcal{N}_i}) = \binom{M-1}{f_i} q^{f_i} (1-q)^{M-1-f_i} \quad (1.41)$$

where

$$q = \frac{e^{\alpha_i + \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} f_{i'}}}{1 + e^{\alpha_i + \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} f_{i'}}} \quad (1.42)$$

The corresponding energy takes the following form

$$U(f) = - \sum_{\{i\} \in \mathcal{C}_1} \ln \binom{M-1}{f_i} - \sum_{\{i\} \in \mathcal{C}_1} \alpha_i f_i - \sum_{\{i,i'\} \in \mathcal{C}_2} \beta_{i,i'} f_i f_{i'} \quad (1.43)$$

It reduces to the auto-logistic model when  $M = 1$ .

An auto-model is said to be an *auto-normal model*, also called a Gaussian MRF (Chellappa 1985), if the label set  $\mathcal{L}$  is the real line and the joint distribution is multivariate normal. Its conditional p.d.f. is

$$p(f_i | f_{\mathcal{N}_i}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} [f_i - \mu_i - \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} (f_{i'} - \mu_{i'})]^2} \quad (1.44)$$

It is the normal distribution with conditional mean

$$E(f_i | f_{\mathcal{N}_i}) = \mu_i - \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} (f_{i'} - \mu_{i'}) \quad (1.45)$$

and conditional variance

$$\text{var}(f_i | f_{\mathcal{N}_i}) = \sigma^2 \quad (1.46)$$

The joint probability is a Gibbs distribution

$$p(f) = \frac{\sqrt{\det(B)}}{\sqrt{(2\pi\sigma^2)^m}} e^{\frac{(f-\mu)^T B (f-\mu)}{2\sigma^2}} \quad (1.47)$$

where  $f$  is viewed as a vector,  $\mu$  is the  $m \times 1$  vector of the conditional means, and  $B = [b_{i,i'}]$  is the  $m \times m$  *interaction matrix* whose elements are unity and off-diagonal element at  $(i, i')$  is  $-\beta_{i,i'}$ , i.e.  $b_{i,i'} = \delta_{i,i'} - \beta_{i,i'}$  with  $\beta_{i,i} = 0$ . Therefore, the single-site and pair-site clique potential functions for the auto-normal model are

$$V_1(f_i) = (f_i - \mu_i)^2 / 2\sigma^2 \quad (1.48)$$

and

$$V_2(f_i, f_{i'}) = \beta_{i,i'}(f_i - \mu_i)(f_{i'} - \mu_{i'}) / 2\sigma^2 \quad (1.49)$$

respectively. A field of independent Gaussian noise is a special MRF whose Gibbs energy consists of only single-site clique potentials. Because all higher order clique potentials are zero, there is no contextual interaction in the independent Gaussian noise.  $B$  is related to the covariance matrix  $\Sigma$  by  $B = \Sigma^{-1}$ . The necessary and sufficient condition for (1.47) to be a valid p.d.f. is that  $B$  is symmetric and positive definite.

A related but different model is the simultaneous auto-regression (SAR) model. Unlike the auto-normal model which is defined by the  $m$  conditional p.d.f.'s, this model is defined by a set of  $m$  equations

$$f_i = \mu_i + \sum \beta_{i,i'}(f_{i'} - \mu_{i'}) + e_i \quad (1.50)$$

where  $e_i$  are independent Gaussian,  $e_i \sim N(0, \sigma^2)$ . It also generates the class of all multivariate normal distributions but with joint p.d.f. as

$$p(f) = \frac{\det(B)}{\sqrt{(2\pi\sigma^2)^m}} e^{\frac{(f-\mu)^T B^T B (f-\mu)}{2\sigma^2}} \quad (1.51)$$

where  $B$  is defined as before. Any SAR model is an auto-normal model with the  $B$  matrix in (1.47) being  $B = B_2 + B_2^T - B_2^T B_2$  where  $B_2 = B_{\text{auto-regressive}}$ . The reverse can also be done, though in a rather unnatural way via Cholesky decomposition (Ripley 1981). Therefore, both models can have their p.d.f.'s in the form (1.47). However, for (1.51) to be a valid p.d.f., it requires only that  $B_{\text{auto-regressive}}$  be non-singular.

### 1.3.2 Multi-Level Logistic Model

The auto-logistic model can be generalized to *multi-level logistic* (MLL) model (Elliott et al. 1984; Derin and Cole 1986; Derin and Elliott 1987), also called Strauss process (Strauss 1977) and generalized Ising model (Geman and Geman 1984). There are  $M$  ( $> 2$ ) discrete labels in the label set,  $\mathcal{L} = \{1, \dots, M\}$ . In this type of models, a clique potential depends on the type  $c$  (related to size, shape and possibly orientation) of the clique and the local configuration  $f_c \triangleq \{f_i \mid i \in c\}$ . For cliques containing more than one site ( $\#c > 1$ ), the MLL clique potentials are defined by

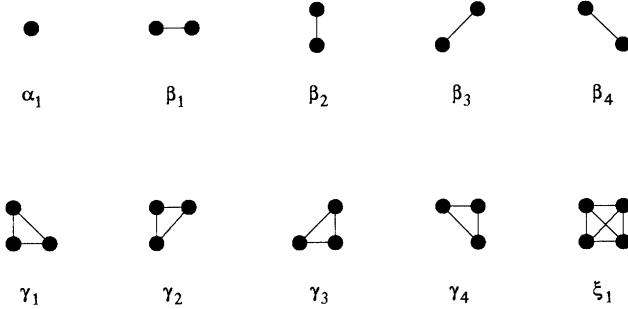


Figure 1.4: Clique types and associated potential parameters for the second order neighborhood system. Sites are shown in dots and neighboring relationships in joining lines.

$$V_c(f) = \begin{cases} \zeta_c & \text{if all sites on } c \text{ have the same label} \\ -\zeta_c & \text{otherwise} \end{cases} \quad (1.52)$$

where  $\zeta_c$  is the potential for type- $c$  cliques; for single site cliques, they depend on the label assigned to the site

$$V_c(f) = V_c(f_i) = \alpha_I \quad \text{if } f_i = I \in \mathcal{L}_d \quad (1.53)$$

where  $\alpha_I$  is the potential for label value  $I$ . Fig.1.4 shows the clique types and the associated parameters in the second order (8-neighbor) neighborhood system.

Assume that an MLL model is of second-order as in (1.37), so that only  $\alpha$  (for single-site cliques) and  $\beta$  (for pair-site cliques) parameters are non-zero. The potential function for pair-wise cliques is written as

$$V_2(f_i, f_{i'}) = \begin{cases} \beta_c & \text{if sites on clique } \{i, i'\} = c \in \mathcal{C}_2 \text{ have the same label} \\ -\beta_c & \text{otherwise} \end{cases} \quad (1.54)$$

where  $\beta_c$  is the  $\beta$  parameter for type- $c$  cliques and  $\mathcal{C}_2$  is set of pair-site cliques. For the 4-neighborhood system, there are four types of pair-wise cliques (cf. Fig.1.4) and so there can be four different  $\beta_c$ 's. When the model is isotropic all the four take the same value. Owing to its simplicity, the pair-wise MLL model (1.54) has been widely used for modeling regions and textures (Elliott et al. 1984; Geman and Geman 1984; Derin and Cole 1986; Derin and Elliott 1987; Murray and Buxton 1987; Lakshmanan and Derin 1989; Won and Derin 1992).

When the MLL model is isotropic, it depicts blob-like regions. In this case, the conditional probability can be expressed as follows (Strauss 1977)

$$P(f_i = I \mid f_{\mathcal{N}_i}) = \frac{e^{-\alpha_I - \beta n_i(I)}}{\sum_{I=1}^M e^{-\alpha_I - \beta n_i(I)}} \quad (1.55)$$

where  $n_i(I)$  is the number of sites in  $\mathcal{N}_i$  which are labeled  $I$ . It reduces to (1.40) when there are only two labels, 0 and 1. In contrast, an anisotropic model tends to generate texture-like patterns. See examples in Section 2.4.

A hierarchical two-level Gibbs model has been proposed to represent both noise-contaminated and textured images (Derin and Cole 1986; Derin and Elliott 1987). The higher level Gibbs distribution uses an isotropic random field, *e.g.* MLL, to characterize the blob-like region formation process. A lower level Gibbs distribution describes the filling-in in each region. The filling-in may be independent noise or a type of texture, both of which can be characterized by Gibbs distributions. This provides a convenient approach for MAP-MRF modeling. In segmentation of noisy and textured image (Derin and Cole 1986; Derin and Elliott 1987; Lakshmanan and Derin 1989; Hu and Fahmy 1987; Won and Derin 1992), for example, the higher level determines the prior of  $f$  for the region process while the lower level Gibbs contributes to the conditional probability of the data given  $f$ . Note that different levels of MRFs in the hierarchy can have different neighborhood systems.

### 1.3.3 The Smoothness Prior

A generic contextual constraint on this world is the *smoothness*. It assumes that physical properties in a neighborhood of space or in an interval of time present some coherence and generally do not change abruptly. For example, the surface of a table is flat, a meadow presents a texture of grass, and a temporal event does not change abruptly over a short period of time. Indeed, we can always find regularities of a physical phenomenon with respect to certain properties. Since its early applications in vision (Grimson 1981; Horn and Schunck 1981; Ikeuchi and Horn 1981) aimed to impose constraints, in addition to those from the data, on the computation of image properties, the smoothness prior has been one of the most popular prior assumptions in low level vision. It has been developed into a general framework, called regularization (Poggio et al. 1985; Bertero et al. 1988), for a variety of low level vision problems.

Smoothness constraints are often expressed as the prior probability or equivalently an energy term  $U(f)$  measuring the extent to which the smoothness assumption is violated by  $f$ . There are two basic forms of such smoothness terms corresponding to situations with discrete and continuous labels, respectively.

The equations (1.52) and (1.54) of the MLL model with negative  $\zeta$  and  $\beta$  coefficients provide a method for constructing smoothness terms for unordered, discrete labels. Whenever all labels  $f_c$  on a clique  $c$  take the same value, which means the solution  $f$  is locally smooth on  $c$ , they incur a negative clique potential (cost); otherwise, if they are not all the same, they incur a positive potential. Such an MLL model tends to give a smooth solution which prefers uniform labels.

For spatially (and also temporally in image sequence analysis) continuous MRFs, the smoothness prior often involves derivatives. This is the case with the analytical regularization (to be introduced in Section 1.5.3). There, the potential at a point is in the form of  $[f^{(n)}(x)]^2$ . The order  $n$  determines the number of sites in the involved cliques; for example,  $[f'(x)]^2$  where  $n = 1$  corresponds to a pair-site smoothness potential. Different orders implies different class of smoothness.

Let us take continuous restoration or reconstruction of non-texture surfaces as an example. Let  $f = \{f_1, \dots, f_m\}$  be the sampling of an underlying “surface”  $f(x)$  on  $x \in [a, b]$  where the surface is one-dimensional for simplicity. The Gibbs distribution  $P(f)$ , or equivalently the energy  $U(f)$ , depends on the type of the surface  $f$  we expect to reconstruct. Assume that the surface is flat – *a priori*. A flat surface which has equation  $f(x) = a_0$  should have zero first-order derivative,  $f'(x) = 0$ . Therefore, we may choose the prior energy as

$$U(f) = \int [f'(x)]^2 dx \quad (1.56)$$

which is called a *string*. The energy takes the minimum value of zero only if  $f$  is absolutely flat or a positive value otherwise. Therefore, the surface which minimizes (1.56) alone has a constant height (grey value for an image).

In the discrete case where the surface is sampled at discrete points  $a \leq x_i \leq b$ ,  $i \in \mathcal{S}$ , we use the first order difference to approximate the first derivative and use a summation to approximate the integral; so the above energy becomes

$$U(f) = \sum_i [f_i - f_{i-1}]^2 \quad (1.57)$$

where  $f_i = f(x_i)$ . Expressed as the sum of clique potentials, we have

$$U(f) = \sum_{c \in \mathcal{C}} V_c(f) = \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}) \quad (1.58)$$

where  $\mathcal{C} = \{(1, 2), (2, 1), (2, 3), \dots, (m-2, m-1), (m, m-1), (m-1, m)\}$  consists of only pair-site cliques and

$$V_c(f) = V_2(f_i, f_{i'}) = \frac{1}{2}(f_i - f_{i'})^2 \quad (1.59)$$

Its 2D equivalent is

$$\int \int \{[f_x(x, y)]^2 + [f_y(x, y)]^2\} dx dy \quad (1.60)$$

and is called a *membrane*.

Similarly, the prior energy  $U(f)$  can be designed for planar or quadratic surfaces. A planar surface,  $f(x) = a_0 + a_1 x$ , has zero second-order derivative,  $f''(x) = 0$ . Therefore, the following may be chosen

$$U(f) = \int [f''(x)]^2 dx \quad (1.61)$$

which is called a *rod*. The surface which minimizes (1.61) alone has a constant gradient. In the discrete case, we use the second-order difference to approximate the second-order derivative and the above energy becomes

$$U(f) = \sum_i [f_{i+1} - 2f_i + f_{i-1}]^2 \quad (1.62)$$

For a quadratic surface,  $f(x) = a_0 + a_1x + a_2x^2$ , the third-order derivative is zero,  $f'''(x) = 0$  and the prior energy may be

$$U(f) = \int [f'''(x)]^2 dx \quad (1.63)$$

The surface which minimizes the above energy alone has a constant curvature. In the discrete case, we use the third-order difference to approximate the second-order derivative and the above energy becomes

$$U(f) = \sum_i [f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2}]^2 \quad (1.64)$$

The above smoothness models can be extended to 2D. For example, the 2D equivalent of the rod, called a plate, comes in two varieties, the quadratic variation

$$\int \int \{[f_{xx}(x, y)]^2 + 2[f_{xy}(x, y)]^2 + [f_{yy}(x, y)]^2\} dx dy \quad (1.65)$$

and the squared Laplacian

$$\int \int \{f_{xx}(x, y) + f_{yy}(x, y)\}^2 dx dy \quad (1.66)$$

The surface which minimizes one of the smoothness prior energy alone has either a constant grey level, a constant gradient or a constant curvature. This is undesirable because constraints from other sources such as the data are not used. Therefore, a smoothness term  $U(f)$  is usually utilized in conjunction with other energy terms. In regularization, an energy consists of a smoothness term and a closeness term and the minimal solution is a compromise between the two constraints. Refer to Section 1.5.3.

The encodings of the smoothness prior in terms of derivatives usually lead to *isotropic* potential functions. This is due to the assumption that the underlying surface is non-textured. *Anisotropic* priors have to be used for texture patterns. This can be done, for example, by choosing (1.37) with direction-dependent  $V_2$ 's. This will be discussed in Section 2.4.

### 1.3.4 Hierarchical GRF Model

A hierarchical two-level Gibbs model has been proposed to represent both noise-contaminated and textured images (Derin and Cole 1986; Derin and Elliott 1987). The higher level Gibbs distribution uses an isotropic random field, *e.g.* MLL, to characterize the blob-like region formation process. A lower level Gibbs distribution describes the filling-in in each region. The filling-in may be independent noise or a type of texture, both of which can be characterized by Gibbs distributions. This provides a convenient approach for MAP-MRF modeling. In segmentation of noisy and textured image (Derin and Cole 1986; Derin and Elliott 1987; Lakshmanan and Derin 1989; Hu and Fahmy 1987; Won and Derin 1992), for example, the higher level determines the prior of  $f$  for the region process while the lower level Gibbs contributes to the conditional probability of the data given  $f$ . Note that different levels of MRFs in the hierarchy can have different neighborhood systems.

Various hierarchical Gibbs models result according to what are chosen for the regions and for the filling-in's, respectively. For example, each region may be filled in by an auto-normal texture (Simchony and Chellappa 1988; Won and Derin 1992) or an auto-binomial texture (Hu and Fahmy 1987); the MLL for the region formation may be substituted by another appropriate MRF. The hierarchical MRF model for textured regions will be further discussed in Section 2.4.1.

A drawback of the hierarchical model is that the conditional probability  $P(d_i | f_i = I)$  for regions given by  $\{i \in \mathcal{S} | f_i = I\}$  can not always be written exactly. For example, when the lower level MRF is a texture modeled as an auto-normal field, its joint distribution over an irregularly shaped region is not known. This difficulty may be overcome by using approximate schemes such as pseudo-likelihood to be introduced in Section 6.1 or by using the eigen-analysis method (Wu and Leahy 1993).

### 1.3.5 The FRAME Model

The FRAME (Filter, Random Fields And Maximum Entropy) model, proposed by (Zhu et al. 1997; Zhu and Mumford 1997; Zhu et al. 1998), is a generalized MRF model which fuses the essence of filtering theory and MRF modeling through the maximum entropy principle. It is generalized in the following two aspects: (1) The FRAME model is defined in terms of statistics, *i.e.* potential functions, calculated from the output of a filter bank by which the image is filtered, instead of the cliques potentials of the image itself. Given an image (a realization of a MRF), the image is filtered by a bank of filters, giving a set of output images. Some statistics are then calculated from the output images. (2) The FRAME model provides a means of learning the model parameters from a set of samples (example images) representative of the MRF to be modeled. Besides, it also gives an algorithm for filter selection.

The joint distribution of the FRAME model is constrained in such a way that the model can reproduce the statistics of the example images. It is found by solving a constrained maximum entropy problem. Let  $G^{(k)}$  ( $k = 1, \dots, K$ ) be a bank of  $K$  filters (such a Gabor filters),  $f^{(k)} = G^{(k)} * f$  the output of filtering  $f$  by  $G^{(k)}$ , and  $H^{(k)} \in \mathcal{L}^S$  (the  $\mathcal{L}$  is assumed to be the same for all the  $K$  filter outputs) the histogram of  $f^{(k)}$  defined by

$$H^{(k)}(I) = \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \delta(I - f_i^{(k)}) \quad (1.67)$$

where  $\delta(t) = 1$  if  $t = 0$  or 0 otherwise. For the filtered sample images, we denote the averaged histogram of the  $k$ -th filter output by  $\overline{H_{\text{sample}}^{(k)}}$  (averaged across all example images). Now, the joint distribution of the FRAME is defined below:

$$p(f) = \arg \max_p \left\{ - \int p(f) \log(p(f)) df \right\} \quad (1.68)$$

$$\text{subject to} \quad \overline{H_{p(f)}^{(k)}}(I) = \overline{H_{\text{sample}}^{(k)}}(I) \quad \forall k, \forall I \quad (1.69)$$

$$\int p(f) df = 1 \quad (1.70)$$

where

$$\overline{H_{p(f)}^{(k)}} = \int H^{(k)}(f) p(f) df \quad (1.71)$$

is the expectation of  $H^{(k)}$  with respect to  $p(f)$ . By using Lagrange multipliers  $\theta_I^{(k)}$  for the constraints of (1.69), we get the Lagrangian

$$\begin{aligned} L(p, \theta) &= - \int p(f) \log(p(f)) df \\ &\quad + \int_I \sum_k \theta_I^{(k)} \left\{ \int_f p(f) \sum_i \delta(I - f_i^{(k)}) df - |\mathcal{S}| \overline{H_{\text{sample}}^{(k)}}(I) \right\} dI \end{aligned} \quad (1.72)$$

(Note that the constraints are multiplied by the factor of  $|\mathcal{S}|$ .) By setting  $\frac{\partial L(p, \theta)}{\partial p} = 0$ , the solution to the constrained optimization (ME) problem can be derived as (consider  $p(f) = p(f | \theta)$  when  $\theta$  is given)

$$\begin{aligned} p(f | \theta) &= \frac{1}{Z(\theta)} e^{-\sum_{k=1}^K \sum_{i \in \mathcal{S}} \{\int \theta^{(k)}(I) \delta(I - f_i^{(k)}) dI\}} \\ &= \frac{1}{Z(\theta)} e^{-\sum_{k=1}^K \sum_{i \in \mathcal{S}} \{\theta^{(k)}(f_i^{(k)})\}} \end{aligned} \quad (1.73)$$

where  $\theta^{(k)}(\cdot)$  are the potential functions of the FRAME model and  $Z$  the normalizing factor.

In the discrete form, assume that  $I^{(k)} = f_i^{(k)}$  is quantized into  $L$  discrete values  $I_1^{(k)}, \dots, I_L^{(k)}$ . The solution in (1.73) can be written as

$$\begin{aligned} p(f \mid \theta) &= \frac{1}{Z(\theta)} e^{-\sum_{k=1}^K \sum_{i \in s} \sum_{\ell=1}^L \{\theta_{\ell}^{(k)} \delta(I_{i \ell}^{(k)} - f_i^{(k)})\}} \\ &= \frac{1}{Z(\theta)} e^{-\sum_{k=1}^K \sum_{\ell=1}^L \theta_{\ell}^{(k)} H_{\ell}^{(k)}} \\ &= \frac{1}{Z} e^{-\langle \theta, H \rangle} \end{aligned} \quad (1.74)$$

where  $\theta_{\ell}^{(k)} = \theta^{(k)}(I_{\ell}^{(k)})$ ,  $H_{\ell}^{(k)} = H^{(k)}(I_{\ell}^{(k)})$ , and  $\langle a, b \rangle$  is the inner product of  $a$  and  $b$ .

The FRAME model provides a means of modeling complicated high order patterns in a tractible way. In the traditional MRF model, the neighborhood is usually small to keep the model tractable, and therefore it is difficult to model patterns in which interaction in a large neighborhood is necessary. In contrast, the FRAME model is able to model more complicated patterns by incorporating larger neighborhood and potential functions of higher order cliques implicitly determined by the filter windows; moreover, it uses an accompanying learning procedure to estimate high order potential functions from the filter outputs. This makes the high order model tractable in formulation albeit expensive in computation. There are two things to learn in the FRAME model: (1) the potential functions  $\theta_I^{(k)}$  and (2) the types of filters  $G^{(k)}$  to use. These will be described in Section 6.1.7.

More recently, Zhu and his colleagues (Wu et al. 2000) have established an equivalence between the FRAME model and another mathematical model of texture called Julesz ensembles (Julesz 1962) when the size of the image lattice goes to infinity. On the other hand, they also proposes fast MCMC algorithms for sampling  $p(f \mid \theta)$  which involves hundreds of parameters to estimate in a large neighborhood (Zhu and Liu 2000).

### 1.3.6 Multiresolution MRF Modeling

The motivation for multiresolution MRF (MRMRF) modeling here is similar to that of FRAME: to model complex and macro patterns by incorporating interactions in a large neighborhood. The approach used in MRMRF modeling is to build an MRF model based on the outputs of multiresolution filters.

From orthogonal wavelet decomposition, such as Haar or Daubechies wavelet, non-redundant subbands can be obtained in different scales and directions. They can be used to represent the original image completely. On the other hand, these subbands are downsampled with the discrete wavelet transform. Therefore, the texture structure represented by the information of two far away pixels in the original image may become immediate neighbors in the subband images on the higher levels. Fig.1.5 shows an example of

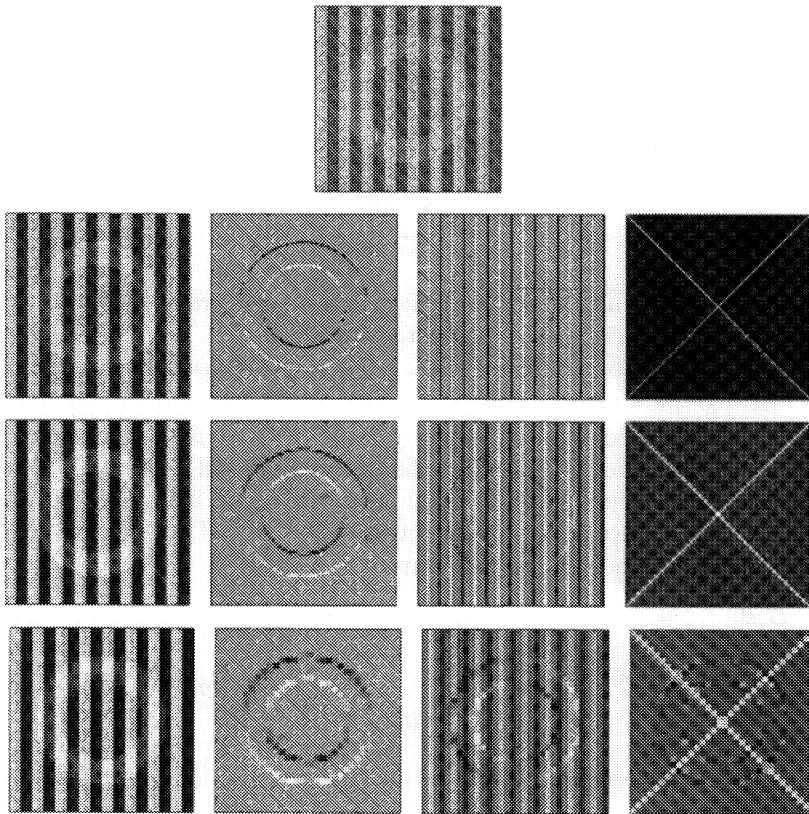


Figure 1.5: The original image (row 0). Haar wavelet decomposition at level 1 (row 1), level 2 (row 2) and level 3 (row 3). From left to right of rows 1 – 3 are the low-pass, horizontal, vertical and diagonal subbands.

multiresolution wavelet decomposition. We can see that with the decomposition and downsampling, the subbands of different scales and directions can reveal different characteristics of the original image. The pixel relationship at different scales is not the same even in the same direction.

Let  $f$  be an image defined on a lattice of sites  $\mathcal{S}$  indexed by  $i \in \mathcal{S}$ ,  $G = \{G^{(1)}, G^{(2)}, \dots, G^{(K)}\}$  a set of multiresolution filters such as wavelet filters, and  $f^{(k)} = G^{(k)} * f$  the  $k$ -th subband output of filtering  $f$  with  $G^{(k)}$ . Assume that the pixel values of the  $K$  filter outputs are quantized into  $M$  levels, giving the label set  $\mathcal{L} = \{1, \dots, M\}$  which is the same for all the  $K$  subbands. The distribution of image  $f$  can be written in the following

form

$$P(f \mid G) = \frac{1}{Z(G)} \exp(-U(f \mid G)) \quad (1.75)$$

where  $Z(G)$  is the normalizing partition function.  $U(f \mid G)$  is the energy function which takes the following form

$$U(f \mid G) = \sum_{k=1}^K \sum_{c \in \mathcal{C}} V_c^{(k)}(f) \quad (1.76)$$

where  $\mathcal{C}$  is the set of all cliques in a neighborhood system, and  $V_c^{(k)}(f)$  is the clique potential associated with the filter output  $f^{(k)}$ .

Consider cliques of up to two sites. Let  $\theta = \{\theta^{(k)}\} = \{\alpha^{(k)}, \beta^{(k)}\}$  be the set of MRMRF parameters where  $\alpha^{(k)} = \{\alpha^{(k)}(I)\}$  consists of  $M$  components for the eight quantized pixel values  $I = f_i^{(k)}$  and  $\beta^{(k)} = \{\beta_c^{(k)}\}$  consists of four components for cliques  $c = (i, i')$  in the 4-neighborhood system. For homogeneous MRMRF, the potential functions are location independent though the pair-site clique potentials are direction dependent. The following energy function is used to include cliques of up to two sites

$$U(f \mid G, \theta) = \sum_{k=1}^K \sum_{i \in \mathcal{S}} \left\{ \alpha_i^{(k)}(f_i^{(k)}) + \sum_{i' \in \mathcal{N}_i, c=(i, i')} \beta_c^{(k)} \left[ 1 - 2 \exp(-(f_i^{(k)} - f_{i'}^{(k)})^2) \right] \right\} \quad (1.77)$$

The corresponding potential functions are  $V_1^{(k)}(i) = \alpha_i^{(k)}(f_i^{(k)})$  for the single-site clique and  $V_2^{(k)}(i, i') = \beta_c^{(k)} \left[ 1 - 2 \exp(-(f_i^{(k)} - f_{i'}^{(k)})^2) \right]$  for  $c = (i, i')$ .

In this model, the exponential form of the pair-site clique potential is similar to that of the multilevel logistic (MLL) model (*cf.* Section 1.3.2). But it is easier to process and more meaningful in texture representation than the MLL model (Liu and Wang 1999). In MLL, if the two labels are not exactly the same, they contribute nothing to the potential even if they are similar; in contrast, the use of the exponential term in MRMRF incorporates the label similarity into the potential in a soft way.

Before giving the conditional probability  $P(f_i \mid f_{\mathcal{N}_i}, G, \theta)$ , we first define a generalized energy of  $f_i$  as

$$U(f_i \mid G, \theta) = \sum_{k=1}^K \left\{ \alpha^{(k)}(f_i^{(k)}) + \sum_{i' \in \mathcal{N}_i, c=(i, i')} \beta_c^{(k)} \left[ 1 - 2 \exp(-(f_i^{(k)} - f_{i'}^{(k)})^2) \right] \right\} \quad (1.78)$$

Thus

$$P(f_i \mid f_{\mathcal{N}_i}, G, \theta) = \frac{\sum_{k=1}^K \exp(-U(f_i^{(k)} \mid \theta^{(k)}))}{\sum_{k=1}^K \sum_{I^{(k)}=1}^M \exp(-U(I^{(k)} \mid \theta^{(k)}))} \quad (1.79)$$

where  $I^{(k)} = f_i^{(k)}$  be the value of pixel  $i$  in the  $k$ -th subband image. To define a pseudo-likelihood, we simplify it by

$$P(f_i | f_{\mathcal{N}_i}, G, \theta) \approx \prod_{k=1}^K \frac{\exp(-U(f_i^{(k)} | \theta^{(k)}))}{\sum_{\substack{M \\ I^{(k)}=1}} \exp(-U(I^{(k)} | \theta^{(k)}))} \quad (1.80)$$

Then the pseudo-likelihood can be written as

$$\begin{aligned} PL(f | G, \theta) &= \prod_{i \in \mathcal{S}} P(f_i | f_{\mathcal{N}_i}, G, \theta) \\ &\approx \prod_{i \in \mathcal{S}} \prod_{k=1}^K \frac{\exp(-U(f_i^{(k)} | \theta^{(k)}))}{\sum_{\substack{M \\ I^{(k)}=1}} \exp(-U(I^{(k)} | \theta^{(k)}))} \\ &= \prod_{k=1}^K PL(f^{(k)} | \theta^{(k)}) \end{aligned} \quad (1.81)$$

where

$$PL(f^{(k)} | \theta^{(k)}) = \prod_{i \in \mathcal{S}} \frac{\exp(-U(f_i^{(k)} | \theta^{(k)}))}{\sum_{\substack{M \\ I^{(k)}=1}} \exp(-U(I^{(k)} | \theta^{(k)}))} \quad (1.82)$$

Thus the pseudo-likelihood  $PL(f | G, \theta)$  can be approximated by the product of individual pseudo-likelihood of the subband outputs  $f^{(k)}$ , in which a assumption is made that the parameters at different subbands are independent of each other. With this simplification, the parameter of the model can be estimated easily.

The parameters are estimated from sample data  $f_{\text{samp}}$  which is given. The estimation can be done by maximum likelihood  $P(f_{\text{samp}}^{(k)} | \theta^{(k)})$  or maximizing the pseudo-likelihood  $PL(f_{\text{samp}}^{(k)} | \theta^{(k)})$  defined in (1.81). According to that definition, each subband can be considered as an independent MRF model and hence the parameters of each subband can be estimated independently without considering the other subbands. Any method for MRF parameter estimation can be used to estimate parameters of each subband. The Markov chain Monte Carlo (MCMC) method (*cf.* Section 6.1.6) used in the parameter estimation for the FRAME model (*cf.* Section 6.1.7) would be a proper choice.

The MRMRF model attempts to incorporate information in a large neighborhood by fusing filtering theory and MRF models, which is similar to the FRAME model (Zhu et al. 1998). Compared with traditional MRF model, the MRMRF model can reveal more information contained in the textures since the original images are decomposed into subbands of different scales and directions and downsampled. For this reason, the MRMRF model is more powerful than the traditional MRF model; however it seems less powerful

than the FRAME model since only up to pair-site interactions are considered in MRMRF. Computationally, it is also between the traditional MRF and the FRAME, the FRAME being very expensive.

## 1.4 Optimization-Based Vision

Optimization has been playing an essential and important role in computer vision. Most vision problems are formulated as optimizing a criteria, explicitly or implicitly. The extensive use of optimization principles is due to various uncertainties in vision processes, such as noise and occlusion in the sensed image and ambiguities in visual interpretation. Exact or perfect solutions hardly exist. Inexact but optimal (in some sense) solutions are usually sought instead.

In the pioneer vision system of Roberts (Roberts 1965), object identification and pose estimation are performed using the simplest least squares (LS) fitting. Nowadays, optimization is pervasive in all aspects of vision, including image restoration and reconstruction (Grimson 1981; Terzopoulos 1983a; Geman and Geman 1984; Leclerc 1989; Hung et al. 1991), shape from shading (Ikeuchi and Horn 1981), stereo, motion and optical flow (Ullman 1979; Horn and Schunck 1981; Hildreth 1984; Murray and Buxton 1987; Barnard 1987), texture (Hassner and Slansky 1980; Kashyap et al. 1982; Cross and Jain 1983), edge detection (Torre and Poggio 1986; Tan et al. 1992), image segmentation (Silverman and Cooper 1988; Li 1990a), perceptual grouping (Lowe 1985; Mohan and Nevatia 1989; Herault and Horaud 1993), interpretation of line drawings (Leclerc and Fischler 1992), object matching and recognition (Fischler and Elschlager 1973; Davis 1979; Shapiro and Haralick 1981; Bhanu and Faugeras 1984; Ben-Arie and Meiri 1987; Modestino and Zhang 1989; Nasrabadi et al. 1990; Wells III 1991; Friedland and Rosenfeld 1992; Li 1992a; Li 1994a), and pose estimation (Haralick et al. 1989).

In all of the above cited examples, the solution is explicitly defined as an optimum of an objective function by which the goodness, or otherwise cost, of the solution is measured. Optimization may also be performed implicitly: the solution may optimize an objective function but in an implicit way which may or may not be realized. Hough transform (Hough 1962; Duda and Hart 1972; Ballard 1981; Illingworth and Kittler 1988) is a well-known technique for detecting lines and curves by looking at peaks of an accumulation function. It is later found to be equivalent to template matching (Stockman and Agrawala 1977) and can be reformulated as a maximizer of some probabilities such as the likelihood (Haralick and Shapiro 1992). Edge detection was performed using some simple operators like derivatives of Gaussian (Rosenfeld and Kak 1976). The operators can be derived by using regularization principles in which an energy function is explicitly minimized (Poggio et al. 1985).

The main reason for the extensive use of optimization is the existence of uncertainties in every vision process. Noise and other degradation fac-

tors, such as caused by disturbances and quantization in sensing and signal processing, are sources of uncertainties. Different appearances and poses of objects, their mutual and self occlusion and possible shape deformation also cause ambiguities in visual interpretation. Under such circumstances, we can hardly obtain exact or perfect solutions and have to resort to inexact yet optimal solutions.

Because of the importance of optimization, it is crucial to study vision problems from the viewpoint of optimization and to develop methodologies for optimization-based vision modeling. The following presents discussions on optimization-based vision.

### 1.4.1 Research Issues

There are three basic issues in optimization-based vision: problem representation, objective function and optimization algorithms. There are two aspects of a representation: descriptive and computational. The former concerns how to represent image features and object shapes, which relates to photometry and geometry (Koenderink 1990; Mundy and Zisserman 1992; Kanatani 1993) and is not an emphasis of this book. The latter concerns how to represent the solution, which relates to the choice of sites and label set for a labeling problem. For example, in image segmentation, we may use a chain of boundary locations to represent the solution; we may alternatively use a region map to do the same job. Comparatively speaking, however, the region map is a more natural representation for MRFs.

The second issue is how to formulate an objective function for the optimization. The objective function maps a solution to a real number measuring the quality of the solution in terms of some goodness or cost. The formulation determines how various constraints, which may be pixel properties like intensity and color and/or context like relations between pixels or object features, are encoded into the function. Because the optimal solution which is the optimum of the objective function, the formulation defines the optimal solution.

The third is how to optimize the objective, *i.e.* how to search for the optimal solution in the admissible space. Two major concerns are (1) the problem of local minima existing in non-convex functions and (2) the efficiency of algorithms in space and time. They are somewhat contradictory and currently there is no algorithms which guarantee the global solution with good efficiency.

These three issues are related to one another. In the first place, the scheme of representation influences the formulation of the objective function and the design of the search algorithm. On the other hand, the formulation of an objective function affects the search. For example, suppose two objective functions have the same point as the unique global optimum but one of them is convex whereas the other is not; obviously the convex one is much more desired because it provides convenience for the search.

In the following presentation, we will be mainly dealing with minimization problems. An objective function is in the form of an energy function and is to be minimized.

### 1.4.2 Role of Energy Functions

The role of an energy function in minimization-based vision is twofold: (1) as the quantitative measure of the global quality of the solution and (2) as a guide to the search for a minimal solution. As the quantitative cost measure, an energy function defines the minimal solution as its minimum, usually a global one. In this regard, it is important to formulate an energy function so that the “correct solution” is embedded as the minimum. We call this the correctness of the formulation.

To understand an optimization approach, one should not mix problems in formulation and those in search. Differentiating the two different kinds of problems helps debug the modeling. For example, if the output of an optimization procedure (assuming the implementation is correct) is not what is expected, there are two possible reasons: (1) the formulation of the objective function is not a correct one for modeling the reality and (2) the output is a low quality local minimum. Due to which one is the problem should be identified before the modeling can be improved.

The role of an energy function as a guide to the search may or may not be fully played. In real minimization, for example, when the energy function is smooth and convex w.r.t. its variables, global minimization is equivalent to local minimization and the gradient of the energy function provides sufficient information about where to search for the global solution. In this case, the role of guiding the search can be fully played. However, when the problem is non-convex, there is no general method which can efficiently utilize the energy function to guide the search. In this case, the role as the search-guide is underplayed.

In certain cases, it may be advantageous to consider the formulation of an energy function and the search simultaneously. This is to formulate the function appropriately to facilitate the search. The work of graduated non-convexity (GNC) (Blake and Zisserman 1987) is an example in this regard. There, the energy function is deformed gradually from a convex form to its target form in the process of approximating the global solution using a gradient-based algorithm.

Local minimization in real spaces is the most mature area in optimization and many formal approaches exist for solving it. This is not so for combinatorial and global minimization. In the latter cases, heuristics become an important and perhaps necessary element in practice. In the heuristic treatment of global minimization, rather restrictive assumptions are made. An example is the bounded model (Baird 1985; Breuel 1992). It assumes that a measurement error is upper-bounded by a certain threshold (within the threshold, the error may be assumed to be evenly distributed). Whether the

assumption is valid depends on the threshold. It is absolutely true when the threshold is infinitely large. But in practice, the threshold is almost always set to a value which is less than that required to entirely validate the bounded-error assumption. The lower the value, the higher the efficiency is, but the less general the algorithm becomes.

In the hypothesis-verification approach, efficient algorithms are used to generate hypothetical solutions, such as Hough transform (Hough 1962; Duda and Hart 1972), interpretation tree search (Grimson and Lozano-Prez 1987) and geometric hashing (Lamdan and Wolfson 1988). The efficiency comes from the fast elimination of infeasible solutions, or pruning of the solution space, by taking advantage of heuristics. In this way, a relatively small number of solution candidates are picked up relatively quickly and are then verified or evaluated thoroughly, for example, by using an energy function. In this strategy, the energy function is used for the evaluation only, not as a guide to the search.

Note that the advantage of formal approaches is in the evaluation and the advantage of heuristic approaches is in the search. A good strategy for the overall design of a specialized system may be the following: use a heuristic algorithm to quickly find a small number of solution candidates and then evaluate the found candidates using an energy function derived formally to give the best solution.

### 1.4.3 Formulation of Objective Functions

In pattern recognition, there are two basic approaches to formulating an energy function: parametric and nonparametric. In the parametric approach, the types of underlying distributions are known and the distributions are parameterized by a few parameters. Therefore, the functional form of the energy can be obtained and the energy function is completely defined when the parameters are specified.

In the nonparametric approach, sometimes called distribution free approach, no assumptions about the distributions are made. There, a distribution is either estimated from the data or approximated by a pre-specified basis functions with several unknown parameters in it to be estimated. In the latter case, the pre-specified basis functions will determine the functional form of the energy.

Despite the terms parametric and nonparametric, both approaches are somewhat parametric in nature. This is because in any case, there are always parameters that must be determined to define the energy function.

The two most important aspects of an energy function are its form and the involved parameters. The form and parameters together define the energy function which in turn defines the minimal solution. The form depends on assumptions about the solution  $f$  and the observed data  $d$ . We express this using the notation  $E(f \mid d)$ . Denote the set of involved parameters by  $\theta$ . With  $\theta$ , the energy is expressed further as  $E(f \mid d, \theta)$ . In general, given the

functional form for  $E$ , a different  $d$  or  $\theta$  defines a different energy function,  $E(f \mid d, \theta)$ , w.r.t.  $f$  and hence a (possibly) different minimal solution  $f^*$ .

Since the parameters are part of the definition of the energy function  $E(f \mid d, \theta)$ , the minimal solution  $f^* = \arg \min_f E(f \mid d)$  is not completely defined if the parameters are not specified even if the functional form is known. These parameters must be specified or estimated by some means. This is an important area of study in the MRF vision modeling.

#### 1.4.4 Optimality Criteria

In formal models, as opposed to heuristic ones, an energy function is formulated based on an established criterion. Because of inevitable uncertainties in vision processes, principles from statistics, probability and information theory are often used as the formal basis.

Because of inevitable uncertainties in imaging and vision processes, principles from statistics, probability and information theory are often used as the formal basis. When the knowledge about the data distribution is available but not about the prior information, the *maximum likelihood* (ML) criterion may be used,  $f^* = \arg \max P(d \mid f)$ . On the other hand, if only the prior information is available, the *maximum entropy* criterion may be chosen,  $f^* = \arg \max -\sum_{i=1}^m P(f_i) \ln P(f_i)$ . The maximum entropy criterion is simply taking this fact into account: Configurations with higher entropy are more likely because nature can generate them in more ways (Jaynes 1982).

When both the prior and likelihood distributions are known, the best result is achieved by that maximizes a Bayes criterion according to Bayes statistics (Therrien 1989). Bayes statistics is a theory of fundamental importance in estimation and decision making. Although there have been philosophical and scientific controversies about their appropriateness in inference and decision making (see (Clark and Yuille 1990) for a short review), Bayes criteria, the MAP principle in particular, are the most popular ones in computer vision and in fact, MAP is the most popular criterion in optimization-based MRF modeling. The equivalence theorem of between Markov random fields and Gibbs distribution established in Section 1.2.4 provides a convenient way for specifying the joint prior probability, solving a difficult issue in MAP-MRF labeling.

In the principle of *minimum description length* (MDL) (Rissanen 1978; Rissanen 1983), the optimal solution to a problem is that needs the smallest set of vocabulary in a given language for explaining the input data. The MDL has close relationships to the statistical methods such as the ML and MAP (Rissanen 1983). For example, if  $P(f)$  is related to the description length and  $P(d \mid f)$  related to the description error, then MDL is equivalent to MAP. However, it is a more natural and intuitive when prior probabilities are not well defined. The MDL has been used for vision problems at different levels such as segmentation (Leclerc 1989; Pentland 1990; Darrell et al. 1990; Dengler 1991; Keeler 1991) and object recognition (Breuel 1993).

## 1.5 Bayes Labeling of MRFs

Bayes statistics is a theory of fundamental importance in estimation and decision making. According to this theory, when both the prior distribution and the likelihood function of a pattern are known, the best that can be estimated from these sources of knowledge is the Bayes labeling. The maximum *a posterior* (MAP) solution, as a special case in the Bayes framework, is sought in many vision works.

The MAP-MRF framework is advocated by Geman and Geman (1984) and others (Geman and McClure 1985; Derin and Elliott 1987; Geman and Graffigne 1987; Dubes and Jain 1989; Besag 1989; Szeliski 1989; Geman and Gidas 1991). Since the paper of (Geman and Geman 1984), numerous vision problems have formulated in this framework. This section reviews related concepts and derives involved probabilistic distributions and energies in MAP-MRF labeling. For more detailed materials on Bayes theory, the reader is referred to books like (Therrien 1989).

### 1.5.1 Bayes Estimation

In Bayes estimation, a risk is minimized to obtain the optimal estimate. The Bayes risk of estimate  $f^*$  is defined as

$$R(f^*) = \int_{f \in \mathbb{F}} C(f^*, f) P(f \mid d) df \quad (1.83)$$

where  $d$  is the observation,  $C(f^*, f)$  is a cost function and  $P(f \mid d)$  is the posterior distribution. First of all, we need to compute the posterior distribution from the prior and the likelihood. According to the Bayes rule, the posterior probability can be computed by using the following formulation

$$P(f \mid d) = \frac{p(d \mid f)P(f)}{p(d)} \quad (1.84)$$

where  $P(f)$  is the prior probability of labelings  $f$ ,  $p(d \mid f)$  is the conditional p.d.f. of the observations  $d$ , also called the likelihood function of  $f$  for  $d$  fixed, and  $p(d)$  is the density of  $d$  which is a constant when  $d$  is given.

The cost function  $C(f^*, f)$  determines the cost of estimate  $f$  when the truth is  $f^*$ . It is defined according to our preference. Two popular choices are the quadratic cost function

$$C(f^*, f) = \|f^* - f\|^2 \quad (1.85)$$

where  $\|a - b\|$  is a distance between  $a$  and  $b$ , and the  $\delta$  (0-1) cost function

$$C(f^*, f) = \begin{cases} 0 & \text{if } \|f^* - f\| \leq \delta \\ 1 & \text{otherwise} \end{cases} \quad (1.86)$$

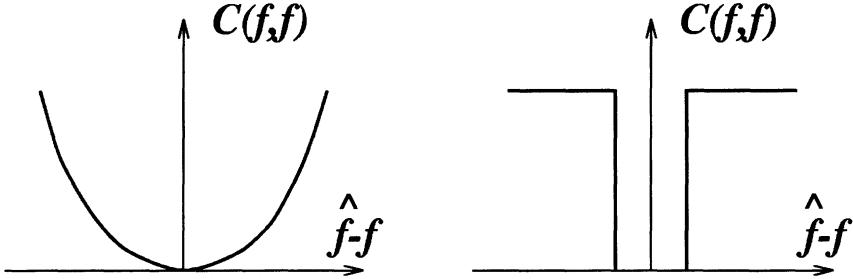


Figure 1.6: Two choices of cost functions.

where  $\delta > 0$  is any small constant. A plot of the two cost functions are shown in Fig.1.6.

The Bayes risk under the quadratic cost function measures the variance of the estimate

$$R(f^*) = \int_{f \in \mathcal{F}} \|f^* - f\|^2 P(f | d) df \quad (1.87)$$

Letting  $\frac{\partial R(f^*)}{\partial f^*} = 0$ , we obtain the minimal variance estimate

$$f^* = \int_{f \in \mathcal{F}} f P(f | d) df \quad (1.88)$$

The above is the mean of the posterior probability.

For the  $\delta$  cost function, the Bayes risk is

$$R(f^*) = \int_{f: \|f^* - f\| > \delta} P(f | d) df = 1 - \int_{f: \|f^* - f\| \leq \delta} P(f | d) df \quad (1.89)$$

When  $\delta \rightarrow 0$ , the above is approximated by

$$R(f^*) = 1 - \kappa P(f | d) \quad (1.90)$$

where  $\kappa$  is the volume of the space containing all points  $f$  for which  $\|f^* - f\| \leq \delta$ . Minimizing the above is equivalent to maximizing the posterior probability. Therefore, the minimal risk estimate is

$$f^* = \arg \max_{f \in \mathcal{F}} P(f | d) \quad (1.91)$$

which is known as the MAP estimate. Because  $p(d)$  in (1.84) is a constant for a fixed  $d$ ,  $P(f | d)$  is proportional to the joint distribution

$$P(f | d) \propto P(f, d) = p(d | f)P(f) \quad (1.92)$$

Then the MAP estimate is equivalently found by

$$f^* = \arg \max_{f \in \mathbb{F}} \{p(d | f)P(f)\} \quad (1.93)$$

Obviously, when the prior distribution,  $P(f)$ , is flat, the MAP is equivalent to the maximum likelihood.

### 1.5.2 MAP-MRF Labeling

In the MAP-MRF labeling,  $P(f | d)$  is the posterior distribution of an MRF. An important step in Bayes labeling of MRFs is to derive this distribution. Here we use a simple example to illustrate the formulation of a MAP-MRF labeling problem. The problem is to restore images from noisy data. Assuming that the image surfaces are flat, then the joint prior distribution of  $f$  is

$$P(f) = \frac{1}{Z} e^{-U(f)} \quad (1.94)$$

where  $U(f) = \sum_i \sum_{i' \in \{i-1, i+1\}} (f_i - f_{i'})^2$  is the *prior energy* for the type of surfaces. Assuming that the observation is the true surface height plus the independent Gaussian noise,  $d_i = f_i + e_i$ , where  $e_i \sim N(\mu, \sigma^2)$ , then the likelihood distribution is

$$p(d | f) = \frac{1}{\prod_{i=1}^m \sqrt{2\pi\sigma^2}} e^{-U(d | f)} \quad (1.95)$$

where

$$U(d | f) = \sum_{i=1}^m (f_i - d_i)^2 / 2\sigma^2 \quad (1.96)$$

is the *likelihood energy*. Now the posterior probability is

$$P(f | d) \propto e^{-U(f | d)} \quad (1.97)$$

where

$$\begin{aligned} U(f | d) &= U(d | f) + U(f) \\ &= \sum_{i=1}^m (f_i - d_i)^2 / 2\sigma_i^2 + \sum_{i=1}^m (f_i - f_{i-1})^2 \end{aligned} \quad (1.98)$$

is the *posterior energy*. The MAP estimate is equivalently found by minimizing the posterior energy function

$$f^* = \arg \min_f U(f | d) \quad (1.99)$$

There is only one parameter in this simple example,  $\sigma_i$ . When it is determined,  $U(f | d)$  is fully specified and the MAP-MRF solution is completely defined.

### 1.5.3 Regularization

The MAP labeling with a prior potential of the form  $[f^{(n)}]^2$  introduced in Section 1.3.3 is equivalent to the regularization of an ill-posed problem. A problem is mathematically an *ill-posed problem* (Tikhonov and Arsenin 1977) in the Hadamard sense if its solution (1) does not exist, (2) is not unique or (3) does not depend continuously on the initial data. An example is surface reconstruction or interpolation. A problem therein is that there are infinitely many ways to determine the interpolated surface values if only the constraint from the data is used. Additional constraints are needed to guarantee the uniqueness of the solution to make the problem well-posed. An important such constraint is *smoothness*. By imposing the smoothness constraint, the analytic regularization method converts an ill-posed problem into a well-posed one. This has been used in solving low level problems such as surface reconstruction from stereo (Grimson 1981), optical flow (Horn and Schunck 1981), shape-from-shading (Ikeuchi and Horn 1981), and motion analysis (Hildreth 1984). A review of earlier work of regularization in vision is given by (Poggio et al. 1985). Relationships between regularization and smoothing splines for vision processing are investigated by (Terzopoulos 1986b; Lee and Pavlidis 1987).

A regularization solution is obtained by minimizing an energy of the following form

$$E(f) = \sum_{i \in \mathcal{A}} [f(x_i) - d(x_i)]^2 + \lambda \int_a^b [f^{(n)}(x)]^2 dx \quad (1.100)$$

where  $\mathcal{A}$  is a set of indices to the sample data points,  $x_i$ 's are the locations of the data points,  $\lambda \geq 0$  is a weighting factor and  $n \geq 1$  is an integer number. The first term on the RHS, called the closeness term, imposes the constraint from the data  $d$ . The second term, called the smoothness term or the regularizer, imposes the *a priori* smoothness constraint on the solution. It is desired to minimize both but they may not be each minimized simultaneously. The two constraints are balanced by  $\lambda$ . Any  $f$  minimizing (1.100) is a smooth solution in the so-called Sobolev space  $W_2^{n,4}$ <sup>4</sup>. Different types of regularizers impose different types of smoothness constraints. Section 1.3.3 has described smoothness priors for some classes of surfaces. A study of the quadratic smoothness constraints is given in (Snyder 1991).

Under certain conditions, the MAP labeling of MRFs is equivalent to a regularization solution (Marroquin et al. 1987). These conditions are (1) that the likelihood energy is due to additive white Gaussian noise and (2) that the prior assumption is the smoothness. Letting  $n = 1$  and discretizing  $x$ , (1.100) becomes the posterior energy (1.98) for the restoration of flat surfaces (in 1D). When the regularizer takes the form of the rod (1.61), it

---

<sup>4</sup>In the Sobolev space, every point  $f$  is a function whose  $n - 1$  derivative  $f^{(n-1)}$  is absolutely continuous and whose  $n$ -th derivative  $f^{(n)}$  is square integrable.

encodes the smoothness of planar surfaces (in 1D). The 2D counterparts are the membrane (1.60), (1.65) and the plate (1.66).

MRFs is more general than regularization in that it can encode not only the smoothness prior but also priors of other constraints. MRF-based models have to be chosen among the two when the priors are due to those other than the smoothness, *e.g.* in texture modeling and analysis. However, regularization can deal with spatially continuous fields whereas MRFs, which in this book are spatially discrete, cannot. When the problem under consideration involves concepts such as discontinuities, the analytical regularization is more suitable for the analysis. This is why we will study the issue of discontinuities, in Chapter 4, from regularization viewpoint.

#### 1.5.4 Summary of MAP-MRF Approach

The procedure of the MAP-MRF approach for solving computer vision problems is summarized in the following:

1. Pose a vision problem as one of labeling in categories LP1-LP4 and choose an appropriate MRF representation  $f$ .
2. Derive the posterior energy to define the MAP solution to a problem.
3. Find the MAP solution.

The process of deriving the posterior energy is summarized as the following four steps:

1. Define a neighborhood system  $\mathcal{N}$  on  $\mathcal{S}$  and the set of cliques  $\mathcal{C}$  for  $\mathcal{N}$ .
2. Define the prior clique potentials  $V_c(f)$  to give  $U(f)$ .
3. Derive the likelihood energy  $U(d | f)$ .
4. Add  $U(f)$  and  $U(d | f)$  to yield the posterior energy  $U(f | d)$ .

The prior model depends on the type of the scene (*e.g.* the type of surfaces) we expect. In vision, it is often one of the Gibbs models introduced in Section 1.3. The likelihood model depends on physical considerations such as the sensor process (transformations, noise, *etc.* ). It is often assumed to be Gaussian. The parameters in both models need be specified for the definitions of the models to be complete. The specifications can be something of arts when done manually and it is desirable that it is done automatically.

In the subsequent chapters, we are concerned with the following issues:

1. Choosing an appropriate representation for the MRF labeling.
2. Deriving the *a posteriori* distribution of the MRF as the criterion function of the labeling solution. It mainly concerns the specification of the forms of the prior and the likelihood distributions. The involved parameters may or may not be specified at this stage.

3. Estimating involved parameters in the prior and the likelihood distributions. The estimation is also based on some criterion, very often, maximum likelihood. In the unsupervised case, it is performed together with MAP labeling.
4. Searching for the MRF configuration to maximize the posterior distribution. This is mainly algorithmic. The main issues are the quality (globalness) of the solution and the efficiency.

## 1.6 Validation of Modeling

Having formulated an energy function based on probability and statistics, with both its form and parameters given, one would ask the following question: Is the formulation correct? In other words, whether the energy minimum  $f^*$  corresponds to the correct solution? Before answering this, let us make some remarks on the question. First, “the energy minimum”  $f^*$  is meant to be a global one. Second, “the correct solution”, denoted  $f_{correct}$ , is said here in the subjective sense of our own judgment. For example, in edge detection, we run a program and get an edge map as a solution; but we find that some edges are missing and some false edges appear. This is the case where the solution differs from the subjectively correct solution of ours.

In the ideal case, the global minimal solution  $f^*$  should correspond to the correct solution  $f_{correct}$

$$f_{correct} = f^* = \min_f E(f \mid d, \theta) \quad (1.101)$$

A formulation is *correct for d* if the above holds. This reflects our desire to encode our ability into the machine and is our ideal in formulating an energy function. For the time being, we do not expect the machine to surpass our ability and therefore we desire that the minimum of an energy function, or the machine’s solution, be consistent with ours. Whenever there is a difference, we assume that our own perception is correct and the difference is due to problems such as incorrect assumptions made in the modeling stage.

Assume that there are  $N$  observation data sets  $d^{(n)}$  ( $n = 1, 2, \dots, N$ ). Supposing that  $f_{correct}[d^{(n)}]$  is the correct solution for  $d^{(n)}$  in our own perception, we say that the functional form of the energy  $E$  and the chosen parameters  $\theta$  are correct for the  $N$  observations if the following

$$f_{correct}[d] = f^* = \arg \min_f E(f \mid d, \theta) \quad (1.102)$$

where  $f^*$  is the energy minimum, holds for each  $d = d^{(n)}$ . Let  $\mathbb{D}$  be the space of all possible observations. If Eq.(1.102) holds for all  $d \in \mathbb{D}$ , we say that  $E$  and  $\theta$ , hence the formulation, is *entirely correct*.

A formulation may not be absolutely correct. For example, a formulation for edge detection may give solutions having some missing or false edges or

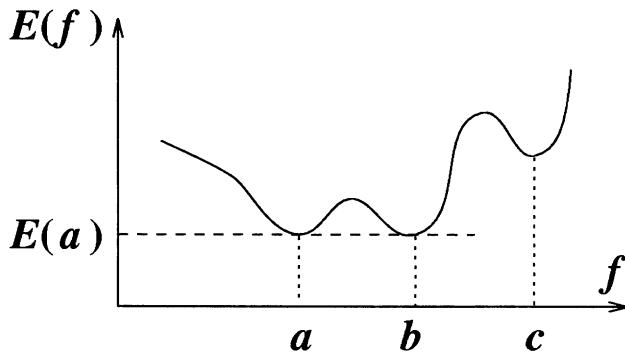


Figure 1.7: Two global minima  $a$  and  $b$  and a local minimum  $c$ .

having some inaccurate edge locations. The quality of a formulation may be measured by some distance between the global energy minimum  $f^*$  and the correct solution  $f_{correct}$ . Ideally, it should be zero.

What about multiple global minima? Suppose there are two global minima for which  $E(a) = E(b)$ , as in Fig.1.7. Both solutions  $a$  and  $b$  should be equally good if the formulation is correct. If in the designer's opinion, one of them is better than the other in our perception, then there is still room to improve the formulation of  $E(f)$ . The improvement is made by adjusting either the functional form or the parameters so that the improved energy function embeds the better solution as its unique global minimum.

What if we run a program which correctly implements an energy minimization based system but find that the result is not what we expect? First of all, we may want to identify whether the fault is due to the local minimum problem caused by the minimization algorithm or due to the incorrect formulation of the energy function. The local minimum problem is identified if a better solution can be obtained with a different initialization (for an iterative algorithm). Otherwise if the algorithm is good enough to find the global minimum or to approximate by a good quality local minimum but such a solution is still far from our subjective judgment, we may need to check the formulation, *i.e.*, the form of the energy function and the involved parameters.

We may try different combinations of parameters to see if we can get the desired results – though this is not a clever way. Assuming that we had exhausted all possible combinations but none of the solutions (again, global minima) are close enough to the desired solution. Then there must be something fundamentally wrong. We have to have a critical look at the problem formulation.

One possible reason may be that some constraints which may be important for solving the problem have not been properly embedded into the energy function – current imaging models are often formulated based on oversimplified assumptions (Pavlidis 1992; Rosenfeld 1994) for tractability rea-

sons. Another possibility is that some assumptions about the prior knowledge and the observed data are incorrectly made. An example is the oversmoothing problem in the MAP restoration and reconstruction. The MAP was blamed for the problem; however, what really caused the problem was the wrong prior assumption of the quadratic smoothness rather than the MAP principle.

# Chapter 2

## Low Level MRF Models

Low level processing is performed on regular lattices of images. The set of sites  $\mathcal{S} = \{1, \dots, m\}$  index image pixels in the image plane. The set  $\mathcal{L}$  contains continuous or discrete labels for the pixels. Therefore the problems fall into categories LP1 and LP2. Most existing MRF vision models are for low level processing. MRF models for image restoration and segmentation have been studied most comprehensively. Surface reconstruction can be viewed as a more general case than restoration in that the data can be sparse, *i.e.* available at certain locations of the image lattice. MRFs can play a fuller role in texture analysis because textured images present anisotropic properties. The treatment of optical flow as MRF is similar to that of restoration and reconstruction. Edge detection is often addressed along with restoration, reconstruction and analysis of other image properties such as texture, flow and motion. We can also find low level applications of MRFs such as active contours, deformable templates, data fusion and visual integration.

In this chapter, we formulate various MAP-MRF models for low level vision, following the procedure summarized in Section 1.5.4. We begin with the prototypical MAP-MRF models for image restoration. The presentation therein introduces the most important concepts in MRF modeling. After that, the formulations for the image restoration are extended to a closely related problem, surface reconstruction, in which the observation may be sparser. The MRF models for boundary detection, texture and optical flow are described subsequently. How to impose the smoothness constraint while allowing discontinuities is an important issue in computer vision (Terzopoulos 1983b; Geman and Geman 1984; Blake and Zisserman 1987) which deserves a thorough investigation; it is the topic of Chapter 4. Another important issue, MRF parameter estimation in low level vision, will be discussed in Chapter 6.

## 2.1 Observation Models

In low level vision, an observation  $d = \{d_1, \dots, d_m\}$  is a rectangular array of pixel values. In some cases, the observation may be sparse, that is, it is available at  $i \in \mathcal{A} \subset \mathcal{S}$ . Every pixel takes a value  $d_i$  in a set  $\mathcal{D}$ . In practice,  $\mathcal{D}$  is often the set of integers encoded by a byte (8 bits), as a result of normalization and quantization, so that  $\mathcal{D} = \{0, 1, \dots, 255\}$ .

An observation  $d$  can be considered as a transformed and degraded version of an MRF realization  $f$ . The transformation may include geometric transformation and blurring and the degradation is due to random factors such as noise. These determine the conditional distribution  $p(d | f)$ , or the likelihood of  $f$ . A general observation model can be expressed as

$$d = \varphi(B(f)) \odot e \quad (2.1)$$

where  $B$  is a blurring effect,  $\varphi$  is a transformation which can be linear or nonlinear, deterministic or probabilistic,  $e$  is the sensor noise and  $\odot$  is an operator of addition or multiplication. In practice, a simple observation model of no blurring, linear transformation and independent additive Gaussian noise is often assumed. Each observed pixel value is assumed to be the sum of the true grey value and independent Gaussian noise

$$d_i = \varphi(f_i) + e_i \quad (2.2)$$

where  $\varphi(\cdot)$  is a linear function and  $e_i \sim N(0, \sigma_i^2)$ . The probability distribution of  $d$  conditional on  $f$ , or the likelihood of  $f$ , is

$$p(d | f) = \frac{1}{\prod_i^m \sqrt{2\pi\sigma_i^2}} e^{-U(d | f)} \quad (2.3)$$

where

$$U(d | f) = \sum_{i \in \mathcal{S}} (\varphi(f_i) - d_i)^2 / [2\sigma_i^2] \quad (2.4)$$

is the likelihood energy. Obviously, the above is a special form of Gibbs distribution whose energy is due purely to single-site cliques in the zero-th order neighborhood system (where the radius  $r = 0$ ), with clique potentials being  $[\varphi(f_i) - d_i]^2 / [2\sigma_i^2]$ . If the noise distribution is also homogeneous, then the deviations are the same for all  $i \in \mathcal{S}$ , i.e.  $\sigma_i = \sigma$ .

The function  $\varphi(\dots)$  maps a label  $f_i$  to a real grey value where  $f_i$  may be numerical or symbolic, continuous or discrete. When  $f_i$  is symbolic, for example, representing a texture type, the ordering of the MRF labels is usually not defined unless artificially. Without loss of generality, we can consider that there is a unique numerical value for a label  $f_i$  and denote  $d_i = \varphi(f_i) + e_i$  simply as  $d_i = f_i + e_i$ . Then the likelihood energy becomes

$$U(d | f) = \sum_{i \in \mathcal{S}} (f_i - d_i)^2 / [2\sigma^2] \quad (2.5)$$

for i.i.d. Gaussian noise.

A special observation model for discrete label set is random replacement. An  $f_i$  is transformed into  $d_i$  according to the following likelihood probability

$$P(d_i = k \mid f_i = k') = \begin{cases} p & = e^{\ln p} \quad \text{if } k = k' \\ \frac{1-p}{M-1} & = e^{\ln \frac{1-p}{M-1}} \quad \text{otherwise} \end{cases} \quad (2.6)$$

where  $\mathcal{L} = \mathcal{D} = \{0, \dots, M-1\}$  contain un-ordered labels for  $f_i$  and  $d_i$ ,  $0 \leq p \leq 1$ ,  $k \in \mathcal{D}$  and  $k' \in \mathcal{L}$ . In this model, a label value remains unchanged with probability  $p$  and changes to any other value with equal probability  $\frac{1-p}{M-1}$ . This describes the transition from one state to another. The simplest case of this is the random flip-over of binary values.

A useful and convenient model for both the underlying MRF and the observation is the hierarchical GRF model (Derin and Cole 1986; Derin and Elliott 1987). There two-levels of Gibbs distributions are used to represent noisy or textured regions. The higher level Gibbs distribution, which is usually an MLL, characterizes the blob-like region formation process while the lower level Gibbs distribution describes the filling-in, such as noise or texture, in each region. This will be described in detail in Section 2.4.1.

## 2.2 Image Restoration and Reconstruction

Image restoration is to recover true pixel values,  $f$ , from the observed (noisy) image pixel values,  $d$ . When the image  $d(x, y)$  is considered as a noise-added surface, the problem is to restore the underlying surface  $f(x, y)$ .

### 2.2.1 MRF Priors for Image Surfaces

The underlying surface from which  $f$  is sampled is a graph surface  $\{f(x, y)\}$  defined on a continuous domain where  $f(x, y)$  is the height of the surface at  $(x, y)$ . The first factor affecting the specification of the MRF prior distributions is whether  $f(x, y)$  takes a continuous or discrete value. We are interested in  $\{f(x, y)\}$  which is either piecewise continuous or piecewise constant, *i.e.* being piecewise because of involved discontinuities. In the image representation,  $\{f(x, y)\}$  is sampled at an image lattice, giving sampled surface heights  $\{f_{i,j}\}$ . In the subsequent presentation, the double subscript  $(i, j)$  is replaced by the single subscript  $i$  and the MRF samples is denoted as  $f = \{f_1, \dots, f_m\}$  unless there is a need for the elaboration.

The set of the sampled heights  $f = \{f_1, \dots, f_m\}$  is assumed to be a realization of an MRF, that is  $f_i$  at a particular location  $i$  depends on those in the neighborhood. According to MRF-Gibbs equivalence, specifying the prior distribution of an MRF amounts to specifying the clique potential functions  $V_c$  in the corresponding Gibbs prior distribution (1.24). The assumption of  $\{f(x, y)\}$  being continuous, piecewise continuous or piecewise constant leads to different Gibbs distributions for the MRF.

### MRF Prior for Piecewise Constant Surfaces

Piecewise constant surfaces, or homogeneous blob-like regions, can be properly characterized by the MLL (multi-level logistic), more specifically the homogeneous and isotropic MLL model described in Section 1.3.2. For cliques containing more than one site, the clique potentials are defined as<sup>1</sup>

$$V_c(f) = \begin{cases} 0 & \text{if all sites in } c \text{ have the same label} \\ -\zeta_c & \text{otherwise} \end{cases} \quad (2.7)$$

where  $\zeta_c < 0$  is a constant dependent on  $c$ . That “all sites in  $c$  have the same label”, that is, all  $\{f_i \mid i \in c\}$  are the same, means the entire smoothness of labels  $f$  on the clique  $c$ . Any violation of the entire smoothness incurs a penalty of the positive number  $-\zeta_c > 0$ . Because the more probable configurations are those with higher  $P(f)$ , or lower  $U(f)$ , values, the MLL model (2.7) favors smooth  $f$ .

For single site cliques, the clique potentials depend on the label assigned to the site

$$V_c(f) = V_1(f_i) = \alpha_l \quad \text{if } f_i = l \in \mathcal{L}_d \quad (2.8)$$

where  $\alpha_l$  is the penalty against that  $f_i$  is labeled  $l$ ; see (1.53). The higher  $\alpha_l$  is, the less pixels will be assigned the value  $l$ . This has an effect of controlling the percentage of the sites labeled  $l$ .

A special case of (2.7) is such that  $v_c$  is nonzero only for the pair-site cliques and zero for all the other types. In this case,  $\zeta_c = 0$  for all  $c$  of size  $\#c > 2$ ; when  $\#c = 2$ , it is

$$V_c(f) = V_2(f_i, f_{i'}) = v_{20}[1 - \delta(f_i - f_{i'})] \quad (2.9)$$

where  $\delta(\cdot)$  is the Kronecker delta function and  $v_{20}$  is the penalty against non-equal labels on two-site cliques. The prior energy is the sum of all the clique potentials, that is,

$$U(f) = \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} v_{20}[1 - \delta(f_i - f_{i'})] \quad (2.10)$$

where “ $\sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i}$ ” is equivalent to “ $\sum_{\{i, i'\} \in \mathcal{C}_2}$ ”. This simple form has been used for the restoration of piecewise constant images by (Geman and Geman 1984; Elliott et al. 1984; Derin and Cole 1986; Derin and Elliott 1987; Leclerc 1989; Li 1990a).

### MRF Prior for Piecewise Continuous Surfaces

In specifying prior clique potentials for continuous surfaces, only pair-site clique potentials are normally used for piecewise continuous surfaces. In the simplest case, that of flat surface, they can be defined by

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<sup>1</sup>The definition in (2.7) is effectively the same as (1.52) for the restoration purpose.

$$V_2(f_i, f_{i'}) = g(f_i - f_{i'}) \quad (2.11)$$

where  $g(f_i - f_{i'})$  is a function penalizing the violation of smoothness caused by the difference  $f_i - f_{i'}$ . For the purpose of restoration, the function  $g$  is generally even, *i.e.*

$$g(\eta) = g(-\eta) \quad (2.12)$$

and nondecreasing, *i.e.*

$$g'(\eta) \geq 0 \quad (2.13)$$

on  $[0, +\infty)$ . The derivative of  $g$  can be expressed as the following form

$$g'(\eta) = 2\eta h(\eta) \quad (2.14)$$

One may choose appropriate  $g$  functions to impose different smoothness assumptions: complete smoothness or piecewise smoothness.

When the underlying  $\{f(x, y)\}$  does not contain discontinuities, *i.e.* continuous everywhere,  $g(\cdot)$  is usually a quadratic function

$$g(\eta) = \eta^2 \quad (2.15)$$

Under the quadratic  $g$ , the penalty to the violation of smoothness is proportional to  $(f_i - f_{i'})^2$ . The quadratic pair-site potential function is not suitable for prior surface models in which the underlying  $\{f(x, y)\}$  is only piecewise continuous. At discontinuities,  $\|f_i - f_{i'}\|$  tends to be very large and the quadratic function brings about a large smoothing force when the energy is being minimized, giving an oversmoothed result.

To encode piecewise smoothness,  $g$  has to satisfy a necessary condition<sup>2</sup>

$$\lim_{\eta \rightarrow \infty} |g'(\eta)| = C < \infty \quad (2.16)$$

where  $C \in [0, \infty)$  is a constant. The above means that  $g$  should saturate at its asymptotic upper-bound when  $\eta \rightarrow \infty$  to allow discontinuities. A possible choice is the truncated quadratic used in the line process model (Geman and Geman 1984; Marroquin 1985; Blake and Zisserman 1987) in which

$$g(\eta) = \min\{\eta^2, \alpha\} \quad (2.17)$$

The above satisfies condition (2.16). How to properly choose the  $g$  (or  $h$ ) function discontinuity-adaptive restoration is the subject to be studied in Chapter 4.

A significant difference between (2.11) and (2.9) is due to the nature of label set  $\mathcal{L}$ . For piecewise constant restoration where labels are considered un-ordered, the difference between any two labels  $f_i$  and  $f_{i'}$  is symbolic, *e.g.* in the set of  $\{yes, no\}$ . In the continuous case where labels are ordered by the relation, *e.g.* “smaller than”, the difference takes a continuous quantitative

<sup>2</sup>This equation will be referred to again as (4.26).

value. We may define a “softer” version of (2.9) to provide a connection between continuous and discrete restoration. For piecewise constant restoration of a two-level image, for example, we may define

$$V_2(f_i, f_{i'}) = v_{20}(1 - e^{-\eta^2/\gamma}) \quad (2.18)$$

which in the limit is  $\lim_{\gamma \rightarrow 0} V_2(f_i, f_{i'}) = v_{20}[1 - \delta(f_i - f_{i'})]$ .

The above instances of prior models, the clique potential is a function of the *first order* difference  $f_i - f_{i'}$  which is an approximation of  $f'_i$ . They have a tendency of producing surfaces of constant or piecewise constant height, though in the continuous restoration, continuous variation in height is allowed. They are not very appropriate models for situations where the underlying surface is not flat or piecewise flat.

According to the discussions in Section 1.3.3, we may design potential function  $g(f'_i)$ ,  $g(f''_i)$  and  $g(f'''_i)$  for surfaces of constant grey level (horizontally flat), constant gradient (planar but maybe slant) and constant curvature, respectively. It is demonstrated, *e.g.* in (Geman and Reynolds 1992), that potentials involving  $f''$  give better results for surfaces of constant gradient. Higher order models are also necessary for reconstruction from sparse data (Grimson 1981; Terzopoulos 1983a; Poggio et al. 1985; Blake and Zisserman 1987).

### 2.2.2 Piecewise Constant Restoration

In piecewise constant restoration,  $\mathcal{L}$  consists of discrete values. The task is to recover the true configuration  $f$  from observed image  $d$ . This is akin to region segmentation, because  $f$  in effect partitions the set of lattice sites into mutually exclusive regions. In the MAP-MRF framework, the optimal  $f$  is that minimizes the posterior energy.

#### Deriving Posterior Energy

Let us derive the posterior energy using the four-step procedure summarized in Section 1.5.4:

1. Define a neighborhood system and the set of cliques for it. The neighborhood system is defined according to (1.13). The set of cliques for the 4-neighborhood system are shown in Fig.1.2(d)-(e). For the 8-neighborhood system, cliques in Fig.1.2(f)-(h) are also included.
2. Define the prior clique potential functions in the Gibbs prior distribution (1.24-1.26). For the MLL prior, it is (2.9).
3. Derive the likelihood energy from the observation model. Assume the i.i.d. additive Gaussian model. The likelihood function  $p(f | d)$  takes the form of (2.5) with  $f_i$  taking a discrete value.

4. Add the prior energy  $U(f)$  and the likelihood energy  $U(d | f)$  to yield the posterior energy

$$E(f) = U(f | d) = \sum_{i \in \mathcal{S}} (f_i - d_i)^2 / [2\sigma^2] + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} v_{20} [1 - \delta(f_i - f_{i'})] \quad (2.19)$$

Note that

$$\sum_{i' \in \mathcal{N}_i} [1 - \delta(f_i - f_{i'})] = \#\{f_{i'} \neq f_i \mid i' \in \mathcal{N}_i\} \quad (2.20)$$

is simply the number of neighboring sites whose label  $f_{i'}$  is different from  $f_i$ .

In this problem, as well as in other labeling problems in this chapter, the parameters, such as  $\sigma^2$  and  $v_{20}$  here, are assumed known. The more advanced topic of labeling with unknown noise and MRF parameters will be discussed in Section 6.2.1.

### Energy Minimization

Because  $\mathcal{L}_d$  is discrete, minimizing (2.19) is a combinatorial problem. The minimal solution  $f^*$  is the optimally restored image in the configuration space  $\mathbb{F} = \mathcal{L}_d^m$ . The simplest algorithm is steepest local energy descent or the “greedy” method. It proceeds as follows: Start with an initial configuration  $f^{(0)} \in \mathbb{F}$ ; for site  $i$ , choose the new label  $f_i^{(t+1)}$  among all  $f_i^{(t+1)} \in \mathcal{L}_d$ ,  $f_i^{(t+1)} \neq f_i^{(t)}$ , that minimizes  $E(f^{(t+1)}) - E(f^{(t)})$  locally; the iteration continues until no further energy descent is possible. An example of such greedy algorithms is the iterative conditional modes (ICM) (Besag 1986); see Section 8.2.1 for details.

This simple algorithm finds a local energy minimum whose quality depends much on the initial estimate  $f^{(0)}$ . Global minimization algorithms such as simulated annealing (Geman and Geman 1984) need be used if global solutions are required. This is the subject in Chapter 9.

### 2.2.3 Piecewise Continuous Restoration

In piecewise continuous restoration,  $\mathcal{L}$  consists of a real interval. The task is to recover the underlying  $\{f(x, y)\}$  which is a piecewise continuous graph surface, being piecewise because discontinuities may exist in  $\{f(x, y)\}$ .

#### Deriving Posterior Energy

The first step is the same as that for piecewise constant restoration. The second step is to define the prior clique potential functions. Here, the single-site clique potentials are set to zero; only the pair-site cliques are considered with potential function (2.11) rewritten below

$$V_2(f_i, f_{i'}) = g(f_i - f_{i'}) \quad (2.21)$$

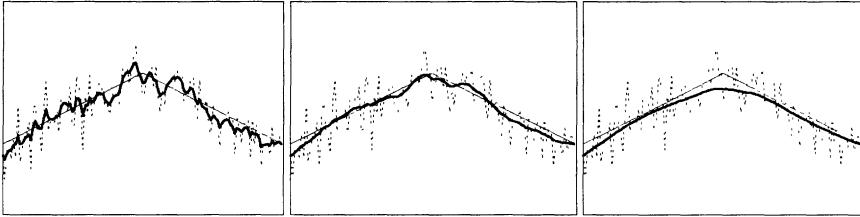


Figure 2.1: Solution  $f^*$  obtained with  $\lambda = 2$  (left), 20 (middle) and 200 (right). The original signals are in thinner solid lines, noisy data in dotted lines, and the minimal solutions in thicker solid lines.

To impose the piecewise smoothness prior, the  $g$  function is required to satisfy condition (2.16).

The third step is to derive the likelihood energy. When the noise is i.i.d. additive Gaussian,  $e_i \sim N(0, \sigma^2)$ , the likelihood energy is given (2.5) with continuously valued  $f_i$ .

In the fourth step, the prior energy  $U(f)$  and the likelihood energy  $U(d | f)$  are added to obtain the posterior energy

$$E(f) = \sum_{i \in \mathcal{S}} (f_i - d_i)^2 / [2\sigma^2] + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} g(f_i - f_{i'}) \quad (2.22)$$

or equivalently

$$E(f) = \sum_{i \in \mathcal{S}} (f_i - d_i)^2 + \lambda \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} g(f_i - f_{i'}) \quad (2.23)$$

where  $\lambda = 2\sigma^2$ . When no noise is present,  $\lambda = 0$ , only the first term is effective and the MAP solution is exactly the same as the data,  $f^* = r$ , which is also the maximum likelihood solution. As  $\lambda$  becomes larger, the solution is more influenced by the second, *i.e.* the smoothness, term. Fig.2.1 shows the effect of  $\lambda$  on  $f^*$  with the quadratic  $g$  function. It can be seen that the larger is  $\lambda$ , the smoother is  $f^*$ .

In the 1D case where  $\mathcal{S} = \{1, \dots, m\}$  can be an ordered set such that the nearest neighbor set for each interior site  $i$  is  $\mathcal{N}_i = \{i - 1, i + 1\}$ , (2.22) can be written as

$$E(f) = \sum_{i=1}^m (f_i - d_i)^2 + 2\lambda \sum_{i=2}^m g(f_i - f_{i-1}) \quad (2.24)$$

(the terms involving the two boundary points are omitted.) When  $f = \{f_1, \dots, f_m\}$  is the sample of a continuous function  $f = f(x)$ ,  $a \leq x \leq b$ , the above is an approximation to the regularization energy

$$E(f) = \sum_{i=1}^m (f_i - d_i)^2 + 2\lambda \int_a^b g(f'(x)) dx \quad (2.25)$$

(cf. Section 1.5.3.) When  $g(\eta)$  is as in (2.15), the above is the energy for the standard quadratic regularization (Tikhonov and Arsenin 1977; Poggio et al. 1985). When it is defined as (2.17), the energy is called *weak string* and its two-dimensional counterpart is called *weak membrane* (Blake and Zisserman 1987). Regularization of ill-posed problem will be discussed further in surface reconstruction.

In the 2D case where  $\mathcal{S} = \{(i, j) \mid 1 \leq i, j \leq n\}$  is a lattice, each site  $(i, j)$  (except at the boundaries) have four nearest neighbors  $\mathcal{N}_{i,j} = \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$ . The corresponding posterior energy is

$$E(f) = \sum_{i,j} (d_{i,j} - f_{i,j})^2 + \lambda \sum_{i,j} \sum_{(i',j') \in \mathcal{N}_{i,j}} g(f_{i,j} - f_{i',j'}) \quad (2.26)$$

This is a direct extension to (2.24).

### Energy Minimization

The MAP solution is defined as  $f^* = \arg \min_f E(f)$ . The simplest way to find the  $f^*$  is to perform gradient descent. Start with an initial configuration; iterate with

$$f^{(t+1)} \leftarrow f^{(t)} - \mu \nabla E(f^{(t)}) \quad (2.27)$$

where  $\mu > 0$  is a step size and  $\nabla E(f)$  is the gradient of the energy function, until the algorithm converges to a point  $f$  for which  $\nabla E(f) = 0$ . Consider  $E(f) = \sum_{i=1}^m (f_i - d_i)^2 + \lambda \sum_i \sum_{i' \in \mathcal{N}_i} g(f_i - f_{i'})$ . The gradient is composed of the following components

$$\begin{aligned} \frac{\partial E(f)}{\partial f_i} &= 2(f_i - d_i) + \lambda \sum_{i' \in \mathcal{N}_i} g'(f_i - f_{i'}) \\ &= 2(f_i - d_i) + 2\lambda \sum_{i' \in \mathcal{N}_i} (f_i - f_{i'}) h(f_i - f_{i'}) \end{aligned} \quad (2.28)$$

where  $g'(\eta) = 2\eta h(\eta)$  as assumed in (2.14). Fig.2.2 shows restored curves at three stages of an iterative minimization process. In the 2D case, it is

$$\frac{\partial E(f)}{\partial f_{i,j}} = 2(f_{i,j} - d_{i,j}) + 2\lambda \sum_{(i',j') \in \mathcal{N}_{i,j}} (f_{i,j} - f_{i',j'}) h(f_{i,j} - f_{i',j'}) \quad (2.29)$$

Alternatively, one may directly solve the system of equations

$$\nabla E(f^*) = \mathbf{0} \quad (2.30)$$

which is composed of  $m$  simultaneous equations  $\frac{\partial E(f)}{\partial f_i} = 0$  ( $i = 1, \dots, m$ ).

When  $E(f)$  is convex w.r.t.  $f$ , there is a unique global energy minimum  $f^*$  and the convergence to the global is guaranteed by the gradient descent

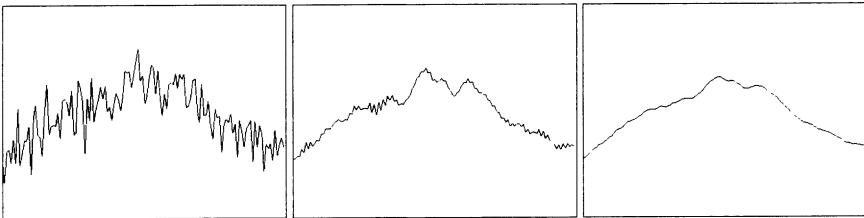


Figure 2.2: Minimizing  $E(f)$ . From left to right: the initial  $f$  set to the observed data  $d$ ,  $f$  after 5 iterations, and the final result  $f^*$ .

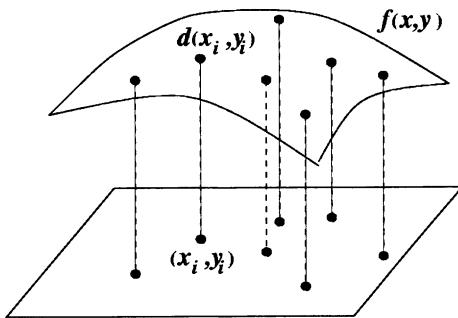


Figure 2.3: Surface reconstruction from sparse data.

algorithm. When it is not, as is with the truncated quadratic potential function (2.17), there are local minima in the energy landscape and the gradient descent algorithm only gives a local minimum. Some global optimization techniques may be used to overcome this difficulty, such as simulated annealing (Geman and Geman 1984) and graduated non-convexity (Blake and Zisserman 1987) (see Chapter 9).

### 2.2.4 Surface Reconstruction

Surface reconstruction also recovers surface values but from sparse data. It consists of two main components: restoration and interpolation. The situation of sparse data is typical with stereopsis (Marr and Poggio 1979; Grimson 1981; Mayhew and Frisby 1981). There, a pair of images taken from two slightly different viewpoints are compared and matched to give the corresponding points. The depth values at the matched points are computed using triangulation. Since the matched points are sparsely distributed, so are the computed depth values. The sparseness brings about more uncertainties which have to be resolved by interpolation.

Let  $d = \{d_i \mid i \in \mathcal{A}\}$  be a set of available depth values where  $d_i = d(x_i, y_i)$  is the depth value at location  $(x_i, y_i)$  and  $\mathcal{A} \subset \mathcal{S}$  is the set of subscripts for

the available data. Assume the observation model

$$d(x_i, y_i) = f(x_i, y_i) + e_i \quad (2.31)$$

where  $e_i$  is identical independent Gaussian noise and  $f(x_i, y_i)$  is the true value of the underlying surface. Fig. 2.3 illustrates  $(x_i, y_i)$ ,  $d(x_i, y_i)$  and  $f(x, y)$ . Our aim is to recover the underlying surface  $\{f(x, y)\}$  from the sparse data  $d$ . When continuous solutions are considered, it may be reasonable to look at the problem from the analytic viewpoint.

### Regularization Solution

From the analytic viewpoint, the reconstruction problem is mathematically *ill-posed* (Tikhonov and Arsenin 1977; Poggio et al. 1985) in at least that the solution is not unique (*cf.* Section 1.5.3). Regularization methods can be used to convert the problem into a well-posed one.

In reconstruction from sparse data, the regularizer, *i.e.* the smoothness term, have to involve the second or higher order derivatives. This is because regularizers with first order derivatives can be creased, *e.g.* in the absence of data at lattice points, without any increase in the associated energy. Regularizers with second or higher order derivatives are sensitive to creases. The smoothness term with the second order derivative is

$$U(f) = \int [f''(x)]^2 dx \quad (2.32)$$

Its 2D counterpart, the plate with quadratic variation, is

$$U(f) = \int \{[f_{xx}(x, y)]^2 + 2[f_{xy}(x, y)]^2 + [f_{yy}(x, y)]^2\} dx dy \quad (2.33)$$

For spatially discrete computation where the continuous domain is sampled at sites a regular lattice, the derivatives are approximated by the finite differences. A discrete form for (2.33) is

$$\begin{aligned} U(f) = & \sum_u \sum_v [f_{i+1,j} - 2f_{i,j} + f_{i-1,j}]^2 + \\ & 2 \sum_u \sum_v [f_{i,j} - f_{i-1,j} - f_{i,j-1} + f_{i-1,j-1}]^2 + \\ & \sum_u \sum_v [f_{i,j+1} - 2f_{i,j} + f_{i,j-1}]^2 \end{aligned} \quad (2.34)$$

For piecewise smoothness, the prior terms may be defined with a  $g$  function having the property (2.16). In the 1D case,

$$U(f) = \int g(f''(x)) dx \quad (2.35)$$

In the 2D case,

$$U(f) = \int [g(f_{xx}^2(x, y)) + g(f_{xy}^2(x, y)) + g(f_{yy}^2(x, y))] \, dx \, dy \quad (2.36)$$

They are called weak rod and weak plate, respectively. The latter has been used by (Geman and Reynolds 1992) to restore damaged film of movies.

Compared with the first order models like strings and membranes, the second order models are more capable of interpolating under sparse data; but they are less flexible in dealing with discontinuities and more expensive to compute. It is also reported that the mixture of 1st and 2nd order prior energies, which is akin to “spring under tension” (Terzopoulos 1983b; Terzopoulos 1986b), performs poorly (Blake and Zisserman 1987).

