NUMERICKE RESENI NELINEARNICH ROVNIC

Jestlize je f spojita na interval <a, b> a funkcni hodnoty v bodech a a b maji opacna znamenka, pak v tomto interval lezi alespon jeden koren rovnice.

NELINEARNI FUNKCE

1.1/ Metoda tecen (Newtonova metoda)

- → Zvolime x0; dalsi aproximace pocitame jako: $X_{K+1} = X_K - \frac{f(xk)}{f'(xk)}$
- → Metoda muze divergovat anebo najit jiny koren
- → Aby byla zarucena konvergence, musime zvolit spravne koren:
  - $f'(x) > 0$ && $f''(x) > 0$ => x0 = b
  - $f'(x) > 0$ && $f''(x) < 0$ => x0 = a
  - $f'(x) < 0$ && $f''(x) > 0$ => x0 = a
  - $f'(x) < 0$ && $f''(x) < 0$ => x0 = b
- → priklad:
  - $e^x + x^2 - 3 = 0$ s presnosti eps = 0.01
  - Vime, ze koren lezi na interval <-2, -1>
  - $f'(x) = e^x + 2x$ v intervalu <-2, -1>:zaporn
  - $f''(x) = e^x + 2$ porad kladne -> x0 = a;
  - x0 = -2
  - x1 = -1,70623 …

1.2/ Metoda proste iterace

- → rovnici upravime na tvar: $x = g(x)$
- → zvolime x0, nasledne dalsi aproximace $X_{K+1} = g(X_K),\ k = 0, 1, 2$
- → podminka: $\propto \in (0, 1),\ |g'(x)| \leq \propto$
- → priklad:
  - $f(x) = e^x + x^2 - 3 = 0$
  - $x^2 = 3^2 - e^x$
  - $x = \pm\sqrt{3 - e^x}$, $x < 0$ … chceme najit zaporny koren
  - $x = -\sqrt{3 - e^x};\ x_{k+1} = -\sqrt{3 - e^{xk}}$
- → Pokud mame vice rovnic, pocitame pro kazdou zvlast

SOUSTAVA LINEARNICH ROVNIC

2.1/ Jacobiho metoda

- → Podminky: matice musi byt radkove/sloupcove ostre diagonalne dominantni, presklada, pokud nelze nutno pouzit $A^TAx = A^Tb$ a G-S metodu
- → Z 1. rovnice vyjadrime 1. neznamou, ze 2. druhou atd.
- → $15x_1 - x_2 + 2x_3 = 30 \Rightarrow x_1 = (30 + x_2 - 2x_3) * \frac{1}{15}$
- → $2x_1 - 10x_2 + x_3 = 23 \Rightarrow x_2 = (23 + 2x_1 - x_3) * (-\frac{1}{10})$
- → $x_1 + 3x_2 + 18x_3 = -22 \Rightarrow x_3 = (-22 - x_1 - 3x_2) * \frac{1}{18}$
- → Zvolime pocatecni aproximaci (pocatecni odhad) (vse na nula), dale pocitame podle vyse uvedenych vzorcu.

2.2/ Gauss-Seidelova metoda

- → Stejne jako Jacobiho akorat v novem kroce vzdy bereme nejaktualnejsi hodnoty x
- → Zpravidla vede k rychlejsimu vysledku, nez Jacobiho

2.3/ Radkove/sloupcove ostre diagonalne dominantni matice

- → Na kazdem radku/sloupci absolutni hodnota prvku na diagonal je vetsi, nez soucet absolutnich hodnot vsech ostatnich prvku v onom radku

$$\begin{matrix} 15 & -1 & 2 & \vdots & 15 > 1 + 2 \\ 2 & -10 & 1 & \vdots & 10 > 2 + 1 \\ 1 & 3 & 18 & \vdots & 18 > 1 + 3 \end{matrix}$$

