0.1 Regional New Keynesian Model

The model is populated by four agents: (1) a representative household, (2) a continuum of firms producing intermediate goods, (3) a firm producing a final good, and (4) the monetary authority.

The representative household maximizes utility based on consumption and labor, subject to a budget constraint composed of wages, capital rental rates, and firm profits.

The final-goods firm produces the final-good consumed by households: it aggregates all intermediate-goods produced by intermediate firms, operates under perfect competition and seeks to maximize profit subject to the bundle technology.

Intermediate firms each produce a single intermediate good, all exhibiting imperfect substitution, thus operating in monopolistic competition. Intermediate-goods firms have two problems to solve: minimize costs subject to production level and choose an optimal price to maximize the intertemporal profit flow.

Periodically, a portion of intermediate-goods firms have the opportunity to adjust prices, while others miss this chance, following to a **calvo_staggered_1983** rule. This mechanism generates nominal price rigidities, altering equilibrium relationships in the system. These rigidities lead to non-neutrality of money in the short term, as explained by **costa_junior_understanding_2016**.

The monetary authority determines the nominal interest rate in response to fluctuations in previous period's inflation and production, aiming to control price levels and growth, following a **taylor_discretion_1993** rule.

Stochastic shocks will be present in the intermediate-goods firms' productivity and in the monetary policy.

For regionalization of the model, an index will be used to differentiate the studied region from the rest of Brazil, resulting in separate households, intermediate- and final-goods firms for each region. Households will lack mobility between regions. The link connecting the two regions will be the final-goods.

Then, equilibrium conditions of the system will be determined. Assuming the system tends toward long-term equilibrium, a steady state will be reached where variables cease to change. Thus, for a given $t \longrightarrow \infty$, we will have $\boldsymbol{X}_t = \boldsymbol{X}_{t+1} = \boldsymbol{X}_{ss} \implies \dot{\boldsymbol{X}} = 0$, where $\boldsymbol{X}$ denotes the vector of system variables, ss indicates the steady state and $\dot{\boldsymbol{X}} = {}^{\partial X}/_{\partial t}$.

After that, the log-linearization method proposed by **uhlig_toolkit_1999** will be employed to convert the system of equations into a linear system, so that this linear system can be solved by the program Dynare, which computes the solution and produces impulse-response graphs based on the stochastic shocks.

Regions

falta revisar esta parte e agrupar por agentes da economia.

colocar estatística descritiva para justificar as variáveis.

Regions will have an index $\eta \in \{1, 2, \ldots, n\}$ representing the variables of each region. Whenever necessary, a second region index $\nu \in \{1, 2, \ldots, n\}$ will be used. For example, the variable C_t represents the total consumption (the grand total of all regions), $C_{\eta t}$ represents the consumption composition of region η and $C_{\eta 2t}$ represents the consumption of final-good of region ν by region η (the first index indicating the origin and the second the destiny of the good). Without loss of generality, the model will have two regions: the main region 1 and the remaining of the country 2, so that $\eta, \nu \in \{1, 2\}$.

For each region, the variables are:

- Consumption $C_{\eta 2t}$: households from region $\eta \in \{1, 2\}$ consume from both regions $\eta \in \{1, 2\}$.
- Labor $L_{\eta t}$: there is no mobility in the labor market, so that households will work for firms in the same region they live.
- Investment and Capital $I_{\eta t}, K_{\eta t}$: there is no mobility in investments and capital rent: households will invest and rent capital in their own region.
- Final-good production $Y_{\eta t}$: there is one representative final-good firm in each region that aggregates all intermediate-goods of that region.
- Final-good price $P_{\eta t}$ and regional inflation $\pi_{\eta t}$: in each region, there is a final-good price and a regional inflation level.
- Intermediate-goods firms $Y_{\eta j t}$: there is a continuum $j \in [0, 1]$ for each region and these firms will demand labor and capital from within the region.

- Productivity level $Z_{A\eta t}$ and capital weight in production α: each region has its own characteristics and because of that has a difference productivity level subject to different shock rule and a different capital weight in production.

Model Diagram

Figure 1: Model Diagram

0.1.1 Household

Utility Maximization Problem

Following the models presented by **costa_junior_understanding_2016** and **solis-garcia_ucb_20**
the representative household problem is to maximize an intertemporal utility function U_η with respect to consumption $C_{\eta t}$ and labor $L_{\eta t}$, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_{\eta t}, L_{\eta t}, B_{\eta t}, K_{\eta,t+1}} : \quad U_\eta(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{0.1}$$

$$\text{s. t.}: \quad P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \tag{0.2}$$

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{0.3}$$

$$C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \tag{0.4}$$

$$C_{\eta t}, L_{\eta t}, K_{\eta,t+1} > 0 \, ; K_0 \text{ given.}$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, P_{1t} and P_{2t} are the prices of goods 1 and 2, respectively, $C_{\eta 1t}$ and $C_{\eta 2t}$ are the goods produced in region 1 and 2, respectively, and consumed in region η, $I_{\eta t}$ is the investment, $B_{\eta t}$ are the bonds, W_t is the wage level, $K_{\eta t}$ is the capital, R_{Kt} is the return on capital, R_t is the return on bonds (which is also the nominal interest rate of the economy) and $\Pi_{\eta t}$ is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate. In the consumption aggregation, $\omega_{\eta 1}$ and $1-\omega_{\eta 1}$ are the weights of goods $C_{\eta 1t}$ and $C_{\eta 2t}$, respectively, in the consumption bundle $C_{\eta t}$.

