

10707

Deep Learning: Spring 2020

Andrej Risteski

Machine Learning Department

Lecture 6: Advanced
optimization

Supervised learning

Empirical risk minimization approach:
minimize a **training** loss l over a class of **predictors** \mathcal{F} :

$$\hat{f} = \operatorname{argmin}_{f \in \mathcal{F}} \sum_{(x,y): \text{training samples}} l(f(x), y)$$

Three pillars:

(1) How expressive is the class \mathcal{F} ? (**Representational power**)

(2) How do we minimize the training loss efficiently? (**Optimization**)

(3) How does \hat{f} perform on unseen samples? (**Generalization**)

Recap: continuous optimization

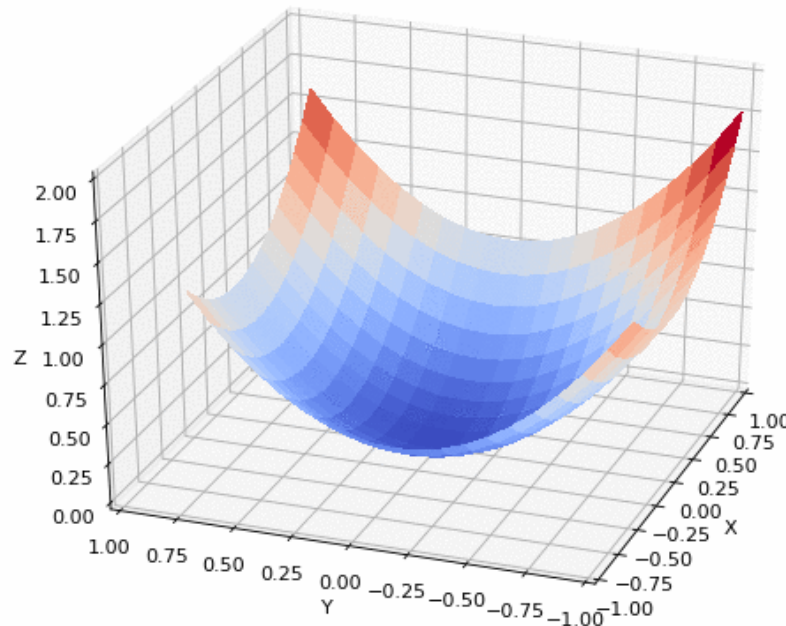
The typical training task in ML can be cast as: $\min_{x \in \mathbb{R}^d} f(x)$

Most algorithms we will look at are iterative: they progressively pick points x_1, x_2, \dots that are supposed to bring “improvement”.

$$x_{t+1} = x_t - \eta \nabla f(x_t)$$

Gradient descent

Recap: continuous optimization

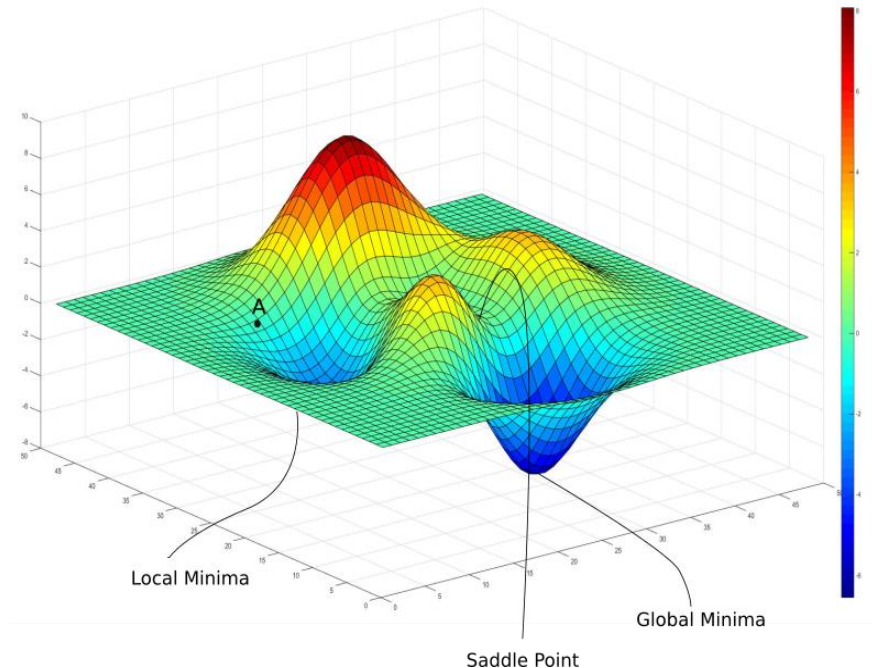


What can we hope for, in the case that $\eta \rightarrow 0$?

We stop moving when $\nabla f(x_t) \approx 0$: these are called **stationary points**.

What kinds of stationary points are there?

Recap: continuous optimization



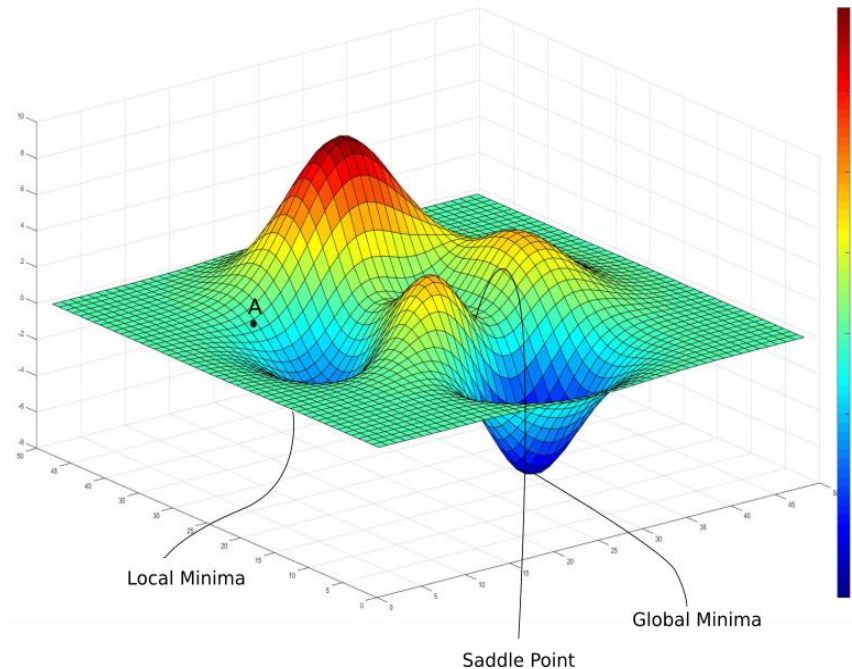
Global minimum: actual minimizer, namely $f(\hat{x}) \leq f(x), \forall x \in \mathbb{R}^d$

Local minimum: $f(\hat{x}) \leq f(x), \forall x$ s. t. $\|x - \hat{x}\| \leq \epsilon$ for some $\epsilon > 0$

Local maximum: $f(\hat{x}) \geq f(x), \forall x$ s. t. $\|x - \hat{x}\| \leq \epsilon$ for some $\epsilon > 0$

Saddle points: stationary point that is *not* a local min/max.

