10707
Deep Learning: Spring 2021

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Lecture 2:
Representational power of neural networks

Neural network basics: the artificial neuron

Neuron **pre-activation:**

$$a(\boldsymbol{x}) = b + \sum_i w_i x_i = b + \boldsymbol{w}^T \boldsymbol{x}$$

Neuron **post-activation:**

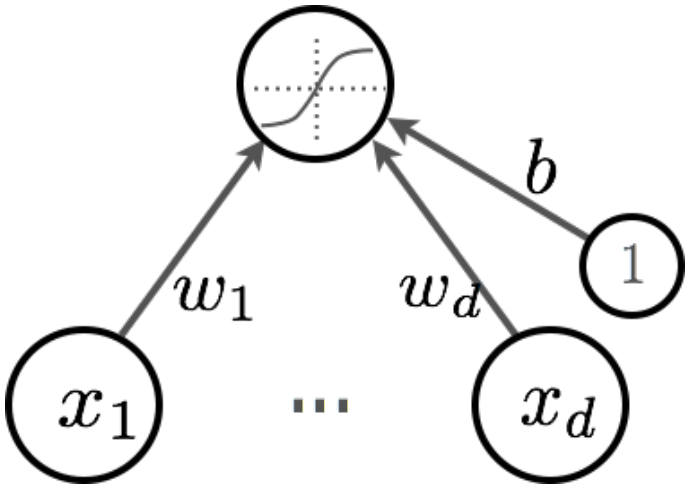
$$h(\boldsymbol{x}) = \sigma(b + \boldsymbol{w}^T \boldsymbol{x})$$

Where:

$\boldsymbol{w}$ are the **weights** (parameters)

b is the **bias** term

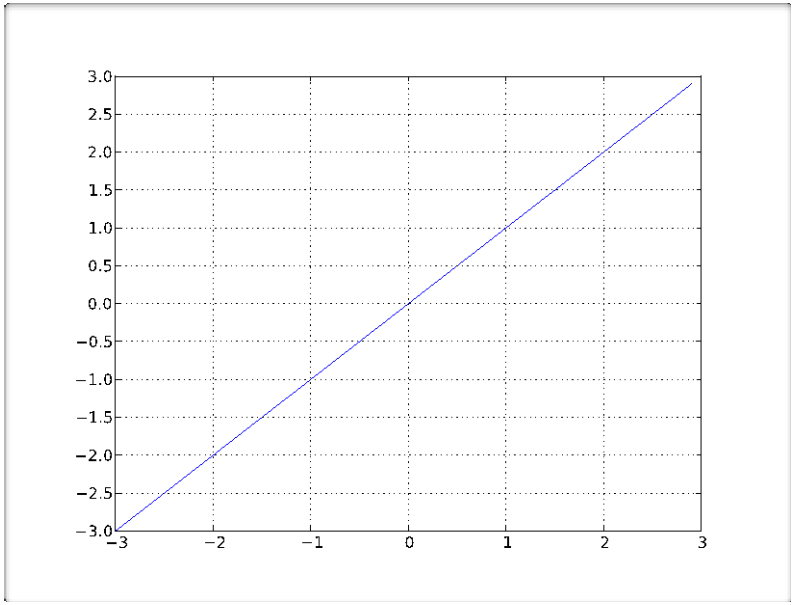
$\sigma(\cdot)$ is called the **activation function**



Popular activations

Linear activation function:

$$\sigma(a) = a$$



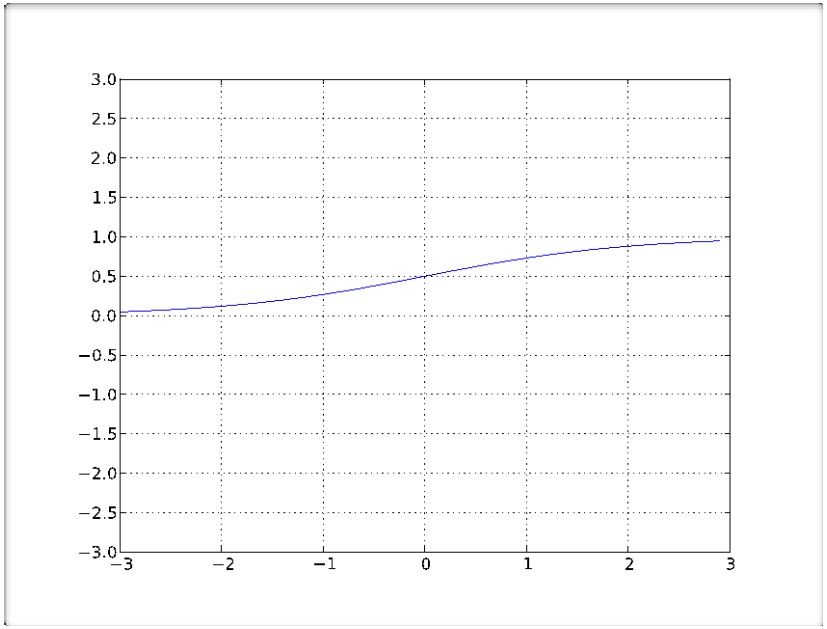
- No nonlinear transformation
- No output squashing
- Poor representational power (linear composed w/ linear = linear)

Popular activations

Sigmoid activation function:

$$\sigma(a) = \frac{1}{1 + \exp(-a)}$$

- Squashes the neuron's output between 0 and 1: can be interpreted as $P(\text{ouput } = 1|a)$ (i.e. **logistic classifier**)
- Always positive
- Bounded
- Strictly Increasing



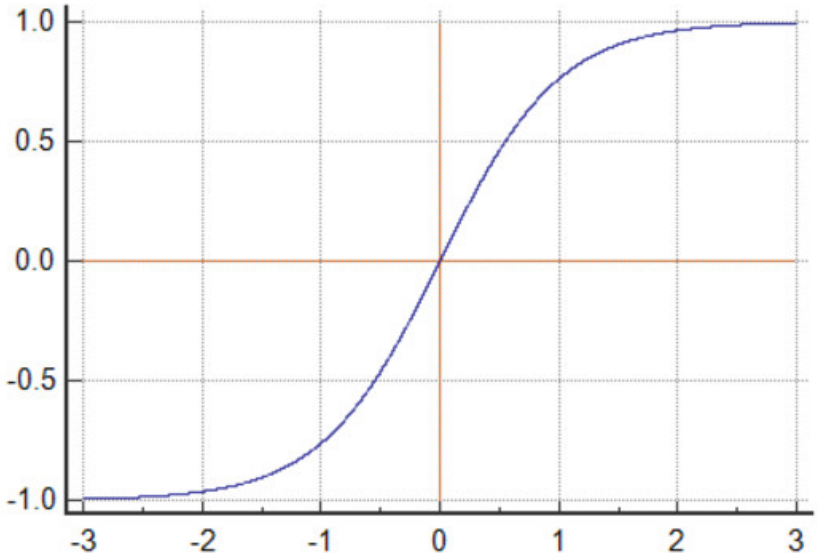
Popular activations

Hyperbolic tangent ("tanh") activation function:

$$\sigma(a) = \tanh(a)$$

$$= \frac{\exp(a) - \exp(-a)}{\exp(a) + \exp(-a)} = \frac{\exp(2a) - 1}{\exp(2a) + 1}$$

- Squashes neuron's output between -1 and 1
- Can be positive or negative
- Bounded
- Strictly increasing

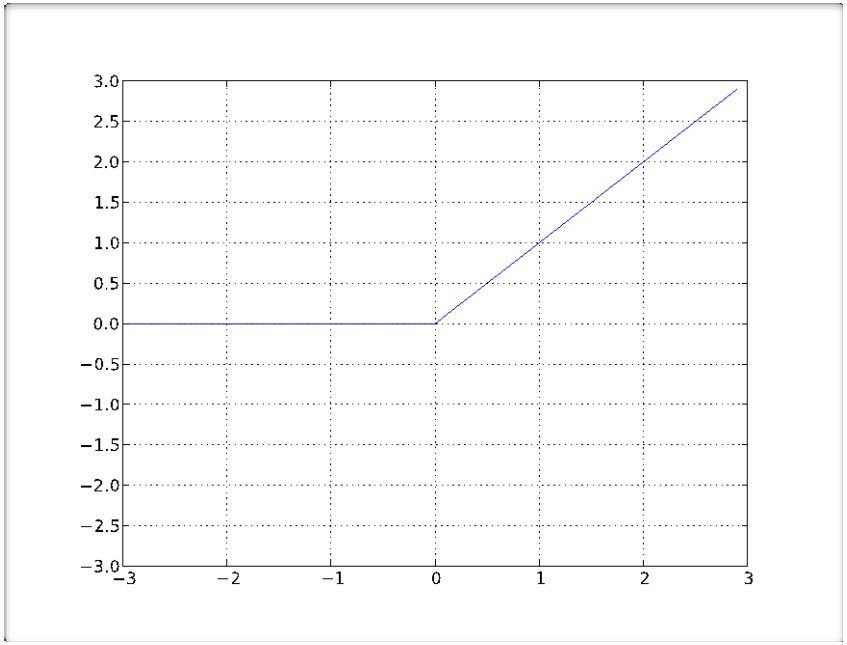


Popular activations

Rectified linear (“ReLU”) activation function:

$$\sigma(a) = \max(a, 0)$$

- Bounded below by 0 (always non-negative)
- Tends to produce units with sparse activities
- Not upper bounded
- Strictly increasing