## 2.3 Edge Detection

Edges correspond to abrupt changes or discontinuities in certain image properties between neighboring areas. The image properties may be of non-texture or of texture. In this section, we are interested in non-texture edges due to changes in image intensity, such as jump edges and roof edges. Jump edges correspond to the discontinuities in the underlying surface  $\{f(x, y)\}$ , or to the maxima and minima<sup>3</sup> of its first (directional) derivative,  $f'$ . Roof edges correspond to the discontinuities in the first derivative,  $f'$ , or to the maxima and minima of its second derivative,  $f''$ . However, pixel sites are spatially quantized (in the  $x - y$  plane) the pixel values are subject to noise. In this case, discontinuities, maxima and minima are not well defined (their definitions becomes part of the solution to the edge detection problem). These cause problems for edge detection.

The first part in edge detection (Torre and Poggio 1986; Canny 1986) is the estimating the derivatives from noisy and spatially quantized data. The second is to detect the zeros and extrema of the estimated derivative function. Another issue is to link edges, which are detected based on local changes, to form boundaries which are coherent in a more global sense. The latter part also relates to perceptual organization (Lowe 1985). This section concerns only the first two parts by which pixel locations where sharp changes occur are marked.

Edge detection is closely related to those for image restoration and surface reconstruction involving discontinuities. There, we are mainly interested in removing noise and getting (piecewise) smooth surfaces; although discontinuities are also taken care of to avoid oversmoothing, they are not required to be marked explicitly. In this subsection, we modify the piecewise continuous restoration model to obtain explicit labeling of edges.

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<sup>3</sup>More exactly, positive and negative impulses

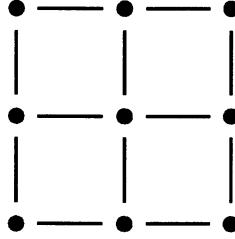


Figure 2.4: A lattice of sites (in dots) and its dual sites (in bars) in the 4-neighborhood system.

### 2.3.1 Edge Labeling Using Line Process

Rewrite the posterior energy (2.23) for the piecewise continuous restoration below

$$E(f) = \sum_{i \in \mathcal{S}} (f_i - d_i)^2 + \lambda \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} g(f_i - f_{i'}) \quad (2.37)$$

To impose the piecewise smoothness prior, the  $g$  function must have the property (2.16). The  $g(f_i - f_{i'})$  is modified for the purpose of explicitly marking edges.

In addition to the existing MRF for pixel values, we introduce another coupled MRF called *line process* (LP) (Geman and Geman 1984; Marroquin 1985) in which each label takes a value on  $\{0, 1\}$ , to signal the occurrences of edges. The two coupled MRFs are defined on two spatially interwoven sets as illustrated in Fig.2.4 (assuming the 4-neighborhood system). One set is the existing lattice for the intensity field and the other is the dual lattice for the introduced edge field. A possible edge may exist between each pair of the neighboring pixel sites.

We use the following notations. Denote the lattice of the  $m$  pixel sites by

$$\mathcal{S}^P = \{i \mid i = 1, \dots, m\} \quad (2.38)$$

(corresponding to the dots in Fig.2.4) and the dual lattice (corresponding to the bars) by

$$\mathcal{S}^E = \{(i, i') \mid i, i' \in \mathcal{S}^P; i \bowtie i'\} \quad (2.39)$$

where  $i \bowtie i'$  means  $i$  and  $i'$  are neighbors. Let  $f_i^P$ ,  $i \in \mathcal{S}^P$ , be an intensity label taking a value in a real interval  $\mathcal{L}^P$ . Let  $f_{i,i'}^E$ ,  $(i, i') \in \mathcal{S}^E$ , be an edge label, also called a line process variable (Geman and Geman 1984), taking a value in  $\mathcal{L}^E = \{0, 1\}$ , with 0 and 1 representing the absence and presence of an edge, respectively.

The interaction between the two coupled MRFs is determined by the joint probability  $P(f^P, f^E)$  or the prior energy  $U(f^P, f^E)$ . Consider the energy

(2.23) with the potential function  $g(f_i^P - f_{i'}^P) = \min\{(f_i^P - f_{i'}^P)^2, \alpha\}$  defined in (2.17). It was for piecewise continuous restoration without explicit labeling of edge. We modify it into a function of both intensity and edge variables:

$$g(f_i^P, f_{i'}^P, f_{i,i'}^E) = (f_i^P - f_{i'}^P)^2(1 - f_{i,i'}^E) + \alpha f_{i,i'}^E \quad (2.40)$$

where  $i$  and  $i'$  are neighbors. To minimize the above  $g(f_i^P, f_{i'}^P, f_{i,i'}^E)$  alone, the intensity labels and the edge labels are determined as follows: If  $[f_i^P - f_{i'}^P]^2 < \alpha$ , then it is cheaper to pay the price  $[f_i^P - f_{i'}^P]^2$  and set  $f_{i,i'}^E = 0$ ; otherwise it is more economical to set  $f_{i,i'}^E = 1$  to insert an edge at the cost of  $\alpha$ .

The prior energy with the  $g$  function in (2.40) is

$$U(f^P, f^E) = \sum_{i \in \mathcal{S}^P} \sum_{i' \in \mathcal{N}_i} g(f_i^P, f_{i'}^P, f_{i,i'}^E) \quad (2.41)$$

It can be expressed as

$$U(f^P, f^E) = \sum_{i \in \mathcal{S}^P} \sum_{i' \in \mathcal{N}_i} V(f_i^P, f_{i'}^P \mid f_{i,i'}^E) + \sum_{i \in \mathcal{S}^P} \sum_{i' \in \mathcal{N}_i} V(f_{i,i'}^E) \quad (2.42)$$

where  $V(f_i^P, f_{i'}^P \mid f_{i,i'}^E) = (f_i^P - f_{i'}^P)^2(1 - f_{i,i'}^E)$  and  $V(f_{i,i'}^E) = \alpha f_{i,i'}^E$ . In terms of probability, the above corresponds to

$$P(f^P, f^E) = P(f^P \mid f^E) P(f^E) \quad (2.43)$$

Adding the prior energy and the likelihood energy yields the posterior energy

$$U(f^P, f^E \mid d) = \sum_{i \in \mathcal{S}^P} (f_i^P - d_i)^2 + \sum_{i \in \mathcal{S}^P} \sum_{i' \in \mathcal{N}_i} \lambda g(f_i^P, f_{i'}^P, f_{i,i'}^E) \quad (2.44)$$

Minimization of the equation (2.44) has to be performed over all  $f^P$  and  $f^E$ . Since the latter field assumes discrete configurations, the minimization is a combination of real and combinatorial problems. This can be converted to a simpler real minimization with the following energy (Blake and Zisserman 1987)

$$E(f^P) = \sum_{i \in \mathcal{S}} (f_i^P - d_i)^2 + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} \lambda_i g(f_i^P - f_{i'}^P) \quad (2.45)$$

where  $g$  is the truncated quadratic potential function (2.40). The above is exactly the same as (2.22). Its minimization can be performed using algorithms working on real numbers, such as the GNC (Blake and Zisserman 1987). By minimizing it with respect to only  $f^P$ , we obtain the restored image  $(f^P)^*$  in which the restored pixel values near edges are properly preserved (not over-smoothed). The edge field  $f^E$  is then determined by thresholding  $(f^P)^*$ :

$$f_{i,i'}^E = \begin{cases} 1 & \text{if } (f_i^P - f_{i'}^P)^2 > \alpha \\ 0 & \text{otherwise} \end{cases} \quad (2.46)$$

$$\begin{array}{ccc}
 0 & 1 & 1 \\
 0 \ 1 \ 0 & 1 \ 0 \ 1 & 1 \ 1 \ 1 \\
 & & 1
 \end{array}
 \quad
 \begin{array}{cc}
 1 & 1 \\
 1 \ 1 & 1
 \end{array}$$

Figure 2.5: Forbidden edge patterns.

where  $i$  and  $i'$  are neighbors.

Do we need the edge field  $f^E$  explicitly? The current trend seems to be using a  $g$  function without the explicit edge labels, *i.e.* performing minimization after eliminating the line process as illustrated previously. This is reflected by *e.g.* (Koch et al. 1986; Blake and Zisserman 1987; Geiger and Girosi 1989; Shulman and Herve 1989; Lange 1990; Li 1990b; Rangarajan and Chellappa 1990; Geman and Reynolds 1992; Geman et al. 1992; Bouman and Sauer 1993; Stevenson et al. 1994). It is less difficult to perform the real minimization than the combinatorial minimization. However, an explicit edge field may be useful for modeling contextual interactions of complex edge configurations, such as forbidden edge patterns which are naturally described by a discrete field.

### 2.3.2 Forbidden Edge Patterns

There are two essential ways to impose constraints: exact and inexact. In the inexact way, any violation of the constraints incurs a *finite* cost. The use of inexact constraints are seen in works on edge and boundary detection using relaxation labeling, *e.g.* (Zucker 1976; Hanson and Riseman 1978; Faugeras and Berthod 1981). This is also what we have done so far, *e.g.* with the  $g$  function. On the other hand, when the constraints are imposed exactly, any violation is forbidden. For example, if edges are required to form a boundary of single pixel width, the adjacent parallel edges are not allowed. A family of four forbidden edge configurations are given in Fig.2.5; they correspond to, respectively, isolated edge (a terminated boundary), sharp turn, quadruple junction, and “small structure”.

Let  $U^E(f^E)$  be the total number of the occurrences of the forbidden configurations in  $f^E$ . When there are no forbidden edge configurations,  $U^E(f^E) = 0$  must hold exactly. Under this constraint, the MAP edge detection problem may be posed as follows (Geman et al. 1990; Heitz and Bouthemy 1993)

$$(f^P, f^E)^* = \arg \min_{(f^P, f^E): U^E(f^E) = 0} E(f^P, f^E) \quad (2.47)$$

where  $E(f^P, f^E)$  is the posterior energy due to the inexact constraints. The minimization is performed over all configurations for which  $U^E(f^E) = 0$ . This is a problem of constrained minimization (*cf.* Section 8.3).

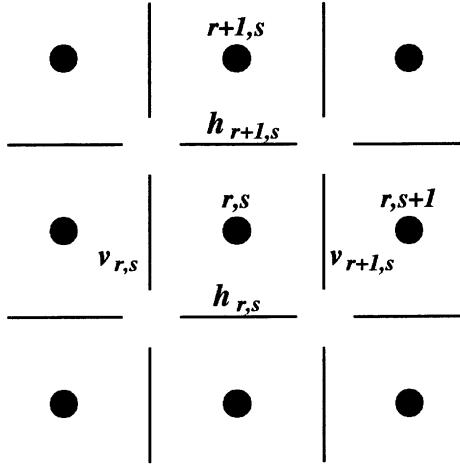


Figure 2.6: Horizontal and vertical edges.

The constrained minimization may be computed by using the penalty method. Any forbidden local configuration is finally penalized by using an infinitely large clique potential; the exact constraints are imposed by gradually increasing the penalties against the violation of inexact constraints. An example is given below which rules out adjacent, horizontally parallel edges. Divide  $f^E = \{f_{i,i'}^E\}$  into a horizontal edge field  $\{h_{i,j}\}$  and a vertical edge field  $\{v_{i,j}\}$  ( $(i, j) \in \mathcal{S}^E$ ), respectively, as in Fig. 2.6. Each  $h_{i,j}$  and  $v_{i,j}$  variable takes a value in  $\{0, 1\}$  as  $f_{i,i'}^E$ . We may define the following energy to penalize adjacent, horizontally parallel edges (Koch et al. 1986)

$$U_h(h) = \lambda_h \sum_{i,j} h_{i,j} h_{i+1,j} \quad (2.48)$$

where  $\lambda_h > 0$  is a constant. The above is non-zero when both  $h_{i,j}$  and  $h_{i+1,j}$  take the value 1, indicating the occurrence of a pair of horizontally parallel edges. This incurs a penalty  $\lambda_h$ . To prohibit the formation of such edge configurations, we can gradually increase  $\lambda_h$  to a sufficiently large value ( $\rightarrow +\infty$ ) so that the final minimal solution does not contain any horizontally parallel edges as a result of energy minimization. In the same way,  $U_v(v) = \lambda_v \sum_{i,j} v_{i,j} v_{i,j+1}$  can be added to rule out adjacent, vertically parallel edges.

## 2.4 Texture Synthesis and Analysis

Three important issues in MRF texture analysis are texture modeling, texture classification and texture segmentation. In MRF based texture modeling,

a texture is assumed to be an MRF and to model a texture is to specify the corresponding conditional probabilities or Gibbs clique potential parameters. Texture classification is an application of pattern recognition techniques. It concerns the extraction of texture features and the design of a decision rule or classifier for classification. In MRF modeling, texture features correspond to the MRF texture parameters and feature extraction is equivalent to parameter estimation. In the supervised case, the estimation is performed using training data. This establishes reference parameters. Textures are classified by comparing texture feature vector extracted from the image and the reference feature vectors, which is basically a pattern recognition problem. Texture segmentation is to partition a textured image into regions such that each region contains a single texture and no two adjacent regions have the same texture. To do this, one has to deal with issues in texture modeling, parameter estimation and labeling. This chapter discusses problems in MRF texture modeling and segmentation.

### 2.4.1 MRF Texture Modeling

An MRF model of texture can be specified by the joint probability  $P(f)$ . The probability determines how likely a texture pattern  $f$  is to occur. Several MRF models, such as auto-binomial (Cross and Jain 1983), auto-normal (or GMRF) (Chellappa 1985; Cohen and Cooper 1987), and multi-level logistic (MLL) model (Derin et al. 1984; Derin and Elliott 1987), have been used for modeling textures. A particular MRF model tends to favor the corresponding class of textures, by associating them with larger probability than others. Here, “a particular MRF model” is a particular probability function  $P(f)$  which is specified by its form and the parameters in it.

In the MLL model, for example, the probability of a texture pattern  $f$  is defined based on the MLL clique potential function (1.52). In the second order (8-neighbor) neighborhood system, there are ten types of cliques as illustrated in Fig.1.2, with the parameters associated with them shown in Fig.1.4. When only the pair-wise clique potentials are non-zero, equation (1.52) reduces to (1.54), rewritten below

$$V_2(f_i, f_{i'}) = \begin{cases} \beta_c & \text{if sites on } \{i, i'\} = c \in \mathcal{C}_2 \text{ have the same label} \\ -\beta_c & \text{otherwise} \end{cases} \quad (2.49)$$

where  $f_i \in \mathcal{L} = \{1, \dots, M\}$  and  $\mathcal{C}_2$  is a set of pair-site cliques, and  $\beta_c$  is a parameter associated with the type- $c$  pair-site cliques. After MLL is chosen, to specify an MRF model is to specify the  $\beta_c$ ’s. When the model is anisotropic, *i.e.* with all  $\beta_c = \beta$ , the model tends to generate texture-like patterns; otherwise it generates blob-like regions. The probability  $P(f)$  can be calculated from  $V_2(f_i, f_{i'})$ ’s using the corresponding Gibbs distribution.

A textured pattern corresponding to a realization of a Gibbs distribution  $P(f) = \frac{1}{Z}e^{-U(f)}$  can be generated by sampling the distribution. Two often

**Sampling Algorithm 1**

Begin Algorithm

- (1) randomly initialize  $f$  to a point in  $\mathcal{L}^S$ ;
- (2) for  $i \in \mathcal{S}$  do
  - (2.1) let  $f'_{i'} = f_{i'}$  for all  $i' \neq i$ ;
  - choose  $f_i \in \mathcal{L}$  at random;
  - (2.2) let  $p = \min\{1, P(f')/P(f)\}$   
where  $P$  is the given Gibbs distribution;
- (3) replace  $f$  by  $f'$  with probability  $p$ ;

End Algorithm

Figure 2.7: Generating a texture using Metropolis sampler.

**Sampling Algorithm 2**

Begin Algorithm

- (1) randomly initialize  $f$  to a point in  $\mathcal{L}^S$ ;
- (2) for  $i \in \mathcal{S}$  do
  - (2.1) compute  $p_l = P(f_i = l \mid f_{\mathcal{N}_i})$  for all  $l \in \mathcal{L}$ ;  
where  $f_{\mathcal{N}_i}$  are the pixel values at neighboring sites;
  - (2.2) set  $f_i$  to  $l$  with probability  $p_l$ ;
- (3) repeat (2) for  $N$  times;

End Algorithm

Figure 2.8: Generating a texture using Gibbs sampler.

used sampling algorithms are the Metropolis sampler (Metropolis et al. ) and the Gibbs sampler (Geman and Geman 1984) (*cf.* Section 6.1.6). Figs.2.7 and 2.8 list two sampling algorithms (Chen 1988). Algorithm 1 is based on the Metropolis sampler (Hammersley and Handscomb 1964) whereas Algorithm 2 is based on the Gibbs sampler of (Geman and Geman 1984). A sampling algorithm generates a texture  $f$  with probability  $P(f)$ .

The differences between the two sampling algorithm is in step (2). Algorithm 1 needs only to evaluate one exponential function because the update is based on the ratio  $P(f')/P(f)$  in step (2.2). Algorithm 2 needs to compute  $M$  exponential functions and when the exponents are very large, this com-

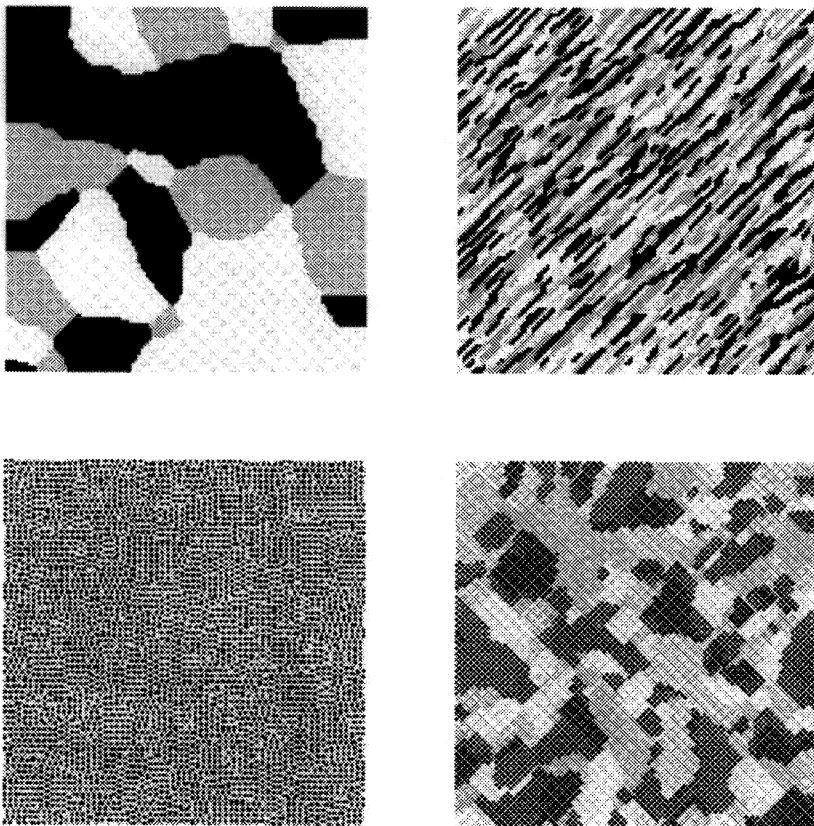


Figure 2.9: Textured images ( $128 \times 128$ ) generated by using MLL models. Upper-left: Number of pixel levels  $M = 3$  and parameters  $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ . Upper-right:  $M = 4$ ,  $\beta_1 = \beta_2 = \beta_3 = 1$ ,  $\beta_4 = -1$ . Lower-left:  $M = 4$ ,  $\beta_1 = \beta_2 = \beta_3 = \beta_4 = -1$ . Lower-right:  $M = 4$ ,  $\beta_1 = \beta_2 = -1$ ,  $\beta_3 = \beta_4 = 2$ .

putation can be inaccurate. The following conclusions on the two algorithms are made by (Chen 1988): (1)  $N = 50$  iterations are enough for both algorithms, (2) Algorithm 2 tends to generate a realization with a lower energy than Algorithm 1 if  $N$  is fixed and (3) Algorithm 1 is faster than Algorithm 2.

Fig.2.9 shows four texture images generated using the MLL model (1.54) and Algorithm 1. It can be seen that when all the  $\beta_c$  parameters are the same such that the model is isotropic, the formed regions are blob-like; when it is anisotropic, the generated pattern looks textured. So, blob regions are

modeled as a special type of textures whose MRF parameters tend to be isotropic.

The above method for modeling a single texture provides the basis for modeling images composed of multiple textured regions. In the so-called hierarchical model (Derin and Cole 1986; Derin and Elliott 1987; Won and Derin 1992), blob-like regions are modeled by a high level MRF which is an isotropic MLL; these regions are filled in by patterns generated according to MRFs at the lower level. A filling-in may be either additive noise or a type of texture, characterized by a Gibbs distribution. Let  $f$  be a realization of the region process with probability  $P(f)$ . Let  $d$  be the textured image data with probability  $P(d | f)$ . A texture label  $f_i$  indicates which of the  $M$  possible textures in the texture label set  $\mathcal{L}$  that pixel  $i$  belongs to.

The MLL model is often used for the higher level region process. Given  $f$ , the filling-in's are generated using the following observation model

$$d = \varphi(f) \quad (2.50)$$

where  $\varphi$  is a random function that implements the mechanism of a noise or texture generator. To generate noisy regions, the observation model is simply  $d_i = \varphi(f_i) = \ell(f_i) + e_i$ , where  $\ell(f_i)$  denotes the true grey level for region type  $f_i$ , as in (2.2). When the noise is also Gaussian, it follows a special Gibbs distribution, that in the zero-th order neighborhood system (where the radius of the neighborhood is zero). When the noise is an identical, independent and zero-mean distribution, the likelihood potential function is  $V(d_i | f_i) = [\ell(f_i) - d_i]^2 / 2\sigma^2$  where  $\sigma^2$  is the variance.

For textured regions,  $\varphi(f)$  implements anisotropic MRFs. Because of contextual dependence of textures, the value for  $d_i$  depends on the neighboring values  $d_{\mathcal{N}_i}$ . Given  $f_i$  and  $d_{\mathcal{N}_i}$ , its conditional probability is  $P(d_i | d_{\mathcal{N}_i}, f)$  where all the sites in  $\{i, \mathcal{N}_i\}$  are assumed to belong to the same texture type,  $f_i$ , and hence some treatment may be needed at and near boundaries. Note that the texture configuration is  $d$  rather than  $f$ ;  $f$  is the region configuration. The configuration  $d$  for a particular texture can be generated using an appropriate MRF model (such as MLL used before for generating a single texture). That is to say,  $d$  is generated according to  $P(d | f)$ ; see further discussions in Section 2.4.2. Fig.2.10 shows an image of textured regions generated by using the hierarchical Gibbs model. There, the region process is the same as that for the upper-left of Fig.2.9 and the texture filling-in's correspond to the other three realizations.

### 2.4.2 Texture Segmentation

Texture segmentation is to segment an image into regions according to the textures of the regions. In the supervised texture segmentation, it is assumed that all the parameters for the textures, and for the noise if it is present, are specified (Hansen and Elliott 1982; Elliott et al. 1984; Geman and Geman

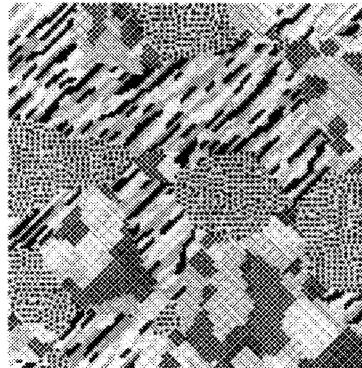


Figure 2.10: Multiple texture regions generated using the hierarchical Gibbs model of textures.

1984; Besag 1986; Derin and Cole 1986; Cohen and Cooper 1987; Derin and Elliott 1987); and the segmentation is to partition the image in terms of the textures whose distribution functions have been completely specified.

Unsupervised segmentation not only does the partition but also needs to estimate the involved parameters (Lakshmanan and Derin 1989; Manjunath and Chellappa 1991; Cohen and Fan 1992; Hu and Fahmy 1987; Won and Derin 1992). Obviously, this is more practical in applications and is also more challenging. There is a chicken-and-egg problem. The estimation should be performed by using realizations of a single class of MRFs, *i.e.* a single type of textures (*cf.* Chapter 6), albeit noisy or noise-free. This requires that the segmentation be done. However, the segmentation depends on the parameters of the underlying textures. A strategy for solving this is to use an iterative algorithm alternating between segmentation and estimation. While the more advanced topic of unsupervised segmentation will be discussed in Section 6.2.2, we focus here on the supervised MRF segmentation with all parameters known.

Texture segmentation, as other labeling problems, is usually performed in an optimization sense, such as MAP. A main step is to formulate the posterior energy. In MAP texture segmentation with the hierarchical texture model, the higher level MRF determines the prior probability  $P(f)$ , where the segmentation  $f$  partitions  $\mathcal{S}$  into regions each of which is assigned a texture type from the label set  $\mathcal{L}$ ; the lower level field contributes to the likelihood function  $P(d | f)$  where  $d$  is the image data composed of multiple textures.

Let the label set be  $\mathcal{L} \in \{1, \dots, M\}$  and  $f$  represent a segmentation in which  $f_i \in \mathcal{L}$  is the indicator of the texture type for pixel  $i$ . Denote the set

of all sites labeled  $I$  by

$$\mathcal{S}^{(I)}(f) = \{i \in \mathcal{S} \mid I = f_i\} \quad (2.51)$$

Then

$$\mathcal{S} = \bigcup_{I \in \mathcal{L}} \mathcal{S}^{(I)}(f) \quad (2.52)$$

and

$$\mathcal{S}^{(I)}(f) \cap \mathcal{S}^{(J)}(f) = \emptyset, \quad I \neq J \quad (2.53)$$

The likelihood energy function can be expressed as

$$U(d \mid f) = \sum_{c \in \mathcal{C}} V_c(d \mid f) = \sum_{I \in \mathcal{L}} \sum_{\forall c \subset \mathcal{S}^{(I)}} V_c^{(I)}(d \mid f) \quad (2.54)$$

where  $V_c^{(I)}(d \mid f)$  is the potential function for the data  $d$  on  $c$  labeled as type  $I$ .

Suppose that type- $I$  texture is modeled as an MLL with parameters

$$\theta^{(I)} = \{\alpha^{(I)}, \beta^{(I)}, \dots\} \quad (2.55)$$

Then according to (1.53), the single-site clique potentials are

$$V_c^{(I)}(d \mid f) = \alpha^{(I)} \quad (2.56)$$

where  $I = f_i$  and according to (1.52), the multi-site clique potentials are

$$V_c^{(I)}(d \mid f) = \begin{cases} \beta_c^{(I)} & \text{if all } d_i, i \in c, \text{ are the same} \\ -\beta_c^{(I)} & \text{otherwise} \end{cases} \quad (2.57)$$

The above is for the cliques in the interior of  $\mathcal{S}^{(I)}$ . At and near region boundaries, a clique  $c$  may ride across two or more  $\mathcal{S}^{(I)}$ 's. In this case, the following rule may be used to determine the type of texture for generating the data: If  $c$  sits mostly in a particular  $\mathcal{S}^{(I)}$ , then choose model parameters  $\theta^{(I)}$ ; if it sits equally in all the involved  $\mathcal{S}^{(I)}$ 's, choose an  $I$  at random from the involved labels. When the grey levels for type  $I$  texture image data are known as  $\mathcal{D}_I = \{\ell_1^{(I)}, \dots, \ell_{M_I}^{(I)}\}$  where  $M_I$  is the number of grey levels for type  $I$  texture, more constraints are imposed on the texture segmentation and better results can be expected. When the grey levels are also subject to noise, then the constraints become inexact.

After  $U(f)$  is also defined for the region process, the posterior energy can be obtained as

$$U(f \mid d) = U(f) + U(d \mid f) \quad (2.58)$$

Minimizing (2.58) with respect to  $f$  is more complicated than the minimization for the restoration and reconstruction formulated in the previous sections because neither term on the RHS can be decomposed into independent sub-terms. In (Derin and Elliott 1987), some assumptions of independence are made to simplify the formulation and a recursive dynamic programming algorithm (see Section 8.2.4) is used to find a sub-optimal solution.

## 2.5 Optical Flow

Optical flow is the velocity field in the image plane caused by the motion of the observer and objects in the scene. It contains important information about cues for region and boundary segmentation, shape recovery, and so on. This section describes methods for computing optical flow and motion from a sequence of time-varying images.

Two major paradigms exist for determining visual motion: feature-based (*e.g.* edge-based) (Nagel 1983; Paquin and Dubios 1983; Boutheny 1989) and gradient-based (Horn and Schunck 1981). In the former paradigm, features are extracted from the sequence of images, matched between two neighboring frames and tracked over time. This gives a sparse flow of which the information is available at the sparse set of the extracted image features. In the gradient-based paradigm, the flow is recovered based on local spatial-temporal changes in image intensity. This gives a dense flow field whose values are available throughout the image plane. The focus of this section will be on finding flow field using gradient-based methods.

### 2.5.1 Variational Approach

Gradient-based methods advocated by (Horn and Schunck 1981) utilize the constraints of intensity constancy, spatial and temporal coherence. Let  $d(x, y, t)$  denote the intensity at  $(x, y)$  of the image plane at time  $t$ . Consider a brightness pattern that is displaced in the direction  $(\delta x, \delta y)$  in time  $\delta t$ . A constraint is that the intensity of the brightness remain constant, *i.e.*  $d(x, y, t) = d(x + \delta x, y + \delta y, t + \delta t)$ . This gives the following equation of *intensity constancy*

$$\frac{\partial d}{\partial x} \frac{dx}{dt} + \frac{\partial d}{\partial y} \frac{dy}{dt} + \frac{\partial d}{\partial t} = 0 \quad (2.59)$$

This equation relates the change in image intensity at a point to the motion of the intensity pattern (the distribution of pixel intensities). Let

$$f(x, y) = [u(x, y), v(x, y)]^T = (dx/dt, dy/dt)^T \quad (2.60)$$

designate the optical flow vector at  $(x, y)$ . Then  $f = \{f(x, y)\}$  is the optical flow field we want to compute. The intensity constancy constraint is a single equation with the two unknowns  $u$  and  $v$

$$ud_x + vd_y + d_t = 0 \quad (2.61)$$

This corresponds to a straight line in the velocity space. The equation shows that only the velocity components parallel to the spatial image gradient can be recovered through the local computation. This is termed the “aperture problem” by (Marr 1982). When noise is taken into consideration, the flow may be computed by minimizing

$$V(d(x, y) | f(x, y)) = (ud_x + vd_y + d_t)^2 \quad (2.62)$$

However, optical flow  $f$  thus computed under the intensity constancy constraint alone is not unique. Additional constraints must be imposed on the flow.

An important constraint is the smoothness, *i.e.* flow at nearby places image will be similar unless discontinuities exist there. The smoothness is imposed in a similar way as in the restoration and reconstruction discussed earlier. The movement of intensity points should present some coherence and so the flow in a neighborhood should change smoothly unless there are discontinuities. One way to impose this constraint is to minimize the squared magnitude of the gradient of the optical flow (Horn and Schunck 1981)

$$V(f(x, y)) = \|\nabla u\|^2 + \|\nabla v\|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \quad (2.63)$$

See also (Snyder 1991) for a study of smoothness constraints for optical flow.

In the MAP-MRF framework, (2.62) corresponds to the likelihood potential due to independent Gaussian noise in the image intensities and (2.63) corresponds to the prior potential due to the prior distribution of the MRF,  $f$ . The posterior potential is obtained by combining (2.62) and (2.63) into a weighted sum

$$V(f(x, y) | d(x, y)) = V(f(x, y) | d(x, y)) + \lambda V(f(x, y)) \quad (2.64)$$

where  $\lambda$  is a weighting factor. Integrating it over all  $(x, y)$  gives the posterior energy

$$E(f) = \int \int [V(f(x, y) | d(x, y)) + \lambda V(f(x, y))] dx dy \quad (2.65)$$

This formulation (Horn and Schunck 1981) not only provides a method for computing optical flow, but also is a pioneer work on the variational approach in low level vision which is later developed into the regularization framework (Poggio et al. 1985; Bertero et al. 1988). Various MAP-MRF formulations for flow estimation can be found in *e.g.* (Murray and Buxton 1987; Konrad and Dubois 1988a; Black and Anandan 1990; Konrad and Dubois 1992; Heitz and Bouthemy 1993).

In discrete computation of the variational solution, it is important that the spatial and temporal intensity partial derivatives of the input image sequence  $d = \{d_{i,j,k}\}$ , which appear in (2.62), are consistent. They may be estimated by averaging the four first differences taken over the adjacent measurements (Horn and Schunck 1981)

$$\begin{aligned} d_x &\approx \frac{1}{4} [(d_{i,j+1,k} - d_{i,j,k}) + (d_{i+1,j+1,k} - d_{i+1,j,k}) + \\ &\quad (d_{i,j+1,k+1} - d_{i,j,k+1}) + (d_{i+1,j+1,k+1} - d_{i+1,j,k+1})] \\ d_y &\approx \frac{1}{4} [(d_{i+1,j,k} - d_{i,j,k}) + (d_{i+1,j+1,k} - d_{i,j+1,k}) + \\ &\quad (d_{i+1,j,k+1} - d_{i,j,k+1}) + (d_{i+1,j+1,k+1} - d_{i,j+1,k+1})] \\ d_t &\approx \frac{1}{4} [(d_{i,j,k+1} - d_{i,j,k}) + (d_{i+1,j,k+1} - d_{i+1,j,k}) + \\ &\quad (d_{i,j+1,k+1} - d_{i,j+1,k}) + (d_{i+1,j+1,k+1} - d_{i+1,j+1,k})] \end{aligned} \quad (2.66)$$

Another formula calculates them using a three-point approximation (Battiti et al. 1991). The spatial derivatives of the flow  $f_{i,j,k}$  in (2.68) are usually approximated by the simple differences

$$\begin{aligned} u_x &= u_{i,j+1,k} - u_{i,j,k} \\ u_y &= u_{i+1,j,k} - u_{i,j,k} \\ v_x &= v_{i,j+1,k} - v_{i,j,k} \\ v_y &= v_{i+1,j,k} - v_{i,j,k} \end{aligned} \quad (2.67)$$

### 2.5.2 Flow Discontinuities

The quadratic smoothness in (2.63) is unable to deal with discontinuities, *i.e.* boundaries between regions moving differently. For the discontinuity-preserving computation of optical flow, the smoothness is more appropriately imposed as minimizing the following

$$V(f(x, y)) = g(\left[\frac{du}{dx}\right]^2 + \left[\frac{dv}{dx}\right]^2) + g(\left[\frac{du}{dy}\right]^2 + \left[\frac{dv}{dy}\right]^2) \quad (2.68)$$

where the  $g$  function satisfies a necessary condition (2.16). When  $g$  is a truncated quadratic, (2.68) gives the line process model (Koch et al. 1986; Murray and Buxton 1987). A more careful treatment, “oriented smoothness” constraint (Nagel and Enkelmann 1986), may be used to avoid smoothing across intensity, rather than flow, discontinuities. In (Shulman and Herve 1989),  $g$  is chosen to be the Huber robust penalty function (Huber 1981). In (Black and Anandan 1993), such a robust function  $g$  is also applied to the data term, giving  $V(d(x, y) \mid f(x, y)) = g(ud_x + vd_y + d_t)$ .

Recently, Heitz and Bouthemy (1993) propose an MRF interaction model which combines constraints from both gradient based and edge based paradigms. It is assumed that motion discontinuities appear with a rather low probability when there is no intensity edge at the same location (Gamble and Poggio 1987). This is implemented as an energy term which imposes a constraint on the interaction between motion discontinuities and intensity edges. The energy is designed to prevent motion discontinuities from appearing at points where there are no intensity edges.

The computation of optical flow discussed so far is based on the assumption of constant intensity. The assumption is valid only for very small displacement or short-range motion. For long-range motion there is a significant time gap between different frames. In this case, the analysis is usually feature based. This now requires to resolve the correspondence between features in successive frames.

## 2.6 Bayesian Deformable Models

Correct location and extraction of objects of interest is an important step in many applications such as industrial inspection, medical image analysis, and object recognition. Deformable models, such as snake (Kass et al. 1988), G-Snake (Lai and Chin 1995), Active Shape Model (ASM) (Cootes et al. 1995) and deformable templates (Jain et al. 1996), are able to fit objects more closely and have proven to be more effective than those based on rigid models in detecting shapes subject to local deformation. Such models can be classified into two classes (Jain et al. 1996; Jain et al. 1998): (i) free-form, and (ii) parametric. In the free-form class, such as the snakes (Terzopolous et al. 1987; Terzopolous et al. 1988), there is no global structure of the template; the template is constrained by only local continuity and smoothness constraints. Such models can be deformed to any arbitrary shapes deviating greatly from the shape of interest.

In the parametric class, a global shape model is designed and encoded by a small number of parameters (Yuille et al. 1989; Grenander and Y. Chow 1991; Staib and Duncan 1992; Lai and Chin 1995; Cootes et al. 1995; Mumford 1996; Jain et al. 1996; Blake et al. 1998). *Prior* knowledge about the structural properties is incorporated into the shape models, assuming that prior information of the geometrical shape or a set of training samples is available. Compared with the free-form class, the parametric class is more robust against to irrelevant structures, occlusion and bad initialization. A quite successful and versatile scheme in this class is that using statistical shape models in the Bayesian framework (Grenander 1976; Staib and Duncan 1992; Mumford 1996; Lai and Chin 1995; Jain et al. 1996; Blake et al. 1998). In these models, both the prior knowledge and observation statistics are used to define the optimal, Bayesian estimate.

In this section, a deformable contour model, called EigenSnake, is presented in the Bayesian framework (Section 2.6.1). Three strategies are combined into the EigenSnake to make the object seeking more robust and accurate: (i) The prior knowledge of a specific object shape is derived from a training set of shape samples, and expressed as a prior distribution for defining a Bayesian estimate as the optimal solution for the object contour. The prior distribution can explain and represent the global and local variation in the training set. (ii) Constraints on object shapes are imposed in a “soft” manner, following the derived prior distribution. The shape encoding can adjust itself dynamically based on the up-to-date knowledge learned during the solution finding process. This adjustment allows the solution to fit more closely to the object shape. (iii) A shape space is constructed based on a set of eigenvectors derived by principal component analysis (PCA) performed on the training set. It is used to restrict and stabilize the search for the Bayesian optimum.

The affine invariant *shape matrix* method in the G-Snake (Lai and Chin 1995) is used for encoding the global shape information. However, we noticed

two drawbacks of the G-Snake model: (i) It encodes the shape information in a “hard” manner in that the shape matrix is fixed. This actually assumes that the considered object shape undergoes only *rigid* (such as affine) transformation. So, only a small amount of local deformation is tolerated, and an incorrect solution results when the shape deformation is large, which is often a problem in many applications such as face detection. (ii) A more fundamental problem is what we call “incorrect formulation” of the G-Snake: A better solution, which is a contour fitting more closely to the object in the image, may have a higher energy value than an inferior solution. The problem is due to the lack of constraints on the optimal solution. The EigenSnake amends these problems in two ways: (i) It modifies the shape matrix dynamically based on the up-to-date information gathered from the solution finding process. (ii) It encodes prior constraints specific about objects of interest in its energy function. More accurate solutions are obtained by using these methods.

The shape space constraint used in the ASM (Cootes et al. 1995; Lanitis et al. 1997) is applied to the solution finding process. The shape space, which is a subspace of the whole arbitrary contour space, explains and represents most of the non-rigid variance in the training set, based on the eigenvectors derived through PCA performed on the training contours. Every intermediate solutions (energy minima) is projected into the subspace and thus the solution is always restricted to be within the subspace. This proves to be effective in stabilizing the solution. However, in the ASM, no priori distribution accounting for global rigid motion is incorporated; therefore, the original ASM is unable to handle large scale and orientational shape-changes. This is amended in the EigenSnake.

Experiments are performed in the application domain of face extraction (Section 2.6.2). Four methods based on the G-Snake and the EigenSnake are compared to evaluate how various constraints and different prior knowledges have affected the results. The results demonstrate that the use of various prior constraints in the EigenSnake helps produce more stable and accurate solutions. The EigenSnake, as compared to the other three methods, effectively overcomes problems due to initialization, occlusion, noise and variations in scale and rotation.

### 2.6.1 Formulation of EigenSnake

The EigenSnake incorporates the prior knowledge about the distribution of training contours. The global and local variations in the training set are well represented by the resulting prior distribution. Moreover, a shape space is constructed to further constrain the solution.

### Joint Prior Distribution

Let  $\mathbb{E} = \{(x, y) \in \mathbb{R}^2\}$  be the image plane. An arbitrary contour can be represented by a vector of  $m$  points in  $\mathbb{E}$ ,  $f = [f_1^T, f_2^T, \dots, f_m^T]^T$  (a  $2 \times m$  matrix) where  $f_i \in \mathbb{E}$  is a point. Given a prototype contour  $\bar{f}$ , a shape matrix  $A = A(\bar{f})$  can be determined by  $A(\bar{f})\bar{f}^T = \mathbf{0}$ . The shape matrix is under affine transformation: If  $f^{aff}$  is an affine-transformed version of  $f$ , then  $A(\bar{f})f^{affT} = \mathbf{0}$  (Lai and Chin 1995).

For the convenience of the following description, we express any contour as a  $2m \times 1$  vector  $f = [f_1, f_2, \dots, f_m]^T \in \mathbb{R}^{2m}$ . Given a prototype contour  $\bar{f}$  and the corresponding shape matrix  $A(\bar{f})$ , the prior distribution of  $f$  is given as

$$p(f \mid \bar{f}) \propto \exp \{-E(f \mid \bar{f})\} \quad (2.69)$$

where  $E(f \mid \bar{f}) = \sum_{i=1}^m \frac{E(f_i \mid \bar{f})}{\sigma_i^2}$  is the internal energy (Lai and Chin 1995).

In the G-Snake,  $\bar{f}$  is chosen to be the mean of the training contours and is fixed. This actually assumes that the considered object shape (that of  $\bar{f}$ ) is subject to *rigid* (such as affine) transformation only. However, it ignores prior knowledge about the fact that local variations exist among the training contours. This is equivalent to the assumption that  $f$  is a distribution with zero variance, which is obviously invalid with the training set. Since  $\bar{f}$ , and hence  $A(\bar{f})$ , is fixed, it is difficult for the solution  $f$  to converge closely to the object contour in the image when the local shape deformation is large (this is often the case for many applications such as face detection). Because of this, a better solution, which is a contour fitting more closely to the object in the image, may have a higher energy value than an inferior solution, as will be illustrated by experimental results. This is referred to as the incorrectness of an energy model.

The EigenSnake amends this problem by allowing  $\bar{f}$  to deform according to a prior distribution derived from the training contour. In this case, the following joint prior is considered

$$P(f, \bar{f}) = p(f \mid \bar{f}) \cdot p(\bar{f}) \propto \exp \{-E(f, \bar{f})\} \quad (2.70)$$

Hence, the shape information is encoded in a “soft” manner, and  $\bar{f}$  and hence  $A(\bar{f})$  can be modified dynamically during the computation. The prior  $p(\bar{f})$  is derived from a set of training contours in the following.

### Learning Prior from Training Set

Let a set of  $L$  face contours  $\{f^1, f^2, \dots, f^L\}$  be given as the training set where each contour  $f^i$  is represented by a sequence of  $m$  evenly placed (done manually) points in  $\mathbb{E}$  (i.e.  $f_k^i \in \mathbb{E}$  and  $f^i \in \mathbb{R}^{2m}$ ). The training contours are aligned at the centers of their gravities, and then scaled and rotated by using an iterative alignment algorithm so as to remove the rigid variation in the

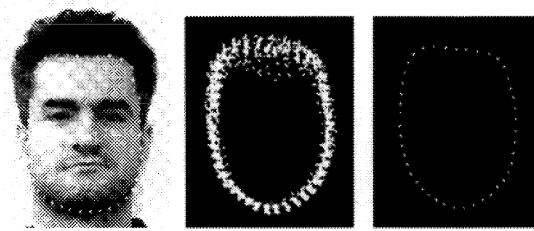


Figure 2.11: **Left:** A sample of face contour with  $m = 44$  points. **Middle:** Scatter of points from normalized and aligned set of face boundary shapes. **Right:** The mean (average) of the training face contours.

training samples as much as possible. Thus pre-processed contours constitute a random distribution. The mean contour  $\bar{f}_{mean}$  is computed by averaging the corresponding contour points. See Fig. 2.11.

Assume that the training samples obey a Gaussian distribution with the mean vector  $\bar{f}_{mean}$  and the covariance matrix  $C$ :

$$p(f) \propto \exp \left\{ -\frac{1}{2} (f - \bar{f}_{mean})^T C^{-1} (f - \bar{f}_{mean}) \right\} \quad (2.71)$$

The distribution represents the prior knowledge about local variation derived from the training contours. However, the covariance matrix  $C$  is often singular, thus  $C^{-1}$  non-existing, because the dimensionality of the samples is high and the number of available training contours is often small. The distribution has to be approximated by some means.

### Approximation of Prior Using PCA

We adopt an optimal method proposed by Moghaddam and Pentland (Moghaddam and Pentland 1997) to approximate the Gaussian density  $p(f)$  by applying a PCA to the training samples to capture the main modes of the variation in the directionally elongated (*i.e.* elliptical) Gaussian distribution.

Let  $\Psi$  be the eigenvector matrix of  $C$ , and  $\Lambda$  be the corresponding diagonal matrix of eigenvalues. We have  $\Lambda = \Psi^T C \Psi$ . The first  $M (\ll 2m)$  principal eigenvectors are selected:  $\Phi = [\phi_1, \phi_2, \dots, \phi_M]$ . They are used as the basis vectors to constitute a specific *shape space* (*i.e.* the  $M$ -dimensional space spanned by the  $M$  orthonormal vectors). The shape space efficiently explains most of the local variation within the training set. Any contour  $f \in \mathbb{R}^{2m}$  can be approximated by a linear combination of the  $M$  eigenvectors (Turk and Pentland 1991)

$$\hat{f} = \bar{f}_{mean} + \Phi \omega \quad (2.72)$$

where  $\omega = \Phi^T(f - \bar{f}_{mean})$  is the vector of weights.  $\hat{f}$  is the projection of  $f$  into the shape space, and  $\omega$  is the coordinate of the projection point in the shape space. It provides an optimum approximation for  $f$  in the sense of least squares error. It discards the variation of  $f$  which is inconsistent with those reflected by training contours, while retaining most of the variation that the shape space can explain. The projection  $\hat{f}$  has the following distribution (Moghaddam and Pentland 1997)

$$p(\hat{f}) = p(\omega) = \frac{\exp \left\{ -\frac{1}{2} \sum_{k=1}^M \frac{\omega_k^2}{e_k} \right\}}{(2\pi)^{M/2} \prod_{k=1}^M e_k^{1/2}} \quad (2.73)$$

where  $e_k$  is the  $k$ -th largest eigenvalue of  $C$ ,  $\omega_k = \omega_k(f) = \phi_k^T(f - \bar{f}_{mean})$ ,  $\varepsilon^2(f) = \|f - \bar{f}_{mean}\|^2 - \sum_{k=1}^M \omega_k^2$  is the residual error.

### Prior Distribution of Prototype Contour

During solution finding process, the prototype contour  $\bar{f}$  is modified dynamically according to the up to date MAP solution, and is confined in the shape space. The method is described below.

Let  $f' = \arg \min_{f \in \mathcal{N}(f^{(t-1)})} E(f, \bar{f} \mid d)$  be the MAP solution found at iteration  $t$  where  $E(f, \bar{f} \mid d)$  is the posterior energy (to be defined later) and  $\mathcal{N}(f^{(t-1)})$  is the neighborhood of the previously found solution  $f^{(t-1)}$ . The MAP estimate  $f'$  is aligned with  $\bar{f}_{mean}$  by minimizing  $\|u + s \cdot R_\theta(f') - \bar{f}_{mean}\|$  where  $u$  (translation),  $s$  (scaling) and  $\theta$  (rotation) are the rigid transformation parameters and  $R_\theta(f)$  is a function that rotates the contour  $f$  by  $\theta$ . The translation vector can be found as  $u = (x_u, y_u) = \frac{1}{m} \sum_{i=1}^m [\bar{f}_{mean, i} - f'_i]$ . The other two parameters can be calculated using a simple alignment algorithm. The aligned contour is given as  $f'' = u + s \cdot R_\theta(f')$ .

The aligned contour  $f''$  is then projected into the shape space using Equ.(2.72), resulting in the projected version  $\hat{f}''$ . The prototype contour  $\bar{f}$  is set to

$$\bar{f} = \hat{f}'' = \bar{f}_{mean} + \Phi \omega \quad (2.74)$$

where  $\omega = \Phi^T(f'' - \bar{f}_{mean})$  is the coordinates of  $f''$  in the shape space.

Therefore,  $\bar{f}$  obtained in such a way follows the same distribution as that of  $\hat{f}''$

$$p(\bar{f}) = p(\hat{f}'') = p(\omega) \quad (2.75)$$

as in Equ.(2.73). The corresponding prototype contour energy is

$$E(\bar{f}) = \frac{1}{2} \sum_{i=1}^M \frac{\omega_i^2}{e_i} \quad (2.76)$$

Note that  $p(f | \bar{f})$  is scale and rotation invariant, and  $p(\bar{f})$  depends only on the local deformation parameter vector  $\omega$  and is independent of the global deformation parameter such as  $u, s, \theta$ . As a consequence,  $p(f, \bar{f})$  is also scale and rotation invariant, retaining the advantage of the prior distribution of the G-Snake.

To take the advantage of the shape space constraint, the solution found in iteration  $t$  is correspondingly revised into

$$f^{(t)} = -u + \frac{1}{s} \cdot R_{-\theta}(\hat{f}'') \quad (2.77)$$

The revised solution  $f^{(t)}$  is in the shape space. It retains the rigid variation of  $f'$  (w.r.t.  $\bar{f}_{mean}$ ), and has the local deformation consistent with the samples in the training set.

Two remarks about the imposition of the shape space constraints follow: (i) The imposition of the shape space constraint makes the EigenSnake more robust against spurious structures and occlusion. The effect will be demonstrated later in the experiments section. (ii) Because  $\omega = \mathbf{0}$  when  $f'' = \bar{f}_{mean}$ ,  $\omega$  can be considered as the local deformation parameters of  $f''$  or  $f'$ . Hence, the rigid transformation parameters ( $u, s$  and  $\theta$ ) and the non-rigid deformation parameters ( $\omega$ ) are separated. Therefore, the prior distribution  $p(f, \bar{f})$  explains not only the global variation but also the local variation, derived from a training set.

## Likelihood Distribution

Edges are detected from the image  $d = \{d(x, y)\}$ . The normalized edge direction at each pixel location  $(x, y)$ , denoted  $d^e(x, y) = (d_x^e(x, y), d_y^e(x, y))$ , is used to provide constraints on the data, with  $\|d^e(x, y)\| \in (0, 1]$  if an edge is detected at  $(x, y)$  or a zero vector  $d^e(x, y) = \mathbf{0}$  if an edge is not detected there.

Denote the unit tangent directions of  $f$  by  $f^e = [f_1^e, \dots, f_m^e]$ . The direction at a point  $f_i = (x_i, y_i)$  can be estimated by using a finite difference formula:  $\frac{f_{i+1} - f_i}{|f_{i+1} - f_i|} + \frac{f_i - f_{i-1}}{|f_i - f_{i-1}|}$ . It is normalized into a unit-length vector:  $f_i^e = (f_{x_i}^e, f_{y_i}^e)$ , with  $\|f_i^e(x, y)\| = 1$ .

Constraints on the image  $d$  are imposed through the differences  $\|f_i^e - d_i^e\|$ . Assuming the differences have the same variance  $\sigma_e^2$  for all  $i$ , we define the likelihood energy function as

$$E(d | f) = \frac{1}{\sigma_e^2} \sum_{i=1}^m \|f_i^e - d_i^e\|^2 \quad (2.78)$$

where  $d_i^e = d^e(x_i + x_u, y_i + y_u)$ . The likelihood distribution can be defined as

$$p(d | f) = \frac{1}{Z'} \exp \{-E(d | f)\} \quad (2.79)$$

where  $Z'$  is the normalizing constant

### Bayesian Estimates

In the Bayesian framework, the problem of extracting a contour with unknown deformation from a given image can be posed as a problem of maximum a posterior (MAP) estimation. The posterior distribution of the Eigen-Snake is then defined as

$$p(f, \bar{f} | d) = \frac{p(d | f) \cdot p(f, \bar{f})}{p(d)} \quad (2.80)$$

where  $p(d | f) = p(d | f, \bar{f})$ . The corresponding energy is

$$E(f, \bar{f} | d) = E(d | f) + E(f | \bar{f}) + E(\bar{f}) \quad (2.81)$$

The MAP estimates  $f_{MAP}$  and  $\bar{f}_{MAP}$  are then defined as

$$\{f_{MAP}, \bar{f}_{MAP}\} = \arg \min_{f, \bar{f}} \{E(f, \bar{f} | d)\} \quad (2.82)$$

### MultiEigenSnake

The EigenSnake is expanded into the MultiEigenSnake for detecting a compound object with multiple separate sub-objects. Suppose a compound object consists of  $L$  simple sub-objects. The shape of each sub-object is represented by a simple contour, and their combination is represented by a compound contour. Therefore, the MultiEigenSnake consists of  $L + 1$  contours. For example, a face, is considered as composed of  $L = 7$  sub-objects: a face outline, a pair of brows, a pair of eyes, a nose and a mouth. Fig.2.12 shows the construction the MultiEigenSnake from individual training contours for the representation of a full face.

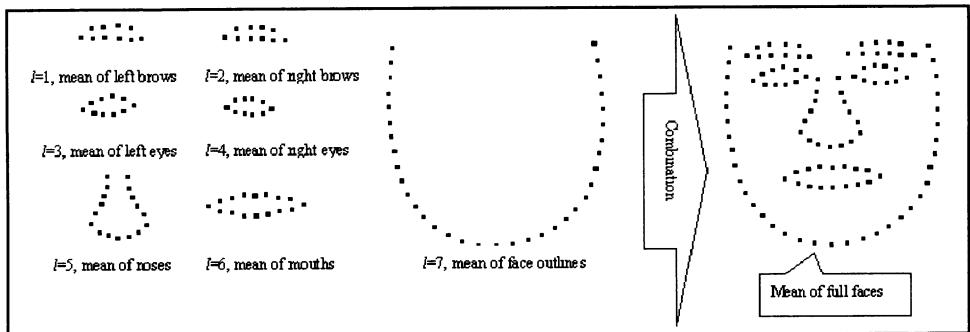


Figure 2.12: Construction of MultiEigenSnake contour.

Let  $\overline{f^{ME}}$  denotes the prototype contours of the compound object, and  $\overline{f}_\ell$  denotes the prototype contour of the  $\ell$ -th sub-object. Corresponding to Equ.(2.81), we define the following posterior energy for the MultiEigenSnake

$$E(\overline{f^{ME}}, f^{ME}, f^1, \dots, f^L, \overline{f^1}, \dots, \overline{f^L} | d) = E(\overline{f^{ME}}) + \sum_{\ell=1}^L [E(d | f^\ell) + E(f^\ell | \overline{f^\ell})] \quad (2.83)$$

The prior energy  $E(\overline{f^{ME}})$  accounts for prior knowledge about local deformation of the compound object and the spatial configuration of its  $L$  sub-objects.

There are two reasons why we partition a compound object into many sub-models instead of using a single EigenSnake model.

1. The prior distribution of the EigenSnake,  $p(\mathbf{f} | \overline{\mathbf{f}})$ , is based on an assumption that every contour point is dependent only on its two direct neighboring points (the Markovian property). However, no neighboring relationship is defined between points belonging to different sub-objects, and so each contour has to be modeled by its one energy  $E_{GS}(\mathbf{f}^\ell, \overline{\mathbf{f}}^\ell, \mathbf{u}^\ell)$ .
2. We assume that the shape of each sub-object is stable, so the prior for each sub-object can be modeled effectively by its shape model and closely fitting to the contour of a sub-object can be achieved. However, the spatial configuration, *i.e.* relations among the sub-objects, is subject to larger changes, and therefore has to be modeled by using the PCA method. The effect of the strategy to design the MultiEigenSnake will be demonstrated in the experiments section.

## 2.6.2 Experiments

Two sets of experiments are presented. The first examines the effects of using the specific prior knowledge and the shape space constraint by comparing the EigenSnake and other methods in extracting face outlines. The second set demonstrates the efficacy of the MultiEigenSnake in the extraction of the full facial features.

### Comparison EigenSnake with Other Methods

Here, four methods are compared: the original G-Snake (GS), enhanced G-Snake with the shape space constraint imposed (GS+), EigenSnake without the imposition of the shape space (ES-), and the full EigenSnake (ES). Their respective performances will tell us how the incorporation of the specific prior knowledge (by comparing ES- vs GS, and ES vs GS+) and the imposition of the shape space constraint (ES vs ES-, and GS+ vs GS) helps improve the extraction of face outline.

The experiment data set contains 60 frontal face images of size  $512 \times 342$  from the face database of Bern university.  $m = 44$  landmark points are manually and evenly placed on the face outline of each training image as

accurately as possible. A subset of 40 images are used as the *training set* and all the 60 images are used as the *testing set*.

The mean contours, which was shown in Fig.2.11:Right, is calculated from the training set. 8 eigenvectors are used to form the shape space.  $M = 35$  eigenvectors are used to estimate the prior energy  $E(\bar{f})$ .

Contour initialization is done manually, as a common practice for snakes. It has the same shape as the mean contour but it is re-scaled randomly to between 70% and 120% of the true mean scale, shifted away randomly from the true position by between  $\pm 20$  pixels, and rotated randomly from the true orientation in the range of  $\pm 15^\circ$ . All the methods have the same initial contour for each test image.

Table 2.1: Accuracy of the four methods.

Method	Init	GS	GS+	ES-	ES
$D(\text{method})$	23.8	17.1	12.7	7.7	4.8

Table 2.1 compares the accuracy, denoted  $D(\text{method})$ , of the face outlines detected by the four methods (plus that of the initial outline). There, the accuracy is measured by the average Euclidean distance (in pixels) between corresponding points in the extracted outline and the manual marked outline, and the statistics are obtained with the results for the 60 test images. The compared results illustrate that both the prior energy  $E(\bar{f})$  and the shape space constraint can improve the accuracy. The improvement in accuracy made by the use of the shape space constraint may be calculated as  $\Delta_S = [(D(\text{GS}) - D(\text{GS+})) + (D(\text{ES-}) - D(\text{ES}))]/2 = 3.65$  (the average of the two improvements, one from GS to GS+, the other from ES- to ES). The improvement in accuracy made by the use of the prior energy  $E(\bar{f})$  may be calculated as  $\Delta_P = [(D(\text{GS}) - D(\text{ES-})) + (D(\text{GS+}) - D(\text{ES}))]/2 = 8.65$  (the average of the two improvements, one from GS to ES-, the other from GS+ to ES). The two numbers,  $\Delta_S$  and  $\Delta_P$ , suggest that the contribution of the prior energy  $E(\bar{f})$  is greater than that of the shape space constraint; in other words, more was done by the imposition of the specific prior knowledge than by the use of the shape space constraint.

Fig.2.13 shows an example of face outline extraction. It can be seen that the GS often fails to localize objects. Although GS+ can modify the results of GS so that it looks more like “face outline”, the improvement is less than that produced by ES- and ES. The two ES methods, which use the new energy functions with abundant prior knowledges incorporated, can effectively resist the disturbance from spurious structures, noisy edge map and improper initial scales.

The final energy values tell us the “correctness” of the four energy-minimization based models (for a correct energy model, a better solution

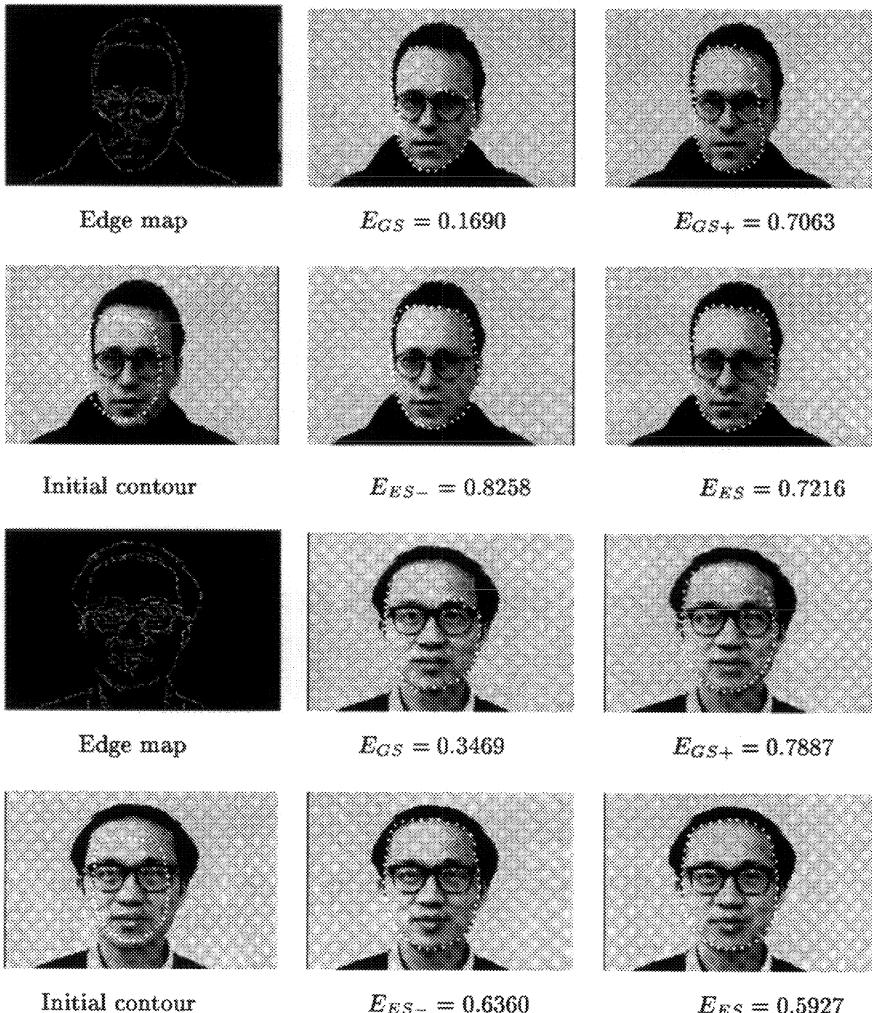


Figure 2.13: Comparison of the four methods.

should have a lower energy). The GS+ produced better results than the GS; however, the final GS+ energies are higher than corresponding GS energies. This means that the GS energy does not really define the optimal solution in its minimum and is therefore “incorrect”. This is due to lack of constraint on the optimal solution. In contrast, the ES model has a proper energy function because a better solution is always associated with a lower energy.

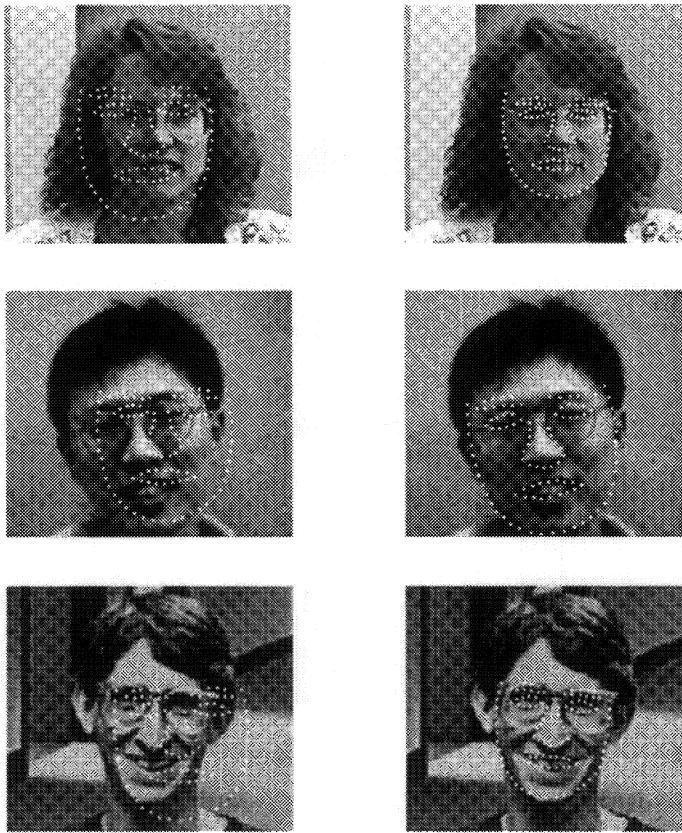


Figure 2.14: Examples of MultiEigenSnake results.

### Extracting Full Face Using MultiEigenSnake

This experiment demonstrates the MultiEigenSnake applied to extract a full face with  $L = 7$  facial features. The training set consists of 120 frontal face images from the face database of Bern university. The testing set is chosen from the face database of Carnegie Mellon University.

A total of 108 landmark points is used to represent a full face, among which 10 for left brow, 10 for right brow, 10 for left eye, 10 for right eye, 17 for nose, 16 for mouth and 35 for the face boundary. The means of the training contours (shown as Fig.2.12) are used as initial prototype contours of  $L$  sub-models and the compound model respectively.

Some experimental results are shown in Fig.2.14. Basically, the MultiEigenSnake can successfully extract the full face contours if the initial distance between the initial prototype contour and the target object is less than

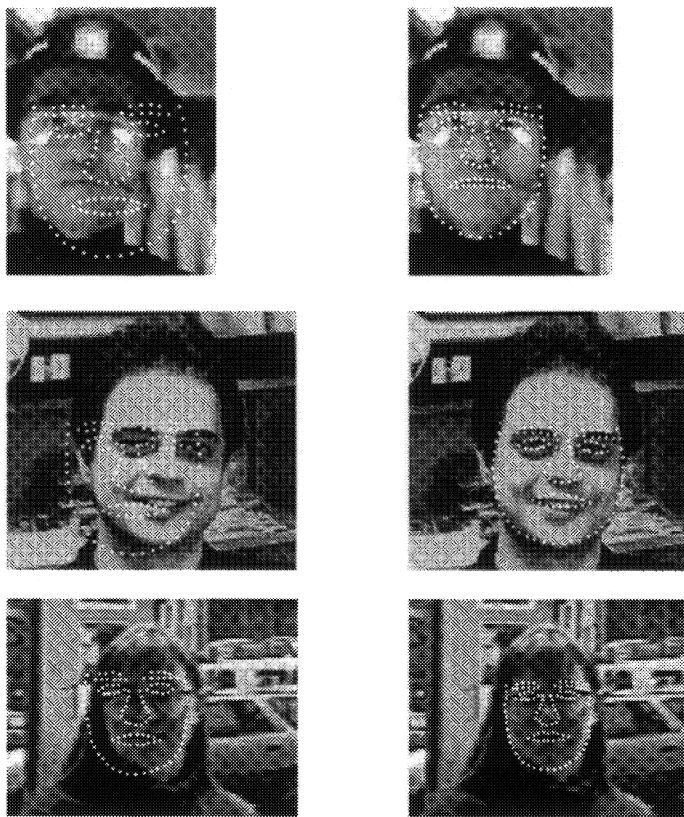


Figure 2.15: Examples of MultiEigenSnake results (cont'd).

one fourth of the initial prototype contour size. The MultiEigenSnake has certain scale insensitive and rotation insensitive; for example, it allows up to  $\pm 20^\circ$  rotation of initial prototype contour from orientation of the true object.

### Conclusion

The EigenSnake is trainable, general and capable of representing any arbitrary shape. It incorporates prior knowledge about both rigid and non-rigid deformations of expected shapes, and deforms to fit the contour in a way which reflects variations in the training set. The EigenSnake has the ability not only to account for global changes due to rigid motions, but also to learn local deformation of contours from examples.

Experimental results show that the EigenSnake is more robust to initialization, spurious features, and occlusion, and can produce more accurate contours than existing methods. The results comparing various schemes show

that both the incorporation of the prior knowledge about the specific shape of interest into the energy function and the shape space constraint can improve the accuracy. Of the two contributions, the improvement made by the use of the prior shape knowledge is greater than that of the shape space constraint.

Although the EigenSnake presented here is for the face contour and facial feature extraction, the method may be applied to other applications such as object tracking in motion, as long as a set of training samples of the object shape is provided.

# Chapter 3

# High Level MRF Models

High level vision tasks, such as object matching and recognition and pose estimation, are performed on features extracted from images. The arrangements of such features are usually irregular and hence the problems fall into categories LP3 and LP4. In this chapter, we present MAP-MRF formulations for solving these problems.

We begin with a study on the problem of object matching and recognition under contextual constraints. An MAP-MRF model is then formulated following the systematic approach summarized in Section 1.5.4. The labeling of a scene in terms of a model<sup>1</sup> object is considered as an MRF. The optimal labeling of the MRF is obtained by using the MAP principle. The matching of different types of features and of multiple objects is discussed. A related issue, MRF parameter estimation for object matching and recognition, will be studied in Chapter 6.

We then derive two MRF models for pose computation. By pose, it means the geometric transformation from one coordinate system to another. In visual matching, the transformation is from the scene (image) to the considered model object (or vice versa). The derived models, the transformation is from a set of object features to a set of image features. They minimize posterior energies derived for the MAP pose estimation, possibly together with an MRF for matching.

## 3.1 Matching Under Relational Constraints

In high level image analysis, we are dealing with image features, such as critical points, lines and surface patches, which are more abstract than image pixels. Such features in a scene are not only attributed by (unary) properties about the features themselves but also related to each other by relations

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<sup>1</sup>In this chapter, the word “model” is used to refer to both mathematical vision models and object models.

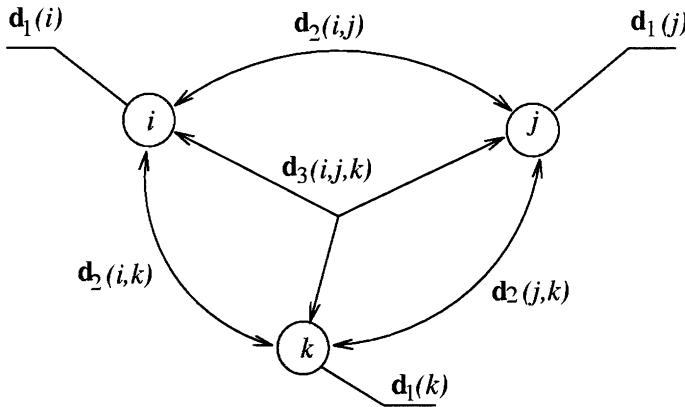


Figure 3.1: Three nodes and their unary, binary and triple relations in the RS representation. From (Li 1992b) with permission; ©1992 Elsevier.

between them. In other words, an object or a scene is represented by features constrained by the properties and relations. It is the bilateral or higher order relations that convey the contextual constraints. They play a crucial role in visual pattern matching.

### 3.1.1 Relational Structure Representation

The features, properties and relations can be denoted compactly as a *relational structure* (RS) (Fischler and Elschlager 1973; Ambler et al. 1973; Cheng and Huang 1984; Radig 1984; Li 1992a; Li 1992b). An RS describes a scene or a (part of) model object. The problem of object recognition is reduced to that of RS matching.

Let us start with a scene RS. Assume there are  $m$  features in the scene. These features are indexed by a set  $\mathcal{S} = \{1, \dots, m\}$  of sites. The sites constitute the nodes of the RS. Each node  $i \in \mathcal{S}$  is associated with it a vector  $d_1(i)$  composed of a number of  $K_1$  *unary properties* or *unary relations*,  $d_1(i) = [d_1^{(1)}(i), \dots, d_1^{(K_1)}(i)]^T$ . A unary property could be *e.g.* the color of a region, the size of an area, or the length of a line. Each pair of nodes  $(i, i' \in \mathcal{S}, i' \neq i)$  are related to each other by a vector  $d_2(i, i')$  composed of a number of  $K_2$  *binary (bilateral) relations*,  $d_2(i, i') = [d_2^{(1)}(i, i'), \dots, d_2^{(K_2)}(i, i')]^T$ . A binary relation could be *e.g.* the distance between two points, or the angle between two lines. More generally, among  $n$  features  $i_1, \dots, i_n \in \mathcal{S}$ , there may be a

vector  $d_n(i_1, \dots, i_n)$  of  $K_n$   $n$ -ary relations. This is illustrated in Fig.3.1. An  $n$ -ary relation is also called a relation, or constraint, of order  $n$ . The scope of relational dependencies can be determined by a neighborhood system  $\mathcal{N}$  on  $\mathcal{S}$ . Now, the RS for the scene is defined by a triple

$$\mathcal{G} = (\mathcal{S}, \mathcal{N}, d) \quad (3.1)$$

where  $d = \{d_1, d_2, \dots, d_H\}$  and  $H$  is the highest order. For  $H = 2$ , the RS is also called a *relational graph (RG)*. The highest order  $H$  cannot be lower than 2 when contextual constraints must be considered.

The RS for a model object is similarly defined as

$$\mathcal{G}' = (\mathcal{L}, \mathcal{N}', D) \quad (3.2)$$

where  $D = \{D_1, D_2, \dots, D_H\}$ ,  $D_1(I) = [D_1^{(1)}(I), \dots, D_1^{(K_1)}(I)]^T$ ,  $D_2(I, I') = [D_2^{(1)}(I, I'), \dots, D_2^{(K_2)}(I, I')]^T$  and so on. In this case, the set of labels  $\mathcal{L}$  replaces the set of sites. Each element in  $\mathcal{L}$  indexes one of the  $M$  model features. In addition, the “neighborhood system” for  $\mathcal{L}$  is defined to consist of all the other elements, that is,

$$\mathcal{N}'_I = \{I' \mid \forall I' \in \mathcal{L}, I' \neq I\} \quad (3.3)$$

This means each model feature is related to all the other model features. The highest considered order,  $H$ , in  $\mathcal{G}'$  is equal to that in  $\mathcal{G}$ . For particular  $n$  and  $k$  ( $1 \leq k \leq K_n$ ;  $1 \leq n \leq H$ ),  $D_n^{(k)}$  represent the same type of constraint as  $d_n$ ; for example, both represent the angle between two line segments.

Relations of various orders impose unary, binary,  $\dots$ ,  $H$ -ary constraints on the features. Inter-contextual constraints are represented by the second or higher order relations. Due to these constraints, a scene, an object or a view of an object is seen as an integrated part rather than as individual features. The higher the order of relations is, the more powerful the constraints are, but the higher the complication and expenses are in the computation.

There can be multiple model RS's in a model base. A model RS describes a whole model object or a part of it. When an RS is used to describe a part (for example a view) of an object, the whole object may be described by several RS's and these RS's may be related by some inter-RS constraints.

Now introduce a virtual model composed of a single node  $\mathcal{L}_0 = \{0\}$ . It is called NULL model. This special model represents everything not modeled by  $\mathcal{G}'$ , such as features due to all the other model objects and the noise. So the actual label set in matching the scene to the model plus NULL contains  $M + 1$  labels. It is denoted by

$$\mathcal{L}^+ = \{0, 1, \dots, M\} \quad (3.4)$$

After the introduction of the NULL, the mapping from  $\mathcal{S}$  to  $\mathcal{L}$  is illustrated in Fig.3.2.

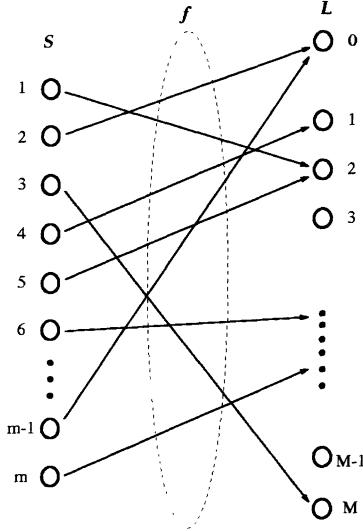


Figure 3.2: Discrete mapping involving the NULL label (numbered 0). All scene nodes (sites) not modeled by the considered object should be matched to this special label.

Figures 3.3 and 3.4 demonstrate two cup images and their segmentations based on the  $H$ - $K$  surface curvatures (Besl and Jain 1985). Suppose we are matching the RG in Fig. 3.4(b) to that in (a). Based on the constraints from the unary properties of surface curvature and region area, and binary relations of distance between regions, the correct matching from (b) to (a) is  $5 \rightarrow 1$ ,  $1 \rightarrow 2$  and the rest to NULL .

Model-based matching can be considered as finding the optimal mapping from the image RS to the model RS (or vice versa). Such a mapping from one RS to another is called a *morphism*, written as

$$f : \mathcal{G}(\mathcal{S}, \mathcal{N}, d) \rightarrow \mathcal{G}'(\mathcal{L}, \mathcal{N}', D) \quad (3.5)$$

which maps each node in  $\mathcal{S}$  to a node in  $\mathcal{L}$

$$f : \mathcal{S} \rightarrow \mathcal{L} \quad (3.6)$$

and thus maps relations  $d_n$  to relations  $D_n$

$$f : d_n \rightarrow D_n \quad (3.7)$$

A morphism is called an *isomorphism* if it is one-to-one and onto. It is called a *monomorphism* if it is one-to-one but not onto. It is called a *homomorphism* if it is many-to-one. We do not allow one-to-many mappings because

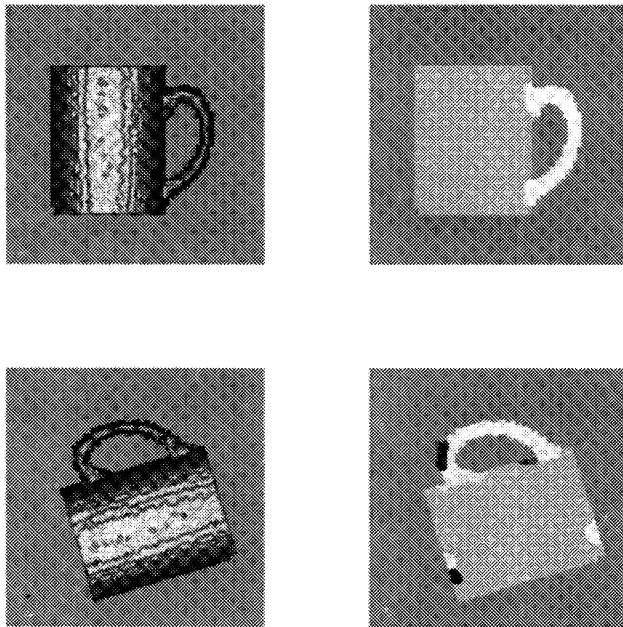


Figure 3.3: Cup images and segmentation based on  $H$ - $K$  surface curvatures. Top: a cup image and its  $H$ - $K$  map. Bottom: a transformed version of the cup image and the  $HK$  map. Some noise is introduced during the transformation due to quantization. From (Li 1992a) with permission; ©1992 Academic Press.

they contradict the definition of functions and, more crucially, increase the difficulties in finding the optimal solutions.

When the properties and relations in the considered RS's include numerical values, the mappings are numerical morphisms which are more difficult to resolve than symbolic morphisms. Since such morphisms do not preserve relations in the exact, symbolic sense, they may be called *weak morphisms* – a term extended from the weak constraint models (Hinton 1978; Blake 1983; Blake and Zisserman 1987).

The goodness of a numerical morphism is usually judged by an objective function such as an energy. It is not very difficult to find a correct one-to-one mapping (isomorphism) between two *identical* RS's. The requirement that the unary properties of two nodes and the binary relations of two links must be exactly the same in order to be matched to each other provides a strong constraint for resolving the ambiguities. For the case where the two RS's have different numbers of nodes and the matched properties and relations are not

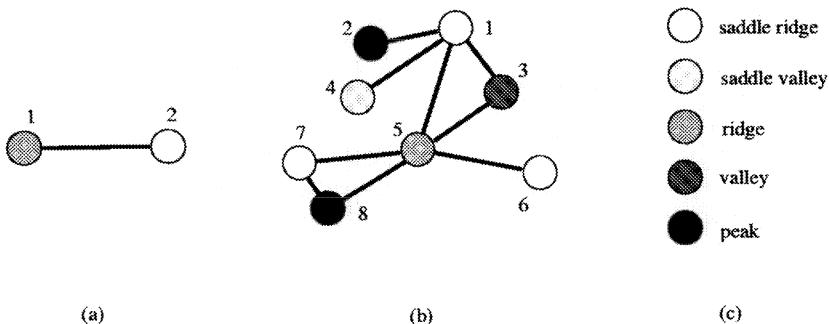


Figure 3.4: Relational graphs built from the cup *HK* segmentation maps. The texture of the nodes denotes different surface types and the links represent the adjacency relations. (a) The RG for the orginal upright cup image. Node 1 corresponds to the body of the cup and node 2 to the handle. (b) The RG for the transformed cup. Node 5 corresponds to the body, node 1 to the handle, and the rest to NULL . (c) Legend for the correspondences between texture types and *H-K* surface types. From (Li 1992a) with permission; ©1992 Academic Press.

exactly the same, the matching is more difficult because it cannot exploit the advantage of the strong constraint of the exact equalities. Only “weak constraints” are available. The subsequent sections use MRFs to establish the objective function for weak homomorphisms for inexact, partial matching.

### 3.1.2 Work in Relational Matching

Two important computational issues in object matching are how to use contextual constraints and how to deal with uncertainties. Contextual constraints in object recognition are often represented using the notion of relational graphs. An object is represented by a set of features, their properties and relations. Relative values between two image features embed context in matching (Fischler and Elschlager 1973). Object matching is then reduced to matching between two relational graphs. On the other hand, noise is inevitably introduced in the process of feature extraction and relation measuring.

In the maximum clique method (Ambler et al. 1973; Ghahraman et al. 1980) for relational graph matching, an associated graph is formed from the two relational graphs, one for the scene and the other for a model object

(Ambler et al. 1973). Each node of the associated graph represents a possible match. The optimal matching is given by the maximal cliques.

A class of constraint satisfaction problems for matching is studied by (Shapiro and Haralick 1981). In their work, the criterion is the weighted number of mis-matched relations. Most of such work involves definition of some criteria to describe the “goodness”, or reversely the cost, of matching with relational constraints. Errors and mistakes in symbolic relations are mapped into a number, and this number is then used as a criterion to decide whether a homomorphism is acceptable or not. The inexact matching problem is viewed as finding the best homomorphism, i.e. the one with minimum number of errors with respect to a given attribute value threshold, a missing part threshold, and a relation threshold.

Relaxation labeling (RL) (Rosenfeld et al. 1976) has been a useful method for solving the matching problem. The constraints are propagated via a compatibility function and the ambiguity of labeling is reduced using an iterative RL algorithm. In our view, the most important part in the RL-based recognition is the definition of the compatibility function. Various RL schemes should be considered as algorithms for finding solutions. This will be further examined in Section 8.2.2.

Typical early works on relational matching using RL include (Davis 1979) and (Bhanu and Faugeras 1984; Bhanu 1984). In (Davis 1979), the objective function consists of four terms. Each term either encodes a particular constraint or penalizes unmatched features, the idea dated back to the work by Fischler and Elschlager (1973) . An association graph (Ambler et al. 1973) is used for matching relational structures. In the search for optimal matching, incompatible nodes for which some evaluation function is below a threshold are deleted from the graph. This generates a sequence of association graphs until a fixed point is reached. In (Bhanu and Faugeras 1984; Bhanu 1984), matching is posed as an optimization problem and the optimization is performed by using an RL algorithm presented in (Faugeras and Berthod 1981).

A feature common to most of the above matching works is that thresholds are used to determine whether two matches are compatible. This effectively converts the weighted-graph matching into symbolic matching. While greatly reducing search space, this may rule out the potential matches. Because of the noise, the observation of the objects, which represents feature properties and relations extracted from the scene, can be considered as a set of random variables. Furthermore, some object features may be missing and spurious features may emerge due to noise and un-modeled objects. The matching strategy has to deal with these uncertainties. It is hard to judge that in the presence of uncertainties, a difference of 1.000001 is an impossible while 0.999999 is a possible.

In the weak constraint satisfaction paradigm, “hard” constraints are allowed to be violated without causing the failure of constraint satisfaction. However, each such violation is penalized by adding an amount to a cost

function that measures the imperfection of the matching. This is usually implemented by using line process in low level vision (Geman and Geman 1984; Marroquin 1985). In a weak notion of graph-matching, Bienenstock (1988) proposes a scheme for an approximation of graph isomorphism in which relation-preserving characteristics of isomorphism can be violated but each violation incurs a small penalty. This is a transplant of the idea of the line process at the lower level to the higher level perception. Nevertheless, at a higher level where more abstract representations are used, the weak constraint satisfaction problem becomes more complicated.

Li makes use of contextual constraints not only on the prior configuration of labelings but also on the observed data into the labeling process (Li 1992a; Li 1992b; Li 1992c; Li et al. 1993). He proposes an energy function, on a heuristic basis, which combines contextual constraints from both the prior knowledge and the observation. Kittler *et al.* (1993) later derive the same compatibility from probabilistic view point used in Li's energy function.

MRFs provide a formal basis for matching and recognition under contextual constraints. Modestino and Zhang (1989) describe an MRF model for image interpretation. They consider an interpretation of a scene as an MRF and define the optimal matching as the MAP estimate of the MRF. Unfortunately, the posterior probability therein is derived not by using the laws of probability but designed directly by using some heuristic rules. This contradicts the promises of MAP-MRF modeling. Cooper (1990) describes a coupled network for simultaneous object recognition and segmentation. MRF is used to encode prior qualitative, and possibly, quantitative knowledge in the non-homogeneous and anisotropic situations. The network is applied to recognize Tinkertoy objects. An interesting development is Markov processes of objects proposed by (Baddeley and van Lieshout 1993). Other works in MRF-based recognition can be found in (Grenander and Y. Chow 1991; Baddeley and van Lieshout 1992; Friedland and Rosenfeld 1992; Kim and Yang 1992; Cooper et al. 1993). The MRF model described in the next section is based on (Li 1994a).

## 3.2 MRF-Based Matching

The labeling of a scene in terms of a model object is denoted by  $f = \{f_i \in \mathcal{L}^+ \mid i \in \mathcal{S}\}$  where elements in  $\mathcal{S}$  index image features and those in  $\mathcal{L}$  do model object features plus the NULL. It is also interpreted as a relational mapping from  $\mathcal{G}(\mathcal{S}, \mathcal{N}, d)$  to  $\mathcal{G}'(\mathcal{L}, \mathcal{N}', D)$  (*cf.* (3.5)). We assume that  $f$  is a realization of an MRF with respect to  $\mathcal{N}$ . Below, we derive its posterior probability using the MAP-MRF approach, in which contextual constraints not only on the prior configuration but also the observation are considered.

### 3.2.1 Posterior Probability and Energy

The prior distribution of  $f$  is defined using MRFs. In RS matching, the neighborhood covers all other related sites (scene features). One may restrict the scope of interaction by defining  $\mathcal{N}_i$  as the set of other features which are within a distance  $r$  from  $i$  (cf. (1.14))

$$\mathcal{N}_i = \{i' \in \mathcal{S} \mid [\text{dist}(\text{feature}_{i'}, \text{feature}_i)]^2 \leq r, i' \neq i\} \quad (3.8)$$

where the function  $\text{dist}$  is a suitably defined distance function for features. The distance threshold  $r$  may be reasonably related to the size of the model object. The set of first order cliques is

$$\mathcal{C}_1 = \{\{i\} \mid i \in \mathcal{S}\} \quad (3.9)$$

The set of second order cliques is

$$\mathcal{C}_2 = \{\{i, i'\} \mid i' \in \mathcal{N}_i, i \in \mathcal{S}\} \quad (3.10)$$

Here, only cliques of up to order two are considered.

The single-site potential is defined as

$$V_1(f_i) = \begin{cases} v_{10} & \text{if } f_i = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.11)$$

where  $v_{10}$  is a constant. If  $f_i$  is the NULL label, it incurs a penalty of  $v_{10}$ ; or the nil penalty is imposed otherwise. The pair-site potential is defined as

$$V_2(f_i, f_{i'}) = \begin{cases} v_{20} & \text{if } f_i = 0 \text{ or } f_{i'} = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.12)$$

where  $v_{20}$  is a constant. If either  $f_i$  or  $f_{i'}$  is the NULL , it incurs a penalty of  $v_{20}$  or nil penalty otherwise. The above clique potentials define the prior energy  $U(f)$ . The prior energy is then

$$U(f) = \sum_{i \in \mathcal{S}} V_1(f_i) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}) \quad (3.13)$$

The definitions of the above prior potentials are a generalization of that penalizing line process variables (Geman and Geman 1984; Marroquin 1985). The potentials may also be defined in terms of stochastic geometry (Baddeley and van Lieshout 1992).