Recap: continuous optimization



Global minimum: finding these in general is very hard (both in theory – NP-hard, as well as in practice)

Local minimum: seem to work quite well often. Some theoretical understanding of why in very restricted cases.

Saddle points: typically bad, arise from invariances in input. Want to avoid these. (Stay tuned.)

Why are local minima good?

In short: we don't know.

Some results:

(Kawaguchi '16): **Linear** neural networks (that is, neural networks with linear activation functions): **all local minima are also global minima.**

(Hardt-Ma '18): **Linear** residual networks (that is, ResNets with linear activation functions): **all stationary points are also global minima.**

Evidence for the other side:

(Safran-Shamir '18, Yun-Sra-Jadbabie '19): **Non-linear** nets with:

- Virtually any non-linearity.
- **Even** Gaussian inputs (about as simple as it gets)
- Labels **generated** by a neural net of the same architecture we are training can have **many** bad local minima.

Overparametrization (training larger net) breaks these examples.

Why are local minima good?

In short: we don't know.

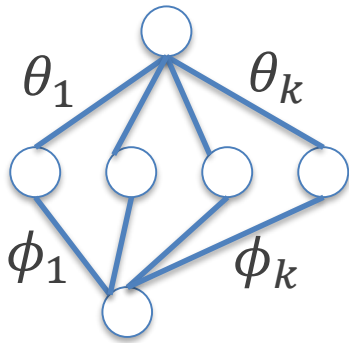
Gradient descent w/ overparametrization:

(Allen-Zhu et al '18, Du et al '18): **Sufficiently overparametrized** neural networks (that is neural nets with # parameters \gg # of training samples, training from **random initialization**, converge to small **training error** solutions.

Interaction with **generalization** is still evolving/unclear.

Intuition: saddle points arise due to symmetry

Warning: intuition only, not formal in any way!



Permutation symmetry: loss function is

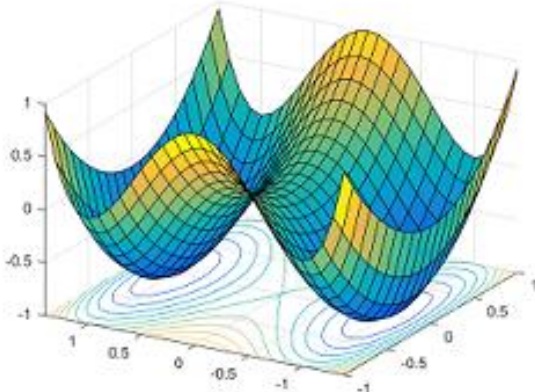
$$f(\theta_1, \theta_2, \dots, \theta_k, \phi_1, \dots, \phi_k).$$

Clearly same loss if we permute neurons, e.g.

$$f(\theta_3, \theta_1, \theta_2, \dots, \theta_k, \phi_3, \phi_1, \phi_2, \dots, \phi_k)$$

However, averaging likely will give different (larger) loss

$$f\left(\frac{\theta_1+\theta_3}{2}, \frac{\theta_2+\theta_1}{2}, \frac{\theta_3+\theta_2}{2}, \dots, \theta_k, \frac{\phi_1+\phi_3}{2}, \frac{\phi_2+\phi_1}{2}, \frac{\phi_3+\phi_2}{2}, \dots, \phi_k\right).$$



Isolated minima corresponding to permutations, connected via a “flat” region inbetween. (We’ll see later this intuition may be deficient in some ways.)

First order methods on the cheap: stochastic gradient descent

In the supervised learning setup, function we are optimizing has the form of a sum:

$$\min_{\theta} \sum_{(x,y): \text{training samples}} l(f_{\theta}(x), y)$$

Stochastic gradient descent

- **Initialize:** $\theta_0 := \{W^{(1)}, b^{(1)}, \dots, W^{(L+1)}, b^{(L+1)}\}$
- For $t=1$ to T
 - Pick a uniformly random training example (x, y) :
 - Set $\theta_{t+1} = \theta_t - \eta \nabla_{\theta} l(f_{\theta}(x), y)$



*Random variable w/ expectation true
gradient*

First order methods on the cheap: stochastic gradient descent

In the supervised learning setup, function we are optimizing has the form of a sum:

$$\min_{\theta} \sum_{(x,y): \text{training samples}} l(f_{\theta}(x), y)$$

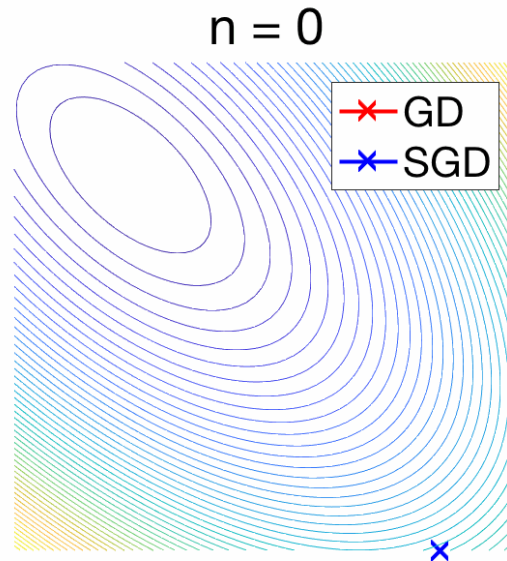
Stochastic gradient descent

- **Initialize:** $\theta_0 := \{W^{(1)}, b^{(1)}, \dots, W^{(L+1)}, b^{(L+1)}\}$
- For $t=1$ to T
 - Pick a uniformly random training example (x, y) :
 - Set $\theta_{t+1} = \theta_t - \eta \nabla_{\theta} l(f_{\theta}(x), y)$

*Actually, typically, one takes a **mini-batch** of B samples, and calculates gradient over those samples*

First order methods on the cheap: stochastic gradient descent

Due to randomness, more jittery than gradient descent:



Bug? Longer convergence time

Feature? Regularization effect / better generalization.

Feature? Can help escape saddles.

Recap: local minima

Second order checks: Hessian approximates a function to second order

Taylor's thm: $f(x + \Delta) \approx f(x) + \Delta^T \nabla f(x) + \frac{1}{2} \Delta^T \nabla^2 f(x) \Delta + O(||\Delta||^3)$

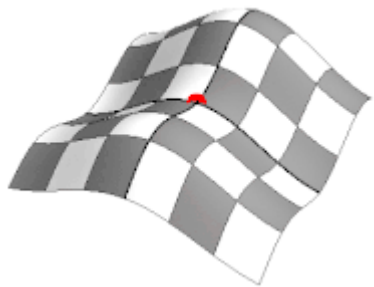
$$\approx f(x) + \frac{1}{2} \Delta^T \nabla^2 f(x) \Delta + O(||\Delta||^3)$$



If $\nabla^2 f(x) \succ 0$: for any direction Δ , and small enough $||\Delta||$

$$\Delta^T \nabla^2 f(x) \Delta + O(||\Delta||^3) \geq 0, \text{ so } f(x + \Delta) > f(x)$$

Local minimum! (Flipped for local maximum)



If $\nabla^2 f(x)$ has both positive and negative eigenvalues:

Saddle point (not a local minimum/maximum)

If neither of these attains, test is inconclusive!