To solve the household problem, first substitute 0.4 in 0.1:

$$U_\eta(C_{\eta 1t}, C_{\eta 2t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{\left[C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{0.5}$$

Isolate $I_{\eta t}$ in 0.3 and substitute in 0.2:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies I_{\eta t} = K_{\eta,t+1} - (1-\delta)K_{\eta t} \tag{0.3}$$

$$\begin{aligned} &P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ &\quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \implies \end{aligned} \tag{0.2}$$

$$\begin{aligned} &P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1-\delta)K_{\eta t}] + B_{\eta t} = \\ &\quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \tag{0.6}$$

Lagrangian

The maximization problem with restrictions can be transformed into one without restriction using the Lagrangian function $\mathscr{L}$ formed by 0.5 and 0.6:

$$\begin{aligned} \mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ &\left(\frac{\left[C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}}\right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) - \\ &- \mu_t \Big[P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1-\delta)K_{\eta t}] + B_{\eta t} - \\ &\quad - (W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t}) \Big] \Bigg\} \end{aligned} \tag{0.7}$$

First Order Conditions

The first order conditions are:

$$
\begin{aligned}
C_{\eta 1t}: &\quad \mu_t = \frac{\omega_{\eta 1} C_{\eta 1t}^{\omega_{\eta 1}(1-\sigma)-1} C_{\eta 2t}^{(1-\omega_{\eta 1})(1-\sigma)}}{P_{1t}} = \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} C_{\eta t}^{1-\sigma} && (0.8)\\
C_{\eta 2t}: &\quad \mu_t = \frac{(1-\omega_{\eta 1}) C_{\eta 1t}^{\omega_{\eta 1}(1-\sigma)} C_{\eta 2t}^{(1-\omega_{\eta 1})(1-\sigma)-1}}{P_{2t}} = \frac{(1-\omega_{\eta 1})}{P_{2t} C_{\eta 2t}} C_{\eta t}^{1-\sigma} && (0.9)\\
L_{\eta t}: &\quad -\phi L_{\eta t}^{\varphi} + \mu_t W_t = 0 \implies \mu_t = \frac{\phi L_{\eta t}^{\varphi}}{W_t} && (0.10)\\
B_{\eta t}: &\quad \beta^t\{-\mu_t\} + \mathbb{E}_t \beta^{t+1}\{-\mu_{t+1}[-(1+R_t)]\} = 0 \implies && \\
&\quad \mu_t = \beta(1+R_t)\mathbb{E}_t \mu_{t+1} && (0.11)\\
K_{\eta,t+1}: &\quad -\mu_t P_{\eta t} + \mathbb{E}_t \beta\{\mu_{t+1}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\} = 0 \implies && \\
&\quad \mu_t P_{\eta t} = \beta \mathbb{E}_t\{\mu_{t+1}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\} && (0.12)\\
\mu_t: &\quad P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1-\delta)K_{\eta t}] + B_{\eta t} = && \\
&\quad = W_t L_{\eta t} + R_{Kt} K_{\eta t} + (1+R_{t-1}) B_{\eta,t-1} + \Pi_{\eta t} && (0.6)
\end{aligned}
$$

Solutions

Match 0.8 and 0.9:

$$
\begin{aligned}
&\mu_t = \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} C_{\eta t}^{1-\sigma} = \frac{(1-\omega_{\eta 1})}{P_{2t} C_{\eta 2t}} C_{\eta t}^{1-\sigma} \implies \\
&\frac{C_{\eta 1t}}{C_{\eta 2t}} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \iff C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta 2t} && (0.13)
\end{aligned}
$$

Equation 0.13 is the relative consumption of regional goods in region η.

Substitute 0.13 in 0.4:

$$
\begin{aligned}
C_{\eta t} &= C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \implies && (0.4)\\
C_{\eta t} &= \left[\frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta 2t}\right]^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \implies && \\
C_{\eta 2t} &= C_{\eta t}\left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} && (0.14)
\end{aligned}
$$

Substitute 0.14 in 0.13:

$$C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta 2t} \qquad \Longrightarrow \tag{0.13}$$

$$C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \Longrightarrow$$

$$C_{\eta 1t} = C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}-1} \tag{0.15}$$

Define the total goods expense $\mathcal{E}_{\eta t}$ of household η:

$$\mathcal{E}_{\eta t} = P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} \tag{0.16}$$

Substitute 0.14 and 0.15 in 0.16:

$$\mathcal{E}_{\eta t} = P_{1t} C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}-1} + P_{2t} C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \Longrightarrow$$

$$\mathcal{E}_{\eta t} = C_{\eta t} \left[\frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}} \right]^{1-\omega_{\eta 1}} \tag{0.17}$$

Equation 0.17 shows that the total expense with goods is proportional to the goods' prices P_{1t} and P_{2t}.

Rewrite 0.14 in terms of 0.17:

$$C_{\eta 2t} = C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \qquad \Longrightarrow \tag{0.14}$$

$$C_{\eta 2t} = C_{\eta t} \left[\frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}} \right]^{-\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}} \cdot \frac{1-\omega_{\eta 1}}{P_{2t}} \right] \Longrightarrow$$

$$C_{\eta 2t} = \mathcal{E}_{\eta t} \frac{1-\omega_{\eta 1}}{P_{2t}} \tag{0.18}$$

Rewrite 0.15 in terms of 0.17 and isolate $(\omega_{\eta 1}/P_{1t}C_{\eta 1t})$:

$$C_{\eta 1t} = C_{\eta t}\left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}-1} \implies \tag{0.15}$$

$$C_{\eta 1t} = C_{\eta t}\left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}}\left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}}\left[\frac{\omega_{\eta 1}}{P_{1t}}\right] \implies$$

$$C_{\eta 1t} = \mathcal{E}_{\eta t}\frac{\omega_{\eta 1}}{P_{1t}} \iff \frac{\omega_{\eta 1}}{P_{1t}C_{\eta 1t}} = \frac{1}{\mathcal{E}_{\eta t}} \tag{0.19}$$

Substitute 0.19 in 0.8:

$$\mu_t = \frac{\omega_{\eta 1}}{P_{1t}C_{\eta 1t}}C_{\eta t}^{1-\sigma} \implies \tag{0.8}$$

$$\mu_t = \frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} \tag{0.20}$$

Match 0.20 and 0.10:

$$\mu_t = \frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \frac{\phi L_{\eta t}^{\varphi}}{W_t} \implies$$

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \tag{0.21}$$

Equation 0.21 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute μ_t and μ_{t+1} from equation 0.20 in 0.11:

$$\mu_t = \beta(1+R_t)\mathbb{E}_t\mu_{t+1} \implies \tag{0.11}$$

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1+R_t)\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}\right\} \tag{0.22}$$

Equation 0.22 is the Euler equation for the bonds return.