Single Hidden Layer Neural Net

Hidden layer **pre-activation**:

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{b}^{(1)} + \boldsymbol{W}^{(1)}\boldsymbol{x}$$

$$\boldsymbol{a}(\boldsymbol{x})_i = \boldsymbol{b}_i^{(1)} + \sum_j \boldsymbol{W}_{i,j}^{(1)}\boldsymbol{x}_j$$

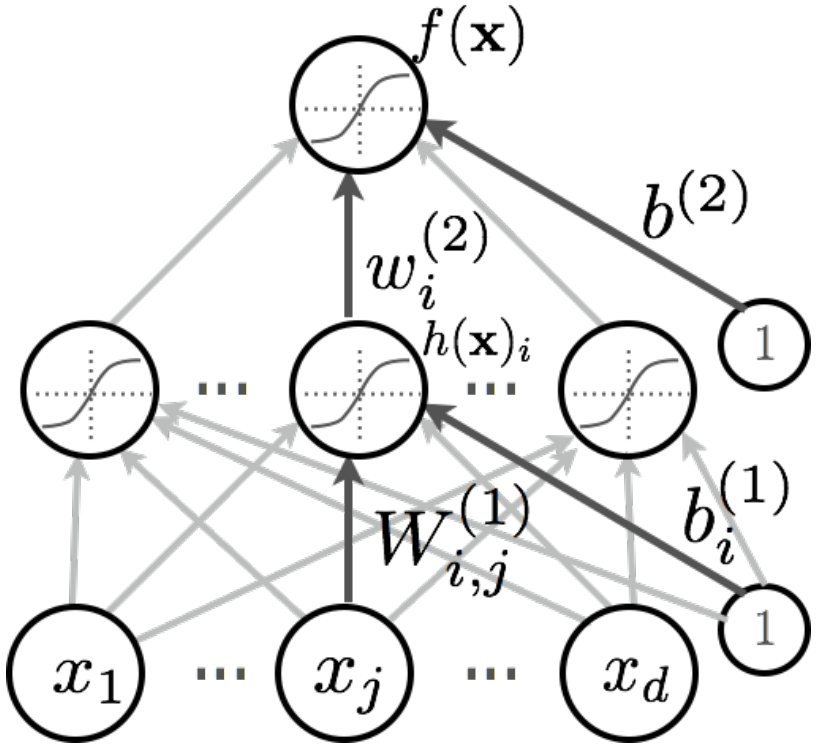
Hidden layer **post-activation**:

$$\boldsymbol{h}(\boldsymbol{x}) = \sigma(\boldsymbol{a}(\boldsymbol{x}))$$

Output layer activation:

$$\boldsymbol{f}(\boldsymbol{x}) = o(b^{(2)} + \boldsymbol{w}^{(2)^T}\boldsymbol{h}^{(1)}(\boldsymbol{x}))$$

Output activation function



Softmax output activation

In **multi-way classification**, we need multiple outputs (1 per class)

Natural: model calculates conditional probabilities P(ouput $= c|\boldsymbol{x}$)

Softmax activation function at the output

$$\mathbf{o}(\mathbf{a}) = \text{softmax}(\mathbf{a}) = \left[\frac{\exp(a_1)}{\sum_c \exp(a_c)} \cdots \frac{\exp(a_C)}{\sum_c \exp(a_c)}\right]^\top$$

- strictly positive
- sums to one

Predict class with the highest estimated class conditional probability.

Multilayer Neural Net

Consider a network with L hidden layers.

Layer **pre-activations**:

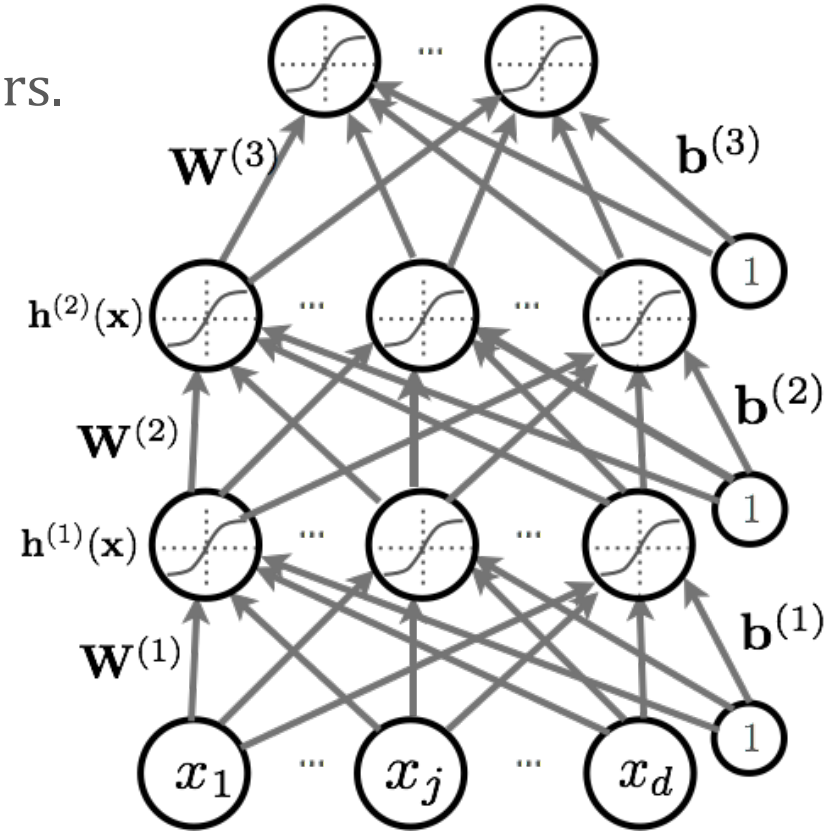
$$\boldsymbol{a}^{(k)}(\boldsymbol{x}) = \boldsymbol{b}^{(k)} + \boldsymbol{W}^{(k)}\boldsymbol{h}^{(k-1)}(\boldsymbol{x})$$

Hidden layer post-activations:

$$\boldsymbol{h}^{(k)}(\boldsymbol{x}) = \sigma(\boldsymbol{a}^{(k)}(\boldsymbol{x}))$$

Output layer activation:

$$\boldsymbol{h}^{(L+1)}(\boldsymbol{x}) = o\left(\boldsymbol{a}^{(L+1)}(\boldsymbol{x})\right) = \boldsymbol{f}(\boldsymbol{x})$$



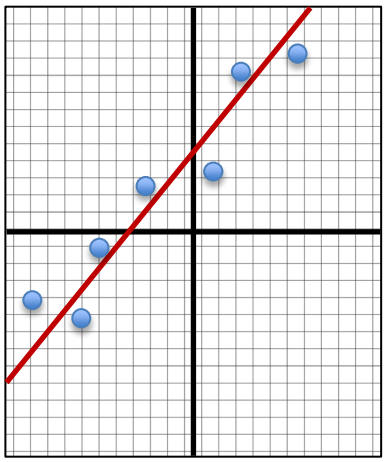
Loss functions

Recall: typical approach is to minimize a training loss l over predictors $\mathcal{F}$:

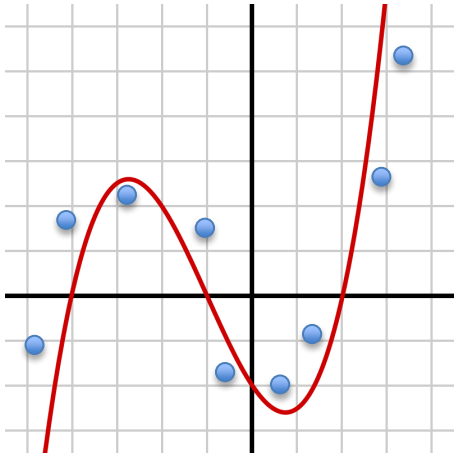
$$\hat{f} = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \sum_{(x,y):\text{training samples}} l(f(x), y)$$

Common losses:

$\boldsymbol{l_2}$: $l(f(x), y) = ||f(x) - y||^2$, more common for **regression**, y can be vector or scalar