First order methods on the cheap: stochastic gradient descent

Feature: can help escape (second-order) saddle points

- ⌘ Consider $f(x) = \frac{1}{2}x^\top Ax$, A is *strict saddle*: has **negative** eval
- ⌘ Gradient descent updates: $x_{t+1} = x_t - \eta \nabla f(x_t) = (I - \eta A)x_t$
- ⌘ Write $x_t = \sum_i \mu_i^t v_i$, for the *eigenbasis* $\{(v_i, \lambda_i)\}_i$ of A .
- ⌘ Then, $x_{t+1} = \sum_i (1 - \eta \lambda_i) \mu_i^t v_i$
- ⌘ In other words, $x_T = \sum_i (1 - \eta \lambda_i)^T \mu_i^0 v_i$

In other words: component in direction of negative eigenvalue grows
– we don't get stuck!

Stochastic noise helps no component to get too small.

[Can be formalized for fresh noise, *Jin, Ge, Netrapalli, Jordan, Kakade '17*]

Recap: continuous optimization

The typical training task in ML can be cast as: $\min_{x \in \mathbb{R}^d} f(x)$

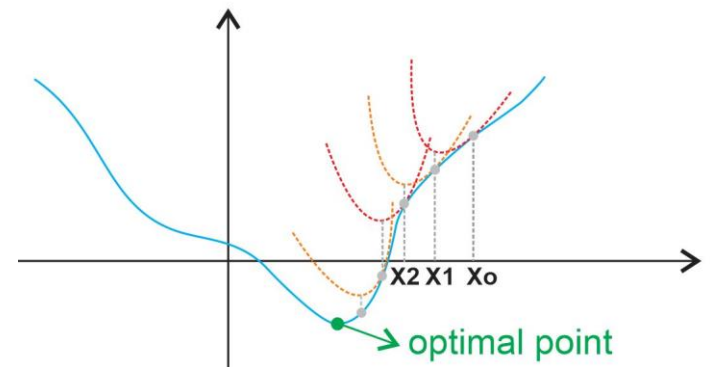
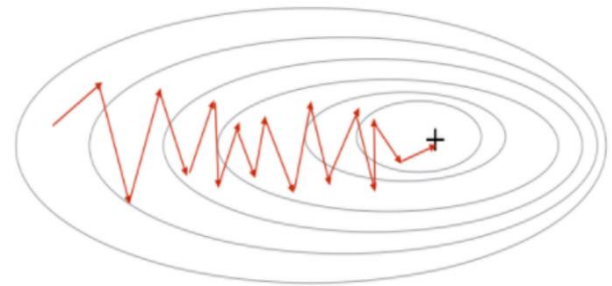
Most algorithms we will look at are iterative: they progressively pick points x_1, x_2, \dots that are supposed to bring “improvement”.

$$x_{t+1} = x_t - \eta \nabla f(x_t)$$

Gradient descent

$$\text{Set } x_{t+1} = x_t - \eta (\nabla^2 f(x_t))^{-1} \nabla f(x_t)$$

Newton's method.



Second order methods on the cheap: AdaGrad

Recall the problem with Newton: inverting large matrices

$$\text{Set } x_{t+1} = x_t - \eta (\nabla^2 f(x_t))^{-1} \nabla f(x_t)$$

Newton's method.

AdaGrad (simplified): have a separate learning rate for every parameter

$$\text{Set } x_{t+1} = x_t - \eta G_t^{-1} \nabla f(x_t),$$

$$(G_t)_{ii} = \sqrt{\sum_{j=1}^{t-1} (\nabla f(x_j)_{ii})^2}, \quad \text{else } 0$$

$$(G_t)^{-1}_{ii} = \frac{1}{\sqrt{\sum_{j=1}^{t-1} (\nabla f(x_j)_{ii})^2}}, \quad \text{else } 0$$

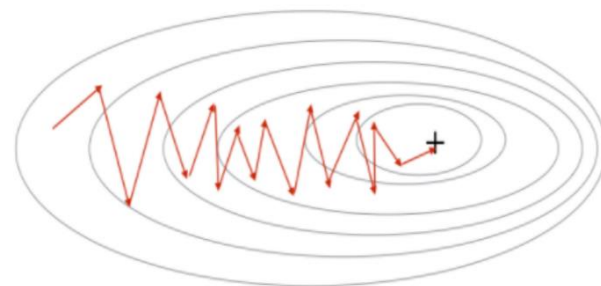
Recap: continuous optimization

The typical training task in ML can be cast as: $\min_{x \in \mathbb{R}^d} f(x)$

Most algorithms we will look at are iterative: they progressively pick points x_1, x_2, \dots that are supposed to bring “improvement”.

$$x_{t+1} = x_t - \eta \nabla f(x_t)$$

Gradient descent



*Linear combination of
prior gradients + current one*

**Momentum
(Polyak and Nesterov)**

*Evaluate gradient at a
“lookahead” point*

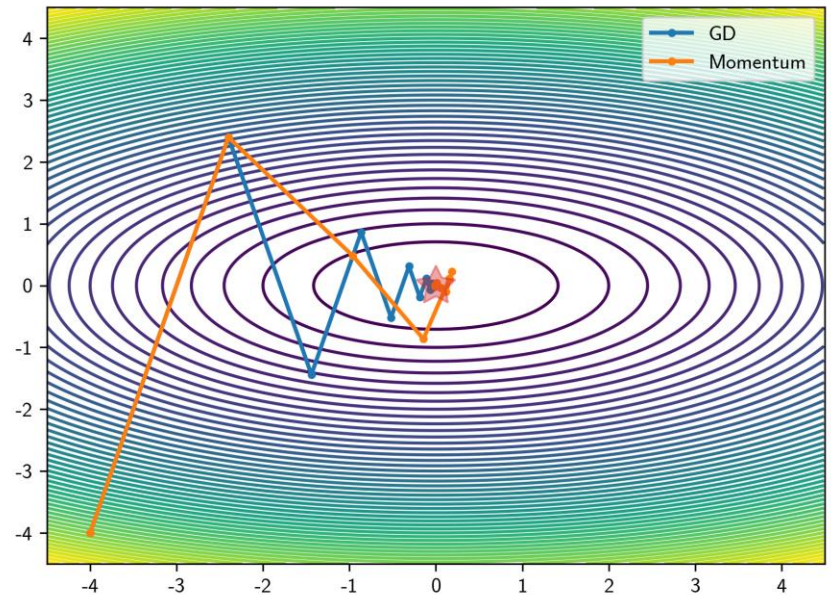
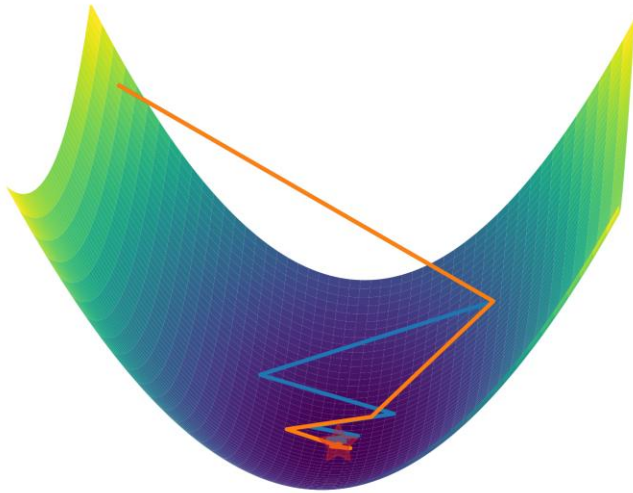
$$v_{t+1} = -\nabla f(x_t) + \beta v_t$$

