

10707

Deep Learning: Spring 2020

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Lecture 2:

Representational power of
neural networks

Supervised learning

Empirical risk minimization approach:
minimize a **training** loss l over a class of **predictors** \mathcal{F} :

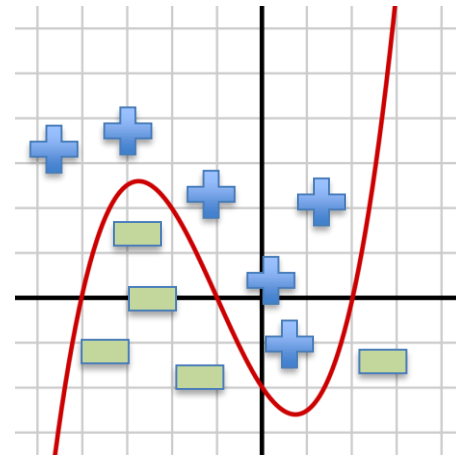
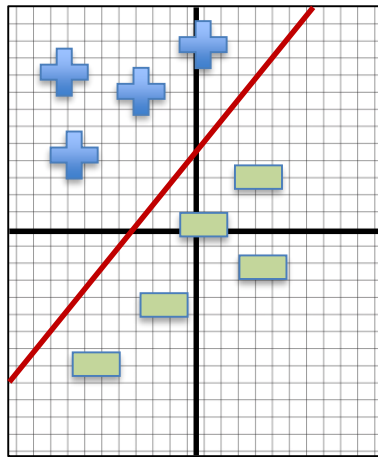
$$\hat{f} = \operatorname{argmin}_{f \in \mathcal{F}} \sum_{(x,y): \text{training samples}} l(f(x), y)$$

Three pillars:

- (1) How expressive is the class \mathcal{F} ? (**Representational power**)
- (2) How do we minimize the training loss efficiently? (**Optimization**)
- (3) How does \hat{f} perform on unseen samples? (**Generalization**)

Expressivity

What do we mean by expressivity?



Expressive = functions in class can represent “complicated” functions

“Universal” expressivity of neural networks

(1): Neural networks are **universal approximators**: given any Lipschitz $f: \mathbb{R}^d \rightarrow \mathbb{R}$, a **shallow** (3-layer) neural network with $\sim \left(\frac{1}{\epsilon}\right)^d$ neurons can approximate it to within ϵ error.

“curse of dimensionality”

(2): Neural networks can **circumvent** the curse of dimensionality for functions w/ decaying Fourier coefficients:

shallow neural networks with $\sim \left(\frac{1}{\epsilon}\right)$ neurons can approximate them to within ϵ error.



Universal approximation I: Lipschitz function are approximable

Recall, a function $f: [0,1]^d \rightarrow \mathbb{R}$ is **L-Lipschitz** (in an l_∞ sense) if:

$$\forall x, y \in [0,1]^d, \quad |f(x) - f(y)| \leq L \max_{i \in [d]} |x_i - y_i|$$

First, we show neural networks are **universal approximators**: given any Lipschitz function $f: [0,1]^d \rightarrow \mathbb{R}$, a **shallow** (3-layer) neural network with $\sim \left(\frac{1}{\epsilon}\right)^d$ neurons can approximate it to within ϵ error.

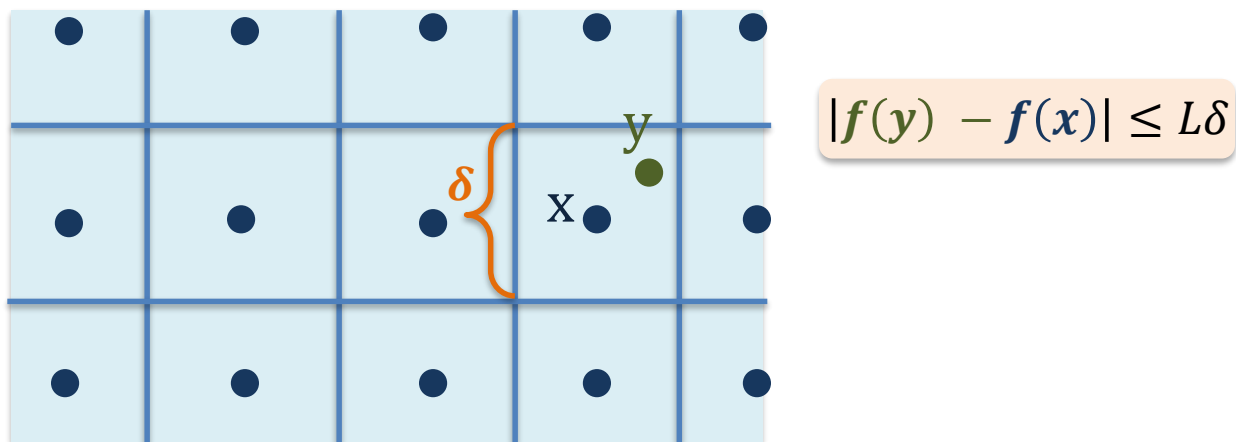
Theorem: For any **L-Lipschitz** function $f: [0,1]^d \rightarrow \mathbb{R}$, there is a **3-layer** neural network \hat{f} with $O\left(d \left(\frac{L}{\epsilon}\right)^d\right)$ ReLU neurons, s.t.

$$\int_{[0,1]^d} |f(x) - \hat{f}(x)| dx \leq \epsilon$$

 l_1 error

Universal approximation I: Proof intuition

Part 1: using Lipschitzness, we can “query” the values of function f approximately by querying its values on a fine grid.