$\boldsymbol{f(x) = \langle w, x \rangle}$



$\boldsymbol{f(x) = \sum_i a_i x^i}$

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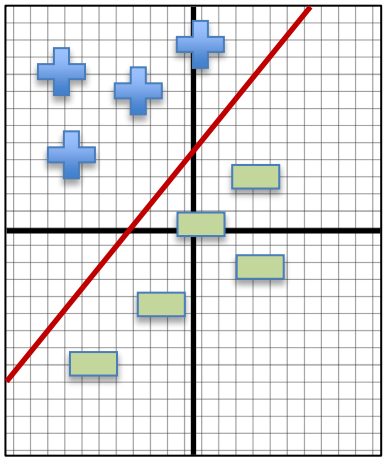
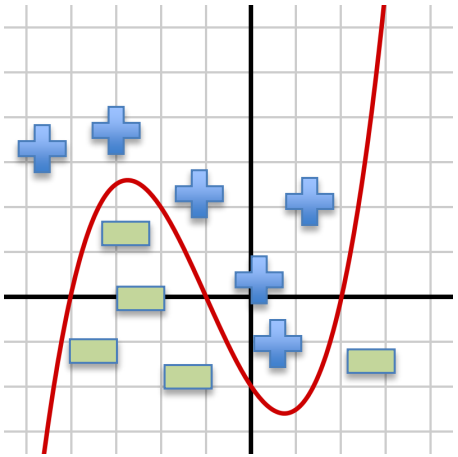


$$f(x) = sgn(\langle w, x \rangle)$$



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Log-loss/ cross entropy: $l(f(x), y) = -\log f(x)_y$, for f using a **softmax** output layer, well-behaved gradients

For softmax, $f(x)_c = P(\text{output} = c|\text{x})$, so we maximize the log-probability of correct label. Generalizes naturally when y **not** deterministic fn of x in $\mathcal{D}$:

$$-\log f(x)_y = -\sum_c 1_{y=c} \log f(x)_c = -\sum_c 1_{y=c} \log P(\text{output} = c|\text{x})$$

Taking expectation of y: $\mathbb{E}_{y|x} l(f(x), y) = -\mathbb{E}_{y|x} \log P(\text{output} = y\,|x)$

Basic optimization algorithm: stochastic gradient descent

Recall: typical approach is to minimize a training loss l over predictors $\mathcal{F}$:

$$\hat{f} = \operatorname*{argmin}_{f \in \mathcal{F}} \sum_{(x,y):\text{training samples}} l(f(x), y)$$

Basic algorithm (Stochastic Gradient Descent)

Glossing over many details. Stay tuned.

- **Initialize**: $\theta_0 := \{W^{(1)}, b^{(1)}, \ldots, W^{(L+1)}, b^{(L+1)}\}$
- For t=1 to T
  - Pick a uniformly random training example (x, y):
    - Set $\theta_{t+1} = \theta_t - \eta \nabla_\theta l(f_\theta(x, y))$

Step size

"Steepest" descent: *direction of most (local) improvement*

Neural nets: gradients can be efficiently calculated, using ***backpropagation***

Supervised learning

Empirical risk minimization approach:
minimize a **training** loss l over a class of **predictors** $\mathcal{F}$:

$$\hat{f} = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \sum_{(x,y):\text{training samples}} l(f(x), y)$$

Three pillars:

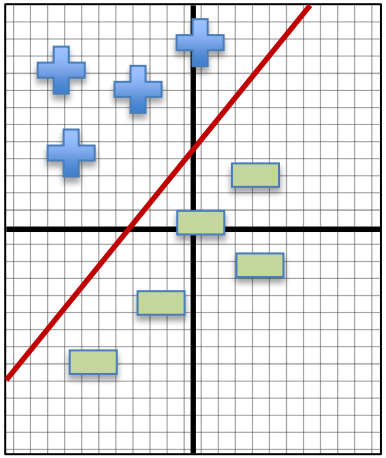
(1) How expressive is the class $\mathcal{F}$? (**Representational power**)

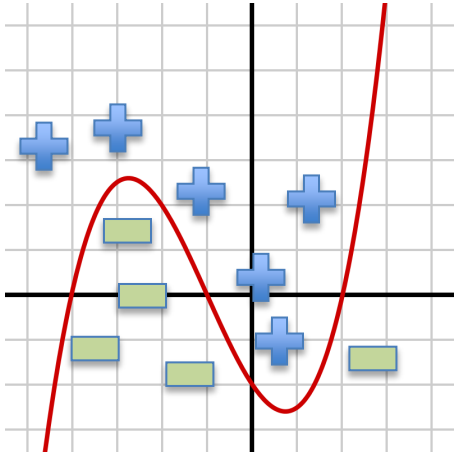
(2) How do we minimize the training loss efficiently? (**Optimization**)

(3) How does $\hat{f}$ perform on unseen samples? (**Generalization**)

Expressivity

What do we mean by expressivity?

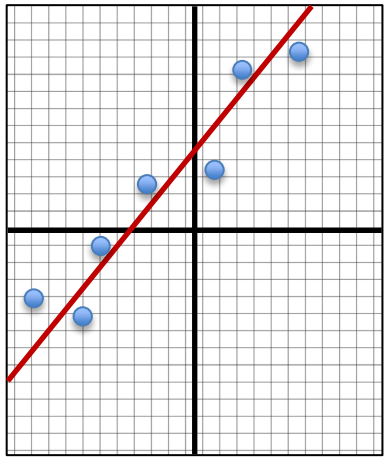




Expressive = functions in class can represent “complicated” functions

Linear classification

The arguably simplest class of classifiers is **linear:**



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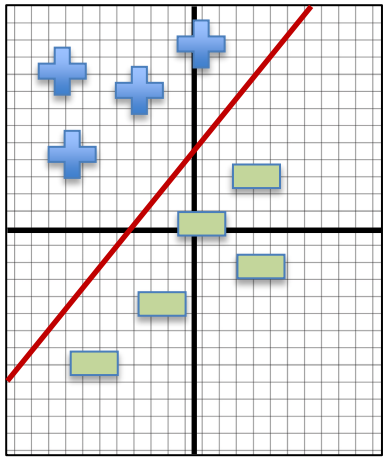
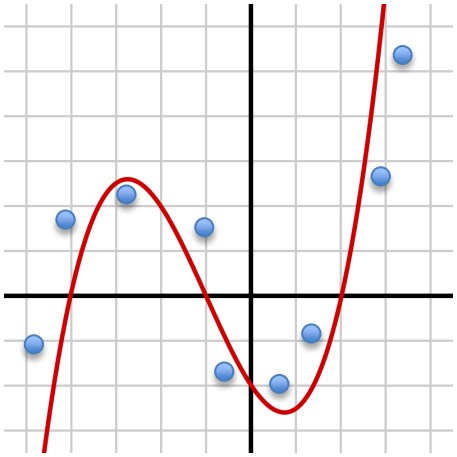


$$f(x) = sgn(\langle w, x \rangle)$$

How do we make classifiers "more expressive"?

One pervasive idea in machine learning (from kernels onward): train a linear classifier on a **feature embedding** of data.



$$f(x) = \sum_{i=0}^{k} a_i x^i$$

For instance, we can write

$f(x) = \langle a, \phi(x) \rangle$, where

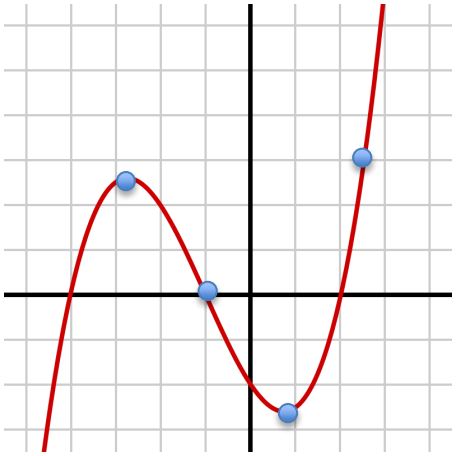
$$a = (a_1, a_2, \dots, a_k)^T, \phi(x) = (1, x, x^2, \dots, x^k)$$

Hence, we first embed x via ϕ from $\mathbb{R}$ into $\mathbb{R}^k$, and train a linear classifier on these new features.

How do we make classifiers "more expressive"?

By increasing degree we can increase expressiveness *a lot*:

For finite set of points $\{(x_1, y_1), \dots, (x_n, y_n)\}$, $x_i \in \mathbb{R}^d, y_i \in \mathbb{R}$ by **Lagrange's interpolation theorem**, we can find a polynomial p of degree $n-1$, s.t. $\forall i, y_i = f(x_i)$



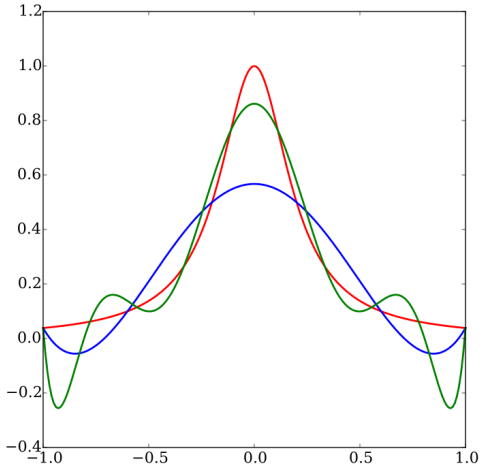
How do we make classifiers "more expressive"?