$$x_{t+1} = x_t + \eta v_{t+1}$$

$$v_{t+1} = -\nabla f(x_t + \beta v_t) + \beta v_t$$

$$x_{t+1} = x_t + \eta v_{t+1}$$

Recap:continuous optimization



Momentum vs grad. descent

<https://tangbinh.github.io/01/04/Optimizers.html>

Second order methods on the cheap: RMSProp

AdaGrad (simplified): have a separate learning rate for every parameter

$$\text{Set } x_{t+1} = x_t - \eta G_t^{-1} \nabla f(x_t), \quad (G_t)_{ii} = \sqrt{\sum_{j=1}^{t-1} (\nabla f(x_t)_{ii})^2}, \quad \text{else } 0$$

RMSProp: since we keep summing the norms of the gradient, the learning rate could keep getting smaller. [**NOTE**: For convex functions, this is standard and in fact works.] Instead, keep a weighted exponential avg.

$$(g_{t+1})_i = \beta (g_t)_i + (1 - \beta) (\nabla f(x_t)_{ii})^2$$

$$(x_{t+1})_i = (x_t)_i - \frac{\eta \nabla f(x_t)_i}{\sqrt{(g_{t+1})_i}}$$

Momentum + AdaGrad = Adam

Adam: Frankenstein momentum and AdaGrad together:

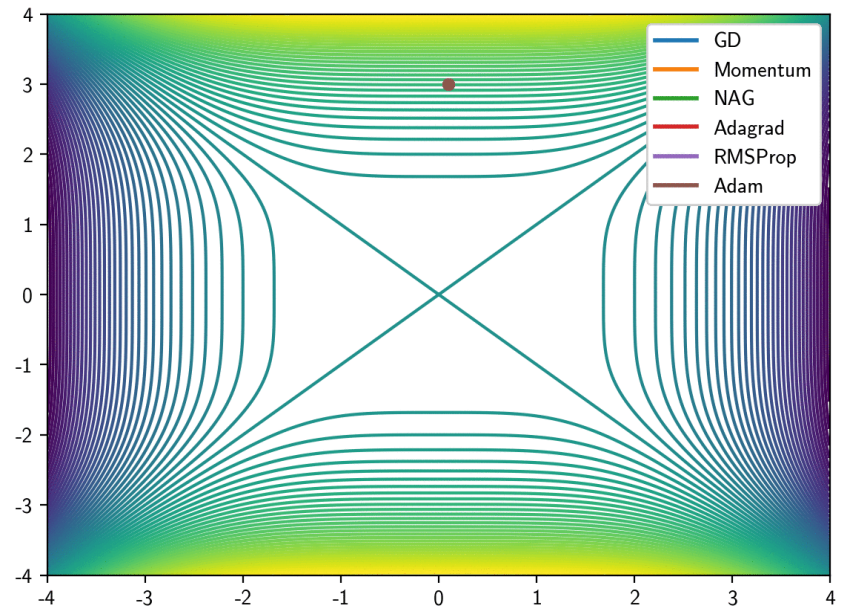
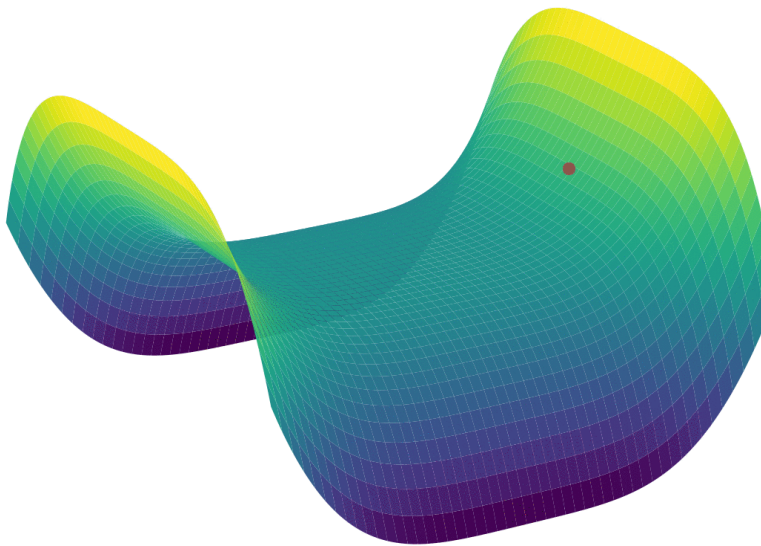
The diagram shows three equations for the Adam optimizer, each with an arrow pointing to a descriptive label on the right:

- The first equation, $v_{t+1} = \beta_1 v_t + (1 - \beta_1) \nabla f(x_t)$, is labeled *Momentum*.
- The second equation, $(g_{t+1})_i = \beta_2 (g_t)_i + (1 - \beta_2) (\nabla f(x_t)_{ii})^2$, is labeled *AdaGrad*.
- The third equation, $(x_{t+1})_i = (x_t)_i - \frac{\eta v_{t+1}}{\sqrt{(g_{t+1})_i}}$, is labeled *Combine the two*.

Default choice nowadays.