Part 2: we can approximate f as linear combination of “queries”.

$$f(x) \approx \sum_{\text{cells } C_i} 1_{x \in C_i} f(x_i)$$

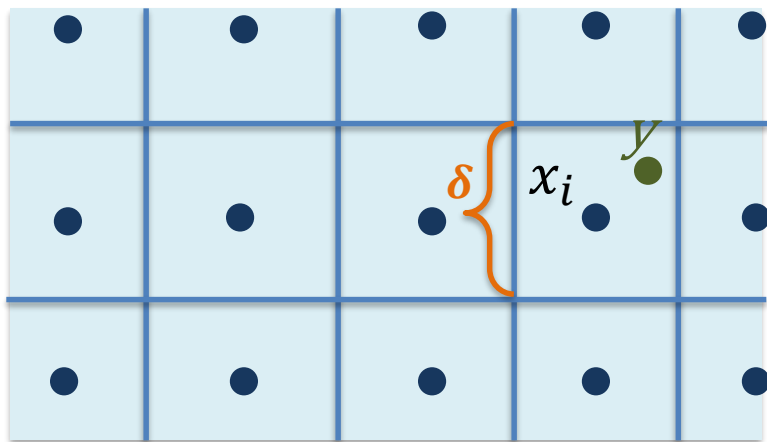
Part 3: Approximate the indicators using ReLUs

Universal approximation I:

Part 1, formally

Lemma: Let $f: [0,1]^d \rightarrow \mathbb{R}$ be L -Lipschitz and $P = (C_1, C_2, \dots, C_N)$ a partition of $[0,1]^d$ into cells of side lengths at most δ . Consider any set $(x_1, x_2, \dots, x_N), x_i \in C_i$. Then:

$$\sup_{i \in N} \sup_{y \in C_i} |f(y) - f(x_i)| \leq L\delta$$



Proof: By Lipschitzness, we have

$$\begin{aligned} \forall i, y \in C_i: |f(y) - f(x_i)| &\leq L \max_{i \in [d]} |y - x_i| \\ &\leq L\delta \quad \square \end{aligned}$$

Universal approximation I:

Part 2, formally

Lemma: Let $f: [0,1]^d \rightarrow \mathbb{R}$ be 1-Lipschitz, $P = (C_1, C_2, \dots, C_N)$ a partition of $[0,1]^d$ into rectangles of side lengths at most δ , and a set (x_1, x_2, \dots, x_N) , $x_i \in C_i$. Then,

$$g(x) = \sum_{i=1}^N 1_{x \in C_i} f(x_i) \text{ satisfies } \sup_{x \in [0,1]^d} |f(x) - g(x)| \leq L \delta$$

Proof: Let $x \in C_i$. Then, $1_{x \in C_i} = 1$, and $1_{x \in C_j} = 0$ for $j \neq i$.

So, $g(x) = f(x_i)$.

By Lemma 1, $|f(x) - g(x)| = |f(x) - f(x_i)| \leq L \delta \quad \square$

Universal approximation I: approximating indicators of cells

Lemma: Let $C \subseteq \mathbb{R}^d$ be a cell, namely $C = \{x: x \in [l_i, r_i], i \in d\}$. Then, there exists a 2-layer network $\tilde{h}(x)$ of size $O(d)$ and ReLU activation, s.t. $\int_{x \in [0,1]^d} |\tilde{h}(x) - 1(x \in [l_i, r_i], i \in d)| dx \rightarrow 0$

Proof: First, write indicator for cell as:

*For any $\gamma > 0$,
we'll take γ small*

$$1(x \in [l_i, r_i], i \in d) = 1 \left(\sum_{i=1}^d (1(x_i \geq l_i) + 1(x_i \leq r_i)) \geq 2d - 1 + \gamma \right)$$

Why? x is in cell iff all the indicators $1(x_i \geq l_i) + 1(x_i \leq r_i)$ are on.

All these indicators are on iff they sum to $2d$.

(If at least one is off, they sum to $2d-1$)

If we can approximate indicators, we're all good!

Universal approximation I: approximating indicators of cells

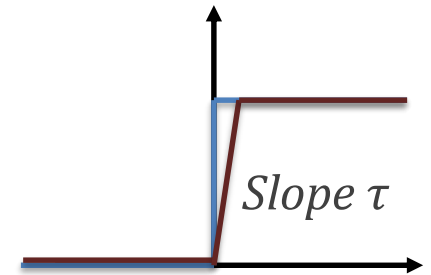
Claim: For $\tau \geq 0$, $x \in \mathbb{R}$:

$$|1(x \geq 0) - (\sigma(\tau x) - \sigma(\tau x - 1))| = \begin{cases} \leq 1, & \text{if } 0 \leq x \leq 1/\tau \\ 0, & \text{otherwise} \end{cases}$$

Proof: Consider several cases:

Case 1, $x \leq 0$: $1(x \geq 0) = 0$ and $\sigma(\tau x) - \sigma(\tau x - 1) = 0$, so

$$1(x \geq 0) - (\sigma(\tau x) - \sigma(\tau x - 1)) = 0$$



Case 2, $x \geq 1/\tau$: $1(x \geq 0) = 1$ and $\sigma(\tau x) - \sigma(\tau x - 1) = 1$, so

$$1(x \geq 0) - (\sigma(\tau x) - \sigma(\tau x - 1)) = 0$$

Case 3, $0 \leq x \leq 1/\tau$: $1(x \geq 0) = 1$ and $\sigma(\tau x) - \sigma(\tau x - 1) = \tau x$, so

$$|1(x \geq 0) - (\sigma(\tau x) - \sigma(\tau x - 1))| = \tau x \leq 1$$

Universal approximation I: approximating indicators of cells

$$h(x) := 1 \left(\sum_{i=1}^d (1(x_i \geq l_i) + 1(x_i \leq r_i)) \geq 2d - 1 + \gamma \right)$$

Replace all indicators by difference of ReLUs. *What is the error?*

For brevity, let $\tilde{1}(x \geq 0) = \sigma(\tau x) - \sigma(\tau x - 1)$, for some τ we will choose.

Let
$$\tilde{h}(x) := 1 \left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) \right) \geq 2d - 1 + \gamma \right)$$

$$\tilde{\tilde{h}}(x) := \tilde{1} \left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) \right) \geq 2d - 1 + \gamma \right)$$

(Change the approximations “iteratively”.)

Universal approximation I: approximating indicators of cells

$$\begin{aligned} h(x) &:= 1\left(\sum_{i=1}^d (1(x_i \geq l_i) + 1(x_i \leq r_i)) \geq 2d - 1 + \gamma\right) \\ \tilde{h}(x) &:= 1\left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right) \\ \tilde{\tilde{h}}(x) &:= \tilde{1}\left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right) \end{aligned}$$

We have:

$$\begin{aligned} \int_{x \in [0,1]^d} |\tilde{\tilde{h}}(x) - h(x)| dx &= \int_{x \in [0,1]^d} |\tilde{\tilde{h}}(x) - \tilde{h}(x) + \tilde{h}(x) - h(x)| dx \\ &\stackrel{\text{Triangle inequality}}{\leq} \int_{x \in [0,1]^d} \left(|\tilde{\tilde{h}}(x) - \tilde{h}(x)| + |\tilde{h}(x) - h(x)| \right) dx \end{aligned}$$

Let's handle two terms one by one.

Universal approximation I: approximating indicators of cells

$$\begin{aligned}\tilde{h}(x) &:= 1 \left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) \right) \geq 2d - 1 + \gamma \right) \\ \tilde{\tilde{h}}(x) &:= \tilde{1} \left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) \right) \geq 2d - 1 + \gamma \right)\end{aligned}$$

Claim: $|1(x \geq 0) - (\sigma(\tau x) - \sigma(\tau x - 1))| = \begin{cases} \leq 1, & \text{if } 0 \leq x \leq 1/\tau \\ 0, & \text{otherwise} \end{cases}$

First: $\tilde{\tilde{h}}(x) - \tilde{h}(x) \neq 0$ only if $\exists i : x_i \in \left(l_i, l_i + \frac{1}{\tau}\right)$ or $x_i \in \left(r_i, r_i - \frac{1}{\tau}\right)$

(If $\tilde{\tilde{h}}(x) - \tilde{h}(x) \neq 0$, $\sum_i \tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) - (2d - 1) \in [\gamma, \gamma + \frac{1}{\tau}]$, and if condition above isn't satisfied, $\tilde{1} = 1$, so $\sum_i \tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) - (2d - 1)$ is integer, so cannot belong to interval for small enough $\gamma > 0$.)

Measure of such x's is bdd by $\sum_i \left(\int_{x_i \in (l_i, l_i + \frac{1}{\tau})} 1 \, dx + \int_{x_i \in (r_i, r_i - \frac{1}{\tau})} 1 \, dx \right) \leq \frac{2d}{\tau}$, so

$$\int_{x \in [0,1]^d} |\tilde{\tilde{h}}(x) - \tilde{h}(x)| \leq \frac{2d}{\tau}$$

Universal approximation I: approximating indicators of cells

$$h(x) := 1(\sum_{i=1}^d (1(x_i \geq l_i) + 1(x_i \leq r_i)) \geq 2d - 1 + \gamma)$$

$$\tilde{h}(x) := 1(\sum_{i=1}^d (\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)) \geq 2d - 1 + \gamma)$$

Claim: $|1(x \geq 0) - (\sigma(\tau x) - \sigma(\tau x - 1))| = \begin{cases} \leq 1, & \text{if } 0 \leq x \leq 1/\tau \\ 0, & \text{otherwise} \end{cases}$

Second: $\int_{x \in [0,1]} |\tilde{h}(x) - h(x)| dx$ *Indicators are equal if inputs are equal*

$$\leq \int_{x \in [0,1]^d} \left| 1 \left(\sum_{i=1}^d (1(x \geq l_i) + 1(x \leq r_i)) \neq \sum_{i=1}^d (\tilde{1}(x \geq l_i) + \tilde{1}(x \leq r_i)) \right) \right| dx$$

$$\leq \int_{x \in [0,1]^d} \sum_{i=1}^d 1(1(x \geq l_i) \neq \tilde{1}(x \geq l_i)) + \sum_{i=1}^d 1(1(x \leq r_i) \neq \tilde{1}(x \leq r_i)) dx$$