By increasing degree we can increase expressiveness *a lot*:

For any function f, we can **approximate it** on any compact set Ω by a sufficiently high degree polynomial: for every $\epsilon > 0, \exists p$ of sufficiently high degree, s.t.

$$\max_{x \in \Omega} |f(x) - p(x)| \leq \epsilon$$

(Stone-Weierstrass)



Vague intuition: think of Taylor series; near point x_0, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots$$

$$\boldsymbol{f(x) = \langle w, \phi(x) \rangle,}$$
$$\boldsymbol{\phi(x) = \left(1, x - x_0, (x - x_0)^2, \dots, \right), w = (f(x_0), f'(x_0), \dots)}$$

Lots of choices!

The name of the game in **kernel methods** was choose a good embedding $\boldsymbol{\phi}$ we explored. Lots of latitude here:

Polynomial kernel (in d dim.):

$$\boldsymbol{\phi}(\boldsymbol{x}) = (\mathbf{1}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_d}, \boldsymbol{x_1^2}, \boldsymbol{x_1 x_2}, \dots, \boldsymbol{x_d^2}, \dots, \boldsymbol{x_d^k})$$

Gaussian kernel (in 1d.):

$$\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{e}^{-\frac{\boldsymbol{x}^2}{2\boldsymbol{\sigma}^2}} \left(\mathbf{1}, \sqrt{\frac{\mathbf{1}}{\mathbf{1!}}} \frac{\boldsymbol{x}}{\boldsymbol{\sigma}}, \sqrt{\frac{\mathbf{1}}{\mathbf{2!}}} \sqrt{\frac{\mathbf{1}}{\mathbf{1!}}} \left(\frac{\boldsymbol{x}}{\boldsymbol{\sigma}}\right)^{\mathbf{2}}, \dots \right)$$

Choices of these kernels is closely related to something called the "kernel trick", which allows for cheap computation of the **kernel** $\langle \boldsymbol{\phi}(\boldsymbol{x}), \boldsymbol{\phi}(\boldsymbol{y}) \rangle$**.** Beyond the scope of this course!

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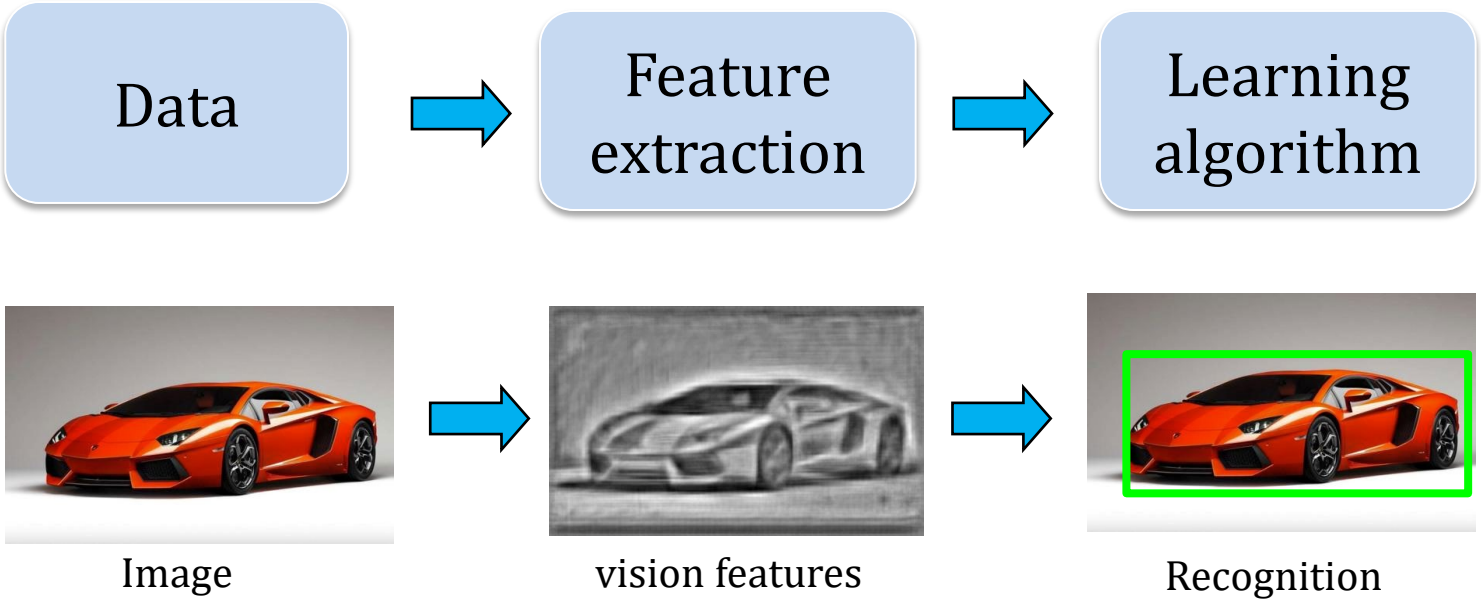
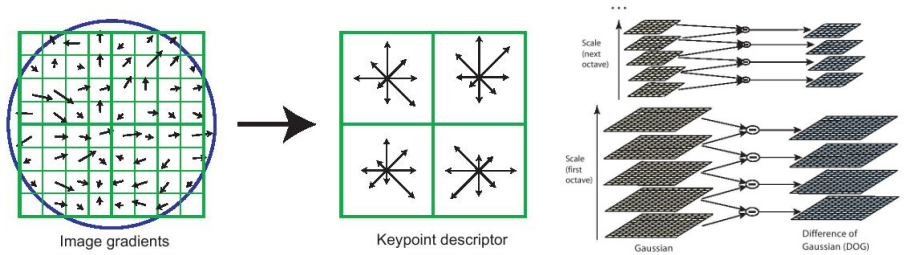
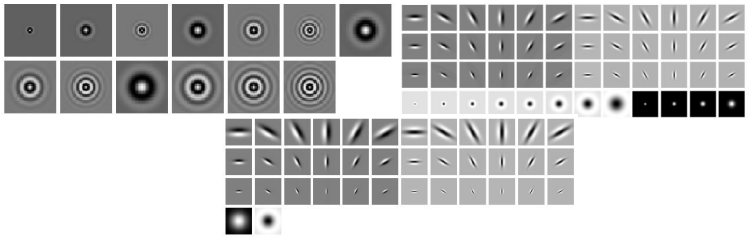
Part of the deep learning story