The conditional p.d.f.,  $p(d \mid f)$ , of the observed data  $d$ , also called the *likelihood function* when viewed as a function of  $f$  given  $d$  fixed, has the following characteristics:

1. It is conditioned on pure non-NULL matches  $f_i \neq 0$ ,
2. It is independent of the neighborhood system  $\mathcal{N}$ , and

3. It depends on how the model object is observed in the scene which in turn depends on the underlying transformations and noise.

Assume (1) that  $D$  and  $d$  are composed of types of features which are invariant under the considered class of transformations<sup>2</sup>; (2) that they are related via the observation model

$$d_1(i) = D_1(f_i) + e_1(i), \quad d_1(i, i') = D_2(f_i, f_{i'}) + e_2(i, i') \quad (3.14)$$

where  $e$  is additive independent zero mean Gaussian noise. The assumptions of the independent and Gaussian noise may not be accurate but offers an approximation when an accurate observation model is not available.

Then the likelihood function is a Gibbs distribution with the energy

$$U(d | f) = \sum_{i \in \mathcal{S}, f_i \neq 0} V_1(d_1(i) | f_i) + \sum_{i \in \mathcal{S}, f_i \neq 0} \sum_{i' \in \mathcal{S} - \{i\}, f_{i'} \neq 0} V_2(d_2(i, i') | f_i, f_{i'}) \quad (3.15)$$

where the constraints,  $f_i \neq 0$  and  $f_{i'} \neq 0$ , restrict the summations to take over the non-NUL matches. The likelihood potentials are

$$V_1(d_1(i) | f_i) = \begin{cases} \sum_{k=1}^{K_1} [d_1^{(k)}(i) - D_1^{(k)}(f_i)]^2 / \{2[\sigma_1^{(k)}]^2\} & \text{if } f_i \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.16)$$

and

$$V_2(d_2(i, i') | f_i, f_{i'}) = \begin{cases} \sum_{k=1}^{K_2} [d_2^{(k)}(i, i') - D_2^{(k)}(f_i, f_{i'})]^2 / \{2[\sigma_2^{(k)}]^2\} & \text{if } i' \neq i \& f_i \neq 0 \& f_{i'} \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.17)$$

where  $[\sigma_n^{(k)}]^2$  ( $k = 1, \dots, K_n$  and  $n = 1, 2$ ) are the variances of the corresponding noise components. The vectors  $D_1(f_i)$  and  $D_2(f_i, f_{i'})$  are the “mean vectors”, conditioned on  $f_i$  and  $f_{i'}$ , for the random vectors  $d_1(i)$  and  $d_2(i, i')$ , respectively, conditioned on the configuration  $f$ .

Using  $U(f | d) = U(f) + U(d | f)$ , we obtain the posterior energy

$$\begin{aligned} U(f | d) = & \sum_{i \in \mathcal{S}} V_1(f_i) + \\ & \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}) + \\ & \sum_{i \in \mathcal{S}: f_i \neq 0} V_1(d_1(i) | f_i) + \\ & \sum_{i \in \mathcal{S}: f_i \neq 0} \sum_{i' \in \mathcal{S} - \{i\}: f_{i'} \neq 0} V_2(d_2(i, i') | f_i, f_{i'}) \end{aligned} \quad (3.18)$$

There are several parameters involved in the posterior energy: the noise variances  $[\sigma_n^{(k)}]^2$  and the prior penalties  $v_{n0}$ . Only the relative, not absolute, values of  $[\sigma_n^{(k)}]^2$  and  $v_{n0}$  are important because the solution  $f^*$  remains the

<sup>2</sup>The discovery and computation of visual invariants is an active area of research. See (Mundy and Zisserman 1992).

same after the energy  $E$  is multiplied by a factor. The  $v_{n0}$  in the MRF prior potential functions can be specified to achieve the desired system behavior. The higher the prior penalties  $v_{n0}$ , the fewer features in the scene will be matched to the NULL for the minimal energy solution.

Normally, symbolic relations are represented internally by a number. The variances  $[\sigma_n^{(k)}]^2$  for those relations are zero. One may set corresponding  $[\sigma_n^{(k)}]^2$  to  $0^+$  (a very small positive number), which is consistent with the concept of discrete distributions. Setting  $[\sigma_n^{(k)}]^2 = 0^+$  causes the corresponding distance to be infinitely large when the compared symbolic relations are not the same. This inhibits symbolically incompatible matches, if an optimal solution is sought, and thus imposes the desired symbolic constraint. A method for learning  $[\sigma_n^{(k)}]^2$  parameters from examples will be presented in Chapter 7.

### 3.2.2 Matching to Multiple Objects

The MAP configuration  $f^*$  derived in the above is the optimal mapping from the scene to the model object under consideration. In other words, it is the optimal labeling of the scene in terms of the model object.

Suppose there are  $L$  potential model objects. Then  $L$  MAP solutions,  $f^{(1)}, \dots, f^{(L)}$ , can be obtained after matching the scene to each of the models in turn<sup>3</sup>. However, any feature in the scene can have only one match of model feature. To resolve this, we use the following method of cost minimization.

Rewrite  $E(f) = U(f \mid d)$  in (3.18) into the following form

$$E(f) = \sum_{i \in \mathcal{S}} E_1(f_i) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{S}, i' \neq i} E_2(f_i, f_{i'}) \triangleq \sum_{i \in \mathcal{S}} E(f_i \mid f_{\mathcal{N}_i}) \quad (3.19)$$

where

$$E_1(f_i) = \begin{cases} v_{10} & \text{if } f_i = 0 \\ V_1(d_1(i) \mid f_i) & \text{otherwise} \end{cases} \quad (3.20)$$

and

$$E_2(f_i, f_{i'}) = \begin{cases} v_{20} & \text{if } i' \in \mathcal{N}_i \text{ \& } (f_i = 0 \text{ or } f_{i'} = 0) \\ V_2(d_2(i, i') \mid f_i, f_{i'}) & \text{if } i' \neq i \text{ \& } f_i \neq 0 \text{ \& } f_{i'} \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.21)$$

are local posterior energy of order one and two, respectively.  $E(f_i \mid f_{\mathcal{N}_i}) = E_1(f_i) + \sum_{i' \in \mathcal{S}} E_2(f_i, f_{i'})$  is the cost incurred by the local match  $i \rightarrow f_i$  given the rest of the matches. It will be used as the basis for selecting the best matched objects for  $i$  in matching to multiple model objects. The image

<sup>3</sup>Fast indexing of model objects (Lamdan and Wolfson 1988) is a topic not studied in this work.

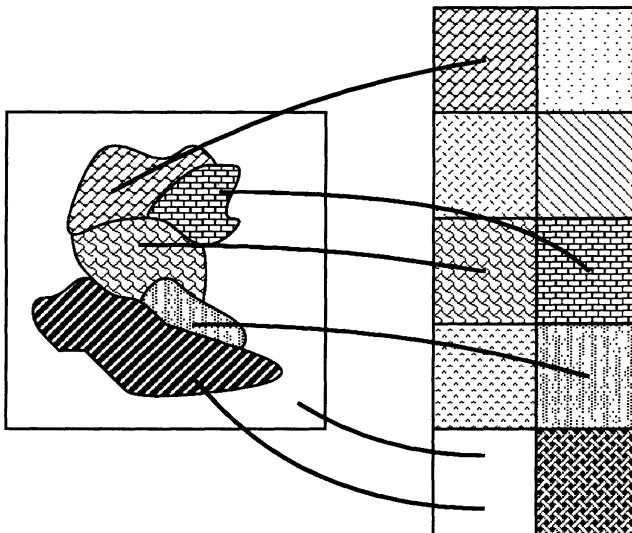


Figure 3.5: Mapping from the scene to multiple model objects. Different textures represent different structures. Bold lines represent sub-mappings. Note that the background and un-model structure are mapped to the NULL structure (the blank square). From (Li 1992c) with permission; ©1992 Elsevier.

feature  $i$  is considered to come from object  $\ell_i$  if the decision incurs the least cost

$$\ell_i = \arg \min_{\ell \in \{1, \dots, L\}} E(f_i^{(\ell)} | f_{\mathcal{N}_i}^{(\ell)}) \quad (3.22)$$

The final label for  $i$  is feature number  $f_i^{(\ell_i)}$  of object  $\ell_i$ . Note, however, the MAP principle is applied to matching to a single model object not to multiple objects; the simple rule (3.22) does not maximizes the posterior since at least the partition functions are different for matching to different objects.

Applying (3.22) to every  $i$  yields an overall mapping from the scene to the models, composed of several sub-mappings, as illustrated in Fig. 3.5. On the right, the squares with different textures represent the candidate model structures to which the scene is to be matched. Among them, the blank square represents the NULL model. On the left is the scene structure. The regions, each corresponding to a subpart of a model structure, are overlapping (un-separated). The recognition of the overlapping scene is (1) to partition the scene structure into parts such that each part is due to a single object and (2) to find correspondences between features in each part and those in the corresponding model object. The parts of the scene corresponding to the background and un-modeled objects should be mapped to the NULL , or in other words assigned the NULL label.

### 3.2.3 Extensions

The above model may be extended in a number of ways. With the assumption that different constraints are independent of each other, embedding a higher constraint can be achieved by adding a new energy term. Matching of different types of object features, *e.g.* points and lines, can be treated as coupled MRFs.

#### Incorporating Higher Constraints

In matching schemes based on invariants, the features chosen to represent the object modeled and the scene are invariant to the expected transformations from the object to the observation for the process of cognition to be accomplished. The more complicated the transformations are, the higher the order of features is needed for the invariant object representation (Li 1992b); the order needed may be higher than two.

In previous subsections, only constraints of up to the second order are considered. Incorporation of higher order constraints can be achieved by adding higher order energy terms. A clique of order  $n > 2$  is an  $n$ -tuple  $\{i_1, \dots, i_n\}$  in which  $i_r$  and  $i_s$  ( $r \neq s$ ) are neighbors to each other. The incorporation is done as follows. Firstly, the following  $n^{th}$  order *a priori* clique potentials are added to the prior energy  $U(f)$

$$V_{n0}(f_{i_1}, \dots, f_{i_n}) = \begin{cases} v_{n0} & \text{if } f_{i_k} = 0 \quad \exists i_k \in \{i_1, \dots, i_n\} \\ 0 & \text{otherwise} \end{cases} \quad (3.23)$$

where  $v_{n0}$  is a prior penalty constant. Secondly, the likelihood energy for the  $n^{th}$  order observation has the following likelihood potentials

$$V_n(d_n(i_1, \dots, i_n) \mid f_{i_1}, \dots, f_{i_n}) = \sum_{k=1}^{K_n} [d_n^{(k)}(i_1, \dots, i_n) - D_n^{(k)}(f_{i_1}, \dots, f_{i_n})]^2 / \{2[\sigma_n^{(k)}]^2\} \quad (3.24)$$

The corresponding posterior can be obtained using the Bayes rule, resulting in the following  $n^{th}$  order energy

$$E_n(f_{i_1}, \dots, f_{i_n}) = \begin{cases} v_{n0} & \text{if } f_{i_k} = 0 \quad \exists i_k \in \{i_1, \dots, i_n\} \\ V_n(d_n(i_1, \dots, i_n) \mid f_{i_1}, \dots, f_{i_n}) & \text{otherwise} \end{cases} \quad (3.25)$$

Adding together all the energy terms yields

$$\begin{aligned} E(f) = \sum_{i \in \mathcal{S}} E_1(f_i) &+ \sum_{i, i' \in \mathcal{S}} E_2(f_i, f_{i'}) + \dots \\ &+ \sum_{i_1, \dots, i_H \in \mathcal{S}} E_H(f_{i_1}, \dots, f_{i_H}) \end{aligned} \quad (3.26)$$

where  $H$  is the highest order.

### Coupled MRFs for Matching of Different Features

Let us consider the situation where an object consists of different types of features such as points and lines. Obviously, a point in the scene should not be matched to a line in an object model. This is a symbolic constraint. In this case, the positivity condition of MRF in Equ.(1.19) does not hold any more if the configuration space  $\mathbb{F}$  is still defined as the simple product as in Eq.(3.4) for a single MRF.

To overcome the limitation, we partition the whole set  $\mathcal{L}$  of labels to a few admissible sets for different types of sites. This results in a few coupled MRFs. These MRFs are coupled to each other via inter-relations  $d_n$  ( $n \geq 2$ ). For example, the distance between a point and a line can constrain the two different types of features. Furthermore, they are also coupled via the label **NULL** which is a “wildcard” compatible to all types of features.

If there are two different types of features, then  $\mathcal{L}$  can be partitioned into two admissible sets, with each set consisting of indices to a single type of features. In the most general case, each of the  $m$  sites has its own set of labels  $\mathcal{L}_i \subseteq \mathcal{L}$  ( $i = 1, \dots, m$ ), each  $\mathcal{L}_i$  being determined using the symbolic unary constraints; and the label for site  $i$  assumes a value  $f_i \in \mathcal{L}_i^+$  where  $\mathcal{L}_i^+ = \{0\} \cup \mathcal{L}_i$ . Then, the configuration space is defined as

$$\mathbb{F} = \mathcal{L}_1^+ \times \mathcal{L}_2^+ \times \dots \times \mathcal{L}_m^+ \quad (3.27)$$

In this situation, the energy  $E(f)$  has the same form as usual and the solution is still found by  $f^* = \arg \min_{f \in \mathbb{F}} E(f)$ . The only difference is in the definition of the configuration space  $\mathbb{F}$  in which the solution is searched for.

### Relationships with Low Level MRF Models

Let us compare the present model with low level vision MRF models prototyped by that of Geman and Geman (Geman and Geman 1984). The present model is similar to the MRF models for piecewise constant image restoration, edge detection and texture segmentation in that labels are discrete. Of course, their prior distributions must be different to cope with different tasks.

In surface reconstruction involving discontinuities (Marroquin 1985; Blake and Zisserman 1987; Chou and Brown 1990; Szeliski 1989; Geiger and Girosi 1991), there are commonly two coupled MRFs, a surface field and a line process field. The former field is defined on  $\mathcal{S}_1$ , the domain of an image grid. It assumes configurations in the space  $\mathcal{L}_1^{\mathcal{S}_1}$  where  $\mathcal{L}_1$  is a real interval. The latter is defined on  $\mathcal{S}_2$ , the dual of  $\mathcal{S}_1$ . It assumes configurations in the space  $\mathcal{L}_2^{\mathcal{S}_2}$  where  $\mathcal{L}_2$  is the set of labels such as {edge, non-edge}. These fields are coupled to each other by the interaction between the line process variable and the neighboring pixels.

The concept of discontinuity in the high level is the relational bond in the scene RS. For example, when none of  $f_i$  and  $f_{i'}$  assumes the **NULL** value,  $i$  and

$i'$  are relationally constrained; otherwise when  $f_i = 0$  or  $f_{i'} = 0$ , the relational bond between  $i$  and  $i'$  is broken. This corresponds to the line process.

The main difference between this high level model and those low level models is in the encoding of higher order relational constraints. Low level models use unary observation only, such as pixel intensity; although intensity difference between neighboring pixels is also used, it is derived directly from the intensity. The present model uses relational measurements of any orders. This is important for high level problems in which contextual constraints play a more important role. Moreover, in the present model, the neighborhood system is non-homogeneous and anisotropic, which also differs from the image case.

The matching method we have presented is based on a pre-requisition that invariants are available for object representation under the concerned group of transformations. If geometric variants are also used as sources of constraints, object poses have to be resolved during the computation of matching. See the next section.

### 3.2.4 Experiments

The following presents some experiments. Given a number of model objects, a scene is generated. Properties and relations in model objects and the scene are measured using the same program. Only the second order energy  $E_2$  is taken into consideration. The energy is minimized by using a relaxation labeling algorithm (see Section 8.2.2). In the computation of the minimal solution, interactions or compatibilities are represented by integers of only 8 bits and good results are achieved; this demonstrates the error-tolerant aspect of the model. The optimal matching result is displayed by aligning the matched object features to the scene while the unmatched are not displayed. The alignment is performed between the matched pairs by using the least squares fitting method (Umeyama 1991). Parameters  $v_{20} = 0.7$  is fixed for all the experiments. Parameters  $[\sigma_2^{(k)}]^2$  vary for different applications.

#### Matching Objects of Point Patterns

Please refer to Fig.3.6. There are three model objects as shown in (a) – (c) of the figure. Each of the objects consists of three types of point features shown in different sizes. The scene in (d) is generated from the model objects as follows: (1) Take a subset of features from each of the three objects, (2) do a transformation (rotation and translation) on each of the subsets, (3) mix the subsets together after that, (4) randomly deviate the locations of the points using either Gaussian or uniform noise, and (5) add spurious point features.

In this case of point matching, there is only one unary property, *i.e.* the point size, denoted  $d_1(i)$ . Each  $d_1(i)$  takes a value  $\{1, 2, 3\}$ . There is only binary relation, *i.e.* the Euclidean distance between two points  $d_2(i, i') = dist(i, i')$ . The points, their sizes and the distances constitute an RS. The

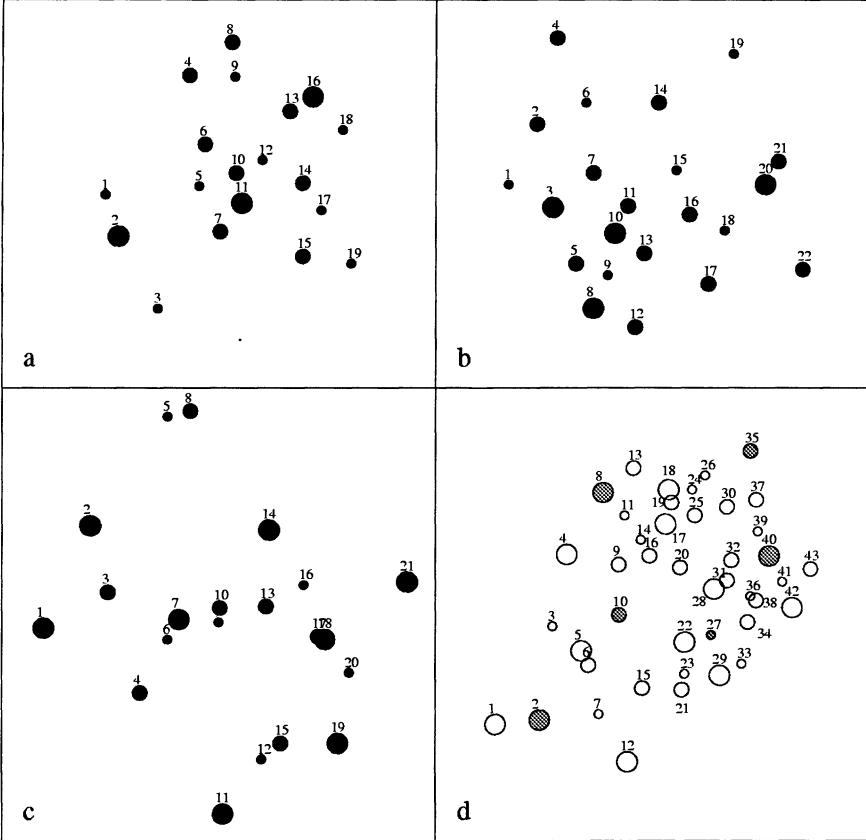


Figure 3.6: Matching objects of points. (a-c) The model objects. (d) The scene.

unary property is symbolic and this restricts the set of admissible labels for each  $i \in \mathcal{S}$  as  $\mathcal{L}_i = \{I \mid D_1(I) = d_1(i), \forall I \in \mathcal{L}\}$ . The parameters is chosen as  $[\sigma_2^{(1)}]^2 = 0.1$ .

Fig.3.7 shows the matching results in which the matched object points are aligned with the scene in (d). The black points in (a) correspond to points 5, 6, 7, 11, 12, 14, 15, 17 and 19 of the object in Fig.3.6(a). Those in (b) correspond to points 3, 5, 8, 9, 10, 11, 12, 13, 16, 17 and 18 of the object in Fig.3.6(b). Those in (c) are points 4, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 and 21 of the object in Fig.3.6(c). In (d) is shown the union of all the individual results. The spurious points 2, 8, 10, 27, 35 and 40 in Fig.3.6(d) have found no counterparts in the three objects. They are correctly classified as the NULL. There is one mis-match: point 19 in Fig.3.6(d) is matched to the NULL while its correct home should be point 10 of Fig.3.6(a).

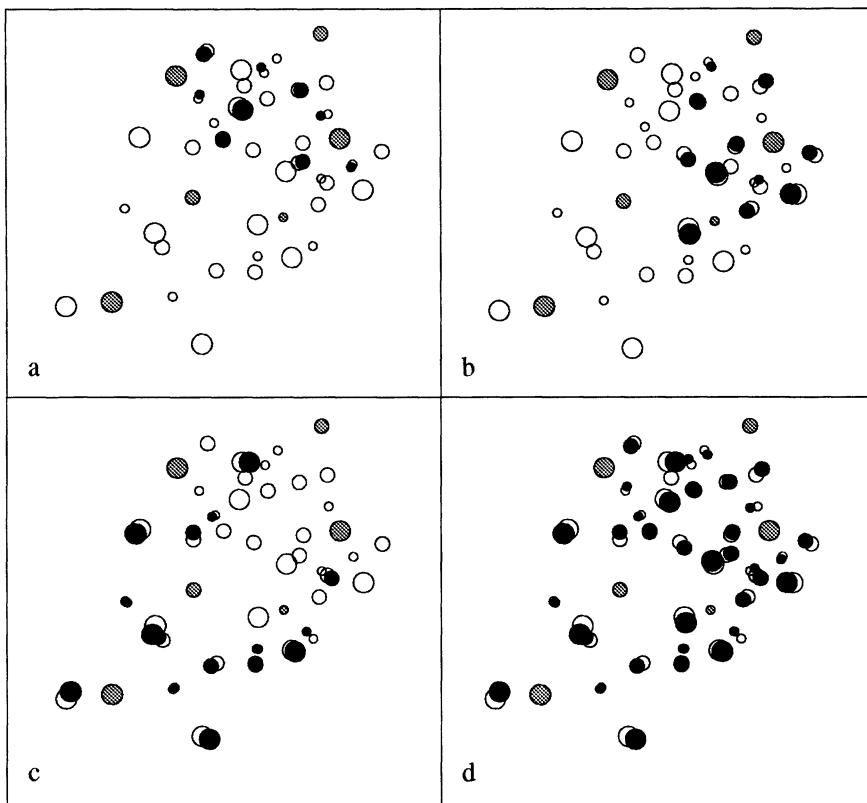


Figure 3.7: Results of matching objects of points. (a-c) Matched points (in black) from the respective models aligned with the scene. (d) All matched points aligned with the scene.

### Matching Objects of Line Patterns

Please refer to Fig.3.8. There are five objects made of lines, as shown in (a) – (e). The scene in (f) consists of subset of deviated lines taken from the first three objects (a) – (c), shown in dotted lines, and spurious line features shown in dashed lines. Four types of binary relations are measured:

- (1)  $d_2^{(1)}(i, i')$ : the angle between lines  $i$  and  $i'$ ,
- (2)  $d_2^{(2)}(i, i')$ : the distance between the mid-points of the lines,
- (3)  $d_2^{(3)}(i, i')$ : the minimum distance between the end-points of the lines, and
- (4)  $d_2^{(4)}(i, i')$ : the maximum distance between the end-points of the lines.

No unary relations are used. The value for the prior clique potential is  $v_{20} = 0.70000$  fixed. The values for weighting the binary measurements are  $1/[\sigma_2^{(1)}]^2 = 0.36422$ ,  $1/[\sigma_2^{(2)}]^2 = 0.20910$ ,  $1/[\sigma_2^{(3)}]^2 = 0.44354$  and  $1/[\sigma_2^{(4)}]^2 = 0.74789$ , which are estimated using a supervised learning procedure to be presented in Chapter 7.

Fig.3.9 shows the matching results in which the matched object lines from the first three objects are aligned with the scene in (d). The solid lines in (a) correspond to lines 11, 13, 14, 16, 17, 18, 19, 21, 25, 26 and 28 of the object in Fig.3.8(a). Those in (b) correspond to lines 5, 8, 10, 11, 13, 14, 15, 16 and 17 of the object in Fig.3.8(b). Those in (c) correspond to points 1, 2, 3, 5, 8, 9, 11, 12 and 14 of the object in Fig.3.8(c). Objects in (d) – (e) do not have matches. In (d) is shown the union of all individual results. The spurious lines 2, 13, 14, 15, 27, 33 and 36 in Fig.3.8(f) have found no counterparts in the object models. They are correctly classified as the NULL .

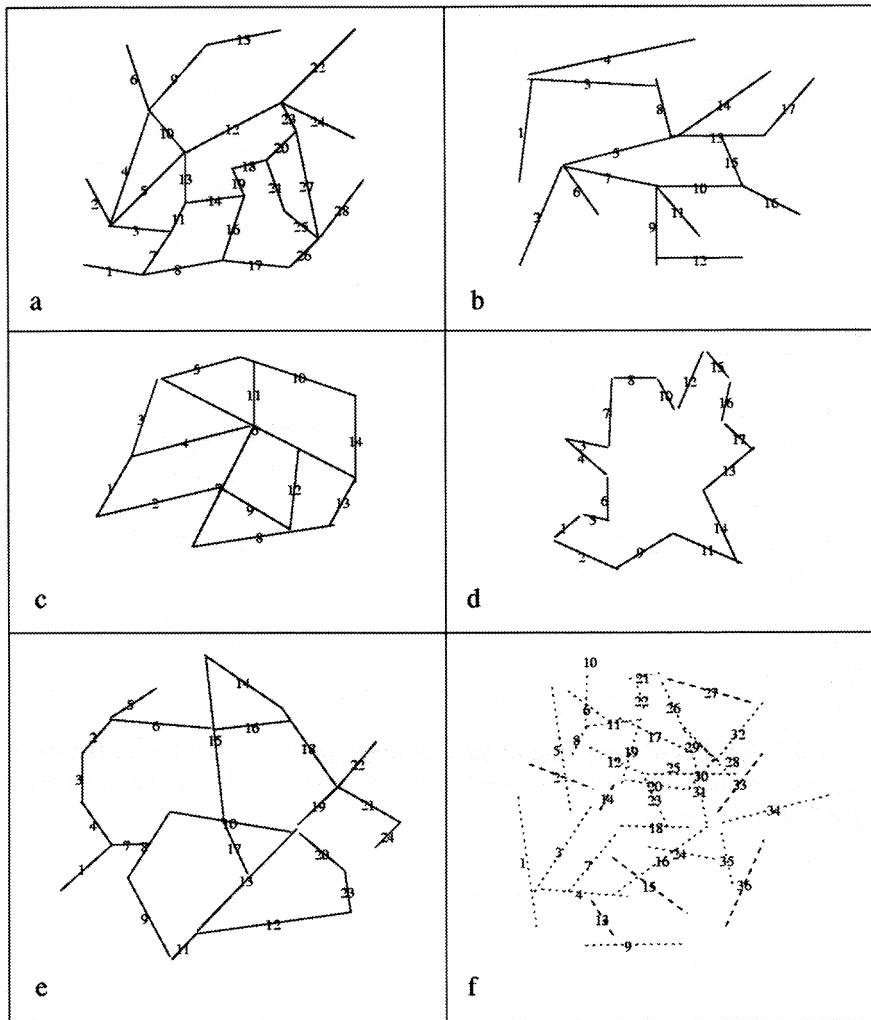


Figure 3.8: Matching objects of lines. (a-e) The five model objects. (f) The scene.

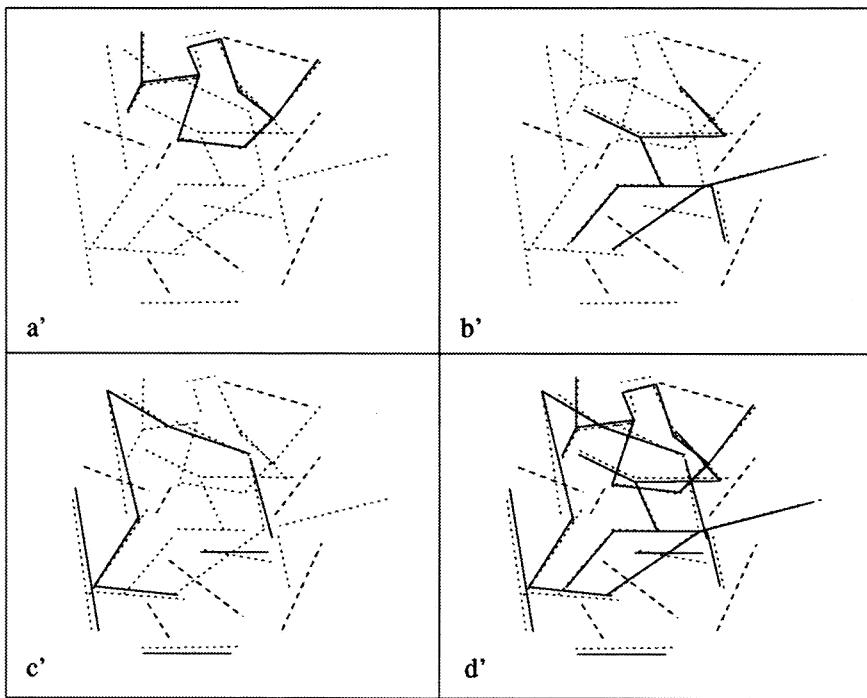


Figure 3.9: Results of matching objects of lines. (a-c) Matched lines (in solid) from the respective models aligned with the scene. (d) All matched lines aligned with the scene.

### Matching Curved Objects Under Similarity Transformations

Please refer to Figure 3.10. There are eight model jigsaw objects in this experiment. The scene contains rotated, translated and scaled parts of the model objects some of which are considerably occluded. Boundaries are computed from the image using the canny detector followed by hysteresis and edge linking. After that, corners of the boundaries are located as  $p_1, \dots, p_m$ .

The sites correspond to the corners on the scene curve and the labels correspond to the feature points on the considered model curve. The neighbors of a site is defined as the 5 forward points and the 5 backward points. Invariant relations are derived from the boundaries as well as the corners based on a similarity-invariant representation of curves (Li 1993). No unary properties are used ( $K_1 = 0$ ). Only binary relations are used, which are the following five types ( $K_2 = 5$ ):

- (1)  $d_2^{(1)}(i, i')$ : ratio of curve arc-length  $\widehat{p_i p_{i'}}$  and chord-length  $\overline{p_i p_{i'}}$ ,
- (2)  $d_2^{(2)}(i, i')$ : ratio of curvature at  $p_i$  and  $p_{i'}$ ,
- (3)  $d_2^{(3)}(i, i')$ : invariant coordinates vector,
- (4)  $d_2^{(4)}(i, i')$ : invariant radius vector, and
- (5)  $d_2^{(5)}(i, i')$ : invariant angle vector

which are derived from the boundaries and the corners using a similarity-invariant representation of curves (Li 1993).

The involved parameters are supplied by an automated optimal estimation procedure (see Chapter 7) with the following values  $v_{20} = 0.7$ ,  $1/\sigma_2^{(1)} = 0.00025$ ,  $1/\sigma_2^{(2)} = 0$ ,  $1/\sigma_2^{(3)} = 0.04429$ ,  $1/\sigma_2^{(4)} = 0.02240$  and  $1/\sigma_2^{(5)} = 0.21060$ . The minimal labeling  $f^*$  is found by using a deterministic relaxation labeling algorithm (Hummel and Zucker 1983). The final result of recognition is shown in Figure 3.11 in which model objects are aligned with the corresponding (sub-) parts in the scene. Note that most of the model objects share common structures of round extrusion and intrusion. This means ambiguities exist extensively and have to be resolved by using context.

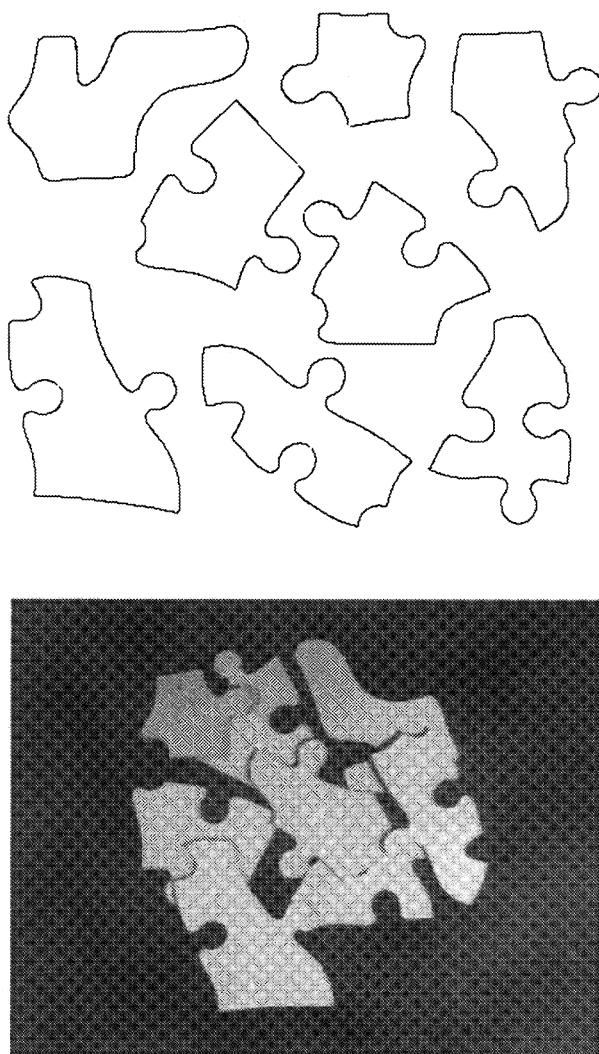


Figure 3.10: (Top) The eight model jigsaw objects; From (Li 1995c) with permission; ©1995 Kluwer. (Bottom) An overlapping scene.

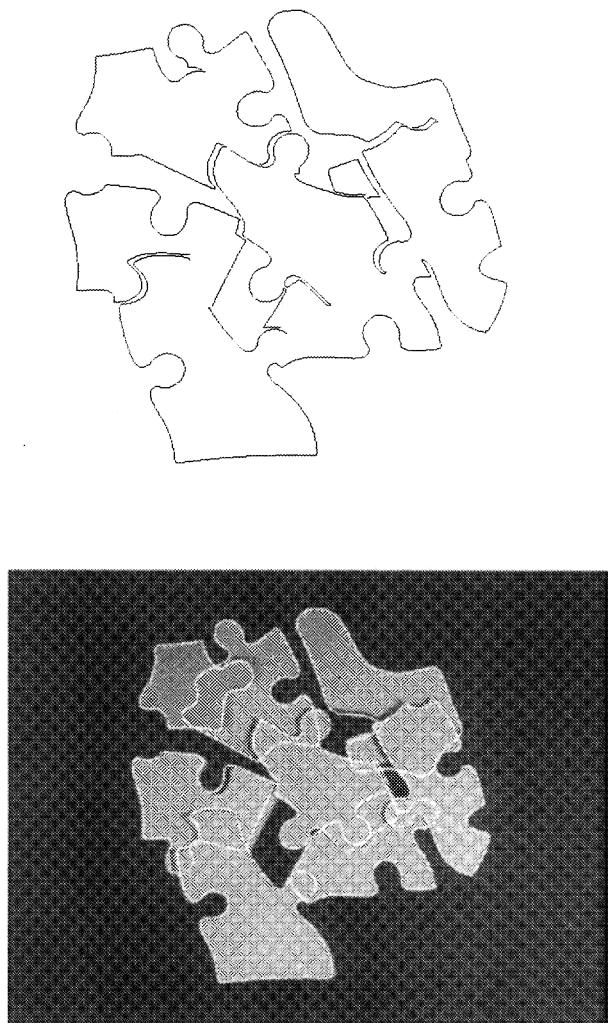


Figure 3.11: (Top) Boundaries detected from the scene. (Bottom) Matched objects are aligned with the scene; From (Li 1994b) with permission; ©1994 IEEE.

### 3.3 Optimal Matching to Multiple Overlapping Objects

Previously, matching between the scene and model objects is performed by considering one model object at a time. Once a part of the scene has been matched to a model object, it is excluded from subsequent matching; or better still, multiple matching results are obtained, each being optimal with respect to a single model object, as done in Section 3.2. Inconsistencies may exist among the individual results because the matching to one model is done independently of the other models. Post-processing may be applied to obtain an overall consistent solution.

When a scene contains multiple, mutually occluded objects, two sources of contextual constraints are required: *between-object constraints* (BOCs) and *within-object constraints* (WOCs). WOCs of an object, which describe the particular structure of the object itself, are used to identify instances of that object. BOCs, which are the constraints on features belonging to different objects, are used to discriminate between objects and un-modeled features.

Here, a statistically optimal, MAP-MRF based, formulation for recognition of multiple, partially occluded objects is presented. The MAP solution is defined with respect to all, not just individual, model objects. Such a solution is overall optimal, and consistent by itself. A two-stage MAP estimation approach is proposed to reduce the computational cost. The first stage finds feature correspondence between the scene and *each* model object. The second stage solves the original, target MAP-MRF estimation problem in a much reduced space constructed from the stage 1 solutions. The energy functions for the two MAP estimation problems are formulated. BOCs are encoded in the prior distribution modeled as a Markov random field (MRF). WOCs are encoded in the likelihood distribution modeled as a Gaussian. By these, both BOCs and WOCs are incorporated into the posterior distribution. Experimental results are incorporated into the presentation to illustrate the theoretical formulation.

#### 3.3.1 Formulation of MAP-MRF Estimation

Fig.3.12 illustrates an example of object and scene representation. Let  $\mathcal{O}^{(all)} = \{\mathcal{O}^{(1)}, \dots, \mathcal{O}^{(L)}\}$  be a number of  $L$  model objects ( $L = 8$  in Fig.3.12:Left). The objects in the scene are rotated, translated, scaled, and partially occluded versions of the model objects (Fig.3.12:Middle). The objects and the scene are represented by features (*e.g.* corners on boundary curves in Fig.3.12:Right), and constraints on the feature such as unary properties and binary relations between features (the interested reader is referred to (Li 1997a) for an invariant representation of curved objects). The task is to recognize (separate and identify) the objects in the scene, optimally in the MAP sense, with respect to the  $L$  given objects. This can be done via feature

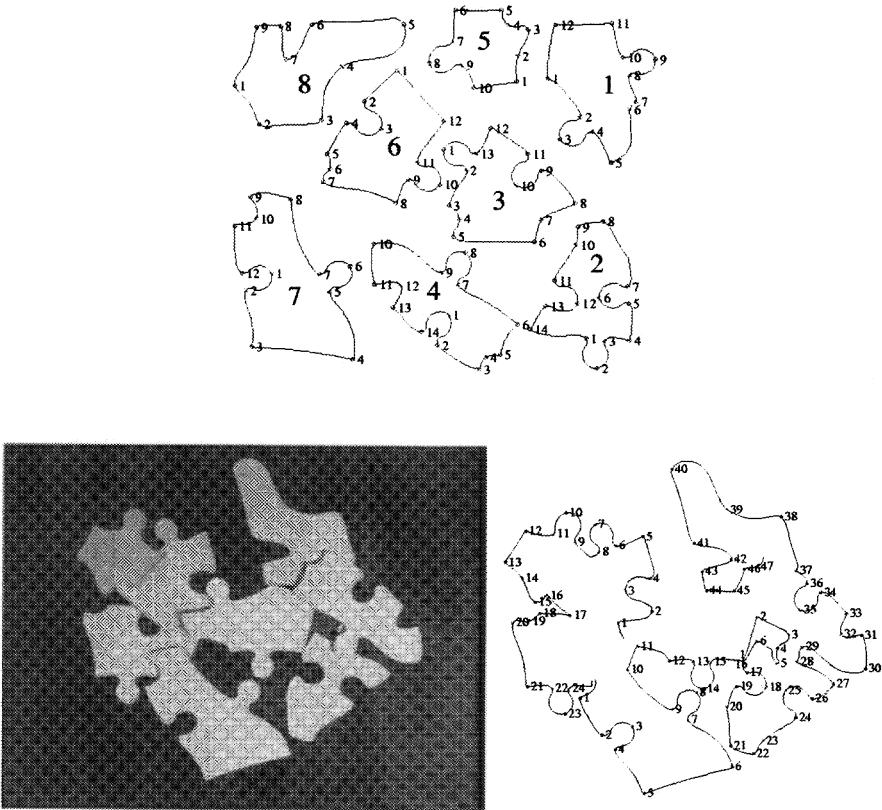


Figure 3.12: Top: Eight model jigsaw objects. Lower-Left: A scene. Lower-Right: Three boundary curves and the corner features extracted from the scene.

matching which is aimed at establishing feature correspondence between the scene and the model objects based on partial observation of the objects.

In this problem, we have  $\mathcal{S} = \{1, \dots, m\}$  corresponds to the set of  $m$  image features. In Fig.3.12:Right, there are three sequences of corner features in the scene, represented by three  $\mathcal{S}$  sets with  $m = 24$ ,  $m = 47$  and  $m = 6$ , respectively.

Let  $\mathcal{L}^{(\alpha)} = \{1, \dots, M^{(\alpha)}\}$  be a set of  $M^{(\alpha)}$  labels corresponding to the set of  $M^{(\alpha)}$  features for model object  $\alpha$  ( $M^{(1)} = 12$  for model object 1 in Fig.3.12:Left). A virtual label, called the NULL and numbered 0, is added to represent everything not belonging to  $\mathcal{L}^{(\alpha)}$  (such as features due to other model objects and those due to background and noise). This augments  $\mathcal{L}^{(\alpha)}$  into  $\mathcal{L}^{(\alpha)+} = \{0, 1, \dots, M^{(\alpha)}\}$ . Without confusion, the notation  $\mathcal{L}$  is still used to denote the augmented set  $\mathcal{L}^+$  unless there is a necessity to elaborate. The

set of all model features plus the NULL is  $\mathcal{L}^{(all)} = \mathcal{L}^{(1)} \cup \mathcal{L}^{(2)} \dots \mathcal{L}^{(L)}$ . It consists of  $\#\mathcal{L}^{(all)}$  elements where  $\#\mathcal{L}^{(all)} = 1 + \sum_{\alpha=1}^L M^{(\alpha)}$ .

The overall matching from  $\mathcal{S}$  to the  $L$  model objects is represented by a label configuration  $f = \{f_1^{(\alpha_1)}, \dots, f_m^{(\alpha_m)}\}$ . It is a mapping from the set of the sites to the set of the labels,  $f : \mathcal{S} \rightarrow \mathcal{L}^{(all)}$ . Three things are told by a label  $f_i^{(\alpha_i)} \in \mathcal{L}^{(\alpha_i)}$ : (i) It separates image features belonging to a model object from those not belonging to any, in the following way: If  $f_i^{(\alpha_i)} \neq 0$  (a non-NULL label), then image feature  $i$  belongs to an object; otherwise if  $f_i^{(\alpha_i)} = 0$ , it belongs to background, noise or an un-modeled object. (ii) If  $f_i^{(\alpha_i)} \neq 0$ ,  $\alpha_i$  indicates that image feature  $i$  belongs to model object  $\alpha_i$ . (iii) If  $f_i^{(\alpha_i)} \neq 0$ ,  $f_i^{(\alpha_i)}$  indexes the corresponding feature of object  $\alpha_i$  that image feature  $i$  is matched to.

The MAP solution for matching the scene to all the objects is defined by

$$f^* = \arg \max_{f \in \mathbb{F}^{(all)}} P(f \mid d, \mathcal{O}^{(all)}) \quad (3.28)$$

where  $P(f \mid d, \mathcal{O}^{(all)})$  is the posterior probability of the labeling  $f$  given the observation  $d$  and the  $L$  object models, and  $\mathbb{F}^{(all)}$  is the space of all admissible configurations (solutions). When all the labels are admissible for all the sites  $\mathbb{F}^{(all)}$  is the Cartesian product of  $m$   $\mathcal{L}^{(all)}$ 's, that is,  $\mathbb{F}^{(all)} = \prod_{i=1}^m \mathcal{L}^{(all)}$ .

Assuming that  $f$ , which is a realization of a family of  $m$  random variables, is a Markov random field (MRF), then its posterior is a Gibbs distribution  $P(f \mid d, \mathcal{O}^{(all)}) \propto e^{-E^{(all)}(f)}$  where

$$E^{(all)}(f) \triangleq U(f \mid \mathcal{O}^{(all)}) + U(d \mid f, \mathcal{O}^{(all)}) \quad (3.29)$$

is the posterior energy consisting of the prior energy  $U(f \mid \mathcal{O}^{(all)})$  and the likelihood energy  $U(d \mid f, \mathcal{O}^{(all)})$  (the derivations of the energies will be given in Section 3). The solution to problem (3.28) equivalently minimizes the posterior energy:  $f^* = \arg \min_{f \in \mathbb{F}^{(all)}} E^{(all)}(f)$ .

The objective of (3.28), finding the minimum  $f^*$  in  $\mathbb{F}^{(all)}$ , is a formidable job since the configuration space  $\mathbb{F}^{(all)}$  consists of a huge number of  $\#\mathbb{F}^{(all)} = (1 + \sum_{\alpha=1}^L M^{(\alpha)})^m$  elements when all the model features (labels) are admissible. In the following, a two stage MAP-MRF estimation approach is proposed to tackle this problem.

### Formulation of Energy Functions

The prior distribution of  $f^{(\alpha)} = \{f_1^{(\alpha)}, \dots, f_m^{(\alpha)}\}$  for matching with respect to object  $\alpha$  is assumed to be an MRF and hence is a Gibbs distribution  $P(f^{(\alpha)} \mid \mathcal{O}^{(\alpha)}) \propto e^{-U(f^{(\alpha)} \mid \mathcal{O}^{(\alpha)})}$ . The prior energy takes the following form

$$U(f^{(\alpha)} \mid \mathcal{O}^{(\alpha)}) = \sum_{i \in \mathcal{S}} V_1(f_i^{(\alpha)}) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i^{(\alpha)}, f_{i'}^{(\alpha)}) \quad (3.30)$$

where  $\mathcal{N}_i$  is the set of neighbors for  $i$ , and  $V_1(f_i^{(\alpha)})$  and  $V_2(f_i^{(\alpha)}, f_{i'}^{(\alpha)})$  are single- and pair-site clique prior potential functions, respectively for  $f^{(\alpha)}$ . The clique potentials are defined based on Equations 3.11 and 3.12 as

$$V_1(f_i^{(\alpha)}) = \begin{cases} 0 & \text{if } f_i \neq 0 \\ v_{10} & \text{if } f_i = 0 \end{cases} \quad (3.31)$$

$$V_2(f_i^{(\alpha)}, f_{i'}^{(\alpha)}) = \begin{cases} 0 & \text{if } f_i^{(\alpha)} \neq 0 \text{ and } f_{i'}^{(\alpha)} \neq 0 \\ v_{20} & \text{if } f_i^{(\alpha)} = 0 \text{ or } f_{i'}^{(\alpha)} = 0 \end{cases} \quad (3.32)$$

where  $v_{10} > 0$  and  $v_{20} > 0$  are penalty constants for NULL labels. These definitions encode BOCs, that is, constraints between different objects and between an object and the background. In a way, this is similar to the line-process model (Geman and Geman 1984) for differentiating edge and non-edge elements.

The likelihood distribution  $p(d \mid f^{(\alpha)}, \mathcal{O}^{(\alpha)})$  describes the statistical properties of the features seen in the scene and is therefore conditioned on pure non-NULL matches ( $f_i^{(\alpha)} \neq 0$ ) only. The likelihood is a Gibbs distribution with the energy function, which is based on Eq.3.15

$$\begin{aligned} U(d \mid f^{(\alpha)}, \mathcal{O}^{(\alpha)}) &= \sum_{i \in \mathcal{S}, f_i^{(\alpha)} \neq 0} V_1(d_1(i) \mid f_i^{(\alpha)}) \\ &\quad \sum_{i \in \mathcal{S}, f_i^{(\alpha)} \neq 0} \sum_{i' \in \mathcal{S} \setminus i, f_{i'}^{(\alpha)} \neq 0} V_2(d_2(i, i') \mid f_i^{(\alpha)}, f_{i'}^{(\alpha)}) \end{aligned} \quad (3.33)$$

where  $d_1(i)$  is the set of unary properties of image feature  $i$ ,  $d_2(i, i')$  is the set of binary relations between  $i$  and  $i'$ , and  $V_1(d_1(i) \mid f_i^{(\alpha)})$  and  $V_2(d_2(i, i') \mid f_i^{(\alpha)}, f_{i'}^{(\alpha)})$  are the potentials in the likelihood distributions (where the distributions may be assumed to be Gaussian). The likelihood potentials encode WOCs, that is, constraints on image features belonging to object  $\alpha$  only. Both BOCs and WOCs are incorporated into the posterior distribution with the posterior energy  $E^{(\alpha)}(f^{(\alpha)}) = U(f^{(\alpha)} \mid \mathcal{O}^{(\alpha)}) + U(d \mid f^{(\alpha)}, \mathcal{O}^{(\alpha)})$ .

The posterior in stage 2 is the target posterior distribution  $P(f \mid d, \mathcal{O}^{(all)})$  of the original problem (1), with the posterior energy  $U(f \mid d, \mathcal{O}^{(all)}) = U(f \mid \mathcal{O}^{(all)}) + U(d \mid f, \mathcal{O}^{(all)})$ . In this stage, if  $f_i$  is non-NULL, then it is associated with a model  $\alpha$ , so it should be read as  $f_i^{(\alpha)}$  (and  $f_{i'}$  as  $f_{i'}^{(\alpha')}$ ). In the following derivation, it is assumed that model objects  $\alpha$  and  $\alpha'$  are independent to each other when  $\alpha \neq \alpha'$ .

The prior energy is

$$U(f \mid \mathcal{O}^{(all)}) = \sum_{i \in \mathcal{S}} V_1(f_i \mid \mathcal{O}^{(all)}) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'} \mid \mathcal{O}^{(all)}) \quad (3.34)$$

The single-site prior potentials are defined as  $V_1(f_i \mid \mathcal{O}^{(all)}) = V_1(f_i^{(\alpha)})$  which is the same as that in Eq.(3.11) for matching to a single model object  $\alpha$ . The pair-site potential  $V_2(f_i, f_{i'} \mid \mathcal{O}^{(all)}) = V_2(f_i^{(\alpha)}, f_{i'}^{(\alpha')} \mid \mathcal{O}^{(\alpha)}, \mathcal{O}^{(\alpha')})$  where

$$V_2(f_i^{(\alpha)}, f_{i'}^{(\alpha')} \mid \mathcal{O}^{(\alpha)}, \mathcal{O}^{(\alpha')}) = \begin{cases} V_2(f_i^{(\alpha)}, f_{i'}^{(\alpha)}) & \text{if } \alpha = \alpha' \\ v_{20} & \text{otherwise} \end{cases} \quad (3.35)$$

where  $V_2(f_i^{(\alpha)}, f_{i'}^{(\alpha)})$  are as defined in (3.12) for matching to object  $\alpha$ . Substituting Eq.(3.12) into Eq.(3.35) yields

$$V_2(f_i, f_{i'} \mid \mathcal{O}^{(all)}) = \begin{cases} 0 & \text{if } (\alpha = \alpha') \text{ and } (f_i \neq 0) \text{ and } (f_{i'} \neq 0) \\ v_{20} & \text{otherwise} \end{cases} \quad (3.36)$$

where  $f_i$  is associated with object  $\alpha$  and  $f_{i'}$  with object  $\alpha'$ . The above definitions are an extension of Eq.(3.12) for dealing with multiple objects: In Eq.(3.12), features due to other objects (not belonging to object  $\alpha$ ) are all labeled as NULL ; Eq.(3.36) simply takes this into consideration.

The likelihood energy is

$$\begin{aligned} U(d \mid f, \mathcal{O}^{(all)}) = & \sum_{i \in \mathcal{S}, f_i \neq 0} V_1(d_1(i) \mid f_i, \mathcal{O}^{(all)}) + \\ & \sum_{i \in \mathcal{S}, f_i \neq 0} \sum_{i' \in \mathcal{S} \setminus i, f_{i'} \neq 0} V_2(d_2(i, i') \mid f_i, f_{i'}, \mathcal{O}^{(all)}) \end{aligned} \quad (3.37)$$

The single-site likelihood potentials are  $V_1(d_1(i) \mid f_i, \mathcal{O}^{(all)}) = V_1(d_1(i) \mid f_i^{(\alpha)})$  defined in the same way as for (3.15). The pair-site likelihood potentials are  $V_2(d_2(i, i') \mid f_i, f_{i'}, \mathcal{O}^{(all)}) = V_2(d_2(i, i') \mid f_i^{(\alpha)}, f_{i'}^{(\alpha')}, \mathcal{O}^{(\alpha)}, \mathcal{O}^{(\alpha')})$  where

$$V_2(d_2(i, i') \mid f_i^{(\alpha)}, f_{i'}^{(\alpha')}, \mathcal{O}^{(\alpha)}, \mathcal{O}^{(\alpha')}) = \begin{cases} V_2(d_2(i, i') \mid f_i^{(\alpha)}, f_{i'}^{(\alpha)}) & \text{if } \alpha = \alpha' \\ 0 & \text{otherwise} \end{cases} \quad (3.38)$$

where  $V_2(d_2(i, i') \mid f_i^{(\alpha)}, f_{i'}^{(\alpha)})$  are defined in the same way as for (3.15) when for matching to object  $\alpha$ .

Some parameters have to be determined, such as  $v_{10}$  and  $v_{20}$  in the MRF prior, in order to define the MAP solutions completely. They may be estimated by using a supervised learning algorithm (see Chapter 7).

### 3.3.2 Computational Issues

#### Finding Solution in Two-Stage

Stage 1 solves  $L$  sub-problems,  $f^{(\alpha)} = \arg \max_{f \in \mathbb{F}^{(\alpha)}} P(f \mid d, \mathcal{O}^{(\alpha)})$  for  $\alpha = 1, \dots, L$ , resulting in  $L$  MAP solutions  $\{f^{(1)}, \dots, f^{(L)}\}$ . Then, a reduced configuration space is constructed from the  $L$  solutions. In stage 2, the

solution of (3.28) with respect to all the  $L$  objects is found in the reduced space.

Stage 1 matches the scene to each of the  $L$  objects, individually (which can be done in parallel for all model objects). Let  $P(f \mid d, \mathcal{O}^{(\alpha)}) \propto e^{-E^{(\alpha)}(f)}$  be the posterior distribution of  $f$  for matching the scene to object  $\alpha$  ( $1 \leq \alpha \leq L$ ). The MAP-MRF solution for this is  $f^{(\alpha)} = \{f_1^{(\alpha)}, \dots, f_m^{(\alpha)}\} = \arg \min_{f \in \mathbb{F}^{(\alpha)}} E^{(\alpha)}(f)$  where  $f_i^{(\alpha)}$  denotes the corresponding model feature in object  $\alpha$ . The configuration space  $\mathbb{F}^{(\alpha)}$  for object  $\alpha$  consists of only  $\#\mathbb{F}^{(\alpha)} = (1 + M^{(\alpha)})^m$  elements. For the  $L$  objects, the total size is  $\sum_1^L (1 + M^{(\alpha)})^m$ , much smaller than  $\#\mathbb{F}^{(all)}$  which is  $(1 + \sum_{\alpha=1}^L M^{(\alpha)})^m$ .

Two things are told in  $f^{(\alpha)}$ : First, it separates image features belonging to the object  $\alpha$  from the other image features in the following way: If image feature  $i$  belongs to the object  $\alpha$ , then  $f_i^{(\alpha)} \neq 0$  (a non-NULL label); otherwise,  $f_i^{(\alpha)} = 0$ . Second, if  $f_i^{(\alpha)} \neq 0$ ,  $f_i^{(\alpha)}$  is the model feature that image feature  $i$  is matched to.

$f^{(\alpha)}$  is optimal with respect to model object  $\alpha$  but not to another one, so inconsistencies may exist among the  $L$  MAP solutions: A feature  $i \in \mathcal{S}$  in the scene may have been matched to more than one model features belonging to different objects; that is, there may exist more than one  $\alpha \in \{1, \dots, L\}$  for which  $f_i^{(\alpha)} \neq 0$ . For example, in Fig.3.13, model instances found by  $f^{(2)}$ ,  $f^{(3)}$  and  $f^{(7)}$  compete for a common part of the scene, which is mostly due to the common structures, such as the round extrusions and intrusions, of the matched objects. (Fig.3.13 also shows that an MAP solution allows multiple instances of a model object: e.g.  $f^{(7)}$  contains two instances of model object 7.)

Stage 2 solves the original MAP problem of (3.28) with respect to all the  $L$  objects, in a reduced solution space constructed from the stage 1 solutions  $\{f^{(1)}, \dots, f^{(L)}\}$  (see next subsection for the construction of the reduced space). This also resolves possible inconsistencies among the  $L$  MAP solutions because only one label  $f_i$  is assigned to  $i$  in the overall solution  $f$ . Fig.3.14 shows the overall optimal result, which is the output of stage 2, for the MAP recognition of the scene with respect to all the models. It is consistent by itself.

## Reduced Solution Space

Consider an illustration in Table 3.1, where a scene with  $m = 12$  features is matched to  $L = 5$  model objects, resulting in five MAP solutions  $f^{(\alpha)}$  ( $\alpha = 1, \dots, 5$ ). Let  $\mathcal{S}' \subset \mathcal{S}$  be the set of sites which according to the stage 1 solution have been matched to more than one non-NULL labels,  $\mathcal{S}' = \{i \in \mathcal{S} \mid f_i^{(\alpha)} \neq 0 \text{ for more than one } \alpha\}$ . For example,  $\mathcal{S}' = \{8, 9, 10, 11\}$  for the  $f^{(\alpha)}$ 's in Table 3.1. We can derive the reduced set of admissible labels for  $i$ , denoted by  $\mathcal{L}_i^{(all)}$ , from the  $f^{(\alpha)}$ 's as follows:

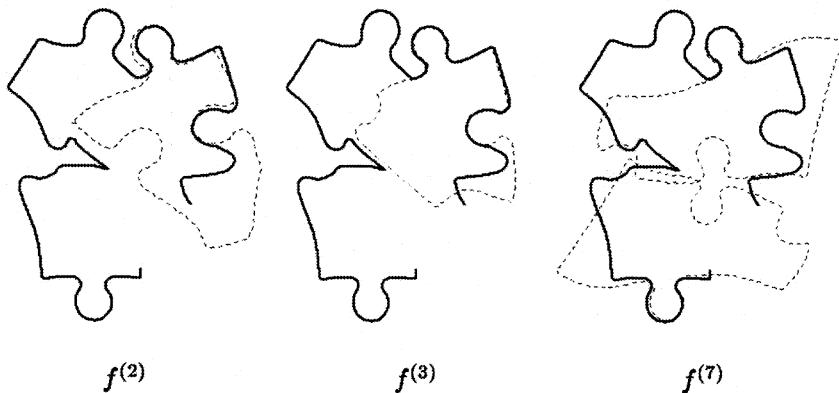


Figure 3.13:  $f^{(2)}$ ,  $f^{(3)}$  and  $f^{(7)}$  compete for a common part of the scene.

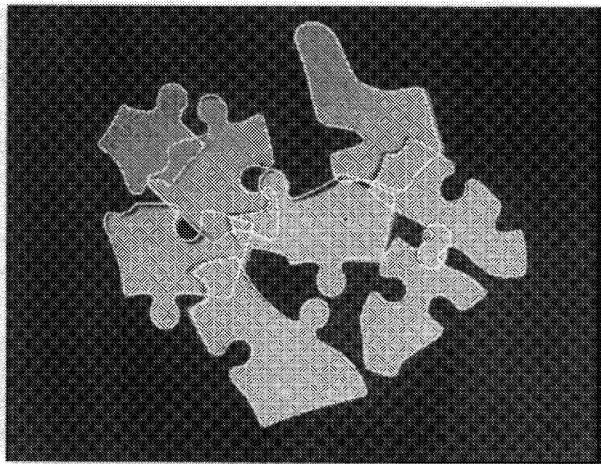


Figure 3.14: The overall matching and recognition result.

- For  $i \in \mathcal{S}'$ ,  $\mathcal{L}_i^{(all)}$  consists of all the non-NULL labels assigned to  $i$  by the  $f^{(\alpha)}$ 's, plus the NULL label; that is  $\mathcal{L}_i^{(all)} = 0 \cup \{f_i^{(\alpha)} \neq 0 \mid \alpha = 1, \dots, L\}$ . For the case in Table 3.1, for example,  $\mathcal{L}_8^{(all)} = \{0, 10^{(2)}, 1^{(4)}, 7^{(5)}\}$ ,  $\mathcal{L}_9^{(all)} = \{0, 9^{(2)}, 6^{(5)}\}$ ,  $\mathcal{L}_{10}^{(all)} = \{0, 7^{(2)}, 3^{(3)}, 5^{(5)}\}$ ,  $\mathcal{L}_9^{(all)} = \{0, 4^{(3)}, 4^{(5)}\}$ .
- For  $i \notin \mathcal{S}'$ ,  $\mathcal{L}_i^{(all)}$  consists in a unique label; e.g.  $\mathcal{L}_6^{(all)} = \{3^{(4)}\}$ , and  $\mathcal{L}_3^{(all)} = \{0\}$ .

$i =$	1	2	3	4	5	6	7	8	9	10	11	12
$f^{(1)}$	0	0	0	0	0	0	0	0	0	0	0	0
$f^{(2)}$	0	0	0	0	0	0	0	10^{(2)}	9^{(2)}	7^{(2)}	0	0
$f^{(3)}$	0	0	0	0	0	0	0	0	0	3^{(3)}	4^{(3)}	0
$f^{(4)}$	0	0	0	5^{(4)}	4^{(4)}	3^{(4)}	2^{(4)}	1^{(4)}	0	0	0	0
$f^{(5)}$	0	0	0	0	0	0	0	7^{(5)}	6^{(5)}	5^{(5)}	4^{(5)}	3^{(5)}
$f^*$	0	0	0	5^{(4)}	4^{(4)}	3^{(4)}	2^{(4)}	7^{(5)}	6^{(5)}	5^{(5)}	4^{(5)}	3^{(5)}

Table 3.1: Matching and recognition of an image containing  $m = 12$  features to  $L = 5$  objects.

Object  $\alpha$  contributes one or no label to  $\mathcal{L}_i^{(all)}$ , as opposed to  $M^{(\alpha)}$  labels before the reduction. For  $i \in S'$ , the size of  $\mathcal{L}_i^{(all)}$  is at most  $L + 1$ ; for  $i \notin S'$ , the size of  $\mathcal{L}_i^{(all)}$  is one; whereas before the reduction, the size is  $1 + \sum_{\alpha=1}^L M^{(\alpha)}$  for every  $i$ .

The reduced space is constructed as  $\mathbb{F}_{reduced}^{(all)} = \mathcal{L}_1^{(all)} \times \mathcal{L}_2^{(all)} \times \cdots \times \mathcal{L}_m^{(all)}$  where  $\times$  is the Cartesian product of sets. Its size is much reduced. For the case in Table 3.1, the previous size of the raw solution space  $\#\mathbb{F}^{(all)} = (\sum_{\alpha=1}^5 M^{(\alpha)} + 1)^{12}$  configurations (e.g. 31384283770 for  $M^{(\alpha)} = 10$ ) is reduced to  $4 \times 3 \times 4 \times 3 = 144$ . It is so small that an exhaustive search is affordable.

Stage 2 performs the target minimization in  $\mathbb{F}_{reduced}^{(all)}$ . In an iterative search algorithm, only those labels on the sites  $i \in S'$  are subject to changes whereas those not in  $S'$  are fixed. This is equivalent to maximizing the conditional posterior:  $f_{S'}^* = \arg \max_{f_{S'} \in \mathbb{F}_{S'}^{(all)}} P(f_{S'} \mid d, f_{S-S'}, \mathcal{O}^{(all)})$  where  $f_{S'} = \{f_i \mid i \in S'\}$  is the set of labels to be updated,  $f_{S-S'} = \{f_i \mid i \in S - S'\}$  is the set of labels that are fixed during the maximization, and  $\mathbb{F}_{S'}^{(all)} = \prod_{i \in S'} \mathcal{L}_i^{(all)}$ .

A crucial question for the validity of the two-stage approach is whether the solution of Eq.(3.28) is contained in the reduced solution space  $\mathbb{F}_{reduced}^{(all)}$ . The necessary and sufficient condition is that  $f^{(\alpha)}$  contains correct matches for object  $\alpha$  (it is allowed to also contain spurious matches). Now that the global solution can be found in  $\mathbb{F}_{reduced}^{(all)}$  (e.g. by an exhaustive search), this means that the two-stage strategy can find the global solution of Eq.(3.28) if  $\mathcal{L}_i^{(\alpha)}$  derived from  $f^{(\alpha)}$  contains the correct matching components for the original problem.

The optimization in MAP matching and recognition is combinatorial. While an optimum is sought in a global sense, many optimization algorithms are based on local optimization. The Hummel-Zucker relaxation labeling algorithm (Hummel and Zucker 1983) is preferable in terms of the minimized energy value and computational costs and is used in the implementation. It converges after dozens of iterations. The computational time is dominated

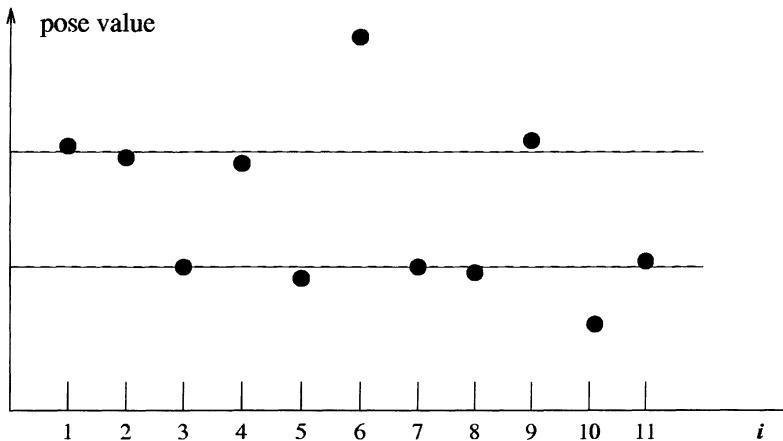


Figure 3.15: Pose clusters in one dimensional parameter space. Poses  $f_1$ ,  $f_2$ ,  $f_4$  and  $f_9$  due to pairs 1, 2, 4 and 9 agree to a transformation and poses  $f_3$ ,  $f_5$ ,  $f_7$ ,  $f_8$  and  $f_{11}$  agree to another. Poses  $f_6$  and  $f_{10}$  forms isolated points so that pair 6 and pair 10 are outliers.

by relaxation labeling in the first stage and is roughly the complexity of the whole system.

## 3.4 Pose Computation

Pose computation aims to estimate the transformation needed to map an object model from the model coordinate system into the sensory data (Ayache and Faugeras 1986; Faugeras and Hebert 1986; Bolles and Horaud 1986; Stockman 1987; Haralick et al. 1989; Grimson 1990; Umeyama 1991). In this section, we derive two MRF models for pose estimation. The first is a model for pose clustering from corresponding point data containing multiple poses and outliers. The second model attempts to solve 3D matching and pose simultaneously from a 2D image without using view-invariants.

### 3.4.1 Pose Clustering and Estimation

The problem of the pose clustering is stated as follows: Let a set of corresponding points be given as the data,  $d = \{(p_i, P_i) \mid i \in \mathcal{S}\}$ , where  $p_i$ 's are the *model* features,  $P_i$  are the *scene* features<sup>4</sup> and  $\mathcal{S} = \{1, \dots, m\}$  indexes the set of the matched pairs. Let  $f_i$  be the geometric transformation

<sup>4</sup>Note that in this section, the upper case notations are for models and the lower case notations for the scene.

from  $p_i$  to  $P_i$  and consider the set  $f = \{f_1, \dots, f_m\}$  as a configuration of the “pose field”. In the case of noiseless, perfect correspondences, the following  $m$  equations, each transforming a model feature to a scene feature, should hold simultaneously

$$P_i = f_i(p_i) \quad i \in \mathcal{S} \quad (3.39)$$

We want to find the optimal pose configuration in the MAP sense, *i.e.*  $f^* = \arg \min_f U(f \mid d)$ .

Assume that each  $f_i$ ’s is confined to a certain class  $\mathcal{L}$  of pose transformations such that the admissible pose space is  $\mathbb{F} = \mathcal{L}^m$ . This imposes constraints on the parameters governing  $f_i$ . The needed number of transformation parameters (degree of freedom) depends on the class of transformations and the adopted representation for the pose transformation. In the case of the 3D-3D Euclidean transformation, for example, it can consist of an orthogonal rotation  $O_i$  followed by a translation  $T_i$ , *i.e.*  $f_i = (O_i, T_i)$ ; the relation between the corresponding points is  $P_i = f_i(p_i) = O_i p_i + T_i$ . The simple matrix representation needs 12 parameters: 9 elements in the rotation matrix  $O_i$  plus 3 elements in the translation vector  $T_i$ . The rotation angle representation needs six parameters: 3 for the three rotation angles and 3 for the translation. Quaternions provide still another choice. A single pair  $(p_i, P_i)$  alone is usually insufficient to determine a pose transformation  $f_i$ ; more are needed for the pose to be fully determined.

If all the pairs in the data,  $d$ , are inliers and are due to a single transformation, then all  $f_i$ ,  $i \in \mathcal{S}$ , which are points in the pose space, must be close to each other; and the errors  $\|P_i - f_i(p_i)\|$ , where  $\|\cdot\|$  is the Euclidean distance, must all be small. Complication increases when there are multiple pose clusters and outlier pairs. When there are multiple poses,  $f_i$ ’s should form distinct clusters. In this case, the set  $f$  is divided into subset, each giving a consistent pose transformation from a partition of  $\{p_i\}$  to a partition of  $\{P_i\}$ . Fig.3.15 illustrates a case in which there are two pose clusters and some outliers. Outlier pairs, if contained in the data, should be excluded from the pose estimation because they can cause large errors. Multiple pose identification with outlier detection has a close affinity to the prototypical problem of image restoration involving discontinuities (Geman and Geman 1984) and to that of matching overlapping objects using data containing spurious features (Li 1994a).

Now we derive the MAP-MRF formulation. The neighborhood system is defined by

$$\mathcal{N}_i = \{i' \in \mathcal{S} \mid [\text{dist}(p_i, p_{i'})]^2 \leq r, i' \neq i\} \quad (3.40)$$

where  $\text{dist}(A, B)$  is some suitably defined measure of distance between model features and the scope  $r$  may be reasonably related to the size of the largest model object. We consider cliques of up to order two and so the clique set  $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$  where  $\mathcal{C}_1 = \{\{i\} \mid i \in \mathcal{S}\}$  is the set of single-site (first order) cliques and  $\mathcal{C}_2 = \{\{i, i'\} \mid i' \in \mathcal{N}_i, i \in \mathcal{S}\}$  pair-site (second order) cliques.

Under a single pose transformation, nearby model features are likely to appear together in the scene whereas distantly apart model features tend less likely. This is the coherence of spatial features. We characterize this using the Markovianity condition  $p(f_i | f_{\mathcal{S} - \{i\}}) = P(f_i | f_{\mathcal{N}_i})$ . The positivity condition  $P(f) > 0$  also hold for all  $f \in \mathbb{F}$  where  $BbbF$  is the set of admissible transformations.

The MRF configuration  $f$  follows a Gibbs distribution. The two-site potentials determine interactions between the individual  $f_i$ 's. They may be defined as

$$V_2(f_i, f_{i'}) = g(\|f_i - f_{i'}\|) \quad (3.41)$$

where  $\|\cdot\|$  is a norm in the pose space and  $g(\cdot)$  is some function. To be able to separate different pose clusters, the function  $g(\cdot)$  should stop increasing as  $\|f_i - f_{i'}\|$  becomes very large. A choice is

$$g_{\alpha, T}(\eta) = \min\{\eta^2, \alpha\} \quad (3.42)$$

where  $\alpha > 0$  is a threshold parameter; this is the same as that used in the line process model for image restoration with discontinuities (Geman and Geman 1984; Marroquin et al. 1987; Blake and Zisserman 1987). It may be any APF defined in (4.28). Its value reflects the cost associated with the pair of pose labels,  $f_i$  and  $f_{i'}$ , and will be large when  $f_i$  and  $f_{i'}$  belong to different clusters. But it cannot be arbitrarily large since a large value, as might be given by a quadratic  $g$ , tends to force the  $f_i$  and  $f_{i'}$  to stay in one cluster, as the result of energy minimization, even when they should not. Using an APF imposes piecewise smoothness.

The single-site potentials  $V_1(f_i)$  may be used to force  $f_i$  to stay in the admissible set  $\mathcal{L}$  if such a force is needed. For example, assume  $f_i = (O_i, T_i)$  is a 2D-2D Euclidean transformation. Then, the rotation matrix  $O_i = [o_{i,r,s} \mid r, s = 1, 2]$  must be orthogonal. The unary potential for the orthogonality constraint can be expressed by  $(o_{i,1,1}o_{i,2,1} + o_{i,1,2}o_{i,2,2})^2 + (o_{i,1,1}o_{i,1,2} + o_{i,2,1}o_{i,2,2})^2$ . It has the value of zero only when  $O_i$  is orthogonal. If no scale change is allowed, then the scaling factor should be exactly one, and an additional term  $[\det(O_i) - 1]^2$  can be added where  $\det(O_i)$  is the determinant. Adding these two gives the single-site potential as

$$\begin{aligned} V_1(f_i) = & a [(o_{i,1,1}o_{i,2,1} + o_{i,1,2}o_{i,2,2})^2 + (o_{i,1,1}o_{i,1,2} + o_{i,2,1}o_{i,2,2})^2] \\ & + b[\det(O_i) - 1]^2 \end{aligned} \quad (3.43)$$

where  $a$  and  $b$  are the weighting factors. In this case,  $V_1$  imposes the orthogonality. It is also possible to define  $V_1(f_i)$  for other classes of transformations. Summing all prior clique potentials yields the following prior energy

$$U(f) = \sum_{i \in \mathcal{S}} V_1(f_i) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}) \quad (3.44)$$

The above defines the prior distribution  $P(f)$ .

The likelihood function is derived below. Assume that the features are point locations and that they are subject to the additive noise model,  $P_i = f_i(p_i) + e_i$ , where  $e_i \sim N(0, \sigma^2)$  is a vector of i.i.d. Gaussian noise. Then the distribution of the data  $d$  conditional on the configuration  $f$  is

$$P(d | f) \propto e^{-U(d | f)} \quad (3.45)$$

where the likelihood energy is

$$U(d | f) = \sum_{i \in \mathcal{S}} \|f_i(p_i) - P_i\|^2 / [2\sigma^2] \quad (3.46)$$

The location  $f_i(p_i)$  is the conditional “mean” of the random variable  $P_i$ . The quantity,  $f_i(p_i) - P_i$ , reflects the error between the location  $f_i(p_i)$  predicated by  $f_i$  and the actual location  $P_i$ .

After that, the posterior energy follows immediately as  $U(f | d) = U(f) + U(d | f)$ . The optimal solution is  $f^* = \arg \min_f U(f | d)$ . As the result of energy minimization, inlier pairs undergo the same pose transformation will form a cluster whereas outlier pairs will form isolated points in the pose space, as illustrated in Fig.3.15.

### 3.4.2 Simultaneous Matching and Pose

In the previous pose estimation formulation, a set of matched pairs is assumed available. Here we assume the situation in which the matching has not been done and pose has to be performed during the matching. Pose estimation during matching is practiced when invariants are unavailable or difficult to compute, *e.g.* due to that the class of transformations is not linear or involves projections. In the following, an MRF model for simultaneous 3D-from-2D matching and pose is derived without using view-invariants. Matching and pose are jointly sought as in (Wells III 1991). The formulation is an extension to that given in Section 3.2.

Let  $\mathcal{S} = \{1, \dots, m\}$  index a set of  $m$  points on a 3D *model* object,  $\{p_i | i \in \mathcal{S}\}$ . Let  $\mathcal{L} = \{1, \dots, M\}$  be the label set indexing a set of  $M$  *scene* points in 2D,  $\{P_I | I \in \mathcal{L}\}$ , and  $\mathcal{L}^+ = \{0\} \cup \mathcal{L}$  be the augmented set with 0 representing the NULL label. Let  $f = \{f_1, \dots, f_m\}$ ,  $f_i \in \mathcal{L}^+$ , denote the matching from the  $\{p_i\}$  to the  $\{P_I\} \cup \text{NULL}$ . When  $i$  is assigned the virtual point 0,  $f_i = 0$ , it means that there is no corresponding point found in the physically existing point set  $\mathcal{L}$ . Let  $\mathcal{T}$  be the *projective* pose transformation from the 3D model points  $p_i$  to the matched 2D image points  $P_{f_i}$ , ( $f_i \neq 0$ ). We have  $P_{f_i} = \mathcal{T}(p_i)$ , for all  $i$  for which  $f_i \neq 0$ , under an exact pose.

Now we derive the MAP-MRF formulation. The neighborhood system is defined by

$$\mathcal{N}_i = \{i' \in \mathcal{S} | \|p_i - p_{i'}\|^2 \leq r, i' \neq i\} \quad (3.47)$$

The single-site potential is an extension to (3.11) as

$$V_1(f_i, \mathcal{T}) = \begin{cases} v_{10} & \text{if } f_i = 0 \\ v(\mathcal{T}(p_i), P_{f_i}) & \text{otherwise} \end{cases} \quad (3.48)$$

where  $v_{10}$  is a constant. The function  $v(\mathcal{T}(p_i), P_{f_i})$  encodes the prior knowledge about  $\mathcal{T}$ . It may include prior terms, such as  $V_1$  in the previous subsection for the admissibility of pose transformations. If the p.d.f. of the pose is known, *e.g.* to be a normal distribution centered at a known mean pose<sup>5</sup>, then  $v(\mathcal{T}(p_i), P_{f_i})$  is a multi-variate Gaussian function. The pair-site potential is defined the same as (3.12)

$$V_2(f_i, f_{i'}, \mathcal{T}) = \begin{cases} v_{20} & \text{if } f_i = 0 \text{ or } f_{i'} = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.49)$$

where  $v_{20}$  is a constant.

The likelihood function characterizes the distribution of the errors and relates to the observation model and the noise in it. Given  $f_i \neq 0$ , the model point  $p_i = (x_i, y_i, z_i)$  is projected to a point  $\mathcal{T}(p_i) \stackrel{\Delta}{=} \hat{P}_{f_i} = (\hat{X}_i, \hat{Y}_i)$  by the projective transformation  $\mathcal{T}$ . In the inexact situation,  $\hat{P}_{f_i} \neq P_{f_i}$  where  $P_{f_i} = (X_i, Y_i) \stackrel{\Delta}{=} d_1(i)$  is the corresponding image point actually observed.

Assume the following additive noise model

$$P_{f_i} = \mathcal{T}(p_i) + e_i = \hat{P}_{f_i} + e_i \quad (3.50)$$

where  $e_i \sim N(0, \sigma^2)$  is a vector of i.i.d. Gaussian noise. Then the likelihood function is

$$p(d_1(i) | f_i, \mathcal{T}) = \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^2 e^{-V_1(d_1(i) | f_i, \mathcal{T})} \quad (3.51)$$

where

$$V_1(d_1(i) | f_i, \mathcal{T}) = [(X_i - \hat{X}_i)^2 + (Y_i - \hat{Y}_i)^2] / [2\sigma^2] \quad (3.52)$$

is the unary likelihood potential. The joint likelihood is then  $p(d_1 | f, \mathcal{T}) = \prod_i p(d_1(i) | f_i, \mathcal{T})$  where  $d_1$  denotes the set of unary properties.

We also make use of the distances as an additional binary constraint. The distance,  $\|p_i - p_{i'}\|$ , between the two model points in 3D is projected to the distance

$$d_2(i, i') = \|\hat{P}_{f_i} - \hat{P}_{f_{i'}}\| = \sqrt{(X_i - X_{i'})^2 + (Y_i - Y_{i'})^2} \quad (3.53)$$

in 2D. Its p.d.f. can be derived, based on the distribution of the projected points given in (3.50), in the following way. Let  $X = (X_i, Y_i, X_{i'}, Y_{i'})$ . These random variables are assumed independent, so their joint conditional p.d.f. is

$$p(X_i, Y_i, X_{i'}, Y_{i'} | f_i, f_{i'}, \mathcal{T}) = \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^4 e^{-[(X_i - \hat{X}_i)^2 + (Y_i - \hat{Y}_i)^2 + (X_{i'} - \hat{X}_{i'})^2 + (Y_{i'} - \hat{Y}_{i'})^2] / [2\sigma^2]} \quad (3.54)$$

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<sup>5</sup>This is assumed in (Wells III 1991).

Introduce new random variables,  $Z(X) = (Z_1, Z_2, Z_3, Z_4)$ , as

$$\begin{aligned} Z_1 &= \sqrt{(X_i - X_{i'})^2 + (Y_i - Y_{i'})^2} \\ Z_2 &= Y_i \\ Z_3 &= X_{i'} \\ Z_4 &= Y_{i'} \end{aligned} \quad (3.55)$$

each of which is a function of the  $X$  variables. Note that we are deriving the p.d.f. of  $Z_1$ . The inverse of  $Z(X)$ , denoted by  $X = X(Z)$ , is determined by

$$\begin{aligned} X_i &= \sqrt{Z_1^2 - (Z_2 - Z_4)^2} + Z_3 \\ Y_i &= Z_2 \\ X_{i'} &= Z_3 \\ Y_{i'} &= Z_4 \end{aligned} \quad (3.56)$$

The Jacobian of the inverse is defined to be the determinant

$$J = \det [\nabla Z(X)] = \frac{Z_1}{\sqrt{Z_1^2 - (Z_2 - Z_4)^2}} \quad (3.57)$$

which is a function of the  $Z$  variables. The joint conditional p.d.f.  $p_Z(Z)$  for  $Z$  can be derived from the joint p.d.f. (3.54) using the following relation (Grimmett 1982)

$$p_Z(Z \mid f_i, f_{i'}, \mathcal{T}) = p_X(X(Z) \mid f_i, f_{i'}, \mathcal{T}) \times |J| \quad (3.58)$$

The conditional distribution of  $Z_1 = d_2(i, i')$  is then the conditional marginal

$$\begin{aligned} p(d_2(i, i') \mid f_i, f_{i'}, \mathcal{T}) &= p_{Z_1}(Z_1 \mid f_i, f_{i'}, \mathcal{T}) = \\ &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_Z(Z_1, Z_2, Z_3, Z_4 \mid f_i, f_{i'}, \mathcal{T}) \, dZ_2 \, dZ_3 \, dZ_4 \end{aligned} \quad (3.59)$$

which is a function of  $X_i, Y_i, X_{i'}, Y_{i'}$ . This gives the binary likelihood potential  $V(d_2(i, i') \mid f_i, f_{i'}, \mathcal{T})$ . The joint p.d.f. of the set of binary features,  $d_2$ , is approximated by the “pseudo-likelihood”

$$p(d_2 \mid f, \mathcal{T}) = \prod_{i \in \mathcal{S}} \prod_{i' \in \mathcal{N}_i} p(d_2(i, i') \mid f_i, f_{i'}, \mathcal{T}) \quad (3.60)$$

The joint p.d.f. of  $d = \{d_1, d_2\}$  is approximated by

$$p(d \mid f, \mathcal{T}) = p(d_1 \mid f, \mathcal{T}) p(d_2 \mid f, \mathcal{T}) \quad (3.61)$$

Now the posterior energy can be obtained as

$$\begin{aligned} U(f, \mathcal{T} \mid d) = & \sum_{i \in \mathcal{S}} V_1(f_i, \mathcal{T}) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}, \mathcal{T}) + \\ & \sum_{i \in \mathcal{S}: f_i \neq 0} V_1(d_1(i) \mid f_i, \mathcal{T}) + \\ & \sum_{i \in \mathcal{S}: f_i \neq 0} \sum_{i' \in \mathcal{S}: f_{i'} \neq 0} V_2(d_2(i, i') \mid f_i, f_{i'}, \mathcal{T}) \end{aligned} \quad (3.62)$$

The optimal solution is  $(f^*, \mathcal{T}^*) = \arg \min_{(f, \mathcal{T})} U(f, \mathcal{T} \mid d)$ . The non-NUL labels in  $f^*$  represents the matching from the considered model object to the scene and  $\mathcal{T}^*$  determines the pose transformation therein. The model points which are assigned the NULL label are either spurious or due to other model objects. Another round matching-pose operation may be formed on these remaining point in terms of another model object.

### 3.4.3 Discussion

Minimizing the energies derived in this section is difficult. The dimensionality of the search space is high. As a guide to the search, the energies are inefficient unless some strong prior constraints are available, such as the normal prior distribution of poses assumed in (Wells III 1991). However, the derived models may be useful for the verification of matching and pose. Assume that pose candidates are found by using some techniques such as Hough transform or geometric indexing. The energies may be used as global cost measures for the matching and pose.

# Chapter 4

## Discontinuities in MRFs

Smoothness is a generic assumption underlying a wide range of physical phenomena. It characterizes the coherence and homogeneity of matter within a scope of space (or an interval of time). It is one of the most common assumptions in computer vision models, in particular, those formulated in terms of Markov random fields (MRFs) (Geman and Geman 1984; Elliott et al. 1984; Marroquin 1985) and regularization (Poggio et al. 1985). Its applications are seen widely in image restoration, surface reconstruction, optical flow and motion, shape from X, texture, edge detection, region segmentation, visual integration and so on.

The assumption of the uniform smoothness implies the smoothness *everywhere*. However, improper imposition of it can lead to undesirable, over-smoothed, solutions. This occurs when the uniform smoothness is violated, for example, at discontinuities where abrupt changes occur. It is necessary to take care of discontinuities when using smoothness priors. Therefore, how to apply the smoothness constraint while preserving discontinuities has been one of the most active research areas in image processing and low level vision (see, *e.g.* (Blake 1983; Terzopoulos 1983b; Geman and Geman 1984; Grimson and Pavlidis 1985; Marroquin 1985; Mumford and Shah 1985; Terzopoulos 1986b; Blake and Zisserman 1987; Lee and Pavlidis 1987; Koch 1988; Leclerc 1989; Shulman and Herve 1989; Li 1990b; Nordstrom 1990; Geiger and Girosi 1991; Li 1991)).

This chapter<sup>1</sup> presents a systematic study on smoothness priors involving discontinuities. Through an analysis of the *Euler equation* associated with the energy minimization in MRF and regularization models, it is identified that the fundamental difference among different models for dealing with discontinuities lies in their ways of controlling the *interaction* between neighboring points. Thereby, an important necessary condition is derived for any regularizers or MRF prior potential functions to be able to deal with discontinuities.

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<sup>1</sup>This chapter is based on (Li 1995b) and (Li 2000).

Based on these findings, a so-called *discontinuity adaptive* (DA) smoothness model is defined in terms of the Euler equation constrained by a class of *adaptive interaction functions* (AIFs). The DA solution is  $C^1$  continuous, allowing arbitrarily large but bounded slopes. Because of the continuous nature, it is stable to changes in parameters and data. This is a good property for regularizing ill-posed problems. The results provide principles for the selection of *a priori* clique potential functions in stochastic MRF models and regularizers in deterministic regularization models. It is also shown that the DA model includes as special instances most of the existing models, such as the line process (LP) model (Geman and Geman 1984; Marroquin 1985), weak string and membrane (Blake and Zisserman 1987), approximations of the LP model (Koch et al. 1986; Yuille 1987), minimal description length (Leclerc 1989), biased anisotropic diffusion (Nordstrom 1990), and mean field theory approximation (Geiger and Girosi 1991).

The study of discontinuities is most sensibly carried out in terms of analytical properties, such as derivatives. For this reason, analytical regularization, a special class of MRF models, is used as the platform for it. If we consider that regularization contains three parts (Boult 1987): the data, the class of solution functions and the regularizer, the following addresses mainly the regularizer part. Regularization models are reviewed in connection to discontinuities. A necessary condition for the discontinuity adaptivity is made explicit. Based on this, the DA model (for step-discontinuities) is defined and compared with other models. The model is then extended for surfaces containing roof-discontinuities. Some experimental results for step- and roof-edge preserving smoothing are presented.

## 4.1 Smoothness, Regularization and Discontinuities

In MRF vision modeling, the smoothness assumption can be encoded into an energy via one of the two routes: analytic and probabilistic. In the analytic route, the encoding is done in the *regularization* framework (Poggio et al. 1985; Bertero et al. 1988). From the regularization viewpoint, a problem is said to be “ill-posed” if it fails to satisfy one or more of the following criteria: the solution exists, is unique and depends continuously on the data. Additional, *a priori*, assumptions have to be imposed on the solution to convert an ill-posed problem into a well-posed one. An important assumption of such assumptions is the *smoothness* (Tikhonov and Arsenin 1977). It is incorporated into the energy function whereby the cost of the solution is defined.