SOUSTAVA NELINEARNICH ROVNIC

3.1 Newtonova metoda

- → $xy = 2$
- → $x^2 + y^2 + 3 = 4y$
- → $xy - 2 = 0$
- → $x^2 + y^2 - 4y + 3 = 0$
- → $f_1(x, y) = xy - 2$
- → $f_2(x, y) = x^2 + y^2 - 4y + 3$
- → $F' = \begin{bmatrix} y & x \\ 2x & 2y - 4 \\ (\text{deriv. podle x}) & (\text{podle y}) \end{bmatrix}$
- → 1. krok: poc. aprox: x0 = 1, y0 = 0 -> $\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$
- → $0\delta_1 + \delta_2 = 2$
- → $2\delta_1 + 4\delta_2 = -4$ … vysledky (2 a -4) ziskame tak, ze poc. aprox. hodime do nejpuvodnejsich rovnic + zmenime znamenko vysledku
- → $\delta_2 = 2$
- → $2\delta_1 - 8 = -4;\ \delta_1 = 2$
- → $x_1 = x_0 + \delta_1;\ x1 = 3$
- → $y_1 = y_0 + \delta_2;\ y1 = 2$
- → Konec: $|\delta_1| < \epsilon$ AND $|\delta_2| < \epsilon$

APROXIMACE FUNKCI

4.1/ Lagrangeuv interpolacni polynom (interpolacni metoda)

- → Funkce p je polynom, ktery temito body prochazi. Mame tyto body
- → $\begin{matrix} x_i & -1 & 0 & 2 & 3 \\ y_i & 5 & 10 & 2 & 1 \end{matrix}$
- → polynom 3. stupne, protoze mame 4 body:
- → $P_3(x) = 5\frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)} + \ldots$ obecne:
- → $P_n(x) = f_0\frac{(x-x1)(x-x2)(x-x3)}{(x0-x1)(x0-x2)(x0-x3)} + f_1\frac{(x-x0)(x-x2)(x-x3)}{(x1-x0)(x1-x2)(x1-x3)} + f_n \ldots$

4.2/ Newtonuv interpolacni polynom (interpolacni metoda)

xi	fi	f[xi, xi+1]	f[xi,xi+1,xi+2]
x0	f0	$f_{0,1} = \frac{f_1 - f_0}{x_1 - x_0}$	$f_{0,1,2} = \frac{f_{1,2} - f_{0,1}}{x_2 - x_0}$
x1	f1	$f_{1,2} = \frac{f_2 - f_1}{x_2 - x_1}$	$f_{1,2,3} = \frac{f_{2,3} - f_{1,2}}{x_3 - x_1}$
x2	f2		

- → $P_n(x) = f(x0) + f[x0, x1](x - x0) + f[x0, x1, x2](x - x0)(x - x1)$

4.3/ Metoda nejmensich ctvercu

- → Mame zadan n bodu xi a n bodu yi
- → Aproximace primkou:
- → $c0(n\ (\text{pocet bodu}) + 1) + c1\sum_{i=0}^{n} xi = \sum_{i=0}^{n} yi$
- → $c0\sum_{i=0}^{n} xi + c1\sum_{i=0}^{n} xi^2 = \sum_{i=0}^{n} xiyi$
- → Primka: y = c0+c1 * x
- → Aproximace parabolou
- → $c0(n + 1) + c1\sum_{i=0}^{n} xi + c2\sum_{i=0}^{n} xi^2 = \sum_{i=0}^{n} yi$
- → $c0\sum_{i=0}^{n} xi + c1\sum_{i=0}^{n} xi^2 + c2\sum_{i=0}^{n} xi^3 = \sum_{i=0}^{n} xiyi$
- → $c0\sum_{i=0}^{n} xi^2 + c1\sum_{i=0}^{n} xi^3 + c2\sum_{i=0}^{n} xi^4 = \sum_{i=0}^{n} xi^2yi$
- → Parabola: $y = c0 + c1x + c2x^2$

NUMERICKE DERIVOVANI

- → Dostanu body (x) a vzdalenost (h)
- → Dva body:
  - $f'(x) = \frac{f(x+h)-f(x)}{h}$
  - $f'(x) = \frac{f(x)-f(x+h)}{h}$
- → Tri body:
  - $f'(x0) = \frac{-3f(x0)+4f(x1)-f(x2)}{2h}$
  - $f'(x1) = \frac{f(x2)-f(x0)}{2h}$
  - $f'(x2) = \frac{f(x0)-4f(x1)+3f(x2)}{2h}$