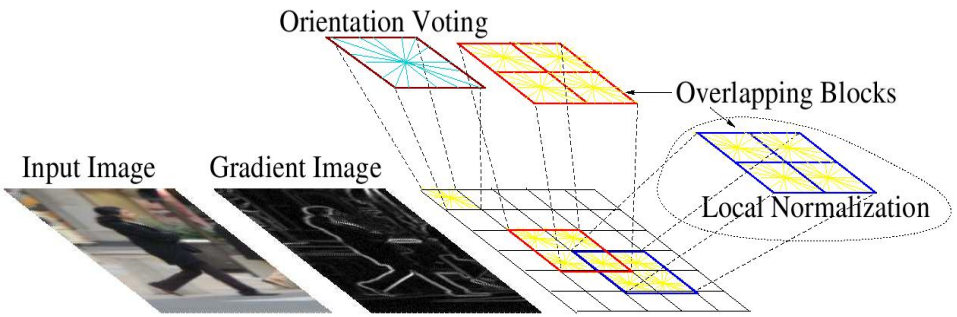
Old school: hand-craft features



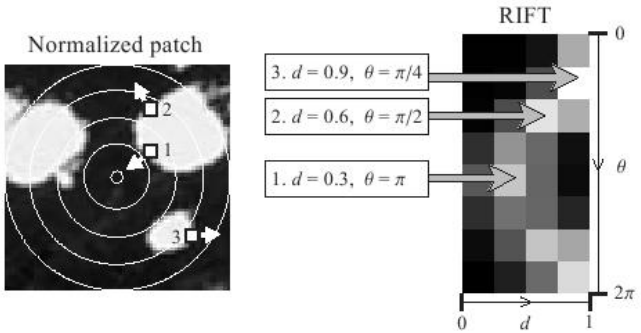
SIFT



Textons

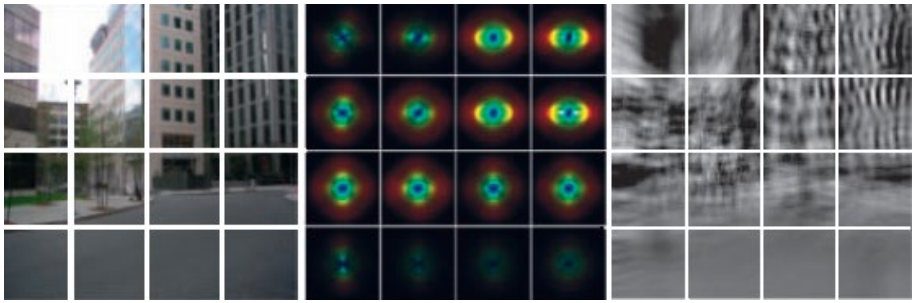


HoG



RIFT

GIST



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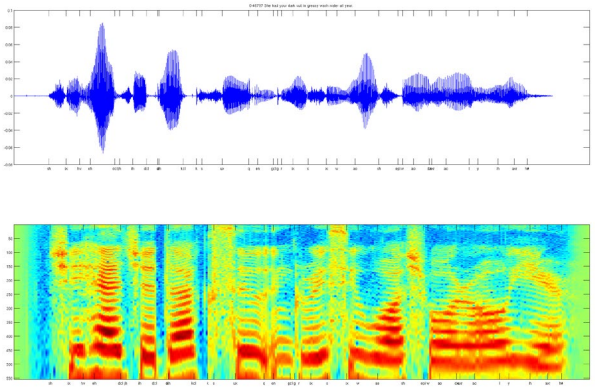
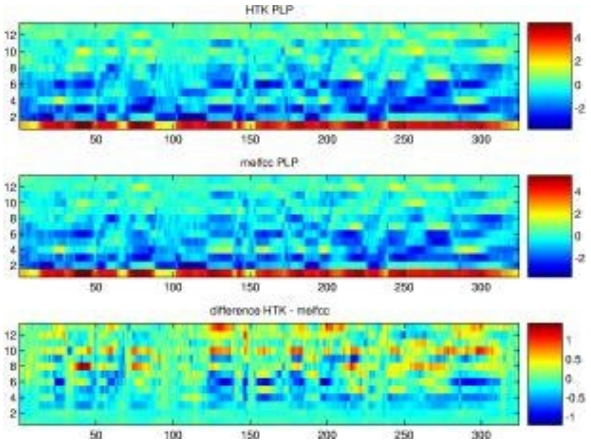
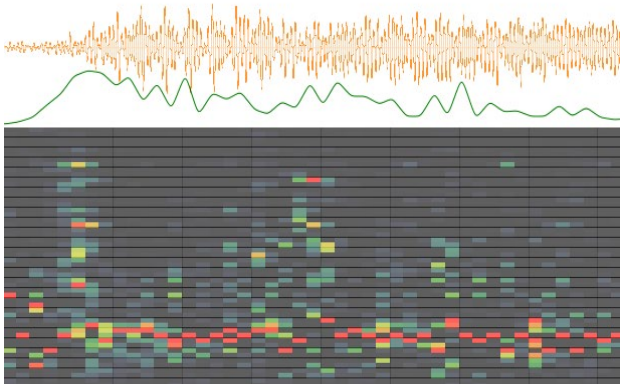


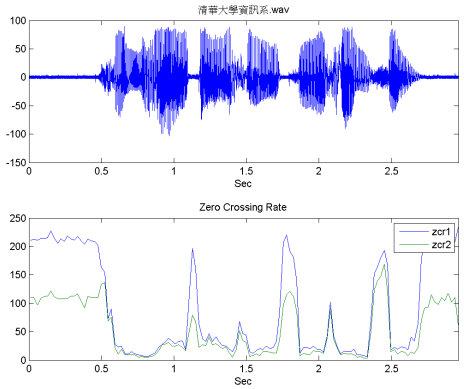
Spectrogram



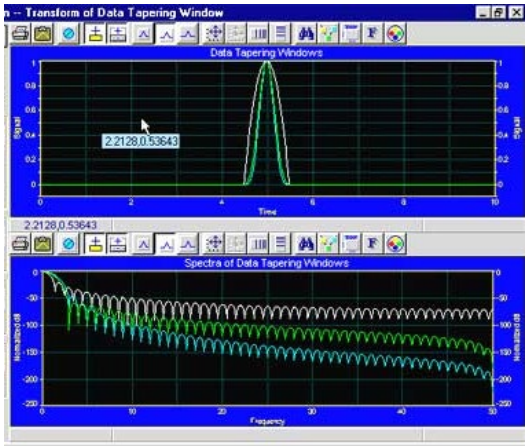
MFCC



Flux

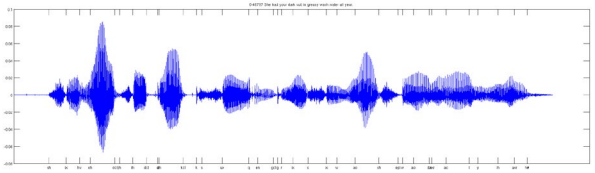


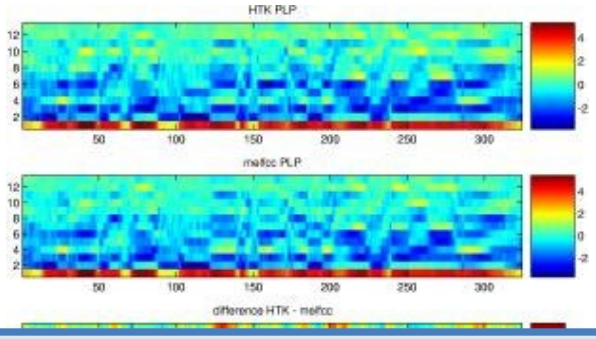
ZCR

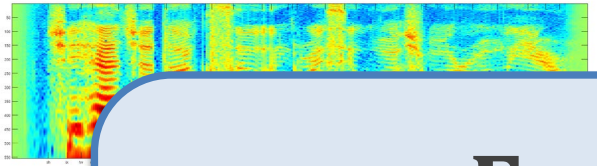


Rolloff

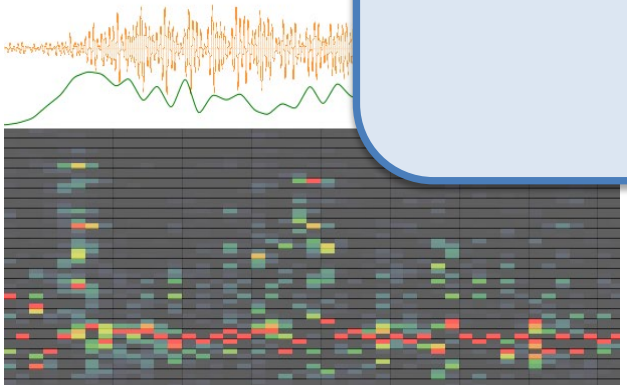
Old school: hand-craft features



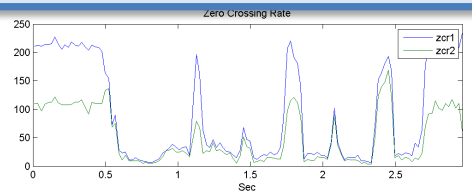




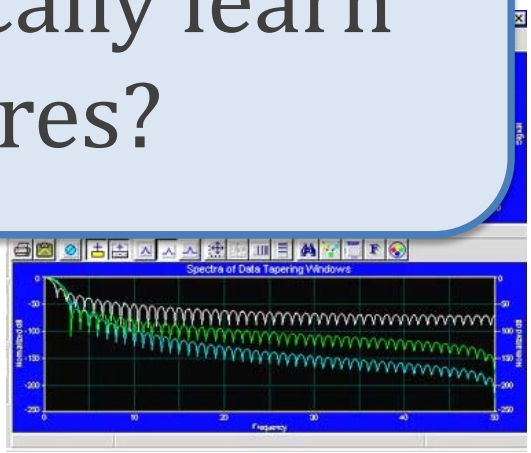
Feature learning:
Can we automatically learn useful features?



Flux

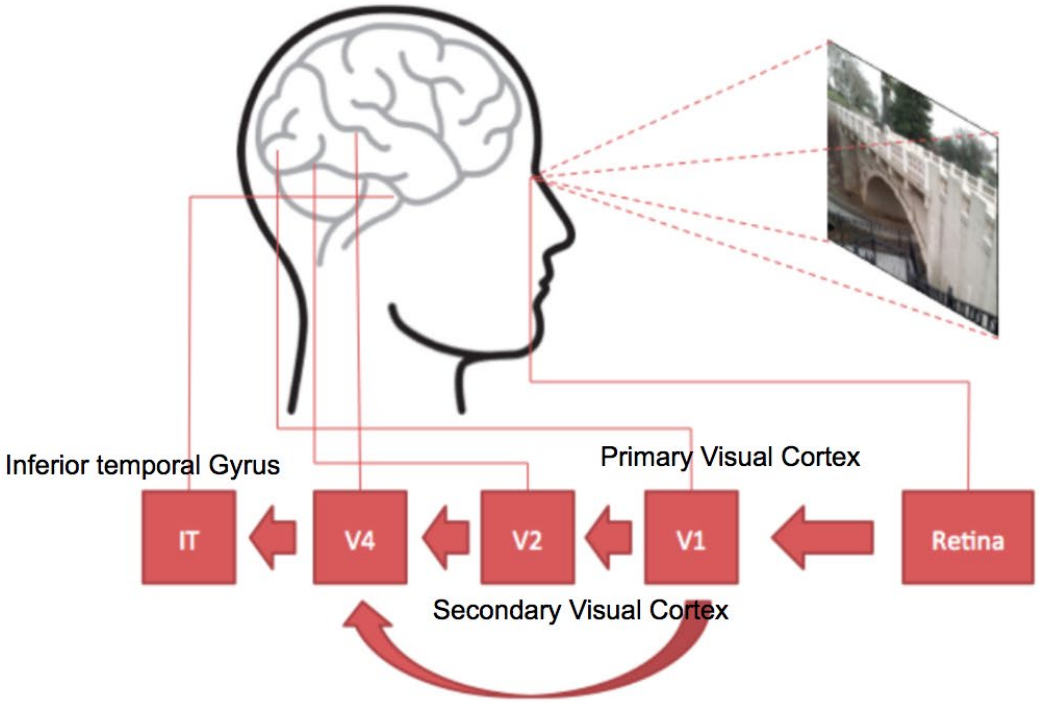


ZCR



Rolloff

Early inspirations from visual cortex



V1: Edge detection, etc.