Substitute μ_t and μ_{t+1} from equation 0.20 in 0.12:

$$\mu_t P_{\eta t} = \beta \mathbb{E}_t \{\mu_{t+1}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\} \qquad \Longrightarrow \tag{0.12}$$

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} = \beta \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} [(1-\delta)P_{\eta,t+1} + R_{K,t+1}] \right\} \tag{0.23}$$

Equation 0.23 is the Euler equation for the capital return.

Firms

Consider two types of firms: (1) a continuum of intermediate-goods firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-goods firm, which aggregates all the varieties into a final bundle and operates in perfect competition.

0.1.2 Final-Goods Firm

Profit Maximization Problem

The role of the final-goods firm is to aggregate all the varieties $Y_{\eta j t}$ produced by the intermediate-goods firms in each region $\eta \in \{1, 2\}$, so that the representative consumer can buy only one good $Y_{\eta t}$, the bundle good, from each region.

The final-goods firm problem is to maximize its profit, considering that its output is the bundle $Y_{\eta t}$ formed by a continuum $j \in [0, 1]$ of intermediate goods $Y_{\eta j t}$, with elasticity of substitution between intermediate goods ψ:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_{\eta t} Y_{\eta t} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{0.24}$$

$$\text{s. t.} : \quad Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \tag{0.25}$$

Substitute 0.25 in 0.24:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_{\eta t} \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{0.26}$$

First Order Condition and Solutions

The first order condition is:

$$
\begin{aligned}
Y_{\eta jt}: \quad & P_{\eta t}\left(\frac{\psi}{\psi-1}\right)\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}-1}\left(\frac{\psi-1}{\psi}\right)Y_{\eta jt}^{\frac{\psi-1}{\psi}-1}-P_{\eta jt}=0 \implies \\
& Y_{\eta jt}=Y_t\left(\frac{P_{\eta t}}{P_{\eta jt}}\right)^{\psi} \qquad (0.27)
\end{aligned}
$$

Equation 0.27 shows that the demand for variety j depends on its relative price.

Substitute 0.27 in 0.25:

$$
\begin{aligned}
Y_{\eta t} &= \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} && \implies \\
Y_{\eta t} &= \left(\int_0^1 \left[Y_{\eta t}\left(\frac{P_{\eta t}}{P_{\eta jt}}\right)^{\psi}\right]^{\frac{\psi-1}{\psi}}\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} && \implies \\
P_{\eta t} &= \left[\int_0^1 P_{\eta jt}^{1-\psi}\,\mathrm{d}\,j\right]^{\frac{1}{1-\psi}} && (0.28)
\end{aligned}
$$

Equation 0.28 is the final-goods firm's markup.

0.1.3 Intermediate-Goods Firms

Cost Minimization Problem

The intermediate-goods firms, denoted by $j \in [0,1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose capital $K_{\eta jt}$ and labor $L_{\eta jt}$ to minimize production costs, subject to a technology rule.

$$
\begin{aligned}
\min_{K_{\eta jt},L_{\eta jt}} \;:\quad & R_{Kt}K_{\eta jt}+W_tL_{\eta jt} && (0.29)\\
\text{s.t.}:\quad & Y_{\eta jt}=Z_{A\eta t}K_{\eta jt}^{\alpha}L_{\eta jt}^{1-\alpha} && (0.30)
\end{aligned}
$$

where $Y_{\eta jt}$ is the output obtained by the production technology level $Z_{A\eta t}$ that transforms capital $K_{\eta jt}$ and labor $L_{\eta jt}$ in proportions α and $(1-\alpha)$, respectively, into intermediate goods.[1]

Lagrangian

Transform the minimization problem with restriction into one without restriction applying the Lagrangian function $\mathscr{L}$:

$$\mathscr{L} = (R_{Kt}K_{\eta jt} + W_t L_{\eta jt}) - \Lambda_{\eta jt}(Z_{A\eta t}K_{\eta jt}^{\alpha}L_{\eta jt}^{1-\alpha} - Y_{\eta jt}) \tag{0.31}$$

where the Lagrangian multiplier $\Lambda_{\eta jt}$ is the marginal cost.[2]

First Order Conditions

The first-order conditions are:

$$\begin{aligned}
K_{\eta jt}: \quad & R_{Kt} - \Lambda_{\eta jt}Z_{A\eta t}\alpha K_{\eta jt}^{\alpha-1}L_{\eta jt}^{1-\alpha} = 0 \qquad \Longrightarrow \\
& K_{\eta jt} = \alpha Y_{\eta jt}\frac{\Lambda_{\eta jt}}{R_{Kt}} && (0.32)\\
L_{\eta jt}: \quad & W_t - \Lambda_{\eta jt}Z_{A\eta t}K_{\eta jt}^{\alpha}(1-\alpha)L_{\eta jt}^{-\alpha} = 0 \qquad \Longrightarrow \\
& L_{\eta jt} = (1-\alpha)Y_{\eta jt}\frac{\Lambda_{\eta jt}}{W_t} && (0.33)\\
\Lambda_{\eta jt}: \quad & Y_{\eta jt} = Z_{A\eta t}K_{\eta jt}^{\alpha}L_{\eta jt}^{1-\alpha} && (0.30)
\end{aligned}$$

Solutions

Divide equation 0.32 by 0.33:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \frac{\alpha Y_{\eta jt}\Lambda_{\eta jt}/R_{Kt}}{(1-\alpha)Y_{\eta jt}\Lambda_{\eta jt}/W_t} \Longrightarrow \frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_{Kt}} \tag{0.34}$$

Equation 0.34 demonstrates the relationship between the technical marginal rate

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[1] the production technology level $Z_{A\eta t}$ will be submitted to a productivity shock, detailed in section 0.1.5.

[2] see Lemma **??**

of substitution (TMRS) and the economical marginal rate of substitution (EMRS).

Substitute $L_{\eta jt}$ from equation 0.34 in 0.30:

$$\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha} L_{\eta jt}^{1-\alpha} && \Longrightarrow \\
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha} \left[\left(\frac{1-\alpha}{\alpha} \right) \frac{R_{Kt} K_{\eta jt}}{W_t} \right]^{1-\alpha} && \Longrightarrow \\
K_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha} && (0.35)
\end{aligned}$$

Equation 0.35 is the intermediate-goods firm demand for capital.