NUMERICKE INTEGROVANI

- → Interval <a,b> si rozdelime na n (m) lichobeznikovych dilku delky h ($n = \frac{b-a}{h}$)
- → Delici body oznacime x0 = a; x1 = a + h; x2 = a + 2h; …
- → V kazdem bode provedeme danou metodu

DIFERENCIALNI ROVNICE

7.1/ Eulerova metoda

- → $y' = x^2 - 4y,\ \ y(0) = -2$; h = 0.5

$$\begin{matrix} xi & yi & ki \\ 0 & -2 & 8 \\ 0.5 & 2 & -7.75 \\ 1 & -1.875 & 8.5 \\ 1.5 & 2.375 & -7.25 \end{matrix}$$

- → Nove y = stare y + (stare k * krok); -1.875 = 2 + (-7.75*0.5)
- → Nove k = podle y'; -7.75 = 0.5^2 – 4 * 2

7.2/ Prvni modifikace Eulerovy metody

- → $y' = x^2 - 4y,\ \ y(0) = -2$; h = 0.5

$$\begin{matrix} xi & yi & k1 & k2 \\ 0 & -2 & 8 & 0.0625 \\ 0.5 & -1.969 & 8.125 & 0.3125 \\ 1 & -1.813 & 8.25 & 0.5625 \\ 1.5 & -1.531 & & \end{matrix}$$

7.3/ Druha modifikace Eulerovy metody

- → $y' = x^2 - 4y,\ \ y(0) = -2$; h = 0.5

$$\begin{matrix} xi & yi & k1 & k2 \\ 0 & -2 & 8 & 0.125 \\ 0.5 & -1.9375 & 8 & 0.375 \\ 1 & -1.75 & 8 & 0.625 \\ 1.5 & & & \end{matrix}$$

PRAVDEPODOBNOST

2 kostky

a) Jaka je pst, ze hodime soucet 4?

$A = \{13,22,31\};\ P(A) = \frac{3}{36}$

b) Jaka je pst, ze 2. hozene cislo je vetsi, nez prvni hozene

$B = \{12,13, \ldots, 56\};\ P(B) = \frac{15}{36}$

3 kostky

a) Pst, ze soucet = 6?

vsechny kombinace: 6; pocet vyhovujicich: 10

b) **Mame 100 losu, 15 je vyhravajicich, koupime 3 losy, jaka je pst, ze zadny nevyhrava?**