From the probabilistic viewpoint, a regularized solution corresponds to the *maximum a posteriori* (MAP) estimate of an MRF (Geman and Geman 1984; Marroquin et al. 1987). Here, the prior constraints are encoded into

the *a priori* MRF probability distribution. The MAP solution is obtained by maximizing the posterior probability or equivalently minimizing the corresponding energy.

The MRF model is more general than the regularization model in (1) that it can encode prior constraints other than the smoothness and (2) that it allows arbitrary neighborhood systems other than the nearest ones. However, the analytic regularization model provides a convenient platform for the study of smoothness priors because of close relationships between the smoothness and the analytical continuity.

#### 4.1.1 Regularization and Discontinuities

Consider the problem of restoring a signal  $f$  from the data  $d = f + e$  where  $e$  denotes the noise. The regularization formulation defines the solution  $f^*$  to be the global minimum of an energy function  $E(f)$ ,  $f^* = \arg \min_f E(f)$ . The energy is the sum of two terms

$$E(f) \doteq U(f \mid d) = U(d \mid f) + U(f) \quad (4.1)$$

The *closeness* term,  $U(d \mid f)$ , measures the cost caused by the discrepancy between the solution  $f$  and the data  $d$

$$U(d \mid f) = \int_a^b \chi(x)[f(x) - d(x)]^2 dx \quad (4.2)$$

where  $\chi(x)$  is a weighting function and  $a$  and  $b$  are the bounds of the integral. The *smoothness* term,  $U(f)$ , measures the cost caused by the irregularities of the solution  $f$ , the irregularities being measured by the derivative magnitudes  $|f^{(n)}(x)|$ . With identical independent additive Gaussian noise,  $U(f \mid d)$ ,  $U(d \mid f)$  and  $U(f)$  correspond to the energies in the posterior, the likelihood and the prior Gibbs distributions of an MRF, respectively (Marroquin et al. 1987).

The smoothness term  $U(f)$ , also called a *regularizer*, is the object of study in this work. It penalizes the irregularities according to the *a priori* smoothness constraint encoded in it. It is generally defined as

$$U(f) \doteq \sum_{n=1}^N U_n(f) = \sum_{n=1}^N \lambda_n \int_a^b g(f^{(n)}(x)) dx \quad (4.3)$$

where  $U_n(f)$  is the  $n^{th}$  order regularizer,  $N$  is the highest order to be considered and  $\lambda_n \geq 0$  is a weighting factor. A *potential function*  $g(f^{(n)}(x))$  is the penalty against the irregularity in  $f^{(n-1)}(x)$  and corresponds to prior clique potentials in MRF models. Regularizers differ in the definition of  $U_n(f)$ , more specifically in the selection of  $g$ .

### Standard Regularization

In the standard regularization (Tikhonov and Arsenin 1977; Poggio et al. 1985), the potential function takes the pure quadratic form

$$g_q(\eta) = \eta^2 \quad (4.4)$$

With  $g_q$ , the more irregular  $f^{(n-1)}(x)$  is at  $x$ , the larger  $|f^{(n)}|$ , and consequently the larger potential  $g(f^{(n)})$  contributed to  $U_n(f)$ . The standard quadratic regularizer can have a more general form

$$U_n(f, w_n) = \lambda_n \int_a^b w_n(x) [f^{(n)}(x)]^2 dx \quad (4.5)$$

where  $w_n(x)$  are the pre-specified non-negative continuous functions (Tikhonov and Arsenin 1977). It may also be generalized to multi-dimensional cases and to include cross derivative terms.

The quadratic regularizer imposes the smoothness constraint *everywhere*. It determines the constant *interaction* between neighboring points and leads to *smoothing strength* proportional to  $|f^{(n)}|$ , as will be shown in the next section. The homogeneous or isotropic application of the smoothness constraint inevitably leads to oversmoothing at discontinuities at which the derivative is infinite.

If the function  $w_n(x)$  can be pre-specified in such a way that  $w_n(x) = 0$  at  $x$  where  $f^{(n)}(x)$  is infinite, then the oversmoothing can be avoided. In this way,  $w_n(x)$  act as *continuity-controllers* (Terzopoulos 1983b). It is further suggested that  $w_n(x)$  may be discontinuous and not pre-specified (Terzopoulos 1986b). For example, by regarding  $w_n(x)$  as unknown functions, one could solve these unknowns using variational methods. But how well  $w_n(x)$  can thus be derived remains unclear. The introduction of *line processes* (Geman and Geman 1984; Marroquin 1985) or *weak continuity constraints* (Blake and Zisserman 1987) provides a solution to this problem.

### Line Process Model and Its Approximations

The LP model assumes *piecewise smoothness* whereby the smoothness constraint is *switched off* at points where the magnitude of the signal derivative exceeds certain threshold. It is defined on a lattice rather than on a continuous domain. Quantize the continuous interval  $[a, b]$  into  $m$  uniformly spaced points  $x_1, \dots, x_m$  so that  $f_i = f(x_i)$ ,  $d_i = d(x_i)$  and  $\chi_i = \chi(x_i)$ . Introduce a set of binary line process variables  $l_i \in \{0, 1\}$  into the smoothness term. If  $w_i = w_1(x_i)$  in (4.5) also takes on a value in  $\{0, 1\}$ , then  $l_i$  is related it by  $l_i = 1 - w_i$ . The on-state,  $l_i = 1$ , of the line process variable indicates that a discontinuity is detected between neighboring points  $i - 1$  and  $i$ ; the off-state,  $l_i = 0$ , indicates that the signal between the two points is continuous. Each turn-on of a line process variable is penalized by a quantity  $\lambda\alpha$ . These give the LP regularizer

$$U(f, l) = \lambda \sum_{i=2}^m [f_i - f_{i-1}]^2 [1 - l_i] + \lambda \alpha \sum_{i=2}^m l_i \quad (4.6)$$

The energy for the LP model is

$$E(f, l) = U(f, l \mid d) = \sum_{i=1}^m \chi_i [f_i - d_i]^2 + \lambda \left( \sum_{i=2}^m [f_i - f_{i-1}]^2 [1 - l_i] + \alpha \sum_{i=2}^m l_i \right) \quad (4.7)$$

This is the *weak string* model whose 2D extension is the *weak membrane* model (Blake and Zisserman 1987). Eq.(4.7) corresponds to the energy in the posterior distribution of the surface field  $f$  and the line process field  $l$ , the distribution being of the Gibbs form

$$P(f, l \mid d) = \frac{1}{Z} e^{-U(f, l \mid d)} \quad (4.8)$$

where  $Z$  is a normalizing constant called the partition function.

The line process variables are determined as follows: If  $[f_i - f_{i-1}]^2 < \alpha$ , then it is cheaper to pay the price  $[f_i - f_{i-1}]^2$  and set  $l_i = 0$ ; otherwise it is more economical to turn on the variable  $l_i = 1$  to insert a discontinuity with the cost of  $\alpha$ . This is an interpretation of the LP model based on the concept of the weak continuity constraint introduced by Blake (Blake 1983) for edge labeling. An earlier idea of weak constraints is seen in Hinton's thesis work (Hinton 1978). In the LP model, the *interaction* is piecewise constant (1 or 0) and the *smoothing strength* at  $i$  is either proportional to  $|f_i - f_{i-1}|$  or zero – see the next section. The concept of discontinuities can be extended to model-based recognition of overlapping objects (Li 1994a). There, the relational bond between any two features in the scene should be broken if the features are ascribed to two different objects.

Finding  $f^* \in \mathbb{R}^m$  and  $l^* \in \{0, 1\}^m$  such that  $U(f, l \mid d)$  is minimized is a mixture of real and combinatorial optimization. Algorithms for this can be classified as categories: stochastic (Geman and Geman 1984; Marroquin 1985) and deterministic (Koch et al. 1986; Yuille 1987; Blake and Zisserman 1987; Geiger and Girosi 1991). Some annealing techniques are often combined into them to obtain global solutions.

In stochastic approaches,  $f$  and  $l$  are updated according to some probability distribution parameterized by a temperature parameter. For example, Geman and Geman propose to use simulated annealing with the Gibbs sampler (Geman and Geman 1984) to find the global MAP solution. Marroquin (1985) minimizes the energy by a stochastic update in  $l$  together with a deterministic update in  $f$  using gradient descent.

Deterministic approaches often use some classical gradient based methods. Before these can be applied, the combinatorial minimization problem has to be converted into one of real minimization. By eliminating the line process, Blake and Zisserman (1987) convert the previous minimization problem into one which minimizes the following function containing only real variables

$$U(f \mid d) = \sum_{i=1}^m \chi_i [f_i - d_i]^2 + \lambda \sum_{i=2}^m g_\alpha(f_i - f_{i-1}) \quad (4.9)$$

where the truncated quadratic potential function (*cf.* Equ.(2.17))

$$g_\alpha(\eta) = \min\{\eta^2, \alpha\} \quad (4.10)$$

shall be referred to as the *line process potential function*. Blake and Zisserman introduce a parameter  $p$  into  $g_\alpha(\eta)$  to control the convexity of  $E$ , obtaining  $g_\alpha^{(p)}(\eta)$ . The parameter  $p$  varies from 1 to 0, which corresponds to the variation from a convex approximation of the function to its original form.

Koch *et al.* (1986) and Yuille (1987) perform the conversion using the Hopfield approach (Hopfield 1984). Continuous variables  $\bar{l}_i$  in the range  $[0, 1]$  are introduced to replace the binary line process variables  $l_i$  in  $\{0, 1\}$ . Each  $\bar{l}_i$  is related to an internal variable  $v_i \in (-\infty, +\infty)$  by a sigmoid function

$$\bar{l}_i = \theta(v_i) = 1/(1 + e^{-v_i/\tau}) \quad (4.11)$$

with  $\tau >$  as the parameter whereby  $\lim_{\tau \rightarrow 0} \bar{l}_i = l_i$ . The energy with this treatment is

$$\begin{aligned} U(f, \bar{l} \mid d) = & \sum_{i=1}^m \chi_i [f_i - d_i]^2 + \lambda \left( \sum_{i=2}^m [f_i - f_{i-1}]^2 [1 - \bar{l}_i] + \alpha \sum_{i=2}^m \bar{l}_i \right) \\ & + \sum_{i=2}^m \int_0^{\bar{l}_i} \theta^{-1}(l) dl \end{aligned} \quad (4.12)$$

It is shown that at stationary points where  $dE/dt = 0$ , there are  $v_i = \lambda([f_i - f_{i-1}]^2 - \alpha)$  and hence the approximated line process variables (Yuille 1987)

$$\bar{l}_i = \frac{1}{1 + e^{-\lambda([f_i - f_{i-1}]^2 - \alpha)/\tau}} \quad (4.13)$$

This gives the effective potential function as

$$g_{\alpha, \tau}(\eta) = -\frac{\tau}{2} \ln[1 + e^{-\lambda(\eta^2 - \alpha)/\tau}] \quad (4.14)$$

As the temperature decreases toward zero,  $\bar{l}_i$  approaches  $l_i$ :  $\lim_{\tau \rightarrow 0+} \bar{l}_i = l_i$ .

Geiger and Girosi (1991) approximate the line process using mean field theory. They introduce a parameter  $\beta$  into (4.8), giving an approximated posterior probability

$$P_\beta(f, l \mid d) = \frac{1}{Z_\beta} [P(f, l \mid d)]^\beta \quad (4.15)$$

Using the saddle point approximation method, they derive mean field equations which yield the approximated line process variables which are identical to (4.13). The solution is in the limit  $f^* = \arg \max_{f, l} \lim_{\beta \rightarrow \infty} P_\beta(f, l \mid d)$ .

Li (1990) proposes a continuous *adaptive regularizer* model. There, the smoothness constraint is applied without the switch-off as the LP model. Its effect is decreased as the derivative magnitude becomes larger and is completely off only at the true discontinuities where the derivative is infinite. This is an earlier form of the DA model in this work.

#### 4.1.2 Other Regularization Models

Grimson and Pavlidis (1985) propose an approach in which the degree of interaction between pixels across edges is adjusted in order to detect discontinuities. Lee and Pavlidis (1987) investigate a class of smoothing splines which are piecewise polynomials. Errors of fit are measured after each successive regularization and used to determine whether discontinuities should be inserted. This process iterates until convergence is reached. Besl *et al.* (1988) propose a smoothing window operator to prevent smoothing across discontinuities based on robust statistics. Liu and Harris (1989) develop, based on a previous work (Harris 1987), a computational network in which surface reconstruction, discontinuity detection and estimation of first and second derivatives are performed cooperatively.

Mumford and Shah (1985) define an energy on a continuous domain

$$U(f, \{a_i\}, k \mid d) = \int_a^b [f(x) - d(x)]^2 dx + \lambda \sum_{i=0}^k \int_{a_i}^{a_{i+1}} [f'(x)]^2 dx + \alpha k \quad (4.16)$$

where  $\lambda$  and  $\alpha$  are constants,  $k$  is the number of discontinuities and the sequence  $a = a_0 < a_1 < \dots < a_k < a_{k+1} = b$  indicates the locations of discontinuities. The minimal solution  $(f^*, \{a_i\}^*, k^*)$  minimizes  $U(f, \{a_i\}, k \mid d)$  over each value of the integer  $k$ , every sequence  $\{a_k\}$ , and every function  $f(x)$  continuously differentiable on each interval  $a_i \leq x \leq a_{i+1}$ . The minimization over  $k$  is a hard problem.

Using the minimal description length principle, Leclerc (1989) presents the following function for restoration of piecewise constant image  $f$  from noisy data  $d$

$$U(f \mid d) = \sum_{i=1}^m [f_i - d_i]^2 + \lambda \sum_{i=2}^m [1 - \delta(f_i - f_{i-1})] \quad (4.17)$$

where  $\delta(\cdot) \in \{0, 1\}$  is the Kronecker delta function. To minimize the function, he approximates the delta function with the exponential function parameterized by  $\tau$

$$U(f \mid d) = \sum_{i=1}^m [f_i - d_i]^2 + \lambda \sum_{i=2}^m [1 - e^{(f_i - f_{i-1})^2 / \tau}] \quad (4.18)$$

and approaches the solution by continuation in  $\tau$  toward 0.

## 4.2 The Discontinuity Adaptive MRF Model

By analyzing the smoothing mechanism in terms of the Euler equation, it will be clear that major difference between different models lies in their way of controlling the interaction between neighboring points and adjusting the smoothing strength. The DA model is defined based on the principle that wherever a discontinuity occurs, the interaction should diminish.

### 4.2.1 Defining the DA Model

We focus on the models which involve only the first order derivative  $f' = f^{(1)}$  and consider the general string model

$$E(f) = U(f \mid d) = \int_a^b u(f \mid d) dx \quad (4.19)$$

where

$$u(f \mid d) = \chi(x)[f(x) - d(x)]^2 + \lambda g(f'(x)) \quad (4.20)$$

Solutions  $f^*$  minimizing  $U(f \mid d)$  must satisfy the following associated Euler-Lagrange differential equation or simply the *Euler equation* (Courant and Hilbert 1953)

$$u_f(f, f') - \frac{d}{dx} u_{f'}(f, f') = 0 \quad (4.21)$$

with the boundary conditions

$$f(a) = f_a, \quad f(b) = f_b \quad (4.22)$$

where  $f_a$  and  $f_b$  are prescribed constants. In the following discussion of solutions to the differential equation, the following assumptions are made: Both  $\chi(x)$  and  $d(x)$  are continuous and  $f(x)$  is continuously differentiable<sup>2</sup>. Writing the Euler equation out yields

$$2\chi(x)[f(x) - d(x)] - \lambda \frac{d}{dx} g'(f'(x)) = 0 \quad (4.23)$$

A potential function  $g$  is usually chosen to be (a) even such that  $g(\eta) = g(|\eta|)$  and (b) the derivative of  $g$  can be expressed as the following form

$$g'(\eta) = 2\eta h(\eta) \quad (4.24)$$

where  $h$  is called an *interaction function*. Obviously,  $h$  thus defined is also even. With these assumptions, the Euler equation can be expressed as

<sup>2</sup>In this work, the continuity of  $\chi(x)$  and  $d(x)$  and differentiability of  $f(x)$  are assumed for the variational problems defined on continuous domains (Courant and Hilbert 1953; Cesari 1983). However, they are not necessary for discrete problems where  $[a, b]$  is quantized into discrete points. For example, in the discrete case,  $\chi(x_i)$  is allowed to take a value in  $\{1, 0\}$ , indicating whether datum  $d(x_i)$  is available or not.

$$\chi(x)[f(x) - d(x)] - \lambda \frac{d}{dx}[f'(x)h(f'(x))] = 0 \quad (4.25)$$

The magnitude  $|g'(f')| = |2f'h(f')|$  relates to the *strength* with which a regularizer performs smoothing; and  $h(f'(x))$  determines the *interaction* between neighboring pixels.

Let  $\eta \doteq f'(x)$ . A *necessary condition* for any regularization model to be adaptive to discontinuities is (Li 1991; Li 1995b)

$$\lim_{\eta \rightarrow \infty} |g'(\eta)| = \lim_{\eta \rightarrow \infty} |2\eta h(\eta)| = C \quad (4.26)$$

where  $C \in [0, \infty)$  is a constant. The above condition with  $C = 0$  entirely prohibits smoothing at discontinuities where  $\eta \rightarrow \infty$  whereas with  $C > 0$  allows limited (bounded) smoothing. In any case, however, the interaction  $h(\eta)$  must be small for large  $|\eta|$  and approaches 0 as  $|\eta|$  goes to  $\infty$ . This is an important guideline for selecting  $g$  and  $h$  for the purpose of the adaptation.

**Definition 1.** An *adaptive interaction function* (AIF)  $h_\gamma$  parameterized by  $\gamma (> 0)$  is a function that satisfies:

- (i)  $h_\gamma \in C^1$
- (ii)  $h_\gamma(\eta) = h_\gamma(-\eta)$
- (iii)  $h_\gamma(\eta) > 0$
- (iv)  $h'_\gamma(\eta) < 0 \quad (\forall \eta > 0)$
- (v)  $\lim_{\eta \rightarrow \infty} |2\eta h_\gamma(\eta)| = C < \infty$

The class of AIFs, denoted by  $\mathbb{H}_\gamma$ , is defined as the collection of all such  $h_\gamma$ .  $\square$

The continuity requirement (i) guarantees the twice differentiability of the integrand  $u(f \mid d)$  in (4.20) with respect to  $f'$ , a condition for the solution  $f$  to exist (Courant and Hilbert 1953). However, this can be relaxed to  $h_\gamma \in C^0$  for discrete problems. The evenness of (ii) is usually assumed for spatially unbiased smoothing. The positive definiteness of (iii) keeps the interaction positive such that the sign of  $\eta h_\gamma(\eta)$  will not be altered by  $h_\gamma(\eta)$ . The monotony of (iv) leads to decreasing interaction as the magnitude of the derivative increases. The bounded asymptote property of (v) provides the adaptive discontinuity control as stated earlier. Other properties follow:  $h_\gamma(0) = \sup_\eta h_\gamma(\eta)$ ,  $h'_\gamma(0) = 0$  and  $\lim_{\eta \rightarrow \infty} h_\gamma(\eta) = 0$ , the last meaning zero interaction at discontinuities. The above definition characterizes the properties AIFs should possess rather than instantiates some particular functions. Therefore, the following definition of the DA model is rather broad.

**Definition 2.** The DA solution  $f$  is defined by the Euler equation (4.25) constrained by  $h = h_\gamma \in \mathbb{H}_\gamma$ .  $\square$

The DA solution is  $C^1$  continuous<sup>3</sup>. Therefore the DA solution and its derivative never have discontinuities in them. The DA overcomes oversmoothing by allowing its solution to be steep, but  $C^1$  continuous, at point  $x$  where data  $d(x)$  is steep. With every possible data configuration  $d$ , every  $f \in C^1$  is possible.

There are two reasons for defining the DA in terms of the constrained Euler equation: First, it captures the essence of the DA problem; the DA model should be defined based on the  $h_\gamma$  therein. Second, some satisfying  $h_\gamma$  may not have their corresponding  $g_\gamma$  (and hence the energy) in closed-form, where  $g_\gamma$  is defined below.

**Definition 3.** The *adaptive potential function* (APF) corresponding to an  $h_\gamma \in \mathbb{H}_\gamma$  is defined by

$$g_\gamma(\eta) = 2 \int_0^\eta \eta' h_\gamma(\eta') d\eta' \quad (4.28)$$

□

The energy function (4.19) with  $g = g_\gamma$  is called an *adaptive string*. The following are some properties of  $g_\gamma$ : Basically,  $g_\gamma$  is one order higher than  $h_\gamma$  in continuity; it is even,  $g_\gamma(\eta) = g_\gamma(-\eta)$ ; its derivative function is odd,  $g'_\gamma(\eta) = -g'_\gamma(-\eta)$ ; however, it is not necessary for  $g_\gamma(\infty)$  to be bounded. Furthermore,  $g_\gamma$  is strictly monotonically increasing as  $|\eta|$  increases because  $g_\gamma(\eta) = g_\gamma(|\eta|)$  and  $g'_\gamma(\eta) = 2\eta h_\gamma(\eta) > 0$  for  $\eta > 0$ . This means larger  $|\eta|$  leads to larger penalty  $g_\gamma(\eta)$ . It conforms to the original spirit of the standard quadratic regularizers determined by  $g_q$ . The line process potential function  $g_\alpha$  does not have such a property: Its penalty is fixed and does not increase as  $|\eta|$  increases beyond  $\sqrt{\alpha}$ . The idea that large values of  $\eta$  are equally penalized is questionable (Shulman and Herve 1989).

In practice, it is not always necessary to know the explicit definition of  $g_\gamma$ . The most important factors are the Euler equation and the constraining function  $h_\gamma$ . Nonetheless, knowing  $g_\gamma$  is helpful for analyzing the convexity of  $E(f)$ .

For a given  $g_\gamma(\eta)$ , there exists a region of  $\eta$  within which the smoothing strength  $|g'_\gamma(\eta)| = |\eta h_\gamma(\eta)|$  increases monotonically as  $|\eta|$  increases and the function  $g_\gamma$  is convex:

$$B_\gamma = \{\eta \mid g''_\gamma(\eta) > 0\} \stackrel{\Delta}{=} (b_L, b_H) \quad (4.29)$$

The region  $B_\gamma$  is referred to as the *band of convexity*, because  $g(\eta)$  is convex in this region. The lower and upper bounds  $b_L, b_H$  correspond to the two extrema of  $g'_\gamma(\eta)$ , which can be obtained by solving  $g''_\gamma(\eta) = 0$ , and we have

<sup>3</sup>See (Cesari 1983) for a comprehensive discussion about the continuity of solutions of the class of problems to which the DA belongs.

AIF	APF	Band
$h_{1\gamma}(\eta) = e^{-\frac{\eta^2}{\gamma}}$	$g_{1\gamma}(\eta) = -\gamma e^{-\frac{\eta^2}{\gamma}}$	$B_{1\gamma} = (-\sqrt{\frac{\gamma}{2}}, \sqrt{\frac{\gamma}{2}})$
$h_{2\gamma}(\eta) = \frac{1}{[1 + \frac{\eta^2}{\gamma}]^2}$	$g_{2\gamma}(\eta) = -\frac{\gamma}{1 + \frac{\eta^2}{\gamma}}$	$B_{2\gamma} = (-\sqrt{\frac{\gamma}{3}}, \sqrt{\frac{\gamma}{3}})$
$h_{3\gamma}(\eta) = \frac{1}{1 + \frac{\eta^2}{\gamma}}$	$g_{3\gamma}(\eta) = \gamma \ln(1 + \frac{\eta^2}{\gamma})$	$B_{3\gamma} = (-\sqrt{\gamma}, \sqrt{\gamma})$
$h_{4\gamma}(\eta) = \frac{1}{1 + \frac{ \eta }{\gamma}}$	$g_{4\gamma}(\eta) = \gamma \eta  - \gamma^2 \ln(1 + \frac{ \eta }{\gamma})$	$B_{4\gamma} = (-\infty, +\infty)$

Table 4.1: Four choices of AIFs, the corresponding APFs and bands.

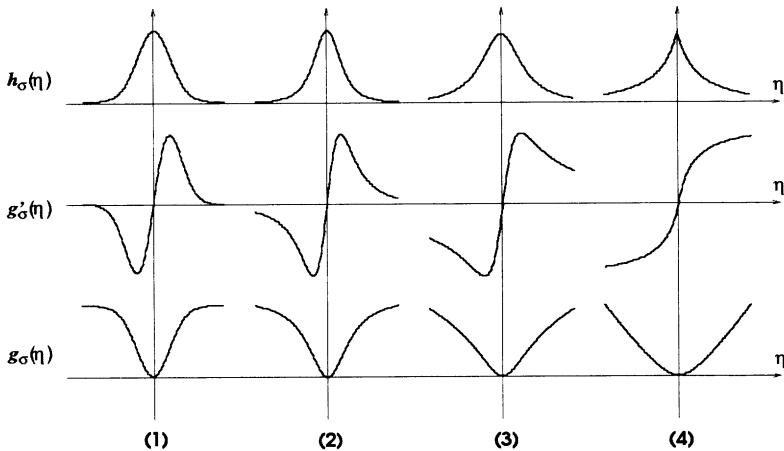


Figure 4.1: The qualitative shapes of the four DA functions. From (Li 1995b) with permission; ©1995 IEEE.

$b_L = -b_H$  when  $g$  is even. When  $b_L < \eta < b_H$ ,  $g''(\eta) > 0$  and thus  $g_\gamma(\eta)$  is strictly convex. For  $h_\gamma$  defined with  $C > 0$ , the bounds are  $b_L \rightarrow -\infty$  and  $b_H \rightarrow +\infty$ .

Table 4.1 instantiates four possible choices of AIFs, the corresponding APFs and the bands. Fig.4.1 shows their qualitative shapes (a trivial constant may be added to  $g_\gamma(\eta)$ ). Fig.4.2 gives a graphical comparison of the  $g(\eta)$ 's for the quadratic, the LP model and the first three DA models listed in Table 4.1 and their derivatives,  $g'(\eta) = 2\eta h(\eta)$ .

The fourth AIF,  $h_{4\gamma} = 1/[1 + |\eta|/\gamma]$  allows bounded but non-zero smoothing at discontinuities:  $\lim_{\eta \rightarrow \infty} \eta h_{4\gamma}(\eta) = \gamma$ . It is interesting because  $g''_{4\gamma}(\eta) = [2\eta h_{4\gamma}(\eta)]' > 0$  for all  $\eta$  (except at  $\eta = 0$ ) and leads to strictly convex minimization. In fact, a positive number  $C$  in (4.27) leads to a convex AIF

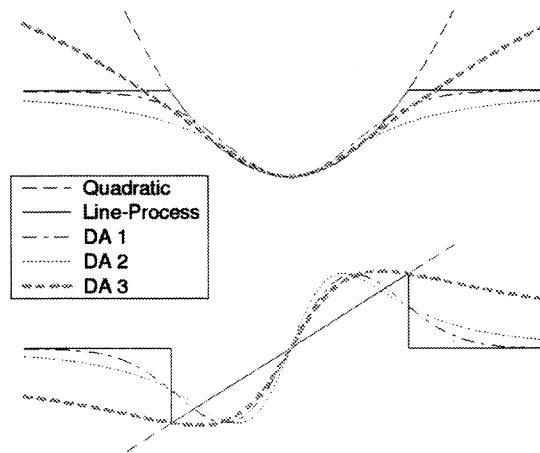


Figure 4.2: Comparison of different potential functions  $g(\eta)$  (top) and their first derivatives  $g'(\eta) = 2\eta h(\eta)$  (bottom). From (Li 1995b) with permission; ©1995 IEEE.

and hence energy function<sup>4</sup>. The convex subset of models for discontinuity adaptivity and M estimation is appealing because they have some inherent advantages over nonconvex models, both in stability and computational efficiency. A discussion on convex models will be given in Section 5.1.5; see also (Li et al. 1995).

Fig.4.2 also helps us visualize how the DA performs smoothing. Like the quadratic  $g_q(\eta) = \eta^2$ , the DA allows the smoothing strength  $|g'_\gamma(\eta)|$  to increase monotonically as  $\eta$  increases within the band  $B_\gamma$ . Outside the band, the smoothing decreases as  $\eta$  increases and becomes zero as  $\eta \rightarrow \infty$ . This differs from the quadratic regularizer which allows boundless smoothing when  $\eta \rightarrow \infty$ . Also unlike the LP model which shuts down smoothing abruptly just beyond its band  $B_\alpha = (-\sqrt{\alpha}, \sqrt{\alpha})$ , the DA decreases smoothing continuously towards zero.

#### 4.2.2 Relations with Previous Models

The first three instantiated models behave in a similar way to the quadratic prior model when  $\eta^2/\gamma \ll 1$  or  $\eta^2 \ll \gamma$ , as noticed by (Li 1990b). This can be understood by looking at the power series expansion of  $g_\gamma(\eta)$ ,  $g_\gamma(\eta) = \sum_{k=1}^{\infty} c_k (\eta^2/\gamma)^k$  where  $c_k$  are constants with  $c_1 > 0$  (the expansion can also involve a trivial additive constant  $c_0$ ). When  $\eta^2/\gamma \ll 1$ ,

<sup>4</sup>This is due to the following theorem: If  $g(\cdot)$  is convex on  $\mathbb{R}$ , a real-valued energy function  $E(f) = \int \chi(x)[f(x) - d(x)]^2 + \lambda g(f'(x))dx$  is convex w.r.t.  $f$  for all  $f \in C^1$  and fixed  $\chi, d \in C^1$ .

$$g_\gamma(\eta) = c_1(\eta^2/\gamma) + O(\eta^4/\gamma^2) \quad (4.30)$$

Thus in this situation of sufficiently small  $\eta^2/\gamma$ , the adaptive model inherits the convexity of the quadratic model.

The interaction function  $h$  also well explains the differences between various regularizers. For the quadratic regularizer, the interaction is constant everywhere

$$h_q(\eta) = 1 \quad (4.31)$$

and the smoothing strength is proportional to  $|\eta|$ . This is why the quadratic regularizer leads to oversmoothing at discontinuities where  $\eta$  is infinite. In the LP model, the interaction is piecewise constant

$$h_\alpha(\eta) = \begin{cases} 1 & |\eta| < \sqrt{\alpha} \\ 0 & \text{otherwise} \end{cases} \quad (4.32)$$

Obviously, it inhibits oversmoothing by switching off smoothing when  $|\eta|$  exceeds  $\sqrt{\alpha}$  in a binary manner.

In the LP approximations using the Hopfield approach (Koch et al. 1986; Yuille 1987) and mean field theory (Geiger and Girosi 1991), the line process variables are approximated by (4.13). This approximation effectively results in the following interaction function

$$h_{\alpha,T}(\eta) = 1 - \bar{l}(\eta) = 1 - \frac{1}{1 + e^{-\lambda(\eta^2 - \alpha)/T}} = \frac{e^{-\lambda(\eta^2 - \alpha)/T}}{1 + e^{-\lambda(\eta^2 - \alpha)/T}} \quad (4.33)$$

As the temperature  $\tau$  decreases toward zero, the above approaches  $h_\alpha/2$ , that is,

$\lim_{\tau \rightarrow 0^+} h_{\alpha,\tau} = h_\alpha/2$ . Obviously,  $h_{\alpha,\tau}(\eta)$  with nonzero  $\tau$  is a member of the AIF family, i.e.  $h_{\alpha,\tau}(\eta) \in \mathbb{H}_\gamma$  and therefore the approximated LP models are instances of the DA model.

It is interesting to note an observation made by Geiger and Girosi: “sometimes a finite  $\beta$  solution may be more desirable or robust” ((Geiger and Girosi 1991), pages 406-407) where  $\beta = \frac{1}{T}$ . They further suggest that there is “an optimal (finite) temperature ( $\beta$ )” for the solution. An algorithm is presented in (Hebert and Leahy 1992) for estimating an optimal  $\beta$ . The LP approximation with finite  $\beta < +\infty$  or nonzero  $T > 0$  is more an instance of the DA than the LP model which they aimed to approximate. It will be shown in Section 4.2.4 that the DA model is indeed more stable than the LP model.

Anisotropic diffusion (Perona and Malik 1990) is a scale-space method for edge-preserving smoothing. Unlike fixed coefficients in the traditional isotropic scale-space filtering (Witkin 1983), anisotropic diffusion coefficients are spatially varying according to the gradient information. A so-called biased anisotropic diffusion (Nordstrom 1990) model is obtained if anisotropic diffusion is combined with a closeness term. Two choices of APFs are used in those anisotropic diffusion models:  $g_{1\gamma}$  and  $g_{2\gamma}$ .

Shulman and Herve (1989) propose to use the following Huber's robust error penalty function (Huber 1981) as the adaptive potential

$$g_\beta(\eta) = \min\{\eta^2, \beta^2 + 2\beta|\eta - \beta|\} \quad (4.34)$$

where  $\beta > 0$  plays a similar role to  $\alpha$  in  $g_\alpha$ . The above is a convex function and has the first derivative as:

$$g'_\beta(\eta) = 2\eta h_\beta(\eta) = 2\eta \quad (4.35)$$

for  $|\eta| \leq \beta$ , and

$$g'_\beta(\eta) = 2\eta h_\beta(\eta) = 2\beta\eta/|\eta| \quad (4.36)$$

for other  $\eta$ . Comparing  $g'_\beta(\eta)$  with (4.24), we find that the corresponding AIF is  $h_\beta(\eta) = 1$  for  $|\eta| \leq \beta$  and  $h_\beta(\eta) = \beta/|\eta|$  for other  $\eta$ . This function allows bounded but nonzero smoothing at discontinuities. The same function has also been applied by (Stevenson and Delp 1990) to curve fitting. A comparative study on the DA model and robust statistics is presented in Chapter 5 (see also (Li 1995a)).

The approximation (4.18) of Leclerc's minimal length model (Leclerc 1989) is in effect the same as the DA with APF 1. Eq.(4.17) may be one of the best cost functions for the piecewise constant restoration; for more general piecewise continuous restoration, one needs to use (4.18) with a *nonzero*  $\tau$ , which is a DA instance. Regarding the continuity property of domains, Mumford and Shah's model (Mumford and Shah 1985) and Terzopoulos' continuity-controlled regularization model (Terzopoulos 1986b) can also be defined on continuous domains as the DA model.

#### 4.2.3 Discrete Data and 2D Cases

When the data  $d$  is available at discrete points:  $d_i = d(x_i)$ ,  $1 \leq i \leq m$ , the Euler equation for the adaptive string is

$$\sum_i \chi(x)[f(x) - d_i]\delta(x - x_i) = \lambda \frac{d}{dx}[f'(x)h(f'(x))] \quad (4.37)$$

where  $\delta(\cdot)$  is the Dirac delta function. Integrating the above equation once yields

$$\sum_{i: x_i < x} \chi(x_i)[f(x_i) - d_i] = \lambda[f'(x)h(f'(x))] + c \quad (4.38)$$

where  $c$  is a constant. From the above, the solution  $f$  is determined by the following

$$f''(x) = 0 \quad x_{i-1} < x < x_i, \quad 1 < i \leq m \quad (4.39)$$

$$f(x_i^+) - f(x_i^-) = 0 \quad 1 < i < m \quad (4.40)$$

$$f'(x_i^+)h(f'(x_i^+)) - f'(x_i^-)h(f'(x_i^-)) = \frac{1}{\lambda}\chi(x_i)[f(x_i) - d_i] \quad 1 < i < m \quad (4.41)$$

with appropriate boundary conditions at  $x = x_1$  and  $x = x_m$ . Obviously,  $f$  in this case is piecewise  $C^1$  continuous; more exactly, it is composed of consecutively joined line segments.

The adaptive string model can be extended to 2D and higher order equivalents. The 2D adaptive membrane has the following

$$\chi[f - d] - \lambda \frac{\partial}{\partial x}[f_x h(f_x)] - \lambda \frac{\partial}{\partial y}[f_y h(f_y)] = 0 \quad (4.42)$$

corresponding to the Euler equation (4.25), where data  $d = d(x, y)$  and solution  $f = f(x, y)$  represent images. Extending the DA to second order, one obtains the following for the adaptive rod on 1D

$$\chi[f - d] + \lambda \frac{d}{dx^2}[f'' h(f'')] = 0 \quad (4.43)$$

and for the adaptive plate on 2D

$$\chi[f - d] + \lambda \frac{\partial}{\partial x^2}[f_{xx} h(f_{xx})] + 2\lambda \frac{\partial}{\partial xy}[f_{xy} h(f_{xy})] + \lambda \frac{\partial}{\partial y^2}[f_{yy} h(f_{yy})] = 0 \quad (4.44)$$

The discrete form of the adaptive membrane will be given in Section 4.2.5.

#### 4.2.4 Solution Stability

The DA solution depends continuously on its parameters and the data whereas the LP solution does not. An informal analysis follows. Consider an LP solution  $f_\alpha$  obtained with  $\alpha$ . The solution is a local equilibrium satisfying

$$\chi(f_\alpha - d) - \lambda f'_\alpha h_\alpha(f'_\alpha) = 0 \quad (4.45)$$

In the above,  $h_\alpha(f'_\alpha) = 1$  or 0, depending on  $\alpha$  and the configuration of  $f_\alpha$ . When  $|f'_\alpha(x)|^2$  is close to  $\alpha$  for some  $x$ , a small change  $\Delta\alpha$  may flip  $h_\alpha$  over from one state to the other. This is due to the binary non-linearity of  $h_\alpha$ . The flip-over leads to a significantly different solution. This can be expressed as

$$\lim_{\Delta\alpha \rightarrow 0} [f_{\alpha+\Delta\alpha} - f_\alpha] \neq 0(x) \quad \exists \alpha \quad (4.46)$$

where  $0(x)$  is a function which is constantly zero in the domain  $[a, b]$ . The variation  $\delta f_\alpha$  with respect to  $\Delta\alpha$  may not be zero for some  $f_\alpha$  and  $\alpha$ , which causes instability. However, the DA solution, denoted  $f_\gamma$ , is stable

$$\lim_{\Delta\gamma \rightarrow 0} [f_{\gamma+\Delta\gamma} - f_\gamma] = 0(x) \quad \forall \gamma \quad (4.47)$$

where  $f_\gamma$  denotes the DA solution. Conclusion on the stability due to changes in parameter  $\lambda$  can be drawn similarly.

The same is also true with respect to the data. Given  $\alpha$  and  $\lambda$  fixed, the solution depends on the data, *i.e.*  $f = f[d]$ . Assume a small variation  $\delta d$  in the data  $d$ . The solution  $f[d]$  must change accordingly to reach a new equilibrium  $f[d + \delta d]$  to satisfy the Euler equation. However, there always exist possibilities that  $h_\alpha(f'_\alpha(x))$  may flip over for some  $x$  when  $|f'_\alpha(x)|^2$  is near the  $\alpha$ , resulting in an abrupt change in the LP solution  $f_\alpha$ . This can be represented by

$$\lim_{\delta d \rightarrow 0(x)} [f_\alpha[d + \delta d] - f_\alpha[d]] \neq 0(x) \quad \exists f_\alpha, d \quad (4.48)$$

That is, the variation  $\delta f_\alpha$  with respect to  $\delta d$  may not be zero for some  $f_\alpha$  and  $d$ . However, the DA model is stable to such changes, *i.e.*

$$\lim_{\delta d \rightarrow 0(x)} [f_\gamma[d + \delta d] - f_\gamma[d]] = 0(x) \quad \forall f_\gamma, d \quad (4.49)$$

because of its continuous nature. From the analysis, it can be concluded that the DA better regularizes ill-posed problems than the LP.

To summarize the above: Through an analysis of the associated Euler equation, a necessary condition is made explicit for MRF or regularization models to be adaptive to discontinuities. On this basis, the DA model is defined by the Euler equation constrained by the class of adaptive interaction functions (AIFs). The definition provides principles for choosing deterministic regularizers and MRF clique potential functions. It also includes many existing models as special instances.

The DA model has its solution in  $C^1$  and adaptively overrides the smoothness assumption where the assumption is not valid, without the switching-on/off of discontinuities in the LP model. The DA solution never contains true discontinuities. The DA model “preserves” discontinuities by allowing the solution to have arbitrarily large but *bounded* slopes. The LP model “preserves true discontinuities” by switching between small and *unbounded* (or large) slopes.

Owing to its continuous properties, the DA model possesses some theoretical advantages over the LP model. Unlike the LP model, it is stable to changes in parameters and in the data. Therefore it is better than the LP model in solving ill-posed problems. In addition, it is able to deal with problems on a continuous domain. Furthermore, it is better suited for analog VLSI implementation; see Section 9.3.

#### 4.2.5 Computational Issues

The Euler equation (4.25) can be treated as a boundary value problem. It can also be solved by minimizing the corresponding energy (4.19) because a minimum of the energy is (sufficiently) a solution of the equation (With  $g'(\eta) = 2\eta h(\eta)$ , the form of the energy needs not to be known in order to minimize it – see below). The energy minimization approach is chosen here.

Because both  $\chi(x)[f(x) - d(x)]^2$  and  $g(f'(x))$  are bounded below, so is the energy. This means that a minimal solution, and hence a solution to the Euler equation, exists. If  $h_\gamma$  is chosen with  $C = 0$ , the corresponding energy  $E(f)$  is non-convex with respect to  $f$ , and the minimization is subject local minima. See (Blake and Zisserman 1987) for an analysis of convexity in string and membrane models. The following presents a discrete method for finding a local minimum.

Sample the integral interval  $[a, b]$  into  $m$  uniformly spaced points:  $x_1 = a, \dots, x_m = b$ . For clarity and without loss of generality, let us assume that the bounds  $a$  and  $b$  are re-scaled in such a way that the point spacing is one unit<sup>5</sup> i.e.  $x_i - x_{i-1} = 1$ . Approximate the first derivative  $f'_i$  by the first order backward difference  $f'_i = \frac{f_i - f_{i-1}}{x_i - x_{i-1}} = f_i - f_{i-1}$ . Then the energy (4.19) is approximated by

$$E(f) = \sum_{i=1}^m \chi_i [f_i - d_i]^2 + \lambda \sum_{i=1}^m \sum_{i' \in \mathcal{N}_i} g(f_i - f_{i'}) \quad (4.50)$$

where  $\mathcal{N}_i$  consists of the neighbors of  $i$ . Using the gradient-descent method, we can obtain the following updating equation: for  $1 < i < m$ ,

$$\begin{aligned} f_i^{(t+1)} \leftarrow f_i^{(t)} & - 2\mu \{ \chi_i [f_i^{(t)} - d_i] \\ & - \lambda \sum_{i' \in \mathcal{N}_i} (f_{i'}^{(t)} - f_i^{(t)}) h(f_{i'}^{(t)} - f_i^{(t)}) \} \end{aligned} \quad (4.51)$$

The solution  $f^{(t)}$  takes prescribed values at the boundary points at  $i = 1$  and  $i = m$  to meet the boundary condition (4.22) and the values may be estimated from data  $d_i$  near the boundaries. With initial  $f^{(0)}$ , the solution is in the limit  $f^* = f^{(\infty)}$ .

Equation (4.51) helps us see more of how the DA works. The smoothing at  $i$  is due to  $\lambda \sum_{i' \in \mathcal{N}_i} (f_{i'} - f_i) h(f_{i'} - f_i)$  where  $\mathcal{N}_i = \{i-1, i+1\}$ . The contributions to smoothing are from the two neighboring points. That from site  $i'$  is proportional to the product of the two factors,  $f_{i'} - f_i$  and  $h_\gamma(f_{i'} - f_i)$ . This is why we relate  $|g'(\eta)| = |2\eta h(\eta)|$  to the strength of smoothing performed by regularizers. On the other hand,  $h(\eta)$  acts as an adaptive weighting function to control the smoothing due to the difference  $\eta = f_{i'} - f_i$ . Therefore we regard it as the interaction. Two more remarks are made: First, the contributions from the two sides,  $i-1$  and  $i+1$  are treated separately or non-symmetrically. Second, the sum of the contributions to smoothing is zero if the three points are aligned when  $f_{i+1} - f_i = f_i - f_{i-1}$  and  $h(\eta)$  is even since in this situation  $\sum_{i' \in \mathcal{N}_i} (f_{i'} - f_i) h(f_{i'} - f_i) = 0$ .

The updating rule for the adaptive membrane on 2D can be easily derived. In the 2D case (4.42), there is an additional term in the  $y$  dimension. This leads to the following energy

<sup>5</sup>More advanced numerical methods using varying spacing, such as (Weiss 1990) may be advantageous in obtaining more accurate solutions.

$$E(f) = \sum_{i,j} \chi_{i,j} [f_{i,j} - d_{i,j}]^2 + \lambda g(f_{i,j} - f_{i-1,j}) + \lambda g(f_{i,j} - f_{i,j-1}) \quad (4.52)$$

Letting  $\frac{\partial f_{i,j}}{\partial t} = -\mu \frac{\partial E}{\partial f_{i,j}}$  leads to the following updating rule

$$f_{i,j}^{(t+1)} \leftarrow f_{i,j}^{(t)} - 2\mu \{ \chi_{i,j} [f_{i,j}^{(t)} - d_{i,j}] - \lambda \sum_{(x,y) \in \mathcal{N}_{i,j}} (f_{x,y}^{(t)} - f_{i,j}^{(t)}) h(f_{x,y}^{(t)} - f_{i,j}^{(t)}) \} \quad (4.53)$$

where  $\mathcal{N}_{i,j} = \{(i-1,j), (i+1,j), (i,j-1), (i,j+1)\}$  is the set of the four neighboring points<sup>6</sup> of  $(i,j)$ . The updating on 2D grid can be performed on the white and black sites of a checkerboard alternately to accelerate convergence.

There are three parameters in (4.51) which shall be determined for the DA model:  $\lambda$ ,  $\gamma$  and  $\mu$ . Parameter  $\mu$  is related to the convergence of the relaxation algorithm. There is an upper bound for  $\mu$  for the system to be stable. There also exists an optimal value for  $\mu$  (Blake and Zisserman 1987). Optimal choices of  $\lambda$  for quadratic regularization may be made using cross-validation (Wahba 1980; Shahraray and Anderson 1989). In (Nadabar and Jain 1992), a least squares method (Derin and Elliott 1987) is presented for estimating MRF clique potentials for a LP model. Automated selection of the  $\lambda$  and  $\gamma$  parameters for the DA model is an unsolved problem. They are currently chosen in an *ad hoc* way.

The DA model with  $C = 0$  leads to non-convex dynamic systems and direct minimization using gradient descent only guarantees to find a local minimum. A GNC-like algorithm can be constructed for approximating the global solution; see Section 9.3.

### 4.3 Modeling Roof Discontinuities

The previous sections dealt with *step-edge* preserving smoothing where the models assume that the underlying surface has zero first order derivatives. For *roof-edges*, one might suggest to use higher order piecewise smoothness, such as weak rod and plate (Blake and Zisserman 1987). However, such models are usually not recommended because of their instability (Blake and Zisserman 1987).

In this section, roof-edge (as well as step-edge) preserving smoothing is performed by using first order piecewise smoothness. The approach is the following: The image surface function is modeled using a parametric polynomial representation. Based on this, the problem of preserving  $C^1$  discontinuities in the image surface function is converted to preserving  $C^0$  discontinuities in

<sup>6</sup>With the 4-neighborhood system, the model considers derivatives in the horizontal and vertical directions. With the 8-neighborhood system, the regularizer also includes the diagonal derivatives weighted by  $1/\sqrt{2}$ .

the governing parameter functions; in other words, the piecewise smoothness is imposed on the parameter functions rather directly on the image surface height function itself. In this way, only the first derivatives are required to preserve roof edges. This avoids the instability of using higher order derivatives and is the advantage of such an MAP-MRF model. The model extends the ability of the existing first order piecewise smoothness models (such as the line process and weak membrane models) in edge preserving smoothing.

### 4.3.1 Roof-Edge Model

An *ideal roof edge* can be modeled as a joining point of two planar surfaces. Fig.4.3 shows a few examples of roof edges, noting that they are not necessarily local extrema in surface height. A planar surface takes the form  $z(x) = a + bx$  if it is on a 1D domain, or  $z(x, y) = a + bx + cy$  if on 2D, where  $a, b$  and  $c$  are some constant parameters. Let the parameters be denoted collectively by  $f = \{a, b, c\}$  and a parameterized plane be denoted by  $z = z_f(x, y) = a + bx + cy$ . Each distinct set of constants  $\{a, b, c\}$  represents a distinct plane. For a slanted plane, at least one of  $b$  and  $c$  is nonzero.

Let us look at Fig.4.4 to see what happens when the line process model is used to smooth  $z(x)$ : the discontinuity in  $b$  (also in  $a$ ) is smoothed out and the information about the roof is lost. However, imposing the first order piecewise smoothness on these parameters can solve this problem; see below.

A *general roof* can be modeled as a joining point of two nonlinear surfaces represented by  $z = z_f(x, y) = a(x, y) + b(x, y)x + c(x, y)y$  where  $a, b$  and  $c$  are now some functions of  $x$  and  $y$ . A step edge is a step ( $C^0$ ) discontinuity in  $z(x, y)$ ; a roof is signaled by a step ( $C^0$ ) discontinuity in either  $b(x, y)$  or  $c(x, y)$ . In addition, there is usually also a discontinuity in  $a(x, y)$  at either a roof or a step discontinuity. If  $a(x, y)$ ,  $b(x, y)$  and  $c(x, y)$  are (piecewise) constant, then  $z_f(x, y)$  degenerates to be (piecewise) planar. If they are (piecewise)  $C^0$  continuous, then  $z_f(x, y)$  consists of (piecewise) nonlinear surface(s).

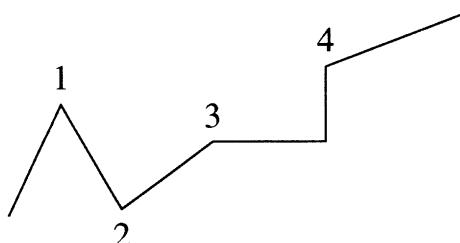


Figure 4.3: Examples of roof edges in 1D. Corner points 1, 2 and 3 are roof edge points. No.4 is a roof to the right but a step to the left.

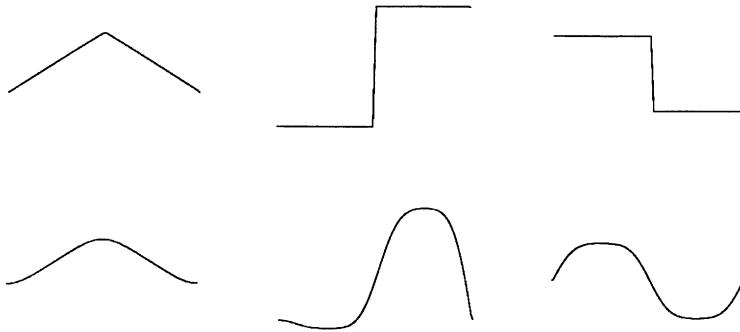


Figure 4.4: A clean ideal roof in 1D before (top) and after (bottom) being smoothed by the line-process model. From left to right are  $z(x) = a(x) + b(x)x$ ,  $a(x)$ , and  $b(x)$  functions, respectively. The smoothing is performed on the  $z(x)$ . The  $a(x)$  and  $b(x)$  in the figure are computed from the respective  $z(x)$  functions. The discontinuities in  $a$  and  $b$ , and hence the roof edge, are smoothed out.

For notational convenience, let the points on the  $x$ - $y$  grid be indexed by a single index  $i \in \mathcal{S} = \{1, \dots, m\}$  where  $\mathcal{S}$  is called the set of *sites* in MRF literature. When all the  $z_i = z_f(x_i, y_i)$  belong to a single planar surface of the form  $z(x, y) = a + bx + cy$ , the problem of finding the underlying  $a$ ,  $b$  and  $c$  parameter constants can be well solved, for example, by using the least squares method. However, when they are due to piecewise surfaces and the segmentation of the data into separate surfaces is unknown, finding the underlying parameters  $f = \{a, b, c\}$  as piecewise functions is a non-trivial problem.

When the segmentation is unknown, we may consider the parameters are some functions of  $(x, y)$ :  $a_i = a(x_i, y_i)$ ,  $b_i = b(x_i, y_i)$  and  $c_i = c(x_i, y_i)$ , collectively denoted by  $f_i = [a_i, b_i, c_i]^T$ . As such, the surface heights is reconstructed as  $z_i = z_{f_i}(x_i, y_i) = a_i + b_i x_i + c_i y_i$ . Consider  $a_i$ ,  $b_i$  and  $c_i$  as random variables,  $f_i$  a random vector, and  $f = \{f_1, \dots, f_m\}$  a random field. We have the following remarks about relationship between the  $a, b, c$  functions and the local configuration of  $f$ :

- In the case of a single planar surface, the parameter functions  $a(x, y)$ ,  $b(x, y)$  and  $c(x, y)$  are constant over the  $x$ - $y$  domain. Therefore, there must be  $f_i = f_{i'}$  for all  $i \neq i'$ .
- In the case of a piecewise planar surface, the functions are piecewise constant over the  $x$ - $y$  domain. Therefore,  $f_i = f_{i'}$  holds only when both

$z_i$  and  $z_{i'}$  belong to the same plane, and  $f_i \neq f_{i'}$  otherwise. Discontinuities occur at boundary locations where two surfaces meet.

- In the case of a single nonlinear surface, the functions  $a(x, y)$ ,  $b(x, y)$  and  $c(x, y)$  are  $C^0$  continuous over the  $x$ - $y$  domain. Therefore, there should be no discontinuity between  $f_i$  and  $f_{i'}$  when  $i$  and  $i'$  are neighboring sites.
- In the case of a piecewise nonlinear surface containing roofs, *i.e.*  $C^1$  discontinuities in  $z$ , the parameter functions are piecewise  $C^0$  continuous over the  $x$ - $y$  domain. Therefore, two neighboring parameter vectors  $f_i$  and  $f_{i'}$  should be similar when  $z_i$  and  $z_{i'}$  belong to a continuous part of the surface; otherwise, there should be a discontinuity between  $f_i$  and  $f_{i'}$ .

### 4.3.2 MAP-MRF Solution

Firstly, let us define the prior for the parameter vector field  $f = \{f_1, \dots, f_m\}$  which is assumed to be Markovian. The  $a(x, y)$ ,  $b(x, y)$  and  $c(x, y)$  are piecewise constant functions when  $z(x, y)$  consists of piecewise planar surfaces, and therefore their partial derivatives with respect to  $x$  and  $y$  should be zero at non-edge locations. These constraints may be imposed by using the following “prior energy”

$$U(f) = \sum_i \sum_{i' \in \mathcal{N}_i} g(\rho(f_i, f_{i'})) \quad (4.54)$$

where  $\mathcal{N}_i$  is the set of neighbors of  $i$  (*e.g.* the 4 or 8 directly adjacent points),  $g$  is an adaptive potential function satisfying certain conditions (Li 1995b) including the necessary condition mentioned earlier (*e.g.*  $g(\eta) = 1/(1 + \eta^2)$  used for experiments in this paper), and

$$[\rho(f_i, f_{i'})]^2 = w_a[a_i - a_{i'}]^2 + w_b[b_i - b_{i'}]^2 + w_c[c_i - c_{i'}]^2 \quad (4.55)$$

is a weighted distance between  $f_i$  and  $f_{i'}$  where  $w > 0$  are the weights.

Then, let us define the observation model. Assume that the data  $d = \{d_1, \dots, d_m\}$ ,  $d_i = z_{f_i}(x_i, y_i) + e_i$ , is the true surface height  $z_i$  corrupted by independently and identically distributed (i.i.d.) Gaussian noise  $e_i$ . Then the conditional density of data  $p(d | f)$  is a Gibbs distribution with the following “likelihood energy”

$$U(d | f) = \sum_i [z_{f_i}(x_i, y_i) - d_i]^2 = \sum_i (a_i + b_i x_i + c_i y_i - d_i)^2 \quad (4.56)$$

This imposes the constraint from the data.

Now, the posterior  $p(f | d)$  is a Gibbs distribution with the “posterior energy” function  $U(f | d) = U(d | f) + U(f)$ . For  $z = a + bx + cy$ , the energy can be written as

$$U(f \mid d) = \sum_i (a_i + b_i x_i + c_i y_i - d_i)^2 + \sum_i \sum_{i' \in \mathcal{N}_i} g(\sqrt{w_a [a_i - a_{i'}]^2 + w_b [b_i - b_{i'}]^2 + w_c [c_i - c_{i'}]^2}) \quad (4.57)$$

The MAP estimate is defined as  $f^* = \arg \min_f U(f \mid d)$ .

The proposed MAP-MRF model generalizes the line process model of (Geman and Geman 1984). The latter is a special case of the former with  $b_i = c_i = 0$ : When  $b_i = c_i = 0$ , then the surface function is reduced to  $z_i = a_i$  and the two models become equivalent.

### 4.3.3 Computational Issues

The posterior energy  $E(f) = U(f \mid d)$  may be minimized by using a deterministic gradient descent algorithm to achieve  $\frac{\partial E(f)}{\partial a_i} = \frac{\partial E(f)}{\partial b_i} = 0$  for all  $i$ . The computation of gradient is illustrated below with the surface function  $z = a + bx + cy$  and the potential function  $g(\eta) = \ln(1 + \eta^2)$ . The gradient components are calculated as

$$\frac{1}{2} \frac{\partial E(f)}{\partial a_i} = (a_i + b_i x_i + c_i y_i - d_i) + \sum_{i' \in \mathcal{N}_i} w_a h(\rho(f_i, f_{i'}))(a_i - a_{i'}) \quad (4.58)$$

$$\frac{1}{2} \frac{\partial E(f)}{\partial b_i} = (a_i + b_i x_i + c_i y_i - d_i) x_i + \sum_{i' \in \mathcal{N}_i} w_b h(\rho(f_i, f_{i'}))(b_i - b_{i'}) \quad (4.59)$$

$$\frac{1}{2} \frac{\partial E(f)}{\partial c_i} = (a_i + b_i x_i + c_i y_i - d_i) y_i + \sum_{i' \in \mathcal{N}_i} w_c h(\rho(f_i, f_{i'}))(c_i - c_{i'}) \quad (4.60)$$

where

$$\rho(f_i, f_{i'}) = \sqrt{w_a [a_i - a_{i'}]^2 + w_b [b_i - b_{i'}]^2 + w_c [c_i - c_{i'}]^2} \quad (4.61)$$

and

$$h(\eta) = \frac{1}{1 + \eta^2} \quad (4.62)$$

determines the interactions between the two pixels  $i$  and  $i'$ . The updating is performed on  $(a_i, b_i, c_i)$  for all  $i \in \mathcal{S}$ . For example, the equation for updating  $b$  is  $b_i^{(t+1)} - b_i^{(t)} \leftarrow \Delta_b$  where

$$\Delta_b = \mu \left\{ (a_i + b_i x_i + c_i y_i - d_i) x_i + \sum_{i' \in \mathcal{N}_i} w_b h(\rho(f_i, f_{i'}))(b_i - b_{i'}) \right\} \quad (4.63)$$

with  $\mu > 0$  being a constant factor.

Annealing can be incorporated into the iterative gradient descent process to help escape from local minima. For example, introduce a “temperature” parameter  $T$  into the  $h$  function so that it becomes

$$h_T(\eta) = \frac{1}{1 + \eta^2/T} \quad (4.64)$$

At the beginning,  $T$  is set to a high value  $T^{(0)}$ . As the iteration continues, it is decreased towards the target value of 1 according to, e.g.  $T^{(t+1)} \leftarrow 0.9T^{(t)}$ . This is shown to be effective.

Randomization can also be incorporated into the gradient descent. In a simple method called randomized gradient descent, we introduce a random weight  $\lambda$  between the two terms in every updating for every  $i$  and  $n$ . For example, for the  $b$  parameter, this gives

$$\Delta_b = \mu \left\{ (a_i + b_i x_i + b_i y_i - d_i) x_i + \lambda \sum_{i' \in \mathcal{N}_i} w_b h(\rho(f_i, f_{i'})) (b_i - b_{i'}) \right\} \quad (4.65)$$

When  $\lambda$  is constant all the time, it reduces to the deterministic gradient descent. The randomized gradient descent strategy helps escaping from local minima and yields better results than the deterministic gradient descent. But it is more efficient than random sampling methods such as Metropolis algorithm (Metropolis et al. ) and Gibbs sampler (Geman and Geman 1984) because the acceptance probability of any update is always one. A randomized neighborhood system (Li 1990a) also works well in avoiding local minima.

## 4.4 Experimental Results

### 4.4.1 Step-Edge Preserving Smoothing

Two experimental results are presented in the following (more can be found in Chapter 3 of (Li 1991)). The first is the reconstruction of a real image of size  $256 \times 256$  (Fig.4.5). Here, APF 1 ( $g_{1\gamma}$ ) is used and the parameters are empirically chosen as  $\lambda = 50$ ,  $\gamma = 2$ . The result shows that the reconstructed image is much cleaner with discontinuities well preserved.

The second experiment is the detection of step and roof edges from a simulated noisy pyramid image of size  $128 \times 128$  (Fig.4.6). The detection process runs in three stages: 1) regularizing the input image and computing images of first derivatives in the two directions from the regularized image using finite difference, 2) regularizing the derivative images and 3) detecting steps and roofs by thresholding the regularized derivative images. APF 2 ( $g_{2\gamma}$ ) is used and the parameters are empirically chosen as  $\lambda = 10$ ,  $\gamma_{target} = 10$  for the first stage of the regularization and  $\lambda = 100$ ,  $\gamma_{target} = 0.002$  for the second stage.

Edges in the horizontal and vertical directions are best detected while those in the diagonal directions are not so well done. This is because only derivatives in the two axes directions are considered in the DA discussed so far; changes in the diagonal directions are largely ignored. Regularizers using the 8-neighborhood system (see the footnote for Eq.(4.53) should help improve the detection of diagonal changes.

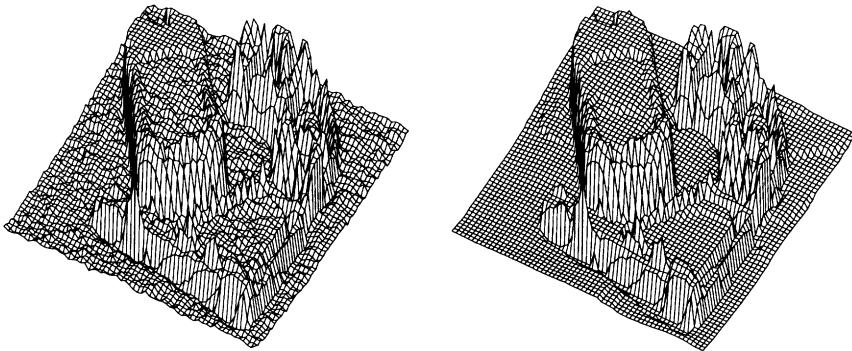


Figure 4.5: 3D plots of an image (left) and its reconstruction using DA (right). The plots are drawn after sampling the images at every 4 pixels in both directions into the size of  $64 \times 64$ . From (Li 1995b) with permission; ©1995 IEEE.

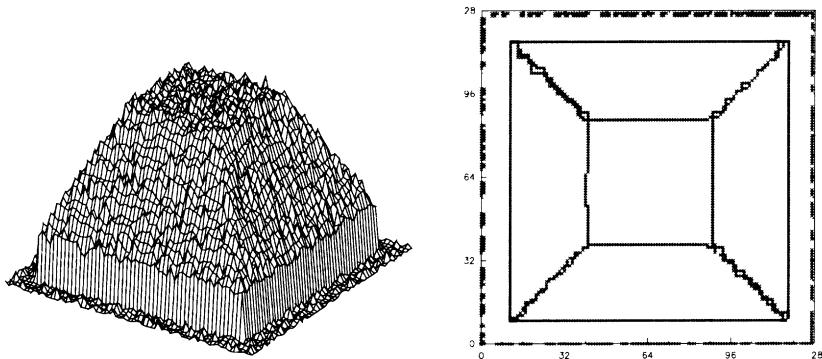


Figure 4.6: Step and roof edges (right) detected from a pyramid image (left). Step edges are shown in dots and roof edges in crosses. From (Li 1995b) with permission; ©1995 IEEE.

#### 4.4.2 Roof-Edge Preserving Smoothing

Range images are used because in such data there are geometrically well defined roof and step edges. The 8-adjacency is used for defining the MRF neighborhood system. The surfaces are assumed to be of the form  $z(x_i, y_i) = a_i + b_i x_i + c_i y_i$ . The  $a_i$ ,  $b_i$  and  $c_i$  parameters are initially estimated from the data by local bilinear fit. The MAP estimates  $a^*$ ,  $b^*$  and  $c^*$  are computed by minimizing  $U(f \mid d)$  (using an iterative gradient descent combined with annealing). The smoothed surface is reconstructed as  $z_i^* = a_i^* + b_i^* x_i + c_i^* y_i$ . Roof edges are detected by thresholding the directional derivatives of  $b^*$  and

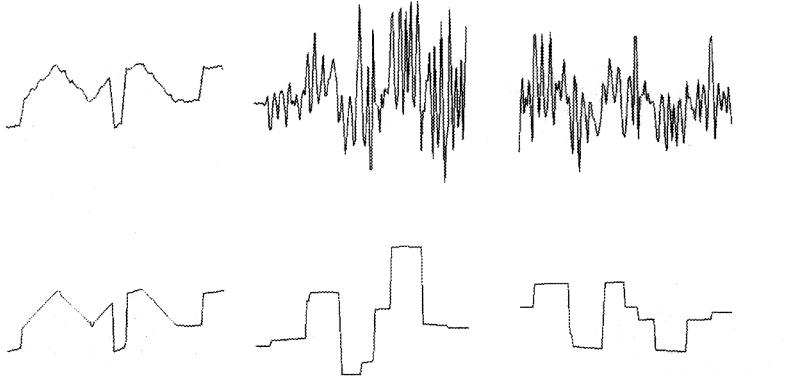


Figure 4.7: Smoothing 1D surface. Top (from left to right): surface data  $d(x)$ , initial estimates for  $a(x)$  and  $b(x)$  obtained by using finite difference. Bottom (from left to right): the smoothed surface  $z^*(x) = a^*(x) + b^*(x) \cdot x$ , and the MAP estimates  $a^*(x)$  and  $b^*(x)$ .

$c^*$ , and step edges are detected by thresholding  $z^*$ ; this is followed by a local maximum selection operation to get the final edge map.

Before presenting results for image surfaces on the 2D domain, let us look at a result for 1D surface. For the clean ideal roof in Fig.4.4, the proposed model produces results which are very close to the analytically calculated shapes shown in the top row of Fig.4.4, with sharp steps in  $a^*$  and  $b^*$  and hence a sharp roof in  $z^*$ . For a noisy input (Fig.4.7), the discontinuities in  $a^*$  and  $b^*$  are well preserved and there are no round-up's over the discontinuities. This demonstrates significant improvement over the line process model result shown at the bottom of Fig.4.4.

Now, let us look at two results for 2D surfaces. The first is obtained from a synthetic pyramid image composed of piecewise planar surfaces (Fig.4.8) containing ideal roof edges. The smoothed image has clean surfaces and clear roof boundaries as can be seen in the 3D plots. The the  $a, b, c$  and the edge maps before and after smoothing demonstrate the removal of noise and the preservation of discontinuities in the  $a, b$  and  $c$  parameter functions. The detection of step edges (the out-most part of the edge map) and roof edges (the inner part) is nearly perfect even with the simple thresholding edge detector.

The second result is obtained on a Renault part image composed of free-form surfaces with roof edges (Fig.4.9), noting that the orginal image is severely quantized in depth. In this case,  $a, b$  and  $c$  are piecewise continuous nonlinear functions (of  $x$  and  $y$ ) rather than piecewise constant. True roof edges emerge as a result of the edge-preserving smoothing. This result demonstrates that the proposed model also works well for free-form surfaces containing general roof edges.

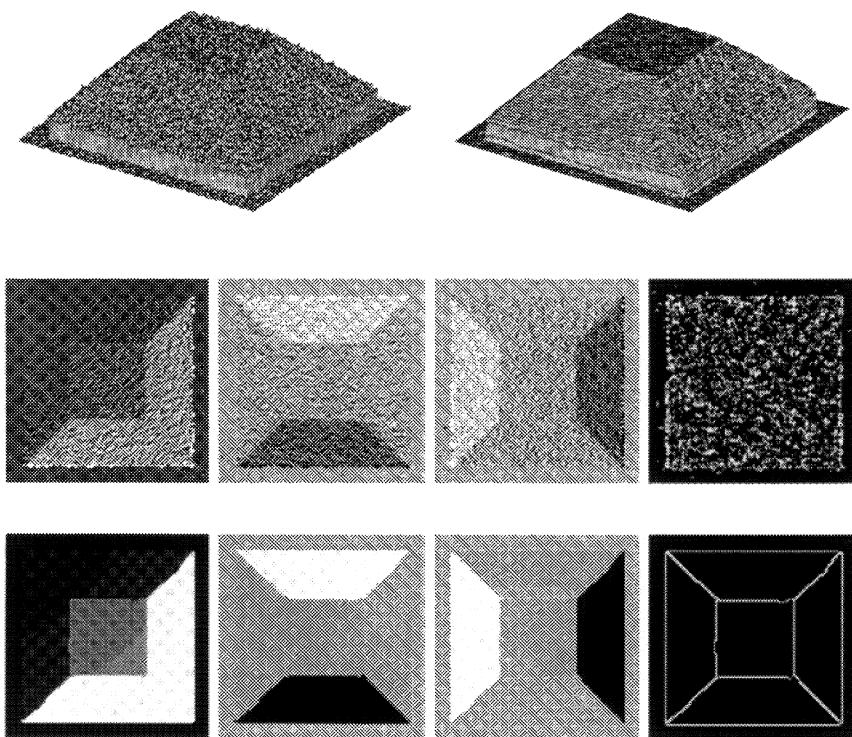


Figure 4.8: Row 1: 3D plots of a pyramid image before and after smoothing. Rows 2 and 3: The parameter and edge maps of the pyramid before (row 2) after (row 3) smoothing; from left to right are: parameter maps *a*, *b* and *c* and the detected edges.

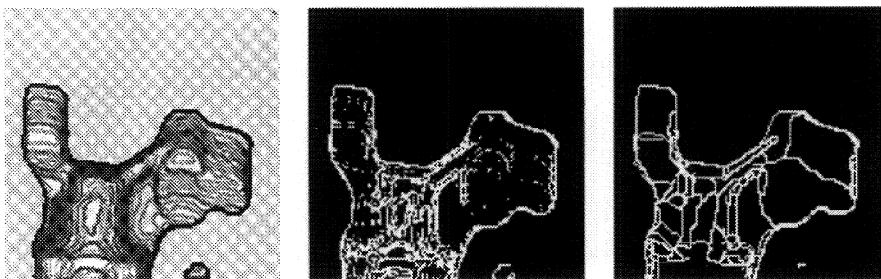


Figure 4.9: The Renault part image (left, shaded for visualization of the shape) and edges detected before (middle) and after (right) roof edge preserving smoothing.

## Chapter 5

# Discontinuity-Adaptivity Model and Robust Estimation

Robust methods (Tukey 1977; Huber 1981; Rousseeuw 1984) are tools for statistics problems in which *outliers* are an issue. It is well known that the least squares (LS) error estimates can be arbitrarily wrong when outliers are present in the data. A robust procedure is aimed to make solutions insensitive to the influence caused by outliers. That is, its performance should be good with all-inlier data and deteriorates gracefully with increasing number of outliers. The mechanism of robust estimators in dealing with outliers is similar to that of the discontinuity adaptive MRF prior model studied in the previous chapter. This chapter provides a comparative study (Li 1995a) of the two kinds of models based on the results about the DA model and presents an algorithm (Li 1996b) to improve the stability of the robust M-estimator to the initialization.

The conceptual and mathematical comparison comes naturally from the parallelism of the two models: Outliers cause violation of a distributional assumption while discontinuities cause violation of the smoothness assumption. Robustness to outliers is in parallel to adaptation to discontinuities. Detecting outliers is corresponding to inserting discontinuities. The likeness of the two models suggests that results in either model could be used for the other.

Probably for this reason, recent years have seen considerable interests in applying robust techniques to solving computer vision problems. Kashyap and Eom (1988) develop an robust algorithm for estimating parameters in an autoregressive image model where the noise is assumed to be a mixture of a Gaussian and an outlier process. Shulman and Herve (1989) (Shulman and Herve 1989) propose to use Huber's robust M-estimator to compute optical flow involving discontinuities. Stevenson and Delp (1990) use the same

estimator for curve fitting. Besl *et al.* (1988) propose a robust M window operator to prevent smoothing across discontinuities. Haralick *et al.* (1989), Kumar and Hanson (1989) and Zhuang *et al.* (1992) use robust estimators to find pose parameters. Jolion and Meer (1991) identify clusters in feature space based on the robust minimum volume ellipsoid estimator. Boyer *et al.* (1994) present a procedure for surface parameterization using a robust M estimator. Black *et al.* (1993,1994) apply a robust operator not only to the smoothness term but also to the data term. Li presents a comparative study on robust models and discontinuity adaptive MRF models (Li 1995a). He also presents a method for stabilizing robust M estimation with respect to the initialization and convergence (Li 1996b).

A robust location estimator, which essentially seeks for the mode of an outlier-contaminated distribution, can be extended to performing data clustering. In this connection, the mean shift algorithm (Fukunaga 1990) has been successfully used in vision problems such as segmentation (Cheng 1995; Comaniciu and Meer 1997; Comaniciu and Meer 1999).

As well known, robust estimation procedures have a serious problem that the estimates are dependent on the initial estimate value; this problem has been successfully overcome by applying the principle of the graduated non-convexity (GNC) method (Blake and Zisserman 1987) for visual reconstruction. The exchange of theoretical results and practical algorithms are useful to the vision community because both MRF and robust models have their applications in vision.

## 5.1 The DA Prior and Robust Statistics

What does it mean by discontinuities and outliers? Unfortunately, their definitions are usually ambiguous. What we are certain of is that the likelihood of a discontinuity between a pair of neighboring pixels is related to the difference in pixel labels (such as pixel values) and an outlier is related to the distance between the location of the datum and the estimated value. Where the label difference is very large, there is likely to be a discontinuity between the pixels; and where the datum is very far from the cluster, it is likely an outlier.

A more concrete comparison can be made by analyzing the adaptation (to discontinuities) and the robustness (to outliers) in mathematical terms (Li 1995a). The adaptation is realized as follows: The interaction between related (*e.g.* neighboring) points must be decreased as the violation of the relational bond between them is increased, and prohibited in the limit. This is true of both the MRF and the robust models. We give the necessary condition for such adaptation and then, based on this condition, a definition of a class of adaptive interaction functions for both models. The definition captures the essence of the adaptation ability and is general enough to offer in theory infinitely many suitable choices of such functions.

The problem of discontinuities and outliers also exists in other areas. In model based object recognition, for example, there are two related sub-problems: first, separating the scene into different parts, each being due to a single object; and second, finding feature correspondences between each separate part of the scene and an object. The two sub-problems somewhat have to be solved simultaneously. The process of matching while separating is similar to reconstruction with discontinuities and estimation with outliers. Indeed, matches to one object can be considered as outliers w.r.t. matches to a different object. Different groups of matches should not be constraining to each other. The separation can be done as inserting “discontinuities” between different groups. This view can be regarded as a generalization of the weak constraint (Hinton 1978; Blake and Zisserman 1987).

### 5.1.1 Robust M Estimator

Robust statistics methods (Tukey 1977; Huber 1981) provide tools for statistics problems in which underlying assumptions are inexact. A robust procedure should be insensitive to departures from underlying assumptions caused by, for example, *outliers*. That is, it should have good performance under the underlying assumptions and the performance deteriorates gracefully as the situation departs from the assumptions. Applications of robust methods in vision are seen in image restoration, smoothing and segmentation (Kashyap and Eom 1988; Jolion et al. 1991; Meer et al. 1991), surface and shape fitting (Besl et al. 1988; Stein and Werman 1992) and pose estimation (Haralick et al. 1989) where outliers are an issue.

There are several types of robust estimators. Among them are M-estimator (maximum likelihood type estimator), L-estimator (linear combinations of order statistics) and R-estimator (estimator based on rank transformation) (Huber 1981); RM estimator (repeated median) (Siegel 1982) and LMS estimator (estimator using the least median of squares) (Rousseeuw 1984). We are concerned with the M-estimator.

The essential form of the M-estimation problem is the following: Given a set of  $m$  data samples  $d = \{d_i \mid 1 \leq i \leq m\}$  where  $d_i = f + \eta_i$ , the problem is to estimate the location parameter  $f$  under noise  $\eta_i$ . The distribution of  $\eta_i$  is not assumed to be known exactly. The only underlying assumption is that  $\eta_1, \dots, \eta_m$  obey a symmetric, independent, identical distribution (symmetric i.i.d.). A robust estimator has to deal with departures from this assumption.

Let the residual errors be  $\eta_i = d_i - f$  ( $i = 1, \dots, m$ ) and the error penalty function be  $g(\eta_i)$ . The M-estimate  $f^*$  is defined as the minimum of a global error function

$$f^* = \arg \min_f E(f) \quad (5.1)$$

where

$$E(f) = \sum_i g(d_i - f) \quad (5.2)$$

Type	$h_\gamma(\xi)$	$g_\gamma(\xi)$	Range of $\xi$
Tukey	$= \begin{cases} (1 - \xi^2)^2 \\ 0 \end{cases}$	$= \begin{cases} \frac{1}{6}[1 - (1 - \xi^2)^3] \\ \frac{1}{6} \end{cases}$	$ \xi  \leq 1$ $ \xi  > 1$
Huber	$= \begin{cases} 1 \\ \tau \frac{\text{sgn}(\xi)}{\xi} \end{cases}$	$= \begin{cases} \xi^2 \\ 2\tau \xi  - \tau^2 \end{cases}$	$ \xi  \leq \tau$ $ \xi  > \tau$
Andrews	$= \begin{cases} \frac{\sin(\pi\xi)}{\pi u} \\ 0 \end{cases}$	$= \begin{cases} \frac{1}{\pi^2}[1 - \cos(\pi\xi)] \\ \frac{1}{\pi^2} \end{cases}$	$ \xi  \leq 1$ $ \xi  > 1$
Hampel	$= \begin{cases} 1 \\ a \frac{\text{sgn}(\xi)}{\xi} \\ a \frac{c -  \xi }{c - b} \frac{\text{sgn}(\xi)}{\xi} \\ 0 \end{cases}$	$= \begin{cases} \frac{1}{2}u^2 \\ a u  - \frac{1}{2}a^2 \\ ab - \frac{1}{2}a^2 + \\ (c - b)\frac{a}{2} \left[1 - \left(\frac{c -  \xi }{c - b}\right)\right] \\ ab - \frac{1}{2}a^2 + (c - b)\frac{a}{2} \end{cases}$	$ \xi  \leq a$ $a <  \xi  \leq b$ $b <  \xi  \leq c$ $ \xi  > c$

Table 5.1: Robust functions.

To minimize above, it is necessary to solve the following equation

$$\sum_i g'(d_i - f) = 0 \quad (5.3)$$

This is based on *gradient descent*. When  $g(\eta_i)$  can also be expressed as a function of  $\eta_i^2$ , its first derivative can take the following form

$$g'(\eta_i) = 2\eta_i h(\eta_i) = 2(d_i - f)h(d_i - f) \quad (5.4)$$

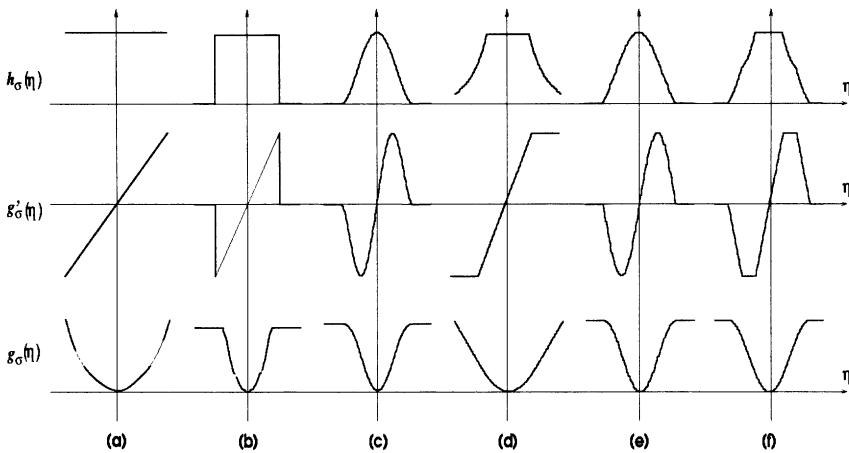
where  $h(\eta)$  is an even function. In this case, the estimate  $f^*$  can be expressed as the following weighted sum of the data samples

$$f^* = \frac{\sum_i h(\eta_i) d_i}{\sum_i h(\eta_i)} \quad (5.5)$$

where  $h$  acts as the weighting function.

In the LS regression, all data points are weighted equally with  $h_\gamma(\eta) = 1$  and the estimate is  $f^* = \frac{1}{m} \sum_{i=1}^m d_i$ . When outliers are weighted equally as inliers, it will cause considerable bias and deterioration in the quality of the estimate. In robust M estimation, the function  $h$  provides adaptive weighting. The influence from  $d_i$  is decreased when  $|\eta_i| = |d_i - f|$  is very large and suppressed when it is infinitely large.

Table 5.1 lists some robust functions used in practice where  $\xi = \eta/\gamma$ . They are closely related to the adaptive interaction function and adaptive potential function defined in (4.27) and (4.28). Fig.5.1 shows their qualitative shapes in comparison with the quadratic and the line process models (note that a trivial constant may be added to  $g_\gamma(\eta)$ ). These robust functions are piecewise as in the line process model. Moreover, the parameter  $\gamma$  in  $\xi$  is dependent on some scale estimate, such as the median of absolute deviation (MAD).



where  $S$  is an estimate of spread,  $c$  is a constant parameter and  $cS$  is the scale estimate. Possible choices are  $S = \text{median}\{\eta_i\}$  with  $c$  set to 6 or 9; or  $S = \text{median}\{|\eta_i - \text{median}\{\eta_i\}|\}$  (median of absolute deviation (MAD)) with  $c = 1.4826$  chosen for the best consistency with the Gaussian distribution. Classical scale estimates such as the median and MAD are not very robust. Design of scale estimates is crucial and needs devoted studies.

Furthermore, the convergence of the M-estimator is often not guaranteed. Divergence can occur when initialization or parameters are not chosen properly. Owing to the above problems, the theoretical breakdown point can hardly be achieved.

In the following, an improved robust M-estimator, referred to as *annealing M-estimator* (AM-estimator), is presented to overcome the above problems. It has two main ingredients: a redefinition of the M-estimator and a GNC-like annealing algorithm.

### 5.1.3 Redefinition of M Estimator

Resemblances between M estimation with outliers and adaptive smoothing with discontinuities have been noted by several authors (Besl et al. 1988; Shulman and Herve 1989; Black and Anandan 1993; Black and Rangarajan 1994; Li 1995a). We can compare the M estimator with the DA model studied in Chapter 4. The influence of the datum  $d_i$  on the estimate  $f$  is proportional to  $\eta_i h(\eta_i)$ . This compares to the smoothing strength  $f' h(f')$  given after equ.(4.25). A very large  $|\eta_i|$  value, due to  $d_i$  being far from  $f$ , are likely an outlier. This is similar to saying that very large  $|f'(x)|$  value is likely due to a step (discontinuity) in the signal there. The resemblance suggests that the definition of the DA model can also be used to define M estimators (Li 1996b).

We replace the scale estimate in the M estimator by a parameter  $\gamma > 0$  and choose to use the adaptive interaction function  $h_\gamma$  and the adaptive potential function  $g_\gamma$  for the M estimation. However,  $h_\gamma$  needs only be  $C^0$  continuous for the location estimation from discrete data. Theoretically, the definitions give an infinite number of suitable choices of the M estimators. Table 4.1 and Fig.4.1 showed four such possibilities. With  $h_\gamma$  and  $g_\gamma$ , we can define the energy under  $\gamma$  as

$$E_\gamma(f) = \sum_i g_\gamma(d_i - f) \quad (5.7)$$

and thereby the minimum energy estimate

$$f_\gamma^* = \arg \min_f E_\gamma(f) = \frac{\sum_i h_\gamma(\eta_i) d_i}{\sum_i h_\gamma(\eta_i)} \quad (5.8)$$

This defines a class of M estimators which are able to deal with outliers as the traditional M Estimators are. Its performance in the solution quality is significantly enhanced by the use of an annealing procedure.

**AM-Estimator**

Begin Algorithm

set  $t = 0$ ,  $f_\gamma^{(1)} = f^{LS}$ ; choose initial  $\gamma$ ;

do {

 $t \leftarrow t + 1$ ;    compute errors  $\eta_i = d_i - f_\gamma^{(t-1)}$ ,  $\forall i$ ;    compute weighted sum  $f_\gamma^{(t)} = \frac{\sum_i h_\gamma(\eta_i) d_i}{\sum_i h_\gamma(\eta_i)}$ ;    if  $(|f_\gamma^{(t)} - f_\gamma^{(t-1)}| < \epsilon)$  /\* converged \*/         $\gamma \leftarrow \text{lower}(\gamma)$ ;    } until  $(\gamma < \delta)$  /\* frozen \*/     $f^* \leftarrow f_\gamma^{(t)}$ ;

End Algorithm

Figure 5.2: The AM-estimation algorithm.

**5.1.4 AM Estimator**

The annealing algorithm for the redefined M-estimator, called the AM-estimator (Li 1996b), is based on the idea of GNC algorithm (Blake and Zisserman 1987). It aims to overcome the local minimum problem in the M estimation, *i.e.* to obtain good estimate regardless of the initialization. It also make the estimation free from parameters or at least insensitive to their choices.

The annealing is performed by continuation in  $\gamma$  and the AM-estimator is defined in the limit

$$f^* = \lim_{\gamma \rightarrow 0^+} f_\gamma^* \quad (5.9)$$

An algorithm which implements the AM-estimator algorithm is given in Fig.5.2. Initially,  $\gamma$  is set to a value high enough to guarantee that the corresponding APF  $g_\gamma(\eta)$  is convex in an interval. With such  $\gamma$ , it is easy to find the unique minimum of the global error function  $E_\gamma(f)$  using the gradient descent method, regardless of the initial value for  $f$ . The minimum is then used as the initial value for the next phase of minimization under a lower  $\gamma$  to obtain the next minimum. As  $\gamma$  is lowered,  $g_\gamma(\eta)$  may be no longer convex and local minima may appear. However, if we track the global minima for decreasing  $\gamma$  values, we may hopefully approximate the global minimum  $f^*$  under  $\gamma \rightarrow 0^+$ .

Obviously, whatever the initialization is, the first iteration always gives a value equal to the LS estimate

$$f^{LS} = \frac{1}{m} \sum_{i=1}^m d_i \quad (5.10)$$

This is because for  $\gamma \rightarrow +\infty$  which guarantees the strict convexity, all weights are the same as  $h_\gamma(\eta_i) = 1$ . The initial  $\gamma$  is chosen to satisfy the following

$$|\eta_i| = |d_i - f^{LS}| < b_H(\gamma) \quad (5.11)$$

where  $b_H(\gamma)$  ( $= -b_L(\gamma)$ ) is the upper bound of the band in (4.29). This guarantees  $g_\gamma''(\eta_i) > 0$  and hence the strict convexity of  $g_\gamma$ . The parameter  $\gamma$  is lowered according to the schedule specified by the function  $\text{lower}(\gamma)$ . The parameters  $\delta$  and  $\epsilon$  in the convergence conditions are some small numbers. An alternative way is to decrease  $\gamma$  according to a fixed schedule regardless of whether or not  $f^{(t)}$  is converged at current  $\gamma$ , which is equivalent to setting a big value for  $\epsilon$ . In this case, the algorithm freezes after dozens of iterations. This quick annealing is used in our experiments.

The advantages of the AM-estimator is summarized below. Firstly, the use of the annealing significantly improves the quality and stability of the estimate. The estimate is made independent of the initialization. Because the start point for obtaining  $f_\gamma^*$  at current  $\gamma$  is the convergence point obtained with the previous  $\gamma$  value, the divergence problem with the traditional M estimator is minimized. Secondly, the definition of the AM-estimator effectively eliminates scale parameters in the M estimation because  $\gamma$  is finally set to zero (or a small number to whose value the final estimate is insensitive). This avoids the instability problem incurred by inappropriate selection of the scale parameters. Furthermore, it needs no order statistics, such as the median, hence no sorting. This improves the computational efficiency.