Substitute 0.35 in 0.34:

$$\begin{aligned}
L_{\eta jt} &= \left(\frac{1-\alpha}{\alpha} \right) \frac{R_{Kt} K_{\eta jt}}{W_t} && \Longrightarrow \\
L_{\eta jt} &= \left(\frac{1-\alpha}{\alpha} \right) \frac{R_{Kt}}{W_t} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha} && \Longrightarrow \\
L_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{-\alpha} && (0.36)
\end{aligned}$$

Equation 0.36 is the intermediate-goods firm demand for labor.

Total and Marginal Costs

Calculate the total cost TC using 0.35 and 0.36:

$$\begin{aligned}
TC_{\eta jt} &= W_t L_{\eta jt} + R_{Kt} K_{\eta jt} && \Longrightarrow \\
TC_{\eta jt} &= W_t \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{-\alpha} + R_{Kt} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha} && \Longrightarrow \\
TC_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} && (0.37)
\end{aligned}$$

Calculate the marginal cost Λ using 0.37:

$$\Lambda_{\eta jt} = \frac{\partial TC_{\eta jt}}{\partial Y_{\eta jt}} \Longrightarrow \Lambda_{\eta jt} = \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \tag{0.38}$$

The marginal cost depends on the technological level $Z_{A\eta t}$, the nominal interest rate R_{Kt} and the nominal wage level W_t, which are the same for all intermediate-goods firms, and because of that, the index j may be dropped:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \tag{0.39}$$

notice that:

$$\Lambda_{\eta t} = \frac{TC_{\eta jt}}{Y_{\eta jt}} \implies TC_{\eta jt} = \Lambda_{\eta t} Y_{\eta jt} \tag{0.40}$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (**calvo_staggered_1983**): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period ($P_{\eta j,t+1} = P_{\eta jt}$), and a probability of $(1-\theta)$ of setting a new optimal price $P^*_{\eta jt}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price.

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \tag{0.41}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Y_{\eta t} \left(\frac{P_{\eta t}}{P_{\eta jt}} \right)^{\psi} \tag{0.27}$$

Substitute 0.40 in 0.41:

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - \Lambda_{\eta,t+s} Y_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \tag{0.42}$$

Substitute 0.27 in 0.42 and rearrange the variables:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} - \Lambda_{\eta,t+s} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt}^{1-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to $P_{\eta jt}$ is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - (-\psi) P_{\eta jt}^{-\psi-1} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} Y_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \\ = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \end{aligned} \tag{0.43}$$

Substitute 0.27 in 0.43:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi-1) Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$(\psi - 1)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \psi P_{\eta jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt}^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}} \tag{0.44}$$

Equation 0.44 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P_{\eta t}^{*} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{0.45}$$

Final-Goods Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1 - \theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 0.28 to determine the aggregate price level:

$$\begin{aligned} P_{\eta t} &= \left[\int_0^{\theta} P_{\eta,t-1}^{1-\psi} \, \mathrm{d}\, j + \int_{\theta}^{1} P_{\eta t}^{*1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies \\ P_{\eta t} &= \left[\theta P_{\eta,t-1}^{1-\psi} + (1 - \theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \end{aligned} \tag{0.46}$$

Equation 0.46 is the aggregate price level.

0.1.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (**taylor_discretion_1993**) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{0.47}$$

where R, π, Y are the variables in steady state, γ_R is the smoothing parameter for the interest rate R_{Kt}, γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively, Z_{Mt} is the monetary shock and π_t is the gross inflation rate, defined by:[3]

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{0.48}$$

where P_t is the national price level, defined by:

$$P_t Y_t = P_{1t} Y_{1t} + P_{2t} Y_{2t} \iff P_t = \frac{P_{1t} Y_{1t} + P_{2t} Y_{2t}}{Y_t} \tag{0.49}$$

Regional Inflation

There is one price level $P_{\eta t}$ in each region, generating a regional inflation rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta, t-1}} \tag{0.50}$$

0.1.5 Stochastic Shocks

Productivity Shock

The production technology level $Z_{A\eta t}$ will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta, t-1} + \varepsilon_{A\eta t} \tag{0.51}$$

where $\rho_{A\eta} \in [0,1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M, t-1} + \varepsilon_{Mt} \tag{0.52}$$

[3] for the monetary shock definition, see section 0.1.5.

where $\rho_M \in [0,1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

0.1.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P^*_{\eta t}, R^*_t, R^*_{Kt}, W^*_t\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C^*_{\eta 1t}, C^*_{\eta 2t}, L^*_{\eta t}, B^*_{\eta t}, K^*_{\eta,t+1}\}$ and allocations for firms $\boldsymbol{\mathcal{A}}_F := \{K^*_{\eta jt}, L^*_{\eta jt}, Y^*_{\eta jt}, Y^*_{\eta t}\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms' problems, and the markets for goods and labor clear:

$$Y_t = Y_{1t} + Y_{2t} \tag{0.53}$$

$$\text{where:} \quad Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \tag{0.54}$$

$$L_{\eta t} = \int_0^1 L_{\eta jt}\, \mathrm{d}\, j \tag{0.55}$$

0.1.7 Model Structure

The model is composed of the preview solutions, forming a square system of 41 variables and 41 equations, summarized as follows:

- Variables:
  - from the household problem: $\langle\, C_{\eta t} \;\; L_{\eta t} \;\; B_{\eta t} \;\; K_{\eta,t+1} \;\; C_{\eta 1t} \;\; C_{\eta 2t} \;\; \mathcal{E}_{\eta t} \,\rangle$;
  - from the final-goods firm problem: $\langle\, Y_{\eta jt} \;\; Y_{\eta t} \;\; P_{\eta t} \,\rangle$;
  - from the intermediate-goods firm problems: $\langle\, K_{\eta jt} \;\; L_{\eta jt} \;\; P^*_{\eta t} \,\rangle$;
  - from the monetary policy: $\langle\, R_t \;\; \pi_t \;\; Y_t \,\rangle$;
  - from the market clearing condition: $\langle\, I_{\eta t} \,\rangle$;
  - prices: $\langle\, W_t \;\; R_{Kt} \;\; \Lambda_{\eta t} \;\; P_t \;\; \pi_{\eta t} \,\rangle$;
  - shocks: $\langle\, Z_{A\eta t} \;\; Z_{Mt} \,\rangle$.
- Equations:
  1. Budget Constraint:

$$\begin{aligned} &P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ &\quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \tag{0.2}$$