By Claim $\leq 2d/\tau$

Universal approximation I: approximating indicators of cells

$$\begin{aligned}h(x) &:= 1\left(\sum_{i=1}^d (1(x_i \geq l_i) + 1(x_i \leq r_i)) \geq 2d - 1 + \gamma\right) \\ \tilde{h}(x) &:= 1\left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right) \\ \tilde{\tilde{h}}(x) &:= \tilde{1}\left(\sum_{i=1}^d \left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)\end{aligned}$$

Putting together, we have:

$$\begin{aligned}\int_{x \in [0,1]^d} |\tilde{\tilde{h}}(x) - h(x)| dx &= \int_{x \in [0,1]^d} |\tilde{\tilde{h}}(x) - \tilde{h}(x) + \tilde{h}(x) - h(x)| dx \\ &\leq 4d/\tau\end{aligned}$$

Also, $\tilde{\tilde{h}}(x)$ is a 2-layer net with ReLU activations and $O(d)$ nodes!

Universal approximation I: Putting everything together

By Part 1+2,
$$\sup_{x \in [0,1]^d} |f(x) - \sum_{i=1}^N 1_{x \in C_i} f(x_i)| \leq L \delta$$

Moreover, the number of cells N can be bounded by $\left(\frac{1}{\delta}\right)^d$

By indicator approximation: can approximate arbitrarily well by taking $\tau \rightarrow \infty$ with a 2-layer ReLU net.

Combining the above two points, we get a $\left(\frac{1}{\delta}\right)^d$ -sized 3-layer net s.t.

$$\int_{[0,1]^d} |f(x) - \hat{f}(x)| dx \leq L \delta$$

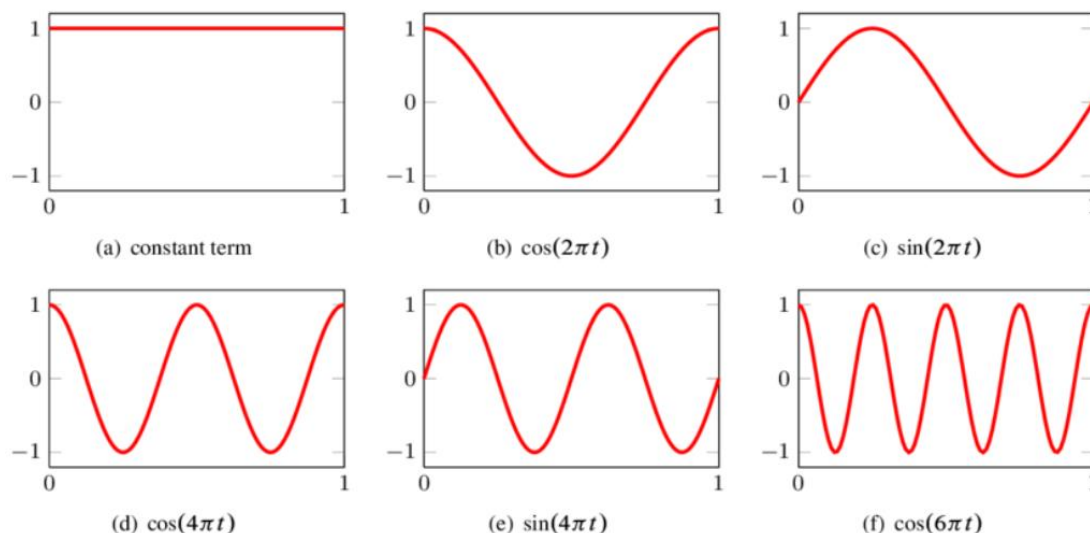
Taking $\delta = \frac{\epsilon}{L}$, the theorem follows.

Escaping the curse of dimensionality: Nuggets of Fourier analysis

Can the $\left(\frac{1}{\epsilon}\right)^d$ dependence be avoided for “nice” functions?

Yes! Relevant property is **decay** of the Fourier coefficients.

Recall: The **Fourier basis** for “nice” functions from $\mathbb{R}^d \rightarrow \mathbb{R}$ consists of basis functions $\{e_w(x) = e^{i\langle w, x \rangle} = \cos(\langle w, x \rangle) + i \sin(\langle w, x \rangle) | w \in \mathbb{R}^d\}$.



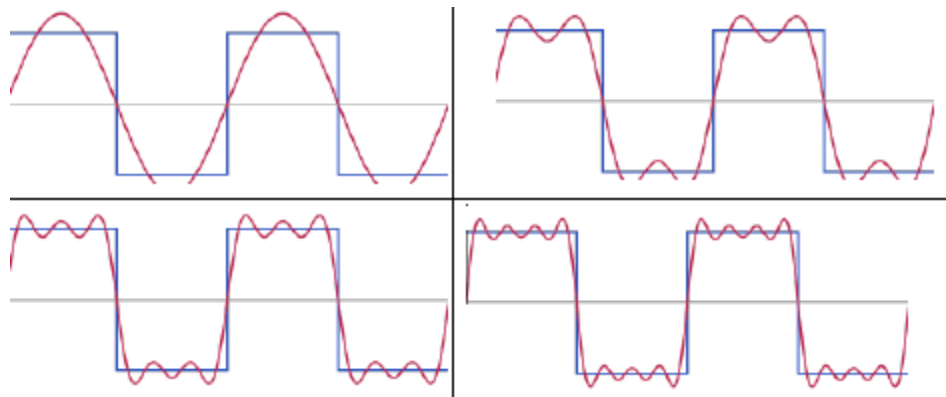
$\|w\|$ is larger: function
oscillates more

Escaping the curse of dimensionality: Nuggets of Fourier analysis

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Higher and higher
frequencies =>
better approximation

Escaping the curse of dimensionality: Nuggets of Fourier analysis

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Recall: The **Fourier basis** for “nice” functions from $\mathbb{R}^d \rightarrow \mathbb{R}$ consists of basis functions $\{e_w(x) = e^{i\langle w, x \rangle} = \cos(\langle w, x \rangle) + i \sin(\langle w, x \rangle) | w \in \mathbb{R}^d\}$.