$0.\ los\ nevyhrava: \frac{100 - 15}{100}$

$1.\ los\ nevyhrava: \frac{99 - 15}{99}$

$2.\ los\ nevyhrava: \frac{98 - 15}{98}$

$vse\ vynasobime; vysledek = \frac{1411}{2310}$

PODMINENA PRAVDEPODOBNOST

$P(B|A) = \frac{P(A \cap B)}{P(A)}$, -> $P(A \cap B) = P(A) * P(B|A)$

1.1/ BAYESUV VZOREC

$P(A) = P(H1) * P(A|H1) + P(H2) * P(A|H2) + \cdots + P(Hn) * P(A * Hn)$

$$P(Hn|A) = \frac{P(H3) * P(A|H3)}{P(A)}$$

- → V obchode prodavaji svestky od 3 dodavatelu. Prodavac to zamicha. Nahodne vyberu svestku, jaka je pst, ze bude cervena (A)?
  - Dodavatel 1 … 50% svestek … z toho 5% cervenych
  - Dodavatel 2 … 30% svestek … z toho 8% cervenych
  - Dodavatel 3 … 20% svestek … z toho 15% cervenych
  - H1 … svestka od 1 $P(H1) = 0.5;\ P(A|H1) = 0.05$
  - H2 … svestka od 2 $P(H2) = 0.3;\ P(A|H1) = 0.08$
  - H3 … svestka od 3 $P(H3) = 0.2;\ P(A|H1) = 0.15$
  - $P(A) = P(H1) * P(A|H1) + P(H2) * P(A|H2) + P(H3) * P(A|H3) = 0.079$
- → Jistou nemoc ma 15% lidi. Clovek ma nemoc, test pozitivni ve 100%. Clovek nema nemoc, test pozitivni v 10%. Test je pozitivni, jaka je pst ze clovek ma onu nemoc?
  - H1 … clovek ma nemoc; $P(H1) = 0.15$
  - H2 … clovek nema nemoc; $P(H1) = 0.85$
  - A … test je pozitivni $P(A|H1) = 1;\ P(A|\ H2) = 0.1$
  - $P(A) = P(H1)*P(A|H1)+P(H2)*P(A|H2) = 0.235$
  - $P(H1|A) = \frac{P(H1)*P(A|H1)}{P(A)} = \frac{0.15*1}{0.235} = 0.638$

UZITECNE VZORCE

Kruznice se stredem S[m,n] a pol. r:

$$(x - m)^2 + (y - n)^2 = r^2$$

Elipsa (hl. osa rovn. s osou x, stred S=[m,n])

$$\frac{(x - m)^2}{a^2} + \frac{(y - n)^2}{b^2} = 1$$

NAHODNE VELICINY

Graf: pouze tecky, znazornuji jaka je pst v danem bode

Histogram: zname

Distrib. funkce: hodnoty 0 .. 1; neklesajici (scitani z grafu), smerem do +inf se blizi k 1, opacnym smerem k 0; $P(a \leq X < b) = F(b) - F(a)$; je zleva spojita

- → Stredni hodnota: $EX = \sum_{xi} xi * p(xi)$; (x * y)
- → Rozptyl: $DX = (\sum_{xi} xi^2 * p(xi)) - (EX)^2$
- → Smer. odchylka: $\sigma = \sqrt{DX}$

→ Geometricke rozdeleni – priklad

- Mame hraci kostku. Hazeme, dokud nepadne sestka. Jaka je pst., ze sestka padne nejpozdeji druhym hodem?
  - 1 .. 5 je uspech, p = 5/6
  - X -> pocet uspesnych hodu (pred tou 6kou)
  - X = Ge(5/6)
  - $p(k) = \left(\frac{5}{6}\right)^k * \frac{1}{6}$
  - Nejpozdeji 2. hodem: $p(0) + p(1) = \frac{1}{6} + \frac{5}{6} * \frac{1}{6} = \frac{11}{36}$
- Stredni hodnota EX
  - $EX = \sum_{k-0}^{inf} k * p(k) = \frac{p}{1-p} = \frac{\frac{5}{6}}{1-\frac{5}{6}} = 5$

    -> v prumeru sestku hodime po 5ti hodech

→ Binomicke rozdeleni – priklad

- Petkrat hodime kostkou. Jaka je pst., ze prave 2x pradne 6?
  - N=5;p=1/6; EX = N*p = 5/6
  - $p(2) = \binom{5}{2} * p^k * (1 - p)^{n-k} = \binom{5}{2} * \left(\frac{1}{6}\right)^k * \left(1 - \frac{1}{6}\right)^{5-2} = \mathbf{10} * \frac{\mathbf{5^3}}{\mathbf{6^5}}$

→ Poissonovo rozdeleni – priklad

- 1 hodina .. 4 lidi. Jaka je pst, ze behem 20 min nikdo neprijde?
  - Cas: 20min; $\lambda = \frac{4}{3} lidi$
  - X = pocet lidi za 20 min: $X \sim Po\left(\frac{4}{3}\right)$
  - $p(k) = P(X = k) = \frac{\lambda^k}{k!} * e^{-\lambda}, k = 0,1,2$
  - $p(0) = e^{-\frac{4}{3}}$
- Jaka je pst, ze behem 20 min prijde vic jak 2 lide?
  - $1 - (p(0) + p(1) + p(2)) = \cdots$