  2. Law of Motion for Capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{0.3}$$

3. Regional Consumption of good 1:

$$C_{\eta 1t} = \mathcal{E}_{\eta t} \frac{\omega_{\eta 1}}{P_{1t}} \tag{0.19}$$

4. Regional Consumption of good 2:

$$C_{\eta 2t} = \mathcal{E}_{\eta t} \frac{1-\omega_{\eta 1}}{P_{2t}} \tag{0.18}$$

5. Price Composition of Consumption Bundle:

$$\mathcal{E}_{\eta t} = C_{\eta t} \left[\frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}} \right]^{1-\omega_{\eta 1}} \tag{0.17}$$

6. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \tag{0.21}$$

7. Euler equation for the bonds return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1+R_t)\mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} \right\} \tag{0.22}$$

8. Euler equation for the capital return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} = \beta \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} [(1-\delta)P_{\eta,t+1} + R_{K,t+1}] \right\} \tag{0.23}$$

9. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}j \right)^{\frac{\psi}{\psi-1}} \tag{0.25}$$

10. Production Function:

$$Y_{\eta j t} = Z_{A\eta t} K_{\eta j t}^{\alpha} L_{\eta j t}^{1-\alpha} \tag{0.30}$$

11. Capital Demand:

$$K_{\eta j t} = \alpha Y_{\eta j t} \frac{\Lambda_{\eta t}}{R_{Kt}} \tag{0.32}$$

12. Labor Demand:

$$L_{\eta jt} = (1-\alpha) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \tag{0.33}$$

13. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha}\right)^{\alpha} \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \tag{0.39}$$

14. Optimal Price:

$$P^*_{\eta t} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \tag{0.45}$$

15. Regional Price Level:

$$P_{\eta t} = \left[\theta P_{\eta,t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{0.46}$$

16. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \tag{0.47}$$

17. National Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{0.48}$$

18. National Price Level:

$$P_t = \frac{P_{1t} Y_{1t} + P_{2t} Y_{2t}}{Y_t} \tag{0.49}$$

19. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{0.50}$$

20. Productivity Shock:

$$\ln Z_{A\eta t} = (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{0.51}$$

21. Monetary Shock:

$$\ln Z_{Mt} = (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{0.52}$$

22. Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \tag{0.53}$$

23. Regional Market Clearing Condition:

$$Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \tag{0.54}$$

0.2 Steady State

The steady state of a variable is defined by its constancy through time. For any given variable X_t, it is in steady state if $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (**costa_junior_understanding_201** For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The model in steady state is:

1. Budget Constraint:

$$\begin{aligned} & P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ & \quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \implies \end{aligned} \tag{0.2}$$

$$P_1 C_{\eta 1} + P_2 C_{\eta 2} + P_\eta I_\eta = W L_\eta + R_K K_\eta + R B_\eta + \Pi_\eta \tag{0.56}$$

2. Law of Motion for Capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies \tag{0.3}$$

$$K_\eta = (1-\delta)K_\eta + I_\eta \implies$$

$$I_\eta = \delta K_\eta \tag{0.57}$$

3. Regional Consumption of good 1:

$$C_{\eta 1t} = \mathcal{E}_{\eta t}\frac{\omega_{\eta 1}}{P_{1t}} \implies \tag{0.19}$$

$$C_{\eta 1} = \mathcal{E}_{\eta}\frac{\omega_{\eta 1}}{P_{1}} \tag{0.58}$$

4. Regional Consumption of good 2:

$$C_{\eta 2t} = \mathcal{E}_{\eta t}\frac{1-\omega_{\eta 1}}{P_{2t}} \implies \tag{0.18}$$

$$C_{\eta 2} = \mathcal{E}_{\eta}\frac{1-\omega_{\eta 1}}{P_{2}} \tag{0.59}$$

5. Price Composition of Consumption Bundle:

$$\mathcal{E}_{\eta t} = C_{\eta t}\left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}}\left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \implies \tag{0.17}$$

$$\mathcal{E}_{\eta} = C_{\eta}\left[\frac{P_{1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}}\left[\frac{P_{2}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \tag{0.60}$$

6. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \implies \tag{0.21}$$

$$\frac{\phi L_{\eta}^{\varphi}}{C_{\eta}^{1-\sigma}} = \frac{W}{\mathcal{E}_{\eta}} \tag{0.61}$$

7. Euler equation for the bonds return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1+R_t)\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}\right\} \implies \tag{0.22}$$

$$\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} = \beta(1+R)\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \implies$$

$$\beta = \frac{1}{(1+R)} \tag{0.62}$$

8. Euler equation for the capital return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}}P_{\eta t} = \beta\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}[(1-\delta)P_{\eta,t+1}+R_{K,t+1}]\right\} \implies \tag{0.23}$$

$$\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}}P_{\eta} = \beta\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}}P_{\eta}\left[(1-\delta)+\frac{R_K}{P_{\eta}}\right] \implies$$

$$1 = \beta\left[(1-\delta)+\frac{R_K}{P_{\eta}}\right] \tag{0.63}$$

9. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \implies Y_{\eta} = \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{0.64}$$

10. Production Function:

$$Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha} L_{\eta jt}^{1-\alpha} \implies Y_{\eta j} = Z_{A\eta} K_{\eta j}^{\alpha} L_{\eta j}^{1-\alpha} \tag{0.65}$$

11. Capital Demand:

$$K_{\eta jt} = \alpha Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_{Kt}} \implies K_{\eta j} = \alpha Y_{\eta j} \frac{\Lambda_{\eta}}{R_K} \tag{0.66}$$

12. Labor Demand:

$$L_{\eta jt} = (1-\alpha) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \implies L_{\eta j} = (1-\alpha) Y_{\eta j} \frac{\Lambda_{\eta}}{W} \tag{0.67}$$