Recall: The **Fourier integral theorem** gives coefficients for this basis:

Defining $\hat{f}(w) = \int_{\mathbb{R}^d} f(x) e^{-i\langle w, x \rangle} dx$, we have:

$$f(x) = \int_{\mathbb{R}^d} \underbrace{\hat{f}(w)}_{\substack{\text{Coefficient for} \\ \text{basis fn } e^{i\langle w, x \rangle}}} e^{i\langle w, x \rangle} dw$$

Escaping the curse of dimensionality: Nuggets of Fourier analysis

Can the $\left(\frac{1}{\epsilon}\right)^d$ dependence be avoided for “nice” functions?
Yes! Relevant property is **decay** of the Fourier coefficients.

Def.: The **Barron constant** of a function f is the quantity

$$C = \int_{\mathbb{R}^d} ||w|| \, |\hat{f}(w)| \, dw$$

Interpretation: the higher-order Fourier coefficients (i.e. high-oscillation parts of f) are small.

We will look for $O_C \left(\frac{1}{\epsilon}\right)$ dependence of the size of the network.

Escaping the curse of dimensionality: Barron's Theorem

Def.: The **Barron constant** of a function f is the quantity

$$C = \int_{\mathbb{R}^d} ||w|| \, |\hat{f}(w)| \, dw$$

$= \{x \in \mathbb{R}^d : ||x|| \leq 1\}$

Theorem (Barron '93): For any $f: \mathbb{B} \rightarrow \mathbb{R}$, there is a **3-layer** neural network \hat{f} with $O\left(\frac{C^2}{\epsilon}\right)$ neurons and sigmoid activation, s.t.

$$\int_{\mathbb{B}} \left(f(x) - \hat{f}(x)\right)^2 dx \leq \epsilon$$

$$= \mathbb{E}_x \left[\left(f(x) - \hat{f}(x)\right)^2 \right]$$

$l_2 \text{ error}$

Barron's theorem: proof idea

Step 1: Show that any continuous function f can be written as an “infinite” convex combination of cosine-like activations .
(**Main tool:** Fourier integral theorem)

Step 2: Show that a function f with small Barron constant can in fact be approximately written as a convex combination of a **small** number of cosine-like activations.
(**Main tool:** subsampling the above infinite combination and concentration bounds.)

Step 3: Show that the cosine non-linearities can be approximated by sigmoid non-linearities.
(**Main tool:** classical approximation theory.)

Step 1: infinite convex combination of cosine-like activations

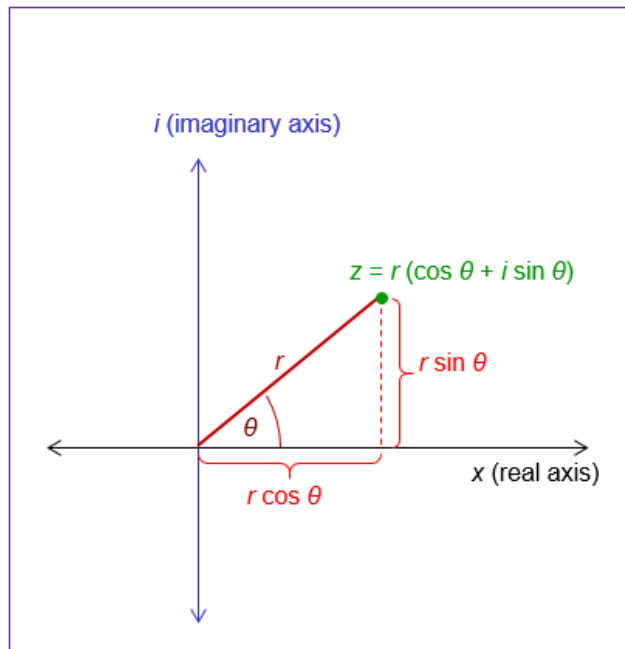
By Fourier integral theorem, we have:

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} \hat{f}(w) e^{i\langle w, x \rangle} dw \\ &= f(0) + \int_{\mathbb{R}^d} \hat{f}(w) (e^{i\langle w, x \rangle} - 1) dw \\ &\quad \swarrow \\ &= \int_{\mathbb{R}^d} \hat{f}(w) dw \end{aligned}$$

Step 1: infinite convex combination of cosine-like activations

By Fourier integral theorem, $f(x) = f(0) + \int_{\mathbb{R}^d} \hat{f}(w)(e^{i\langle w, x \rangle} - 1)dw$

Recall the **polar form** of a complex number:



$$\begin{aligned} z &= |z| e^{i \phi_z} \\ &= |z| (\cos \phi_z + i \sin \phi_z) \end{aligned}$$