It actually does **not** converge, even on convex instances without some modification of updates. (Reddi, Kale, Kumar, Best paper at ICLR '18)

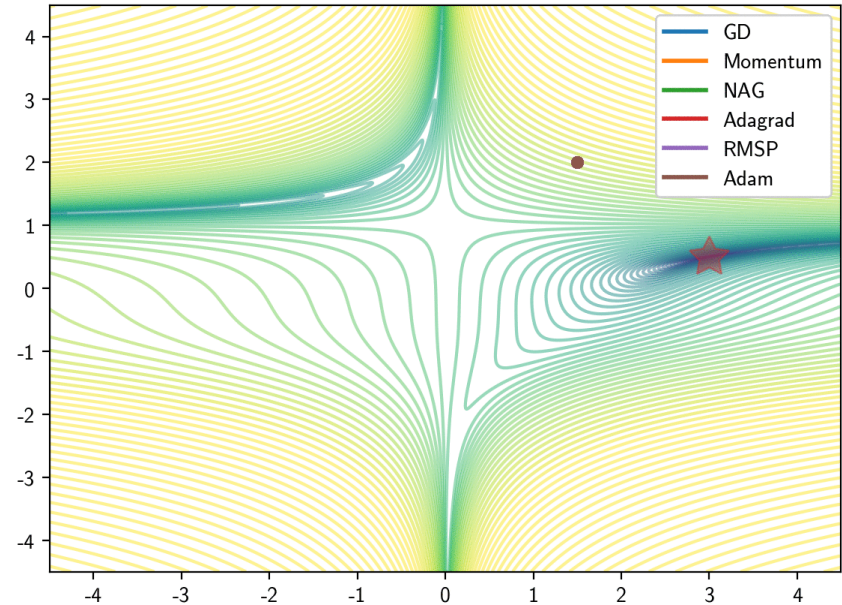
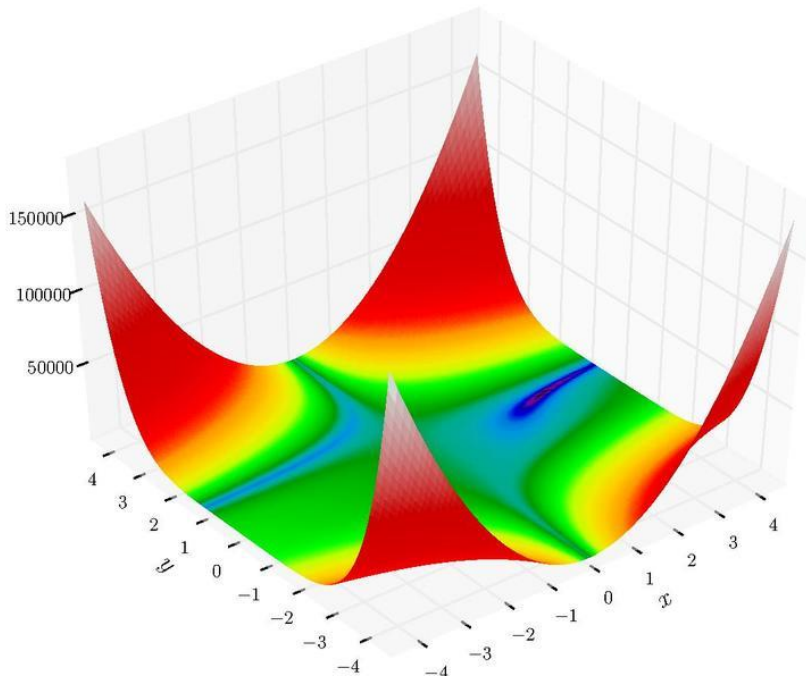
Moving around saddles with various methods



<https://tangbinh.github.io/01/04/Optimizers.html>

Loss function $f(x, y) = y^4 - x^4$, saddle point at (0,0), minimized at $x = \infty$.

Beale's function using various methods



<https://tangbinh.github.io/01/04/Optimizers.html>

Beale's function minimized at $(3, 0)$.

Are these actually helpful?

Remember, what we are trying to do is minimize *test error*.
There's still the question of whether the solution generalizes well.

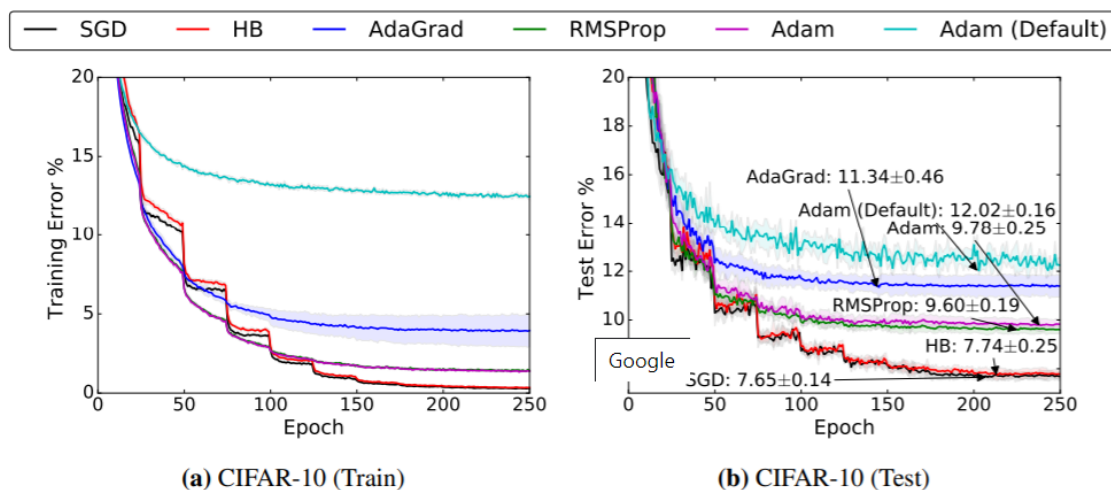


Figure 1: Training (left) and top-1 test error (right) on CIFAR-10. The annotations indicate where the best performance is attained for each method. The shading represents \pm one standard deviation computed across five runs from random initial starting points. In all cases, adaptive methods are performing worse on both train and test than non-adaptive methods.

(Wilson, Roelofs, Stern, Srebro, Recht '18)

Batch Normalization

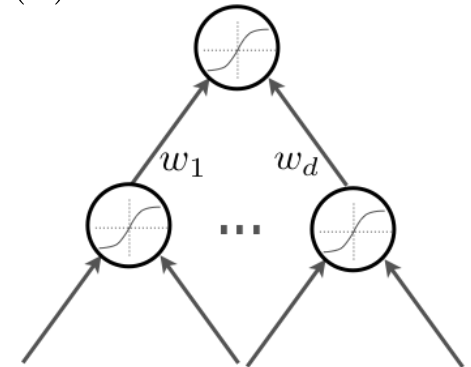
Normalizing/whitening (mean = 0, variance = 1) the inputs speeds up training (*Lecun et al. 1998*)

- Could normalization be useful at the level of the hidden layers?