13. Marginal Cost:

$$\begin{aligned} \Lambda_{\eta t} &= \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha}\right)^{\alpha} \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \implies \\ \Lambda_{\eta} &= \frac{1}{Z_{A\eta}} \left(\frac{R_K}{\alpha}\right)^{\alpha} \left(\frac{W}{1-\alpha}\right)^{1-\alpha} \end{aligned} \tag{0.68}$$

14. Optimal Price:

$$P_{\eta t}^{*} = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \tag{0.45}$$

$$P_{\eta}^{*} = \frac{\psi}{\psi-1} \cdot \frac{Y_{\eta j}\Lambda_{\eta} / [1-\theta(1-R)]}{Y_{\eta j} / [1-\theta(1-R)]} \implies$$

$$P_{\eta}^{*} = \frac{\psi}{\psi-1} \Lambda_{\eta} \tag{0.69}$$

15. Regional Price Level:

$$\begin{aligned} P_{\eta t} &= \left[\theta P_{\eta t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi}\right]^{\frac{1}{1-\psi}} && \implies \\ P_{\eta}^{1-\psi} &= \theta P_{\eta}^{1-\psi} + (1-\theta) P_{\eta}^{*1-\psi} && \implies \\ (1-\theta) P_{\eta}^{1-\psi} &= (1-\theta) P_{\eta}^{*1-\psi} && \implies P_{\eta} = P_{\eta}^{*} \end{aligned} \tag{0.70}$$

16. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \tag{0.71}$$

17. National Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \pi = 1 \tag{0.72}$$

18. National Price Level:

$$P_t = \frac{P_{1t}Y_{1t} + P_{2t}Y_{2t}}{Y_t} \implies P = \frac{P_1 Y_1 + P_2 Y_2}{Y} \tag{0.73}$$

19. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \implies \pi_\eta = 1 \tag{0.74}$$

20. Productivity Shock:

$$\begin{aligned} \ln Z_{A\eta t} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} &&\implies \\ \ln Z_{A\eta} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta} + \varepsilon_{A\eta} &&\implies \\ \varepsilon_{A\eta} &= 0 \end{aligned} \tag{0.75}$$

21. Monetary Shock:

$$\begin{aligned} \ln Z_{Mt} &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} &&\implies \\ \ln Z_M &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_M + \varepsilon_M &&\implies \\ \varepsilon_M &= 0 \end{aligned} \tag{0.76}$$

22. Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \implies \tag{0.53}$$

$$Y = Y_1 + Y_2 \tag{0.77}$$

23. Regional Market Clearing Condition:

$$Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \implies \tag{0.54}$$
$$Y_{\eta} = C_{\eta 1} + C_{\eta 2} + I_{\eta} \tag{0.78}$$

0.2.1 Variables at Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It is assumed that one regional price level and both productivity levels are normalized to one:[4]

$$\langle P_1 \quad Z_{A1} \quad Z_{A2} \rangle = \vec{\mathbf{1}} \tag{0.79}$$

From 0.71, 0.72 and 0.74, the monetary shock, the national and regional gross inflation rates are:

$$\langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle = \vec{\mathbf{1}} \tag{0.80}$$

From 0.75 and 0.76, the productivity and monetary shocks are:

$$\langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle = \vec{\mathbf{0}} \tag{0.81}$$

From 0.62, the return on bonds is:

$$\beta = \frac{1}{(1+R)} \implies \tag{0.62}$$
$$R = \frac{1}{\beta} - 1 \tag{0.82}$$

From 0.63, the return on capital R_K for $\eta = 1$ is:

$$1 = \beta \left[(1-\delta) + \frac{R_K}{P_1} \right] \implies \tag{0.63}$$
$$R_K = P_1 \left[\frac{1}{\beta} - (1-\delta) \right] \tag{0.83}$$

[4] where $\vec{\mathbf{1}}$ is the unit vector.

Divide 0.83 for $\eta = 1$ by 0.83 for $\eta = 2$:

$$\frac{R_K}{R_K} = \frac{P_1\left[\frac{1}{\beta} - (1-\delta)\right]}{P_2\left[\frac{1}{\beta} - (1-\delta)\right]} \implies$$

$$1 = \frac{P_1}{P_2} \implies P_1 = P_2 \tag{0.84}$$

From 0.70 and 0.84, the regional optimal price P_η^* is:

$$P_\eta^* = P_\eta \implies \langle P_1^* \quad P_2^* \rangle = \langle P_1 \quad P_2 \rangle = \langle P_1 \quad P_1 \rangle \tag{0.70}$$

Substitute 0.84 and then 0.78 in 0.73 for the national price level:

$$P = \frac{P_1 Y_1 + P_2 Y_2}{Y} = \frac{P_1 Y_1 + P_1 Y_2}{Y} \implies \tag{0.73}$$

$$P = P_1 = P_2 \tag{0.85}$$

Substitute 0.85 in 0.60 for the price composition of consumption bundle $\mathcal{E}_\eta$:

$$\mathcal{E}_\eta = C_\eta \left[\frac{P_1}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \left[\frac{P_2}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} = C_\eta \left[\frac{P}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \left[\frac{P}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \implies \tag{0.60}$$

$$\mathcal{E}_\eta = \frac{PC_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{0.86}$$

Substitute 0.70 and 0.85 in 0.69 for the marginal cost Λ_η:

$$P_\eta^* = \frac{\psi}{\psi - 1}\Lambda_\eta \implies \tag{0.69}$$

$$\Lambda_\eta = P\frac{\psi - 1}{\psi} = \Lambda \tag{0.87}$$

Substitute 0.79 and 0.87 in 0.68 for the nominal wage W in terms of region 1:

$$\Lambda_\eta = \frac{1}{Z_{A\eta}}\left(\frac{R_K}{\alpha}\right)^\alpha \left(\frac{W}{1-\alpha}\right)^{1-\alpha} \implies \tag{0.68}$$

$$W = (1-\alpha)\left[\Lambda Z_{A1}\left(\frac{\alpha}{R_K}\right)^\alpha\right]^{\frac{1}{1-\alpha}} \tag{0.88}$$