### 5.1.5 Convex Models for DA and M-Estimation

Encoding the edge-preserving ability into prior distributions may lead to *nonconvex* energy functions. This is the case in many models such as the well-known line-process model (Geman and Geman 1984; Blake and Zisserman 1987). Models with nonconvex energy functions have two disadvantages. (i) The first is the instability of the solution, *i.e.* the energy minimum, with respect to the data (Bouman and Sauer 1993; Stevenson et al. 1994). A small change in the input might result in a drastic difference in the solutions. The phenomenon is also due to a hard decision-making property of nonconvex models (Blake and Zisserman 1987). As such, the solution often depends substantially on the method used to perform the minimization. (ii) The second disadvantage is the high computational cost associated with the solution finding procedure. An annealing process, either deterministic or stochastic, is incorporated into a local search algorithm in order to locate the global minimum (Geman and Geman 1984; Blake and Zisserman 1987). This makes the minimization procedure inefficient.

In recent years, there has been considerable interest in convex energy models with edge-preserving ability (Shulman and Herve 1989; Green 1990; Lange 1990; Bouman and Sauer 1993; Stevenson et al. 1994; Li et al. 1995). This class of models overcome the above mentioned problems with nonconvex functions. Firstly, in terms of defining the minimal solution, the convexity guarantees the stability with respect to the input and makes the solution less sensitive to changes in the parameters (Bouman and Sauer 1993). The second advantage is the computational efficiency in searching for the global solution. Because there is only one unique solution, gradient-based minimization techniques can be efficiently utilized. Time consuming techniques for tackling the local minimum problem, such as continuation or annealing, is not necessary in convex minimization.

Shulman and Herve (Shulman and Herve 1989) propose to use

$$g(\eta) = \begin{cases} \eta^2 & |\eta| \leq \gamma \\ 2\gamma|\eta| - \gamma^2 & |\eta| > \gamma \end{cases} \quad (5.12)$$

for computing optical flow involving discontinuities. This is the error function used in Huber's robust M-estimator (Huber 1981). The function has also been applied to curve fitting (Stevenson and Delp 1990), surface reconstruction (Stevenson et al. 1994) and image expansion (Schultz and Stevenson 1994). It is shown to be advantageous in terms of computational complexity and reconstruction quality (Stevenson et al. 1994).

Similar to Huber function, Green's function (Green 1990)

$$g(\eta) = \ln(\cosh(\eta/\gamma)) \quad (5.13)$$

is approximately quadratic for small  $\eta$ , and linear for large values, Lange suggests to use

$$g(\eta) = \frac{1}{2} \left( |\eta/\gamma| + \frac{1}{1 + |\eta/\gamma|} - 1 \right) \quad (5.14)$$

which can be described by seven properties (Lange 1990).

Bouman and Sauer (Bouman and Sauer 1993) construct a scale-invariant Gaussian MRF model by using

$$g(\eta) = |\eta|^p \quad (5.15)$$

where  $1.0 \leq p \leq 2.0$ . When  $p = 2$ , it becomes the standard quadratic function. When  $p = 1$ , the corresponding estimator is the sample median and allows discontinuities. The results show that edges are best preserved when  $p = 1$  and deteriorated for  $p > 1$ . The reason will be given in the next section.

Stevenson *et al.* (Stevenson et al. 1994) present a systematic study on both convex and nonconvex models and give the following four properties for a function  $g$  to have good behaviors: (i) convex, (ii) symmetric, i. e.  $g(\eta) = g(-\eta)$ , (iii)  $g(\eta) < \eta^2$  for  $|\eta|$  large to allow discontinuities, (iv) controlled continuously by a parameter  $\gamma$ . They define a class of convex potential functions

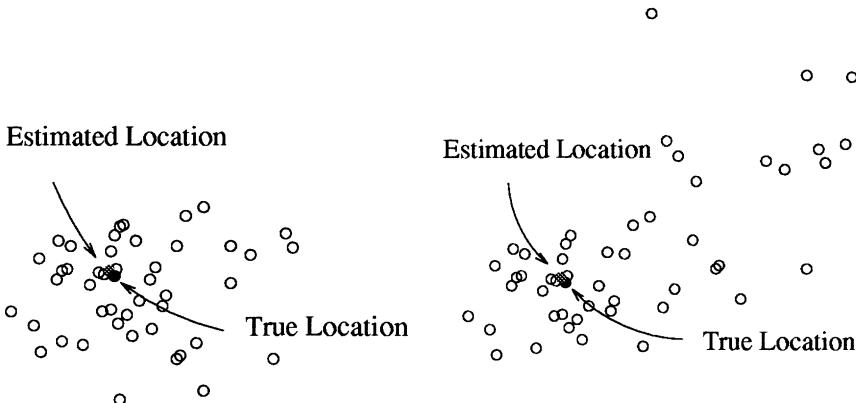


Figure 5.3: The AM estimate of location. From (Li 1995a) with permission; ©1995 Elsevier.

$$g(\eta) = \begin{cases} |\eta|^p & |\eta| \leq \gamma \\ (|\eta| + (\frac{p}{q}\gamma^{p-1})^{\frac{1}{q-1}} - \gamma)^q & +\gamma^p - (\frac{p}{q}\gamma^{p-1})^{\frac{q}{q-1}} & |\eta| > \gamma \end{cases} \quad (5.16)$$

with three parameters,  $\gamma$ ,  $p$  and  $q$ . When  $1.0 \leq p \leq 2.0$  and  $\gamma = \infty$ , it is the same as that used by Bouman and Sauer; when  $p = 2.0$  and  $q = 1.0$ , it is the Huber error function. The values  $p = 1.8$  and  $q = 1.2$  are suggested that paper. We point out that (iii) is too loose and inadequate for preserving discontinuities. An example is  $g(\eta) = \eta^2 - g_0$ , where  $g_0 > 0$  is a constant; it satisfies (iii) but is unable to preserve discontinuities. As pointed out in the next section, it is the derivative of  $g(\eta)$  that determines how a model respond to discontinuities.

## 5.2 Experimental Comparison

Two experiments are presented. The first is a general comparison of two estimators, the AM-estimator and the M-estimator with Tukey's biweight function, with simulated data. The second deals with an application. Experimental results demonstrate that the AM-estimator is significantly better than the traditional M-estimator in the estimation accuracy, stability, and breakdown point.

### 5.2.1 Location Estimation

Simulated data points in 2D locations are generated. The data set is a mixture of true data points and outliers. First,  $m$  true data points  $\{(x_i, y_i) \mid i = 1, \dots, m\}$  are randomly generated around  $\bar{f} = (10, 10)$ . The values of  $x_i$  and

$y_i$  obey an identical, independent Gaussian distribution with a fixed mean value of 10 and a variance value  $V$ . After that, a percentage  $\lambda$  of the  $m$  data points are replaced by random outlier values. The outliers are uniformly distributed in a square of size  $100 \times 100$  centered at  $(b, b) \neq \bar{f}$ . There are four parameters to control the data generation. Their values are:

1. the number of data points  $m \in \{50, 200\}$ ,
2. the noise variance  $V \in \{0, 2, 5, 8, 12, 17, 23, 30\}$ ,
3. the percentage of outliers  $\lambda$  from 0 to 70 with step 5, and
4. the outlier square center parameter  $b = 22.5$  or  $50$ .

The experiments are done with different combination of the parameter values. The AIF is chosen to be  $h_{3\gamma}(\eta) = 1.0/(1 + \eta^2/\gamma)$ . The schedule in lower( $T$ ) is:  $\gamma \leftarrow \left(\frac{100}{t^2}\right)^{1.5} - 1$ ; when time  $t \rightarrow \infty$ ,  $\gamma \rightarrow 0^+$ . It takes about 50 iterations to converge for each of these data sets.

Fig. 5.3 shows two typical data distributions and estimated locations. Each of the two data sets contains 32 Gaussian-distributed true data points and 18 uniformly distributed outliers. The two sets differ only in the arrangement of outliers while the true data points are common to both sets. The algorithm takes about 50 iterations to converge for each of these data sets. The estimated locations for the two data sets is marked in Fig. 5.3. The experiments show that the estimated locations are very stable regardless of the initial estimate though the outliers arrangement is quite different in the two sets. Without the use of AM, the estimated location would have been very much dependent on the initialization.

In a quantitative comparison, two quantities are used as the performance measures: (1) the mean error  $\bar{e}$  versus the percentage of outliers (PO)  $\lambda$  and (2) the mean error  $\bar{e}$  versus the noise variance (NV)  $V$ . Let the Euclidean error by  $e = \|f^* - \bar{f}\| = \sqrt{(x^* - 10)^2 + (y^* - 10)^2}$  where  $f^*$  is the estimate and  $\bar{f}$  is the true location.

Fig. 5.4 and 5.5 shows the mean error of the AM-estimator and the M-estimator, respectively. Every statistic for the simulated experiment is made based on 1000 random tests and the data sets are exactly the same for the two compared estimators. Outliers are uniformly distributed in a square centered at  $b = 22.5$  (the left columns) or  $b = 50$  (the right columns). The plots show the mean error vs. percentage of outliers with  $m = 50$  (row 1) and  $m = 200$  (row 2) and the mean error vs. noise variance with  $m = 50$  (row 3) and  $m = 200$  (row 4). It can be seen that the AM-estimator has a very stable and elegant behavior as the percentage of outliers and the noise variance increase; in contrast, the M-estimator not only gives higher error but also has an unstable behavior.

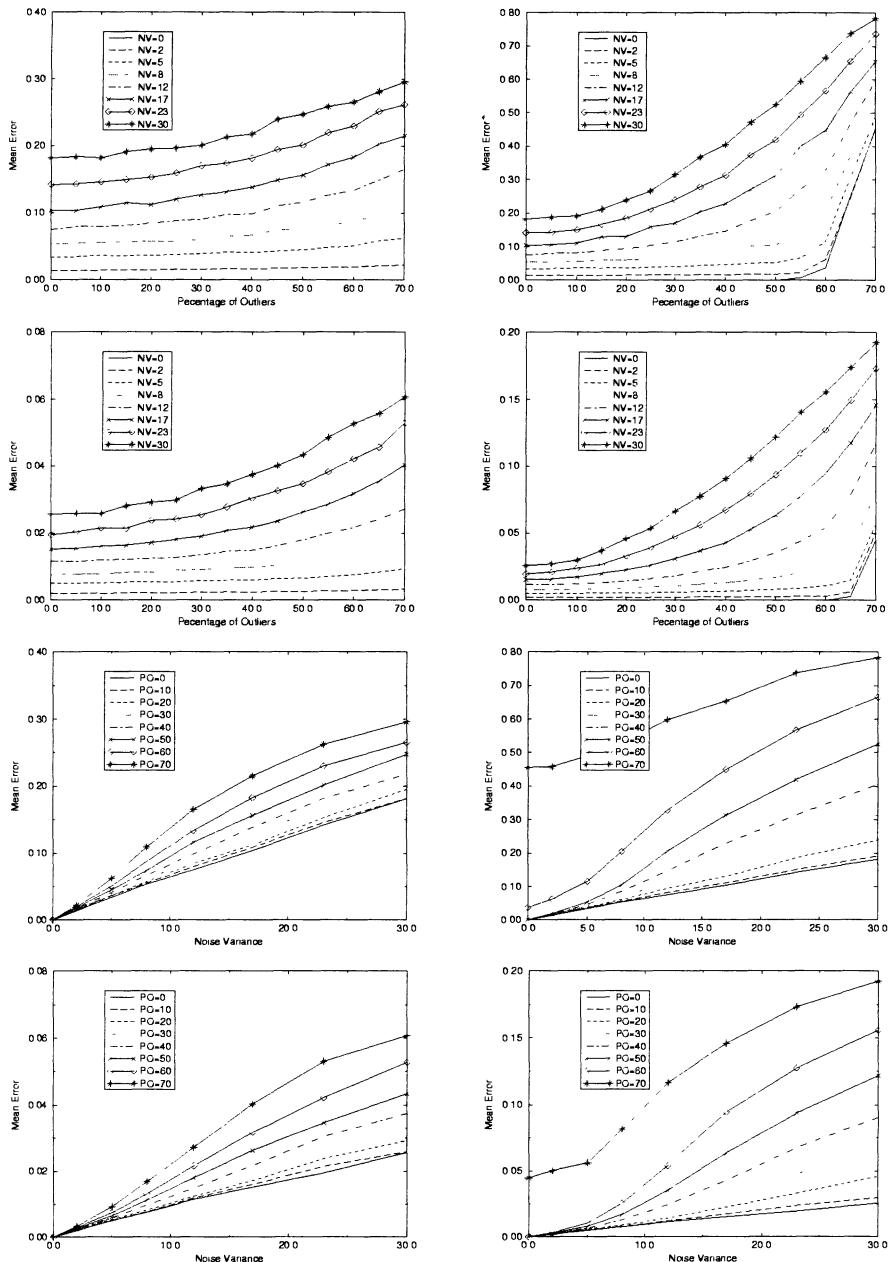


Figure 5.4: Mean error of the AM-estimate. From (Li 1995e) with permission; ©1995 Elsevier.

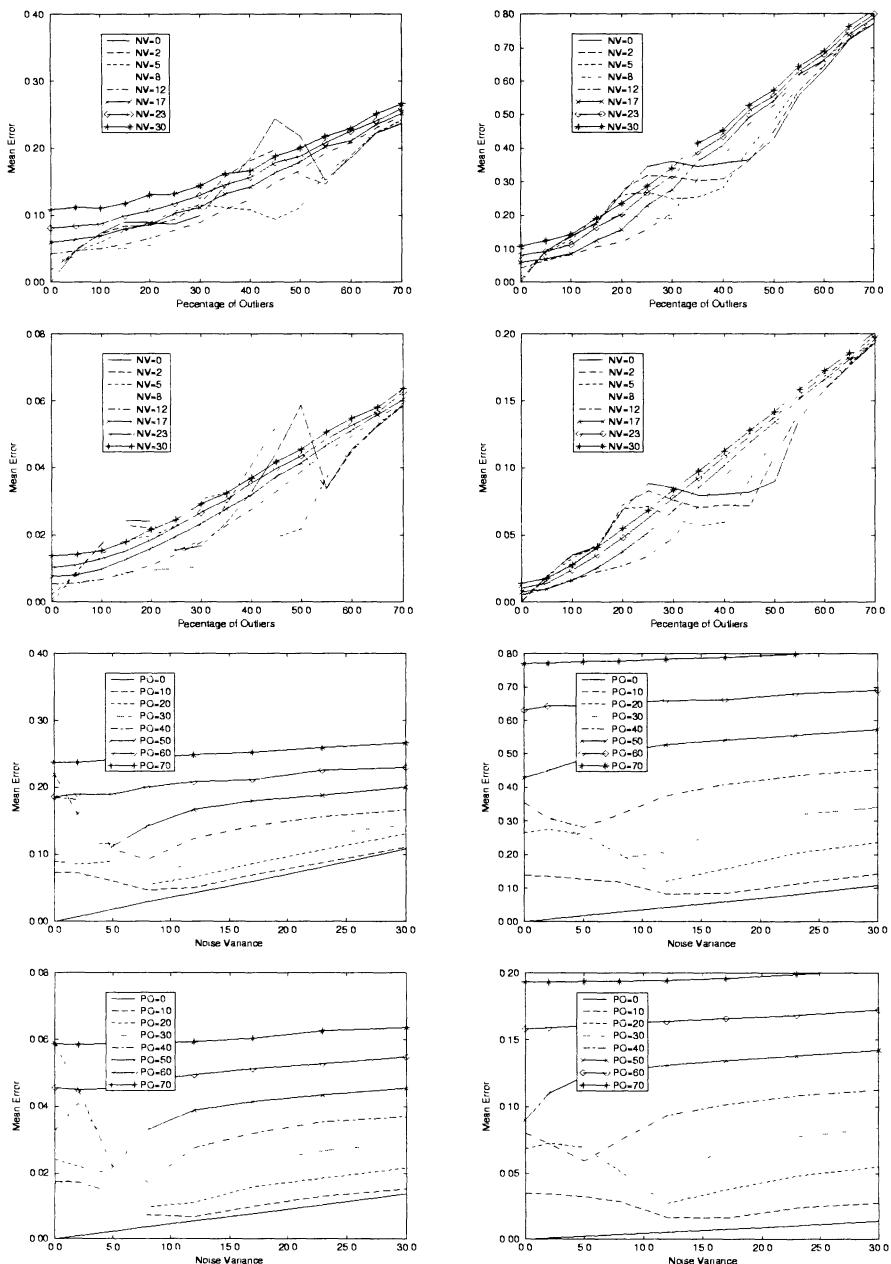


Figure 5.5: Mean error of the M-estimate. From (Li 1995e) with permission; ©1995 Elsevier.

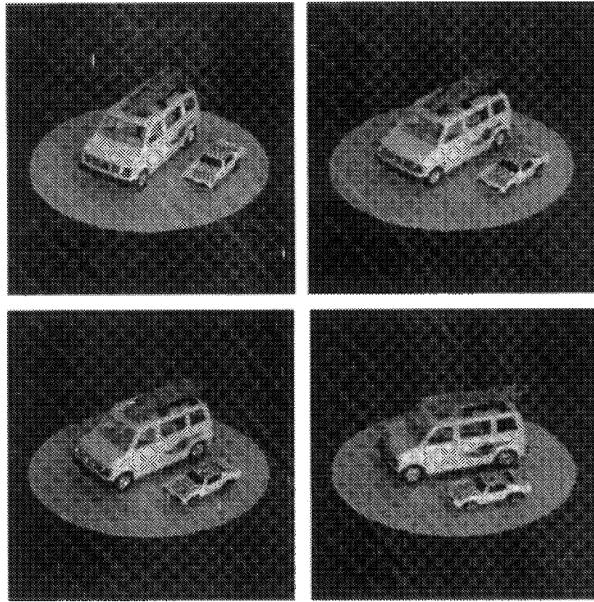


Figure 5.6: Part of a sequence of images rotating at 10 degrees between adjacent frames. The image size is  $256 \times 256$ . Courtesy of H. Wang.

### 5.2.2 Rotation Angle Estimation

This experiment compares the AM-estimator with the M-estimator in computing relative rotation of motion sequences. Consider a sequence of images in Fig.5.6. Corners can be detected from these images as in Fig.5.7 using the Wang-Brady detector (Wang and Brady 1991). The data

$$d = \{(p_i, p'_i) \mid i = 1, \dots, m\} \quad (5.17)$$

where  $p_i = (x_i, y_i)$  and  $p'_i = (x'_i, y'_i)$ , is a set of matched point pairs between two images. A previous work (Wang and Li 1994) shows that when the rotation axis  $\ell = (\ell_x, \ell_y, \ell_z)^T$  is known, a unique solution can be computed using only one pair of corresponding points and the LS solution can be obtained using  $m$  pairs by minimizing

$$E(f) = \sum_{i=1}^m \{A_i \tan(f/2) + B_i\}^2 \quad (5.18)$$

where

$$\begin{aligned} A_i &= \ell_z(\ell_y(x_i + x'_i) - \ell_x(y_i + y'_i)) \\ B_i &= \ell_x(x_i - x'_i) + \ell_y(y_i - y'_i). \end{aligned} \quad (5.19)$$

A unique solution exists for the LS problem (Wang and Li 1994). It is determined by the equation

$$\sum_{i=1}^m \{[A_i \tan(f/2) + B_i] \cdot A_i/2 \cdot \sec^2(f/2)\} = 0 \quad (5.20)$$

where  $f^* \neq 180^\circ$ , that is,

$$f^* = 2 \arctan\left(-\frac{\sum A_i B_i}{\sum A_i^2}\right) \quad (5.21)$$

The above formulation is based on an assumption that all the pairs  $\{(x_i, y_i), (x'_i, y'_i)\}$  are correct correspondences. This may not be true in practice. For example, due to acceleration and deceleration, turning and occlusion, the measurements can change drastically and false matches, *i.e.* outliers, can occur. The LS estimate can get arbitrarily wrong when outliers are present in the data  $d$ . When outliers are present, the M-estimator can produce the more reliable estimate than the LS estimator. The AM-estimator further improve the M-estimator to a significant extent.

The AM-estimator minimizes, instead of (5.18), the following

$$E(f) = \sum_{i=1}^m g_\gamma(A_i \tan(f/2) + B_i) \quad (5.22)$$

where  $g_\gamma$  is an adaptive potential function. By setting  $\frac{dE}{df} = 0$  and using  $g'_\gamma(\eta) = 2\eta h_\gamma(\eta)$ , one obtains

$$\sum_{i=1}^m \{[A_i \tan(f/2) + B_i] \cdot h_\gamma(A_i \tan(f/2) + B_i) \cdot A_i/2 \cdot \sec^2(f/2)\} = 0 \quad (5.23)$$

Rearranging the above gives the following fixed-point equation

$$f = 2 \arctan\left(-\frac{\sum_{i=1}^m h_i A_i B_i}{\sum_{i=1}^m h_i A_i^2}\right) \quad (5.24)$$

where  $h_i = h_\gamma(A_i \tan(f/2) + B_i)$ . It is solved iteratively with decreasing  $\gamma$  value.

Figs.5.8 and 5.9 show the estimated rotation angles (in the vertical direction) between consecutive frames (the label on the horizontal axis is the frame number) and the corresponding standard deviations (in vertical bars) computed using the LS-, M- and AM-estimators. From Fig.5.8, we see that with 20% outliers, the M-estimator still works quite well while the LS estimator has broken down. In fact, the breakdown point of the LS estimator is less than 5%.

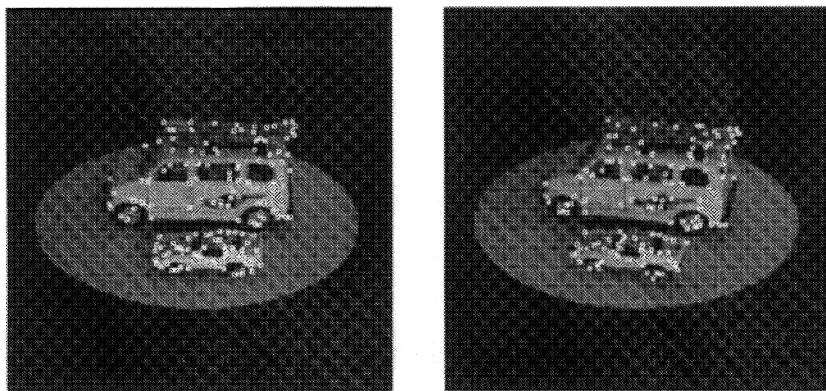


Figure 5.7: Corners detected. Courtesy of H. Wang.

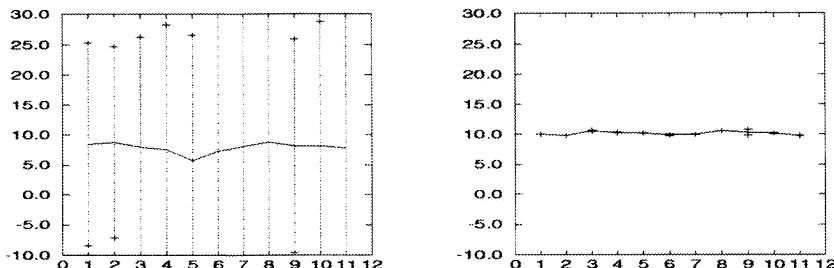


Figure 5.8: Rotation angles computed from the correspondence data containing 20% of outliers using the LS estimator (left) and the M-estimator (right).

From Fig.5.9, we see that M- and AM-estimators are comparable when the data contains less than 20% of outliers. Above this percentage, the AM-estimator demonstrates its enhanced stability. The AM-estimator continues to work well when the M-estimate is broken down by outliers. The AM-estimator has the breakdown point of 60%. This illustrates that the AM-estimator has a considerably higher actual breakpoint than the M-estimator.

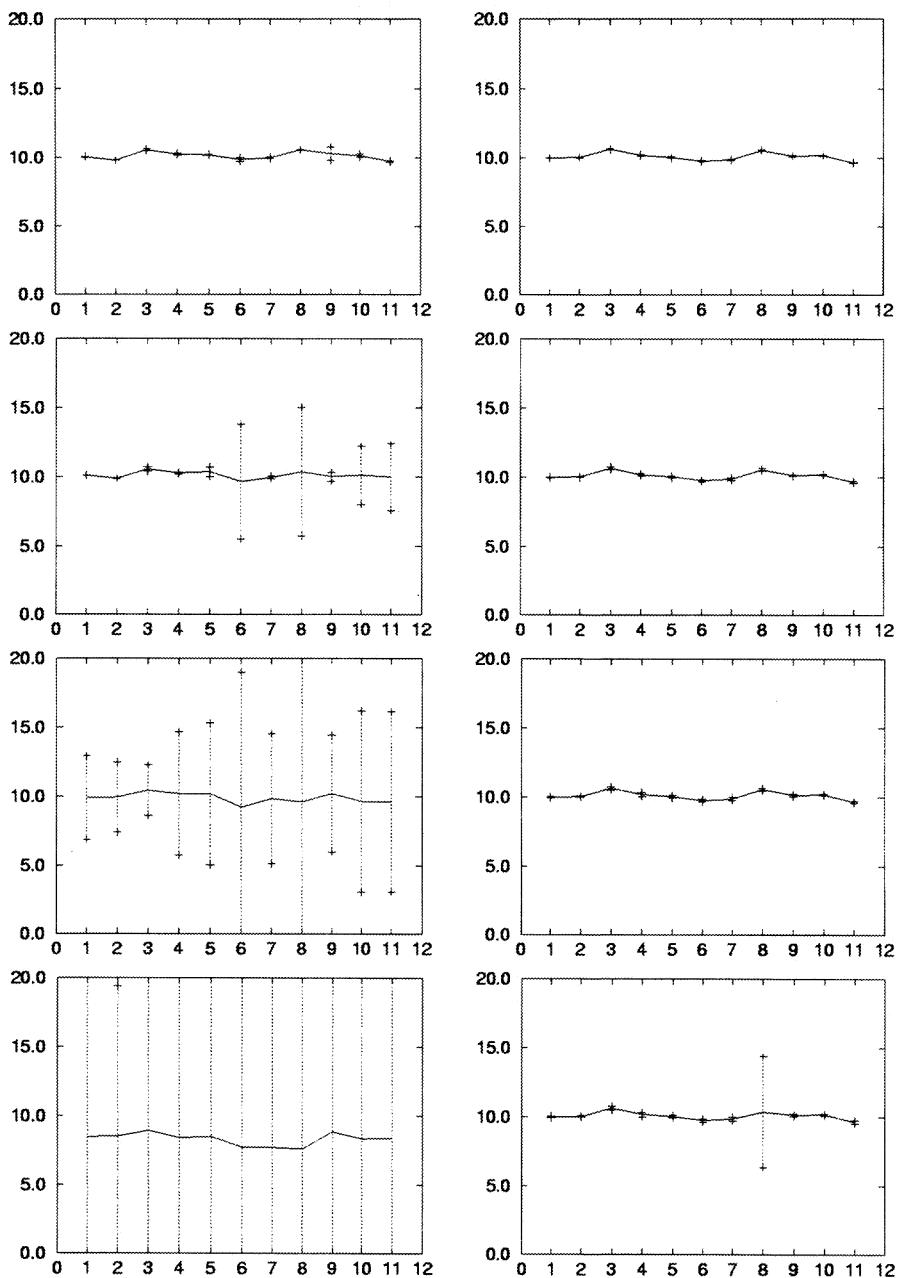


Figure 5.9: Results computed from the correspondence data containing 20% (row 1), 40% (row 2), 50% (row 3) and 60% (row 4) of outliers using the M-estimator (left) and the AM-estimator (right).

## Chapter 6

# MRF Parameter Estimation

A probabilistic distribution function has two essential elements: the form of the function and the involved parameters. For example, the joint distribution of an MRF is characterized by a Gibbs function with a set of clique potential parameters; and the noise by a zero-mean Gaussian distribution parameterized by a variance. A probability model is incomplete if the involved parameters are not all specified even if the functional form of the distribution is known. While formulating the forms of objective functions such as the posterior distribution has long been a subject of research for in vision, estimating the involved parameters has a much shorter history. Generally, it is performed by optimizing a statistical criterion, *e.g.* using existing techniques such as maximum likelihood, coding, pseudo-likelihood, expectation-maximization, Bayes.

The problem of parameter estimation can have several levels of complexity. The simplest is to estimate the parameters, denoted by  $\theta$ , of a single MRF,  $F$ , from the data  $d$  which is due to a clean realization,  $f$ , of that MRF. Treatments are needed if the data is noisy. When the noise parameters are unknown, they have to be estimated, too, along with the MRF parameters. The complexity increases when the given data is due to a realization of more than one MRF, *e.g.* when multiple textures are present in the image data, and the data is unsegmented. Since the parameters of an MRF have to be estimated from the data, partitioning the data into distinct MRFs becomes a part of the problem. The problem is even more complicated when the number of the underlying MRFs is unknown and has to be determined. Furthermore, the order of the neighborhood system and the largest size of cliques for a Gibbs distribution can also be part of the parameters to be estimated.

The chief difficulty in ML estimation for MRF is the following: The partition function  $Z$  in the Gibbs distribution  $P(f | \theta)$  is also a function of  $\theta$  and

has to be taken into consideration; since  $Z$  is calculated by summing over all possible configurations, maximizing  $P(f | \theta)$  becomes intractable, in general, even for small problems.

The example in Fig.6.1 illustrates the significance of getting correct model parameters for MRF labeling procedures to produce good results. The binary textures in row 1 are generated using the MLL model (1.52) with the  $\beta$  parameters being  $(0.5, 0.5, 0.5, -0.5)$  (for the left) and  $(-0.5, -0.5, -0.5, 0.5)$  (for the right), respectively. They have pixel values of 100 and 160. Identical independently distributed Gaussian noise  $N(0, 60^2)$  is added to the pixel values, giving degraded images in Row 2. Row 3 shows the result obtained using the true parameters of the texture and the noise. Since such accurate information is usually not available in practice, results can not be so good as that. Rows 4 and 5 show results obtained by using incorrect parameters  $(0.4, 0.4, 0.4, 0.4)$  and  $(1.5, 1.5, 1.5, 1.5)$ , respectively. These results are unacceptable.

## 6.1 Supervised Estimation with Labeled Data

Labeled data means realization(s) of only a single MRF such as a certain type of texture. For a data set containing more than one MRFs, the data should be partitioned into distinct subsets (*e.g.* texture regions) such that each subset is due to a distinct MRF. Such data is considered as the known-class data. From the viewpoint of pattern recognition, such estimation is *supervised*. The set of parameters,  $\theta$ , for each MRF,  $F$ , are estimated using the data which is a clean realization<sup>1</sup>,  $f$ , of that MRF. The algorithms described this section also provide the basis for more complicated unsupervised estimation with unlabeled data.

### 6.1.1 Maximum Likelihood

Given a realization  $f$  of a single MRF, the maximum likelihood (ML) estimate maximizes the conditional probability  $P(f | \theta)$  (the likelihood of  $\theta$ ), that is,

$$\theta^* = \arg \max_{\theta} P(f | \theta) \quad (6.1)$$

or its log likelihood  $\ln P(f | \theta)$ . Note that in this case,  $f$  is the data used for the estimation. When the prior p.d.f. of the parameters,  $p(\theta)$ , is known, the MAP estimation which maximizes the posterior density

$$P(\theta | f) \propto p(\theta)P(f | \theta) \quad (6.2)$$

---

<sup>1</sup>Some methods are needed to combine estimates obtained from more than one realization.

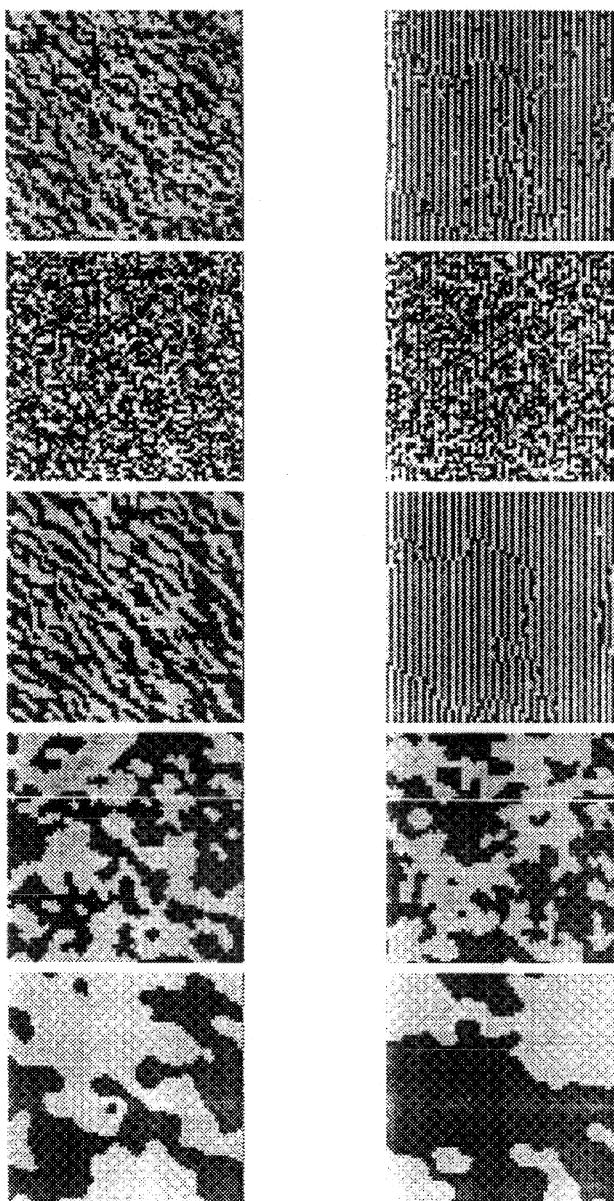


Figure 6.1: Good prior information produces good results. (Row 1) True texture. (Row 2) Images degraded by noise. (Row 3) Restoration results with the exact parameters for generating the images in row 2. (Rows 4 and 5) Images restored with incorrect parameters (see text). From (Dubé and Jain 1989) with permission; ©1989 Carfax.

can be sought. The ignorance of the prior p.d.f. leads the user to a rather diffuse choice. The prior p.d.f. is assumed to be flat when the prior information is totally unavailable. In this case the MAP estimation reduces to the ML estimation.

Let us review the idea of the ML parameter estimation using a simple example. Let  $f = \{f_1, \dots, f_m\}$  be a realization of an identical independent Gaussian distribution with the two parameters, the mean  $\mu$  and the variance  $\sigma^2$ , that is,  $f_i \sim N(\mu, \sigma^2)$ . We want to estimate the two parameters,  $\theta = \{\mu, \sigma^2\}$ , from  $f$ . The likelihood function of  $\theta$  for fixed  $f$  is

$$p(f | \theta) = \frac{1}{(\sqrt{2\pi\sigma^2})^m} e^{-\sum_{i \in S} (f_i - \mu)^2 / 2\sigma^2} \quad (6.3)$$

A necessary condition for maximizing  $p(f | \theta)$ , or equivalently maximizing  $\ln p(f | \theta)$ , is  $\frac{\partial \ln p}{\partial \mu} = 0$  and  $\frac{\partial \ln p}{\partial \sigma} = 0$ . Solving this, we find the ML estimate as

$$\mu^* = \frac{1}{m} \sum_{i=1}^m f_i \quad (6.4)$$

and

$$(\sigma^2)^* = \frac{1}{m} \sum_{i=1}^m (f_i - \mu^*)^2 \quad (6.5)$$

A more general result can be obtained for the  $m$ -variate Gaussian distribution, which when restricted by the Markovianity, is also known as the auto-normal (Besag 1974) MRF (cf. Section 1.3.1). Assuming  $\mu = 0$  in (1.47) and thus  $\theta$  consisting only of  $B = \{\beta_{i,i'}\}$ , the log likelihood function for the auto-normal field is

$$\ln p(f | \theta) \propto -\frac{1}{m} \ln[\det(B)] + \ln(f^T B f) \quad (6.6)$$

where  $f$  is considered as a vector. Maximizing this needs to evaluate the determinant  $\det(B)$ . The ML estimation for the Gaussian cases is usually less involved than for general MRFs.

The ML estimation for an MRF in general needs to evaluate the normalizing partition function in the corresponding Gibbs distribution. Consider a homogeneous and isotropic auto-logistic model in the 4-neighborhood system with the parameters  $\theta = \{\alpha, \beta\}$  (this MRF will also be used in subsequent sections for illustrations). According to (1.39) and (1.40), its energy function and conditional probability are, respectively,

$$U(f | \theta) = \sum_{\{i\} \in \mathcal{C}_1} \alpha f_i + \sum_{\{i, i'\} \in \mathcal{C}_2} \beta f_i f_{i'} \quad (6.7)$$

and

$$P(f_i | f_{\mathcal{N}_i}) = \frac{e^{\alpha f_i + \sum_{i' \in \mathcal{N}_i} \beta f_i f_{i'}}}{1 + e^{\alpha + \sum_{i' \in \mathcal{N}_i} \beta f_{i'}}} \quad (6.8)$$

The likelihood function is in the Gibbs form

$$P(f \mid \theta) = \frac{1}{Z(\theta)} \times e^{-U(f \mid \theta)} \quad (6.9)$$

where the partition function

$$Z(\theta) = \sum_{f \in \mathbb{F}} e^{-U(f \mid \theta)} \quad (6.10)$$

is also a function of  $\theta$ . Maximizing  $P(f \mid \theta)$  with respect to  $\theta$  needs to evaluate the partition function  $Z(\theta)$ . However, the computation of  $Z(\theta)$  is intractable even for moderately sized problems because there are a combinatorial number of elements in the configuration space  $\mathbb{F}$ . This is a main difficulty in parameter estimation for MRFs. Approximate formulae will be used for solving this problem.

The approximate formulae are based on the conditional probabilities  $P(f_i \mid f_{\mathcal{N}_i})$ ,  $i \in \mathcal{S}$ . (The notation of parameters,  $\theta$ , on which the probabilities are conditioned, is dropped temporarily for clarity.) Write the energy function into the form

$$U(f) = \sum_{i \in \mathcal{S}} U_i(f_i, f_{\mathcal{N}_i}) \quad (6.11)$$

Here,  $U_i(f_i, f_{\mathcal{N}_i})$  depends on the configuration on the cliques involving  $i$  and  $\mathcal{N}_i$ , in which the labels  $f_i$  and  $f_{\mathcal{N}_i}$  are mutually dependent. If only single- and pair-site cliques are considered, then

$$U_i(f_i, \mathcal{N}_i) = V_1(f_i) + \sum_{i' : \{i, i'\} \in \mathcal{C}} V_2(f_i, f_{i'}) \quad (6.12)$$

The conditional probability (1.29) can be written as

$$P(f_i \mid f_{\mathcal{N}_i}) = \frac{e^{-U_i(f_i, f_{\mathcal{N}_i})}}{\sum_{f_i \in \mathcal{L}} e^{-\sum_i U_i(f_i, f_{\mathcal{N}_i})}} \quad (6.13)$$

Based on this, approximate formulae for the joint probability  $P(f)$  are given in the following subsections.

### 6.1.2 Pseudo-Likelihood

A simple approximate scheme is pseudo-likelihood (Besag 1975; Besag 1977). In this scheme, each  $U_i(f_i, f_{\mathcal{N}_i})$  in (6.11) is treated as if  $f_{\mathcal{N}_i}$  are given and the pseudo-likelihood is defined as the simple product of the conditional likelihood

$$PL(f) = \prod_{i \in \mathcal{S} - \partial \mathcal{S}} P(f_i \mid f_{\mathcal{N}_i}) = \prod_{i \in \mathcal{S} - \partial \mathcal{S}} \frac{e^{-U_i(f_i, f_{\mathcal{N}_i})}}{\sum_{f_i} e^{-U_i(f_i, f_{\mathcal{N}_i})}} \quad (6.14)$$

1	2	1	2	1
2	1	2	1	2
1	2	1	2	1
2	1	2	1	2
1	2	1	2	1

1	2	1	2	1
3	4	3	4	3
1	2	1	2	1
3	4	3	4	3
1	2	1	2	1

Figure 6.2: Codings for the 4-neighborhood (left) and 8-neighborhood (right) systems. Pixels marked number  $k$  belong to  $\mathcal{S}^{(k)}$ .

where  $\partial\mathcal{S}$  denotes the set of points at the boundaries of  $\mathcal{S}$  under the neighborhood system  $\mathcal{N}^2$ . Substituting the conditional probability (6.8) into (6.14), we obtain the pseudo-likelihood for the homogeneous and isotropic auto-logistic model as

$$PL(f) = \prod_i \frac{e^{\alpha_i f_i + \sum_{i' \in \mathcal{N}_i} \beta f_i f_{i'}}}{1 + e^{\alpha_i + \sum_{i' \in \mathcal{N}_i} \beta f_{i'}}} \quad (6.15)$$

The pseudo-likelihood does not involve the partition function  $Z$ .

Because  $f_i$  and  $f_{\mathcal{N}_i}$  are not independent, the pseudo-likelihood is not the true likelihood function, except in the trivial case of nil neighborhood (contextual independence). In the large lattice limit, the consistency of the maximum pseudo-likelihood (MPL) estimate is proven by (Geman and Graffigne 1987), that is, it converges to the truth with probability one.

The following example illustrates the MPL estimation. Consider the homogeneous and isotropic auto-logistic MRF model described by (6.7) and (6.8). Express its pseudo-likelihood (6.15) as  $PL(f | \theta)$ . The logarithm is

$$\ln PL(f | \theta) = \sum_{i \in \mathcal{S} - \partial\mathcal{S}} \left\{ \alpha f_i + \beta f_i \sum_{i' \in \mathcal{N}_i} f_{i'} - \ln(1 + e^{\alpha + \beta \sum_{i' \in \mathcal{N}_i} f_{i'}}) \right\} \quad (6.16)$$

The MPL estimates  $\{\alpha, \beta\}$  can be obtained by solving

$$\frac{\partial \ln PL(f | \alpha, \beta)}{\partial \alpha} = 0, \quad \frac{\partial \ln PL(f | \alpha, \beta)}{\partial \beta} = 0 \quad (6.17)$$

### 6.1.3 Coding Method

Using the *coding method* (Besag 1974), the likelihood formula of the form (6.14) can be made a genuine one. The idea is the following: Partition the set  $\mathcal{S}$  into several disjoint sets  $\mathcal{S}^{(k)}$ , called codings, such that no two sites

<sup>2</sup>The boundary sites are excluded from the product to reduce artificiality.

in one  $\mathcal{S}^{(k)}$  are neighbors. Fig.6.2 illustrates the codings for the 4- and 8-neighborhood systems. For the 4-neighborhood system, two sets  $\mathcal{S}^1$  and  $\mathcal{S}^2$  are needed, one composed of the black sites and the other of the white sites in a checkerboard. Four codings are needed for the 8-neighborhood system.

The fact that the sites within each  $\mathcal{S}^{(k)}$  are not neighboring to each other provides computational advantages. Under the Markovian assumption, the variables associated with the sites in an  $\mathcal{S}^{(k)}$ , given the labels at all other sites, are mutually independent. This gives the following simple product for the likelihood

$$P^{(k)}(f \mid \theta) = \prod_{i \in \mathcal{S}^{(k)}} P(f_i \mid f_{\mathcal{N}_i}, \theta) \quad (6.18)$$

in which the need to evaluate  $Z$  is also dispensed with. Unlike the pseudo-likelihood, the above definition is a genuine expression for the likelihood function. Maximizing (6.18) gives coding estimates  $\theta^{(k)}$ .

A disadvantage is the inefficiency: Each of the ML estimates  $\theta^{(k)}$  is estimated based on information on only one coding. It is not clear how to combine the results optimally. The simplest scheme of the arithmetic average

$$\theta^* = \frac{1}{K} \sum_k \arg \max_{\theta} P^{(k)}(f \mid \theta) \quad (6.19)$$

is usually used.

Consider the homogeneous and isotropic auto-logistic MRF model described by (6.7) and (6.8). Its log coding likelihood take a similar form to that for the pseudo-likelihood

$$\ln P^{(k)}(f \mid \theta) = \sum_{i \in \mathcal{S}^{(k)}} \left\{ \alpha f_i + \beta f_i \sum_{i' \in \mathcal{N}_i} f_{i'} - \ln(1 + e^{\alpha + \beta \sum_{i' \in \mathcal{N}_i} f_{i'}}) \right\} \quad (6.20)$$

where for  $k = 1, 2$ . The ML estimates  $\{\alpha^{(k)}, \beta^{(k)}\}$  ( $k = 1, 2$ ) can be obtained by solving

$$\frac{\partial \ln P^{(k)}(f \mid \alpha, \beta)}{\partial \alpha} = 0, \quad \frac{\partial \ln P^{(k)}(f \mid \alpha, \beta)}{\partial \beta} = 0 \quad (6.21)$$

Taking the arithmetic averages, we obtain the final estimate as

$$\hat{\theta} = \left\{ \frac{\alpha^1 + \alpha^2}{2}, \frac{\beta^1 + \beta^2}{2} \right\} \quad (6.22)$$

#### 6.1.4 Mean Field Approximations

Mean field approximation from statistical physics (Chandler 1987; Parisi 1988) provides other possibilities. It originally aims to approximate the behavior of interacting spin systems in thermal equilibrium. We use it to approximate the behavior of MRFs in equilibrium. Generally, the mean of a

random variable  $X$  is given by  $\langle X \rangle = \sum_X X P(X)$ . So we define the mean field  $\langle f \rangle$  by the mean values

$$\langle f_i \rangle = \sum_{f \in \mathbb{F}} f_i P(f) = Z^{-1} \sum_f f_i e^{-\frac{1}{T} U(f)} \quad (6.23)$$

The mean field approximation makes the following assumption for calculating  $\langle f_i \rangle$ : The actual influence of  $f_{i'}$  ( $i' \neq i$ ) can be approximated by the influence of  $\langle f_{i'} \rangle$ . This is reasonable when the field is in equilibrium. It leads to a number of consequences. Equation (6.12) is approximated by the mean field local energy expressed as

$$U_i(f_i \mid \langle f_{\mathcal{N}_i} \rangle) = V_1(f_i) + \sum_{i' : \{i, i'\} \in \mathcal{C}} V_2(f_i, \langle f_{i'} \rangle) \quad (6.24)$$

and the conditional probability is approximated by the mean field local probability

$$P(f_i \mid \langle f_{\mathcal{N}_i} \rangle) = Z'_i^{-1} e^{-\frac{1}{T} U_i(f_i \mid \langle f_{\mathcal{N}_i} \rangle)} \quad (6.25)$$

where

$$Z'_i = \sum_{f_i \in \mathcal{L}} e^{-\frac{1}{T} U_i(f_i \mid \langle f_{\mathcal{N}_i} \rangle)} \quad (6.26)$$

is called the mean field local partition function. Note in the above mean field approximations, the values  $\langle f_{\mathcal{N}_i} \rangle$  are assumed given. The mean field values are approximated by

$$\langle f_i \rangle \approx \sum_{f_i \in \mathcal{L}} f_i P(f_i \mid \langle f_{\mathcal{N}_i} \rangle) \quad (6.27)$$

The mean field local probability (6.25) is an approximation to the marginal distribution

$$P(f_i) = \sum_{f_{\mathcal{S} - \{i\}}} Z^{-1} e^{-\frac{1}{T} U(f)} \quad (6.28)$$

Because the mean field approximation “decouples” the interactions in equilibrium, the mean field approximation of the joint probability is the product of the mean field local probabilities

$$P(f) \approx \prod_{i \in \mathcal{S}} P(f_i \mid \langle f_{\mathcal{N}_i} \rangle) = \prod_{i \in \mathcal{S}} Z'_i^{-1} e^{-\frac{1}{T} U_i(f_i \mid \langle f_{\mathcal{N}_i} \rangle)} \quad (6.29)$$

and the mean field partition function is the product of the mean field local partition functions

$$Z \approx Z' = \sum_f e^{-\frac{1}{T} \sum_{i \in \mathcal{S}} U_i(f_i \mid \langle f_{\mathcal{N}_i} \rangle)} = \prod_{i \in \mathcal{S}} \sum_{f_i} e^{-\frac{1}{T} U_i(f_i \mid \langle f_{\mathcal{N}_i} \rangle)} \quad (6.30)$$

When  $\mathcal{L}$  is a continuous, such as the real line, all the above probability functions are replaced by the corresponding p.d.f.’s and the sums by integrals.

$i_1$	$i_2$	$i_3$
$i_8$	$i$	$i_4$
$i_7$	$i_6$	$i_5$

Figure 6.3: Pixel  $i$  and its eight neighbors in the second order neighborhood system.

The mean field approximation (6.29) bears the form of the pseudo-likelihood function. Indeed, if  $\langle f_{i'} \rangle$  ( $i' \in \mathcal{N}_i$ ) in the mean field computation is replaced by the current  $f_{i'}$ , the two approximations become the same. However, unlike the pseudo-likelihood, the mean values  $\langle f_i \rangle$  and the mean field conditional probabilities  $P(f_i | \langle f_{\mathcal{N}_i} \rangle)$  are computed iteratively using (6.25), (6.26) and (6.27) given an initial  $\langle f^{(0)} \rangle$ .

The above is one of various approaches for mean field approximation. In Section 9.2, we will present an approximation for labeling with a discrete label set using saddle point approximation (Peterson and Soderberg 1989). The reader is also referred to (Bilbro and Snyder 1989; Yullie 1990; Geiger and Girosi 1991) for other approaches. A study of mean field approximations of various GRFs and their network implementations is presented in a recent Ph.D thesis (Elfadel 1993). Its application in MRF parameter estimation is proposed by (Zhang 1992; Zhang 1993).

### 6.1.5 Least Squares Fit

MRF parameter estimation can also be done by using a least squares (LS) fit procedure (Derin and Elliott 1987). The procedure consists of the following steps: (1) Find the relationship between the joint probabilities  $P(f_i, f_{\mathcal{N}_i})$  and the parameters  $\theta$ ; (2) use histogram techniques to estimate all the  $P(f_i, f_{\mathcal{N}_i})$ ; (3) construct an over-determined system of equations in terms of the probabilities and the parameters; and (4) solve it using the LS method.

Consider the MLL model with the label set  $\mathcal{L} = \{1, \dots, M\}$ . For the 8-neighborhood system shown in Fig.6.3, there are ten type of clique potential parameters shown in Fig.1.4. Denote all the parameters as a  $K \times 1$  vector

$$\theta = [\alpha_1, \dots, \alpha_M, \beta_1, \dots, \beta_4, \gamma_1, \dots, \gamma_4, \xi]^T = [\theta_1, \dots, \theta_K]^T \quad (6.31)$$

Let

$$U_i(f_i, f_{\mathcal{N}_i}, \theta) = \sum_{c: i \in c} V_c(f) \quad (6.32)$$

be the sum of the potential functions over all the cliques containing  $i$ . Define indicator functions

$$\chi(f_1, \dots, f_k) = \begin{cases} 1 & \text{if } f_1 = f_2 \dots = f_k \\ -1 & \text{otherwise} \end{cases} \quad (6.33)$$

for expressing (1.52) and

$$\chi_I(f_i) = \begin{cases} 1 & \text{if } f_i = I \\ -1 & \text{otherwise} \end{cases} \quad (6.34)$$

for (1.53). In terms of  $\theta$  and  $\chi$ 's, (6.32) can be written as

$$U_i(f_i, f_{\mathcal{N}_i}, \theta) = \theta^T N_i(f_i, f_{\mathcal{N}_i}) = \sum_{k=1}^K \theta_k n_k \quad (6.35)$$

where

$$\begin{aligned} N_i(f_i, f_{\mathcal{N}_i}) &= [ \chi_1(f_i), \chi_2(f_i), \dots, \chi_M(f_i), \\ &\quad \chi(f_i, f_{i_4}) + \chi(f_i, f_{i_8}), \chi(f_i, f_{i_2}) + \chi(f_i, f_{i_6}), \\ &\quad \chi(f_i, f_{i_3}) + \chi(f_i, f_{i_7}), \chi(f_i, f_{i_1}) + \chi(f_i, f_{i_5}), \\ &\quad \chi(f_i, f_{i_2}, f_{i_4}) + \chi(f_i, f_{i_6}, f_{i_8}) + \chi(f_i, f_{i_1}, f_{i_8}), \\ &\quad \chi(f_i, f_{i_2}, f_{i_3}) + \chi(f_i, f_{i_4}, f_{i_6}) + \chi(f_i, f_{i_7}, f_{i_8}), \\ &\quad \chi(f_i, f_{i_2}, f_{i_8}) + \chi(f_i, f_{i_3}, f_{i_4}) + \chi(f_i, f_{i_6}, f_{i_7}), \\ &\quad \chi(f_i, f_{i_1}, f_{i_2}) + \chi(f_i, f_{i_4}, f_{i_6}) + \chi(f_i, f_{i_6}, f_{i_8}), \\ &\quad \chi(f_i, f_{i_1}, f_{i_2}, f_{i_8}) + \chi(f_i, f_{i_2}, f_{i_3}, f_{i_4}) + \\ &\quad \chi(f_i, f_{i_4}, f_{i_5}, f_{i_6}) + \chi(f_i, f_{i_6}, f_{i_7}, f_{i_8}) ]^T \\ &\triangleq [ n_1, \dots, n_K ]^T \end{aligned} \quad (6.36)$$

is a vector whose elements count the number of each type of cliques in the  $3 \times 3$  window. The RHS of the first row in the above equation correspond to the potentials weighted by  $\alpha_1, \dots, \alpha_M$ , the second row by  $\beta_1$  and  $\beta_2$ , the third row by  $\beta_3$  and  $\beta_4$ , the fourth to seventh by  $\gamma_1, \dots, \gamma_4$ , and the last two rows by  $\xi_1$  (refer to Fig.1.4).

The conditional likelihood  $P(f_i | f_{\mathcal{N}_i})$  is related to the potential  $U_i(f_i, f_{\mathcal{N}_i}, \theta)$  by

$$P(f_i | f_{\mathcal{N}_i}) = \frac{e^{-U_i(f_i, f_{\mathcal{N}_i}, \theta)}}{\sum_{f_i \in \mathcal{L}} e^{-U_i(f_i, f_{\mathcal{N}_i}, \theta)}} \quad (6.37)$$

It can be expressed in terms of the joint probability  $P(f_i, f_{\mathcal{N}_i})$  of the labels on the  $3 \times 3$  window and the joint probability  $P(f_{\mathcal{N}_i})$  of the labels at the neighboring sites

$$P(f_i | f_{\mathcal{N}_i}) = \frac{P(f_i, f_{\mathcal{N}_i})}{P(f_{\mathcal{N}_i})} \quad (6.38)$$

The reason for this expression is that  $P(f_i, f_{\mathcal{N}_i})$  can be estimated using histogram techniques (see shortly). Comparing the above two equations yield the following equation

$$\frac{e^{-U_i(f_i, f_{\mathcal{N}_i}, \theta)}}{P(f_i, f_{\mathcal{N}_i})} = \frac{\sum_{f_i \in \mathcal{L}} e^{-U_i(f_i, f_{\mathcal{N}_i}, \theta)}}{P(f_{\mathcal{N}_i})} \quad (6.39)$$

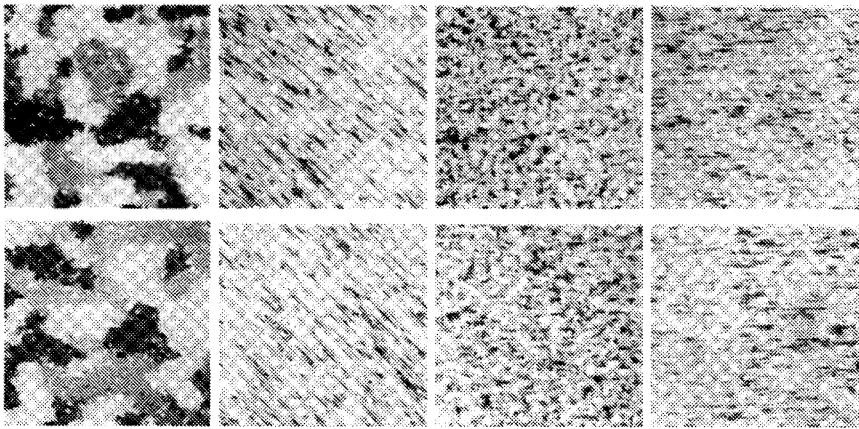


Figure 6.4: Realizations from a Gibbs distribution with specified (upper) and estimated (lower) parameters. See Table 6.1 for parameter values. From (Derin and Elliott 1987) with permission; ©1987 IEEE.

Note that the RHS of the above is independent of the value of  $f_i$  and therefore, so is the LHS.

Consider the LHS of (6.39) for any two distinct values,  $I$  and  $I'$ , for  $f_i$ . Because the LHS is independent of  $f_i$  values, the following holds

$$\frac{e^{-U_i(I, f_{N_i}, \theta)}}{P(I, f_{N_i})} = \frac{e^{-U_i(I', f_{N_i}, \theta)}}{P(I', f_{N_i})} \quad (6.40)$$

Rearranging the above and using the relationship (6.35), we obtain

$$\theta^T [N_i(I', f_{N_i}) - N_i(I, f_{N_i})] = \ln \left( \frac{P(I, f_{N_i})}{P(I', f_{N_i})} \right) \quad (6.41)$$

For any  $I \in \mathcal{L}$  and  $f_{N_i}$ , the vector  $N_i(I, f_{N_i})$  can be determined using (6.36) and  $P(I, f_{N_i})$  can be estimated using histogram techniques. Therefore, (6.41) represents a linear equation with unknown  $\theta$ . One such equation can be obtained for each distinct combination of  $I$ ,  $I'$  and  $N_i$ . An over-determined linear system of equations is obtained from all possible combinations. It can be solved using standard LS method. This procedure is also used by (Nadabar and Jain 1992) for estimating line process parameters in edge detection.

Histogram techniques are proposed to estimate  $P(f_i, f_{N_i})$ , for all combinations of  $(f_i, f_{N_i})$ , from one or several realizations (images). Assume that there are a total of  $N$  distinct  $3 \times 3$  blocks in the image lattice and that a particular  $3 \times 3$  configuration  $(f_i, f_{N_i})$  occurs  $H(f_i, f_{N_i})$  times. Then  $P(f_i, f_{N_i}) = \frac{H(f_i, f_{N_i})}{N}$ . The relationship between the histogramming and the ML estimation is established in (Gurelli and Onural 1994), and also used in (Gimel'farb 1996).

Row-Column	Description	Size	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$
1-1	specified	$128 \times 128$	0.3	0.3	0.3	0.3
2-1	estimated	$128 \times 128$	0.326	0.308	0.265	0.307
1-2	specified	$256 \times 256$	1.0	1.0	-1.0	1.0
2-2	estimated	$256 \times 256$	0.752	0.805	-0.762	0.909
1-3	specified	$256 \times 256$	2.0	2.0	-1.0	-1.0
2-3	estimated	$256 \times 256$	1.728	1.785	-0.856	-0.851
1-4	specified	$128 \times 128$	1.0	-1.0	-1.0	-1.0
2-4	estimated	$128 \times 128$	0.861	0.055	-1.156	-0.254

Table 6.1: Specified and estimated parameters. From (Derin and Elliott 1987) with permission; ©1987 IEEE.

A few remarks follow: Any  $(f_i, f_{\mathcal{N}_i})$  combination for which  $H(f_i, f_{\mathcal{N}_i}) = 0$  cannot be used to obtain a linear equation because of the logarithm in (6.41). Derin and Elliott propose to use only those combinations  $(f_i, f_{\mathcal{N}_i})$  for which  $H(f_i, f_{\mathcal{N}_i})$  are largest. This selection strategy is quite robust (Derin and Elliott 1987). They also suggest to discard the three and four cliques (set their potentials to zero), according to some prior or empirical knowledge, to reduce the length of  $\theta$  and  $N_i(f_i, f_{\mathcal{N}_i})$  to simplify the procedure. The normalized form of GRFs (Griffeth 1976) described in Section 1.2.5 provides a foundation for the reduction. This will be discussed in Section 6.3.2.

Fig.6.4 shows estimation results obtained by using the LS fitting method. In this example, the only nonzero parameters are  $\beta_1, \dots, \beta_4$ . In the upper row are the texture images generated with specified parameter values. These values are estimated from the images. The estimated values are then used to generate textures in the lower row. Table 6.1 compares the specified values and the estimated values. Note that the estimate for the image in column 4 is numerically not so good; Derin and Elliott explanation for this is that the parameter representation for a realization is not unique.

The LS fitting procedure has the following advantages over the coding method and MPL estimation. It does not require numerical maximization, avoiding solving nonlinear equations. Nor is the final solution dependent on the initial estimate. Further, it is more efficient and faster. A disadvantage is that the number of equations (6.41) in the system increases

exponentially as the sizes of  $\mathcal{S}$  and  $\mathcal{L}$  increase. Zero histogramming counts also cause problems with the RHS of (6.41).

### 6.1.6 Markov Chain Monte Carlo Methods

Often in the computation or estimation of random models, one needs to draw random samples from a multivariate distribution  $P(f) = P(f_1, \dots, f_m)$ , or to calculate an expectation of a function of  $f$  with respect to a distribution

$P(f)$ . This cannot be done easily by using standard analytical or numerical methods when  $P(f)$  is complex enough. In this case, Markov chain Monte Carlo (MCMC) methods (Smith and Robert 1993; Besag and Green 1993; Gilks et al. 1993; Neal 1993; Besag et al. 1995) provide an effective, tractable means for the sampling and the calculation.

In MCMC, a time homogeneous (ergodic) sequence of states  $\{f^{(0)}, f^{(1)}, \dots\}$ , or *Markov chain*, is generated, where a state in a state space  $\mathbb{F}$  may correspond to a labeling configuration or parameter vector. At each time  $t \geq 0$ , the next random state  $f^{(t+1)}$  is generated according to a Markov transition probability  $P(f^{(t+1)} | f^{(t)})$ . The transition probability should be constructed in such a way that the chain converges to the target distribution  $P(f)$ ; thus after a (usually long) “burn-in” period, the samples along the path of the chain can be used as samples from  $P(f)$ , which enables *Monte Carlo* methods. Sampling according to  $P(f^{(t+1)} | f^{(t)})$  is much easier than direct sampling from the target  $P(f)$ , which is the reason for using MCMC techniques.

For such a Markov chain to converge to the target distribution  $P(f)$ , it is sufficient to choose the transition probability so that the following time reversibility condition, or detailed balance, is satisfied

$$P(f^{(t)})P(f^{(t+1)} | f^{(t)}) = P(f^{(t+1)})P(f^{(t)} | f^{(t+1)}) \quad \forall f^{(t)}, f^{(t+1)} \in \mathbb{F} \quad (6.42)$$

There can be more than one choices of  $P(f^{(t+1)} | f^{(t)})$  for a given  $P(f)$ . Different choices lead to different MCMC algorithms such as Metropolis-Hastings algorithms and Gibbs sampler which will be described shortly.

---

```

set  $t = 0$ ; generate  $f^{(0)}$  at random;
repeat
    generate next  $f^{(t+1)}$  according to  $P(f^{(t+1)} | f^{(t)})$ ;
     $t = t + 1$ ;
until (maximum  $t$  reached);
return  $\{f^{(t)} | t > t_0\}$ ;

```

---

Figure 6.5: MCMC sampler of  $P(f)$  where  $P(f^{(t+1)} | f^{(t)})$  satisfies (6.42).

Fig.6.5 shows a MCMC procedure for sampling from  $P(f)$ . There,  $f^{(t)}$  (for  $t \leq 0$ ) is assumed to be from the distribution  $P(f)$  already. This is generally not true initially with MCMC starting with an arbitrary state  $f^{(0)}$ . A solution is to start with an arbitrary state  $f^{(0)}$ , run the MCMC procedure for an initial “burn-in” period (for  $0 \leq t \leq t_0$ ), and use  $\{f^{(t)}\}$  (for  $t > t_0$ ) as samples from  $P(f)$  (Besag et al. 1995).

### Metropolis-Hastings Algorithm

In the Metropolis algorithm, a new configuration  $f'$  is accepted with the acceptance ratio

$$P = \alpha(f' | f) = \min\{1, \frac{P(f')}{P(f)}\} \quad (6.43)$$

Fig.6.6 illustrates the Metropolis algorithm for sampling a Gibbs distribution  $P(f) = Z^{-1}e^{-U(f)/T}$  given the  $T$  parameter. In each iteration, the next configuration  $f'$  is randomly chosen from  $\mathcal{N}(f)$  (the vicinity of  $f$ ) by applying a perturbation on  $f$ , for instance, by changing one of the  $f_i$ 's into a new label  $f'_i$ .  $\text{Random}[0,1]$  is a random number from a uniform distribution in  $[0, 1]$ . A new configuration  $f'$  for which  $\Delta U \leq 0$  is accepted for sure whereas one for which  $\Delta U > 0$  is accepted with probability  $P$ . The convergence condition “equilibrium is reached” may be judged by “a prescribed number of iterations have been performed”.

---

```

initialize f;
repeat
  generate  $f' \in \mathcal{N}(f)$ ;
   $\Delta U \leftarrow U(f') - U(f)$ ;
   $P = \alpha(f' | f) = \min\{1, e^{-\Delta U/T}\}$ ;
  if  $\text{random}[0,1] < P$  then  $f \leftarrow f'$ ;
until (equilibrium is reached);
return f;

```

---

Figure 6.6: Metropolis sampling of Gibbs distribution.

The Metropolis-Hastings algorithm (Hastings 1970) is a generalization of the Metropolis algorithm in that the acceptance is calculated as

$$\alpha(f' | f) = \min\left\{1, \frac{P(f')q(f | f')}{P(f)q(f' | f)}\right\} \quad (6.44)$$

where  $q(f' | f)$  is a (arbitrary) density for proposing  $f'$  given  $f$ . If  $q(f | f') = q(f' | f)$ , the Metropolis-Hastings algorithm reduces to the Metropolis algorithm.

The transition probability corresponding to the acceptance probability  $\alpha(f' | f)$  of (6.44) is the following

$$P(f' | f) = \begin{cases} q(f' | f)\alpha(f' | f) & \text{if } f' \neq f \\ 1 - \sum_{f''} q(f'' | f)\alpha(f'' | f) & \text{if } f' = f \end{cases} \quad (6.45)$$

where  $f' \neq f$  means that the proposed  $f'$  is accepted and  $f' = f$  means  $f'$  rejected. It can be checked that the detailed balance of (6.42) is satisfied by such a transition probability.

### Gibbs Sampler

The Gibbs sampler (Geman and Geman 1984) generates the next configuration based on a conditional probability: A candidate  $f'_i$  for the next  $f_i$  is randomly drawn from the conditional distribution  $P(f'_i | f_{S-\{i\}})$ . Given  $f = \{f_1, \dots, f_m\}$ , the transition to  $f' = \{f'_1, \dots, f'_m\}$  is performed by successively drawing samples from the conditional probabilities as follows:

---

$f'_1$	from	$P(f_1   f_2, \dots, f_m);$
$f'_2$	from	$P(f_2   f'_1, f_3, \dots, f_m);$
$f'_3$	from	$P(f_3   f'_1, f'_2, f_4, \dots, f_m);$
⋮		
$f'_m$	from	$P(f_m   f'_1, \dots, f'_{m-1}, f_m);$

---

The transition probability from  $f$  to  $f'$  is given by

$$P(f' | f) = \prod_{i=1}^m P(f'_i | f'_j, f_k, j < l < k) \quad (6.46)$$

The scenario of the who Gibbs sampler procedure is similar to that of the Metropolis algorithm illustrated in Fig.6.6; the difference is in how the next configuration  $f'$  is generated.

When  $f$  is an MRF,  $P(f'_i | f_{S-\{i\}}) = P(f'_i | f_{\mathcal{N}_i})$  which is given by.

$$P(f_i | f_{\mathcal{N}_i}) = \frac{e^{-[V_1(f_i) + \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'})]/T}}{\sum_{f_i \in \mathcal{L}} e^{-[V_1(f_i) + \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'})]/T}} \quad (6.47)$$

when up to pairwise cliques are involved. Compared with (1.29), this formula contains a temperature parameter.

Like the Metropolis algorithms, the Gibbs sampler also produces a Markov chain  $\{f^{(0)}, f^{(1)}, \dots\}$  with the target distribution  $P(f)$  as the equilibrium (Geman and Geman 1984). Comparatively speaking, Metropolis is easier to program and runs more efficiently than Gibbs. However, other considerations also affect a choice between samplers (Besag et al. 1995). In vision research, (Cross and Jain 1983) on MRF texture modeling is the first to use the Metropolis algorithm for sampling MRFs and (Geman and Geman 1984) is the earliest to use the Gibbs sampler and annealing procedures for finding the global estimates of MRFs.

### Calculating Expectation Using MCMC

Many statistics, such as the mean and variance of  $P(f)$  as well as the normalizing constant  $Z$  in a Gibbs distribution, can be expressed as the expectation of a function  $\phi(f)$  with respect to  $P(f)$

$$\bar{a}_{P(f)} = \int a(f)P(f)df \quad (6.48)$$

A direct evaluation would be infeasible. MCMC enables one to approximate the value using the following method: Choose the transition probability  $P(f' | f)$  so that the chain converges to the target distribution  $P(f)$ . Generate a Markov chain flowing  $P(f' | f)$ . Get  $f^{(0)}$  after a long enough burn-in period. Collect samples  $f^{(0)}, f^{(1)}, \dots, f^{(N-1)}$ . Then use their arithmetic average

$$\bar{a} = \frac{1}{N} \sum_{n=0}^{N-1} a(f^{(n)}) \quad (6.49)$$

to approximate the expectation. The average value converges to the expectation  $\bar{a}_{P(f)}$  as  $N \rightarrow \infty$ .

#### 6.1.7 Learning in the FRAME Model

As mentioned in Section 1.3.5, there are two things to learn in the FRAME model: (1) the potential functions  $\theta_I^{(k)}$  and (2) the types of filters  $G^{(k)}$ .

Given  $H_{samp, \ell}^{(k)}$  computed from example images  $f_{samp}$ , the problem is to estimate the potential functions  $\theta_\ell^{(k)}$  of the FRAME model from which  $f_{samp}$  is sampled (generated). The problem is posed as one of maximizing (w.r.t.  $\theta$ ) the likelihood  $p(f_{samp} | \theta)$  of (1.74) or the log-likelihood

$$\ln p(f_{samp} | \theta) = -\log Z(\theta) - \langle \theta, \overline{H_{samp}} \rangle \quad (6.50)$$

where  $Z(\theta) = \sum_f e^{-\langle \theta, \overline{H_{samp}} \rangle}$ . Setting  $\frac{\partial \ln p(f_{samp} | \theta)}{\partial \theta} = 0$ , the minimum  $\theta^*$  is obtained by solving the equation

$$\sum_f p(f | \theta^*) H(f) = \overline{H_{samp}} \quad (6.51)$$

The LHS is the expectation  $\overline{H_{p(f|\theta^*)}}$ . Because the log-likelihood is strictly convex w.r.t.  $\theta$ , the minimum solution can be found by applying gradient descent

$$\frac{d\theta_\ell^{(k)}}{dt} = \overline{H_{p(f|\theta), \ell}^{(k)}} - \overline{H_{samp, \ell}^{(k)}} \quad \forall k, \forall \ell \quad (6.52)$$

This provides an algorithm for learning the parameters  $\theta$  of a specific FRAME model from the sample data  $f_{samp}$ .

While  $\overline{H_{samp,\ell}^{(k)}}$  are calculated from the sample data,  $\overline{H_{p(f|\theta),\ell}^{(k)}}$  have to be estimated with respect to  $p(f \mid \theta)$ . Given  $\theta$  fixed, the estimation may be done by using a Markov chain Monte Carlo (MCMC) procedure (*cf.* Section 6.1.6) as follows: (1) Draw a number of  $N$  samples, *i.e.* synthesize  $N$  images  $f_{syn(1)}, \dots, f_{syn(N)}$ , according to the distribution  $p(f \mid \theta)$ , using say a Gibbs sampler (Geman and Geman 1984). (2) Filter the synthesized images to obtain  $H_{syn(1)}^{(k)}, \dots, H_{syn(N)}^{(k)}$ . (3) Assuming that  $N$  is large enough, we approximate the expectation using the sample mean,  $\overline{H_{p(f|\theta)}^{(k)}} = \frac{1}{N} \sum_{n=1}^N H_{syn(n)}^{(k)}$ . Refer Section 6.1.6 for a MCMC approximation of probabilistic expectation.

Given a fixed model complexity  $K$ , the selection of filters  $G^{(k)}$  may be done in a stepwise way using the greedy strategy (Zhu et al. 1998). Filters are added to the filter bank one at a time until the number  $K$  is reached. Given the current selection of  $k$  filters, the next filter  $k + 1$  is selected such that the entropy of  $p(f)$  is *minimized*. Note however, this is not necessarily a globally optimal selection.

The above can be summarized as a three part algorithm: (1) the FRAME algorithm, (2) the Gibbs sampler and (3) the filter selection. More details of the three parts can be found in (Zhu et al. 1998).

## 6.2 Unsupervised Estimation with Unlabeled Data

The methods described in the previous section are applicable to situations where the data contains either a single MRF or several but *segmented* MRFs so that the estimation can be done for each MRF separately. In practice, it is normal that the data is an observation, such as an image, of the underlying, unobservable MRFs. There can be several complications compared to the clean, single MRF case. Firstly, the data is a transformed version of the true MRF, for example, due to the additive noise observation model  $d_i = f_i + e_i$ . Secondly, the noise parameters can be unknown and hence also a part of the parameters to be estimated. In this case, the set of unknown parameters is composed of the two parts, those for the MRF and those due to the noise, collectively denoted as  $\theta = \{\theta_f, \theta_d\}$ . Thirdly, the data  $d$  can contain realizations of more than one MRF and the origin of each data point is unknown. Since the parameters for an MRF should be estimated by using only realizations of that MRF, the data has to be segmented before used for the estimation. However, to perform good segmentation, the parameter values of the MRFs have to be available. This is a dilemma between segmentation and estimation. The problem of restoration-estimation is similar: The estimation should be performed using smoothed or restored data; however, to obtain good restoration, the parameter values of the MRF have to be known. We consider these as the dilemma between labeling and estimation.

One strategy is to perform parameter estimation after labeling (segmentation or restoration). For example, to perform segmentation using clustering techniques (Silverman and Cooper 1988; Hu and Fahmy 1987). This partitions the input data into blocks; the blocks are then merged into clusters according to some optimization criterion, yielding a segmentation; each cluster contains only blocks due to a single MRF (or due to “boundary blocks”); now parameters for each MRF can be estimated separately. However, such strategy may not yield the best result because optimal labeling relies on correct parameter values.

In a more sophisticated paradigm, iterative labeling-estimation (Besag 1986; Kelly et al. 1988; Lakshmanan and Derin 1989; Qian and Titterington 1992; Pappas 1992; Won and Derin 1992; Zhang 1992; Bouman and Shapiro 1994) is adopted; labeling and estimation are performed alternately. Initial labeling is chosen by using some scheme possibly with initial parameters; a parameter estimate is computed based on this labeling; it is then used to get a hopefully better labeling; a hopefully better estimate is computed after that; and so on. The expectation-maximization algorithm (Dempster et al. 1977) provides a justifiable formalism for such labeling-estimation schemes.

### 6.2.1 Simultaneous Restoration and Estimation

Let us consider a heuristic procedure for simultaneous image restoration and parameter estimation (Besag 1986). Our interest is to restore the underlying MRF realization,  $f$ , from the observed data,  $d$ . During this process, two sets of parameters are to be estimated: a set of the MRF parameters  $\theta_f$  and a set of the observation parameters  $\theta_d$ . The procedure is summarized below:

1. Start with an initial estimate  $\hat{f}$  of the true MRF, and guess for  $\theta_f$  and  $\theta_d$ .
2. Estimate  $\theta_d$  by  $\hat{\theta}_d = \arg \max_{\theta_d} P(d | f, \theta_d)$ .
3. Estimate  $\theta_f$  by  $\hat{\theta}_f = \arg \max_{\theta_f} P(f | \theta_f)$ .
4. Update  $\hat{f}$  by  $\hat{f} = \arg \max_f P(f | d, \theta_f, \theta_d)$  based on the current  $\hat{f}$ ,  $\theta_f$  and  $\theta_d$ .
5. Go to 2 for a number of iterations or until  $\hat{f}$  is approximately converged.

This is illustrated below.

Assume that the data  $d = \{d_1, \dots, d_m\}$  is determined by the additive noise model

$$d_i = f_i + e_i \quad (6.53)$$

where  $e_i \sim N(0, \sigma^2)$  is the independent zero-mean Gaussian noise corrupting the true label  $f_i$ . Then, for step 2, we can maximize

$$\ln p(d \mid f, \theta_d) = -n \ln(2\pi) - n \ln \sigma - \sum_i \frac{(f_i - d_i)^2}{2\sigma^2} \quad (6.54)$$

where  $\theta_d = \sigma^2$ . Step 2, which is not subject to the Markovianity of  $f$ , amounts to finding the ML estimate of  $\theta_d = \sigma^2$  which is, according to (6.5),

$$(\sigma^2)^* = \frac{1}{m} \sum_{i=1}^m (f_i - d_i)^2 \quad (6.55)$$

Steps 3 and 4 have to deal with the Markovianity. Consider again the homogeneous and isotropic auto-logistic MRF, characterized by (6.7) and (6.8). For step 3, we can maximize

$$\ln P(f \mid \theta_f) = -\ln Z(\alpha, \beta) + \sum_i \alpha f_i + \sum_i \sum_{j \in \mathcal{N}_i} \beta f_i f_j \quad (6.56)$$

where  $\theta_f = \{\alpha, \beta\}$ . Heuristics may be used to alleviate difficulties therein. For example, the likelihood function  $P(f \mid \theta_d)$  in step 3 may be replaced by the pseudo-likelihood (6.14) or the mean field approximation. A method for computing the maximum pseudo-likelihood estimate of  $\theta = \{\alpha, \beta\}$  has been given in Section 6.1.2. The maximization in step 4 may be performed by using a local method such as ICM to be introduced in Section 8.2.1.

Fig. 6.7 shows an example. The true MRF,  $f$ , (shown in upper-left) is generated by using a multi-level logistic model, a generalization to the bi-level auto-logistic MRF, with a number of  $M = 6$  (given) level labels in  $\mathcal{L}$  and the parameters being  $\alpha = 0$  (known throughout the estimation) and  $\beta = 1.5$ , respectively. The MRF is corrupted by the addition of Gaussian noise with variance  $\sigma^2 = 3.6$ , giving the observed data,  $d$ , (upper-right). With the correct parameter values  $\sigma^2 = 3.6$  and  $\beta = 1.5$  known, no parameters are to be estimated and only step 3 needs to be done, e.g. using ICM. ICM, with  $\beta = 1.5$  fixed, converges after six iterations and the misclassification rate of the restored image is 2.1%. The rate reduces to 1.2% (lower-left) after eight iterations when  $\beta$  is gradually increased to 1.5 over the first six cycles. With  $\sigma^2$  known  $\beta$  estimated, the procedure consists of all the steps but step 2. The error rate is 1.1% after eight iterations (lower-right), with  $\beta^* = 1.80$ . When  $\sigma^2$  is also to be estimated, the error rate is 1.2% with  $\beta^* = 1.83$  and  $\sigma^2 = 0.366$ .

### 6.2.2 Simultaneous Segmentation and Estimation

In the previous simultaneous restoration and estimation, it is assumed there is only a single MRF present in the data. Segmentation has to be performed when the data contains realizations of more than one MRF. For the said reasons, there is a need for simultaneous segmentation and estimation. The optimal parameters are estimated during the computation of the optimal segmentation.

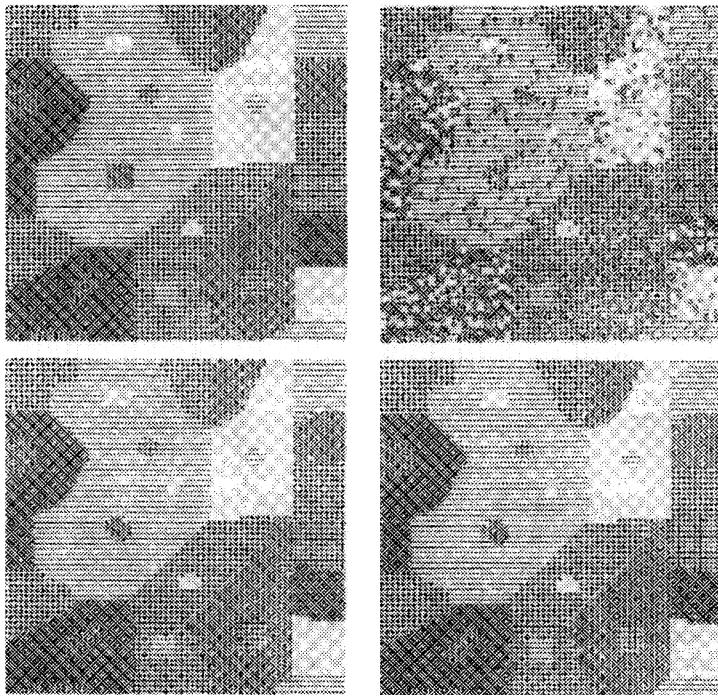


Figure 6.7: Image restoration with unknown parameters. (Upper-left) True MRF. (Upper-right) Noisy observation. (Lower-left) Restoration with the exact parameters. (Lower-right) Restoration with  $\beta$  estimated. From (Besag 1986) with permission; ©1986 Royal Statistical Society.

Let  $\mathcal{L} \in \{1, \dots, M\}$  denote the possible texture types and  $f$  represent a labeling or segmentation with  $f_i \in \mathcal{L}$  indicating the texture type for pixel  $i$ . The segmentation  $f$  partitions  $\mathcal{S}$  into  $M$  types of disjoint regions<sup>3</sup>

$$\mathcal{S}^{(I)}(f) = \{i \in \mathcal{S} \mid I = f_i\}, \quad I \in \mathcal{L} \quad (6.57)$$

(cf. Equations (2.51)-(2.53) The problem may be formulated in terms of the MAP principle as

$$(f^*, \theta_f^*, \theta_d^*) = \arg \max_{f, \theta_f, \theta_d} P(f, \theta_f, \theta_d \mid d) \quad (6.58)$$

Assume that  $\theta_f$  and  $\theta_d$  are independent of each other and that they are uniformly distributed when no prior knowledge about their distributions is available. Then the above is reduced to the following

<sup>3</sup>In completely unsupervised procedures, the number of texture types,  $M$ , is also an unknown parameter to be estimated. See Section 6.3.1.

$$\begin{aligned}
(f^*, \theta_f^*, \theta_d^*) &= \arg \max_{f, \theta_f, \theta_d} P(d | f, \theta_d) P(f | \theta_f) P(\theta_f) P(\theta_d) \\
&= \arg \max_{f, \theta_f, \theta_d} P(d | f, \theta_d) P(f | \theta_f)
\end{aligned} \tag{6.59}$$

In general, this maximization problem is computationally intractable. However, it is solvable when  $\theta_d$  can be expressed in closed-form as function  $\theta_d(f, d)$  (Won and Derin 1992).

Assume that the regions are piecewise constant valued and are governed by an MRF model, and that the observation model  $d_i = \varphi(f_i) + e_i$  where  $e_i$  is additive identical independent zero-mean Gaussian noise and  $\varphi(f_i)$  is the grey level for type  $I = f_i$  regions. Then given  $f$  and  $d$ , the ML estimate of the noise variance for type  $I$  MRF regions can be expressed explicitly as a function of  $f$  and  $d$

$$\theta_d^*(f, d) = (\sigma_I^2)^* = \frac{1}{\#\mathcal{S}^{(I)}} \sum_{i \in \mathcal{S}^{(I)}} (d_i - \varphi(f_i))^2 \tag{6.60}$$

where  $\#\mathcal{S}^{(I)}$  is the number of pixels in type  $I$  regions. When  $\theta_d$  given as  $\theta_d^*(f, d)$ , (6.59) is reduced to

$$(f^*, \theta_f^*) = \arg \max_{f, \theta_f} P(d | f, \theta_d^*(f, d)) P(f | \theta_f) \tag{6.61}$$

However, the minimization over the  $f$ - $\theta_f$  space is a still difficult one. Therefore, it is divided into the two sub-problems

$$f^* = \arg \max_f P(d | f, \theta_d^*(f, d)) P(f | \theta_f^*) \tag{6.62}$$

$$\theta_f^* = \arg \max_{\theta_f} P(d | f, \theta_d^*(f^*, d)) P(f^* | d, \theta_f) \tag{6.63}$$

Thus, the procedure iteratively alternates between the two equations. The estimate  $(f^*, \theta_f^*)$  thus found is suboptimal with respect to (6.61).

As in the case for simultaneous restoration and estimation, there are several options for choosing methods for solving (6.62) and (6.63). For example, one may choose a simulated annealing procedure if he can afford to find a good  $f^*$  for (6.62) or use heuristic ICM to find a local solution. Pseudo-likelihood, the coding method or the mean field method may be used for approximating  $P(f^* | d, \theta_f)$  in (6.63).

Now we turn to the case of textured images. Suppose that the hierarchical Gibbs model of (Derin and Cole 1986; Derin and Elliott 1987) is used to represent textured images. The higher level MRF corresponds to a region process with distribution  $P(f | \theta_f)$ . When modeled as a homogeneous and isotropic MLL model,  $\theta_f = \{\alpha_1, \dots, \alpha_M, \beta\}$  in which  $\alpha_I$  have an effect of controlling the relative percentage of type- $I$  pixels and  $\beta$  controls the interaction between neighboring regions. At the lower level, there are  $M = \#\mathcal{L}$  types of MRFs for the filling-in textures with p.d.f.  $p(d | f, \theta_d)$  with  $\theta_d = \{\theta_d^{(I)} | I \in \mathcal{L}\}$ . When

a filling-in is modeled as an auto-normal MRF (Besag 1974), its parameters consists of the mean and the interaction matrix  $\theta_d^{(I)} = \{\mu^{(I)}, B^{(I)}\}$ . After all, note that the region process is a much “slower varying” MRF than the texture MRFs.

Because it is difficult to compute the true joint conditional p.d.f.  $p(d | f, \theta_d)$ , we use the pseudo-likelihood,  $pl(d | f, \theta_d)$ , to approximate it.<sup>4</sup> The conditional p.d.f. for  $d_i$  takes the form (1.44). When the MRF is also homogeneous, the conditional p.d.f. can be written as

$$p(d_i | d_{\mathcal{N}_i}, f, \theta_d) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} [d_i - \mu - \sum_{i' \in \mathcal{N}_i} \beta_{i,i'} (d_{i'} - \mu)]^2} \quad (6.64)$$

where  $I = f_i$ . The above is a meaningful formula only when all the sites in the block  $\{i, \mathcal{N}_i\}$  have the same texture label. To approximate, we may pretend that the all the sites in  $\mathcal{N}_i$  have the same label as  $i$  even when this is not true; in this case, the formula is only an precise one.

The conditional p.d.f.  $p(d_i | d_{\mathcal{N}_i}, f, \theta_d)$  has a very limited power for texture description because it is about the datum on a single site  $i$  given the neighborhood. Therefore, a scheme based on a block of data may be used to enhance the capability (Won and Derin 1992). Define a block centered at  $i$ ,

$$\mathcal{B}_i = \{i\} \cup \mathcal{N}_i \quad (6.65)$$

for each  $i \in \mathcal{S} - \partial\mathcal{S}$ . Regard the data in  $\mathcal{B}_i$  as a vector of  $n = \#\mathcal{B}_i$  dependent random variables, denoted  $d_{\mathcal{B}_i}$ . Assume that that the all the sites in  $\mathcal{N}_i$  have the same label,  $I = f_i$ , as  $i$ . Then the joint p.d.f. of  $d_{\mathcal{B}_i}$  given  $f$  and  $\theta_d^{(I)}$  is multivariate normal

$$p(d_{\mathcal{B}_i} | f, \theta_d^{(I)}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2} [(d_{\mathcal{B}_i} - \mu^{(I)})^T [\Sigma^{(I)}]^{-1} (d_{\mathcal{B}_i} - \mu^{(I)}))] \quad (6.66)}$$

where  $\mu^{(I)}$  and  $\Sigma^{(I)}$  are the mean vector and covariance matrix of  $d_{\mathcal{B}_i}$ , respectively.