Batch normalization is an attempt to do that (*Ioffe and Szegedy, 2014*)

- each unit's pre-activation is normalized (mean subtraction, stddev division)
- during training, mean and stddev is computed for each minibatch
- backpropagation takes into account the normalization
- at test time, the global mean / stddev is used

$$\mathbf{a}^{(k)}(\mathbf{x}) = \mathbf{b}^{(k)} + \mathbf{W}^{(k)}\mathbf{h}^{(k-1)}(\mathbf{x})$$



Batch Normalization

Input: Values of x over a mini-batch: $\mathcal{B} = \{x_1 \dots x_m\}$;

Parameters to be learned: γ, β

Output: $\{y_i = \text{BN}_{\gamma, \beta}(x_i)\}$

$$\mu_{\mathcal{B}} \leftarrow \frac{1}{m} \sum_{i=1}^m x_i \quad // \text{ mini-batch mean}$$

$$\sigma_{\mathcal{B}}^2 \leftarrow \frac{1}{m} \sum_{i=1}^m (x_i - \mu_{\mathcal{B}})^2 \quad // \text{ mini-batch variance}$$

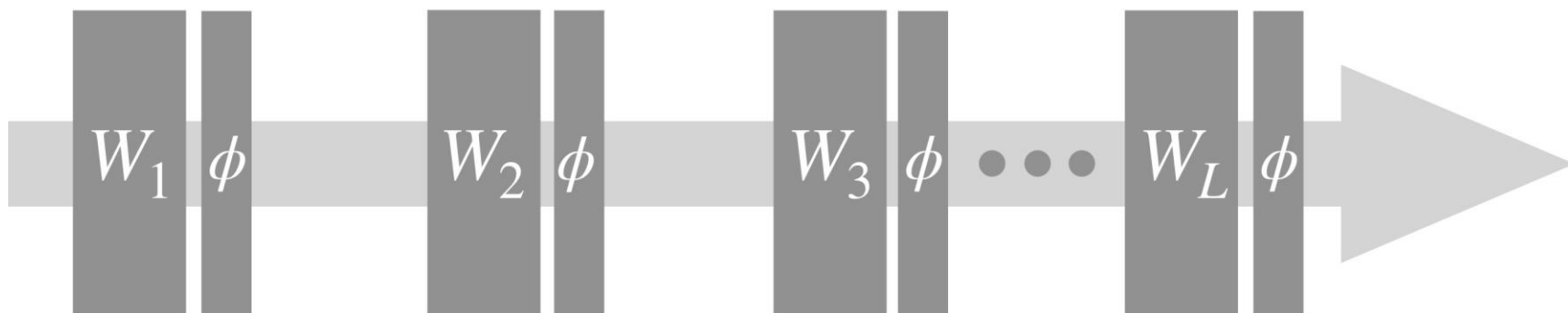
$$\hat{x}_i \leftarrow \frac{x_i - \mu_{\mathcal{B}}}{\sqrt{\sigma_{\mathcal{B}}^2 + \epsilon}} \quad // \text{ normalize}$$

$$y_i \leftarrow \gamma \hat{x}_i + \beta \equiv \text{BN}_{\gamma, \beta}(x_i) \quad // \text{ scale and shift}$$

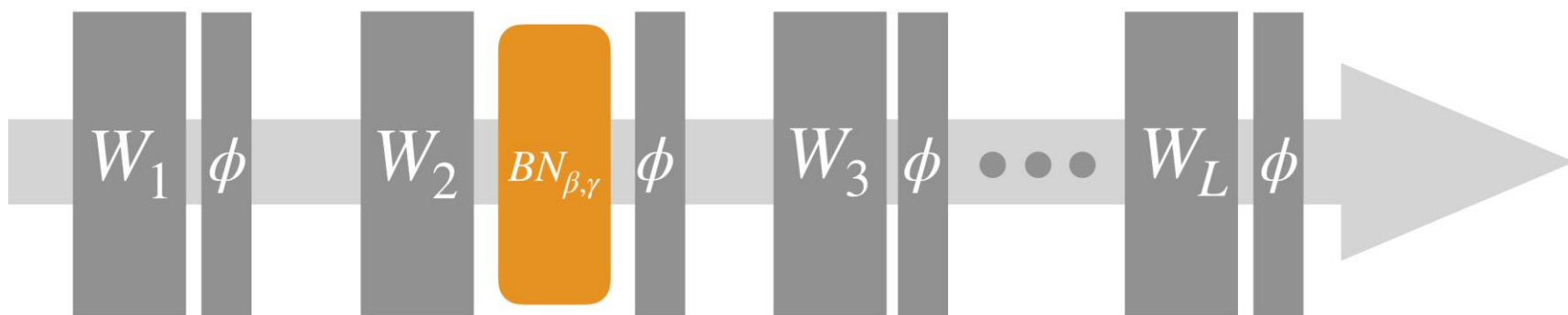
Learned linear transformation to adapt to non-linear activation function (γ and β are trained)

Batch Normalization

Standard Network

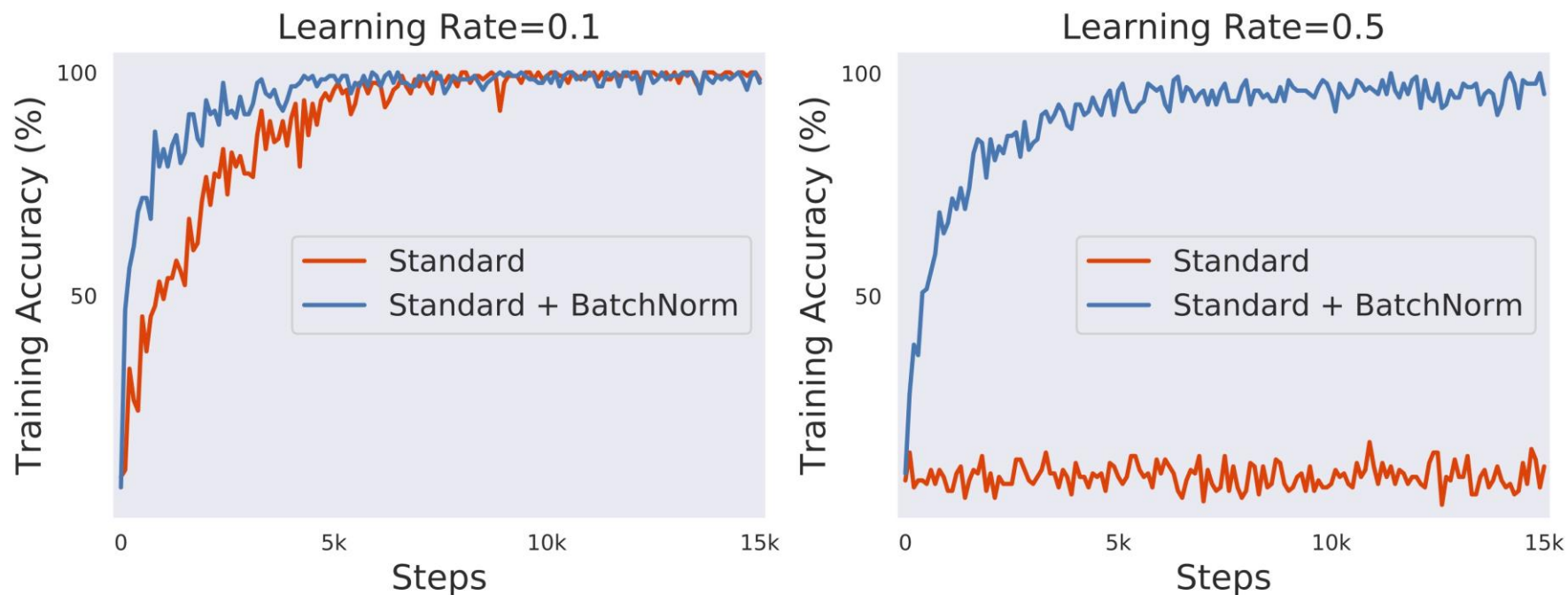


Adding a BatchNorm layer (between weights and activation function)



<http://gradientscience.org/batchnorm/>

Batch Normalization



<http://gradientscience.org/batchnorm/>

Batch Normalization

Why normalize the pre-activation?