→ Exponencialni rozdeleni – priklad

- Na urad prichazeji lidi chaoticky, ale prumerne 5 lidi za hodinu. Urednik chce kavu, potrebuje na ni 15 minut. Jaka je pst, ze behem 15ti minut neprijde nikdo?
  - X = doba do prichodu prvniho cloveka
  - $P(X > \frac{1}{4})$ … ctvrt hodiny
  - $X \sim Exp(\lambda);\ X \sim Exp(5)$
  - $P\left(X > \frac{1}{4}\right) = 1 - P\left(X < \frac{1}{4}\right) = 1 - F\left(\frac{1}{4}\right) = 1 - \left(1 - e^{-5*\frac{1}{4}}\right) = \boldsymbol{e^{-\frac{5}{4}}}$

→ Normalni rozdeleni – priklad

- 1kg balicky masa. Balicek je v norme pokud se hmotnost lisi max o 10 gramu.
  - $\mu = 998[grams]\ (str. hodnota)$
  - $\sigma = 6[g]\ (smerodatna\ odchylka)$
  - Jaka je pst, ze jedno nahodne vybrane baleni je ok?
  - $X \sim No(998, 6^2)$
  - $P(990 < X < 1010) = musime\ prevest = P\left(\frac{990-998}{6} < \frac{X-998}{6} < \frac{1010-998}{6}\right) = P\left(-\frac{4}{3} < U < 2\right) = \Phi(2) - \Phi\left(-\frac{4}{3}\right) = 0.97725 - \left(1 - \Phi\left(\frac{4}{3}\right)\right) = 0.97725 - (1 - 0.90824) = 0.88549$

$P(U < b) = \Phi(b)$

$P(U \geq a) = 1 - \Phi(a)$

$P(a \leq U < b) = \Phi(b) - \Phi(a)$

$\Phi(-u) = 1 - \Phi(u)$

→ Testovani hypotez – priklad

- Pumerna vyska ditete v 6 letech je $\mu = 129[cm]$; $\sigma = 4.5[cm]$. Do prvni tridy prichazi 6 deti: 130, 132, 135, 135, 139, 143
  - H0 .. je to nahoda; H1 .. neni to nahoda
  - $\alpha = 0.01$
  - $\bar{X}$ … $prumer\ 6\ hodnot\ veliciny\ X$
  - $\bar{X} \sim No\left(\mu, \frac{\sigma^2}{N}\right); \bar{\mu} = 129; \overline{\sigma^2} = \frac{4.5^2}{6}$
  - $\bar{X} \sim No\left(129, \frac{4.5^2}{6}\right)$
  - $prumer\ z\ 6ti\ hodnot: 135.667$
  - $P(\bar{X} < T) = 0.99$
  - $P\left(\frac{\bar{X}-129}{1.837} < \frac{T-129}{1.837}\right) = 0.99$
  - $P\left(U < \frac{T-129}{1.837}\right) = 0.99$
  - $\Phi\left(\frac{T-129}{1.837}\right) = 0.99$
  - $\frac{T-129}{1.837} = 2.33$
  - $T = 133.28021$
  - -> vse pod touto hranici je H0, my ale mame prumer 135, takze se jedna o H1
- → Dalsi typy podobnych vypoctu
  - $P\left(\frac{\bar{X}-129}{1.837} > \frac{T-129}{1.837}\right) = 0.99$
  - $P\left(\frac{\bar{X}-129}{1.837} < \frac{T-129}{1.837}\right) = 0.01$
  - $\Phi\left(\frac{T-129}{1.837}\right) = 0.01$
  - $\Phi(a) = 0.01$
  - $1 - \Phi(-a) = 0.01$
  - $\Phi(-a) = 0.99$
  - $-a = 2.33$