The parameters,  $\mu^{(I)}$  and  $\Sigma^{(I)}$ , represent texture features for type- $I$  texture. They can be estimated using sample mean and covariance. For only a single block centered at  $i$ , they are estimated as  $\mu_i^{(I)} = d_{\mathcal{B}_i}$  and  $\Sigma^{(I)} = (d_{\mathcal{B}_i} - \mu_i^{(I)})^T (d_{\mathcal{B}_i} - \mu_i^{(I)})$ . For all blocks in  $\mathcal{S}^{(I)}$ , they are taken as the averages

$$\hat{\mu}^{(I)} = \frac{1}{\#\mathcal{S}^{(I)}} \sum_{i \in \mathcal{S}^{(I)}} d_{\mathcal{B}_i} \quad (6.67)$$

and

$$\hat{\Sigma}^{(I)} = \frac{1}{\#\mathcal{S}^{(I)}} \sum_{i \in \mathcal{S}^{(I)}} [d_{\mathcal{B}_i} - \hat{\mu}^{(I)}]^T [d_{\mathcal{B}_i} - \hat{\mu}^{(I)}] \quad (6.68)$$

<sup>4</sup>The lower case letters  $pl$  is used to denote the PL for the continuous random variables  $d$ .

Obviously, the estimates have larger errors at or near boundaries. Obviously,  $\Sigma^{(I)}$  is symmetric. When  $\#\mathcal{S}^{(I)} > n$ , which is generally true,  $\Sigma^{(I)}$  is positive definite with probability 1. Therefore, the validity of  $p(d_{\mathcal{B}_i} | f, \theta_d^{(I)})$  is generally guaranteed. It is not difficult to establish relationships between the elements in the inverse of the covariance matrix  $[\Sigma^{(I)}]^{-1}$  and those in the interaction matrix  $B^{(I)}$  (Won and Derin 1992).

Based on the conditional p.d.f.'s for the blocks, the pseudo-likelihood for type-*I* texture can be defined as

$$pl^{(I)}(d | f, \theta_d^{(I)}) = \prod_{i \in \mathcal{S}^{(I)}} [p(d_{\mathcal{B}_i} | f, \theta_d^{(I)})]^{1/n} \quad (6.69)$$

where  $\mathcal{S}^{(I)}$  is defined in (2.51), and  $1/n == 1/\#\mathcal{S}^{(I)}$  is there to compensate for the fact that  $p(d_{\mathcal{B}_i} | \cdot)$  is the joint p.d.f. of  $n$  elements in  $d_{\mathcal{B}_i}$ . The overall pseudo-likelihood is then

$$pl(d | f, \theta_d) = \prod_{I \in \mathcal{L}} pl^{(I)}(d | f, \theta_d^{(I)}) \quad (6.70)$$

Given  $f, \theta_d^{(I)}$  can be explicitly expressed as (6.67) and (6.68) and hence  $pl(d | f, \theta_d)$  is  $\theta_d$ -free. Then, the problem reduces to finding optimal  $f$  and  $\theta_f$ .

Given initial values for the parameters  $\theta_f$  and the segmentation, such a procedure alternately solves the two sub-problems:

$$\theta_f^{(t+1)} = \arg \max_{\theta_f} \ln P(f^{(t)} | \theta_f) \quad (6.71)$$

$$f^{(t+1)} = \arg \max_f \ln P(f | \theta_f^{(t+1)}) + \ln pl(d | f, \hat{\theta}_d) \quad (6.72)$$

until convergence. In the above,  $P(f^{(t)} | \theta_f)$  may be replaced by the corresponding pseudo-likelihood. Fig.6.8 show a segmentation result.

### 6.2.3 Expectation-Maximization

The expectation-maximization (EM) algorithm (Dempster et al. 1977) is a general technique for finding maximum likelihood (ML) estimates with *incomplete data*. In the classic ML, the estimate is obtained from the complete data as

$$\theta^* = \arg \max_{\theta} \ln P(d_{com} | \theta) \quad (6.73)$$

In EM, the complete data is considered to consist of the two parts,  $d_{com} = \{d_{obs}, d_{mis}\}$ , of which only  $d_{obs}$  is observed while  $d_{mis}$  is missing (or unobservable, hidden). With only the incomplete data  $d_{obs}$ , an EM procedure attempts to solve the following ML estimation problem

$$\theta^* = \arg \max_{\theta} \ln P(d_{obs} | \theta) \quad (6.74)$$

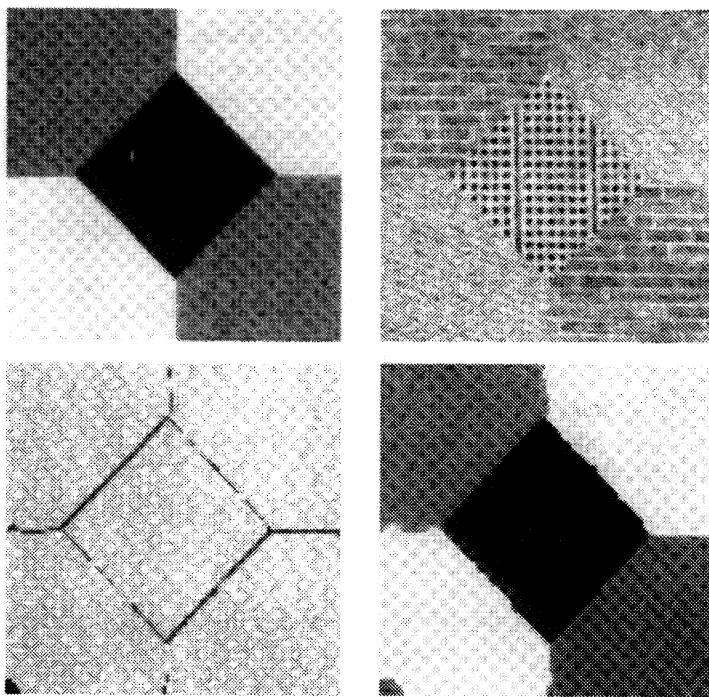


Figure 6.8: Texture segmentation with unknown parameters. (Upper-left) True regions. (Upper-right) Observed noisy textured regions. (Lower-left) Segmentation error. (Lower-right) Segmentation result. From (Won and Derin 1992) with permission; ©1992 Academic Press.

which is more general than the classic ML.

Intuitively, we may envisage a maximum likelihood procedure which after some initialization for  $d_{mis}$  and  $\theta$ , iterates between the following two steps until convergence: (1) estimate the missing part as  $\hat{d}_{mis}$  given the current  $\theta$  estimate and then use it to augment the observed data  $d = d_{obs}$  to form the *complete data* set  $d_{com} = \{d_{obs}, \hat{d}_{mis}\}$ ; and (2) estimate, with  $d_{com}$ ,  $\theta$  by maximizing the complete-data log likelihood  $\ln P(\hat{d}_{mis}, d_{obs} | \theta)$ . The simultaneous labeling-estimation algorithm described in the previous two subsections is based on such an idea.

However, we cannot work directly with this log likelihood because it is a random function of the missing variables  $f$  (the reason why the above procedure is *ad hoc*). The idea of the EM algorithm is to use the expectation of the complete-data log likelihood,  $E \left[ \ln P(\hat{d}_{mis}, d_{obs} | \theta) \right]$ , which formalizes the above intuitive procedure.

Related to parameter estimation in MRF models, the missing part corresponds to the unobservable labeling  $f$ ,  $f = d_{mis}$ , and the observed part is the given data,  $d = d_{obs}$ . The complete data log likelihood is then denoted  $\ln P(f, d | \theta)$ . The EM algorithm consists of the following two steps for each iteration:

- (1) *The E-step.* Computing the following conditional expectation of the log likelihood

$$Q(\theta | \theta^{(t)}) = E \left[ \ln P(f, d | \theta) | d, \theta^{(t)} \right] \quad (6.75)$$

- (2) *The M-step.* Maximizing  $Q(\theta | \theta^{(t)})$  to obtain the next estimate

$$\theta^{(t+1)} = \arg \max_{\theta} Q(\theta | \theta^{(t)}) \quad (6.76)$$

If in the M-step, the next estimate  $\theta^{(t+1)}$  is chosen only to ensure

$$Q(\theta^{(t+1)} | \theta^{(t)}) > Q(\theta^{(t)} | \theta^{(t)}) \quad (6.77)$$

(not necessarily to maximize  $Q$ ), then the algorithm is then called a generalized EM (GEM).

The E-step computes the conditional expectation of the unobservable labels  $f$  given the observed data  $d$  and the current estimate  $\theta^{(t)}$ ; and then substitutes the expectations for the labels. The M-step performs maximum likelihood estimation as if there were no missing data, *i.e.* as if had been filled in by the expectations.

The above describes just an instance of the EM procedure. More generally, the probability  $P(f, d | \theta)$  can be replaced by any complete-data sufficient statistics. When the prior distribution  $p(\theta)$  is known, the M-step can be modified to maximize the expected posterior  $E \left[ \ln p(\theta, f | d) | d, \theta^{(t)} \right]$  or  $E \left[ \ln P(f, d | \theta) | d, \theta^{(t)} \right] + \ln p(\theta)$ , where  $p(\theta, f | d) = \frac{p(f, d | \theta) p(\theta)}{P(d)}$ , instead of the likelihood.

EM is shown to converge to the ML estimates at least locally under some moderate conditions (Wu 1983). Global convergence is also proven for special cases *e.g.* in (Redner and Walker 1984; Lange 1990). However, the convergence of the EM algorithm can be painfully slow.

Let us be more specific about the expectation  $E \left[ \ln P(f, d | \theta) | d, \theta^{(t)} \right]$ . For  $f$  taking discrete values, which is the case for discrete restoration and for segmentation, the expectation is

$$E \left[ \ln P(f, d | \theta) | d, \theta^{(t)} \right] = \sum_{f \in \mathcal{F}} [\ln P(f, d | \theta)] P(f | d, \theta^{(t)}) \quad (6.78)$$

For continuous  $f$ , which is for continuous restoration, it is calculated from the p.d.f.'s

$$E \left[ \ln p(f, d | \theta) | d, \theta^{(t)} \right] = \int_{\mathbb{F}} [\ln p(f, d | \theta)] p(f | d, \theta^{(t)}) df \quad (6.79)$$

Consider the homogeneous and isotropic auto-logistic MRF model described by (6.7) and (6.8). We have

$$\ln P(f | \theta) = -\ln Z(\alpha, \beta) + \sum_i \alpha f_i + \sum_i \sum_{j \in \mathcal{N}_i} \beta f_i f_j \quad (6.80)$$

Assume that the data is obtained according to  $d_i = f_i + e_i$  with independent Gaussian noise  $e_i \sim N(0, \sigma^2)$ . Then

$$\ln p(d | f, \theta) = -n \ln(2\pi) - n \ln \sigma - \sum_i \frac{(f_i - d_i)^2}{2\sigma^2} \quad (6.81)$$

where  $\theta = \{\theta_f, \theta_d\} = \{\alpha, \beta, \sigma^2\}$ . The expectation is

$$\begin{aligned} Q(\theta | \theta^{(t)}) &= E \left[ \ln p(f, d | \theta) | d, \theta^{(t)} \right] \\ &= \sum_{f \in \mathcal{F}} [\ln p(d | f, \theta) + \ln p(f | \theta)] P(f | d, \theta^{(t)}) \end{aligned} \quad (6.82)$$

A main difficulty in applying EM to MRF parameter estimation is again the complexity of  $P(f | d, \theta^{(t)})$  in the computation of  $Q(\theta | \theta^{(t)})$ , due to the need to evaluate the partition function for  $P(f | \theta)$ . This difficulty may be overcome by using the pseudo-likelihood method (Chalmond 1989) or the mean field approximation (Zhang 1992; Zhang 1993).

### 6.2.4 Cross Validation

Lastly in this section, we describe an analytical technique called “cross validation” for estimating the smoothing parameter  $\lambda$  in the regularization model. Consider the energy of the form

$$E(f) = \sum_{i \in \mathcal{A}} [f(x_i) - d_i]^2 + \lambda \int_a^b [f''(x)]^2 dx \quad (6.83)$$

(cf. (1.100)) under additive white noise

$$d_i = f(x_i) + e_i \quad (6.84)$$

where  $e_i \sim N(0, \sigma^2)$  with unknown variance  $\sigma^2$ . It requires no *a priori* knowledge and is entirely data driven. The method has been studied in the context of regression and smoothing splines (see Chapter 4 of (Wahba 1990)). See also (Geiger and Poggio 1987; Shahraray and Anderson 1989; Thompson et al. 1991; Galatsanos and Katsaggelos 1992; Reeves 1992) for its applications in image and vision processing.

The idea of cross validation for parameter estimation is the following: Divide the data into an estimation subset and a validation subset. The former

subset is used to obtain a parameter estimate and the latter is used to validate the performance under the estimate. However, cross validation does not use one subset exclusively for one purpose (estimation or validation); it allows all the data to be used for both purposes. For instance, we can divide the data into  $m$  subsets; compute an estimate from all the subsets but one; and validate the estimate the left-out subset. Then, we perform the estimation-validation with a different left-out. We repeat this  $m$  times.

Ordinary cross validation (OCV) uses “leaving-out-one” strategy. Each point is left out in turn and its value is predicated from the rest of the data using the fitted covariance function. The prediction errors are summed. The best  $\lambda$  minimizes the summed error. This is expressed as follows.

Let  $f_{\lambda}^{[k]}$  be the minimizer of the following energy

$$E^{[k]}(f) = \sum_{i \neq k} \frac{1}{m} [f(x_i) - d_i]^2 + \lambda \int_a^b [f''(x)]^2 dx \quad (6.85)$$

in which  $d_k$  is left out. The OCV function is defined as

$$V_o(\lambda) = \frac{1}{m} \sum_{k=1}^m [f_{\lambda}^{[k]}(x_k) - d_k]^2 \quad (6.86)$$

In the above,  $[f_{\lambda}^{[k]}(x_k) - d_k]^2$  measures the error incurred by using  $f_{\lambda}^{[k]}(x_k)$  to predicate  $d_k$ . Therefore,  $V_o(\lambda)$  is the average prediction error. The OCV estimate of  $\lambda$  is the minimizer of  $V_o(\lambda)$ .

Let  $f_{\lambda}^*$  be the minimizer of the complete energy (6.83) under the parameter value  $\lambda$ . The relationship between  $f_{\lambda}^*$  and  $f_{\lambda}^{[k]}$  can be derived. Define the  $m \times m$  influence matrix  $A(\lambda)$  as satisfying

$$f_{\lambda}^* = A(\lambda)d \quad (6.87)$$

where  $f_{\lambda}^*$  and  $d$  are  $m \times 1$  vectors. The OCV function can be expressed in terms of  $A(\lambda)$  as follows (Craven and Wahba 1979)

$$V_o(\lambda) = \frac{1}{m} \sum_{k=1}^m \frac{[f_{\lambda}^*(x_k) - d_k]^2}{(1 - a_{kk})^2} \quad (6.88)$$

where  $f_{\lambda}^*(x_k) = \sum_{j=1}^m a_{kj} d_j$ . By comparing (6.86) with (6.88), one obtains

$$[f_{\lambda}^{[k]}(x_k) - d_k]^2 = \frac{[f_{\lambda}^*(x_k) - d_k]^2}{1 - a_{kk}(\lambda)} \quad (6.89)$$

OCV is modified to generalize cross validation (GCV) to achieve certain desirable invariance properties that do not generally hold for OCV. Let  $\Gamma$  be any  $m \times m$  orthogonal matrix and consider a new vector of data  $\tilde{d} = \Gamma d$ . In general, the OCV estimate can give a different value of  $\lambda$ ; in contrast, the

GCV estimate is invariant under this (rotation) transformation (Golub et al. 1979).

The GCV function is a weighted version of  $V_o(\lambda)$ :

$$V(\lambda) = \frac{1}{m} \sum_{k=1}^m [f_{\lambda}^{[k]}(x_k) - d_k]^2 w_k(\lambda) \quad (6.90)$$

where the weights

$$w_k(\lambda) = \left[ \frac{1 - a_{kk}(\lambda)}{1 - \frac{1}{m} \sum_{j=1}^m a_{jj}} \right]^2 \quad (6.91)$$

give the relative effect of leaving out  $d_k$ . It is obtained by replacing  $a_{kk}$  in (6.88) by  $\frac{1}{m} \sum_{j=1}^m a_{jj}$ . The GCV estimate  $\lambda_{GCV}$  is the minimizer of  $V(\lambda)$ .

## 6.3 Further Issues

### 6.3.1 Estimating the Number of MRFs

In the previous discussions on parameter estimation of multiple MRFs, the number of distinct MRF models (regions, textures, *etc.* ), here denoted by  $K$ , is assumed to be known. In completely unsupervised estimation and clustering schemes, this number is an unknown parameter and has also to be estimated along other parameters. The selection of the model number is generally a difficult issue in clustering analysis (Jain and Dubes 1988). Various criteria have been proposed for the estimation of  $K$ . They can be classified as two broad categories: heuristic approaches and statistical approaches. An example of the former category is the bootstrap technique (Jain and Moreau 1987); there, the value of  $K$  which provides the most stable partitions (in terms of a heuristic criterion) is the estimate of the number of clusters in the data set. In the following, a particular class of statistically based criteria are discussed.

#### Akaike Information Criterion(AIC)

Let  $K$  be the number to be chosen from the set of numbers  $\mathbf{K} = \{1, \dots, K_{\max}\}$ ,  $\hat{\theta}^{[K]}$  the ML estimate of the model parameter vector of the  $K$  models,  $L(K) = p_K(d \mid \hat{\theta}^{[K]})$  the maximized likelihood,  $n(K)$  the number of independent adjustable parameters in the  $K$  models, and  $N$  the sample size. Various maximization criteria take the form (Sclove 1987)

$$G(K) = \ln[L(K)] - a(N) n(K) - b(K, N) \quad (6.92)$$

Basically, the value of  $G(K)$  increases with the maximized likelihood  $L(K)$  but due to other terms, decreases with the value  $M$ . Criteria  $G(K)$  differ in the choices of  $a(N)$  and  $b(K, N)$ .

For  $a(N) = b(K, N) = 0$ , the  $K$  which maximizes  $G(K)$  is reduced to the maximum likelihood estimate. Such an estimate has been shown to be biased (Sakamoto et al. 1987). The Akaike information criterion (AIC) (Akaike 1974) is obtained as an expected entropy, giving  $a(N) = 1$  and  $b(K, N) = 0$ . The AIC is shown asymptotically unbiased (Sakamoto et al. 1987). From Bayes viewpoint (Schwartz 1987) and (Kashyap 1988),  $G(K)$  can be obtained by expanding the posterior probability  $P(K \mid \text{data})$ . In (Schwartz 1987),  $a(N) = \frac{1}{2} \ln(N)$  and  $b(K, N) = 0$ . Kashyap has the same  $a(N)$  as Schwartz but with  $b(K, N) = \frac{1}{2} \ln[\det B(K, N)]$  where  $B(K, N)$  is the Hessian of  $L(K)$  evaluated at the maximize likelihood estimates of  $K$ . Rissanen (1978) formulates  $G(K)$  in terms of minimum description length and gives  $a(N) = \frac{1}{2} \ln(\frac{N+1}{24})$  and  $b(K, N) = \ln(K+1)$ . Some results comparing the performance of the criteria are presented in (Sclove 1987), showing that Schwartz's gives better results than the AIC.

Sclove (1983) use the AIC to verify the number of region classes in image segmentation. Zhang and Modestino (1990) present a cluster validation scheme based on AIC (Akaike 1974) and argue that the AIC approach is advantageous over other heuristic schemes for cluster validation. Some modified criteria based on (6.92) have also been proposed. Bouman and Liu (1991) adopt a modified criterion for texture segmentation where a texture is modeled as a Gaussian AR random field (Jain 1981) with a number of  $H$  nonzero prediction coefficients. They replace the likelihood function with the joint function  $p_K(d, f \mid \theta^{[K]})$  where  $f$  is the labeling which determines the partition of  $S$  into regions,  $G(\theta^{[K]}, f, K) = \ln[p_K(d, f \mid \theta^{[K]})] - K(H+3) + 1$ . The criterion is optimized with respect to the parameters  $\theta$ , the partition  $f$  and the number of texture  $K$ . Won and Derin (1992) propose  $G(K) = \ln[L(K)] - N^c n(K)$ , where  $c$  is a prespecified constant, for determining the number of texture regions.

## Reversible Jump MCMC

Reversible jump MCMC, previously studied as jump-diffusion sampler (Grenander and Miller 1994) and then elaborated by (Green 1995), is a MCMC method adapted for general state spaces of variable dimensionality. Assume that there are a number of  $n(K)$  parameters to estimate when the model number is  $K$ , where  $K$  may be constrained to take a value in  $\mathbb{K} = \{1, \dots, K_{\max}\}$ . Assuming that every parameter is a real number, then the parameter subspace for  $\omega = (K, \theta^{[K]})$  is  $\Omega_K = \{K\} \times \mathbb{R}^{n(K)}$ . With variable  $K$ , the entire parameter space is the collection of all subspaces,  $\Omega = \cup_{K \in \mathbb{K}} \Omega_K$ . In order to implement sampling over  $\Omega$ , a reversible Markov chain is constructed so that the sampling is allowed to jump between parameter subspaces of variable dimensionality.

A reversible jump MCMC procedure here is aimed to sample a target joint distribution, which could be a likelihood  $p(d \mid K, \theta^{[K]})$  or joint posterior  $p(K, \theta^{[K]} \mid d)$ , over  $\Omega$ . A Markov chain is generated according to a transi-

tion probability  $p(\omega' | \omega)$  designed in such a way that the detailed balance equation of (6.42) is satisfied and the chain has the target distribution as its equilibrium.

If a move from  $\omega$  to  $\omega'$  does not jump so that the dimensionality of  $\theta^{[K]}$  remains unchanged, the reversible jump MCMC reduces to the conventional MCMC (to be specific, Metropolis-Hastings algorithm). For dimension-changing moves, something extra should be done for switching between subspaces.

Consider moves between two subspaces  $\Omega_1 = \{K_1\} \times \mathbb{R}^{n_1}$  and  $\Omega_2 = \{K_2\} \times \mathbb{R}^{n_2}$ . A move from  $\Omega_1$  to  $\Omega_2$  could be implemented by drawing a vector of random variables  $u^{[1]}$  of length  $m_1$  from some distribution  $p_1(u^{[1]})$ , independently of  $\theta^{[1]} \in \mathbb{R}^{n_1}$ , and then setting  $\theta^{[2]} \in \mathbb{R}^{n_2}$  to some deterministic function of  $\theta^{[1]}$  and  $u^{[1]}$ . Similarly, to move back,  $u^{[2]}$  of length  $m_2$  will be generated from  $p_2(u^{[2]})$ , and  $\theta^{[1]}$  is set to some function of  $\theta^{[2]}$  and  $u^{[2]}$ . The dimensions must match,  $n_1 + m_1 = n_2 + m_2$  and there must be a bijection between  $(\theta^{[1]}, u^{[1]})$  and  $(\theta^{[2]}, u^{[2]})$ , that is,  $(\theta^{[2]}, u^{[2]}) = \phi_2(\theta^{[1]}, u^{[1]})$  and  $(\theta^{[1]}, u^{[1]}) = \phi_2^{-1}(\theta^{[2]}, u^{[2]}) = \phi_1(\theta^{[2]}, u^{[2]})$ .

The acceptance probability for a move from  $f = (K_1, \theta^{[1]})$  to  $f' = (K_2, \theta^{[2]})$  can be derived as (Green 1995)

$$\alpha(f' | f) = \min \left\{ 1, \frac{P(K_2, \theta^{[2]} | d) p_2(u^{[2]}) q(K_2, \theta^{[2]})}{P(K_1, \theta^{[1]} | d) p_1(u^{[1]}) q(K_1, \theta^{[1]})} \left| \frac{\partial \phi_2(\theta^{[1]}, u^{[1]})}{\partial (\theta^{[1]}, u^{[1]})} \right| \right\} \quad (6.93)$$

where  $P(K, \theta | d)$  is the target distribution of states  $(K, \theta) \in \Omega$  given data  $d$ ,  $q(K, \theta)$  is the probability of choosing the move to state  $f = (K, \theta)$ , and  $\left| \frac{\partial \phi_2(\theta^{[2]}, u^{[2]})}{\partial (\theta^{[1]}, u^{[1]})} \right|$  is the Jacobian of the mapping  $\phi_2$ . The acceptance probability for the reverse move can be formulated similarly.

In (Richardson and Green 1997) and (Barker and Rayner 1997), the reversible jump MCMC is incorporated into simultaneous segmentation and parameter estimation procedure. A joint posterior distributions  $p(K, \theta^{[K]} | d)$  is formulated where  $K$  represents the number of distinct regions and  $\theta^{[K]}$  are the collection of parameters of the MRF models for the regions and observation noise. Given  $K$ ,  $p(K, \theta^{[K]} | d)$  may be formulated (Richardson and Green 1997; Barker and Rayner 1997), and the formulation is in principle similar to what is detailed in Section 6.2.2.

### 6.3.2 Reduction of Nonzero Parameters

This is an issue in MRF representation which affects parameter estimation. If an MRF can be represented using fewer parameters, then algorithms for it can be more efficient. It is known that Gibbs clique potential parameter representations for an MRF are not unique (Griffiths 1976; Kindermann and Snell 1980). For the sake of economy, it is preferred to use a representation which requires the smallest possible number of nonzero clique parameters.

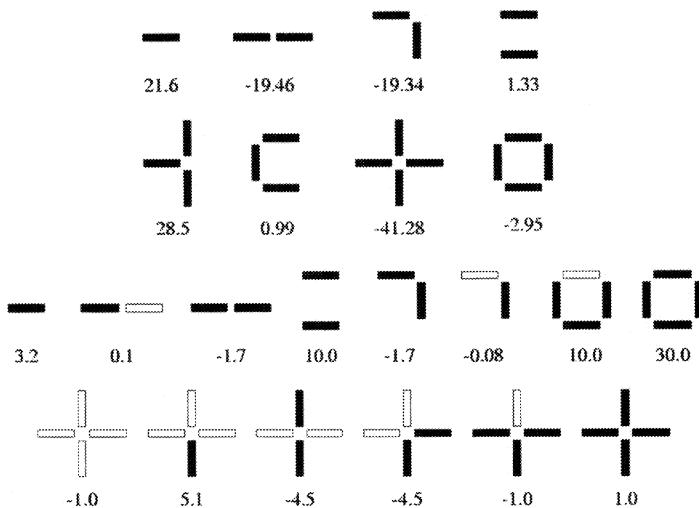


Figure 6.9: Normalized nonzero clique potentials estimated by using the least squares method (top) and nonzero clique potentials chosen *ad hoc* (bottom). Filled bars are edges and non-filled bars non-edges.

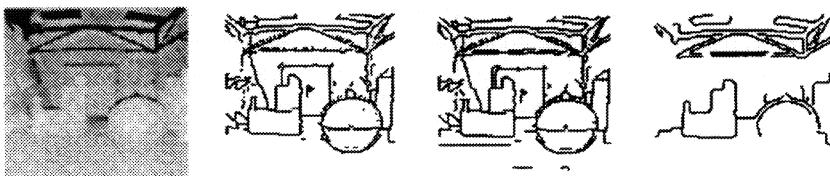


Figure 6.10: Step edge detection using various schemes. From left to right: the input image; edges detected by using estimated canonical clique potentials; edges obtained using the *ad hoc* clique potentials; edges detected using the Canny edge detector. From (Nadabar and Jain 1992) with permission; ©1992 IEEE.

The normalized clique potential parameterization described in Section 1.2.5 can be used for this purpose.

Given a countable set  $\mathcal{L}$  of labels and a set  $\mathcal{C}$  of all cliques, one may choose an arbitrary label  $\ell^0 \in \mathcal{L}$  and let  $V_c(f) = 0$  whenever  $f_i = \ell^0$  for some  $i \in c$ . Then he can specify nonzero clique parameters for the other configurations to define an MRF.

Nadabar and Jain (1991) use a normalized clique potential representation for MRF-based edge detection. Assuming the 4-neighborhood system on

the image lattice, there can be eight different types of edge cliques whose shapes are shown in the top part of Fig.6.9. The label at each edge site takes one of the two values, non-edge and edge, in the label set  $\mathcal{L}=\{\text{NE}, \text{E}\}$ . These gives thirty-five (35) {edge-label, clique-type} combinations. In the normalized representation, clique potentials  $V_c(f)$  are set to zero whenever  $f_i$  takes a particular value, say,  $f_i=\text{NE}$  for some  $i \in c$ . In this way, only 8 clique potentials are nonzero; thus the number of nonzero parameters are reduced from 35 to 8.

In (Zhang 1995), parameters reduction is performed for the compound Gauss-Markov model of (Jeng and Wood 1990; Jeng and Wood 1991). The model contains line process variables on which clique potential parameters are dependent. By imposing reasonable symmetry constraints, the 80 inter-dependent parameters of the model is reduced to 7 independent ones. This not only reduces the parameter number but also guarantees the consistency of the model parameters.

After the normalized representation is chosen, the parameter estimation can be formulated using any method such as one of those described in the previous sections. In (Jain and Nadabar 1990) for example, the Derin-Elliott LS method is adopted to estimate line process parameters for edge detection. The estimated normalized clique potentials is shown in Fig.6.9 in comparison with an *ad hoc* choice made in their previous work (Jain and Nadabar 1990). The corresponding results of edges detected using various schemes are shown in Fig.6.10.

After a set of parameters are estimated, one would evaluate the estimate. The evaluation is to test the goodness of fit: How well does the sample distribution fit the estimated MRF model with parameter  $\theta$ . This is a statistical problem. Some validation schemes are discussed in (Cross and Jain 1983; Chen 1988) for testing the fit between the observed distribution and the assumed distribution with the estimated parameters. One available technique is the well-known  $\chi^2$  test for measuring the correspondence between two distributions.

# Chapter 7

# Parameter Estimation in Optimal Object Recognition

Object recognition systems involve parameters such as thresholds, bounds and weights. These parameters have to be tuned before the system can perform successfully. A common practice is to choose such parameters manually on an *ad hoc* basis, which is a disadvantage. This chapter<sup>1</sup> presents a theory of parameter estimation for optimization-based object recognition where the optimal solution is defined as the global minimum of an energy function. The theory is based on supervised learning from examples. *Correctness* and *instability* are established as criteria for evaluating the estimated parameters. A correct estimate enables the labeling implied in each exemplary configuration to be encoded in a unique global energy minimum. The instability is the ease with which the minimum is replaced by a non-exemplary configuration after a perturbation. The optimal estimate minimizes the instability. Algorithms are presented for computing correct and minimal-instability estimates. The theory is applied to the parameter estimation for MRF-based recognition and promising results are obtained.

## 7.1 Motivation

Object recognition systems almost inevitably involve parameters such as thresholds, bounds and weights (Grimson 1990). In optimization-based object recognition, where the optimal recognition solution is explicitly defined as the global extreme of an objective function, these parameters can be part of the definition of the objective function by which the global cost (or gain)

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<sup>1</sup>This chapter is based on (Li 1997b).

of the solution is measured. The selection of the parameters is crucial for a system to perform successfully.

Among all the admissible parameter estimates, only a subset of them lead to the desirable or correct solutions to the recognition. Among all the correct estimates, a smaller number of them are better in the sense that they lead to correct solutions for a larger variety of data sets. One of them may be optimal in the sense that it makes the vision procedure the most stable to uncertainties and the least prone to local optima in the search for the global optimum.

The manual method performs parameter estimation in an *ad hoc* way by trial and error: A combination of parameters is selected to optimize the objective function, then the optimum is compared with the desirable result in the designer's perception and the selection is adjusted; this process is repeated until a satisfactory choice, which makes the optimum consistent with the desirable result, is found. This is a process of *supervised learning from examples*. When the objective function takes a right functional form, a correct manual selection may be made for a small number of data sets. However, there is no reason to believe that the manual selection is an optimal or even a good one. Such empirical methods have been criticized for its *ad hoc* nature.

This chapter aims to develop an automated, optimal approach for parameter estimation<sup>2</sup> applied to optimization-based object recognition schemes. A theory of parameter estimation based on supervised learning is presented. The learning is “supervised” because *exemplar* is given. Each exemplary instance represents a desirable recognition result where a recognition result is a labeling of the scene in terms of the model objects. *Correctness* and *optimality* are proposed as the two-level criteria for evaluating parameter estimates.

A correct selection of parameter enables the configuration given by each exemplary instance to be embedded as a unique global energy minimum. In other words, if the selection is incorrect, the exemplary configuration will not correspond to a global minimum. While a correct estimate can be learned from the exemplar, it is generally not the only correct solution. *Instability* is defined as the measure of the ease with which the global minimum is replaced by a non-exemplar labeling after a perturbation to the input. The optimality minimizes the *instability* so as to maximize the ability to generalize the estimated parameters to other situations not directly represented by the exemplar.

Combining the two criteria gives a constrained minimization problem: minimize the instability subject to the correctness. A non-parametric algorithm is presented for learning an estimate which is optimal as well as correct. It does not make any assumption about the distributions and is useful for cases where the size of training data obtained from the exemplar is small and

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<sup>2</sup>In this chapter, parameter “selection”, “estimation” and “learning” are used interchangeably.

where the underlying parametric models are not accurate. The estimate thus obtained is optimal with respect to the training data.

The theory is applied to a specific model of MRF recognition proposed in (Li 1994a). The objective function in this model is the posterior energy of an MRF. The form of the energy function has been derived but it involves parameters that have to be estimated. The optimal recognition solution is the maximum *a posteriori* (MAP) configuration of an MRF. Experiments conducted show very promising results in which the optimal estimate serves well for recognizing other scenes and objects.

A parametric method based on *maximum likelihood* is also described for computing the optimal parameter estimate under the Gaussian-MRF assumption. It takes the advantage of the assumption and may be useful when the size of training data is sufficiently large. The parameter estimate thus computed is optimal with respect to the assumption.

Although automated and optimal parameter selection for object recognition in high level vision is an important and interesting problem which has been existing for a long time, reports on this topic are rare. Works have been done in related areas: In (Poggio and Edelman 1990) for recognizing 3D objects from different viewpoints, a function mapping any viewpoint to a standard view is learned from a set of perspective views. In (Weng et al. 1993), a network structure is introduced for automated learning to recognize 3D objects. In (Pope and Lowe 1993), a numerical graph representation for an object model is learned from features computed from training images. A recent work (Pelillo and Refice 1994) proposes a procedure of learning compatibility coefficients for relaxation labeling by minimizing a quadratic error function. Automated and optimal parameter estimation for low level problems has achieved significant progress. MRF parameter selection has been dealt with in statistics (Besag 1974; Besag 1975) and in applications such as image restoration, reconstruction and texture analysis (Cross and Jain 1983; Cohen and Cooper 1987; Derin and Elliott 1987; Qian and Titterington 1989; Zhang 1988; Nadabar and Jain 1992). The problem is also addressed from regularization viewpoint (Wahba 1980; Geiger and Poggio 1987; Shahraray and Anderson 1989; Thompson et al. 1991).

The chapter is organized as follows: Section 7.2 presents the theory. Section 7.3 applies the theory to an MRF recognition model. Section 7.4 presents the experimental results. Finally, conclusions are made in Section 7.5.

## 7.2 Theory of Parameter Estimation for Recognition

In this section, correctness, instability and optimality are proposed for evaluating parameter estimates. Their relationships to non-parametric pattern recognition are discussed and non-parametric methods for computing correct

and optimal estimates are presented. Prior to these, necessary notations for optimization-based object recognition are introduced.

### 7.2.1 Optimization-Based Object Recognition

In optimization-based recognition, the optimal solution is explicitly defined as the extreme of an objective function. Let  $f$  be a configuration representing a recognition solution. The cost of  $f$  is measured by a global objective function  $E(f | \theta)$ , also called the energy. The definition of  $E$  is dependent on  $f$  and a number of  $K + 1$  parameters  $\theta = [\theta_0, \theta_1, \dots, \theta_K]^T$ . As the optimality criterion for model-based recognition, it also relates to other factors such as the observation, denoted  $\mathcal{G}$ , and model references, denoted  $\mathcal{G}'$ . Given  $\mathcal{G}$ ,  $\mathcal{G}'$  and  $\theta$ , the energy maps a solution  $f$  to a real number by which the cost of the solution is evaluated. The optimal solution corresponds to the global energy minimum, expressed as:

$$f^* = \arg \min_f E(f | \theta) \quad (7.1)$$

In this regard, it is important to formulate the energy function so that the “correct solution” is embedded as the global minimum. The energy may also serve as a guide to the search for a minimal solution. In this aspect, it is desirable that the energy should differentiate the global minimum from other configurations as much as possible.

The energy function may be derived using one of the following probabilistic approaches: fully parametric, partially parametric and non-parametric. In the fully parametric approach, the energy function is derived from probability distributions in which all the involved parameters are known. Parameter estimation is a problem only in the partially parametric and the non-parametric cases.

In the partially parametric case, the forms of distributions are given but some involved parameters are unknown. An example is the Gaussian distribution with an unknown variance and another is the Gibbs distribution with unknown clique potential parameters. In this case, the problem of estimating parameters in the objective function is related to estimating parameters in the related probability distributions.

In the nonparametric approach, no assumptions are made about distributions and the form of the objective function is obtained based on experiences or pre-specified “basis functions” (Poggio and Edelman 1990). This also applies to situations where the data set is too small to have statistical significance.

An important form for  $E$  in object recognition is the weighted sum of various terms, expressed as

$$E(f | \theta) = \theta^T U(f) = \sum_{k=0}^K \theta_k U_k(f) \quad (7.2)$$

where  $U(f) = [U_0(f), U_1(f), \dots, U_K(f)]^T$  is a vector of *potential functions*. A potential function is dependent on  $f$ ,  $\mathcal{G}$  and  $\mathcal{G}'$ , where the dependence can be nonlinear in  $f$ , and often measures the violation of a certain constraint incurred by the solution  $f$ . This linear combination of (nonlinear) potential functions is not an unusual form. It has been used in many matching and recognition works; see (Duda and Hart 1973; Fischler and Elschlager 1973; Davis 1979; Ghahraman et al. 1980; Jacobus et al. 1980; Shapiro and Haralick 1981; Oshima and Shirai 1983; Bhanu and Faugeras 1984; Wong and You 1985; Fan et al. 1989; Nasrabadi et al. 1990; Wells III 1991; Weng et al. 1992; Li 1994a). Note that when  $E(f | \theta)$  takes the linear form, multiplying  $\theta$  by a positive factor  $\kappa > 0$  does not change the minimal configuration

$$\arg \min_f E(f | \theta) = \arg \min_f E(f | \kappa\theta) \quad (7.3)$$

Because of this equivalent, an additional constraint should be imposed on  $\theta$  for the uniqueness. In this work,  $\theta$  is confined to have a unit Euclidean length

$$\|\theta\| = \sqrt{\sum_{k=0}^K \theta_k^2} = 1 \quad (7.4)$$

Given an observation  $\mathcal{G}$ , a model reference  $\mathcal{G}'$  and the form of  $E(f | \theta)$ , it is the  $\theta$  value that completely specifies the energy function  $E(f | \theta)$  and thereby defines the minimal solution  $f^*$ . It is desirable to learn the parameters from exemplars so that the minimization-based recognition is performed in a right way. The criteria for this purpose are established in the next subsection.

### 7.2.2 Criteria for Parameter Estimation

An exemplary instance is specified by a triple  $(\bar{f}, \mathcal{G}, \mathcal{G}')$  where  $\bar{f}$  is the exemplary configuration (recognition solution) telling how the scene ( $\mathcal{G}$ ) should be labeled or interpreted in terms of the model reference ( $\mathcal{G}'$ ). The configuration  $\bar{f}$  may be a structural mapping from  $\mathcal{G}$  to  $\mathcal{G}'$ . Assume that there are  $L$  model objects; then at least  $L$  exemplary instances have to be used for learning to recognize the  $L$  object. Let the instances be given as

$$\{(\bar{f}^\ell, \mathcal{G}^\ell, \mathcal{G}'^\ell) \mid \ell = 1, \dots, L\} \quad (7.5)$$

We propose two-level criteria for learning  $\theta$  from examples:

1. *Correctness.* This defines a parameter estimate which encodes constraints into the energy function in a correct way. A correct estimate, denoted  $\theta_{correct}$ , should embed each  $\bar{f}^\ell$  into the minimum of the corresponding energy  $E(f^\ell | \theta_{correct})$ , that is

$$\bar{f}^\ell = \arg \min_f E^\ell(f | \theta_{correct}) \quad \forall \ell \quad (7.6)$$

where the definition of  $E^\ell(f | \theta)$  is dependent upon the given scene  $\mathcal{G}^\ell$  and the model  $\mathcal{G}'^\ell$  of a particular exemplary instance  $(\bar{f}^\ell, \mathcal{G}^\ell, \mathcal{G}'^\ell)$  as well as  $\theta$ . Briefly speaking, a correct  $\theta$  is one which makes the minimal configuration  $f^*$  defined in (7.1) coincide with the exemplary configuration  $\bar{f}^\ell$ .

2. *Optimality.* This is aimed to maximize the generalizability of the parameter estimate to other situations. A measure of *instability* is defined for the optimality and is to be minimized.

The correctness criterion is necessary for a vision system to perform correctly and is of fundamental importance. Only when this is met does the MRF model make a correct use of the constraints. The optimality criterion is not necessary in this regard but makes the estimate most generalizable.

### Correctness

In (7.6), a correct estimate  $\theta_{correct}$  enables the  $\bar{f}$  to be encoded as the global energy minimum for this particular pair  $(\mathcal{G}, \mathcal{G}')$ . Therefore, it makes any energy change due to a configuration change from  $\bar{f}$  to  $f \neq \bar{f}$  positive, that is,

$$\begin{aligned}\Delta E^\ell(f | \theta) &= E^\ell(f | \theta) - E^\ell(\bar{f}^\ell | \theta) \\ &= \sum_k \theta_k [U_k(f) - U_k(\bar{f}^\ell)] > 0 \quad \forall f \in \mathcal{F}^\ell\end{aligned}\tag{7.7}$$

where

$$\mathcal{F}^\ell = \{f \mid f \neq \bar{f}^\ell\}\tag{7.8}$$

is the set of all non- $\bar{f}^\ell$  configurations. Let

$$\Theta_{correct} = \{\theta \mid \Delta E^\ell(f | \theta) > 0, \forall f \in \mathcal{F}^\ell, \ell = 1, \dots, L\}\tag{7.9}$$

be the set of all correct estimates. The set, if non-empty, usually comprises not just a single point but a region in the allowable parameter space. Some of the points in the region are better in the sense of stability to be defined below.

### Instability

The value of the energy change  $\Delta E^\ell(f | \theta) > 0$  can be used to measure the (local) stability of  $\theta \in \Theta_{correct}$  with respect to a certain configuration  $f \in \mathcal{F}^\ell$ . Ideally, we want  $E^\ell(\bar{f}^\ell | \theta)$  to be very low and  $E^\ell(f | \theta)$  to be very high, such that  $\Delta E^\ell(f | \theta)$  is very large, for all  $f \neq \bar{f}^\ell$ . In such a situation,  $\bar{f}^\ell$  is expected to be a stable minimum where the stability is said w.r.t. perturbations in the observation and w.r.t. the local minimum problem with the minimization algorithm.

The smaller is  $\Delta E^\ell(f \mid \theta)$ , the larger is the chance with which a perturbation to the observation will cause  $\Delta E^\ell(f \mid \theta_{\text{correct}})$  to become negative to violate the correctness. When  $\Delta E^\ell(f \mid \theta_{\text{correct}}) < 0$ ,  $f^\ell$  no longer corresponds to the global minimum. Moreover, we assume that configurations  $f$  whose energies are slightly higher than  $E^\ell(\bar{f}^\ell \mid \theta)$  are possibly *local energy minima* at which an energy minimization algorithm is most likely to get stuck.

Therefore, the energy difference, *i.e.* the local stabilities should be enlarged. One may define the global stability as the sum of all  $\Delta E^\ell(f \mid \theta)$ . For reasons to be explained later, *instability*, instead of stability, is used for evaluating  $\theta$ .

The local instability for a correct estimate  $\theta \in \Theta_{\text{correct}}$  is defined as

$$c^\ell(\theta, f) = \frac{1}{\Delta E^\ell(f \mid \theta)} \quad (7.10)$$

where  $f \in \mathcal{F}^\ell$ . It is “local” because it considers only one  $f \in \mathcal{F}^\ell$ . It is desirable to choose  $\theta$  such that the value of  $c^\ell(\theta, f)$  is small *for all*  $f \in \mathcal{F}^\ell$ . Therefore, we defined the following global *p-instability* of  $\theta$

$$C_p^\ell(\theta) = \left\{ \sum_{f \in \mathcal{F}^\ell} [c^\ell(\theta, f)]^p \right\}^{1/p} \quad (7.11)$$

where  $p \geq 1$ . The total global *p-instability* of  $\theta$  is

$$C_p(\theta) = \sum_{\ell=1}^L C_p^\ell(\theta) \quad (7.12)$$

In the limit as  $p \rightarrow \infty$ , we have<sup>3</sup>

$$C_\infty^\ell(\theta) = \max_{f \in \mathcal{F}^\ell} c^\ell(\theta, f) = \frac{1}{\min_{f \in \mathcal{F}^\ell} \Delta E^\ell(f \mid \theta)} \quad (7.13)$$

is due solely to  $f$  having the smallest  $c^\ell(\theta, f)$  or largest  $\Delta E^\ell(f \mid \theta)$  value.

Unlike the global stability definition, the global instability treats each item in the following manner: those  $f$  having smaller  $\Delta E^\ell(f \mid \theta)$  (larger  $c^\ell(\theta, f)$ ) values affect  $C_p^\ell(\theta)$  in a more significant way. For  $p = 2$ , for example, the partial derivative is

$$\frac{\partial C_2^\ell(\theta)}{\partial \theta_k} = \frac{\partial C_2^\ell(\theta)}{\partial \Delta E^\ell(f \mid \theta)} \frac{\partial \Delta E^\ell(f \mid \theta)}{\partial \theta_k} = \quad (7.14)$$

$$= \frac{1}{[\Delta E^\ell(f \mid \theta)]^3} [U_k(f) - U_k(\bar{f}^\ell)] \quad (7.15)$$

<sup>3</sup>This is because for the *p*-norm defined by  $\|y\|_p = (|y_1|^p + |y_2|^p + \cdots + |y_n|^p)^{1/p}$ , we have  $\lim_{p \rightarrow \infty} \|y\|_p = \max_j |y_j|$ .

where  $E^\ell(f | \theta)$  takes the linear form (7.2). The smaller the  $\Delta E^\ell(f | \theta)$  is, the more it affects  $\theta$ . This is desirable because such  $f$  are more likely than the others to violate the correctness, because their  $\Delta E^\ell(f | \theta)$  values are small, and should be more influential in determining  $\theta$ .

## Optimality

The optimal estimate is defined as the one in  $\Theta_{correct}$  that minimizes the instability

$$\bar{\theta} = \arg \min_{\theta \in \Theta_{correct}} C_p(\theta) \quad (7.16)$$

Obviously,  $C_p(\theta)$  is positive for all  $\theta \in \Theta_{correct}$  and hence the minimal solution always exists. The minimal solution tends to increase  $\Delta E(f | \theta)$  values in the global sense and thus maximizes the extent to which an exemplary configuration  $\bar{f}$  remains to be the global energy minimum when the observation  $d$  is perturbed. It is also expected that with such a  $\bar{\theta}$ , local minima corresponding to some low energy-valued  $f$  are least likely to occur in minimization.

The correctness in (7.7), instability in (7.11) and optimality in (7.16) are defined without specifying the form of the energy  $E(f | \theta)$ . Therefore, the principle established so far is general for any optimization-based recognition models. Minimizing the instability with the constraint of the correctness is a nonlinear programming problem when the instability is nonlinear in  $\theta$ .

For conciseness, in the following, the superscript  $\ell$  will be omitted most of the time unless when necessary.

### 7.2.3 Linear Classification Function

In this work, we are interested in cases where the  $E(f | \theta)$  is linear in  $\theta$ . Assume that with a non-parametric or partially parametric modeling method, the energy is derived to take the linear form (7.2). With the linear form, the energy change can be written as

$$\Delta E(f | \theta) = \theta^T x(f) \quad (7.17)$$

where

$$x(f) = [x_0(f), x_1(f), \dots, x_K(f)]^T = U(f) - U(\bar{f}) \quad (7.18)$$

is the potential changes. Denote the set of all potential changes by

$$\mathcal{X} = \{x(f) \mid f \in \mathcal{F}\} \quad (7.19)$$

Note that  $\mathcal{X}$  excludes  $x(\bar{f})$ , the vector of zeros. The set  $\mathcal{X}$  will be used as the *training data* set. When there are  $L > 1$  exemplar instances,

$$\mathcal{X} = \{x^\ell(f) = U^\ell(f) - U^\ell(\bar{f}^\ell) \mid f \in \mathcal{F}^\ell, \forall \ell\} \quad (7.20)$$

contains training data from all the instances.  $\mathcal{X}$  is the  $K + 1$  dimensional data space in which  $x$ 's are points.

The correctness (7.7) can be written as

$$\theta^T x(f) > 0 \quad \forall x \in \mathcal{X} \quad (7.21)$$

In pattern recognition,  $\theta^T x$  is called a *linear classification function* when considered as a function of  $x$ ;  $\theta^T x(f)$  is called a *generalized linear classification function* when considered as a function of  $f$ . There exist useful theories and algorithms for linear pattern classification functions (Duda and Hart 1973).

Equation  $\theta^T x = 0$  is a hyperplane in the space  $\mathcal{X}$ , passing through the origin  $x(f) = \mathbf{0}$ . With a correct  $\theta$ , the hyperplane divides the space into two parts, with all  $x(f)$  ( $f \in \mathcal{F}$ ) in one side, more exactly the “positive” side, of the hyperplane. The Euclidean distance from  $x$  to the hyperplane is equal to  $\theta^T x / \|\theta\|$ , a *signed* quantity. After the normalization (7.4), the point-to-hyperplane distance is just  $\theta^T x(f)$ .

The correctness can be judged by checking the minimal distance from the point set  $\mathcal{X}$  to the hyperplane. We define the “separability” as the smallest distance value

$$S(\theta) = \min_f \theta^T x(f) / \|\theta\| \quad (7.22)$$

The above can also be considered as the stability of the system with a given  $\theta$ . The correctness is equivalent to the positivity of the separability.

It is helpful to visually illustrate the optimality using  $C_\infty$ . With  $C_\infty$ , the minimal-instability solution is the same as the following minimax solution

$$\bar{\theta} = \arg \min_{\theta \in \Theta_{\text{correct}}} C_\infty(\theta) = \arg \min_{\theta \in \Theta_{\text{correct}}} \left[ \max_{f \in \mathcal{F}} \Delta E(f \mid \theta) \right] \quad (7.23)$$

In this case,  $S(\theta) = 1/C_\infty(\theta)$  and the above is maximal-separability.

Fig.7.1 qualitatively illustrates correct/incorrect and the maximal-separability parameters. It is an example in two dimensional space where an energy change takes the form of  $\Delta E(f \mid \theta) = \theta^T x(f) = \theta_1 x_1 + \theta_2 x_2$ . The point  $x(f) = \mathbf{0}$  coincides with the origin of the  $x_1$ - $x_2$  space. Data  $x(f)$  ( $f \in \mathcal{F}$ ) are shown in filled dots. The three lines  $L^{(1)}$ ,  $L^{(2)}$  and  $L^{(3)}$  represent three hyperplanes, corresponding to three different estimates of parameters  $\theta$ . Parameter estimates for  $L^{(1)}$  and  $L^{(2)}$  are correct ones because they make all the data points in the positive side of the hyperplane and thus satisfy Eq.(7.7). However,  $L^{(3)}$  is not a correct one. Of the two correct estimates in Fig.7.1, the one corresponding to  $L^{(1)}$  is better than the others in terms of  $C_\infty$ .

The separability determines the range of disturbances in  $x$  within which the exemplary configuration  $\bar{f}$  remains minimal. Refer to point  $x' = x(f')$  in the figure. Its distance to  $L^{(2)}$  is the smallest. The point may easily deviate across  $L^{(2)}$  to the negative half space due to some perturbation in the observation  $d$ . When this happens,  $\Delta E(f' \mid \theta) < 0$ , causing  $E(f' \mid \theta)$  to be lower than  $E(\bar{f} \mid \theta)$ . This means that  $\bar{f}$  is no longer the energy minimum. If

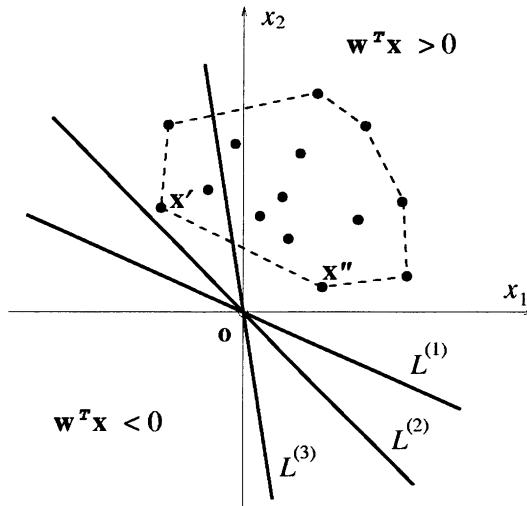


Figure 7.1: Correct and incorrect parameter estimates. A case where the  $C_\infty$ -optimal hyperplane is parallel to  $\overline{x'x''}$ . From (Li 1995c) with permission; ©1995 Kluwer.

the parameters are chosen as those corresponding to  $L^{(1)}$ , the separability is larger and the violation of the correctness is less likely to happen.

The dashed polygon in the figure forms the convex hull (polytope) of the data set. Only those data points which form the hull affect the minimax solution whereas those inside the hull are ineffective. The ineffective data points inside the hull can be removed and only the effective points, the number of which may be small compared to the whole data set, need be considered in the solution finding process. This increases the efficiency.

There are two possibilities for the orientation of the maximal-separability hyperplane with respect to the polytope. The first is that the optimal hyperplane is parallel to one of the sides (faces in cases of 3D or higher dimension parameter space) of the polytope, which is the case in Fig.7.1. The other is that the optimal hyperplane is perpendicular to the line linking the origin  $\emptyset$  and the nearest data point, which is the case in Fig.7.2. This is a property we may use to find the minimax solution. When the points constituting the polytope are identified, all the possible solutions can be enumerated; when there are only a small number of them, we can find the minimax solution by an exhaustive comparison. In Fig.7.2,  $L^{(2)}$  is parallel to  $\overline{x'x''}$ , and  $L^{(3)}$  is parallel to  $\overline{x'x'''}$  but  $L^{(1)}$  is perpendicular to  $\overline{0x'}$ . Let  $\theta_1, \theta_2$  and  $\theta_3$  be the sets of parameters corresponding to  $L^{(1)}, L^{(2)}$  and  $L^{(3)}$ . Suppose in the figure, the following relations hold:  $S(\theta_1) > S(\theta_2)$  and  $S(\theta_1) > S(\theta_3)$ ; then  $\theta_1$  is the best estimate because it maximizes the separability.

The minimax solution (7.23) with  $p = \infty$  was used in the above only for illustration. With  $p = \infty$ , a continuous change in  $x$  may lead to a discontin-

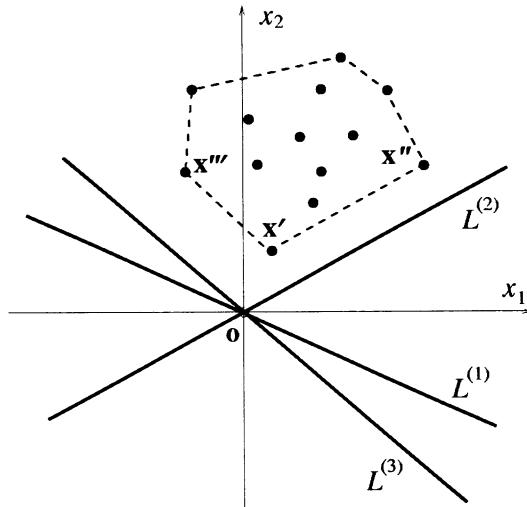


Figure 7.2: A case where the optimal hyperplane is perpendicular to  $\overrightarrow{ox'}$ . From (Li 1995c) with permission; ©1995 Kluwer.

uous change in the  $C_\infty$ -optimal solution. This is due to the decision-making nature of the definition which may cause discontinuous changes in the minimum. Other  $p$ -instability definitions, with  $p = 1$  or 2 for example, are used in our practice because they give more stable estimates.

#### 7.2.4 A Non-Parametric Learning Algorithm

Consider the case when  $p = 2$ . Because  $\theta^T x \geq 0$ , minimizing  $C_2(\theta)$  is equivalent to minimizing its square

$$[C_2(\theta)]^2 = \sum_{x \in \mathcal{X}} \frac{1}{(\theta^T x)^2} \quad (7.24)$$

The problem is given formally as

$$\begin{aligned} & \text{minimize} && \sum_{x \in \mathcal{X}} \frac{1}{(\theta^T x)^2} \\ & \text{subject to} && \|\theta\| = 1 \\ & && \theta^T x > 0 \quad \forall x \in \mathcal{X} \end{aligned} \quad (7.25)$$

This is a nonlinear programming problem and can be solved using the standard techniques. In this work, a gradient-based, non-parametric algorithm is used to obtain a numerical solution.

The perceptron algorithm (Rosenblatt 1962) has already provided a solution for learning a correct parameter estimate. It iterates on  $\theta$  to increase the objective of the form  $\theta^T x$  based on the gradient information where the gradient is simply  $x$ . In one cycle, all  $x \in \mathcal{X}$  are tried in turn to adjust  $\theta$ . If

---

**Algorithm** learning( $\theta$ ,  $\mathcal{X}$ )  
*/\* Learning correct and optimal  $\theta$  from the training data  $\mathcal{X}$  \*/*  
 Begin Algorithm  
 initialize( $\theta$ );  
 do {  
      $\theta_{last} \leftarrow \theta$ ;  
     if ( $\theta^T x > 0$   $\exists x \in \mathcal{X}$ ) {  
         correct( $\theta$ );  
          $\theta \leftarrow \theta / \|\theta\|$ ;  
     }  
      $\theta \leftarrow \theta + \mu \sum_{x \in \mathcal{X}} \frac{x}{(\theta^T x)^3}$ ;  
      $\theta \leftarrow \theta / \|\theta\|$ ;  
   } until ( $\|\theta_{last}^T - \theta\| < \epsilon$ );  
 return( $\theta = \theta$ );  
 End Algorithm

---

Figure 7.3: Algorithm for finding the optimal combination of parameters. From (Li 1995c) with permission; ©1995 Kluwer.

$\theta^T x > 0$ , meaning  $x$  is correctly classified under  $\theta$ , no action is taken. Otherwise if  $\theta^T x \leq 0$ ,  $\theta$  is adjusted according to the following gradient-ascent rule:

$$\theta \leftarrow \theta + \mu x \quad (7.26)$$

where  $\mu$  is a small constant. The updating is followed by a normalization operation

$$\theta \leftarrow \theta / \|\theta\| \quad (7.27)$$

when  $\|\theta\| = 1$  is required. The cycle is repeated until a correct classification is made for all the data and thus a correct  $\theta$  is learned. With the assumption that the solution exists, also meaning that the data set  $\mathcal{X}$  is linearly separable from the origin of the data space, the algorithm is guaranteed to converge after a finite number of iterations (Duda and Hart 1973).

The objective function  $[C_2(\theta)]^2 = \sum_{x \in \mathcal{X}} \frac{1}{(\theta^T x)^2}$  can be minimized in a similar way. The gradient is

$$\nabla [C_2(\theta)]^2 = -2 \sum_{x \in \mathcal{X}} \frac{x}{(\theta^T x)^3} \quad (7.28)$$

An updating goes as follows

$$\theta \leftarrow \theta + \mu \sum_{x \in \mathcal{X}} \frac{x}{(\theta^T x)^3} \quad (7.29)$$

where  $\mu$  is a small constant. If  $\theta$  were unconstrained, the above might diverge when  $(\theta^T x)^3$  becomes too small. However, the updating is again followed by the normalization  $\|\theta\| = 1$ . This process repeats until  $\theta$  converges to  $\bar{\theta}$ .

In our implementation, the two stages are combined into one procedure as shown in Fig. 7.3. In the procedure, initialize( $\theta$ ) sets  $\theta$  at random, correct( $\theta$ ) learns a correct  $\theta$  from  $\mathcal{X}$  and  $(\|\theta_{last}^T - \theta\| < \epsilon)$ , where  $\epsilon > 0$  is a small number, verifies the convergence. The algorithm is very stable.

The amount of change in each minimization iteration is  $\mu \sum_{x \in \mathcal{X}} \frac{x}{(\theta^T x)^3}$ . The influence from  $x$  is weighted by  $1/(\theta^T x)^3$ . This means that those  $x$  with smaller  $\theta^T x$  values (closer to the hyperplane) have bigger force in pushing the hyperplane away from themselves; whereas those with big  $\theta^T x$  values (far away from the hyperplane) have small influence. This effectively stabilizes the learning process.

When  $p = \infty$ , we are facing the minimax problem (7.23). An algorithm for solving this is the “Generalized Portrait Technique” (Vapnik 1982) which is designed for constructing hyperplanes with maximum separability. It is extended by Boser *et al.* (Boser *et al.* 1992) to train classifiers of the form  $\theta^T \varphi(x)$  where  $\varphi(x)$  is a vector of functions of  $x$ . The key idea is to transform the problem into the dual space by means of the Lagrangian. This gives a quadratic optimization with constraints. The optimal parameter estimate is expressed as a linear combination of supporting patterns where the supporting patterns correspond to the data points nearest to the hyperplane. Two benefits are gained from this method: there are no local minima in the quadratic optimization and the obtained maximum separability is insensitive to small changes of the learned parameters.

The  $\theta$  computed using the non-parametric procedure is optimal with respect to the training data  $\mathcal{X}$ . It is the best result that can be obtained from  $\mathcal{X}$  for generalization to other data. Better results may be obtained, provided that more knowledge about the training data is available.

### 7.2.5 Reducing Search Space

The data set  $\mathcal{X}$  in (7.20) may be very large because there are a combinatorial number of possible configurations in  $\mathcal{F} = \{f \neq \bar{f}\}$ . In principle, all  $f \neq \bar{f}$  should be considered. However, we assume that the configurations which are neighboring  $\bar{f}$  have the largest influence on the selection of  $\theta$ . Define the neighborhood of  $\bar{f}$  as

$$\mathcal{N}_{\bar{f}} = \{f = (f_1, \dots, f_m) \mid f_i \neq \bar{f}_i, f_i \in \mathcal{L}, \exists^1 i \in \mathcal{S}\} \quad (7.30)$$

where  $\exists^1$  reads “exists one and only one” and  $\mathcal{L}$  is the set of admissible labels for every  $f_i$ . The  $\mathcal{N}_{\bar{f}}$  consists of all  $f \in \mathcal{F}$  that differ from  $\bar{f}$  by one and only one component. This confinement reduces the search space to an enormous extent. After the configuration space is confined to  $\mathcal{N}_{\bar{f}}$ , the set of training data is computed as

$$\mathcal{X} = \{x = U(f) - U(\bar{f}) \mid f \in \mathcal{N}_f\} \quad (7.31)$$

which is much smaller than the  $\mathcal{X}$  in Eq.(7.20).

## 7.3 Application in MRF Object Recognition

The theory is applied to the parameter estimation for MRF object recognition where the form of the energy is derived based on MRFs. MRF modeling provides one approach to optimization-based object recognition (Modestino and Zhang 1989; Cooper 1990; Baddeley and van Lieshout 1992; Kim and Yang 1992; Li 1994a). The MAP solution is usually sought. The posterior distribution of configurations  $f$  is of Gibbs

$$P(f \mid d) = Z^{-1} e^{-E(f \mid \theta)} \quad (7.32)$$

where  $Z$  is a normalizing constant called the partition function and  $E(f \mid \theta)$  is the posterior energy function measuring the global of  $f$ . In the following,  $E(f \mid \theta)$  is defined and converted to the linear form  $\theta^T U(f)$ .

### 7.3.1 Posterior Energy

An object or a scene is represented by a set of features where the features are attributed by their properties and constrained to one another by contextual relations. Let a set of  $m$  features (sites) in the scene be indexed by  $\mathcal{S} = \{1, \dots, m\}$ , a set of  $M$  features (labels) in the considered model object by  $\mathcal{L} = \{1, \dots, M\}$ , and everything in the scene not modeled by labels in  $\mathcal{L}$  by  $\{0\}$  which is a virtual NULL label. The set union  $\mathcal{L}^+ = \mathcal{L} \cup \{0\}$  is the augmented label set. The structure of the scene is denoted by  $\mathcal{G} = (\mathcal{S}, d)$  and that of the model object by  $\mathcal{G}' = (\mathcal{L}, D)$  where  $d$  denotes the visual constraints on features in  $\mathcal{S}$  and  $D$  describes the visual constraints on features in  $\mathcal{L}$  where the constraints can be *e.g.* properties and relations between features.

Let object recognition be posed as assigning a label from  $\mathcal{L}^+$  to each of the sites in  $\mathcal{S}$  so as to satisfy the constraints. The labeling (configuration) of the sites is defined by  $f = \{f_1, \dots, f_m\}$  in which  $f_i \in \mathcal{L}^+$  is the label assigned to  $i$ . A pair  $(i \in \mathcal{S}, f_i \in \mathcal{L}^+)$  is a match or correspondence. Under contextual constraints, a configuration  $f$  can be interpreted as a mapping from the structure of the scene  $\mathcal{G} = (\mathcal{S}, d)$  to the structure of the model object  $\mathcal{G}' = (\mathcal{L}, D)$ . Therefore, such a mapping is denoted as a triple  $(f, \mathcal{G}, \mathcal{G}')$ .

The observation  $d = (d_1, d_2)$ , which is the features extracted from the image, consists of two sources of constraints, unary properties  $d_1$  for single-site features such as color and size, and binary relations  $d_2$  for pair-site features such as angle and distance. More specifically, each site  $i \in \mathcal{S}$  is associated with a set of  $K_1$  properties  $\{d_1^{(k)}(i) \mid k = 1, \dots, K_1, i \in \mathcal{S}\}$  and each pair of sites with a set of  $K_2$  relations  $\{d_2^{(k)}(i, i') \mid k = 1, \dots, K_2; i, i' \in \mathcal{S}\}$ . In the model object library, we have model features  $\{D_1^{(k)}(I) \mid k = 1, \dots, K_1, I \in \mathcal{L}\}$

and  $\{D_2^{(k)}(I, I') \mid k = 1, \dots, K_2; I, I' \in \mathcal{L}\}$  (note that  $\mathcal{L}$  excludes the NULL label). According to (3.14), under the labeling  $f$ , the observation  $d$  is a noise contaminated version of the corresponding model features  $D$

$$d_1(i) = D_1(f_i) + e(i), \quad d_1(i, i') = D_2(f_i, f_{i'}) + e(i, i') \quad (7.33)$$

where  $f_i, f_{i'} \neq 0$  are non-NUL labels and  $e$  is a white Gauss noise; that is,  $d_1(i)$  and  $d_2(i, i')$  are white Gaussian distributions with conditional means  $D_1(f_i)$  and  $D_2(f_i, f_{i'})$ , respectively.

The posterior energy  $E(f) = U(f \mid d)$  takes the form shown in (3.18), rewritten below

$$\begin{aligned} E(f) = & \sum_{i \in \mathcal{S}} V_1(f_i) + \\ & \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'}) + \\ & \sum_{i \in \mathcal{S}: f_i \neq 0} V_1(d_1(i) \mid f_i) + \\ & \sum_{i \in \mathcal{S}: f_i \neq 0} \sum_{i' \in \mathcal{S} - \{i\}: f_{i'} \neq 0} V_2(d_2(i, i') \mid f_i, f_{i'}) \end{aligned} \quad (7.34)$$

The first and second summations are due to the joint prior probability of the MRF labels  $f$ ; the third and fourth are due to the conditional p.d.f. of  $d$ , or the likelihood of  $f$ , respectively. Refer to (3.11), (3.12), (3.16) and (3.17).

### 7.3.2 Energy in Linear Form

Parameters involved in  $E(f)$  are the noise variances  $[\sigma_n^{(k)}]^2$  and the prior penalties  $v_{n0}$  ( $n = 1, 2$ ). Let the parameters be denoted uniformly by  $\theta = \{\theta_n^{(k)} \mid k = 0, \dots, K_n, n = 1, 2\}$ . For  $k = 0$ ,

$$\theta_n^{(0)} = v_{n0} \quad (7.35)$$

and for  $k \geq 1$ ,

$$\theta_n^{(k)} = (2[\sigma_n^{(k)}]^2)^{-1} \quad (7.36)$$

Note all  $\theta_n^{(k)} \geq 0$ . Let the different energy components be uniformly denoted by  $U = \{U_n^{(k)} \mid k = 0, \dots, K_n, n = 1, 2\}$ . For  $k = 0$ ,

$$U_1^{(0)}(f) = N_1 = \#\{f_i = 0 \mid i \in \mathcal{S}\} \quad (7.37)$$

is the number of NULL labels in  $f$  and

$$U_2^{(0)}(f) = N_2 = \#\{f_i = 0 \text{ or } f_{i'} = 0 \mid i \in \mathcal{S}, i' \in \mathcal{N}_i\} \quad (7.38)$$

is the number of label pairs at least one of which is NULL. For  $k \geq 1$ ,  $U_n^{(k)}(f)$  relates to the likelihood energy components; They measure how much the observations  $d_n^{(k)}$  deviate from the should-be true values  $D_n^{(k)}$  under  $f$ :

$$U_1^{(k)}(f) \triangleq U_1^{(k)}(d \mid f) = \sum_{i \in \mathcal{S}, f_i \neq 0} [d_1^{(k)}(i) - D_1^{(k)}(f_i)]^2 / \{2[\sigma_1^{(k)}]^2\} \quad (7.39)$$

and

$$\begin{aligned} U_2^{(k)}(f) &\triangleq U_2^{(k)}(d \mid f) \\ &= \sum_{i \in \mathcal{S}, f_i \neq 0} \sum_{i' \in \mathcal{S}, i' \neq i, f_{i'} \neq 0} [d_2^{(k)}(i, i') - D_2^{(k)}(f_i, f_{i'})]^2 / \{2[\sigma_2^{(k)}]^2\} \end{aligned} \quad (7.40)$$

After some manipulation, the energy can be written as

$$E(f \mid \theta) = \sum_{n=1}^2 \sum_{k=0}^{K_n} \theta_n^{(k)} U_n^{(k)}(f) = \theta^T U(f) \quad (7.41)$$

where  $\theta$  and  $U(f)$  are column vectors of  $K_1 + K_2 + 2$  components. Given an instance  $(\bar{f}, \mathcal{G}, \mathcal{G}')$ , the  $U(\bar{f})$  is a *known* vector of real numbers. The  $\theta$  is the vector of *unknown* weights to be determined. The stability follows immediately as

$$\Delta E(f \mid \theta) = \theta^T x(f) \quad (7.42)$$

where  $x(f) = U(f) - U(\bar{f})$ .

Some remarks on  $\theta$ ,  $U(f)$  and  $E$  are in order. Obviously, all  $\theta_n^{(k)}$  and  $U_n^{(k)}$  are non-negative. In the ideal case of exact (possibly partial) matching, all  $U_n^{(k)}(f)$  ( $k \geq 1$ ) are zeros because  $d_1^{(k)}(i)$  and  $D_1^{(k)}(f_i)$  are exactly the same and so are  $d_2^{(k)}(i, i')$  and  $D_2^{(k)}(f_i, f_{i'})$ . In the general case of inexact matching, the sum of the  $U_n^{(k)}$  should be as small as possible for the minimal solution. The following are some properties of  $E$ :

- Given  $f$ ,  $E(f \mid \theta)$  is linear in  $\theta$ . Given  $\theta$ , it is linear in  $U(f)$ .
- For  $\kappa > 0$ ,  $\theta$  and  $\kappa\theta$  are equivalent, as have been discussed.
- The values of  $\theta_n^{(0)}$  relative to those of  $\theta_n^{(k)}$  ( $k \geq 1$ ) affect the rate of NULL labels. The higher the penalties  $\theta_n^{(0)}$  are, the more sites in  $\mathcal{S}$  will be assigned non-NUL labels; and vice versa.

The first property enables us to use the results we established on linear classifiers for learning correct and optimal  $\theta$ . According to the second property, a larger  $\theta_n^{(k)}$  relative to the rest makes the constraint  $d_n^{(k)}$  and  $D_n^{(k)}$  play a more important role. Useless and mis-leading constraints  $d_n^{(k)}$  and  $D_n^{(k)}$  should be weighted by 0. Using the third property, one can decrease  $\theta_n^{(0)}$  values to increase the number of the NULL labels. This is because for the minimum energy matching, lower cost for NULL labels makes more sites be labeled NULL, which is equivalent to discarding more not-so-reliable non-NUL labels into the NULL bin.

### 7.3.3 How Minimal Configuration Changes

The following analysis examines how the minimum  $f^* = \arg \min_f E(f \mid \theta)$  changes as the observation changes from  $d0$  to  $d = d0 + \delta d$  where  $\delta d$  is a perturbation. In the beginning when  $\|\delta d\|$  is close to 0,  $f^*$  should remain as the minimum for a range of such small  $\delta d$ . This is simply because  $E(f \mid \theta)$  is continuous with respect to  $d$ . When the perturbation becomes larger and larger, the minimum has to give way to another configuration.

When should a change happens? To see the effect more clearly, assume the perturbation is in observation components relating to only a particular  $i$  so that the only changes are  $d0_1^{(k)}(i) \rightarrow d_1^{(k)}(i)$  and  $d0_2^{(k)}(i, i') \rightarrow d_2^{(k)}(i, i')$ ,  $\forall i' \in \mathcal{N}_i$ . First, assume that  $f_i^*$  is a non-NULL label ( $f_i^* \neq 0$ ) and consider such a perturbation  $\delta d$  that incurs a *larger* likelihood potential. Obviously, as the likelihood potential (conditioned on  $\{f_{i'}^* \neq 0 \mid i' \in \mathcal{N}_i\}$ )

$$\begin{aligned} V(d \mid f_i^*) &= \sum_{k=1}^{K_1} \theta_1^{(k)} (d_1^{(k)}(i) - D_1^{(k)}(f_i^*))^2 + \\ &\quad \sum_{i' \in \mathcal{S}, i' \neq i, f_{i'}^* \neq 0} \sum_{k=1}^{K_2} \theta_2^{(k)} (d_2^{(k)}(i, i') - D_2^{(k)}(f_i^*, f_{i'}^*))^2 \end{aligned} \quad (7.43)$$

increases, it will eventually become cheaper for  $f_i^* \neq 0$  to change to  $f_i = 0$ . More accurately, this should happen when

$$V(d \mid f_i^*) > \theta_1^{(0)} + \sum_{i' \in \mathcal{N}_i, f_{i'}^* \neq 0} \theta_2^{(0)} = \theta_1^{(0)} + N_2^i \theta_2^{(0)} \quad (7.44)$$

where

$$N_2^i = \#\{i' \in \mathcal{N}_i \mid f_{i'}^* \neq 0\} \quad (7.45)$$

is the number of non-NULL labeled sites in  $\mathcal{N}_i$  under  $f^*$ .

Next, assume  $f_i^* = 0$  and consider such a perturbation that incurs a *smaller* likelihood potential. The perturbation has to be such a coincidence that  $d$  and  $D$  become more resemblant to each other. As the conditional likelihood potential

$$\begin{aligned} V(d \mid f_i) &= \sum_{k=1}^{K_1} \theta_1^{(k)} (d_1^{(k)}(i) - D_1^{(k)}(f_i))^2 + \\ &\quad \sum_{i' \in \mathcal{S}, i' \neq i, f_{i'}^* \neq 0} \sum_{k=1}^{K_2} \theta_2^{(k)} (d_2^{(k)}(i, i') - D_2^{(k)}(f_i, f_{i'}^*))^2 \end{aligned} \quad (7.46)$$

decreases, it will eventually become cheaper for  $f_i^* = 0$  to change to one of the non-NULL labels,  $f_i \neq 0$ . More accurately, this should happen when

$$V(d \mid f_i) < \theta_1^{(0)} + \sum_{i' \in \mathcal{N}_i, f_{i'}^* \neq 0} \theta_2^{(0)} = \theta_1^{(0)} + N_2^i \theta_2^{(0)} \quad (7.47)$$

The above analysis shows how the minimal configuration  $f^*$  adjusts as  $d$  changes when  $\theta$  is fixed. On the other hand, the  $f^*$  can be maintained unchanged by adjusting  $\theta$ ; this means a different encoding of constraints into  $E$ .

### 7.3.4 Parametric Estimation Under Gaussian Noise

Assuming the functional form of the noise distribution is known, then we can take the advantage of (partial) parametric modeling for the estimation. When the noise is additive white Gaussian with unknown  $[\sigma_n^{(k)}]^2$ , the estimate can be obtained in closed-form. The closed-form estimation is performed in two steps: (1) to estimate the noise variances  $[\bar{\sigma}_n^{(k)}]^2$  ( $k \geq 1$ ); then to compute the weights  $\bar{\theta}_n^{(k)}$  ( $k \geq 1$ ) using the relationship in Eq.(7.36); and (2) to compute allowable  $\bar{\theta}_n^{(0)}$ , relative to  $\bar{\theta}_n^{(k)}$  ( $k \geq 1$ ), to satisfy the correctness in (7.7). Optimization as (7.16) derived using a non-parametric principle may not be applicable in this case.

Given  $d$ ,  $D$  and  $f$ , the Gaussian noise variances can be estimated by maximizing the joint likelihood function  $p(d | \bar{f})$  (ML estimation). The ML estimates are simply

$$[\bar{\sigma}_1^{(k)}]^2 = \frac{1}{N'_1} \sum_{i \in \mathcal{S}, \bar{f}_i \neq 0} [d_1^{(k)}(i) - D_1^{(k)}(\bar{f}_i)]^2 \quad (7.48)$$

and

$$[\bar{\sigma}_2^{(k)}]^2 = \frac{1}{N'_2} \sum_{i \in \mathcal{S}, \bar{f}_i \neq 0} \sum_{i' \in \mathcal{S}, i' \neq i, \bar{f}_{i'} \neq 0} [d_2^{(k)}(i, i') - D_2^{(k)}(\bar{f}_i, \bar{f}_{i'})]^2 \quad (7.49)$$

where

$$N'_1 = \#\{i \in \mathcal{S} \mid f_i \neq 0\} \quad (7.50)$$

is the number of non-NUL labels in  $f$  and

$$N'_2 = \#\{(i, i') \in \mathcal{S}^2 \mid i' \neq i, f_i \neq 0, f_{i'} \neq 0\} \quad (7.51)$$

is the number of label pairs neither of which is the NUL. The optimal weights for  $k \geq 1$  can be obtained immediately by

$$\bar{\theta}_n^{(k)} = 1/2[\bar{\sigma}_n^{(k)}]^2 \quad (7.52)$$

So far, only the exemplary configurations  $\bar{f}$ , not others, are used in computing the  $\bar{\theta}_n^{(k)}$ .

Now the remaining problem is to determine  $\bar{\theta}_n^{(0)}$  to meet the correctness (7.7). Because of  $\theta_n^{(0)} = v_{n0}$ , this is to estimate the MRF parameters  $v_{n0}$  in the prior distributions implied in the given exemplar. There may be a *range* of  $\bar{\theta}_n^{(0)}$  under which each  $\bar{f}$  is correctly encoded. The range is determined by the lower and upper bounds.

In doing so, only those configurations in  $\mathcal{N}_f$ , which reflect transitions from a non-NUL to the NUL label and the other way around, are needed; the other configurations, which reflect transitions from one non-NUL to another non-NUL label, are not. This subset is obtained by changing each of the

non-NUL labels in  $\bar{f}$  to the NUL label or changing each of the NUL labels to a non-NUL label.

First, consider label changes from a non-NUL label to the NUL label. Assume a configuration change from  $\bar{f}$  to  $f$  is due to the change from  $\bar{f}_i \neq 0$  to  $f_i = 0$  for just one  $i \in \mathcal{S}$ . The corresponding energy change is given by

$$\begin{aligned} \frac{1}{2} \Delta E(f \mid \theta) = & \theta_1^{(0)} - \sum_{k=1}^{K_1} \bar{\theta}_1^{(k)} [d_1^{(k)}(i) - D_1^{(k)}(\bar{f}_i)]^2 + \\ & \sum_{i' \in \mathcal{N}_i, \bar{f}_{i'} \neq 0} \theta_2^{(0)} - \\ & \sum_{i' \in \mathcal{S}, i' \neq i, \bar{f}_{i'} \neq 0} \sum_{k=1}^{K_2} \bar{\theta}_2^{(k)} [d_2^{(k)}(i, i') - D_2^{(k)}(\bar{f}_i, \bar{f}_{i'})]^2 \end{aligned} \quad (7.53)$$

The above change must be positive,  $\Delta E(f \mid \theta) > 0$ . Suppose there are  $N$  non-NUL labeled sites under  $\bar{f}$  and therefore  $\bar{f}$  has  $N$  such neighboring configurations. Then  $N$  such inequalities of  $\Delta E(f \mid \theta) > 0$  can be obtained. The two unknowns,  $\theta_1^{(0)}$  and  $\theta_2^{(0)}$ , can be solved for and used as the lower bounds  $(\theta_1^{(0)})_{min}$  and  $(\theta_2^{(0)})_{min}$ .

Similarly, the upper bounds can be computed by considering label changes from the NUL label to a non-NUL label. The corresponding energy change due to a change from  $\bar{f}_i = 0$  to  $f_i \neq 0$  is given by

$$\begin{aligned} \frac{1}{2} \Delta E(f \mid \theta) = & \sum_{k=1}^{K_1} \bar{\theta}_1^{(k)} [d_1^{(k)}(i) - D_1^{(k)}(f_i)]^2 - \theta_1^{(0)} \\ & \sum_{i' \in \mathcal{S}, \bar{f}_{i'} \neq 0} \sum_{k=1}^{K_2} \bar{\theta}_2^{(k)} [d_2^{(k)}(i, i') - D_2^{(k)}(f_i, f_{i'})]^2 - \\ & \sum_{i' \in \mathcal{N}_i, \bar{f}_{i'} \neq 0} \theta_2^{(0)} \end{aligned} \quad (7.54)$$

The above change must also be positive,  $\Delta E(f \mid \theta) > 0$ . Suppose there are  $N$  NUL labeled sites under  $\bar{f}$  and recall that there are  $M$  possible non-NUL labels in  $\mathcal{L}$ . Then  $N \times M$  inequalities can be obtained. The two unknowns,  $\theta_1^{(0)}$  and  $\theta_2^{(0)}$ , can be solved and used as the upper bounds  $(\theta_1^{(0)})_{max}$  and  $(\theta_2^{(0)})_{max}$ . If the exemplary configurations  $\bar{f}$  are minimal for the corresponding  $\mathcal{G}$  and  $\mathcal{G}'$ , then the solution must be consistent, that is  $(\theta_n^{(0)})_{min} < (\theta_n^{(0)})_{max}$ , for each of the instances.

Now, the space of all correct parameters is given by

$$\Theta_{correct} = \{\theta_n^{(k)} \mid \theta_n^{(0)} \in [(\theta_n^{(0)})_{min}, (\theta_n^{(0)})_{max}]; \theta_n^{(k)} = \bar{\theta}_n^{(k)}, k \geq 1\} \quad (7.55)$$

A correct  $\theta$  makes  $\theta^T x > 0$  for all  $x \in \mathcal{X}$ . The hyperplane  $\theta^T x = 0$  partitions  $\mathcal{X}$  into two parts, with all  $x \in \mathcal{X}$  in the positive side of it. The value for  $\bar{\theta}_n^{(0)}$  may be simply set to the average  $[(\theta_n^{(0)})_{min} + (\theta_n^{(0)})_{max}]/2$ .

When there are  $L > 1$  instances,  $\theta_n^{(k)}$  ( $k \geq 1$ ) are obtained from the data set computed from all the instances. Given the common  $\theta_n^{(k)}$ ,  $L$  correct ranges can be computed. The correct range for the  $L$  instances as a whole is the intersection of the  $L$  ranges. As the result, the overall  $(\theta_n^{(0)})_{min}$  is the maximum of all the lower bounds and  $(\theta_n^{(0)})_{max}$  the minimum of all the upper bounds. Although each range can often be consistent, *i.e.*  $(\theta_n^{(0)})_{min} <$

$(\theta_n^{(0)})_{max}$  for each  $n$ , there is less chance to guarantee that they as a whole are consistent for all  $\ell = 1, \dots, L$ : The intersection may be empty when  $L > 1$ . This inconsistency means a correct estimate does not exist for all the instances as a whole. There are several reasons for this. First of all, the model assumptions such as Gaussian is not verified by the data set, especially when the data set is small. In this case, the noise in different instances has different variances; when the ranges are computed under the assumption that the ML estimate is common to all the instances, they may not be consistent to each other. This is the most direct reason for the inconsistency. Second,  $\bar{f}$  in some exemplary instances can not be embedded as the minimal energy configuration to satisfy the given constraints. Such instances are mis-leading and also causes the inconsistency.

## 7.4 Experiments

The following experiments demonstrate: (i) the computation (learning) of the optimal parameter  $\bar{\theta}$  from the exemplary instance given in the form of a triplet  $(\bar{f}, \mathcal{G}, \mathcal{G}')$ , and (ii) the use of the learned estimate  $\bar{\theta}$  to recognize other scenes and models. The non-parametric learning algorithm is used because the data size is too small to assume a significant distribution. The convergence of the learning algorithm is demonstrated.

### 7.4.1 Recognition of Line Patterns

This experiment performs the recognition of simulated objects of line patterns under 2D rotation and translation. There are six possible model objects shown in Fig.7.4. Fig.7.5 gives the exemplary instance used for parameter estimation. The scene is given in the dotted and dashed lines which are generated as follows: (1) Take a subset of lines from each of the three objects in Fig.7.4(a)-(c); (2) rotate and translate each of the subsets, respectively; (3) mix the transformed subsets; (4) randomly deviate the position of the endpoints of the lines, which results in the dotted lines; and (5) add spurious lines, shown in the dashed lines. The generated scene consists of several subsets of model patterns plus spurious lines as shown in Fig.7.5(b). The exemplary configuration  $\bar{f}$  is shown in Fig.7.5(a). It maps the scene to one of the models given in Fig.7.4. The alignment between the dotted lines of the scene and the solid lines of the model gives the non-NUL labels of  $\bar{f}$  whereas the un-aligned lines of the scene are labeled as NUL.

The following four types of bilateral relations are used ( $n = 2; K_2 = 4$ ):

- (1)  $d_2^{(1)}(i, i')$ : the angle between lines  $i$  and  $i'$ ,
- (2)  $d_2^{(2)}(i, i')$ : the distance between the mid-points of the lines,
- (3)  $d_2^{(3)}(i, i')$ : the minimum distance between the end-points of the lines, and

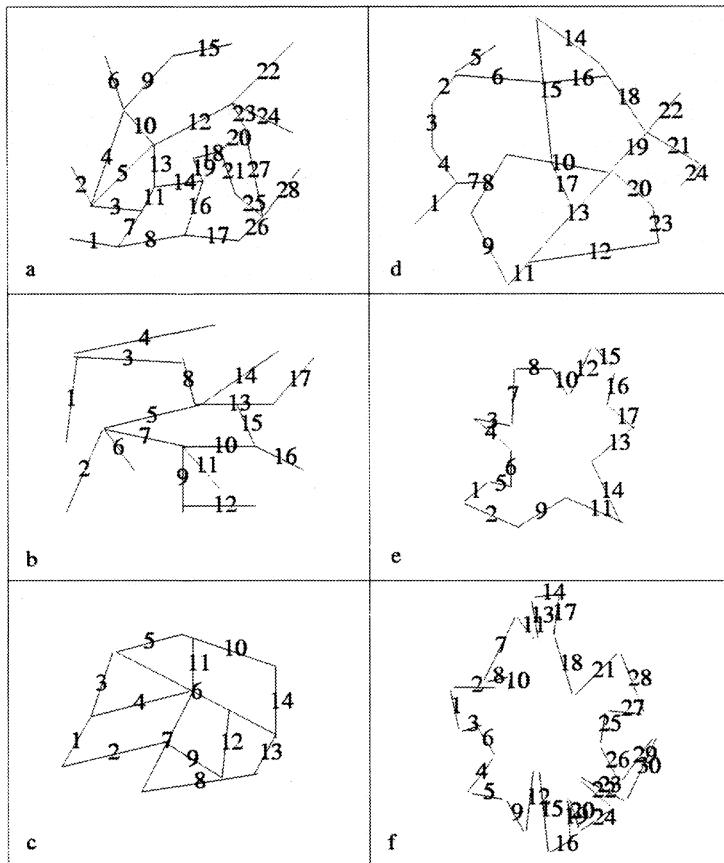


Figure 7.4: The six objects of line patterns in the model base. From (Li 1997) with permission; ©1997 Kluwer.