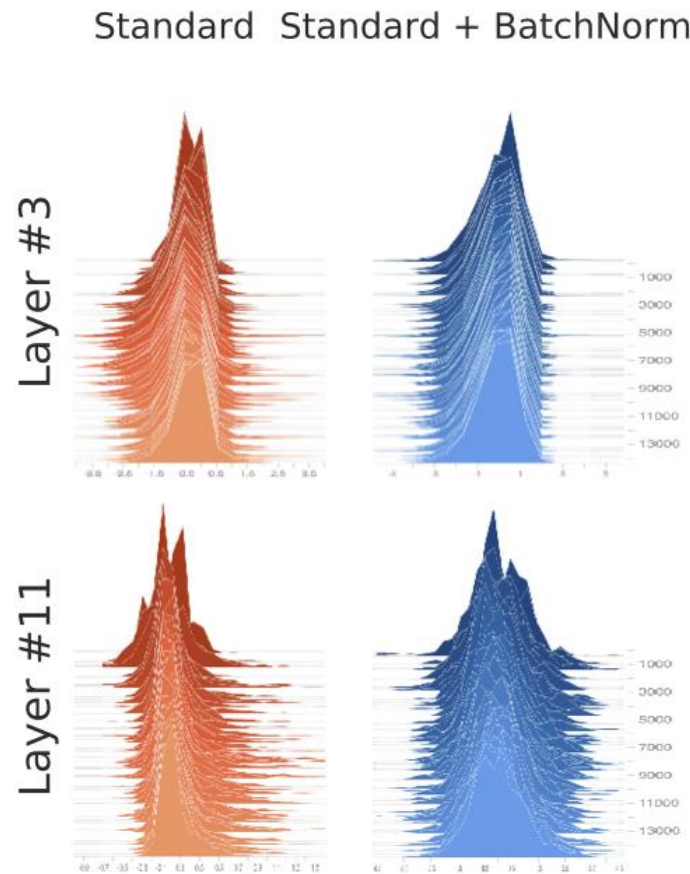
- Can help keep the pre-activation in a non-saturating regime
- Could condition Hessian better (in linear 1-layer nets, *Lecun et al. 1998*)

Why use minibatches?

- since hidden units depend on parameters, can't compute mean/stddev once and for all
- adds stochasticity to training, which might regularize

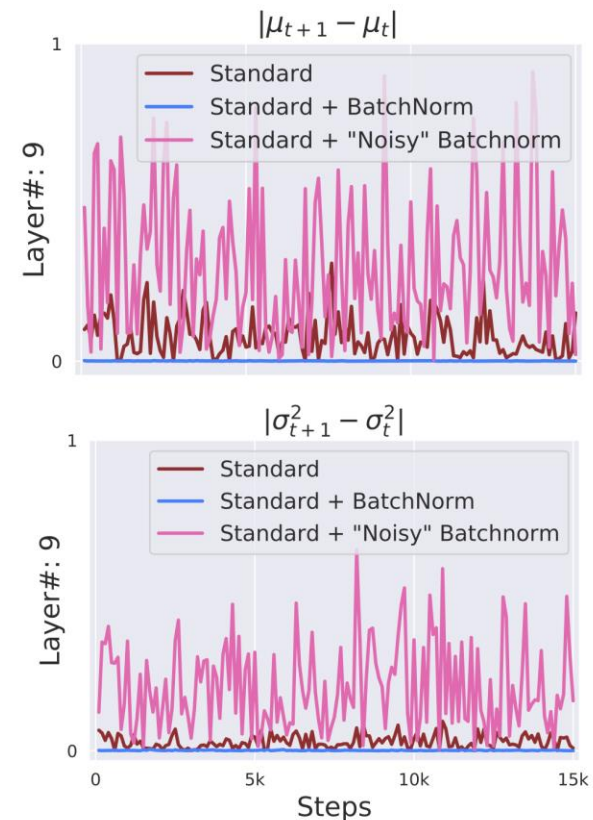
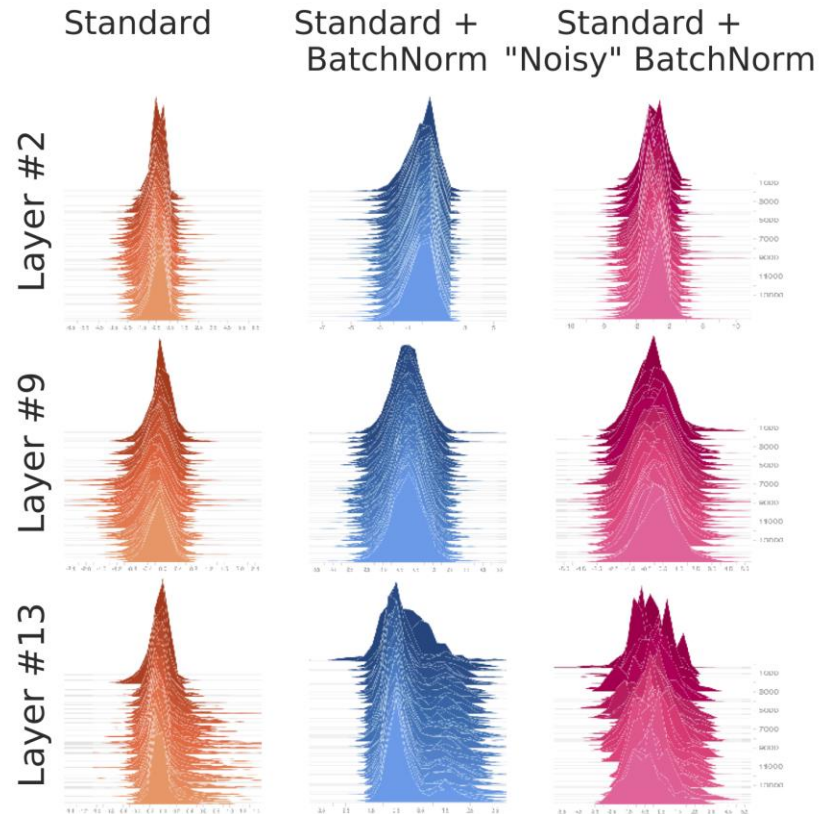
Batch Normalization