Step 1: infinite convex combination of cosine-like activations

By Fourier integral theorem, $f(x) = f(0) + \int_{\mathbb{R}^d} \hat{f}(w)(e^{i\langle w, x \rangle} - 1)dw$

Recall the **polar form** of a complex number: $z = |z| e^{i \phi_z}$

Hence, we can rewrite the Fourier integral formula as:

$$f(x) = f(0) + \int_{\mathbb{R}^d} |\hat{f}(w)| (e^{i(b_w + \langle w, x \rangle)} - e^{ib_w}) dw$$

Step 1: infinite convex combination of cosine-like activations

$$f(x) = f(0) + \int_{\mathbb{R}^d} |\hat{f}(w)| (e^{i(b_w + \langle w, x \rangle)} - e^{ib_w}) dw$$

Recall the expansion of complex exponentials: $e^{iy} = \cos(y) + i \sin(y)$

As f is a real-valued function, only the real part of the above expression will survive. Hence,

$$f(x) = f(0) + \int_{\mathbb{R}^d} |\hat{f}(w)| (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) dw$$

Linear combination of cosine functions, but not *convex*!

(As $\int_{\mathbb{R}^d} |\hat{f}(w)|$ integrates potentially to > 1)

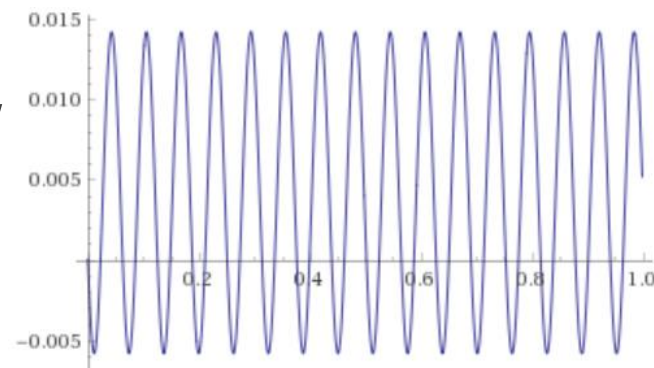
Step 1: infinite convex combination of cosine-like activations

We will rewrite: $f(x) = f(0) + \int_{\mathbb{R}^d} |\hat{f}(w)| (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) dw$

$$\begin{aligned} f(x) &= f(0) + \int_{\mathbb{R}^d} |\hat{f}(w)| (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) dw \\ &= f(0) + \int_{\mathbb{R}^d} \frac{|\hat{f}(w)| \|w\|}{C} \left(\frac{C}{\|w\|} (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) \right) dw \end{aligned}$$

Convex combination of cosine-like activations!

$$\left(\text{As } \int_{\mathbb{R}^d} \frac{|\hat{f}(w)| \|w\|}{C} = 1 \right)$$



Step 2: convex combination of small number of cosine-like activations

Recall: $f(x) = f(0) + \int_{\mathbb{R}^d} \frac{|\hat{f}(w)| ||w||}{C} \left(\frac{C}{||w||} (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) \right)$

We will prove that there is a set S of w 's, s.t.


$$f(x) \approx f(0) + \frac{1}{|S|} \sum_{w \in S} \frac{C}{||w||} (\cos(b_w + \langle w, x \rangle) - \cos(b_w))$$

Natural idea: **subsampling**!

Remember, these *integrate* to 1, so form a **distribution** over w 's.

Repeat **r** times:

Choose a new $w \in \mathbb{R}$ to add to S with probability $\frac{|\hat{f}(w)| ||w||}{C}$



Step 2: convex combination of small number of cosine-like activations


$$\text{Recall: } f(x) = f(0) + \int_{\mathbb{R}^d} \frac{|\hat{f}(w)| \|w\|}{C} \left(\frac{C}{\|w\|} (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) \right)$$

Repeat r times:

Choose a new $w \in \mathbb{R}$ to add to S with probability $\frac{|\hat{f}(w)| \|w\|}{C}$

Let g_i be a random variable, denoting the i -th selected w .

Let $g = \frac{1}{r} \sum_{i=1}^r g_i$. Then, we have:

$$\mathbb{E}_x \mathbb{E}_g [(g(x) - f(x))^2] = \mathbb{E}_x \mathbb{E}_{g_i} \left[\left(\sum_i \left(\frac{1}{r} g_i - \frac{1}{r} f \right) \right)^2 \right] = \frac{1}{r^2} \mathbb{E}_x \mathbb{E}_{g_i} [(\sum_i (g_i - f))^2]$$


Direct substitution

All $g_i - f$ are mean-0, (since $\mathbb{E}[g_i] = f$), and independent.