(4)  $d_2^{(4)}(i, i')$ : the maximum distance between the end-points of the lines.

Similarly, there are four model relations  $D_2^{(k)}(I, I')$  ( $k = 1, \dots, 4$ ) of the same types. No unary properties are used ( $K_1 = 0$ ). The  $\mathcal{G}$  and  $\mathcal{G}'$  are composed of these 4 relational measurements. Therefore, there are five components ( $k = 0, 1, \dots, 4$ ) in  $x$  and  $\theta$ .

The  $C_2$ -optimal parameters are computed as  $\bar{\theta} = \{\bar{\theta}_2^{(0)}, \bar{\theta}_2^{(1)}, \dots, \bar{\theta}_2^{(4)}\} = \{0.58692, 0.30538, 0.17532, 0.37189, 0.62708\}$  which satisfies  $\|\theta\| = 1$ . The computation takes a few seconds on an HP series 9000/755 workstation. To be used for recognition,  $\bar{\theta}$  is multiplied by a factor of  $0.7/\bar{\theta}_2^{(0)}$ , yielding the final

weights  $\theta^* = \{0.70000, 0.36422, 0.20910, 0.44354, 0.74789\}$  (our recognition system requires  $(\theta_2^{(0)})^* = 0.7$ ).

The  $\theta^*$  is used to define the energy for recognizing other objects and scenes. The recognition results are shown in Fig.7.6. There are two scenes, one in the upper row and the other in the lower row, composed the dotted and dashed lines. The one in the upper was the one used in the exemplar whereas the scene in the lower contains sub-parts of the three model objects in Fig.7.4(d)-(f). Each scene are matched against the six model objects. The optimally matched object lines are shown in solid lines aligned with the scenes. The objects in the scenes are correctly matched to the model objects.

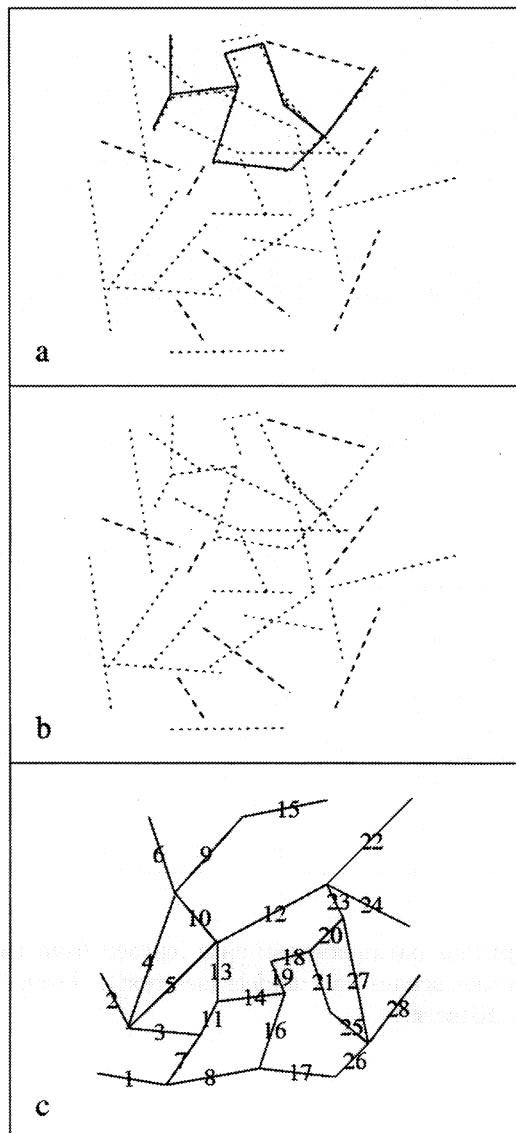


Figure 7.5: An exemplary instance consisting of (a) exemplary configuration  $\tilde{f}$ , (b) scene  $\mathcal{G}$  and (c) model  $\mathcal{G}'$ . From (Li 1997) with permission; ©1997 Kluwer.

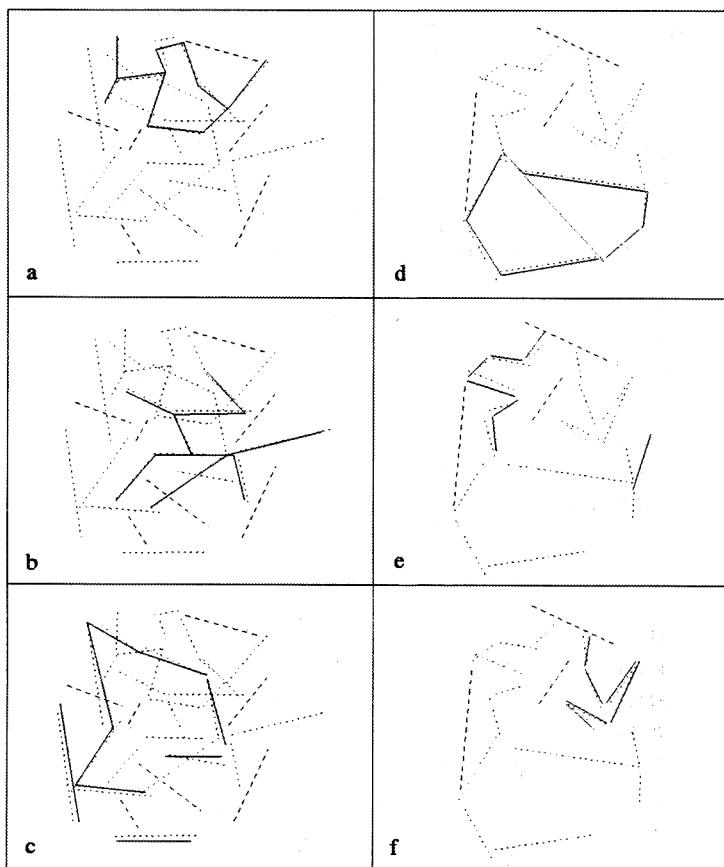


Figure 7.6: The optimal parameter estimate learned from the exemplar is used to recognize other scenes and models (see text). From (Li 1997) with permission; ©1997 Kluwer.

### 7.4.2 Recognition of Curved Objects

This experiment deals with jigsaw objects under 2D rotation, translation and uniform scaling. There are eight model jigsaw objects shown in Fig.3.10. In this case of curved objects, the features of an object correspond to the corner points of its boundary. For both the scene and the models, the boundaries are extracted from the images using the canny detector followed by hysteresis and edge linking. Corners are detected after that. No unary relations are used ( $K_1 = 0$ ). Denoting the corners by  $p_1, \dots, p_m$ , the following five types of bilateral relations are used ( $n = 2$ ;  $K_2 = 5$ ) based on a similarity-invariant curve representation of curves (Li 1993):

- (1)  $d_2^{(1)}(i, i')$ : ratio of curve arc-length  $\widehat{p_i p_{i'}}$  and chord-length  $\overline{p_i p_{i'}}$ ,
- (2)  $d_2^{(2)}(i, i')$ : ratio of curvature at  $p_i$  and  $p_{i'}$ ,
- (3)  $d_2^{(3)}(i, i')$ : invariant coordinates vector,
- (4)  $d_2^{(4)}(i, i')$ : invariant radius vector, and
- (5)  $d_2^{(5)}(i, i')$ : invariant angle vector

They are computed using information about both the boundaries and the corners. Similarly, there are five model relations  $D_2^{(k)}(I, I')$  ( $k = 1, \dots, 5$ ) of the same types. Therefore, there are six components ( $k = 1, \dots, 5$ ) in each  $\boldsymbol{x}$  and  $\boldsymbol{\theta}$ , one for the NULL and five for the above relational quantities.

Fig.7.7 gives the exemplary instance used for parameter estimation. The scene in Fig.7.7(b) contains rotated, translated and scaled parts of one of the model jigsaws. Some objects in the scene are considerably occluded. The alignment between the model jigsaw (the highlighted curve in (a)) and the scene gives non-NUL labels of  $\bar{f}$  whereas the un-aligned boundary corners of the scene are labeled as NULL.

The  $C_2$ -optimal parameters are computed as  $\bar{\boldsymbol{\theta}} = \{\bar{\theta}_2^{(0)}, \bar{\theta}_2^{(1)}, \dots, \bar{\theta}_2^{(6)}\} = \{0.95540, 0.00034, 0.00000, 0.06045, 0.03057, 0.28743\}$  which satisfies  $\|\boldsymbol{\theta}\| = 1$ . It takes a few seconds on the HP workstation. Note that the weight  $\bar{\theta}_2^{(2)}$  for  $d_2^{(2)}(i, i')$  and  $D_2^{(2)}(I, I')$  (ratio of curvature) is zero. This means that this type of feature is not reliable enough to be used. Because our recognition system has a fixed value of  $\theta_2^{(0)} = 0.7$ ,  $\bar{\boldsymbol{\theta}}$  is multiplied by a factor of  $0.7/\theta_2^{(0)}$ , yielding the final weights  $\boldsymbol{\theta}^* = \{0.70000, 0.00025, 0.00000, 0.04429, 0.02240, 0.21060\}$ .

The  $\boldsymbol{\theta}^*$  is used to define the energy function for recognizing other objects and scenes. The recognition results are shown in Fig.7.8. There are two scenes, one in the left column and the other in right column. The scene on the left was the one used in the exemplar and on the right is a new scene. The optimally matched model objects are shown in the highlighted curves aligned with the scenes. The same results can also be obtained using the  $C_\infty$ -optimal estimate which is  $\{0.7, 0.00366, 0.00000, 0.09466, 0.00251\}$ .

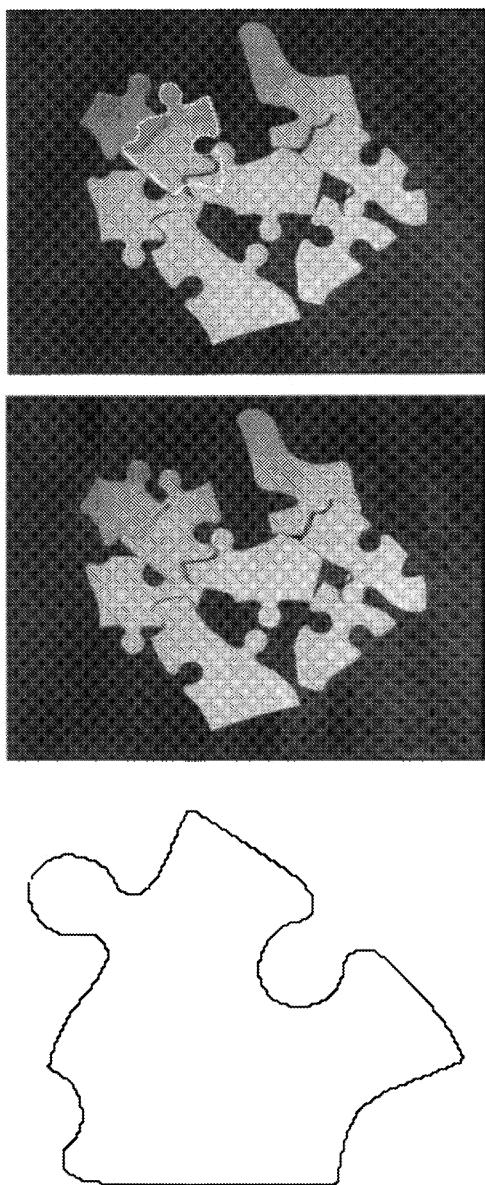


Figure 7.7: An exemplary instance consisting of (a) exemplary configuration  $\bar{f}$ , (b) scene  $\mathcal{G}$  and (c) model  $\mathcal{G}'$ . From (Li 1997) with permission; ©1997 Kluwer.

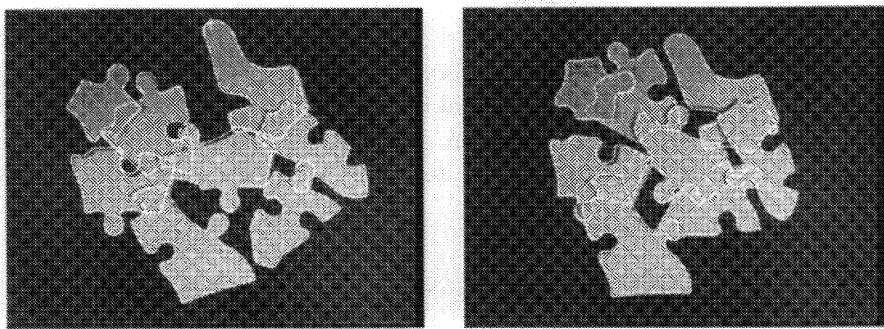


Figure 7.8: The learned estimate is used to recognize other scenes and models. The matched model jigsaw objects are aligned with the scene. From (Li 1997) with permission; ©1997 Kluwer.

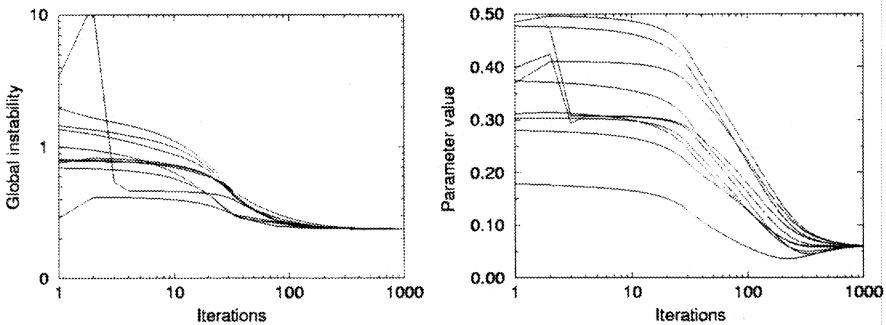


Figure 7.9: Convergence of the algorithm with different starting points. Left: trajectories of  $C_2$ . Right: trajectories of  $\bar{\theta}_2^{(3)}$ . From (Li 1997) with permission; ©1997 Kluwer.

### 7.4.3 Convergence

The parameter estimation algorithm is very stable and has a nice convergence property. Fig.7.9 shows how the global instability measure  $C_2$  and one of the learned parameters  $\hat{\theta}_2^{(3)}$  evolve given different starting points. The values become stabilized after hundreds of iterations and different starting points converge to the same point.

## 7.5 Conclusion

While manual selection is a common practice in object recognition systems, this paper has presented a novel theory for automated, optimal parameter estimation in optimization-based object recognition. The theory is based on learning from examples. Mathematical principles of correctness and instability are established and defined for the evaluation of parameter estimates. A learning algorithm is presented for computing the optimal, *i.e.* minimal-instability, estimate. An application to the MRF-based recognition is given. Experiments conducted show very promising results. Optimal estimates automatically learned from examples can well be generalized for recognizing other scenes and objects.

The exemplary instances are given to reflect the designer's judgment of desirable solutions. However, a recognizer with a given functional form can not be trained by arbitrary exemplar. The exemplar should be selected properly to reflect the correct semantics, in other words, they should be consistent with the constraints with which the functional form is derived. Assuming the form of the objective function is right and the exemplar contains useful information, then the more exemplary instances are used to train, the more generalizable is the learned parameter estimate.

The learning procedure also provides a means of checking the validity of the energy function derived from mathematical models. An improper mathematical model leads to an improper functional form. If no correct parameter estimates can be learned, it is a diagnostic symptom that the assumptions used in the model are not suitable for modeling the reality of the scene. The procedure also provides useful information for feature selection. Components of the optimal parameter estimate will be zero or near zero for instable features.

# Chapter 8

# Minimization – Local Methods

After the energy function  $E(f)$ , including both the functional form and the involved parameters, is given and thus the optimal solution  $f^* = \min_f E(f)$  is entirely defined, the remaining problem is to find the solution. It is most desirable to express the solution in closed-form but generally, this is very difficult in vision problems due to the complexity caused by interactions between labels. Therefore, optimal solutions are usually computed by using some iterative search techniques. This chapter describes techniques for finding local minima and discusses related issues.

## 8.1 Problem Categorization

A minimization problem can be categorized as continuous or combinatorial according to whether the label set  $\mathcal{L}$  is continuous or discrete; it can be further categorized as constrained or unconstrained according to whether  $f$  is constrained within a subspace of the entire search space. A vision problem may also be modeled as a combination of several minimization processes; for example, continuous restoration with line process is a combination of continuous and combinatorial minimization processes (*cf.* Chapters 2 and 4). As far as computational algorithms are concerned, a view of local vs. global methods is important when there are multiple local minima in the solution space.

Because classical unconstrained, continuous minimization methods are most mature, it is sometimes desirable to convert a combinatorial and constrained problems into forms suitable for such classical methods. A combinatorial problem may be converted into a continuous one, for example, using the notion of relaxation labeling (Rosenfeld et al. 1976) or mean field approximation (Peterson and Soderberg 1989). A constrained minimization may be

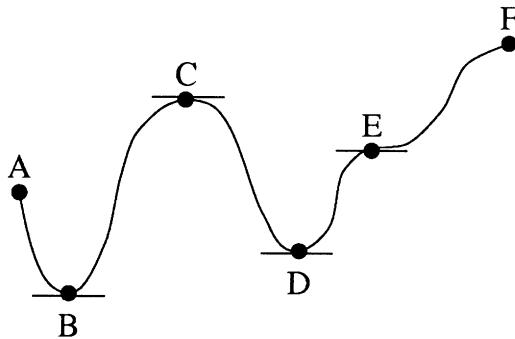


Figure 8.1: Local and global extrema.

converted into an unconstrained one, for example, using classical methods such as the penalty method or the Lagrange multiplier method (Fletcher 1987). This not only is for mathematical convenience but also brings about possibilities for analog implementations.

Fig.8.1 illustrates extremal points of a continuous function in  $\mathbb{R}$ .  $A$ ,  $C$  and  $F$  are local maxima but only  $F$  is the global maximum. Points  $B$  and  $D$  are local minima of which  $B$  is the global one.  $B$ ,  $C$ ,  $D$  and  $E$  are stationary points where the gradient vanishes but  $E$  is an inflection point (or saddle point in higher dimensional cases) rather than an extremum. A gradient-descent search algorithm will stop at  $B$  or  $D$ . Theoretically, it can also stop at a saddle point or even a local maximum because the gradient there is exactly zero. But such chance is small in numerical computation because of the quantization, and the algorithm is likely to pass by these points.

The localness and globalness of a minimum  $f$  are said with respect to a neighborhood system  $\mathcal{N} = \{\mathcal{N}(f) \mid f \in \mathbb{F}\}$  where  $\mathcal{N}(f)$  is the set of neighboring configurations of  $f$  in the solution space. For  $\mathbb{F} = \mathbb{R}^S$ , the neighborhood of  $f$  can be suitably defined as

$$\mathcal{N}_\epsilon(f) = \{x \mid x \in \mathbb{F}, 0 < \|x - f\| \leq \epsilon\} \quad (8.1)$$

where  $\|\cdot\|$  is a norm and  $\epsilon > 0$  is a real number. For discrete problem where  $\mathcal{L}_d$  is a discrete label set and  $\mathbb{F}_d = \mathcal{L}_d^S$ , we may define the following “ $k$ -neighborhood” of  $f \in \mathbb{F}_d$  as

$$\mathcal{N}^k(f^*) = \{x \in \mathbb{F}_d \mid x \text{ differs from } f^* \text{ by at most } k \text{ labels}\} \quad (8.2)$$

For example, assuming that  $\mathcal{L} = \{a, b, c\}$  is the set of allowable label for every  $f_i$  ( $i = 1, \dots, 5$ ), then  $\{a, b, a, a, a\}$  is a 1-neighbor of  $\{a, a, a, a, a\}$  and  $\{b, a, a, a, c\}$  is a 2-neighbor. Note that  $\{a, b, a, a, a\}$  is also a 2-neighbor of

$\{a, a, a, a, a\}$  (due to the phrase “at most” in the definition) but not vice versa.

A point  $f^*$  is a *local minimum* of  $E$  with respect to  $\mathcal{N}$  if

$$E(f^*) \leq E(f) \quad \forall f \in \mathcal{N}(f^*) \quad (8.3)$$

A local minimum is also a *global minimum* if the neighborhood is defined to include all the other configurations,

$$\mathcal{N}(f^*) = \{f \mid \forall f \in \mathbb{F}, f \neq f^*\} = \mathbb{F} - \{f^*\} \quad (8.4)$$

When  $E(f^*)$  is strictly the lowest,  $E(f^*) < E(f)$  for all  $f \neq f^*$ ,  $f^*$  is the unique global minimum. Multiple global minima occur if there exist more than one point  $\xi \in \mathbb{F}$  for which  $E(\xi)$  take the globally minimum value  $E(f^*)$ .

*Local search* is the basis for many minimization algorithms. Fig.8.2 describes the idea of deterministic local search. The idea is very simple and natural: At a point  $f \in \mathbb{F}$ , we look for where to go within the neighborhood  $\mathcal{N}(f)$ ; if  $f'$  in the neighborhood leads to an improvement,  $E(f') < E(f)$ , we replace  $f$  by  $f'$ ; this process continues until no further improvement can be made. When  $E(f)$  is lower bounded, the algorithm converges to a local minimum.

---

```

Set  $f = f(0)$ ;
Repeat
  Generate  $f'$  from  $\mathcal{N}(f)$  such that  $E(f') < E(f)$ ;
   $f \leftarrow f'$ ;
Until  $E(f') \geq E(f)$  for all  $f' \in \mathcal{N}(f)$ ;

```

---

Figure 8.2: Deterministic local search algorithm.

There are different ways of generating (or searching for) a candidate  $f'$ . In the continuous case,  $f'$  may be chosen based on the gradient:  $f' = f - \mu \nabla E(f)$ , where  $\mu > 0$  is a step size. This leads to a zero gradient point, which is necessary for the minimization. In the discrete case, some enumerative method may be used to generate  $f' \in \mathcal{N}(f)$  for which  $E(f') < E(f)$ .

The local search discussed so far is deterministic. In stochastic local search,  $f'$  is generated at random. It is not necessary that  $E(f')$  be lower than  $E(f)$ . Whether to accept  $f'$  for which  $E(f')$  is higher than  $E(f)$  is decided according to some probabilistic rules.

## 8.2 Classical Minimization with Continuous Labels

Let  $\mathcal{L} = [X_l, X_h]$  be a continuous label set. Consider the prototypical problem of (piecewise) continuous restoration in which the energy takes the following form

$$E(f) = \sum_{i \in \mathcal{S}} \chi_i (f_i - d_i)^2 + \lambda \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} g(f_i - f_{i'}) \quad (8.5)$$

where  $\chi_i \in \{1, 0\}$  indicates the presence or absence of the data  $d_i$  and  $g(\eta)$  is the potential function. Because a minimum is necessarily a stationary point, the gradient  $\nabla E(f)$  must be a zero vector at the point. Therefore, the following system of equations must be satisfied by  $f^*$ :

$$\frac{1}{2} \frac{\partial E}{\partial f_i} = \chi_i (f_i - d_i) + \lambda \sum_{i' \in \mathcal{N}_i} (f_i - f_{i'}) h(f_i - f_{i'}) = 0, \quad \forall i \in \mathcal{S} \quad (8.6)$$

where  $g'(\eta) = 2\eta h(\eta)$  is assumed (cf. (4.24)). The system is nonlinear if  $h(\eta)$  is a nonlinear function. Higher order derivatives have to be examined in order to determine whether such a stationary point is a local minimum, maximum or a saddle point. We assume that it is a local minimum.

Solving the system of equations (8.6) is a classical problem (Dahlquist and Bjorck 1974). When  $h$  is a linear function, the system is linear and  $f$  can be solved for by using matrix operations. For a nonlinear system, a numerical iterative method is generally utilized. In a simple gradient based algorithm, for example, a stationary point satisfying (8.6) is computed by iteratively updating  $f$  using the following rule

$$\begin{aligned} f_i^{(t+1)} \leftarrow f_i^{(t)} & - 2\mu \{ \chi_i [f_i^{(t)} - d_i] \\ & + \lambda \sum_{i' \in \mathcal{N}_i} (f_i^{(t)} - f_{i'}^{(t)}) h(f_i^{(t)} - f_{i'}^{(t)}) \} \end{aligned} \quad (8.7)$$

where  $\mu$  is a small constant.

The fixed-point iteration method provides another method for solving the problem. Rearrange (8.6) into the following form

$$f = \varphi(f) \quad (8.8)$$

which is composed of  $m$  fixed-point equations  $f_i = \varphi_i(f)$  ( $i = 1, \dots, m$ ). The following is called fixed-point iteration

$$f^{(t+1)} \leftarrow \varphi(f^{(t)}) \quad (8.9)$$

Let  $J_\varphi(f)$  be the Jacobian matrix of  $\varphi$  and  $\mathbb{D}$  be a region in  $\mathbb{R}^S$ . Suppose (1)  $\varphi$  is a continuous mapping,  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ , and (2)  $\varphi$  is a contraction mapping, that is,  $\|J_\varphi(f)\| \leq L < 1$  for all  $f \in \mathbb{D}$  where  $\|\cdot\|$  is a norm and

$L$  is a constant. Then the fixed-point iteration (8.8) converges to a unique fixed-point  $f^* = \lim_{t \rightarrow \infty} f^{(t)}$  for every  $f^{(0)} \in \mathbb{D}$ . The way to construct  $\varphi$  is not unique; some lead to fast convergence and some diverge. The rate of convergence depends linearly on  $L$ .

To solve (8.6) using fixed-point iteration, we may construct a mapping as the following

$$\begin{aligned}\varphi_i(f) = & \frac{1}{2\lambda C + \chi_i} \{ \chi_i d_i + \lambda C(f_{i-1} + f_{i+1}) + \\ & \lambda [g'(f_{i-1} - f_i) + g'(f_{i+1} - f_i) - C(f_{i-1} - 2f_i + f_{i+1})] \}\end{aligned}\quad (8.10)$$

where  $C$  is a constant. It is not difficult to verify that  $f_i = \varphi_i(f)$  is equivalent to (8.6). It can be shown that (8.10) is a contraction mapping when  $C = \max_\eta g''(\eta)$  and  $\chi_i = 1$  ( $\forall i$ ). In this case, the fixed-point iteration (8.9) converges to a unique fixed-point for which  $f^* = \varphi(f^*)$  where  $\varphi = (\varphi_1, \dots, \varphi_m)$ .

## 8.3 Minimization with Discrete Labels

When the label set is discrete, the minimization of the posterior energy is combinatorial. In this section, we consider methods for minimizing  $E(f)$  when each  $f_i$  takes a value in  $\mathcal{L} = \{1, \dots, M\}$ .

### 8.3.1 Iterated Conditional Modes

Since it is difficult to maximize the joint probability of an MRF, Besag (1986) proposed a deterministic algorithm called iterated conditional modes (ICM) which maximizes local conditional probabilities sequentially. The ICM algorithm uses the “greedy” strategy in the iterative local maximization. Given the data  $d$  and the other labels  $f_{S-\{i\}}^{(k)}$ , the algorithm sequentially updates each  $f_i^{(k)}$  into  $f_i^{(k+1)}$  by maximizing  $P(f_i | d, f_{S-\{i\}})$ , the conditional (posterior) probability, with respect to  $f_i$ .

Two assumptions are made in calculating  $P(f_i | d, f_{S-\{i\}})$ : First, the observation components  $d_1, \dots, d_m$  are conditionally independent given  $f$  and each  $d_i$  has the same known conditional density function  $p(d_i | f_i)$  dependent only on  $f_i$ . Thus

$$p(d | f) = \prod_i p(d_i | f_i) \quad (8.11)$$

The second assumption is that  $f$  depends on the labels in the local neighborhood, which is the Markovianity. From the two assumptions and the Bayes theorem, it follows that

$$P(f_i | d, f_{S-\{i\}}) \propto p(d_i | f_i) P(f_i | f_{N_i}) \quad (8.12)$$

Obviously,  $P(f_i | d_i, f_{N_i}^{(k)})$  is much easier to maximize than  $P(f_i | d)$ , which is the point of ICM.

Maximizing (8.12) is equivalent to minimizing the corresponding posterior potential using the following rule

$$f_i^{(k+1)} \leftarrow \arg \min_{f_i} V(f_i \mid d_i, f_{\mathcal{N}_i}^{(k)}) \quad (8.13)$$

where

$$V(f_i \mid d_i, f_{\mathcal{N}_i}^{(k)}) = \sum_{i' \in \mathcal{N}_i} V(f_i \mid f_{i'}^{(k)}) + V(d_i \mid f_i) \quad (8.14)$$

For example, for the discrete restoration formulated in Section 2.2.2, the posterior potential for (2.19) is

$$V(f_i \mid d_i, f_{\mathcal{N}_i}) = (f_i - d_i)^2 / \sigma + v_{20} \sum_{i' \in \mathcal{N}_i} [1 - \delta(f_i - f_{i'})] \quad (8.15)$$

where  $\sum_{i' \in \mathcal{N}_i} [1 - \delta(f_i - f_{i'})] = \#\{f_{i'} \neq f_i \mid i' \in \mathcal{N}_i\}$  is the number of neighboring sites whose labels  $f_{i'}$  differ from  $f_i$ .

For discrete  $\mathcal{L}$ ,  $V(f_i \mid d_i, f_{\mathcal{N}_i})$  is evaluated with each  $f_i \in \mathcal{L}$  and the label causing the lowest  $V(f_i \mid d_i, f_{\mathcal{N}_i})$  value is chosen as the value for  $f_i^{(k+1)}$ . When applied to each  $i$  in turn, the above defines an updating cycle of ICM. The iteration continues until convergence. The convergence is guaranteed for the serial updating and is rapid (Besag 1986).

The result obtained by ICM depends very much on the initial estimator  $f^{(0)}$ , as widely reported. Currently, it is not known how to set the initialization properly to obtain a good solution. A natural choice for  $f^{(0)}$  is the maximum likelihood estimate

$$f^{(0)} = \arg \max_f p(d \mid f) \quad (8.16)$$

when the noise is identically, independently distributed Gaussian.

ICM can also be applied to problems where  $f_i$  takes a continuous value. In minimizing (2.22) for continuous restoration, for example, one needs to maximize

$$V(f_i \mid d_i, f_{\mathcal{N}_i}) = (f_i - d_i)^2 + \lambda \sum_{i' \in \mathcal{N}_i} g(f_i - f_{i'}) \quad (8.17)$$

for each  $i \in \mathcal{S}$ . To do this, it is necessary to solve  $\frac{dV(f_i \mid d_i, f_{\mathcal{N}_i})}{df_i} = 0$ .

In a genuine MRF algorithm, no two neighboring sites should be updated simultaneously. The “coding method” (Besag 1974) may be incorporated into ICM to parallelize the iteration. Using codings,  $\mathcal{S}$  are partitioned into several sets such that no two sites in one set are neighbors (*cf.* Section 6.1.3). Therefore, all  $f_i$  on a single coding can be updated in parallel.

### 8.3.2 Relaxation Labeling

In this section, the *relaxation labeling* (RL) method is described as an approach for minimizing an energy  $E(f)$  over a discrete space, where  $E(f)$  may

be, for example, the posterior energy  $U(f | d)$  for the MAP-MRF restoration of multi-level image. RL is originally proposed as a class of parallel iterative numerical procedures which uses contextual constraints to reduce ambiguities in image analysis (Rosenfeld et al. 1976). It has been widely used in image and vision community. It is known that RL has a polynomial complexity yet empirically produces good results.

Formally, Faugeras and Berthod (1981) define a class of global criteria in terms of transition probabilities. Peleg (1980) interprets RL in terms of Bayes analysis. Haralick (1983) illustrates RL as minimizing the expected loss from a Bayes viewpoint. Hummel and Zucker (1983) formulate the probabilistic relaxation approach as a global optimization problem in which a so called average local consistency is defined and maximized. They show that, computationally, finding consistent labeling is equivalent to solving a variational inequality. Using the notion of continuous RL, the combinatorial minimization is converted into a real minimization subject to linear constraints, and the combinatorial minimization is performed by constrained real minimization.

A development in relaxation labeling is the use of contextual constraints about the observed data. Li formulates an objective function for relaxation labeling in which contextual constraints from both the prior knowledge and the observation are considered (Li 1992a; Li 1992b; Li 1992c). Kittler *et al.* (1993) later justify the formulation from probabilistic view point.

### Representation of Continuous RL

In continuous RL, the labeling state for each site is represented as a  $\#\mathcal{L}$  position vector; for  $i$ , it is

$$p_i = [p_i(I) \mid I \in \mathcal{L}] \quad (8.18)$$

subject to the following *feasibility* constraints

$$\begin{cases} \sum_{I \in \mathcal{L}} p_i(I) = 1 & \forall i \\ p_i(I) \geq 0 & \forall i, I \end{cases} \quad (8.19)$$

This was interpreted as a fuzzy assignment model (Rosenfeld et al. 1976). The real value  $p_i(I) \in [0, 1]$  reflects the strength with which  $i$  is assigned label  $I$ . Each  $p_i$  lies in a hyperplane in the non-negative quadrant portion (simplex) of the multi-dimensional real space  $\mathbb{R}^{\mathcal{L}}$  as shown in the shaded area of Fig.8.3. The feasible labeling assignment space is

$$\mathbb{P}_i = \{p_i(I) > 0 \mid I \in \mathcal{L}, \sum_{I \in \mathcal{L}} p_i(I) = 1\} \quad (8.20)$$

The set  $p = \{p_i \mid i \in \mathcal{S}\}$  is called a *labeling assignment*. The feasible space for labeling assignments is the following product

$$\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \cdots \times \mathbb{P}_m \quad (8.21)$$

The final solution  $p^*$  must be *unambiguous*

$$p_i^*(I) = \{0, 1\} \quad \forall i, I \quad (8.22)$$

with  $p_i^*(I) = 1$  meaning that  $i$  is unambiguously labeled  $I$ . This gives the unambiguous spaces for  $p_i$  as

$$\mathbb{P}_i^* = \{p_i(I) \in \{0, 1\} \mid I \in \mathcal{L}, \sum_{I \in \mathcal{L}} p_i(I) = 1\} \quad (8.23)$$

So every  $p_i$  in  $\mathbb{P}_i^*$  is one of the vectors  $(1, 0, \dots, 0)$ ,  $(0, 1, \dots, 0)$  and  $(0, 0, \dots, 1)$  corresponding to the “corners” of the simplex  $\mathbb{P}_i$  (*cf.* Fig. 8.3). The unambiguous labeling assignment space is

$$\mathbb{P}^* = \mathbb{P}_1^* \times \mathbb{P}_2^* \cdots \times \mathbb{P}_m^* \quad (8.24)$$

which consists of the corners of the space  $\mathbb{P}$ . Non-corner points in the continuous space  $\mathbb{P}$  provide routes to the corners of  $\mathbb{P}$ , *i.e.* the points in the discrete space  $\mathbb{P}^*$ . An unambiguous labeling assignment is related to the corresponding discrete labeling by

$$f_i = I \quad \text{if} \quad p_i^*(I) = 1 \quad (8.25)$$

The unambiguity may be satisfied automatically upon convergence of some iterative algorithms or if not, has to be enforced *e.g.* using a maximum selection (winner-take-all) operation. However, the forced unambiguous solution is not necessarily a local minimum. How to ensure the result of a continuous RL algorithm to be an unambiguous one has been largely ignored in the literature.

## Maximization Formulation

Although RL can be used for both minimization and maximization, it is sometimes algorithmically more suitable for maximization. Therefore, in practice, we convert the minimization of the MRF energy into the maximization of a corresponding gain function. The gain is the sum of compatibility functions.

The RL compatibility functions can be defined based on the Gibbs clique potential functions as follows: The unary compatibility function is defined by

$$r_i(I) = \text{Const}_1 - V_1(I \mid d) \quad (8.26)$$

where  $V_1(I \mid d)$  is the function of potentials incurred by single-site cliques. The binary compatibility function is defined by

$$r_{i,i'}(I, I') = \text{Const}_2 - V_2(I, I' \mid d) \quad (8.27)$$

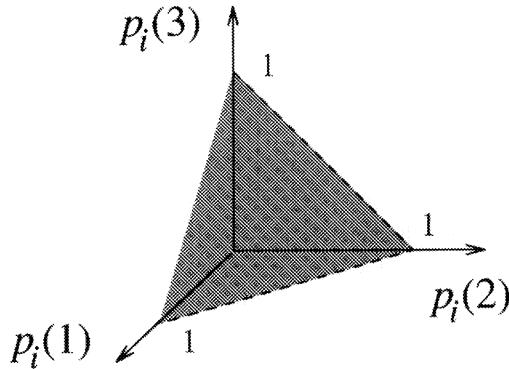


Figure 8.3: The labeling space for  $p_i$  with three labels. The feasible space is shown as the shaded area (simplex) and the unambiguous space consists of the three corner points.

where  $V_2(I, I' | d)$  is the function of potentials incurred by pair-site cliques. The constants  $Const_1$  and  $Const_2$  are chosen so that all the compatibility functions are non-negative.

The gain with the labeling assignment representation can now be written as

$$\begin{aligned} G(p) = & \sum_{i \in \mathcal{S}} \sum_{I \in \mathcal{L}} r_i(I) p_i(I) + \\ & \sum_{i \in \mathcal{S}} \sum_{I \in \mathcal{L}} \sum_{i' \in \mathcal{S}, i' \neq i} \sum_{I' \in \mathcal{L}} r_{i,i'}(I, I') p_i(I) p_{i'}(I') \end{aligned} \quad (8.28)$$

which is to be *maximized*. The above is the standard form of objective functions in RL formulations. Obviously, there is one-one correspondence between the maxima of  $G(f)$  and the minima of  $E(f)$  because of the relationship  $G(f) \propto Const - E(f)$  where  $Const$  is some constant.

### Iterative Updating Equations

In maximizing  $G(f)$  under the feasibility and unambiguity constraints, an RL algorithm iteratively updates  $p$  based on the gradient  $q = -\nabla E(f) = \{q_i(I) | i \in \mathcal{S}, I \in \mathcal{L}\}$ . The gradient components are given by

$$q_i(I) = \frac{\partial G}{\partial p_i(I)} = r_i(I) + 2 \sum_{i' \in \mathcal{S}} \sum_{I' \in \mathcal{L}} r_{i,i'}(I, I') p_{i'}(I') \quad (8.29)$$

for symmetric  $r_{i,i'}(I, I')$ . Let  $p = p^{(t)}$  represent the labeling state at time  $t$ . The initial assignment is set to  $p^{(0)} \in \mathbb{P}$ . A current  $p^{(t)}$  is updated into a new state  $p^{(t+1)}$  based on the information in  $p^{(t)}$  and  $q^{(t)}$

$$p^{(t+1)} \leftarrow \Phi(p^{(t)}, q^{(t)}) \quad (8.30)$$

where  $\Phi$  is an updating operator. RL algorithms differ from one another in the choice of  $\Phi$ . Various update equations are described in the following.

Rosenfeld *et al.* (1976) propose to use the following fixed-point iteration

$$p_i^{(t+1)}(I) \leftarrow \frac{p_i^{(t)}(I)[1 + q_i^{(t)}(I)]}{\sum_J p_i^{(t)}(J)[1 + q_i^{(t)}(J)]} \quad (8.31)$$

Obviously, the feasibility constraints (8.19) are satisfied by the normalization. The above iteration equation is only one of many choices. For example, when the compatibility coefficients are non-negative, we have  $q_i^{(t)}(I) \geq 0$ ; in this case, the following updating scheme also maintains the feasibility while performing gradient ascent

$$p_i^{(t+1)}(I) \leftarrow \frac{p_i^{(t)}(I)q_i^{(t)}(I)}{\sum_J p_i^{(t)}(J)q_i^{(t)}(J)} \quad (8.32)$$

In the continuous RL algorithms by (Faugeras and Berthod 1981) and (Hummel and Zucker 1983), the feasibility is maintained using a technique called *gradient projection* (GP) (Rosen 1960). GP is an extended steepest descent procedure for solving programming problems under linear constraints. It finds a feasible direction  $u$ , from the gradient  $q$  and the current  $p^{(t)}$ , for the following updating scheme

$$p^{(t+1)} \leftarrow p^{(t)} + \mu u \quad (8.33)$$

where  $\mu > 0$  is a scalar factor. The vector  $u = \frac{1}{\mu}[p^{(t+1)} - p^{(t)}]$  must satisfy the condition

$$\sum_{I \in \mathcal{L}} u_i(I) = \frac{1}{\mu} \sum_{I \in \mathcal{L}} [p_i^{(t+1)}(I) - p_i^{(t)}(I)] = \frac{1}{\mu}(1 - 1) = 0 \quad (8.34)$$

in order to maintain the feasibility constraints. If  $p_i^{(t)}(I) = 0$ , then there must be  $u_i(I) \geq 0$  so that  $p_i^{(t+1)}(I)$  would not be negative. For convenience, a normalization of  $u$  may be imposed so that  $\|u\| = 1$ . Then, the feasible space for  $u$  is defined by

$$\mathbb{U} = \{u \in \mathbb{R}^{\mathcal{S} \times \mathcal{L}} \mid \sum_{I \in \mathcal{L}} u_i(I) = 0, \|u\| = 1, \text{ and } u_i(I) \geq 0 \text{ whenever } p_i(I) = 0\} \quad (8.35)$$

The maximum value for  $\mu$  is chosen so that the updated vector  $p^{(t+1)}$  still lies in the space  $\mathbb{P}$ . A detailed implementation of such a projection operator used in (Hummel and Zucker 1983) is described in (Mohammed *et al.* 1983). A discussion on parallel distributed implementation of the Hummel-Zucker algorithm in multiprocessor architecture environment is given in (McMillin and Ni 1989).

### Local vs. Global Solutions

Gradient-based optimization methods guarantees only local solutions when the cost function is non-convex. RL algorithms are local minimizers by nature. Given an RL iteration scheme and the data, two factors affect the solution: the initial assignment of labels and the compatibility function (Rosenfeld et al. 1976). However, it is shown that for certain choices of compatibility function, an RL algorithm can have a *single* convergence point regardless of the initial assignment (Bozma and Duncan 1988; O’Leary and Peleg 1983).

In some earlier RL works (Rosenfeld et al. 1976; Peleg and Rosenfeld 1978), the dependence of solutions on initializations is regarded as a desirable property. There, compatibility functions are defined *without* referring to the observations and therefore they should be considered as containing *a priori* information only. Constraints due to the observation are encoded in the initial assignment. Therefore, traditional RL solutions rely much on the initialization; convergence to a solution regardless of the initialization is considered undesirable. However, the posterior MRF energy already contains both sources of information, *i.e.* on the prior and the observation. Therefore the global energy minimum is the desirable solution (Geman and Geman 1984). An “annealing labeling” algorithm, a GNC-like algorithm for the MAP-MRF matching will be described in Section 9.3.2.

#### 8.3.3 Highest Confidence First

Highest Confidence First (HCF) (Chou and Brown 1990; Chou et al. 1993) is a deterministic algorithm for combinatorial minimization in which the label set is discrete,  $\mathcal{L} = \{1, \dots, M\}$ . Its feature is the introduction of a special *uncommitted* label and the strategy for “committing” a site.

Denote the *uncommitted* label by 0. The original label set is augmented by this label into  $\mathcal{L}^+ = \{0, 1, \dots, M\}$ . A label  $f_i$  is said to be uncommitted if  $f_i = 0$  or committed if  $f_i \in \mathcal{L}$ . Initially, all label are set uncommitted,  $f = \{0, \dots, 0\}$ . A rule is that once a site is committed, its label cannot go back to 0 but can change to another value in  $\mathcal{L}$ .

The commitment and label changes are based on a stability measure defined in the following. The two assumptions made in ICM are also made in HCF. Therefore we have the conditional posterior potential

$$E_i(f_i) = V(f_i \mid d_i, \hat{f}_{\mathcal{N}_i}) = V(d_i \mid f_i) + \sum_{c: i \in c} V_c(f_i \mid f_{\mathcal{N}_i}) \quad (8.36)$$

where  $c : i \in c$  means any clique  $c$  containing site  $i$ . After augmenting the label set, we define the conditional potential for a *committed* label  $f_i \in \mathcal{L}$  as<sup>1</sup>

<sup>1</sup>Our definition has a different appearance from Chou’s  $E_s(l)$  in Equ.(14) of (Chou and Brown 1990), but is equivalent to it. Note that the definition of  $E_s(l)$  given in (Chou et al. 1993) contains a bug.

$$E_i(f_i) = V(f_i \mid d_i, \hat{f}_{N_i}) = V(d_i \mid f_i) + \sum_{c: i \in c} V'_c(f_i \mid f_{N_i}) \quad (8.37)$$

where

$$V'_c = \begin{cases} 0 & \text{if } f_j = 0 \exists j \in c \\ V_c & \text{otherwise} \end{cases} \quad (8.38)$$

Therefore, a site has no effect on its neighbors unless it has committed. When no neighbor is active, the local energy measure reduces to the likelihood of the label. A remark is that the *uncommitted* label is different from the NULL label described in Chapter 3 for object recognition in that any clique involving a site labeled NULL incurs a non-zero penalty.

The stability of  $i$  with respect to  $f$  is defined as

$$S_i(f) = \begin{cases} -\min_{l \in \mathcal{L}, l \neq l_{\min}} [E_i(l) - E_i(l_{\min})] & \text{if } f_i = 0 \\ \min_{l \in \mathcal{L}, l \neq f_i} [E_i(l) - E_i(f_i)] & \text{otherwise} \end{cases} \quad (8.39)$$

where  $l_{\min} = \arg \min_{l \in \mathcal{L}} E_i(l)$ . The stability  $S$  of an uncommitted site is the negative difference between the lowest and the second lowest local energies (conditional potentials). The stability  $S$  of a committed site is the difference between the current local energy  $E_i(f_i)$  and the lowest possible energy due to any other label. Note that the range of a stability is  $-\infty < S_i < +\infty$ . A negative stability means there is room to improve. All uncommitted sites have non-positive  $S_i$ . The magnitude of  $S_i$  is equal to the change in energy due to the change in the label  $f_i$ . A lower value of  $S_i(f)$  indicates a more stable configuration and a negative value with larger magnitude gives us more confidence to change  $f$  to a new configuration.

The following rule is imposed for deciding the order of update: At each step, only the least stable site is allowed to change its label or to make its commitment. Suppose that  $k = \arg \max_i S_i(f)$  is the least stable site. Then if  $f_k = 0$ , change  $f_k$  to

$$f'_k = \arg \min_{l \in \mathcal{L}} E_k(l) \quad (8.40)$$

otherwise, change  $f_k$  to

$$f'_k = \arg \min_{l \in \mathcal{L}, l \neq f_k} [E_k(l) - E_k(f_k)] \quad (8.41)$$

Therefore, the first committed label  $f_i$  is that yields the maximum local likelihood  $V(d_i \mid f_i)$ . The HCF algorithm is described in Fig. 8.4. Create\_Heap creates a heap in which the  $S$ -values are sorted, the least stable site is placed at the *top*. Change\_State( $k$ ) changes  $f_k$  to  $f'_k$  according to the rule described earlier. Update\_S( $k$ ) updates the stability value of  $k$  using the current  $f$ . Adjust\_Heap( $k$ ) adjusts the heap according to the current  $S$ -values.

A parallel version of HCF, called “local HCF”, is described in (Chou et al. 1993). In local HCF, the updating on each iteration is performed according

---

```

 $f = \{0, \dots, 0\};$ 
Create_Heap;
while ( $S_{top}(f) < 0$ ) {
     $k = top;$ 
    Change_State( $k$ );
    Update_S( $k$ );
    Adjust_Heap( $k$ );
    for each  $j \in \mathcal{N}_k$  {
        Update_S( $j$ );
        Adjust_Heap( $j$ );
    }
}
return( $f^* = f$ )

```

---

Figure 8.4: The highest confidence first algorithm.

to the following rule: For each site in parallel, change the state of the site if its stability is negative and lower than the stabilities of its neighbors.

Some results are given in Chou *et al.* 's paper to compare HCF with other algorithms including simulated annealing and ICM. The results show that HCF is, on the whole, better in terms of the minimized energy  $E(f^*)$ . The initialization is not a problem in HCF because it is always set to 0's.

### 8.3.4 Dynamic Programming

Dynamic programming (DP) (Bellman and Dreyfus 1962) is an optimization technique for problems where not all variables (labels in MRFs) are inter-related simultaneously. This is useful for the computation of MRFs because contextual constraints therein, causal or non-causal, are localized. The technique is based on the *principle of optimality*. Bellman states it as follows: “An optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”

Suppose that the global energy can be decomposed into the following form

$$E(f_1, \dots, f_m) = E_1(f_1, f_2) + E_2(f_2, f_3) + \dots + E_{m-1}(f_{m-1}, f_m) \quad (8.42)$$

DP generates a sequence of functions of one variable

$$\begin{aligned}
D_1(f_2) &= \min_{f_1} E_1(f_1, f_2) \\
D_2(f_3) &= \min_{f_2} [D_1(f_2) + E_2(f_2, f_3)] \\
&\vdots
\end{aligned} \tag{8.43}$$

The first function is obtained as follows: for each  $f_2$  value, choose the minimum of  $E_1(f_1, f_2)$  over all  $f_1$  values. The sequence can be written in the recursive form: for  $k = 1, \dots, m-1$ ,

$$D_k(f_{k+1}) = \min_{f_k} [D_{k-1}(f_k) + E_k(f_k, f_{k+1})] \tag{8.44}$$

with  $D_0(f_1) = 0$ . The minimal energy solution is obtained by

$$\min_f E(f_1, \dots, f_m) = \min_{f_m} D_{m-1}(f_m) \tag{8.45}$$

This is a valid formula if the problem can really be decomposed into the form of (8.42).

With regards the validity, we have the following relationship

$$\min_{f_1, f_2, f_3} [E_1(f_1, f_2) + E_2(f_2, f_3)] \leq \min_{f_3} \left\{ \min_{f_2} \left[ \min_{f_1} E_1(f_1, f_2) + E_2(f_2, f_3) \right] \right\} \tag{8.46}$$

If the equality holds, it means the DP algorithm gives the true minimum of  $E$ . The condition for the equality to hold is that  $E_2(f_2, f_3)$  is independent of  $f_1$ . If this is not true, then the DP solution is not a global one. This conclusion can be generalized to situations with more variables.

If each  $f_k$  takes on  $M$  discrete values in  $\mathcal{L}$ , then to compute  $D_{k-1}(f_k)$  for each  $f_k$  value, one must evaluate the minimand  $[D_{k-2}(f_{k-1}) + E_{k-1}(f_{k-1}, f_k)]$  for the  $M$  different  $f_{k-1}$  values in order to determine the minimal value of the minimand. Therefore, the overall minimization involves  $(m-1)M^2 + M$  evaluations of the minimands. This is an enormous reduction from the exhaustive minimization; for the latter case,  $|\mathcal{L}|^{|S|}$  evaluations of  $E(f_1, \dots, f_m)$  have to be made where  $|\cdot|$  denotes the number of elements in the set.

Applications of DP in vision include curve detection (Ballard and Brown 1982), matching (Fischler and Elschlager 1973), energy minimization for MRF-based image segmentation (Derin and Elliott 1987) and for active contours (Amini et al. 1990). The Derin-Elliott algorithm is described below.

The algorithm is designed for the MLL region model with additive white Gaussian noise. The MRF is defined on an image lattice  $\mathcal{S} = \{(i, j) \mid 1 \leq i, j \leq n\}$ . Each true label  $f_{i,j}$  takes a value in  $\mathcal{L} = \{\ell_1, \dots, \ell_M\}$ . The posterior energy is expressed as

$$E(f) = U(f \mid d) = \sum_{c \in \mathcal{C}} V_c(f) + \sum_{I \in \mathcal{L}} \sum_{(i,j) \in \mathcal{S}^{(I)}} \frac{1}{\sigma^2} (d_{i,j} - \ell_I)^2 \tag{8.47}$$

where  $V_c$  are the MLL clique potentials on the 8-neighborhood system given in Fig.1.4,  $\mathcal{S}^{(I)} = \{(i, j) \in \mathcal{S} \mid f_{i,j} = \ell_I\}$  is the set of sites whose labels take the value  $\ell_I \in \mathcal{L}$  and  $\sigma^2$  is the noise variance. Similar to (8.42), a decomposition of energy is done as

$$E(f) = \sum_{k=1}^{m-1} E_k(f_k, f_{k+1}) \quad (8.48)$$

where  $f_k$  denotes the  $k$ -th row. Function  $E_k(f_k, f_{k+1})$  concerns only the two rows,

$$E_k(f_k, f_{k+1}) = \sum_{c \in \mathcal{C}^{k, k+1}} V_c(f) + \sum_{I \in \mathcal{L}} \sum_{S_I^k} \frac{1}{\sigma^2} (d_{k,j} - \ell_I)^2 \quad (8.49)$$

where

$$\begin{aligned} \mathcal{C}^{k, k+1} &= \{c \mid c \text{ is a clique with pixels only in columns } k \text{ and } k+1\} \\ \mathcal{S}_I^k &= \{(k, j) \mid f_{k,j} = \ell_I, 1 \leq i \leq m\} \end{aligned} \quad (8.50)$$

In this decomposed form,  $E(f)$  can be minimized using the DP technique just described. Because (8.49) is a valid formula for the MLL model on the 8-neighborhood system, the DP solution should give exact MAP estimate.

When the observation model is due to textures, the likelihood function cannot be decomposed into independent terms for the application of DP. Derin and Elliott (1987) make some assumptions to simplify the involved conditional probabilities before applying the DP algorithm. This gives sub-optimal solution.

## 8.4 Constrained Minimization

In constrained minimization, extra constraints, either equalities or inequalities, are imposed on the solution  $f$ . The problem with  $K$  equality constraints is stated as

$$\begin{aligned} \min_f \quad & E(f) \\ \text{subject to} \quad & C_k(f) = 0 \quad k = 1, \dots, K \end{aligned} \quad (8.51)$$

where  $C_k(f) = 0$  denotes the equality constraints. For an inequality constraint of the form  $D(f) \leq 0$ , we can introduce a nonnegative slack function of one variable, e.g.  $y^2$ , and convert it into an equality constraint  $D(f) + y^2 = 0$ . So only problems with equality constraints are considered here. The constraints define the feasible region  $\mathbb{F}$  for  $f$ .

The problem is termed linear programming when the energy function and the constraint functions are all linear in  $f$ ; otherwise it is nonlinear. Methods for linear and nonlinear programming have quite different features. Because our energy functions  $E(f)$  can be nonlinear in  $f$ , our discussion will be on nonlinear programming problems.

When the constraint functions are linear, the gradient projection method (Rosen 1960) can be used to find feasible directions for the steepest descent. Given that  $f \in \mathbb{F}$  is in the feasible region, the gradient  $\nabla E(f)$  is computed. Direct gradient descent gives  $f - \mu \nabla E(f)$ . But this vector may no longer be in  $\mathbb{F}$ . To keep the feasibility, the new configuration is obtained by projecting  $f - \mu \nabla E(f)$  onto  $\mathbb{F}$ . By this, the updated configuration still lies in the feasible region. The computation is not difficult when the constraints are linear such that  $\mathbb{F}$  corresponds to a hyperplane. This is the GP operator used in RL described earlier.

In classical methods for nonlinear programming, constrained minimization problems are converted into unconstrained ones, *e.g.* by using penalty functions and Lagrange multipliers. In RL, the feasibility constraints are respected throughout the update process. However, in the numerical computation in penalty and Lagrange methods, this may not necessarily be so.

#### 8.4.1 Penalty Functions

Using penalty functions, we can force  $f$  to stay inside or close to the feasible region determined by  $C_k(f) = 0$  while minimizing  $E(f)$ . A simply penalty function for  $C_k(f) = 0$  can be taken as  $C_k^2(f)$ . Adding them to the original energy yields a new energy

$$E(f, \beta) = E(f) + \sum_k \beta_k C_k^2(f) \quad (8.52)$$

where  $\beta_k > 0$  determines the amount of penalty. Minimizing the above w.r.t.  $f$  for a fixed  $\beta$  is an unconstrained problem. The minimum in the limit  $\beta_k \rightarrow \infty$  ( $\forall k$ )

$$f^* = \arg \lim_{\beta_k \rightarrow \infty (\forall k)} \min_f E(f, \beta) \quad (8.53)$$

will be one satisfying the constraints  $C_k(f) = 0$ . This suggests an algorithm which solves a sequence of unconstrained minimization problems:

1. Choose a fixed sequence  $\beta^{(0)}, \beta^{(1)}, \dots$  with  $\beta_k^{(t)} < \beta_k^{(t+1)}$  and  $\beta_k^{(t)} \rightarrow \infty$ .
2. For each  $\beta^{(t)}$ , find a local minimum  $f^*(\beta^{(t)}) = \arg \min_f E(f, \beta^{(t)})$ .
3. Terminate when  $C_k^2(f^*(\beta^{(t)}))$  is small enough.

In practice, step (2) is usually done numerically using an iterative method. Moreover,  $f^*$  obtained previously with  $\beta^{(t)}$  is used as the initial point for the next minimization with  $\beta^{(t+1)}$ .

Consider the problem of pose estimation from a set of corresponding points discussed in Section 3.4.1. Assume  $\mathbb{F}^0$  is the unconstrained solution space in which a pose  $f$  consists of an arbitrary rotation matrix and a translation vector in 3D. The energy  $E(f)$  without constraints measures squared errors. If poses are constrained to be Euclidean, as assumed, any rotation

matrix must be orthogonal. The penalty function for orthogonality was given in (3.43) as the single-site potential function. A similar penalty function has been used by (Haralick et al. 1989).

As described in Section 2.3.2, (Geman et al. 1990) in which boundary detection is formulated as constrained minimization; certain edge patterns, such as isolated edges, sharp turns, quadruple junctions and “small” structures, are forbidden in forming desirable boundaries. These are expressed as  $U^E(f^E) = 0$ . In the penalty method, the penalty term  $U^E(f^E)$  is added to the energy  $E(f^P, f^E)$ , yielding  $E'(f^P, f^E) = E(f^P, f^E) + \lambda_E U^E(f^E)$ . Sending  $\lambda_E$  to  $+\infty$  in the minimization will rule out occurrences of forbidden edge configurations.

To overcome the problem of local minima, an annealing process can be incorporated with the penalty method. In constrained simulated annealing (Geman et al. 1990), an annealing parameter  $T$  is introduced into  $E$ , yielding

$$E(f, \beta, T) = [E(f) + \sum_k \beta_k C_k^2(f)]/T \quad (8.54)$$

Using a stochastic sampling scheme such as Metropolis algorithm or Gibbs sampler (*cf.* Section 9.1), one can generate a Gibbs distribution  $f$  with energy functions  $E(f, \beta, T)$ . With an annealing scheme in which  $\beta^{(t)} \nearrow \infty$  and  $T^{(t)} \searrow 0$  at a suitably coupled rate,  $f^{(t)}$  will converge to the global minimum of  $E(f)$  with probability approaching 1. Although this is computationally more demanding than ICM (Besag 1986) but invariably arrives at a better labeling.

The penalty method has a number of advantages : It is easy to use; it allows inexact constraints – constraints are not necessary to fulfill exactly; it converges to a feasible solution when  $\beta_k \rightarrow \infty$ . However, allowing inexact constraints is disadvantageous for those problems in which the exact fulfillment of constraints is required. Moreover, when  $\beta_i$  are very large, the system becomes *stiff* and the problem becomes ill-conditioned (Fletcher 1987). The Lagrange multiplier method may be better in this regard.

### 8.4.2 Lagrange Multipliers

The Lagrange multiplier method converts the constrained minimization into an unconstrained minimization of the following Lagrange function of  $m + K$  variables

$$L(f, \gamma) = E(f) + \sum_k \gamma_k C_k(f) \quad (8.55)$$

where  $\gamma_k$  are called Lagrange multipliers. For  $f^*$  to be a local minimum subject to the constraints, it is necessary that  $(f^*, \gamma^*)$  be a stationary point of the Lagrange function:

$$\begin{aligned} \nabla_f L(f^*, \gamma^*) &= 0 \\ \nabla_\gamma L(f^*, \gamma^*) &= 0 \end{aligned} \quad (8.56)$$

If  $(f^*, \gamma^*)$  is a saddle point for which

$$L(f^*, \gamma) \leq L(f^*, \gamma^*) \leq L(f, \gamma^*) \quad (8.57)$$

then  $f^*$  is a local minimum of  $E(f)$  satisfying  $C_k(f^*) = 0$  (Gottfried 1973).

The following dynamics can be used to find such a saddle point

$$\begin{aligned} \frac{df_i}{dt} &= -\frac{\partial L(f, \gamma)}{\partial f_i} = -\frac{\partial E(f)}{\partial f_i} - \sum_k \gamma_k \frac{\partial C_k(f)}{\partial f_i} \\ \frac{d\gamma_k}{dt} &= +\frac{\partial L(f, \gamma)}{\partial \gamma_k} = +C_k(f) \end{aligned} \quad (8.58)$$

It performs energy descent on  $f$  but ascent on  $\gamma$ . Convergence results of this system have been obtained by (Arrow et al. 1958). The dynamics is recently used for neural computing (Platt and Barr 1988).

The penalty terms in the penalty function (8.52) can be added to the Lagrange function (8.55) to give an augmented Lagrange function (Powell 1969; Hestenes 1969)

$$L(f, \gamma, \beta) = E(f) + \sum_k \gamma_k C_k(f) + \sum_k \beta_k [C_k(f)]^2 \quad (8.59)$$

where  $\beta_k$  are *finite* weights. The addition of the penalty terms do not alter the stationary point and sometimes help damp oscillations and improve convergence. Because  $\beta_k$  are finite, the ill-conditioning of the problem in the penalty method when  $\beta_k \rightarrow \infty$  is alleviated. After the penalty term is added, the dynamics for  $f_i$  is

$$\frac{df_i}{dt} = -\frac{\partial E(f)}{\partial f_i} - \sum_k \gamma_k \frac{\partial C_k(f)}{\partial f_i} - 2 \sum_k \beta_k C_k(f) \frac{\partial C_k(f)}{\partial f_i} \quad (8.60)$$

### 8.4.3 Hopfield Method

The graded Hopfield neural network (HNN) (Hopfield 1984) provides another way of applying penalty functions. Let us use the constrained minimization in RL as an example to illustrate this point. There, the feasibility (8.19) must be satisfied by the labeling assignment at any time and the unambiguity (8.22) should be satisfied by the final labeling assignment. The unambiguity constraint can be imposed by the following term

$$E_a(f) = \sum_i \sum_I \int_0^{f_i(I)} \psi_T^{-1}(f) df \quad (8.61)$$

where  $\psi_T^{-1}$  is the inverse of a function  $\psi_T$  to be illustrated below. A local state  $f_i(I) \in [0, 1]$  is related to an *internal variable*  $u_i(I) \in (-\infty, +\infty)$  by

$$f_i(I) = \psi_T(u_i(I)) \quad (8.62)$$

where  $\psi_T(u)$  is usually a sigmoid function

$$\psi_T(u) = 1/[1 + e^{-u/T}] \quad (8.63)$$

controlled by the parameter  $T$ . In very high gain when  $T \rightarrow 0^+$ ,  $f_i(I)$  is forced to be 0 or 1 depending on whether  $u_i(I)$  is positive or negative (Hopfield 1984). The term  $E_a(f)$  reaches the minimum of zero only when all  $f_i(I)$ 's are either 0 or 1. This means that minimizing this term with  $T \rightarrow 0^+$  in effect leads to unambiguous labeling.

On the other hand, The feasibility can be imposed by

$$E_b(f) = \sum_i \left[ \sum_I f_i(I) - 1 \right]^2 = 0 \quad (8.64)$$

This term has its minimum value of zero when the feasibility is satisfied. Now the constrained optimization minimizes the following functional

$$E'(f \mid T) = E(f) + aE_a(f \mid T) + bE_b(f) \quad (8.65)$$

where  $a$  and  $b$  are weights. A minimal solution which is feasible and unambiguous is given by

$$f^* = \arg \min_f \lim_{T \rightarrow 0^+} E'(f \mid T) \quad (8.66)$$

To derive a dynamic system for the minimization, introduce a time variable into  $f$  such that  $f = f^{(t)}$ . The energy change due to state change  $\frac{df_i(I)}{dt}$  is

$$\frac{dE'}{dt} = - \sum_{i,I} \frac{df_i(I)}{dt} \left\{ q_i(I) - a u_i(I) - 2b \left[ \sum_I f_i(I) - 1 \right] \right\} \quad (8.67)$$

where

$$q_i(I) = \frac{dE}{df_i(I)} = V_1(I \mid d_1(i)) + 2 \sum_{i',I'} V_2(I', I' \mid d_2(i, i')) f_{i'}(I') \quad (8.68)$$

is a gradient component. The following dynamics minimizes the three term energy  $E'$ :

$$C \frac{du_i(I)}{dt} = q_i(I) - a u_i(I) - 2b \left[ \sum_I f_i(I) - 1 \right] \quad (8.69)$$

where capacitance  $C > 0$  controls the convergence of the system. With the above dynamics, the energy change

$$\frac{dE'}{dt} = -C \sum_{i,I} \frac{df_i(I)}{dt} \frac{du_i(I)}{dt} - C \sum_{i,I} \frac{d\psi_T^{-1}(f_i(I))}{df_i(I)} \left[ \frac{df_i(I)}{dt} \right]^2 \quad (8.70)$$

is non-positive since  $\psi_T^{-1}$  is a monotonically increasing function and  $C$  is positive. This updating rule will lead to a minimal solution which is both feasible and unambiguous when  $T \rightarrow 0$ ,  $a \rightarrow \infty$  and  $b \rightarrow \infty$ .

To solve combinatorial optimization, Wacholder *et al.* (1989) develop an algorithm which combines the HNN of (Hopfield 1984) and the Lagrange multiplier dynamics of (Platt and Barr 1988). Good convergence to feasible solutions is reported. In Section 8.5, we will describe a method which combines the HNN and the augmented Lagrangian function to perform relaxation labeling.

The HNN approach has been used to convert binary-valued line process variables to real ones for low level process (Koch *et al.* 1986; Yuille 1987) (*cf.* Section sec:DA.LP). It has also been used as an alternative to relaxation labeling for matching; this is seen as symbolic image labeling (Jamison and Schalkoff 1988), subgraph isomorphism for object recognition (Nasrabadi *et al.* 1990), and matching to multiple view models (Lin *et al.* 1991).

Comparing RL and HNN for combinatorial minimization like object matching, our experience is that the former performs better. Due to the influence of the extra penalty function added in the HNN approach, the modified energy may no longer solve the original matching problem (Horn 1988). Moreover, it easily leads to unfavorable local solutions. The RL algorithm is much more careful in this regard. It uses gradient projection to choose the best direction and magnitude for the state to evolve. In other words, the problem of local optima is more significant in the HNN approach. Computationally, the RL needs a smaller number of iterations before convergence.

## 8.5 Augmented Lagrange-Hopfield Method

A combinatorial optimization can often be converted into a constrained real optimization with equality and inequality constraints. The penalty and the Lagrange multiplier methods can be used for coping with equality constraints and the barrier method for coping with inequality constraints. However, the penalty method suffers from the ill-conditioning and the Lagrange method suffers from the zigzagging problem (Fletcher 1987). The augmented Lagrange (AL) method (Powell 1969) combines both the Lagrange and the penalty methods and effectively overcomes the associated problems. In AL, the relative weight for the penalty terms need not be infinitely large, this not only overcoming the ill-conditioning problem but also beneficial for obtaining better quality solutions because the relative importance of the original objective function is more emphasized; on the other hand, its use of quadratic penalty terms “convexifies” and hence stabilizes the system, overcoming the zigzagging problem (Fletcher 1987).

Mean field annealing (MFA) (Peterson and Soderberg 1989) provides still another continuous method. Assuming that the minima of the original energy and the corresponding mean field effective energy coincide, the MFA aims to

approximate the global minimum of the original energy by tracking that of the effective energy with decreasing temperature. A recent analysis shows that the effective energy of MFA is identical to a combination of the original energy, a particular barrier term and a standard Lagrange term (Yullie and Kosowsky 1994).

Here, another deterministic method, called the augmented Lagrange-Hopfield (ALH) method, is presented for the combinatorial optimization in the MAP-MRF image restoration and segmentation. In solving the converted constrained real optimization, the ALH method uses the augmented Lagrangian multiplier method (Powell 1969) to satisfy the equality constraints and the Hopfield network encoding (Hopfield 1984) to impose the inequality constraints. The use of AL effectively overcomes instabilities inherent in the penalty method and the Lagrange multiplier method. The resulting algorithm solves a system of differential equations. Experimental results in both image restoration and segmentation are shown to compare the ALH method with the ICM, HCF and SA. The results show that the ALH outperforms ICM and HCF, and is comparable to SA in terms of the solution quality; it quickly yields a good solution after a dozen of iterations, a number similar to that required by ICM and HCF but much smaller than SA. A discussion on MFA results is also provided.

### 8.5.1 MAP-MRF Estimation as Constrained Optimization

The underlying image signal is denoted as  $f = \{f_i \mid i \in \mathcal{S}\}$  where  $\mathcal{S} = \{1, \dots, m\}$  indexes the set of sites corresponding to image pixels. Each pixel takes on a discrete value  $f_i$  in the label set  $\mathcal{L} = \{1, \dots, M\}$ ;  $f$  is an underlying image to be restored or a segmented map to be computed. The spatial relationship of the sites, each of which is indexed by a single number in  $\mathcal{S}$ , is determined by a neighborhood system  $\mathcal{N} = \{\mathcal{N}_i \mid i \in \mathcal{S}\}$  where  $\mathcal{N}_i$  is the set of sites neighboring  $i$ . A single site or a set of neighboring sites form a clique denoted by  $c$ . In this paper, only up to pair-site cliques defined on the 8-neighborhood system are considered.

The underlying image  $f$  can consist of blob-like regions or a texture pattern. Different types are due to different ways that pixels interact with each other, *i.e.* due to different contextual interactions. Such contextual interactions can be modeled as MRFs or Gibbs distributions of the form  $P(f) = Z^{-1} \times e^{-\sum_{c \in \mathcal{C}} V_c(f)}$  where  $V_c(f)$  is the potential function for clique  $c$ ,  $\mathcal{C}$  is the set of all cliques, and  $Z$  is the normalizing constant. Among various MRFs, the multi-level logistic (MLL) model is a simple and yet powerful mechanism for encoding a large class of spatial patterns such as textured or non-textured images. In MLL, the pair-site clique potentials take the form:  $V_2(f_i, f_{i'}) = \beta_c$  if sites in clique  $\{i, i'\} = c$  have the same label or  $V_2(f_i, f_{i'}) = -\beta_c$  otherwise, where  $\beta_c$  is a parameter for type- $c$  cliques; the single site potentials are defined by  $V_1(f_i) = \alpha_I$  where  $\alpha_I$  is the potential for

the label  $I = f_i$ . When the true pixel values are contaminated by identical independently distributed (i.i.d.) Gaussian noise, the observed data, or the likelihood model, is  $d_i = f_i + e_i$  where  $e_i \sim N(0, \sigma^2)$  is the zero-mean Gaussian distribution with standard deviation  $\sigma$ . With these prior and likelihood models, the energy in the posterior distribution  $P(f | d) \propto e^{-E(f)}$  is

$$E(f) = \sum_{i \in \mathcal{S}} (f_i - d_i)^2 / [2\sigma^2] + \sum_{\{i\} \in \mathcal{C}} V_1(f_i) + \sum_{\{i, i'\} \in \mathcal{C}} V_2(f_i, f_{i'}) \quad (8.71)$$

In this paper, we assume that the MRF and noise parameters  $\alpha, \beta$  and  $\sigma$  are known. The MAP estimate for the restoration or segmentation is defined as  $f^* = \arg \min_{f \in \mathcal{L}^m} E(f)$ . The minimization of the energy function  $E(f)$  is in the discrete space  $\mathcal{L}^m$  and hence combinatorial.

The original combinatorial optimization is converted into a constrained optimization in a real space, using the notion of continuous relaxation labeling can be used for this. Let real value  $p_i(I) \in [0, 1]$  represent the strength with which label  $I$  is assigned to  $i$ , the  $M$ -element vector  $p_i = [p_i(I) | I \in \mathcal{L}]$  represent the state of the assignment for  $i \in \mathcal{S}$ , and the matrix  $p = [p_i(I) | i \in \mathcal{S}, I \in \mathcal{L}]$  the state of the labeling assignment. The energy with the  $p$  variables is given by

$$E(p) = \sum_{i \in \mathcal{S}} \sum_{I \in \mathcal{L}} r_i(I) p_i(I) + \sum_{i \in \mathcal{S}} \sum_{I \in \mathcal{L}} \sum_{i' \in \mathcal{S}, i' \neq i} \sum_{I' \in \mathcal{L}} r_{i,i'}(I, I') p_i(I) p_{i'}(I') \quad (8.72)$$

where  $I = f_i$  and  $I' = f_{i'}$ ,  $r_i(I) = V_1(I | d) = (I - d_i)^2 / 2\sigma + V_1(I)$  is the single-site clique potential function in the posterior distribution  $P(f | d)$ , and  $r_{i,i'}(I, I') = V_2(I, I' | d) = V_2(I, I')$  is the pair-site clique potential function in  $P(f | d)$ .

With such a representation, the combinatorial minimization is reformulated as the following constrained minimization

$$\min_p \quad E(p) \quad (8.73)$$

$$\text{subject to} \quad C_i(p) = \sum_I p_i(I) - 1 = 0 \quad i \in \mathcal{S} \quad (8.74)$$

$$p_i(I) \geq 0 \quad \forall i \in \mathcal{S}, \forall I \in \mathcal{L} \quad (8.75)$$

The final solution  $p^*$  is subject to additional ambiguity constraints:  $p_i^*(I) \in \{0, 1\}$ .

### 8.5.2 The ALH Method

The ALH method aims to solve the above constrained minimization problem. It uses the augmented Lagrange technique to satisfy the equality constraints of (8.74) and the Hopfield encoding to impose the inequality constraints of

(8.75). In the Hopfield encoding, internal variables  $u_i(I) \in (-\infty, +\infty)$  are introduced and related to  $p_i(I)$  via the sigmoid function

$$p_i(I) = \psi_T(u_i(I)) = \frac{1}{1 + e^{-u_i(I)/T}} \quad (8.76)$$

controlled by a temperature parameter  $T > 0$ . This treatment confines  $p_i(I)$  to the range  $(0, 1)$  so as to impose the inequality constraints. When  $T \rightarrow 0^+$ ,  $p_i(I)$  is forced to be 0 or 1 depending on whether  $u_i(I)$  is positive or negative. Thereby, the ambiguity constraints  $p_i^*(I) \in \{0, 1\}$  are imposed. With the free  $u$  variables, the energy function can be considered as  $E(u) = E(p(u))$ .

The sigmoid function of Hopfield is one of many ways for imposing the inequality constraints. Yuille and Kosowsky (Yullie and Kosowsky 1994) have used a barrier term of the form  $T \sum_{i,I} p_i(I) \log p_i(I)$  for imposing  $p_i(I) > 0$ . However, care must be taken in the numerical implementation of a gradient descent algorithm involving the barrier term in order to avoid  $p_i(I)$  from decreasing to non-positive values. A truncation operation  $\min\{0, p_i(I)\}$  may be used to truncate negative  $p_i(I)$  values to 0. But this inevitably causes oscillations and hence non-convergence. We find it easier and more stable to use the sigmoid function in that the  $u$  variables can be updated freely.

The augmented Lagrange (AL) function (Powell 1969) takes the form

$$L_\beta(p, \gamma) = E(p) + \sum_{k \in S} \gamma_k C_k(p) + \frac{\beta}{2} \sum_{k \in S} [C_k(p)]^2 \quad (8.77)$$

where  $\gamma_k$  are the Lagrange multipliers and  $\beta > 0$  is the weight for the quadratic penalty term. It is a function of the  $M \times m$  variables of  $p$  and the  $m$  variables of  $\gamma$ . When  $\gamma_i = 0$  for all  $i$ , the AL reduces to the penalty method  $L_\beta(p) = E(p) + \frac{\beta}{2} \sum_k [C_k(p)]^2$ . When  $\beta = 0$ , it reduces to the standard Lagrange  $L(p, \gamma) = E(p) + \sum_k \gamma_k C_k(p)$ . With the use of Lagrange multipliers, the weight  $\beta$  for the penalty terms need not be infinitely large and this overcomes the ill-conditioning problem in the penalty method. Smaller  $\beta$  is also beneficial for better quality solution because the original objective function  $E$  is more emphasized. The use of quadratic penalty terms “convexifies” the system and hence stabilizes it, and this overcomes the zigzagging problem in the standard Lagrange.

When  $(p^*, \gamma^*)$  is a saddle point for which  $L_\beta(p^*, \gamma) \leq L_\beta(p^*, \gamma^*) \leq L_\beta(p, \gamma^*)$ , then  $p^*$  is a local minimum of  $E(p)$  satisfying (8.74) (Gottfried 1973). The dynamic equations for minimizing  $L_\beta$  are

$$\frac{dp_i(I)}{dt} = -\frac{\partial L_\beta(p, \gamma)}{\partial p_i(I)} = -\frac{\partial E(p)}{\partial p_i(I)} - \sum_k \gamma_k \frac{\partial C_k(p)}{\partial p_i(I)} - \beta \sum_k C_k(p) \frac{\partial C_k(p)}{\partial p_i(I)} \quad (8.78)$$

$$\frac{d\gamma_i}{dt} = +\frac{\partial L_\beta(p, \gamma)}{\partial \gamma_i} = +C_i(p) \quad (8.79)$$

where

$$\frac{\partial C_k(p)}{\partial p_i(I)} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \quad (8.80)$$

for  $C_i(p)$  defined in (8.74). The system performs energy descent on  $p$  but ascent on  $\gamma$ .

In the ALH method, the updating of label assignment is performed on  $u$ , rather than on  $p$  variables. The first part of Equation (8.78) is replaced by

$$\frac{du_i(I)}{dt} = -\frac{\partial L_\beta(p(u), \gamma)}{\partial p_i(I)} = -[q_i(I) + \gamma_i + \beta C_i(p)] \quad (8.81)$$

where  $q_i(I) = \frac{\partial E}{\partial p_i(I)}$ . Because  $\frac{\partial p_i(I)}{\partial u_i(I)} = \frac{e^{-u_i(I)/T}}{T(1+e^{-u_i(I)/T})^2}$  is always positive,  $\frac{dp_i(I)}{dt} = \frac{\partial p_i(I)}{\partial u_i(I)} \frac{du_i(I)}{dt}$  has the same sign as  $\frac{du_i(I)}{dt}$ .

Numerically, the ALH algorithm consists of the following three equations corresponding to (8.81), (8.76) and (8.79):

$$u_i^{(t+1)}(I) \leftarrow u_i^{(t)}(I) - \mu \left\{ q_i^{(t)}(I) + \gamma_i^{(t)} + \beta C_i(p^{(t)}) \right\} \quad (8.82)$$

$$p_i^{(t+1)}(I) \leftarrow \psi_T(u_i^{(t+1)}(I)) \quad (8.83)$$

$$\gamma_i^{(t+1)} \leftarrow \gamma_i^{(t)} + \beta C_i(p^{(t)}) \quad (8.84)$$

In the above,  $\mu$  is the step size; during the update,  $\beta$  is increased to speed up convergence. In practice,  $T$  needs not be very low to impose the ambiguity constraints on the final solution  $p^*$ . The competitive mechanism in (8.74) will make the winner take all. We set  $T = 10^5$  in our experiments and obtained good convergence. In our implementation, the updating is performed synchronously in parallel for all  $i$  and  $I$ .

To summarize: The advantages of the augmented Lagrange-Hopfield (ALH) method are due to the use of the augmented Lagrange method. The ALH method not only overcomes the ill-conditioning problem of the penalty method and the zigzagging problem of the standard Lagrangian but also improves convergence. With the augmented Lagrangian, the weights for the penalty terms can be much smaller than those required by the penalty method and the relative weight for the original objective can be increased. This helps yielding lower minimized energy values. The use of ALH in image restoration and segmentation will be demonstrated in Section 9.5.2. The ALH method as a general method has also been used for solving other combinatorial optimization problems. For example in solving traveling salesman problem, the ALH finds remarkably better solutions than Hopfield type neural networks (Li 1996a).

# Chapter 9

## Minimization – Global Methods

The minimal solution is usually defined as the global one or one of them when there are multiple global minima. Finding a global minimum is non-trivial if the energy function contains many local minima. Whereas methods for local minimization are quite mature with commercial software on market, the study of global minimization is still young. There are no efficient algorithms which guarantee to find globally minimal solutions as are there for local minimization.

In analytical studies, local and global minima may be studied via convexity analysis. Let  $E(f)$  be a real function defined on  $\mathbb{F} = \mathbb{R}^n$ .  $E(f)$  is said to be convex in  $\mathbb{F}$  if

$$E(\lambda x + (1 - \lambda)y) \leq \lambda E(x) + (1 - \lambda)E(y) \quad (9.1)$$

for any two points  $x, y \in \mathbb{F}$  and any real number  $\lambda \in [0, 1]$ . It is strictly convex if strict inequality “ $<$ ” holds in the above. When plotted, a convex  $E(f)$  will always lie below the line segment connecting any two points on itself. The neighboring points of a minimum are always above the tangent at the minimum. A local minimum is also a global one if  $E(f)$  is convex on  $\mathbb{F}$ . It is the unique global minimum if  $E(f)$  is strictly convex. Therefore, when  $E(f)$  is convex, we can find a global minimum by finding a local minimum using the gradient descent method.

Global minimization requires: (1) to find all (a finite number of) local minima and (2) to prove that there are no more local minima. Without efficient algorithm, this amounts to exhaustive search. In reality, one is always facing one of the two choices: (1) to find the exact global minimum with possibly intolerable expenses or (2) to find some approximations to it with much less cost.

Two methods may be used to deal with the local minimum problem: random search and annealing. In random search methods, a new configuration does not always make energy descent; occasional energy increase is allowed. The probability with which a configuration is generated is proportional to  $e^{-E(f)/T}$  where  $T$  is a control parameter called the temperature. Therefore, a lower energy configuration is generated with a larger probability.

Annealing is incorporated into a local search method to overcome the problem of local minima. It is performed by decreasing the temperature parameter in the Gibbs distribution, from a high value to a low value during the iterative minimization. At one extreme *e.g.* a high temperature, the energy landscape is convex and smooth and thus the unique minimum can be located easily; the minimum is tracked as temperature is gradually decreased to a sufficiently low value. This is so-called the continuation method (Wassermann 1973). Such a technique can significantly overcome the local minimum problem and improve the quality of the solution. A disadvantage is that they take more time since an annealing algorithm has to decrease the parameter gradually over a range of values and at each value some convergence has to be reached.

There are two types of annealing: deterministic and stochastic. In MRF vision work, the stochastic simulated annealing algorithm (Geman and Geman 1984) and the deterministic graduated non-convexity (GNC) algorithm of (Blake and Zisserman 1987) enjoy their popularity. Other deterministic algorithms include mean field annealing (Peterson and Soderberg 1989; Yullie 1990; Geiger and Girosi 1991) and the Hopfield network approach (Hopfield and Tank 1985; Koch et al. 1986; Yullie 1987).

While stochastic annealing such as simulated annealing is theoretically justified (Geman and Geman 1984), deterministic annealing remains heuristic. There is no guarantee that the minimum at high temperature can always be tracked to the minimum at low temperature.

## 9.1 Simulated Annealing

Simulated annealing (SA), introduced independently by (Cerny 1982; Cerny 1985) and (Kirkpatrick et al. 1982; Kirkpatrick et al. 1983), is a stochastic algorithm for combinatorial optimization. It simulates the physical annealing procedure in which a physical substance is melted and then slowly cooled down in search of a low energy configuration. Consider a system in which any  $f$  in the configuration space  $\mathbb{F}$  has probability

$$P_T(f) = [P(f)]^{1/T} \quad (9.2)$$

where  $T > 0$  is the temperature parameter. When  $T \rightarrow \infty$ ,  $P_T(f)$  is a uniform distribution on  $\mathbb{F}$ ; for  $T = 1$ ,  $P_T(f) = P(f)$ ; as  $T \rightarrow 0$ ,  $P_T(f)$  is concentrated on the peak(s) of  $P(f)$ . This gives an intuition of how the samples of  $f$  distribute in  $\mathbb{F}$ .

---

```

initialize  $T$  and  $f$ ;
repeat
    randomly sample  $f$  from  $\mathcal{N}(f)$  under  $T$ ;
    decrease  $T$ ;
until ( $T \rightarrow 0$ );
return  $f$ ;

```

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Figure 9.1: The simulated annealing algorithm.

The SA algorithm is described in Fig.9.1. SA applies a sampling algorithm, such as Metropolis algorithm (Metropolis et al. ) or Gibbs sampler (Geman and Geman 1984) (*cf.* Section 6.1.6), successively at decreasing values of temperature  $T$ . Initially,  $T$  is set very high and  $f$  is set to a random configuration. At a fixed  $T$ , the sampling is according to the Gibbs distribution  $P_T(f) = e^{-E(f)/T} / \sum_f e^{-E(f)/T}$ . After the sampling converges to the equilibrium at current  $T$ ,  $T$  is decreased according to a carefully chosen schedule. This continues until  $T$  is close to 0 when the system is “frozen” near the minimum of  $E(f)$ . The cooling schedule, specified by a decrement function and a final value, plays an important part; see below.

Two convergence theorems have been developed (Geman and Geman 1984). The first concerns the convergence of the Metropolis algorithm. It says that if every configuration is visited infinitely often, the distribution of generated configurations is guaranteed to converge to the Boltzmann (*i.e.* Gibbs) distribution. The second is about SA. It states that if the decreasing sequence of temperatures satisfy

$$\lim_{t \rightarrow \infty} T^{(t)} = 0 \quad (9.3)$$

and

$$T^{(t)} \geq \frac{m \times \Delta}{\ln(1 + t)} \quad (9.4)$$

where  $\Delta = \max_f E(f) - \min_f E(f)$ , then the system converges to the global minimum regardless of the initial configuration  $f^{(0)}$ . Note that the above conditions are sufficient but not necessary for the convergence.

Unfortunately, the schedule (9.4) is too slow to be of practical use. In practice, heuristic, faster schedules have to be used instead. Geman and Geman (1984) adopt the following

$$T^{(t)} = \frac{C}{\ln(1 + t)} \quad (9.5)$$

where the constant  $C$  is set  $C = 3.0$  or  $C = 0.4$  for their problem. Kirkpatrick *et al.* (1983) choose

$$T^{(t)} = \kappa T^{(t-1)} \quad (9.6)$$

where  $\kappa$  typically takes a value between 0.8 and 0.99. The initial temperature is set high enough so that essentially all configuration changes are accepted. At each temperature, enough configurations are tried that either there are  $10m$  (10 times the number of sites) accepted configuration transitions or the number of tries exceeds  $100m$ . The system is frozen and annealing stops if the desired number of acceptances is not achieved at three successive temperatures. In (Kato *et al.* 1993a), a multi-temperature annealing scheme is proposed for annealing in multi-resolution computation; there, the temperature in the Gibbs distribution is related not only to the time but also to the scale.

The two sampling procedures, *i.e.* Metropolis algorithm and Gibbs sampler, are proven to be asymptotically equivalent for SA performed on lattices, but this is not generally true for non-lattice structures (Chiang and Chow 1992). The interested reader is referred to (Kirkpatrick *et al.* 1982; Geman and Geman 1984; Aarts 1989) for devoted discussions on SA.

A performance comparison between SA and GNC for edge preverving surface reconstruction is given in (Blake 1989). Based on the results of the experiments under controlled conditions, Blake argues that GNC excels SA both in computational efficiency and problem-solving power. An experimental comparison between SA and deterministic RL algorithms for combinatorial optimization will be presented in Section 9.5.1.