Recent observations cast some doubts on distribution shift explanation
(*Santurka, Tspiras, Ilyas, Madry '18*)



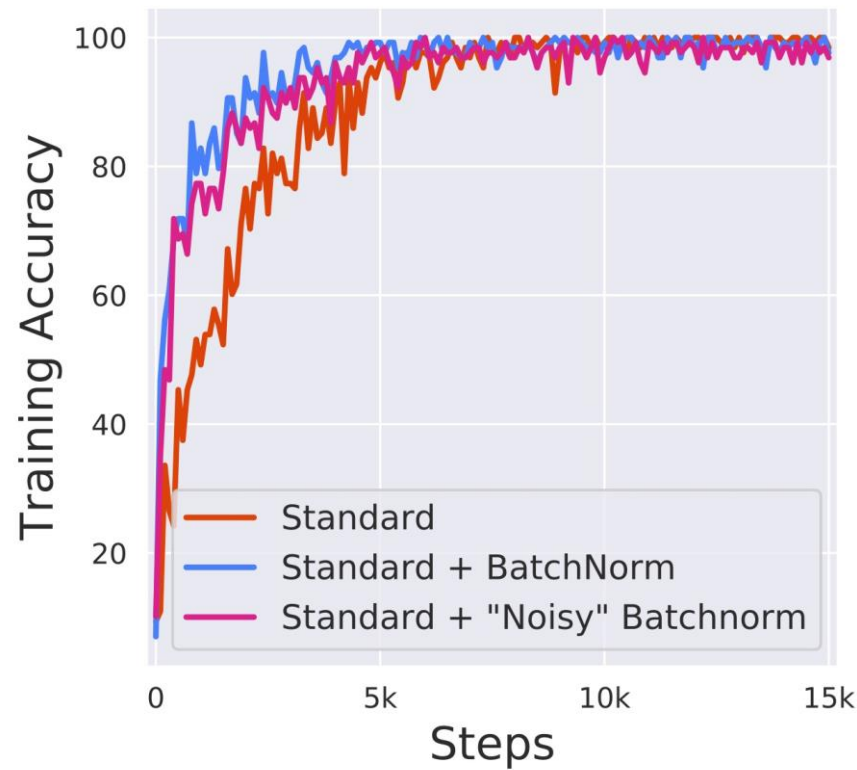
Batch Normalization

Simple experiment: inject non-stationary Gaussian noise after BN layer. This removes the alleged whitening effect of BN layers.



Batch Normalization

Simple experiment: inject non-stationary Gaussian noise after BN layer. This removes the alleged whitening effect of BN layers.



Performance doesn't suffer...

Batch Normalization

Authors suggestions (can be formalized):

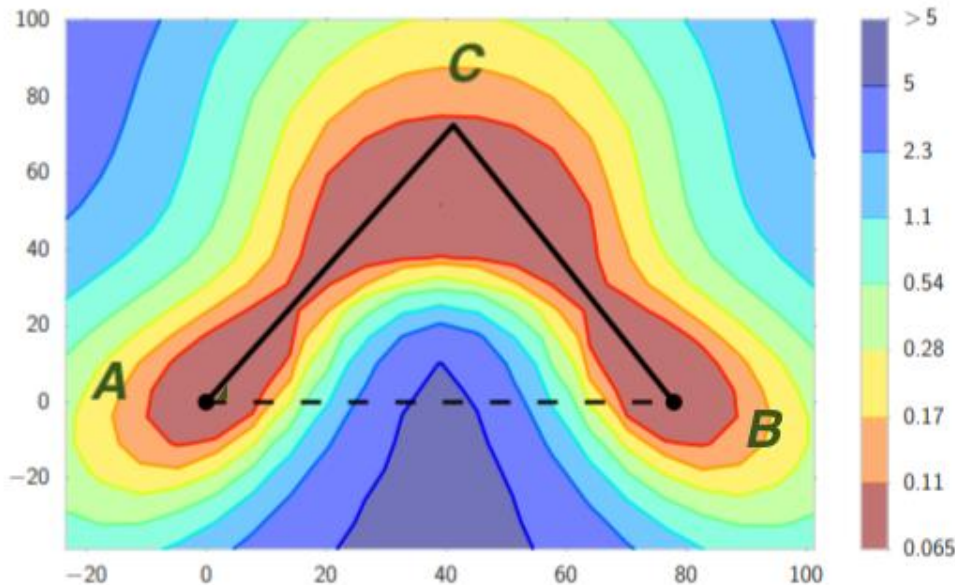
Lipschitz constant of the loss improves.

The **Hessian** evaluated in the direction of the gradient (i.e. second-order term in the Taylor expansion *decreases* in magnitude.

[i.e. first-order term gets comparatively bigger]

Surprises: mode connectivity

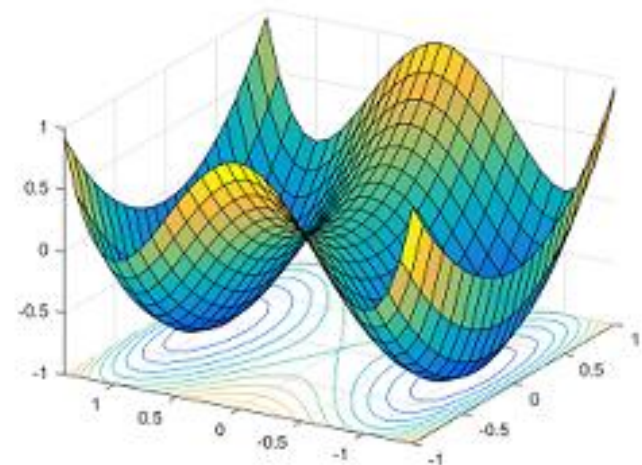
(*Freeman and Bruna, 2016, Garipov et al. 2018, Draxler et al. 2018*):
Local minima can be connected by simple paths of near-same cost.



Surprising!

Remember our intuition,
permutations should give
rise to isolated solutions

Some explanations due to [*Kuditipudi et al '19*]: robustness of the network to “dropping out” some neurons can ensure this.



Surprises: lottery hypothesis

(*Frankle, Carbin ICLR '20, Best paper award*): Typical neural nets have **much** smaller subnetworks that “could have been trained” in isolation to give comparable results.

More precisely, the following works reliably:

1. Initialize a network and train it.
2. “Prune” superfluous structure. (e.g. smallest weights)
3. Reset unpruned weights to values in 1.
4. Repeat 2+3.

B/w 15% and 1% of the original network!

The small network won the “**initialization lottery**” – it could’ve been trained to a good network w/o rest of weights.

We’re training networks that are way larger than in principle necessary!