Step 2: convex combination of small number of cosine-like activations

Repeat r times:

Choose a new $w \in \mathbb{R}$ to add to S with probability $\frac{|\hat{f}(w)||w|}{c}$

Then, we have:

$$\mathbb{E}_x \mathbb{E}_g [(g - f)^2] = \frac{1}{r^2} \mathbb{E}_x \mathbb{E}_{g_i} [(\sum_i (g_i - f))^2]$$

$$\mathbb{E}_x \mathbb{E}_{g_i, g_j} [(g_i - f)(g_j - f)] = \mathbb{E}_x [\mathbb{E}_{g_i} [(g_i - f)] \mathbb{E}_{g_j} [(g_j - f)]] = 0$$

$$\begin{aligned} \mathbb{E}[g_i] = f & \quad \mathbb{E}_x \mathbb{E}_g [(g - f)^2] = \frac{1}{r^2} \left(\sum_i \mathbb{E}_x \mathbb{E}_g [(g_i - f)^2] + \sum_{i \neq j} \mathbb{E}_x \mathbb{E}_{g_i, g_j} [(g_i - f)(g_j - f)] \right) \\ & = \frac{1}{r^2} \left(\sum_i \mathbb{E}_x \mathbb{E}_g [(g - \mathbb{E}[g])^2] \right) = \frac{1}{r} \mathbb{E}_x \mathbb{E}_g [(g - \mathbb{E}_g[g])^2] \end{aligned}$$

$$= \frac{1}{r} (\mathbb{E}_x \mathbb{E}_g [g^2] - \mathbb{E}_x \mathbb{E}_g [g]^2) \leq \frac{1}{r} \mathbb{E}_g \mathbb{E}_x [g^2]$$

$$\leq \frac{1}{r} \max_w \mathbb{E}_x [g_w^2]$$

Slight abuse of notation

Change order of expectations

Step 2: convex combination of small number of cosine-like activations

Repeat r times:

Choose a new $w \in \mathbb{R}$ to add to S with probability $\frac{|\hat{f}(w)||w|}{C}$

Let g_i denote the i -th selected w . Let $g = \frac{1}{r} \sum_{i=1}^r g_i$

Then, we have: $\mathbb{E}_x \mathbb{E}_g [(g - f)^2] \leq \frac{1}{r} \max_w \mathbb{E}_x [g_w^2]$

Writing out $\mathbb{E}_x [g_w^2]$ explicitly, we will show that:

$$\forall w: \int_{x \in \mathbb{B}} \left(\frac{C}{||w||} (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) \right)^2 dx \leq C^2$$

Step 2: convex combination of small number of cosine-like activations

Claim: $\forall w: \int_{x \in \mathbb{B}} \left(\frac{C}{||w||} (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) \right)^2 dx \leq C^2$

Note, \cos is **1-Lipschitz** (show this if you don't see it!). Hence:

$$|(\cos(b_w + \langle w, x \rangle) - \cos(b_w))| \leq |\langle w, x \rangle| \leq ||w|| ||x||$$

So, $\left(\frac{C}{||w||} (\cos(b_w + \langle w, x \rangle) - \cos(b_w)) \right)^2 \leq C^2 ||x||^2 \leq C^2$

Integrating, the claim follows.

Step 2: convex combination of small number of cosine-like activations

Repeat **r** times:

Choose a new $w \in \mathbb{R}$ to add to S with probability $\frac{|\hat{f}(w)||w|}{C}$

Let g_i denote the i -th selected w . Let $g = \frac{1}{r} \sum_{i=1}^r g_i$

Plugging in previous bound: $\mathbb{E}_g \mathbb{E}_x [(g - f)^2] \leq \frac{C^2}{r}$

If the **expectation** of a random variable is $\leq \frac{C^2}{r}$, there must be some **realization** of it w/ value $\leq \frac{C^2}{r}$. Hence:

There exist some g , s.t. $\mathbb{E}_x [(g(x) - f(x))^2] \leq \frac{C^2}{r}$

Almost there! g is a width r network, with cosine-like activation.

Step 3: approximating the cosines

Finally, we approximate the cosine-like activations using sigmoids.

Let us denote $g_w(x) = \frac{C}{||w||} (\cos(b_w + \langle w, x \rangle) - \cos(b_w))$

Namely, we show that: there exists a 2-layer neural net G_0 of size $O\left(\frac{1}{\epsilon}\right)$ with sigmoid activations, s.t. $\sup_{x \in \mathbb{B}} |G_0(x) - g_w(x)| \leq \epsilon$

Step 3: approximating the cosines

$$g_w(x) = \frac{C}{||w||} (\cos(b_w + \langle w, x \rangle) - \cos(b_w))$$

Exists 2-layer neural net G_0 of size $O\left(\frac{1}{\epsilon}\right)$ with sigmoid activations, s.t.

$$\sup_{x \in \mathbb{B}} |G_0(x) - g_w(x)| \leq \epsilon$$

First, we rewrite $g_w(x)$ slightly:

$$g_w(x) = \frac{C}{||w||} \left(\cos \left(b_w + ||w|| \underbrace{\left\langle \frac{w}{||w||}, x \right\rangle}_y \right) - \cos(b_w) \right) \\ := h_w(y)$$

Hence, $g_w(x) = h_w \left(\left\langle \frac{w}{||w||}, x \right\rangle \right)$, i.e. a composition of a **linear function** and **h_w** , and the domain of h_w is $[-1,1]$ (univariate!). Suffices to approx. h_w using sigmoids.