Due to price parity in steady state, where prices are identical ($P_\eta = P^*_\eta$) and resulting in a gross inflation level of one ($\pi_\eta = 1$), all firms produce the same output level ($Y_{\eta j} = Y_\eta$) (**solis-garcia_ucb_2022**). As a consequence, they uniformly demand the same amount of factors (K_η, L_η), and equations 0.65, 0.66 and 0.67 become:

$$Y_\eta = Z_{A\eta} K_\eta^\alpha L_\eta^{1-\alpha} \tag{0.89}$$

$$K_\eta = \alpha Y_\eta \frac{\Lambda}{R_K} \tag{0.90}$$

$$L_\eta = (1-\alpha) Y_\eta \frac{\Lambda}{W} \tag{0.91}$$

Substitute 0.90 in 0.57:

$$I_\eta = \delta K_\eta \implies I_\eta = \delta\alpha \frac{\Lambda}{R_K} Y_\eta \implies \tag{0.57}$$

$$I_\eta = b_\eta Y_\eta \tag{0.92}$$

$$\text{where:} \quad b_\eta = \delta\alpha \frac{\Lambda}{R_K} \tag{0.93}$$

Isolate C_η in 0.61 and then substitute 0.86 and 0.91:

$$\frac{\phi L_\eta^\varphi}{C_\eta^{1-\sigma}} = \frac{W}{\mathcal{E}_\eta} \implies C_\eta^{\sigma-1} = \frac{W}{\mathcal{E}_\eta \phi L_\eta^\varphi} \implies$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{0.94}$$

$$\text{where:} \quad a_\eta = \left[\frac{W^{1+\varphi} \omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}{\phi P \left[(1-\alpha)\Lambda\right]^\varphi} \right]^{\frac{1}{\sigma}} \tag{0.95}$$

Substitute 0.94 in 0.86:

$$\mathcal{E}_\eta = \frac{P C_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \implies \tag{0.86}$$

$$\mathcal{E}_\eta = \frac{P a_\eta Y_\eta^{\frac{-\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{0.96}$$

Substitute 0.96 in 0.58:

$$C_{\eta 1} = \mathcal{E}_\eta \frac{\omega_{\eta 1}}{P_1} \qquad \Longrightarrow \tag{0.58}$$

$$C_{\eta 1} = \frac{P a_\eta Y_\eta^{\frac{-\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \cdot \frac{\omega_{\eta 1}}{P} \Longrightarrow$$

$$C_{\eta 1} = \left(\frac{\omega_{\eta 1}}{1-\omega_{\eta 1}}\right)^{1-\omega_{\eta 1}} a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{0.97}$$

Substitute 0.96 in 0.59:

$$C_{\eta 2} = \mathcal{E}_\eta \frac{1-\omega_{\eta 1}}{P_2} \qquad \Longrightarrow \tag{0.59}$$

$$C_{\eta 2} = \frac{P a_\eta Y_\eta^{\frac{-\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \cdot \frac{1-\omega_{\eta 1}}{P} \Longrightarrow$$

$$C_{\eta 2} = \left(\frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right)^{\omega_{\eta 1}} a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{0.98}$$

Substitute 0.92, 0.97 and 0.98 in 0.78:

$$Y_\eta = C_{\eta 1} + C_{\eta 2} + I_\eta \qquad \Longrightarrow \tag{0.78}$$

$$Y_\eta = \left(\frac{\omega_{\eta 1}}{1-\omega_{\eta 1}}\right)^{1-\omega_{\eta 1}} a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} + \left(\frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right)^{\omega_{\eta 1}} a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} + b_\eta Y_\eta \Longrightarrow$$

$$Y_\eta = \left[\left(\frac{a_1}{1-b_1}\right)\left(\frac{1}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}\right)\right]^{\frac{\sigma}{\sigma+\varphi}} \tag{0.99}$$

The result of 0.99 determines $Y, I_\eta, C_\eta, \mathcal{E}_\eta, C_{\eta 1}, C_{\eta 2}, K_\eta, L_\eta$ in 0.77 0.92, 0.94, 0.86, 0.58, 0.59, 0.90 and 0.91, respectively.

0.2.2 Steady State Solution

$$\vec{\mathbf{1}} = \langle P \quad P_1 \quad P_2 \quad Z_{A1} \quad Z_{A2} \quad Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle \tag{0.100}$$

$$\vec{\mathbf{0}} = \langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle \tag{0.81}$$

$$R = \frac{1}{\beta} - 1 \tag{0.82}$$

$$R_K = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{0.83}$$

$$\Lambda = P\frac{\psi - 1}{\psi} \tag{0.87}$$

$$W = (1-\alpha)\left[\Lambda Z_{A1}\left(\frac{\alpha}{R_K}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} \tag{0.88}$$

$$a_\eta = \left[\frac{W^{1+\varphi}\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}{\phi P\left[(1-\alpha)\Lambda\right]^{\varphi}}\right]^{\frac{1}{\sigma}} \tag{0.95}$$

$$b_\eta = \delta\alpha\frac{\Lambda}{R_K} \tag{0.93}$$

$$Y_\eta = \left[\left(\frac{a_1}{1-b_1}\right)\left(\frac{1}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}\right)\right]^{\frac{\sigma}{\sigma+\varphi}} \tag{0.99}$$

$$Y = Y_1 + Y_2 \tag{0.77}$$

$$I_\eta = b_\eta Y_\eta \tag{0.92}$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{0.94}$$

$$\mathcal{E}_\eta = \frac{PC_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{0.86}$$

$$C_{\eta 1} = \mathcal{E}_\eta\frac{\omega_{\eta 1}}{P} \tag{0.58}$$

$$C_{\eta 2} = \mathcal{E}_\eta\frac{1-\omega_{\eta 1}}{P} \tag{0.59}$$

$$K_\eta = \alpha Y_\eta\frac{\Lambda}{R_K} \tag{0.90}$$

$$L_\eta = (1-\alpha)Y_\eta\frac{\Lambda}{W} \tag{0.91}$$

0.3 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. Dynare is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation 0.45. For this purpose, Uhlig's rules of log-linearization (**uhlig_toolkit_1999**) will be applied to all equations in the model.[5]