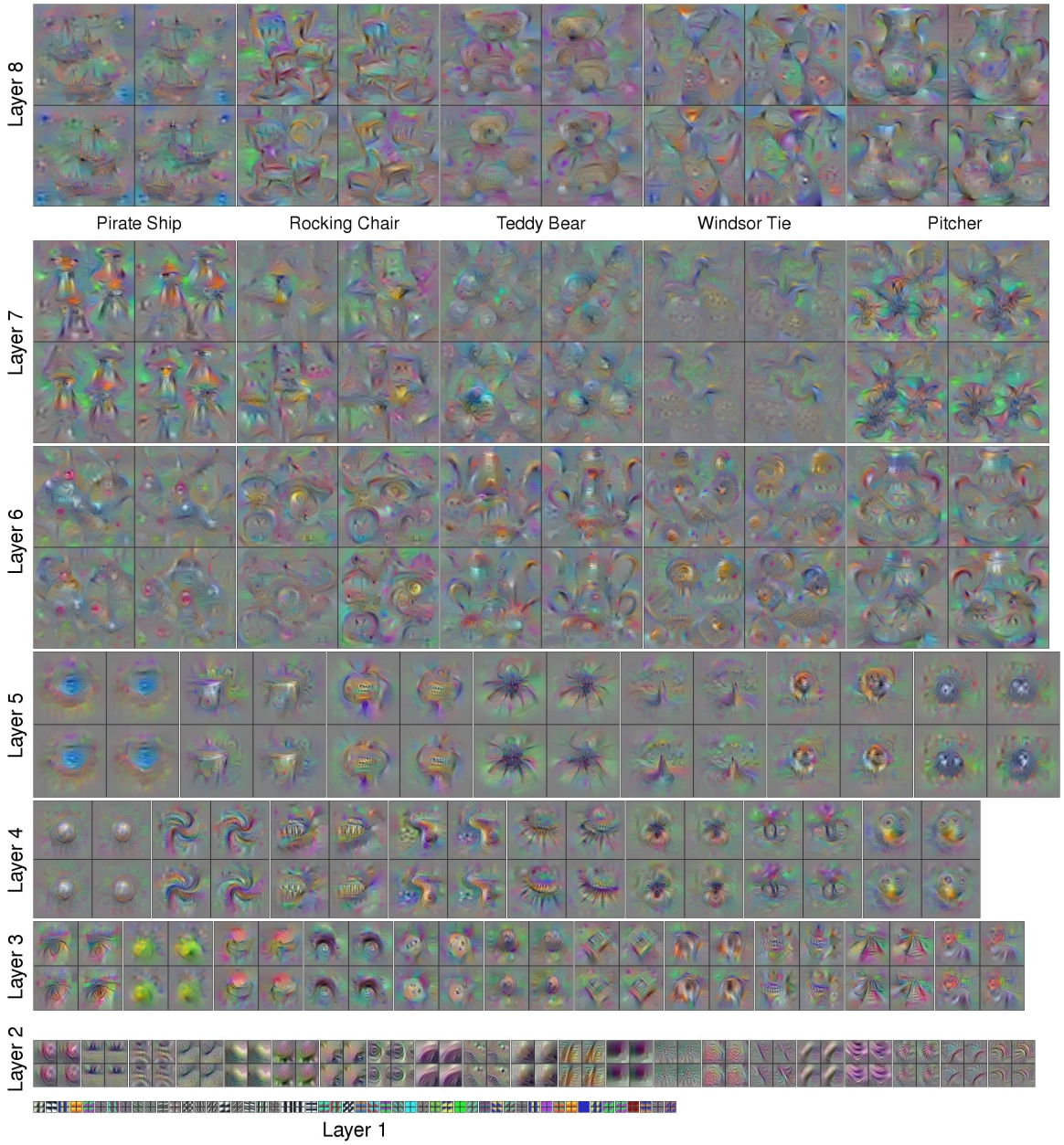
V2: Extract simple visual properties (orientation, spatial frequency, color, etc)

V4: Detect object features of intermediate complexity

TI: Object recognition.

Image: Wang, Raj "On the Origin of Deep Learning."

What do deep networks learn?



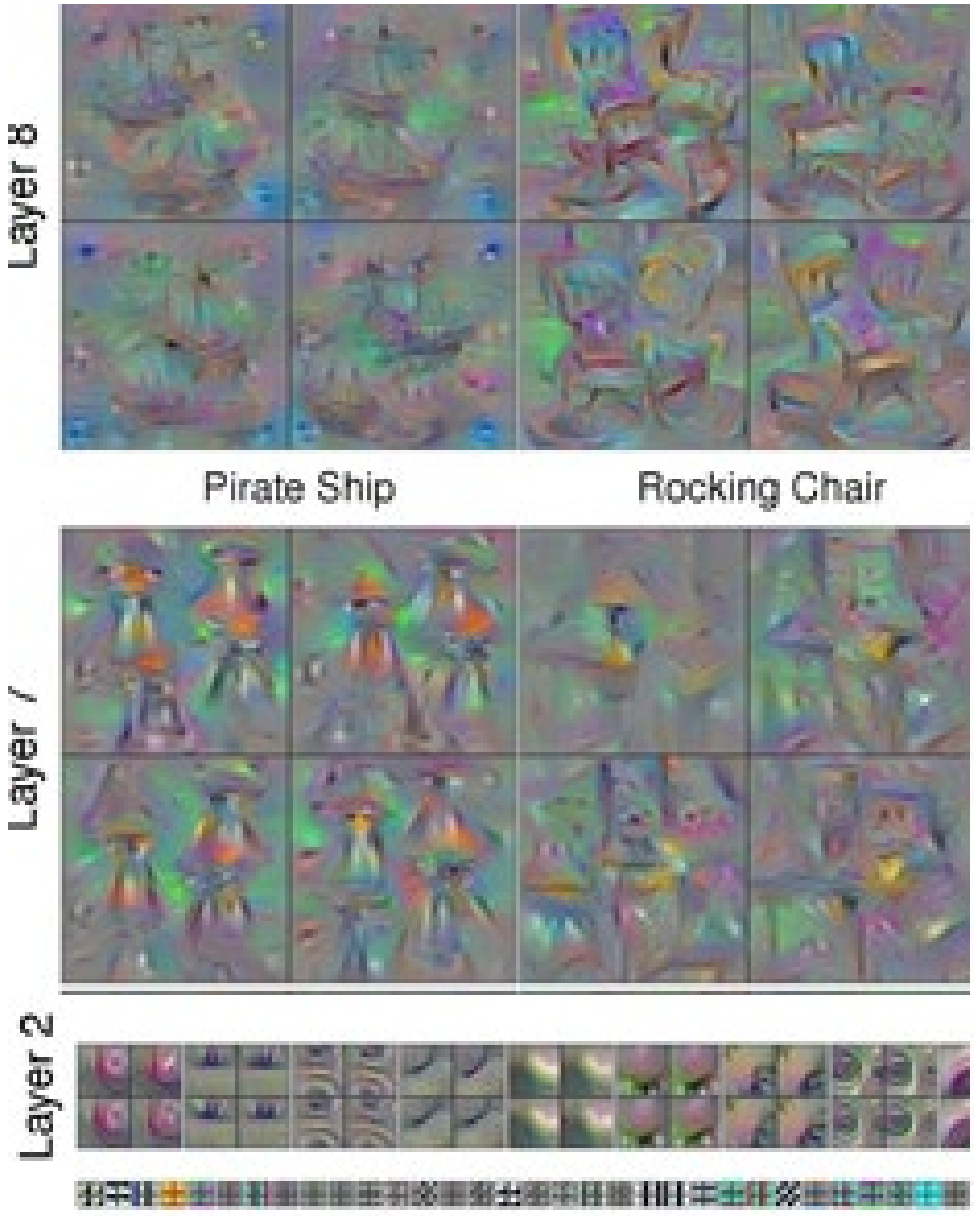
Yosinski et al '15:
http://yosinski.com/deepvis

Q: What "patterns" do neurons respond to?

A: From random start, do gradient descent to find an input for which neuron activation* is high.