→ Binomicke na normalni – priklad

- Hrajeme ruletu, sazime pouze na sude/liche castky - cisla 0.1-36. 200x sazim sude/liche
- Jaka je pst, ze 100 – 110 krat vyhraju?
  - $X = pocet\ vyher$
  - $X \sim Bi\left(200, \frac{18}{37}\right)$
  - $\mu = 200 * \frac{18}{37} = 97.297$
  - $\sigma = 200 * \frac{18}{37} * \left(1 - \frac{18}{37}\right) = 49.963$
  - $X \sim No(97{,}297; 49{,}963);\ X \sim (97{,}297; 7.068)$
  - $P(100 \leq X \leq 110) = P(99.5 \leq X \leq 110.5) = P\left(\frac{99.5-97.297}{7.068} \leq U \leq \frac{110.5-97.297}{7.068}\right) = P(0.312 \leq U \leq 1.868) = \Phi(1.868) - \Phi(0.312) = 0.347$
- Jaka je pst, ze vyhraju vice nez 130 krat?
  - $X = pocet\ vyher$
  - $P(X > 131) = P(131 < X) = P\left(\frac{131.5-97.297}{7.068} < U\right) = 1 - \Phi(4.839) = 0.00003$

INTEGROVANI

- → Hustota f(t) exp rozdeleni:

$$f(t) = \begin{cases} 0 & pro\ t < 0 \\ \lambda * e^{-\lambda t} & pro\ t \geq 0 \end{cases}$$

- Je dana nahodna velicina $X \sim Exp(2)$. Jednoduchou Simpsonovou metodou vypoctete P(X<2).
  - $X \sim Exp(\lambda); \lambda = 2$
  - $P(X < 2) = \int_0^2 2 * e^{-2*t} dt$
  - Sampsonova metoda:
  - $\int_a^b f(x)dx = \cdots = \frac{2-0}{6}\left(f(0) + 4f\left(\frac{2}{2}\right) + f(2)\right) = \frac{1}{3}(2 + 4 * 2e^{-2} + 2e^{-4}) \sim 1.0398$

→ Binomicke na normalni – priklad 2

- Test ma 100 otazek, kazda 5 moznosti, 1 spravne. Jaka je pst., ze ziskame alespon polovinu b.?
  - N = 100; p =1/5; EX = 100*1/5=20
  - $X \sim Bi\left(100, \frac{1}{5}\right)$
  - $\mu = 100 * \frac{1}{5} = 20$
  - $\sigma = 100 * \frac{1}{5} * \left(1 - \frac{1}{5}\right) = 16$
  - $P\left(X > \frac{EX}{2}\right) = P(X > 10)$
  - $X \sim No(20{,}16)$;
  - $P(X > 10) = P\left(\frac{X-20}{4} > \frac{10-20}{4}\right) = P\left(U > -\frac{5}{2}\right) = P\left(U > -\frac{5}{2}\right) = 1 - \Phi\left(-\frac{5}{2}\right) = 1 - \left(1 - \Phi\left(\frac{5}{2}\right)\right) = 1 - (1 - 0.9938903) = 0.99379$

→ Pst funkce (priklad 7)

- Je dana pst funkce p(x) nejake nahodne veliciny X:

$$p(x) = \begin{cases} k * 0.7^x\ pro\ x \in \{0,1,2,3 \ldots\} \\ 0 \qquad\qquad jinak \end{cases}$$

- Desetinnym cislem vyjadrete F(3,5):
  - Soucet geom rady:
  - $\frac{k}{1-0.7} = 1 \ \Rightarrow k = \frac{3}{10}$
  - $F(3.5) = \sum_{x=0}^{3} \frac{3}{10} * 0.7^x = 0.7599$

→ MITOZA

- Doba procesu zvaneho mitoza ma normalni rozdeleni se stredni hodnotou $\mu = 50$ minut a smer odchylkou $\sigma = 5$ minut. Urcete jaka je pst, ze se bunka rozdeli pomaleji nez za 55 minut?
  - $P(X > 55) = ?$
  - $P(X > 55) = P\left(U > \frac{55-50}{5}\right) = P(U > 1) = 1 - \Phi(1) = 0.1587$
- Urcete takove x0, ze u 90% bunek bude deleni trvat dele nez x0 minut od jeho zacatku.
  - $P(X > x_0) = 0.9$
  - $P\left(U > \frac{x_0-50}{5}\right) = 0.9$
  - $1 - \Phi\left(\frac{x_0-50}{5}\right) = 0.9$
  - $\Phi\left(\frac{x_0-50}{5}\right) = 0.1$
  - $x_0 = 43.6$ minut