## 9.2 Mean Field Annealing

Mean field annealing (Peterson and Soderberg 1989) provides another approach for solving combinatorial optimization problems using real computation. It can be considered as a special RL algorithm incorporating an annealing process. Let  $f = \{f_i(I) \in \{0, 1\} \mid i \in \mathcal{S}, I \in \mathcal{L}\}$  be an unambiguous labeling assignment in  $\mathbb{P}^*$  defined in (8.24); each  $f_i$  takes the value of one of the vectors  $(1, 0, \dots, 0)$ ,  $(0, 1, \dots, 0)$ , and  $(0, 0, \dots, 1)$ . Denote the energy with the RL representation still by  $E(f)$ . The Gibbs distribution of the energy under temperature  $T > 0$  is given as

$$P_T(f) = Z^{-1} e^{-\frac{1}{T} E(f)} \quad (9.7)$$

where the partition function

$$Z = \sum_{f \in \mathbb{P}^*} e^{-\frac{1}{T} E(f)} \quad (9.8)$$

sums over  $\mathbb{P}^*$ . The statistical mean of the distribution is defined as

$$\langle f \rangle_T = \sum_f f_i P_T(f) = \sum_f f_i Z^{-1} e^{-\frac{1}{T} E(f)} \quad (9.9)$$

As the temperature approaches zero, the mean of the Gibbs distribution approaches its mode or the global energy minimum  $f^*$

$$\lim_{T \rightarrow 0} \langle f \rangle_T = \lim_{T \rightarrow 0} \sum_f f P_T(f) = f^* \quad (9.10)$$

This suggests that instead of minimizing  $E(f)$  directly, we could try to evaluate the mean field  $\langle f \rangle_T$  at a sufficiently high temperature and then track it down using the continuation method (Wasserstrom 1973) as the temperature is lowered towards zero.

The analysis of the partition function is central in mean field theory because once it is calculated, all statistic information about the system can be deduced from it. The mean field trick for calculating the partition function is to (1) rewrite the sum over a discrete space as an integral over a pair of continuous variables  $p$  and  $q$ ,  $p = \{p_i(I) \in \mathbb{R} \mid i \in \mathcal{S}, I \in \mathcal{L}\}$  and  $q = \{q_i(I) \in \mathbb{R} \mid i \in \mathcal{S}, I \in \mathcal{L}\}$  having the same dimensionality as  $f$ , and then (2) evaluate its integrand at the saddle point (saddle point approximation). As can be seen later,  $p$  and  $q$  corresponds to a labeling assignment as defined in (8.20) and a gradient as defined in (8.29), respectively. In the following,  $f$ ,  $p$  and  $q$  are considered as  $m \times M$  matrices. To do step (1), we begin by rewriting any function  $g(f)$  as the following integral

$$g(f) = \int_{\mathbb{D}_R} g(p) \delta(f - p) dp \quad (9.11)$$

where  $\delta$  is the Dirac delta function and  $\mathbb{D}_R$  is the  $|\mathcal{S}| \times |\mathcal{L}|$  dimensional real space. The following definition of the Dirac delta function is used in the above

$$\delta(q) = C \int_{\mathbb{D}_I} e^{p^T q} dp dq \quad (9.12)$$

where  $T$  denotes transpose,  $\mathbb{D}_I$  is the  $|\mathcal{S}| \times |\mathcal{L}|$  dimensional imaginary space and  $C$  is a normalizing constant. Therefore,

$$g(f) = C \int_{\mathbb{D}_R} \int_{\mathbb{D}_I} g(p) e^{(f^T q - p^T q)} dp dq \quad (9.13)$$

Using the above formula, we can rewrite the partition function, which is the sum of the function  $e^{-\frac{1}{T} E(p)}$  as

$$\begin{aligned} Z &= \sum_{f \in \mathbb{P}^*} C \int_{\mathbb{D}_R} \int_{\mathbb{D}_I} e^{-\frac{1}{T} E(p)} e^{(f^T q - p^T q)} dp dq \\ &= C \int_{\mathbb{D}_R} \int_{\mathbb{D}_I} e^{-\frac{1}{T} E(p)} e^{(-p^T q)} \left[ \sum_{f \in \mathbb{P}^*} e^{(f^T q)} \right] dp dq \end{aligned} \quad (9.14)$$

The summation inside the integrand can be written as

$$\sum_{f \in \mathbb{P}^*} e^{(f^T q)} = \prod_{i \in \mathcal{S}} \sum_{I \in \mathcal{L}} e^{q_i(I)} \quad (9.15)$$

Therefore,

$$Z = C \int_{\mathbb{D}_R} \int_{\mathbb{D}_r} e^{-\frac{1}{T} E_{eff}(p, q)} dp dq \quad (9.16)$$

where

$$E_{eff}(p, q) = E(p) + T p^T q - T \sum_{i \in \mathcal{S}} \ln \sum_{I \in \mathcal{L}} (e^{q_i(I)}) \quad (9.17)$$

is called the *effective energy*.

The above integral expression of  $Z$  is exact but too complicated for precise calculation. The calculation may be approximated based on the following heuristic argument: The double integral is dominated by saddle points in the integration intervals and therefore the integral is approximated by

$$Z \approx e^{-\frac{1}{T} E_{eff}(p^*, q^*)} \quad (9.18)$$

where  $(p^*, q^*)$  is a saddle point of  $E_{eff}$ . The saddle points are among the roots of the equations

$$\nabla_p E_{eff}(p, q) = 0 \quad \& \quad \nabla_q E_{eff}(p, q) = 0 \quad (9.19)$$

where  $\nabla_p$  is the gradient with respect to  $p$ . This yields the mean field theory equations

$$q_i(I) = -\frac{1}{T} \frac{\partial E(p)}{\partial p_i(I)} \quad \& \quad p_i(I) = \frac{e^{q_i(I)}}{\sum_J e^{q_i(J)}} \quad (9.20)$$

The  $p$  matrix represents a labeling assignment in the feasibility space defined by (8.19). The mean field theory equations for combinatorial minimization have been derived in many places *e.g.* in (Yullie 1990; Simic 1990; Elfadel 1993; Haykin 1994; Yang and Kittler 1994).

Note that in terms of the RL representation, the energy can be written as  $E(f) = Const - G(f)$  and so  $q$  can be computed as

$$q_i(I) = \frac{1}{T} [r_i(I) + 2 \sum_{i' \in \mathcal{S}} \sum_{I' \in \mathcal{L}} r_{i,i'}(I, I') p_{i'}(I')] \quad (9.21)$$

(*cf.* 8.29). Now we obtain the following fixed-point iteration

$$p_i^{(t+1)}(I) \leftarrow \frac{e^{q_i^{(t)}(I)}}{\sum_J e^{q_i^{(t)}(J)}} \quad (9.22)$$

In iterative computation, the temperature  $T$  in  $q$  is decreased towards  $0^+$  as iteration goes. The unambiguity of  $p$  is achieved when  $T \rightarrow 0^+$ . In (Peterson and Soderberg 1989), the convergence is judged by a quantity defined as

$$S = \frac{1}{m} \sum_I p_i^2(I) \quad (9.23)$$

also called the “saturation” of  $f$ . The iteration (9.22) is considered converged if  $S > 0.99$ . Analog network implementation schemes for mean field algorithms are investigated in (Elfadel 1993).

It is worth pointing out that the idea of mean field annealing for global optimization has been proposed earlier by (Pinkus 1968) as the integral limit method. There, the global minimum is expressed in closed form as a limit of an integral. Assume that  $\mathbb{F}$  is the closure of a bounded domain in  $\mathbb{R}^m$ . Suppose that  $E(f)$  is continuous on  $\mathbb{F}$  and has a unique global minimum  $f^* = \{f_1^*, \dots, f_m^*\} \in \mathbb{F}$ . Pinkus shows that the following integral

$$\bar{f}_i = \frac{\int_{\mathbb{F}} f_i e^{-\frac{1}{T} E(f)} df}{\int_{\mathbb{F}} e^{-\frac{1}{T} E(f)} df} \quad i = 1, \dots, m \quad (9.24)$$

approaches to the global minimum  $f^*$  as  $T \rightarrow 0^+$ . Obviously, the above integral is the continuous version of the mean field defined in (9.9) and the idea of the integral limit is similar to that of the mean field annealing. Theoretical analysis shows that this approximation has rather good asymptotic properties but its realization may need some heuristic trials. Computationally, the values  $\bar{f}_i$  can be obtained analytically only in exceptional cases and a numerical method has to be used in general.

## 9.3 Graduated Non-Convexity

Graduated non-convexity (GNC) (Blake and Zisserman 1987) is a deterministic annealing method for approximating a global solution for nonconvex minimization of unconstrained, continuous problems such as (8.5). It finds good solutions with much less cost than stochastic simulated annealing. Energies of the following form will be considered

$$E(f) = \sum_{i=1}^m \chi_i (f_i - d_i)^2 + \lambda \sum_{i=1}^m \sum_{i' \in \mathcal{N}_i} g_\gamma(f_i - f_{i'}) \quad (9.25)$$

in the subsequent discussion of GNC. When  $g$  is a non-convex function, a gradient-based method such as (8.7) finds only a local minimum.

### 9.3.1 GNC Algorithm

The idea of GNC is the following: Initially,  $\gamma$  is set to a sufficiently large value  $\gamma^{(0)}$  such that  $E(f | \gamma^{(0)})$  is strictly convex. It is easy to find the unique minimum of  $E(f | \gamma^{(0)})$  regardless of the initial  $f$  using the gradient descent method. The minimum  $f_{\gamma^{(0)}}$  found under  $\gamma^{(0)}$  is then used as the initial

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Choose a convex  $\gamma^{(0)}$ ; Set  $f^{(0)} \leftarrow d$  and  $t = 0$ ;
Do 1) Update  $f^{(t)}$  using Eq.(8.7);
    2) Set  $t \leftarrow t + 1$ ;
    3) If  $(f^{(t)} - f^{(t-1)} < \epsilon)$ 
        set  $\gamma^{(t)} \leftarrow \max\{\gamma_{target}, \kappa\gamma^{(t-1)}\}$ ;
Until  $(f^{(t)} - f^{(t-1)} < \epsilon)$  and  $(\gamma^{(t)} = \gamma_{target})$ ;
Set  $f^* \leftarrow f^{(t)}$ 

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Figure 9.2: A GNC algorithm for finding the DA solution.

value for the next phase of minimization under a lower value  $\gamma^{(1)} < \gamma^{(0)}$  to obtain the next minimum  $f_{\gamma^{(1)}}^*$ . As  $\gamma$  is lowered,  $E(f | \gamma)$  becomes non-convex and local minima appear. However, if we track the sequence of minima as  $\gamma$  decreases from the high value  $\gamma^{(0)}$  to the target value  $\gamma$ , we may approximate the global minimum  $f_\gamma^*$  under the target value  $\gamma$ , *i.e.* the global minimum  $f^*$  of the target energy  $E(f)$ .

The first term on the RHS of (9.25) is quadratic and hence strictly convex with respect to  $f$ . The second term may or may not be convex, depending on  $f$  and  $g$ . Its non-convexity could be partly compensated by the convexity of the closeness term. If  $g(\eta) = g_g(\eta) = \eta^2$ , then the second term is strictly convex and so is  $E(f)$ . If  $g(\eta) = g_\alpha(\eta) = \min\{\eta^2, \alpha\}$  (*cf.* Eq.(4.10)), the second term is nonconvex and so is  $E(f)$ . However if the function  $g$  together with the involved parameters are chosen to satisfy

$$g''(f_i - f_{i-1}) \geq -c^* \quad \forall i \quad (9.26)$$

where  $c^* > 0$  is some constant, then the convexity of  $E(f)$  is guaranteed. The value of  $c^*$  depends on the type of the model and on whether the data  $d$  is complete. Provided  $d_i$  are available for all  $i$ , then the value is  $c^* = 1/2$  for string,  $1/4$  for membrane,  $1/8$  for rod and  $1/32$  for plate, respectively. See (Blake and Zisserman 1987). If  $d_i$  are missing altogether, it requires  $g''(f_i - f_{i-1}) > 0$ , so that the second term is convex by itself, to ensure the convexity of  $E(f)$ . The practical situation is generally better than this worst case because the data cannot be missing altogether.

Therefore, to construct an initially convex  $g_\gamma$ , we can choose an initial parameter  $\gamma = \gamma^{(0)}$  for which  $g_{\gamma^{(0)}}''(f_i^{(0)} - f_{i-1}^{(0)}) \leq 0$  for all  $i$ . This is equivalent to choosing a  $\gamma^{(0)}$  such that  $f_i^{(0)} - f_{i-1}^{(0)} \in B_{\gamma^{(0)}}$  where  $B_\gamma$  is the band introduced in the section. For APFs 1, 2 and 3, the  $\gamma^{(0)}$  must be larger than  $2v$ ,  $3v$  and  $v$ , respectively, where  $v = \max_i [f_i^{(0)} - f_{i-1}^{(0)}]^2$ .

The GNC algorithm is outlined in Fig.9.2. Given  $d$ ,  $\lambda$  and a target value  $\gamma_{target}$  for  $\gamma$ , the algorithm aims to construct a sequence  $\{\gamma^{(t)}\}$  ( $\gamma^{(\infty)} \rightarrow$

$\gamma_{target}$ ) and thus  $\{f_{\gamma^{(t)}}^{(t)}\}$  to approach the global minimum  $f^* = \lim_{t \rightarrow \infty} f_{\gamma^{(t)}}^{(t)}$  for which  $E(f^*) = \min$ . In the algorithm,  $\epsilon$  is a constant for judging the convergence, and  $\kappa$  is a factor for decreasing  $\gamma^{(t)}$  towards  $\gamma_{target}$ ,  $0 < \kappa < 1$ . The choice of  $\kappa$  controls the balance between the quality of the solution and the computational time. In principle,  $\gamma^{(t)}$  should vary continuously to keep a good track of the global minimum. In discrete computation, we choose  $0.9 \leq \kappa \leq 0.99$ . More rapid decrease of  $\gamma^{(t)}$  (with smaller  $\kappa$ ) is likely to lead the system to an unfavorable local minimum. Witkin *et al.* (1987) present a more sophisticated scheme for decreasing  $\gamma$  by relating the step to the energy change:  $\gamma^{(t+1)} = \gamma^{(t)} - c_1 e^{-c_2 |\nabla E|}$  where  $c_1$  and  $c_2$  are constants. This seems reasonable.

To approximate the truncated quadratic function  $\lambda g_\alpha(\eta) = \lambda \min\{\eta^2, \alpha\}$ , Blake and Zisserman (1987) construct the following function

$$\lambda g_{\alpha, \lambda}^{(p)}(\eta) = \begin{cases} \lambda \eta^2 & |\eta| < q \\ \alpha - c(|\eta| - r)^2/2 & q \leq |\eta| < r \\ \alpha & \text{otherwise} \end{cases} \quad (9.27)$$

where  $p \in [0, 1]$  is a continuation parameter,  $c = c^*/p$ ,  $r^2 = \alpha(2/c + 1/\lambda)$  and  $q = \alpha/(\lambda r)$ . The corresponding interaction function is

$$\lambda h_\alpha^{(p)}(\eta) = \begin{cases} \lambda \eta & |\eta| < q \\ -c(|\eta| - r) \text{sign}(\eta) & q \leq |\eta| < r \\ 0 & \text{otherwise} \end{cases} \quad (9.28)$$

The continuation parameter  $p$  is decreased from 1 to 0. When  $p = 1$ , the corresponding  $E_\alpha^{(p=1)}$  is convex where  $E_\alpha$  is the energy with  $g_\alpha$ . When  $p = 0$ ,  $E_\alpha^{(p=0)} = E_\alpha$  equals to the original energy. A performance comparison between GNC and SA for reconstruction is given in (Blake 1989).

However, the complexity of the above convexity treatment can be avoided. Instead of modifying the definition of  $g_\alpha$ , we can simply modify the parameter  $\alpha$  in it. To achieve graduated non-convexity, we can use  $\alpha$  as the control parameter and decrease it from a sufficiently high value,  $\alpha^{(0)} > (\max |d_i - d_{i-1}|)^2$ , towards the target value. Let  $f$  be initialized so that  $\max |f_i^{(0)} - f_{i-1}^{(0)}| \leq \max |d_i - d_{i-1}|$ . Then the energy is convex at such  $\alpha^{(0)}$ . An assumption is that if so initialized,  $\max |f_i^{(t)} - f_{i-1}^{(t)}| < \max |d_i - d_{i-1}|$  always holds at any  $t$  during the iteration. Parameter values for the convexity in this way are related to the band  $B$  defined in (4.29) for the DA (discontinuity-adaptive) model. In a similar way, parameter values for the convexity can be derived for the LP (line process) approximation models given by (Koch *et al.* 1986; Yuille 1987; Geiger and Girosi 1991).

The computation can be performed using an analog network. Let  $f_i$  be the potential of neural cell  $i$ . Let  $C_i = C = 1/2\mu$  be the membrane capacitance and  $R_i = 1/\chi_i$  the membrane resistance. Let

$$T_{i,i'} = T_{i',i} = \lambda h_\gamma(f_i - f_{i'}) \quad (9.29)$$

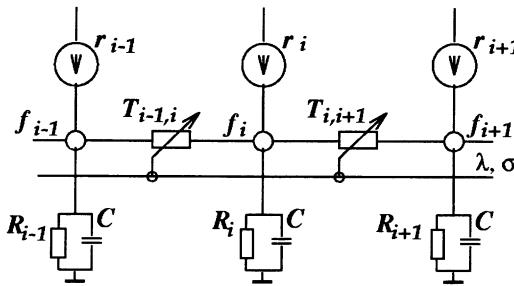


Figure 9.3: Schematic diagram of the analog network circuit.

be the conductance or synaptic efficacy between neurons  $i$  and  $i'$  where  $|i - i'| = 1$  and  $h_\gamma \in \mathbb{H}_\gamma$ . If the exponential  $g_{1\gamma}$  is used, then

$$T_{i,i'} = T_{i',i} = \lambda \exp\{-[f_i - f_{i'}]^2/\gamma\} \quad (9.30)$$

Let  $d_i$  be the external current input to  $i$ , with  $d_i = 0$  when  $\chi_i = 0$ . Now (8.7) can be written as

$$C \frac{\partial f_i}{\partial t} = -\frac{1}{R_i} f_i + d_i + T_{i-1,i}[f_{i-1} - f_i] + T_{i+1,i}[f_{i+1} - f_i] \quad (9.31)$$

The above is the dynamic equation at neuron  $i$  of the network. The diagram of the network circuit is shown in Fig.9.3. The synaptic current from  $i$  to  $i'$  is

$$I_{i,i'} = \lambda g'_\gamma(f_i - f_{i'}) = \lambda[f_i - f_{i'}]h_\gamma(f_i - f_{i'}) \quad (9.32)$$

If the exponential  $g_{1\gamma}$  is used, then

$$I_{i,i'} = \lambda[f_i - f_{i'}] \exp\{-[f_i - f_{i'}]^2/\gamma\} \quad (9.33)$$

A plot of current  $I_{i,i'}$  versus potential difference  $f_i - f_{i'}$  was shown at the bottom of Fig.4.2. The voltage-controlled nonlinear synaptic conductance  $T_{i,i'}$ , characterized by the  $h$  function in (4.27), realizes the adaptive continuity control; the corresponding nonlinear current  $I_{i,i'}$  realizes the adaptive smoothing. The current  $I_{i,i'}$  diminishes asymptotically to zero as the potential difference between neurons  $i$  and  $i'$  reaches far beyond the band  $B_\gamma$ .

In comparison of the DA and the LP models, the former is more suitable for VLSI implementation than the latter. This is because the continuous shape of the adaptive conductance  $T_{i,i'} = \lambda h_\gamma(f_i - f_{i'})$  in the DA is easier to implement using analog circuits than the piecewise  $\lambda h_\alpha$  in the LP. This advantage of the DA model is also reflected in the work by Harris *et al.* (Harris *et al.* 1990) and (Lumsdaine *et al.* 1990). There, the piecewise current-voltage characteristics  $I = \Delta V h_\alpha(\Delta V)$  is implemented by a resistive fuse. Interestingly, the resistive fuse is of *finite gain* and thus does not give rise

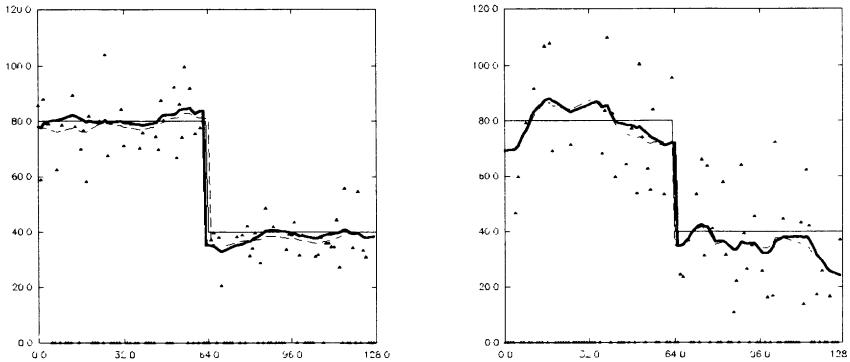


Figure 9.4: Stability of the DA solution under disturbances in parameters.

to a sharp switch-off as described by the piecewise  $h_\alpha$ . The actual current-voltage characteristics of the resistive fuse is more like  $I = \Delta V h_\gamma(\Delta V)$  than  $I = \Delta V h_\alpha(\Delta V)$ . It seems to satisfy all the requirements in (4.27) except for the  $C^1$  continuity. The definition of  $\mathbb{H}_\gamma$  offers guidelines for the DA circuit design.

Fig. 9.4 shows the behavior of the analog DA network under component defects such as manufacturing inadequacy, quality changes, *etc.* The defects are simulated by adding  $\pm 25\%$  evenly distributed random noise into  $R$ ,  $C$ , and  $T$  in (9.31). The data  $d$  is shown in triangles with 50% missing rate; the locations of the missing data, for which  $\chi_i = 0$ , are indicated by triangles at the bottom. The noise in the data is white Gaussian with standard deviation  $\sigma = 10$  (left) and  $\sigma = 20$  (right). The interaction function is chosen to be  $h_{2\gamma}(\eta) = \frac{1}{[1+\eta^2/\gamma^2]}$ . Solutions obtained with simulated component defects are shown in dashed lines in comparison with those obtained without such noise shown in thicker solid lines. The ideal signal is shown in thinner solid lines. As can be seen, there is only a little difference between the solutions obtained with and without such noise. This demonstrates not only the stability of the network circuit but also the error-tolerance property of the DA model.

### 9.3.2 Annealing Labeling for MAP-MRF Matching

Here we describe a special GNC algorithm, called annealing labeling (Li et al. 1994), which may be used to improve the optimization of the posterior energy (3.18) formulated for the MAP matching.

We first do an energy-gain conversion (*cf.* Section 8.2.2). Substituting (3.11), (3.12), (3.16) and (3.17) into (3.18), we have

$$U(f \mid d) = \{v_{10}N_1 + v_{20}N_2\} + \sum_{i \in \mathcal{S}: f_i \neq 0} V_1(d_1(i) \mid f_i) + \sum_{i \in \mathcal{S}: f_i \neq 0} \sum_{i' \in \mathcal{S} - \{i\}: f_{i'} \neq 0} V_2(d_2(i, i') \mid f_i, f_{i'}) \quad (9.34)$$

where  $N_1 = \#\{f_i = 0 \mid i \in \mathcal{S}\}$  is the number of NULL labels in  $f$ , and  $N_2 = \#\{f_i = 0 \text{ or } f_{i'} = 0 \mid i \in \mathcal{S}, i' \in \mathcal{N}_i\}$  is the number of label pairs at least one of which is NULL. The constants  $v_{10}$  and  $v_{20}$  control the number of sites assigned the NULL label. The smaller they are, the more sites will be assigned the NULL. The corresponding gain function is

$$G(f) = \{g_{10}N_1 + g_{20}N_2\} + \sum_{i \in \mathcal{S}: f_i \neq 0} r_i(f_i) + \sum_{i \in \mathcal{S}: f_i \neq 0} \sum_{i' \in \mathcal{S} - \{i\}: f_{i'} \neq 0} r_{i, i'}(f_i, f_{i'}) \quad (9.35)$$

In the above,  $g_{10} = Const_1 - v_{10}$ ,  $g_{20} = Const_2 - v_{20}$ , and  $r_i(f_i)$  and  $r_{i, i'}(f_i, f_{i'})$  are related to  $V_1(d_1(i) \mid f_i)$  and  $V_2(d_2(i, i') \mid f_i, f_{i'})$  by (8.26) and (8.27). These determines the compatibilities in RL. The gain is to be maximized.

The parameters  $g_{10}$  and  $g_{20}$  not only affect the NULL labels but also the local behavior of the maximization algorithm. When both are zero, there will be no NULL labels. The larger their values, the more sites will be assigned the NULL and the more serious the problem of local maxima becomes. In other words, we found that a labeling  $f$ , in which  $f_i = 0$  for some  $i$ , is likely a local optimum and that the labeling  $f$ , in which  $f_i = 0$  for all  $i$ , is a deep local optimum. This has motivated us to incorporate a heuristic annealing procedure into the labeling process, in a way similar to the graduated non-convexity (GNC) algorithm (Blake and Zisserman 1987), to overcome the local optimum problem.

Introduce a temperature parameter into the gain

$$G(f \mid T) = \{g_{10}N_1 + g_{20}N_2\} / T + \sum_{i \in \mathcal{S}: f_i \neq 0} r_i(f_i) + \sum_{i \in \mathcal{S}: f_i \neq 0} \sum_{i' \in \mathcal{S} - \{i\}: f_{i'} \neq 0} r_{i, i'}(f_i, f_{i'}) \quad (9.36)$$

with  $G(f) = \lim_{T \rightarrow 1} G(f \mid T)$ .  $T$  is initially set to a very high value  $T^{(0)} \rightarrow \infty$  and gradually lowered towards  $T^{(\infty)} = 1$ . The maximum  $f^*$  obtained at  $T^{(t-1)}$  is used as the initial value for the next new phase of maximization at  $T^{(t)}$ . In this way, the  $f^*$  are tracked from high  $T$  to low  $T$ . Because  $g_{10}$  and  $g_{20}$  are weighted by  $\frac{1}{T}$ , the optimum  $f^*$  obtained at high  $T$  is less affected by the local optimum problem; when it is tracked down, a better quality solution may be obtained than one found by a non-annealing algorithm. The improvement of annealing labeling ICM, a procedure incorporating annealing labeling into ICM, over the simple ICM will be shown shortly.

Note that the  $T$  in the annealing labeling acts on only the prior part whereas in SA,  $T$  acts on both the prior and likelihood parts. So the present annealing procedure is more like GNC (Blake and Zisserman 1987) than SA (Geman and Geman 1984).

---

```

generate initial population  $P = \{f^1, \dots, f^n\}$ ;
compute  $E(f) \forall f \in P$ ;
repeat
  select two individuals from  $P$ ;
  recombine the individuals to produce two offsprings;
  compute  $E(f)$  of the offsprings;
  update  $P$  with the offsprings;
until (converged);
return  $f = \arg \min_{f \in P} E(f)$ ;

```

---

Figure 9.5: A standard genetic algorithm.

## 9.4 Genetic Algorithms

*Genetic algorithms* (GAs) (Holland 1975; Goldberg 1989) are another class of heuristic procedures for global optimization. Inspired by the principle of natural evolution in the biological world, the procedures simulate the evolutionary process in which a population of individuals who have highest goodness-of-fit values survive.

### 9.4.1 Standard GA

Fig.9.5 illustrates a standard genetic algorithm for unconstrained, combinatorial optimization<sup>1</sup>. Initially, a number of  $n$  (say,  $n = 50$ ) individuals are *generated*, yielding a population  $P$ . An individual is determined by his chromosome  $f$  which is a string of  $m$  genes (labels)  $f_i$  ( $i = 1, \dots, m$ ).

At a point of the evolution, two individuals are randomly *selected* for mating. The selection is done according to a scheme which favors the fitter individuals (with lower  $E$  value); the lower the value of  $E(f)$ , the more likely  $f$  will be selected. For example,  $f$  may be selected with a probability proportional to  $1 - \frac{E(f)}{\sum_{f \in P} E(f)}$  or a monotonically increasing function of it, assuming  $E(f)$  is non-negative. There are numerous selection schemes (Goldberg 1989).

*Recombination* takes place between the two selected individuals. The mechanisms of *crossover* and *mutation* are typically used for this. Fig.9.6 illustrates such two basic operations in which a gene can take any alphanumeric value. Crossover takes two individuals, cuts their chromosome strings at some randomly chosen position, recombines the opposite segments to create two offsprings. Crossover is not always invoked; it is applied with a proba-

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<sup>1</sup>Special coding of solutions into chromosomes needs to be done when GA is used for (piecewise) continuous optimization problems.

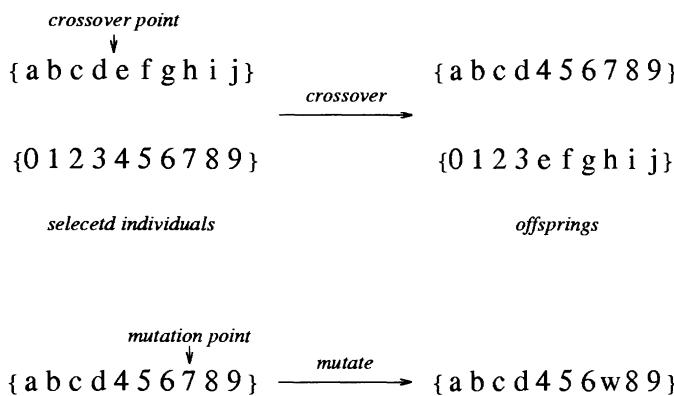


Figure 9.6: The two basic operations in recombination: crossover and mutation.

bility  $p_c$  typically between 0.6 and 1. If it is not applied, the offsprings are simply the duplicates of the selected individuals. Mutation is applied to each offspring after the crossover. It randomly alters a randomly chosen gene with a small probability  $p_m$ , typically 0.001. As usual in GA practice, there are many crossover and mutation operators.

The offsprings are then added to  $P$  and the two least fit individuals, *i.e.* those with the highest  $E(f)$  values, are removed from the population  $P$ . As the evolution continues in this way, the fitness of the best individual as well as the average fitness increases towards the global optimum. *Convergence* is judged by uniformity: A gene is said to have converged when most (say 95%) of the population share the same value. The population is said to have converged when all the genes have converged. The convergence of GAs is obtained but no theoretical proof is given.

Impressive empirical results for solving real and combinatorial, unconstrained and constrained optimization problems, such as traveling salesman problem, neural network optimization and scheduling, are reported (see (Goldberg 1989) for a review). Applications in computer vision are also seen (Bhanu et al. 1989; Ankenbrandt et al. 1990; Hill and Taylor 1992). Currently, there is no theory which explains why GAs work. However, some hypotheses exist. Of these are the schema theorem (Holland 1975) and building block hypothesis (Goldberg 1989).

### 9.4.2 Hybrid GA: Comb Algorithm

Combining local search with a GA yields a hybrid GA also called memetic algorithm (Moscato 1989; Radcliffe and Surry 1994). Here, a new random search method, called the Comb method, is described for combinatorial optimization. Assume that an energy function has been given which in the paper is formulated based on the MRF theory for image restoration and segmentation. The Comb method maintains a number of best local minima found so far, as a population based method. It uses common structure of the local minima to infer the structure of the global minimum. In every iteration, it derives one or two new initial configurations based on the Common structure (common labels) of the Best local minima (hence “Comb”): if two local minima have the same label (pixel value) in a pixel location, the label is copied to the corresponding location in the new configuration, otherwise a label randomly chosen from either local minimum is set to it. The configuration thus derived contains about the same percentage of common labels as the two local minima (assuming the two have about the same percentage of common labels). But the derived configuration is no longer a local minimum, and thus further improvement is possible. The new local minimum then updates the existing best ones. This process is repeated until some termination conditions are satisfied.

The resulting Comb algorithm is equivalent to a GA hybridized with steepest descent, in which the Comb initialization therein works like a uniform crossover operator. There have been various interpretations for the crossover operation. The idea of encouraging common structures in the Comb initialization provides a new perspective for interpreting the crossover operation in GA.

Experiment results in both image restoration and segmentation are provided to compare the Comb method with the ICM, HCF (Chou et al. 1993) and SA. The results show that the Comb yields better quality solutions than the ICM and HCF and comparable to SA.

The Comb method maintains a number of  $N$  best local minima found so far, denoted  $F = \{f^{[1]}, \dots, f^{[N]}\}$ , as a population based method. In every iteration, it derives a new initial configuration from  $F$ , and perform steepest descent using the derived initial configurations. If the found local minimum is better than an existing one in  $F$ , it replaces it.

Ideally, we desire that all local minima in  $F$  converge towards the global minimum  $f_i^{[global]}$ , in which case, there must be

$$f_i^{[n]} = f_i^{[global]} \quad 1 \leq n \leq N \quad (9.37)$$

for all  $i \in \mathcal{S}$ . We call  $f_i^{[global]}$  the *minimal label* at  $i$ . To achieve (9.37), all the labels at  $i$ ,  $\{f_i^{[n]} \mid \forall n\}$ , should finally converge to the minimal label  $f_i^{[global]}$ . The Comb is performed towards this objective.

The following heuristic is the basis for deriving new initial configurations: Although  $f^{[1]}, \dots, f^{[N]}$  are local minima, they share some structure with the

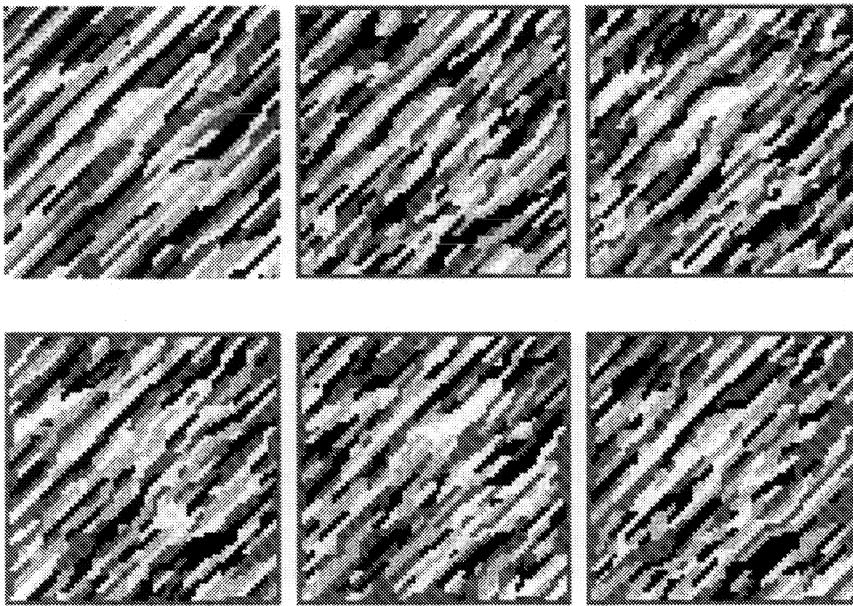


Figure 9.7: The global minimum (upper-left) and 5 local minima. The local minima share some structure with the global minimum.

global minimum  $f^{[global]}$ . More specifically, some local minima  $f^{[n]}$  have the minimal label  $f_i^{[n]} = f_i^{[global]}$  for some  $i \in \mathcal{S}$ . Fig.9.7 shows the (approximate) global minimum for an MAP-MRF restoration problem and some local minima found by using the multi-start method with initially random configurations. A statistic over a number of  $N = 10$  local minima is made to see how many minimal labels they have. Table 9.1 shows the statistic in terms of the percentile of the sites (pixels)  $i \in \mathcal{S}$  at which at least  $k$  local minima  $f^{[n]}$  have the minimal label  $f_i^{[n]} = f_i^{[global]}$ .

The Comb initialization is aimed to derive configurations having a substantial number of minimal labels so as to improve  $F$  towards the objective of (9.37). Although a configuration with a larger number of minimal labels does not necessarily have a lower energy value, we hope that it provides a good basis to start with, *i.e.* it can serve as a good initial configuration.

The Comb algorithm is described in Fig.9.8. The initialization at the beginning of the Comb algorithm is done according to a uniform distribution as the multi-start method. This is followed by iterations of four steps. Firstly, two local minima in  $F$ ,  $f^{[a]}$  and  $f^{[b]}$ , ( $a \neq b$ ) are randomly selected according to a uniform distribution. Secondly, a new initial configuration  $f^{[0]}$  is derived from  $f^{[a]}$  and  $f^{[b]}$  using the standard Comb initialization, which will be ex-

$k$	0	1	2	3	4	5	6	7	8	9	10
%	100	98	95	87	75	60	43	28	16	7	2

Table 9.1: Percentile (rounded-up to integers) of the sites (pixels)  $i \in \mathcal{S}$  at which *at least*  $k$  local minima  $f^{[n]}$  have the same label  $f_i^{[n]} = f_i^{[global]}$  as the global minimum  $f^{[global]}$ .

---

```

comb_initialization( $f^{[a]}, f^{[b]}, F$ )
begin
  for (each  $i \in \mathcal{S}$ ) do:
    if ( $f_i^{[a]} == f_i^{[b]}$  &&  $rand[0, 1] < 1 - \tau$ )
      then  $f_i^{[0]} = f_i^{[a]}$ ;
    else
       $f_i^{[0]} = rand(\mathcal{L})$ ;
end

```

---

```

Comb_Algorithm
begin
  initialize the set  $F = \{f^{[1]}, \dots, f^{[N]}\}$ ;
  do {
    random_selection( $f^{[a]}, f^{[b]}, F$ );
    comb_initialization( $f^{[0]}, f^{[a]}, f^{[b]}$ );
    steepest_descent( $f^*$ ,  $f^{[0]}$ );
    update( $F, f^*$ );
  } until (termination condition satisfied);
  return( $\arg \min_{f \in F} E(f)$ );
end

```

---

Figure 9.8: The Comb algorithm.

plained shortly. Then, steepest descent is applied to  $f^{[0]}$  to produce a local minimum  $f^*$ . Finally, the set  $F$  is updated by  $f^*$ : If  $E(f^*) < \max_{f \in F} E(f)$ , then the configuration  $\arg \max\{f \mid f \in F\}$  which has the highest energy value, higher than  $E(f^*)$ , is replaced by  $f^*$ . The termination condition may be that all configurations in  $F$  are the same or that a certain number of local minima have been performed. The algorithm returns the best local minimum in  $F$ , *i.e.* the one having the lowest energy.

---

```

GA_comb_initialization( $f^{[a]}$ ,  $f^{[b]}$ ,  $F$ )
begin
    for each  $i \in \mathcal{S}$ 
        /* uniform crossover */
        if ( $f_i^{[a]} == f_i^{[b]}$ )
            then  $f_i^{[01]} = f_i^{[02]} = f_i^{[a]}$ ;
        else if ( $rand[0, 1] < 0.5$ )
             $f_i^{[01]} = f_i^{[a]}$  and  $f_i^{[02]} = f_i^{[b]}$ ;
        else
             $f_i^{[01]} = f_i^{[b]}$  and  $f_i^{[02]} = f_i^{[a]}$ ;

        /* mutation */
        if ( $rand[0, 1] < \tau$ )
             $f_i^{[01]} = rand(\mathcal{L})$ ;
        if ( $rand[0, 1] < \tau$ )
             $f_i^{[02]} = rand(\mathcal{L})$ ;
end

```

---

```

GA_Comb_Algorithm
begin
    initialize  $F = \{f^{[1]}, \dots, f^{[N]}\}$ ;
    do {
        random_selection( $f^{[a]}$ ,  $f^{[b]}$ ,  $F$ );
        GA_comb_initialization( $f^{[01]}$ ,  $f^{[02]}$ ,  $f^{[a]}$ ,  $f^{[b]}$ );
        steepest_descent( $f^{*1}$ ,  $f^{[01]}$ );
        steepest_descent( $f^{*2}$ ,  $f^{[02]}$ );
        update( $F$ ,  $f^{*1}$ ,  $f^{*2}$ );
    } until (termination condition satisfied);
    return( $\arg \min_{f \in F} E(f)$ );
end

```

---

Figure 9.9: A GA-like Comb algorithm.

The central part of the Comb method is the derivation of new initial configurations. The Comb is aimed to derive  $f^{[0]}$  in such a way that  $f^{[0]}$  contains as many minimal labels as possible. Because the minimal labels are not known *a priori*, the Comb attempts to use common structure, or *common labels*, of  $f^{[a]}$  and  $f^{[b]}$  to infer the minimal labels. We say that

$f_i^{comm}$  is a common label of  $f^{[a]}$  and  $f^{[b]}$  if  $f_i^{comm} = f_i^{[a]} = f_i^{[b]}$ . The Comb makes a hypothesis that  $f_i^{comm}$  is a minimal label if  $f_i^{[a]} = f_i^{[b]}$ . The Comb initialization schemes are illustrated as follows:

1. The *basic Comb* initialization. For each  $i \in \mathcal{S}$ , if  $f_i^{[a]}$  and  $f_i^{[b]}$  are identical, then set  $f_i^{[0]} = f_i^{[a]}$ ; otherwise set  $f_i^{[0]} = \text{rand}(\mathcal{L})$  which is a label randomly drawn from  $\mathcal{L}$ . The use of the common label in the initial configuration encourages the enlargement of common structure in the local minimum to be found subsequently.
2. The *standard Comb* initialization. The basic Comb initialization is accepted with a probability  $1 - \tau$  where  $0 < \tau < 1$ . The probabilistic acceptance of common labels diversifies the search and prevents  $F$  from converging to a local minimum too soon. The standard Comb initialization is shown in the upper part of Fig.9.8 where  $\text{rand}[0, 1]$  stands for an evenly distributed random number in  $[0, 1]$ .

Then, how many minimal labels are there in  $f^{[0]}$  as the result of copying common labels, *i.e.* as the result of inferring minimal labels using common labels? In supervised tests where the (near) global minimum is known, we find that the percentages of minimal labels in  $f^{[0]}$  is usually only slightly (about 1.0-2.0%) lower than those in  $f^{[a]}$  and  $f^{[b]}$ . That is, the number of minimal labels retained in  $f^{[0]}$  is about the same as those in  $f^{[a]}$  and  $f^{[b]}$ . Given this and that  $f^{[0]}$  is no longer a local minimum as  $f^{[a]}$  and  $f^{[b]}$ , there is room to improve  $f^{[0]}$  using a subsequent local minimization. This makes it possible to yield a better local minimum from  $f^{[0]}$ .

There are two parameters in the Comb algorithm,  $N$  and  $\tau$ . The solution quality increases, *i.e.* the minimized energy value decreases, as the size of  $F$ ,  $N$ , increases from 2 to 10, but remains about the same (probabilistically) for greater  $N$  values; and a larger  $N$  leads to more computational load. Therefore, we choose  $N = 10$ . Empirically, when  $\tau = 0$ , the algorithm converges sooner or later to a unique configuration, and choosing a smaller  $N$  makes such convergence quicker. But  $\tau = 0$  often gives a pre-matured solution. The value of  $\tau = 0.01$  is empirically a good choice.

The Comb algorithm corresponds to a hybrid GA as described in Fig.9.9. The standard Comb initialization is effectively the same as a crossover operation followed by a mutation operation, the major and minor operations in genetic algorithms (GA) (Goldberg 1989). More exactly,

- the basic Comb corresponds to uniform crossover and
- the probability acceptance of in the standard Comb corresponds to mutation.

In GA, two offspring,  $f_i^{[01]}$  and  $f_i^{[02]}$ , are produced as the result of crossover. In the uniform crossover, either of the following two settings are accepted with equal probability:

(i)  $f^{[01]} = f_i^{[a]}$  and  $f^{[02]} = f_i^{[b]}$ ,

(ii)  $f^{[01]} = f_i^{[b]}$  and  $f^{[02]} = f_i^{[a]}$ .

So, if  $f_i^{[a]} = f_i^{[b]}$ , there must be  $f_i^{[01]} = f_i^{[02]} = f_i^{[a]}$ , just as in the Comb initialization. This encourages common labels because common labels are copied to the new initial configurations; in contrast, non-common labels are subject to swap. The above discussion is about the uniform crossover. It should be noted that even the simplest one-point crossover also works in a way that encourages common labels.

The above suggests that the essence of both the Comb and GA is captured by the use of common structure of local minima. This is supported by the fact that the original Comb and the GA-like Comb yield comparable results: In the GA-like Comb algorithm (Fig.9.9), when  $f_i^{[a]} \neq f_i^{[b]}$ ,  $f_i^{[01]}$  and  $f_i^{[02]}$  inherit the values of  $f_i^{[a]}$  and  $f_i^{[b]}$ , as does a crossover operator. However, setting  $f_i^{[01]}$  and  $f_i^{[02]}$  to a random label  $rand(\mathcal{L})$ , *i.e.* not necessarily inheriting  $f_i^{[a]}$  and  $f_i^{[b]}$ , leads to comparable results, as long as the common labels are copied to  $f_i^{[0]}$  when  $f_i^{[a]} = f_i^{[b]}$ . Moreover, whether to derive one initial configuration  $f^{[0]}$  or two initial configurations  $f^{[01]}$  and  $f^{[02]}$  does not matter; both schemes yield comparable results. In summary, the Comb and the GA-like Comb produce comparable results; and this suggests that retaining common labels is important and provides an interpretation for the crossover operation in GA.

The Comb is better than the multi-start method. Running a steepest descent algorithm a number of times using the Comb initialization gets a better solution than running it the same number of times using the independent initialization of multi-start. The Comb has a much higher efficiency in descending to good local minima because it makes use of the best local minima.

To summarize: The Comb attempts to derive good initial configurations from the best local minima found so far in order to achieve better solutions. To do so, it uses common structure of the local minima to infer label values in the global minimum. An initial configuration thus derived has about the same number of minimal labels as the two local minima from which it is derived. However, it is no longer a local minimum and thus its quality can be improved by a subsequent local minimization. This makes it possible to yield a better local minimum and thus increments the solution quality step by step. The comparisons shows that the Comb produces better results than the ICM and HCF though at higher computational costs and comparable results to the SA at lower costs (Section 9.5.3). This suggests that the Comb can provide a good alternative to the well-known global minimizer SA. Further, the Comb algorithm is applicable, in principle, to many optimization problems of vision and pattern recognition.

## 9.5 Experimental Comparisons

### 9.5.1 Comparing Various Relaxation Labeling Algorithms

In the following experiments, we compare several algorithms for combinatorial minimization and for constrained minimization:

1. ICM of (Besag 1986),
2. RL of (Hummel and Zucker 1983),
3. RL of (Rosenfeld et al. 1976),
4. MF annealing of (Peterson and Soderberg 1989),
5. SA of (Geman and Geman 1984),
6. Annealing ICM (Section 9.3.2).

The comparison is in terms of (1) the solution quality measured by the maximized gain  $G(f^*)$ , (2) the computational cost measured by the number of required iterations, and (3) the need for heuristics to tune the algorithm.

The schedules for the annealing algorithms are as follows: There are two annealing schedules for MFA. In the first schedule which is given in (Peterson and Soderberg 1989),  $T$  is decreased according to  $T^{(t+1)} \leftarrow 0.99T^{(t)}$ ; we set the initial temperature as  $T^{(0)} = 10^9 \times M \times m$ . The second is also in the form  $T^{(t+1)} \leftarrow \kappa^{(t+1)}T^{(t)}$  but  $\kappa^{(t+1)}$  is chosen on an *ad hoc* basis of trial and error to get the best result, which is

$$\kappa^{(t+1)} = \begin{cases} 0.9 & \text{if } S < 0.9 \text{ \& } \kappa^{(t)} \leq 0.9 \\ 0.95 & \text{if } S < 0.95 \text{ \& } \kappa^{(t)} \leq 0.95 \\ 0.99 & \text{otherwise} \end{cases} \quad (9.38)$$

where  $S$  is the saturation defined in (9.23). We refer to these two schedules as MFA-1 and MFA-2, respectively. For the SA, it is as  $T^{(t+1)} \leftarrow 0.99T^{(t)}$  but with the initial temperature set to  $10^9 \times M \times m$ ; annealing stops if the desired number of acceptances is not achieved at 100 successive temperatures (Kirkpatrick et al. 1983). These are purposely tuned for the best result possible. The schedule for the annealing ICM is quite tolerant; it is chosen as  $T^{(t+1)} \leftarrow 0.9T^{(t)}$  with  $T^{(0)} = 10$ .

Initial labeling is assigned as follows: First, we set  $p_i^{(0)}(I) = 1 + 0.001 * rnd$  ( $\forall i, I$ ) as the start point common to all the compared algorithms, where  $rnd$  is a random number evenly distributed between 0 and 1. For the continuous algorithms (2, 3 and 4), we normalized  $p^{(0)}$  to satisfy (8.19). For the discrete algorithms (1, 5 and 6),  $i$  is initially labeled according to maximal selection,  $f_i^{(0)} = I^* = \arg \max_I p_i^{(0)}(I)$ . The convergence of the continuous algorithms is judged by checking whether the saturation,  $S$ , is larger than 0.99.

The test bed is the MRF matching of weighted graphs. Here, a node corresponds to a point in  $X - Y$  plane. Two random graphs, each containing  $m = M$  nodes, are randomly generated as follows. First, a number of  $\lfloor \frac{2}{3}m \rfloor$  nodes, where  $\lfloor \cdot \rfloor$  is the “floor” operation, are generated using random numbers uniformly distributed within a box of size  $100 \times 100$ . They are for the first graph. Their counterparts in the other graph are generated by adding Gaussian noise of  $N(0, \sigma^2)$  to the  $x$  and  $y$  coordinates of each of the nodes in the first graph. This gives  $\lfloor \frac{2}{3}m \rfloor$  deviated locations of the nodes. Then, the rest of the  $\lceil \frac{1}{3}m \rceil$  nodes in each of the two graphs are generated independently by using random numbers uniformly distributed in the box, where  $\lceil \cdot \rceil$  is the “ceiling” operation. This simulates outlier nodes.

The above steps generate two weighted graphs  $\mathcal{G} = (\mathcal{S}, d)$  and  $\mathcal{G}' = (\mathcal{L}, D)$  where  $\mathcal{S} = \{1, \dots, m\}$  and  $\mathcal{L} = \{1, \dots, M\}$  index the sets of nodes and  $d = [d(i, i')]_{i, i' \in \mathcal{S}}$  and  $D = [D(I, I')]_{I, I' \in \mathcal{L}}$  are the distances between nodes in the first and second graph, respectively. The basis of matching is the weights in  $d$  and  $D$  which reflect bilateral relations between nodes; unary properties of nodes are not used in the test. We augment  $\mathcal{L}$  by a special node, called the NULL node and indexed by 0, into  $\mathcal{L}' = \{0, 1, \dots, M\}$ . The purpose is to cater for the matching of outlier nodes. Refer also to Section 3.1 for the graph representation. The matching is to assign a node from  $\mathcal{L}'$  to each of the nodes in  $\mathcal{S}$  so that some goodness (cost) of matching is maximized (minimized).

The posterior energy formulated in Section 3.2 is used to measure the cost of a matching  $f = \{f_1, \dots, f_m\}$ . To be suitable for real, as opposed to combinatorial, computation, the problem is then reformulated in terms of relaxation labeling (cf. Section 8.2.2). From the two weighted graphs, the compatibility matrix  $[r_{i, i'}(I, I')]$  is defined in the following way

$$r_{i, i'}(I, I') = \begin{cases} Const_2 - [d(i, i') - D(I, I')]^2 & \text{if } I' \neq 0 \text{ & } I' \neq 0 \\ Const_2 - v_{20} & \text{otherwise} \end{cases} \quad (9.39)$$

where  $v_{20} > 0$  is a constant. The gain function is then

$$G(f) = \sum_{i \in \mathcal{S}} \sum_{I \in \mathcal{L}'} \sum_{i' \in \mathcal{S}, i' \neq i} \sum_{I' \in \mathcal{L}'} r_{i, i'}(I, I') p_i(I) p_{i'}(I') \quad (9.40)$$

To reduce storage, we used only one byte (8 bits) to represent the compatibility coefficients  $r_{i, i'}(I, I')$ . Positive values are truncated to an integer between 0 and 255 while negative values are truncated to zeros. In this case, we set  $Const_2 = 255$ . Although the precision of the compatibility coefficients is low, good results are still obtained; this demonstrates the error-tolerant aspect of the RL minimization approach.

The purpose of the MRF and RL formulations is to provide the compatibility matrix. Our ultimate objective here is more abstract: to compare the ability of the algorithms to solve the minimization problem expressed, given the posterior probability function or the corresponding compatibility matrix.

The following results are obtained after 200 runs: Fig.9.10 illustrates the solution quality in terms of the maximized gain as a function of the noise level  $\sigma$ . SA provides the best result when carefully tuned, followed by the Hummel-Zucker algorithm, whereas the quality of the ICM solution is the poorest. The annealing procedure (*cf.* Section 9.3.2) significantly improves the quality of the ICM solution.

The MFT annealing performs next to the Hummel-Zucker algorithm when  $m = M = 10$ . However, for “harder” problems, *e.g.* when  $m = M = 20$ , the results produced by using the simple schedule MFA-1 deteriorate quickly as the noise level  $\sigma$  increases, dropping to even lower than that of ICM. For large  $m$  and  $M$  and high  $\sigma^2$  values, the algorithm is often trapped to the all-NULL solution, which is a significant local optimum. We also found that a lower schedule, *i.e.* smaller  $\kappa$  in  $T \leftarrow \kappa T$ , does not necessarily yield better solutions.

Fig.9.11 demonstrates the cost in terms of the number of iterations as a function of the noise level  $\sigma$ . ICM converges very fast, after just a few iterations. In contrast, the number for SA with the specified heuristic schedule is two to four order higher than all the deterministic algorithms. Fig.9.12 shows the efficiency measured by the maximized-gain/iteration-number ratio. The ordering by the efficiency is roughly consistent with the inverse ordering by the iteration number.

Besides the solution quality and the cost, an important factor that must be taken into account is the need for, and difficulties in, the tuning of the annealing schedule. It is well known that the schedule is critical to the success of SA and is an area of study (Aarts 1989). We add that the schedule is also critical to the mean field annealing. How to choose an optimal schedule depends not only on the type, but also on the size, of the problem. Finding a good heuristic schedule based on trial and error can be tedious.

The comparison leads to a conclusion in favor of the Hummel-Zucker algorithm. It yields good quality solutions, quite comparable with the time-consuming simulated annealing; yet, it is much more efficient than SA, thus well balancing between the quality and the cost. Further, it avoids cumbersome needs for the heuristic tuning of annealing schedules in annealing algorithms. MFT annealing would also be a good choice if the rapid deterioration of solutions to “harder” problems could be remedied. A recent experimental result (Li 1995c) shows that the Lagrange-Hopfield algorithm described in Section 8.4 yields solutions of quality similar to those produced by the Hummel-Zucker algorithm.

An earlier experimental comparison of several RL algorithms, in terms of the number of iterations (cost) and the number of errors (quality), is done by (Price 1985). Price made the following conclusion: The algorithm of Faugeras and Price (Faugeras and Price 1981) is the best, that of Hummel and Zucker (Hummel and Zucker 1983) is about as well, that by Peleg (Peleg 1980) converges too fast to yield good result, and that of Rosenfeld *et al.* performs

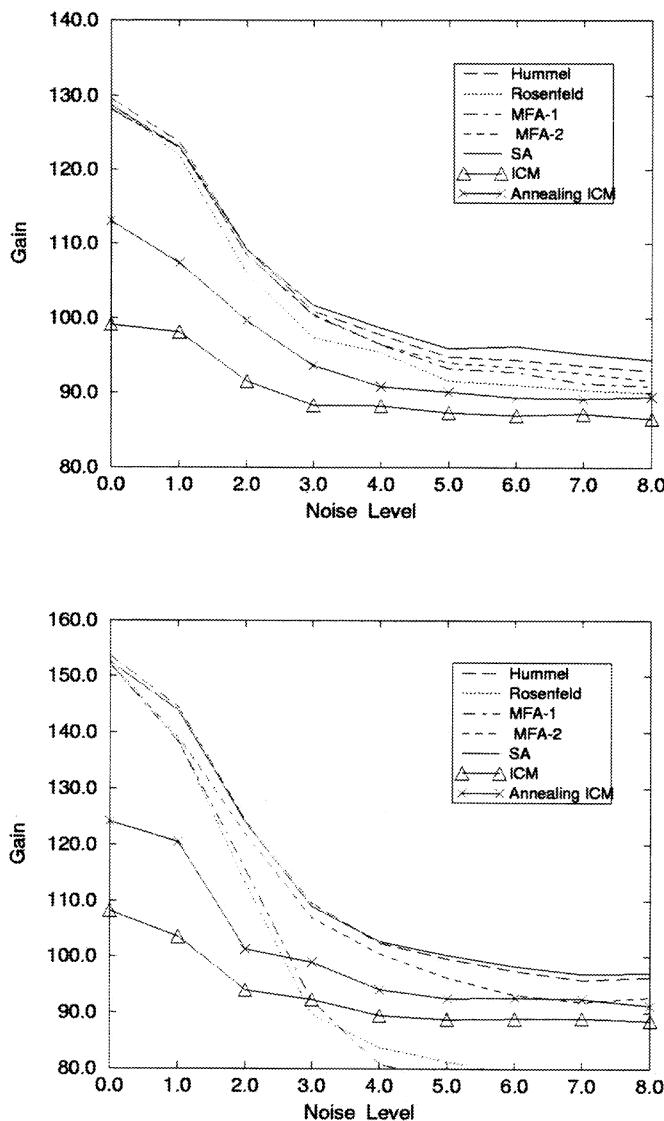


Figure 9.10: The maximized gain  $G(f^*)$  (after divided by  $m^2$ ), for  $m = 10$  (top) and  $m = 20$  (bottom). The higher, the better.

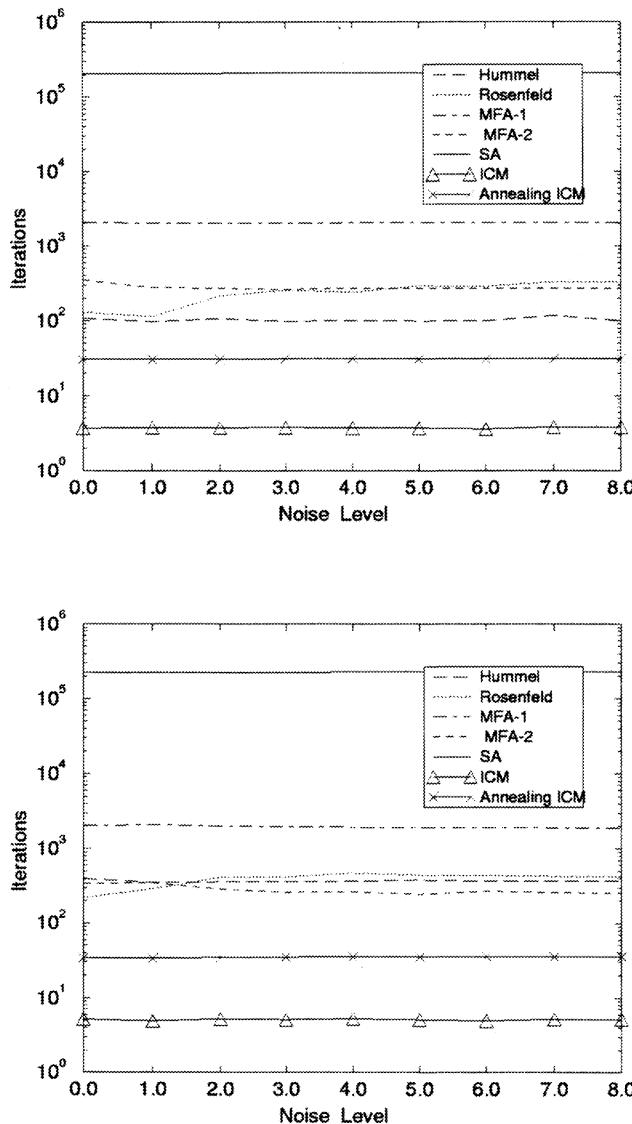


Figure 9.11: The number of iterations, for  $m = 10$  (top) and  $m = 20$  (bottom). The lower, the better.

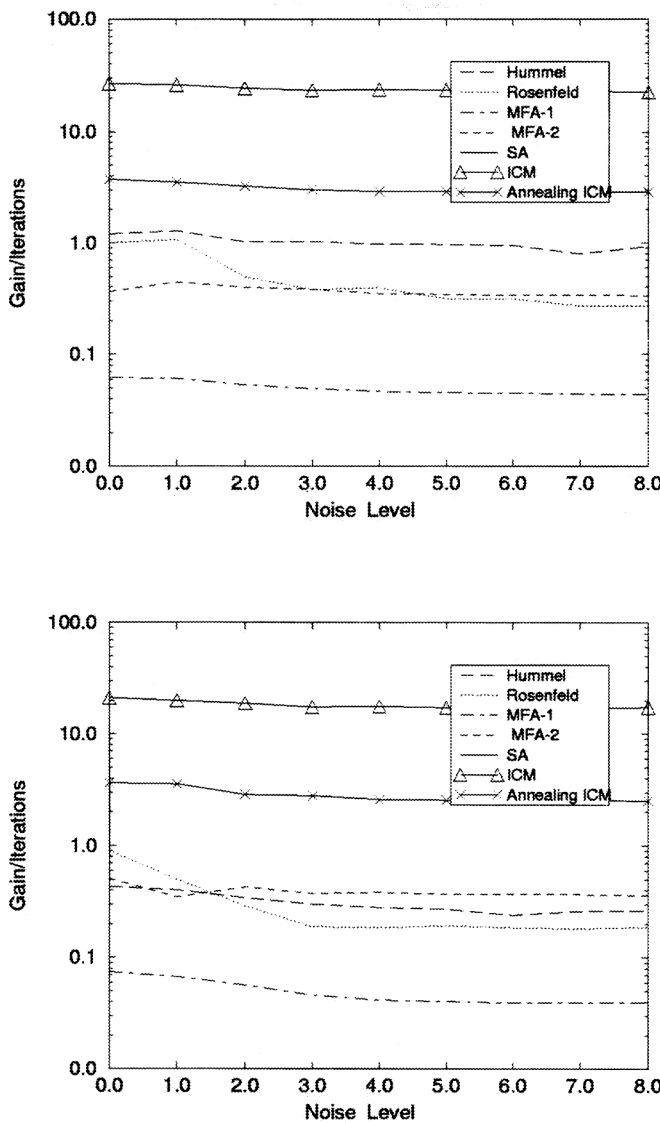


Figure 9.12: The efficiency measured by the maximized-gain/iteration-number ratio, for  $m = 10$  (top) and  $m = 20$  (bottom). The higher, the better. The efficiency of SA is near zero.

only adequately. The goodness of interpretation in (Price 1985) relates not only to algorithms themselves but also to how the problem is formulated (*e.g.* the definition of compatibilities), which is a combination of the two issues, problem formulation and computational algorithm. Here, we measure the solution quality by the quantitative gain  $G(f^*)$  and regard RL as just a mechanism of minimization rather than one of interpretation. The interest in our comparison is how well an algorithm can minimize an energy regardless of the problem formulation.

### 9.5.2 Comparing the ALH Algorithm with Others

	$\sigma$	$\alpha_I$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$
Image No.1	1	0	-1	-1	-1	1
Image No.2	1	0	-2	-2	1	1
Image No.3	1	0	1	1	1	1

Table 9.2: The MRF parameters ( $\alpha$  and  $\beta$ ) and noise parameter  $\sigma$  for generating the three images.

	ALH	ICM	HCF	MFA	SA
Image No.1	-11049	-10003	-10049	-10824	-11988
Image No.2	-10191	-8675	-9650	-10073	-11396
Image No.3	-26974	-25881	-26629	-26716	-27526
Image No.1	0.268	0.349	0.367	0.291	0.164
Image No.2	0.360	0.438	0.414	0.381	0.306
Image No.3	0.212	0.327	0.273	0.181	0.125
Image No.1	3654	7	31	553	83034
Image No.2	1456	6	29	553	68804
Image No.3	2789	7	36	560	92721

Table 9.3: The minimized energy values (top block), error rates (middle block), and iteration numbers (bottom block) for image Nos 1–3.

In the following, two experiments are presented, one for MAP-MRF image restoration and the other for segmentation, to compare the performance of the following algorithms: (1) ALH of this paper, (2) ICM (Besag 1986), (3) HCF (Chou et al. 1993) (the parallel version), (4) MFA (Peterson and Soderberg 1989; Yullie and Kosowsky 1994) (*cf.* Section 9.2), and (5) SA

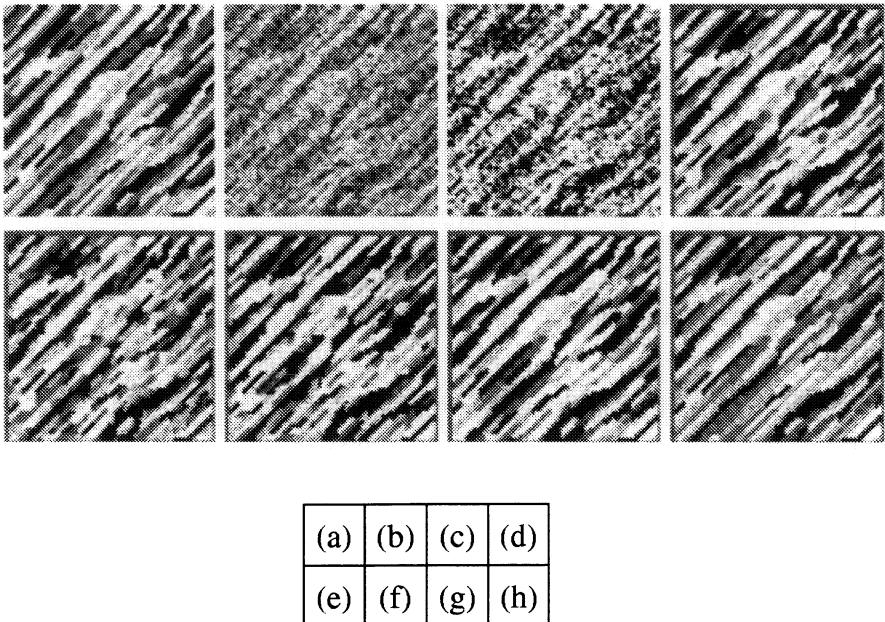


Figure 9.13: Restoration of image No.1. (a) Original image. (b) Observed noisy image. (c) Maximum likelihood estimate. (d) ALH solution. (e) ICM solution. (f) HCF solution. (g) MFA solution. (h) SA solution.

with the Metropolis sampler (Kirkpatrick et al. 1983) implemented based on a procedure given in (Press et al. 1988). The comparison is in terms of (i) the solution quality measured by the minimized energy values and (ii) the convergence rate measured by the number of iterations. In calculating the energy,  $E(f)$  of (8.71) is used where for the continuous algorithms of ALH and MFA,  $f$  is obtained from  $p$  by a maximum selection (winner-take-all) operation.

For ALH,  $\mu = 100$  and  $T = 10^5$  are fixed,  $\beta$  is increased from 1 to 100 according to  $\beta \leftarrow 1.01\beta$ . The convergence criterion is  $\|u^{(t)} - u^{(t-1)}\|_\infty < 0.0001$ . The schedule for SA is  $T^{(t+1)} \leftarrow 0.999T^{(t)}$  (with  $T^{(0)} = 10^4$ ) and for MFA is  $T^{(t+1)} \leftarrow 0.99T^{(t)}$  (with  $T^{(0)} = 10^3$ ).

The first set of experiments is for MAP-MRF restoration performed on three synthetic images of  $M = 4$  gray levels and Figures 9.13–9.15 show the results. In each figure, (a) is the true image with  $M = 4$  gray levels, the label set  $\mathcal{L} = \{1, 2, 3, 4\}$ , and the pixel gray values also in  $\{1, 2, 3, 4\}$ . The clique potential parameters  $\alpha_I$  and  $(\beta_1, \dots, \beta_4)$  for generating the three images are shown in Table 9.2. (b) is the observed image in which every pixel takes a real value which is the true pixel value plus zero-mean i.i.d. Gaussian noise with

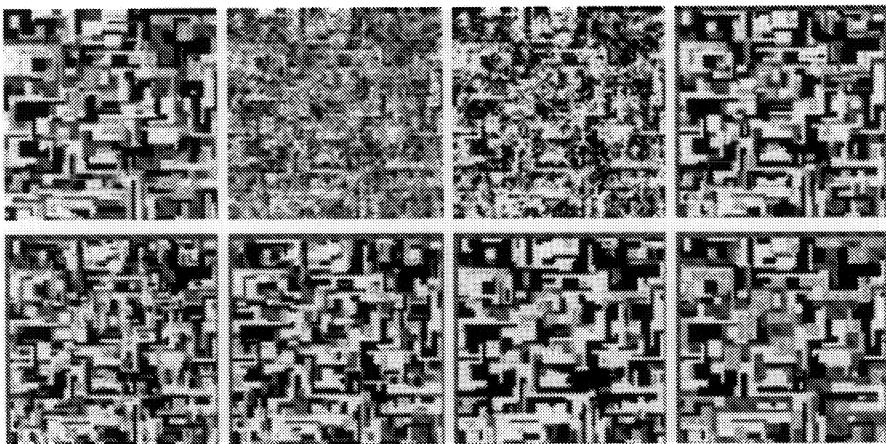


Figure 9.14: Restoration of image No.2 (refer to Fig.9.13 for legend).

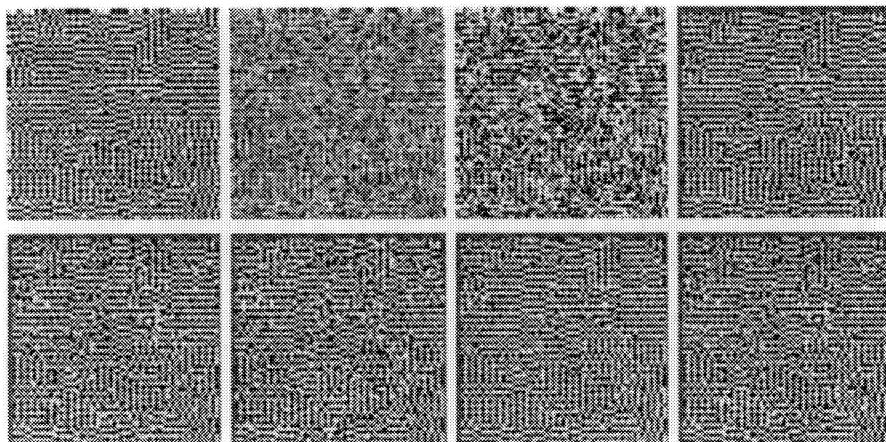


Figure 9.15: Restoration of image No.3 (refer to Fig.9.13 for legend).

standard deviation  $\sigma = 1$ . (c) is the maximum likelihood estimate which was used as the initial labeling. (d) to (h) are the solutions found by the compared algorithms.

Table 9.3 shows the minimized energy values, the error rates, and the required iteration numbers. It can be seen from the table that objectively, ALH performs the best out of the three deterministic algorithms in terms of both the minimized energy values. Overall, the solution quality is ranked as “ICM < HCF < MFA < ALH < SA”, which is in agreement with a subjective evaluation of the results. We also implemented a parallel ICM using codings but the results were not as good as the serial ICM.

	ALH	ICM	HCF	MFA	SA
Min. Energy	-180333	-171806	-176167	-180617	-173301
Iterations	1255	7	38	545	593916

Table 9.4: Numerical comparison of the algorithms on the segmentation of the Lenna image.



Figure 9.16: Segmentation of the Lenna image. Top raw: Input image, maximum likelihood segmentation, ALH solution. Bottom raw: ICM solution, MFA solution, SA solution.

The second experiment compares the algorithms in performing MAP-MRF segmentation on the Lenna image of size  $256 \times 240$  into a tri-level segmentation map. The results are illustrated in Figure 9.16. The input image (a) is the original Lenna image corrupted by the i.i.d. Gaussian noise with standard deviation 10. The observation model is a Gaussian distribution of standard deviation 10 with mean values 40, 125 and 200 for the three-level segmentation. An isometric MRF prior is used with the four  $\beta$  parameters being  $(-1, -1, -1, -1)$ . (b) is the maximum likelihood estimate which was used as the initial segmentation. (c)–(f) are the segmentation results of ALH, ICM, MFA and SA, respectively. Table 9.4 illustrates the minimized energy, *i.e.* maximized posterior probability, values and the iteration numbers numerically. The solution quality is ranked as “HCF < ICM < SA < ALH < MFA”.

According to the iteration numbers, the ALH method takes much fewer iterations than SA, about 0.2% – 4.4%, to converge. Although taking more iterations than the other deterministic algorithms, it is not so slow as it might seem. That the ALH converged after thousands of iterations was due to the stringent convergence criterion  $\|u^{(t)} - u^{(t-1)}\|_\infty < 0.0001$ . However, looking at the energy evolution curve in Figure 9.17 (SA is not included there because it is far from the convergence within a thousand of iterations), we see that the ALH reached a solution better than ICM and HCF after only a dozen of iterations and becomes nearly convergent after a hundred of iterations. The MFA needs five hundred of iterations to converge.

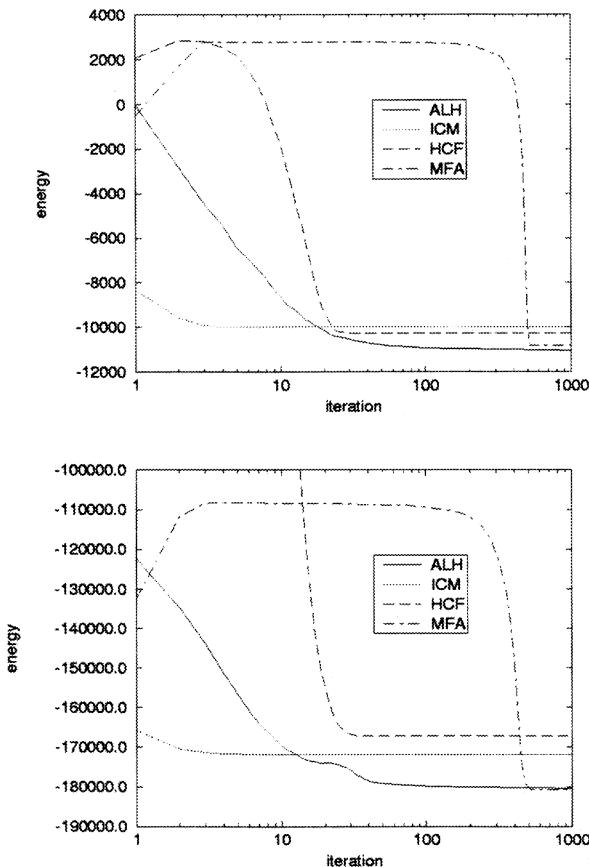


Figure 9.17: The solution evolution curves for image no.1 (left) and the Lenna image.

### 9.5.3 Comparing the Comb Algorithm with Others

In the following, two experiments are presented, one for MAP-MRF image restoration and the other for segmentation, to compare the performance of the following algorithms: (1) the Comb of this paper; (2) the ICM (Besag 1986); (3) the HCF (Chou et al. 1993) (the parallel version); (4) the SA with the Metropolis sampler (Kirkpatrick et al. 1983). For the Comb, the parameters are  $N = 10$  and  $\tau = 0.01$ . The implementation of SA is based on a procedure given in (Press et al. 1988). The schedules for SA are set to  $T^{(t+1)} \leftarrow 0.999T^{(t)}$  with  $T^{(0)} = 10^4$ . The initial configuration for ICM, HCF and SA is taken as the ML estimate whereas those in  $F$  for Comb is entirely random. The termination condition for Comb is that all configurations in  $F$  are the same or that a number of 10000 new local minima have been generated.

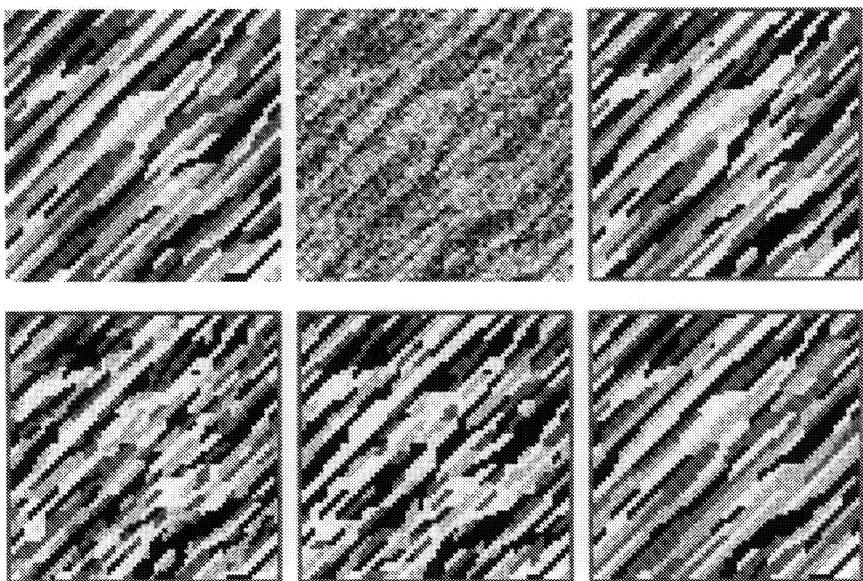
Image	$\sigma$	$\alpha_I$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$
No.1	1	0	-1	-1	-1	1
No.2	1	0	-2	-2	1	1
No.3	1	0	1	1	1	1

Table 9.5: The MRF parameters ( $\alpha$  and  $\beta$ ) and noise parameter  $\sigma$  for generating the three images.

The first set of experiments is for MAP-MRF restoration performed on three synthetic images of  $M = 4$  gray levels, shown in Figs.9.18-9.20. The original have the label set  $\mathcal{L} = \{1, 2, 3, 4\}$ , and the pixel gray values also in  $\{1, 2, 3, 4\}$ . Table 9.5 gives the clique potential parameters  $\alpha_I$  and  $\beta_1, \dots, \beta_4$  for generating the three types of textures and the standard deviation  $\sigma$  of the Gaussian noise.

The second experiment compares the algorithms in performing MAP-MRF segmentation on the Lenna image of size  $256 \times 240$  into a tri-level segmentation map. The results are illustrated in Fig.9.21. The input image is the original Lenna image corrupted by the i.i.d. Gaussian noise with standard deviation 10. The observation model is assumed to be the Gaussian distribution superimposed on the mean values of 40, 125 and 200 for the three-level segmentation. An isometric MRF prior is used with the four  $\beta$  parameters being  $(-1, -1, -1, -1)$ .

Table 9.6 compares the quality of restoration and segmentation solutions in terms of the minimized energy values. We can see that the Comb outperforms the ICM and the HCF and is comparable to SA. A subjective evaluation of the resultant images would also agree to the objective numerical comparison. The quality of the Comb solutions is generally also better than that



a	b	c
d	e	f

Figure 9.18: Restoration of image 1. (a) Original clean image. (b) Observed noisy image (input data). (c) Comb solution. (d) ICM solution. (e) HCF solution. (f) SA solution.

produced by using a continuous method of augmented Lagrange developed previously (Li 1998b).

The Comb as a random search method needs many iterations to converge, the number being increasing as  $\tau$  decreasing. All the Comb solutions with  $\tau = 0.01$  are obtained when the limit of generating 10000 local minima is reached. This is about 1000 times more than the fast converging ICM and HCF. Nonetheless, the Comb takes about 1/20 of the computational effort needed by the SA.

The Comb algorithm does not rely on initial configurations at the beginning of the algorithm to achieve better solutions; the maximum likelihood estimate can lead to a better solution for algorithms which operate on a single configuration, such as the ICM, HCF and SA, but not necessarily for the Comb.

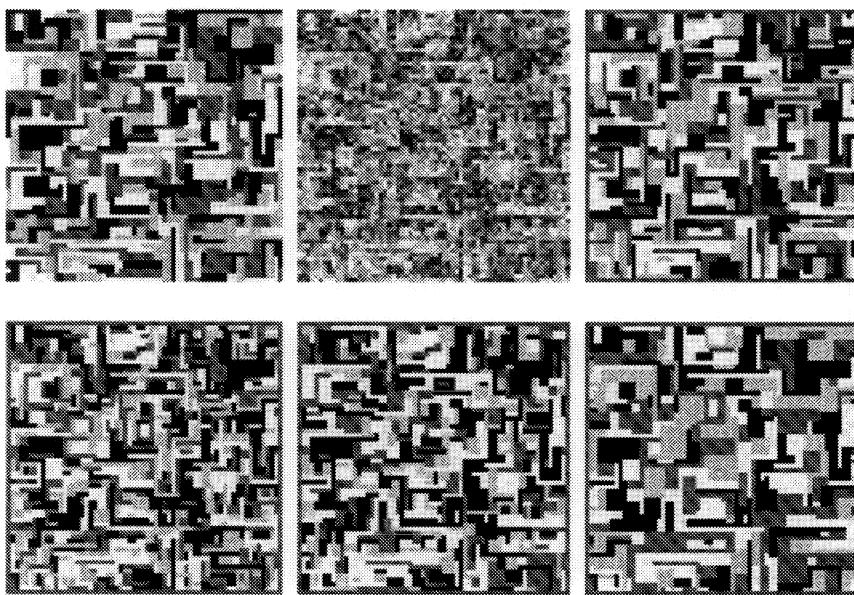


Figure 9.19: Restoration of image 2. Legends same as Fig.9.18.

	Comb	ICM	HCF	SA
No.1	-12057	-10003	-10269	-11988
No.2	-10944	-8675	-9650	-11396
No.3	-27511	-25881	-26629	-27526
Lenna	-175647	-171806	-167167	-173301

Table 9.6: The minimized energy values for the restoration of images 1–3 and the segmentation of the Lenna image.

## 9.6 Accelerating Computation

The MRF configuration space for image and vision analysis is generally large and the search for an energy minimum is computationally intensive. When the global solution is required, the computation is further increased by complications incurred by the problem of local minima and can well become intractable. Unrealistic computational demand has been a criticism on the MAP-MRF framework. Efforts have been made to design efficient algorithms.

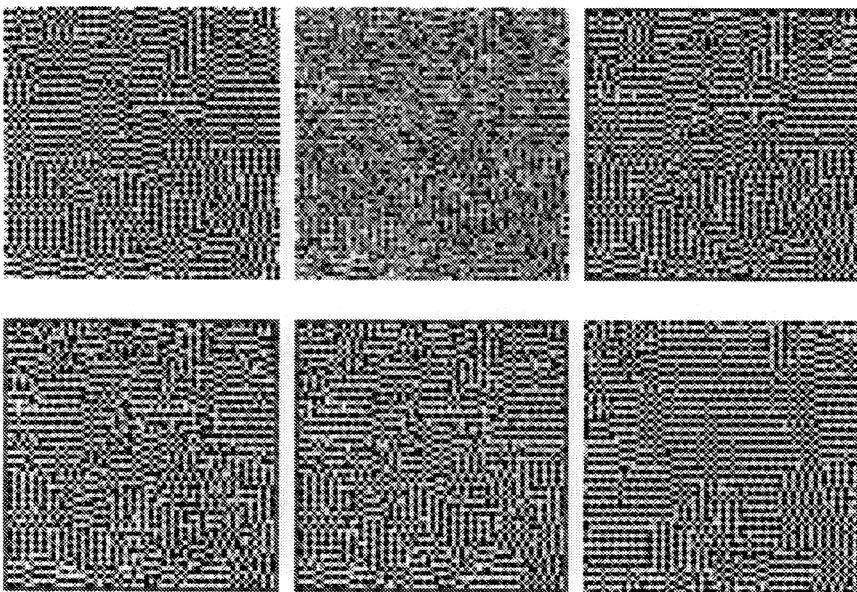


Figure 9.20: Restoration of image 3. Legends same as Fig.9.18.

### 9.6.1 Multi-Resolution Methods

*Multi-resolution methods* provide a means for improving the convergence of iterative relaxation procedures (Hackbusch 1985). It is shown by (Terzopoulos 1986a) that multi-resolution relaxation can be used to solve efficiently a number of low level vision problems. This class of techniques has been used for MRF computation by many authors (Konrad and Dubois 1988b; Barnard 1989; Bouman and Liu 1991; Kato et al. 1993b; Bouman and Shapiro 1994). Gidas (1990) proposes a method which uses the renormalization group theory, MRFs and Metropolis algorithm for global optimization. How to preserve the inter-relationships between objects in the scene using the renormalization group transformation are studied in (Geiger and Kogler Jr. 1993; Hurn and Gennison 1993; Petrou 1993).

An important issue in multi-resolution computation of MRFs is how to preserve the Markovianity and define consistent model descriptions at different resolutions. In general, the local Markovian property is not preserved at the coarse levels after a sub-sampling. In (Jeng 1992), two theorems are given on a periodic sub-sampling of MRFs. One gives necessary and sufficient conditions for preserving the Markovianity and the other state that there is at least one sub-sampling scheme by which the Markovianity is preserved. A multi-resolution treatment is presented by (Lakshmanan and Derin 1993) in which (possibly) non-Markov Gaussian fields are approximated by linear Gaussian MRFs. In (Heitz and Bouthemy 1994), a consistent set of parame-



a	b	c
d	e	f

Figure 9.21: Segmentation of the Lenna image. (a) The noisy Lenna image. (b) ML configuration. (c) Comb solution. (d) ICM solution. (e) HCF solution. (f) SA solution.

ters are determined for objective functions at different resolutions; a nonlinear multi-resolution relaxation algorithm, which has fast convergence towards quasi-optimal solutions, is developed. A general transformation model is considered in (Perez and Heitz 1994) as the “restriction” of an MRF, defined on a finite arbitrary non-directed graph, to a subset of its original sites; several results are derived for the preservation of the Markovianity which may be useful for designing consistent and tractable multi-resolution relaxation algorithms.

### 9.6.2 Use of Heuristics

Apart from theory-supported methods, heuristics are often combined into search. Solution candidates are quickly located using some efficient means. They are then evaluated by the derived energy function. Because the number of such candidates is usually small, the energy values can be compared exhaustively to give the minimal solution.

Not to miss the true global solution is crucial for the success of applying heuristics. One has to balance between efficiency and danger of missing the

true solution. More restricting heuristics reduce the number of generated hypotheses and increase the efficiency, but they also cause a greater chance to miss the true minimum.

The *bounded noise model* (Baird 1985; Breuel 1992) is a simple heuristic for approximating noise distributions to quickly reduce the search space. By checking whether an error is within the allowed bound, numerical constraints are converted into symbolic ones so that yes-or-no decision can be made to prune the search space. This strategy has been used in many methods where numerical constraints have to be converted into symbolic one. For example, in maximal cliques (Ambler et al. 1973), dynamic programming (Fischler and Elschlager 1973), and constraint search (Faugeras and Hebert 1986; Grimson and Lozano-Prez 1987), symbolic compatibilities are determined; in Hough transform (Hough 1962; Duda and Hart 1972) and geometric hashing (Lamdan and Wolfson 1988), numerical values are quantized to vote accumulators.

*Hypothesis-verification* is another approach for efficient search. Hypotheses are generated, which may correspond to peaks in Hough transform space or geometric hashing space, or leaves of an interpretation tree (Grimson and Lozano-Prez 1987), or random samples in random sampling (Fischler and Bolles 1981; Roth and Levine 1993), or result from minimal sets of image-model correspondences in the alignment method (Huttenlocher and Ullman 1987). Because the number of generated hypotheses is much smaller than the number of points in the original solution space, costs incurred by hypotheses evaluation is much reduced. In (Lowe 1992), matching and measurement errors are used to determine the probability of correctness for individual labels. Techniques presented therein may be used to speed verification. All these have potential applications in MRFs to tackle the problem of low computational efficiency.

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# List of Notation

$\alpha, \beta$	parameters for single- and pair- site clique potentials
$c$	clique
$\mathcal{C}$	set of cliques
$d$	observed data
$\mathcal{D}$	set of admissible values for data
$E(f)$	energy function
$f$	MRF configuration, set of assigned labels, or mapping from $\mathcal{S}$ to $\mathcal{L}$
$f_{\mathcal{N}_i}$	set of labels at sites in $\mathcal{N}_i$
$\mathbb{F}$	solution space
$g(\cdot)$	potential function
$G(f)$	gain function
$\mathcal{G}$	relational structure or graph
$h(\cdot)$	interaction function
$i$	index to site
$i'$	neighbor of $i$
$I$	index to labels
$(i, j)$	index to site in a lattice
$\mathcal{L}$	set of labels
$\mathcal{N}_i$	set of sites neighboring $i$
$N(\mu, \sigma^2)$	Gaussian distribution

$P(x), p(x)$	probability, density, function of random variable $x$
$P_i, p_i$	vector of point
$\mathcal{S}$	set of sites
$\sigma^2$	noise variance
$\mathcal{T}$	pose transformation
$\theta$	set of parameters in MRF model
$\theta_d$	set of parameters in $P(d \mid f)$
$\theta_f$	set of parameters in $P(f)$
$U(f)$	energy function for prior distribution
$U(f \mid d)$	energy function for posterior distribution
$U(d \mid f)$	energy function for likelihood distribution
$V_1(f)$	clique potential function
$V_1(f_i)$	single-site potential function
$V_2(f_1, f_{i'})$	pair-site potential function
$Z$	partition function

There are a few exceptions to the interpretation of the symbols. But they can be identified easily from the context.

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