*: This produces completely unrecognizable images – they are regularized w/ an image prior.

What do deep networks learn?



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"Universal" expressivity of neural networks

(1): Neural networks are **universal approximators**: given any Lipschitz f: $\mathbb{R}^d \to \mathbb{R}$, a **shallow** (3-layer) neural network with $\sim \left(\frac{1}{\epsilon}\right)^{\mathrm{d}}$ neurons can approximate it to within ϵ error.

"curse of dimensionality"

Universal approximation I: Lipschitz function are approximable

Recall, a function f: $[0,1]^d \to \mathbb{R}$ is **L-Lipschitz** (*in an l_∞ sense*) if:
$\forall x, y \in [0,1]^d,\ |f(x) - f(y)| \leq L \max_{i \in [d]} |x_i - y_i|$

First, we show neural networks are **universal approximators**: given any Lipschitz function f: $[0,1]^d \to \mathbb{R}$, a **shallow** (3-layer) neural network with ~ $\left(\frac{1}{\epsilon}\right)^{\mathrm{d}}$ neurons can approximate it to within ϵ error.

Theorem: For any L-Lipschitz function f: $[0,1]^d \to \mathbb{R}$, there is a 3-layer neural network $\hat{\mathrm{f}}$ with $\mathrm{O}\left(\mathrm{d}\left(\frac{\mathrm{L}}{\epsilon}\right)^{d}\right)$ ReLU neurons, s.t.

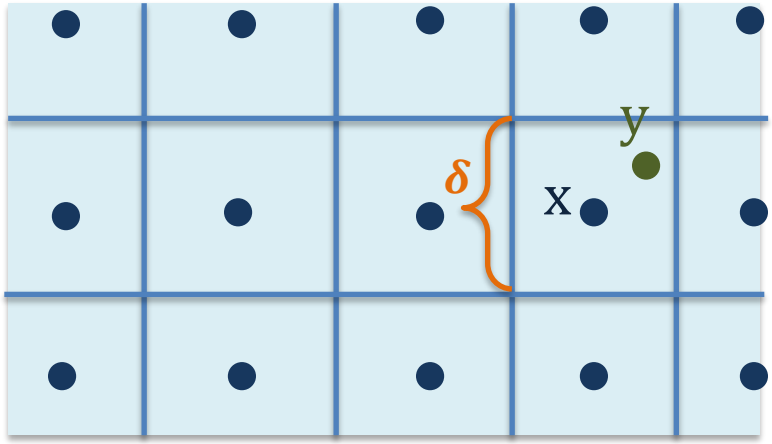
$\mathbb{E}_{x \sim [0,1]^d} |f(x) - \hat{f}(x)|$

$\int_{[0,1]^d} |f(x) - \hat{f}(x)|\, dx \leq \epsilon$

l_1 *error*

Universal approximation I: Proof intuition

Part 1: using Lipschitzness, we can "query" the values of function f approximately by querying its values on a fine grid.



$$|f(y) - f(x)| \le L\delta$$

Part 2: we can approximate f as linear combination of "queries".

$$f(x) \approx \sum_{\text{cells } C_i} 1_{x \in C_i} f(x_i)$$

$$f(x) \approx \langle w, \phi(x) \rangle, \quad \phi(x) = (1(x \in C_i))_i \quad w = \left(f(x_i)\right)_i$$

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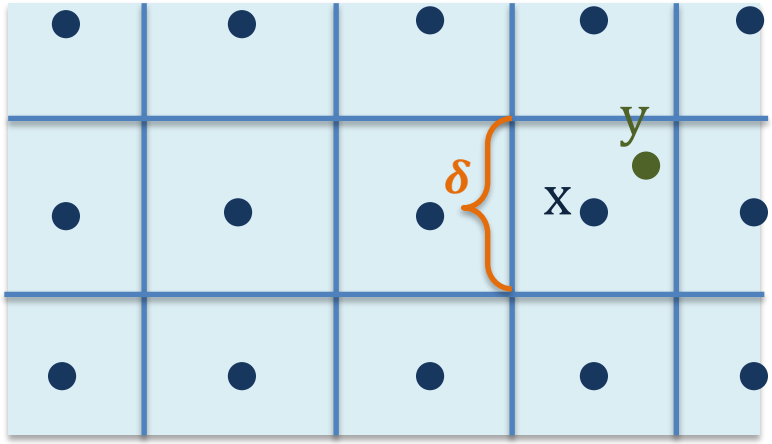

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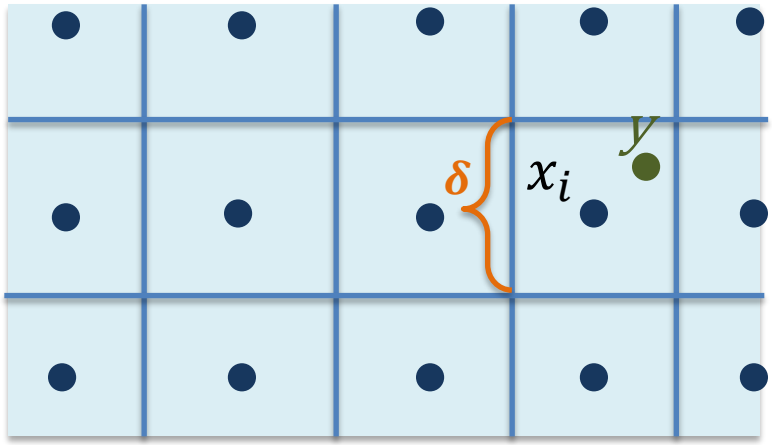
Part 3: Approximate the indicators using ReLUs

Universal approximation I: Part 1, formally

Lemma: Let f: $[0,1]^d \to \mathbb{R}$ be L-Lipschitz and $P = (C_1, C_2, \dots, C_N)$ a partition of $[0,1]^d$ into cells of side lengths at most δ. Consider any set $(x_1, x_2, \dots, x_N), x_i \in C_i$. Then:

$$\sup_{i \in N} \sup_{y \in C_i} |f(y) - f(x_i)| \leq L\delta$$



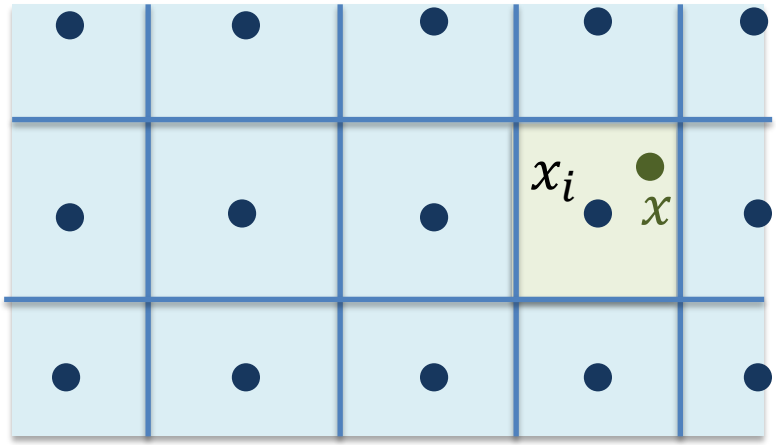
Proof: By Lipschitzness, we have

$$\forall i, y \in C_i: |f(y) - f(x_i)| \leq L \max_{i \in [d]} |y - x_i|$$
$$\leq L\delta \qquad \square$$

Universal approximation I: Part 2, formally

Lemma: Let f: $[0,1]^d \to \mathbb{R}$ be 1-Lipschitz, $P = (C_1, C_2, \dots, C_N)$ a partition of $[0,1]^d$ into rectangles of side lengths at most δ, and a set $(x_1, x_2, \dots, x_N), x_i \in C_i$. Then,

$$g(x) = \sum_{i=1}^{N} 1_{x \in C_i} f(x_i) \text{ satisfies } \sup_{x \in [0,1]^d} |f(x) - g(x)| \leq L\,\delta$$



Proof: Let $x \in C_i$.
Then, $1_{x \in C_i} = 1$, and $1_{x \in C_j} = 0$ for $j \neq i$.

So, $g(x) = f(x_i)$.

By Lemma 1,
$|f(x) - g(x)| = |f(x) - f(x_i)| \leq L\,\delta$ □

Universal approximation I: approximating indicators of cells

Lemma: Let $C \subseteq \mathbb{R}^d$ be a cell, namely $C = \{x\colon\ x\ \in [l_i, r_i], i \in [d]\}$. Then, there exists a 2-layer network $\tilde{\tilde{h}}(x)$ of size O(d) and ReLU activation, s.t. $\int_{x\in[0,1]^d} \left|\tilde{\tilde{h}}(x) - 1(x\ \in [l_i, r_i], i \in [d])\right|\ dx \to 0$

Proof: First, write indicator for cell as:

For any $\gamma > 0$, we'll take γ small

$$1(x\ \in C) = 1\left(\sum_{i=1}^{d}\left(1(x_i \geq l_i) + 1(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$

Why? x is in cell iff all the indicators $1(x_i \geq l_i) + 1(x_i \leq r_i)$ are on.
All these indicators are on iff they sum to $2d$.
(If at least one is off, they sum to $2d$-1)

If we can approximate (scalar) indicators, we're all good!