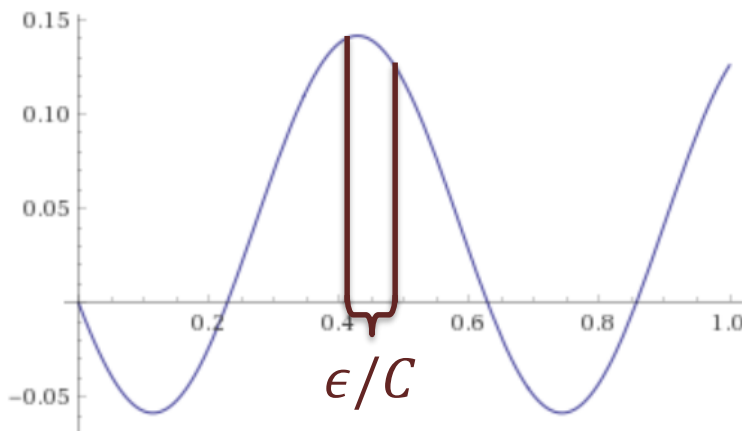
Step 3: approximating the cosines

$$h_w(y) = \frac{C}{||w||} (\cos(b_w + ||w||y) - \cos(b_w))$$

Exists 2-layer neural net G_0 of size $O\left(\frac{1}{\epsilon}\right)$ with sigmoid activations, s.t.

$$\sup_{x \in [-1,1]} |G_0(y) - h_w(y)| \leq \epsilon$$

*Check derivative bd
gives Lipschitzness!*



1. h_w is **C-Lipschitz**:

$$h'_w(y) = C \sin(b_w + ||w||y)$$

2. Grid the interval $[-1,1]$ into intervals $[l_i, r_i]$ of size ϵ/C . Pick arbitrary $y_i \in [l_i, r_i]$

Same as in the first theorem, we have

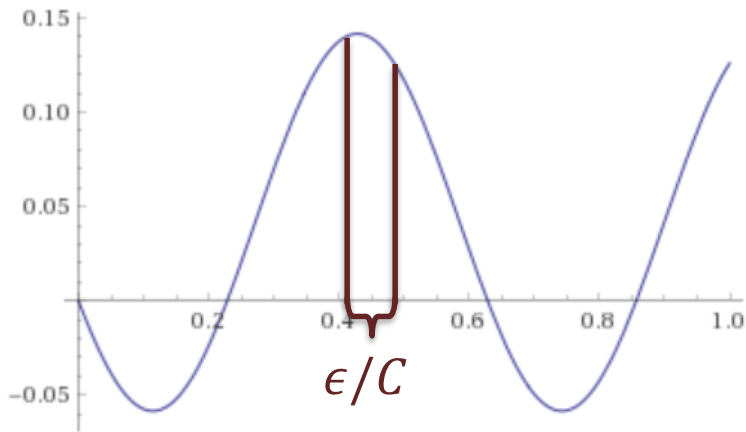
$$\sup_{x \in [-1,1]} \left| \sum_i 1(y \in [l_i, r_i]) h_w(y_i) - h_w(y) \right| \leq \epsilon$$

Step 3: approximating the cosines

$$h_w(y) = \frac{C}{||w||} (\cos(b_w + ||w||y) - \cos(b_w))$$

Exists 2-layer neural net G_0 of size $O\left(\frac{1}{\epsilon}\right)$ with sigmoid activations, s.t.

$$\sup_{x \in [-1,1]} |G_0(y) - h_w(y)| \leq \epsilon$$



$$\sup_{x \in [-1,1]} \left| \sum_i 1(y \in [l_i, r_i]) h_w(y_i) - h_w(y) \right| \leq \epsilon$$

3. We can write the indicators as differences of step functions:

$$1(y \in [l_i, r_i]) = 1(y \geq l_i) - 1(y \geq r_i)$$

Hence, it suffices to approximate a step function using a sigmoid.

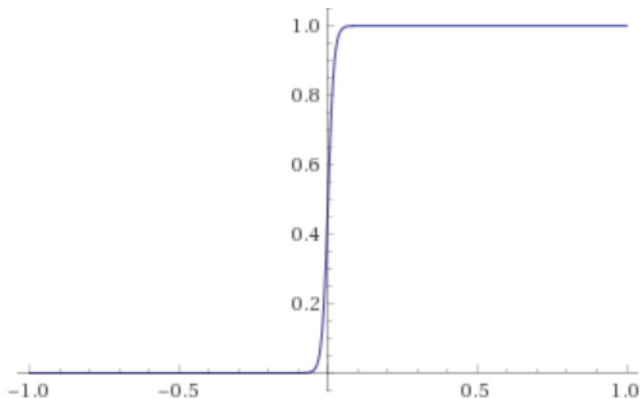
Step 3: approximating the cosines

$$h_w(y) = \frac{C}{||w||} (\cos(b_w + ||w||y) - \cos(b_w))$$

Exists 2-layer neural net G_0 of size $O\left(\frac{1}{\epsilon}\right)$ with sigmoid activations, s.t.

$$\sup_{x \in [0,1]^d} |G_0(x) - h_w(x)| \leq \epsilon$$

Approximating a step function using a sigmoid:



$$\lim_{\tau \rightarrow \infty} \sup_{x \in [0,1]^d} \left| 1(x \geq a) - \frac{1}{1 + e^{-\tau(x-a)}} \right| = 0$$

Hence, taking τ large enough, we can drive the error as small as possible (in the l_∞ sense.)

Putting everything together, the claim follows.

Parting thoughts

All results we proved are **existential**: they prove that a good approximator exists. Finding one efficiently (much less so using gradient descent) is a different matter.

The choices of non-linearities are usually very **flexible**: most results of the type we saw can be re-proven using different non-linearities. (Examples in homework.)

Many other results of similar flavor. For instance, there are also results that deep, but narrow networks are universal approximators.