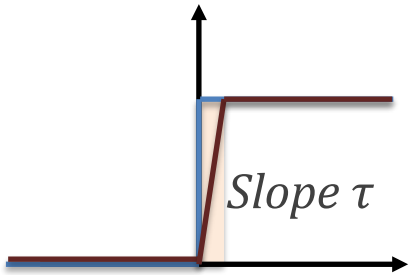
Universal approximation I: approximating indicators of cells

$\sigma(x) = \max(0, x)$

Claim: For $\tau \geq 0,\ x \in \mathbb{R}$:

$$\left|1(x \geq 0) - \left(\sigma(\tau x) - \sigma(\tau x - 1)\right)\right| = \begin{cases} \leq 1, & \text{if } 0 \leq x \leq 1/\tau \\ 0, & \text{otherwise} \end{cases}$$

Proof: Consider several cases:



Case 1, $x \leq 0$: $1(x \geq 0) = 0$ and $\sigma(\tau x) - \sigma(\tau x - 1) = 0$, so

$$1(x \geq 0) - \left(\sigma(\tau x) - \sigma(\tau x - 1)\right) = 0$$

Case 2, $x \geq 1/\tau$: $1(x \geq 0) = 1$ and $\sigma(\tau x) - \sigma(\tau x - 1) = 1$, so

$$1(x \geq 0) - \left(\sigma(\tau x) - \sigma(\tau x - 1)\right) = 0$$

Case 3, $0 \leq x \leq 1/\tau$: $1(x \geq 0) = 1$ and $\sigma(\tau x) - \sigma(\tau x - 1) = \tau x$, so

$$\left|1(x \geq 0) - \left(\sigma(\tau x) - \sigma(\tau x - 1)\right)\right| = 1 - \tau x \leq 1$$

Universal approximation I: approximating indicators of cells

$$h(x) := 1\left(\sum_{i=1}^{d}\left(1(x_i \geq l_i) + 1(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$

Replace all indicators by difference of ReLUs. *What is the error?*

For brevity, let $\tilde{1}(x \geq 0) = \sigma(\tau x) - \sigma(\tau x - 1)$, for some τ we will choose.

Let $\tilde{h}(x) := 1\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$

$\tilde{\tilde{h}}(x) := \tilde{1}\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$

(Change the approximations “iteratively”.)

Universal approximation I: approximating indicators of cells

$$h(x) := 1\left(\sum_{i=1}^{d}\left(1(x_i \geq l_i) + 1(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$
$$\tilde{h}(x) := 1\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$
$$\tilde{\tilde{h}}(x) := \tilde{1}\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$

We have:

$$\int_{x\in[0,1]} \left|\tilde{\tilde{h}}(x) - h(x)\right| dx = \int_{x\in[0,1]^d} \left|\tilde{\tilde{h}}(x) - \tilde{h}(x) + \tilde{h}(x) - h(x)\right| dx$$

$$\leq \int_{x\in[0,1]^d} \left(\left|\tilde{\tilde{h}}(x) - \tilde{h}(x)\right| + \left|\tilde{h}(x) - h(x)\right|\right) dx$$

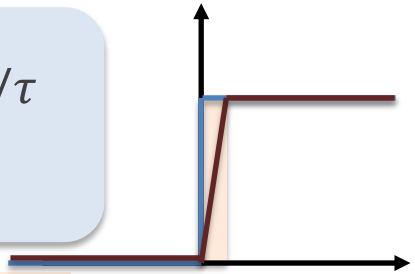
Triangle inequality

Let's handle two terms one by one.

Universal approximation I: approximating indicators of cells

$$\tilde{h}(x) := 1\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$
$$\tilde{\tilde{h}}(x) := \tilde{1}\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$

$$\left|1(x \geq 0) - \left(\sigma(\tau x) - \sigma(\tau x - 1)\right)\right| = \begin{cases} \leq 1, & \text{if } 0 \leq x \leq 1/\tau \\ 0, & \text{otherwise} \end{cases}$$



First: $\tilde{\tilde{h}}(x) - \tilde{h}(x) \neq 0$ only if $\exists i$: $x_i \in \left(l_i, l_i + \frac{1}{\tau}\right)$ or $x_i \in \left(r_i, r_i - \frac{1}{\tau}\right)$

Why: If $\tilde{\tilde{h}}(x) - \tilde{h}(x) \neq 0, \sum_i \tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) - (2d - 1) - \gamma \in (0, \frac{1}{\tau})$;

If condition isn't satisfied, $\tilde{1} = 1$, so $\sum_i \tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i) - (2d - 1)$ is integer, so cannot belong to interval for small enough $\gamma > 0$.

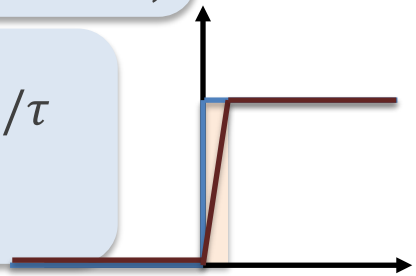
Hence, $\int_{x\in[0,1]^d} |\tilde{\tilde{h}}(x) - \tilde{h}(x)| \leq \sum_i \left(\int_{\substack{x\in[0,1]^d \\ x_i \in \left(l_i, l_i + \frac{1}{\tau}\right)}} 1\, dx + \int_{\substack{x\in[0,1]^d \\ x_i \in \left(r_i, r_i - \frac{1}{\tau}\right)}} 1\, dx\right) \leq \frac{2d}{\tau}$

Universal approximation I: approximating indicators of cells

$$h(x) := 1\left(\sum_{i=1}^{d}\left(1(x_i \geq l_i) + 1(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$

$$\tilde{h}(x) := 1\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$

$$\left|1(x \geq 0) - \left(\sigma(\tau x) - \sigma(\tau x - 1)\right)\right| = \begin{cases} \leq 1, & \text{if } 0 \leq x \leq 1/\tau \\ 0, & \text{otherwise} \end{cases}$$



Second: $\int_{x\in[0,1]} \left|\tilde{h}(x) - h(x)\right| dx$

Indicators are equal if inputs are equal

$$\leq \int_{x\in[0,1]^d} \left| 1\left(\sum_{i=1}^{d}\left(1(x \geq l_i) + 1(x \leq r_i)\right) \neq \sum_{i=1}^{d}\left(\tilde{1}(x \geq l_i) + \tilde{1}(x \leq r_i)\right) \right) \right| dx$$

$$\leq \int_{x\in[0,1]^d} \sum_{i=1}^{d} 1\left(1(x \geq l_i) \neq \tilde{1}(x \geq l_i)\right) + \sum_{i=1}^{d} 1\left(1(x \leq r_i) \neq \tilde{1}(x \leq r_i)\right) dx$$

$$\leq 2d/\tau$$

Universal approximation I: approximating indicators of cells

$$h(x) := 1\left(\sum_{i=1}^{d}\left(1(x_i \geq l_i) + 1(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$
$$\tilde{h}(x) := 1\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$
$$\tilde{\tilde{h}}(x) := \tilde{1}\left(\sum_{i=1}^{d}\left(\tilde{1}(x_i \geq l_i) + \tilde{1}(x_i \leq r_i)\right) \geq 2d - 1 + \gamma\right)$$

Putting together, we have:

$$\int_{x\in[0,1]^d} \left|\tilde{\tilde{h}}(x) - h(x)\right| dx = \int_{x\in[0,1]^d} \left|\tilde{\tilde{h}}(x) - \tilde{h}(x) + \tilde{h}(x) - h(x)\right| dx$$
$$\leq 4d/\tau$$

Also, $\tilde{\tilde{h}}(x)$ is a 2-layer net with ReLU activations and $O(d)$ nodes!

Universal approximation I: Putting everything together

By Part 1+2, $\sup_{x\in[0,1]^d} \left|\mathrm{f(x)} - \sum_{i=1}^{N} 1_{x\in C_i} f(x_i)\right| \leq L\,\delta$

Moreover, the number of cells N can be bounded by $\left(\frac{1}{\delta}\right)^d$

By indicator approximation: can approximate arbitrarily well by taking $\tau \to \infty$ with a 2-layer ReLU net.

Combining the above two points, we get a $\left(\frac{1}{\delta}\right)^d$ −sized 3-layer net s.t.

$$\int_{[0,1]^d} |f(x) - \hat{f}(x)|\, dx \leq L\,\delta$$

Taking $\delta = \frac{\epsilon}{L}$, the theorem follows.

Parting thoughts

All results we proved are **existential**: they prove that a good approximator exists. Finding one efficiently (much less so using gradient descent) is a different matter.

The choices of non-linearities are usually very **flexible**: most results of the type we saw can be re-proven using different non-linearities. (Examples in homework.)

Many other results of similar flavor. For instance, there are also results that deep, but narrow networks are universal approximators.