Regional Price Level

Log-linearize equation 0.46:

$$
\begin{aligned}
P_{\eta t}^{1-\psi} &= \theta P_{\eta,t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi} && \Longrightarrow \qquad (0.46)\\
P^{1-\psi}(1+(1-\psi)\hat{P}_{\eta t}) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{\eta,t-1}) + \\
&\quad + (1-\theta)P^{1-\psi}(1+(1-\psi)\hat{P}^{*}_{\eta t}) && \Longrightarrow\\
\hat{P}_{\eta t} &= \theta \hat{P}_{\eta,t-1} + (1-\theta)\hat{P}^{*}_{\eta t} && (0.101)
\end{aligned}
$$

Regional Gross Inflation Rate

Log-linearize 0.50 and define the level deviation of regional inflation rate $\hat{\pi}_{\eta t}$:

$$
\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{0.50}
$$

$$
\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{0.102}
$$

5 see lemma **??** for details.

New Keynesian Phillips Curve

In order to log-linearize equation 0.45, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma **??**:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P^*_{\eta t} Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{0.45}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P^*_{\eta t} Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\qquad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{0.103}$$

First, log-linearize the left hand side of equation 0.103 with respect to $P^*_{\eta t}, Y_{\eta jt}, \widetilde{R}_t$:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P^*_{\eta t} Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies$$

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P^*_{\eta} Y_{\eta j} \left(1 + \hat{P}^*_{\eta t} + \hat{Y}_{\eta j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies$$

$$P^*_{\eta} Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}^*_{\eta t} + \hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\}$$

Separate the terms not dependent on s:

$$\begin{aligned} &P^*_{\eta} Y_{\eta j}(1 + \hat{P}^*_{\eta t}) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \right\} + \\ &\qquad + P^*_{\eta} Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \implies \end{aligned}$$

Apply definition **??** on the first term:

$$\frac{P^*_{\eta} Y_{\eta j}(1 + \hat{P}^*_{\eta t})}{1 - \theta/(1+R)} + P^*_{\eta} Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\}$$

Second, log-linearize the left hand side of 0.103 with respect to $\Lambda^*_{\eta t}, Y_{\eta jt}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\frac{\theta^s Y_{\eta j,t+s}\Lambda_{\eta,t+s}}{(1+R)^s\left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)}\right\} \implies$$

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\frac{Y_{\eta j}\Lambda_\eta(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s})}{1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right\} \implies$$

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\right\}+ \\ +\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Apply definition **??** on the first term:

$$\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+ \\ +\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Join both sides of the equation again:

$$\frac{P^*_\eta Y_{\eta j}(1+\hat{P}^*_{\eta t})}{1-\theta/(1+R)}+P^*_\eta Y_{\eta j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}= \\ =\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+ \\ +\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} \quad (0.104)$$

Substitute 0.62 in 0.104:

$$\beta = \frac{1}{(1+R)} \tag{0.62}$$

$$\frac{P_\eta^* Y_{\eta j}(1+\hat{P}_{\eta t}^*)}{1-\theta\beta} + P_\eta^* Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} =$$
$$= \frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta\beta} + \tag{0.105}$$
$$+ \frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{Y}_{\eta j,t+s} + \hat{\Lambda}_{\eta,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Substitute 0.87 in 0.105 and simplify all common terms:

$$\cancel{\frac{P_\eta^* Y_{\eta j}}{1-\theta\beta}} + \frac{\cancel{P_\eta^* Y_{\eta j}}\hat{P}_{\eta t}^*}{1-\theta\beta} + \cancel{P_\eta^* Y_{\eta j}}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\cancel{\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right)\right\} =$$
$$= \cancel{\frac{P_\eta^* Y_{\eta j}}{1-\theta\beta}} + \cancel{P_\eta^* Y_{\eta j}}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\cancel{\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}} + \hat{\Lambda}_{\eta,t+s}\right)\right\} \Longrightarrow$$
$$\frac{\hat{P}_{\eta t}^*}{1-\theta\beta} = \mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{\Lambda}_{\eta,t+s}\right)\right\} \tag{0.106}$$

Define the real marginal cost $\lambda_{\eta t}$:

$$\lambda_{\eta t} = \frac{\Lambda_{\eta t}}{P_{\eta t}} \Longrightarrow \Lambda_{\eta t} = P_{\eta t}\lambda_{\eta t} \Longrightarrow$$
$$\hat{\Lambda}_{\eta t} = \hat{P}_{\eta t} + \hat{\lambda}_{\eta t} \tag{0.107}$$

Substitute 0.107 in 0.106:

$$\hat{P}_{\eta t}^* = (1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}\right) \tag{0.108}$$

Substitute 0.108 in 0.101:

$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)\hat{P}_{\eta t}^* \tag{0.101}$$
$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}\right) \tag{0.109}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\beta\mathbb{L}^{-1})$ in 0.109:[6]

$$
\begin{aligned}
(1-\theta\beta\mathbb{L}^{-1})\hat{P}_{\eta t} = (1-\theta\beta\mathbb{L}^{-1})\,\Big[\theta\hat{P}_{\eta,t-1} + \\
+(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right)\Big] \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + \\
(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) - \\
-\,\theta\beta(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)
\end{aligned}
\tag{0.110}
$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\beta$ within the operator. Then, cancel the summations and rearrange the terms:

$$
\begin{aligned}
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + \\
(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) - \\
-\,\theta\beta(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right) \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + (1-\theta)(1-\theta\beta)(\hat{P}_{\eta t}+\hat{\lambda}_{\eta t}) + \\
+\,(1-\theta)(1-\theta\beta)\cancel{\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)} - \\
-\,(1-\theta)(1-\theta\beta)\cancel{\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)} \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta^2\beta\hat{P}_{\eta t} + \\
+\,(1-\theta-\theta\beta+\theta^2\beta)\hat{P}_{\eta t} + (1-\theta)(1-\theta\beta)\hat{\lambda}_{\eta t} \implies \\
(\hat{P}_{\eta t} - \hat{P}_{\eta,t-1}) = \beta(\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_{\eta t}) + \frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{\eta t}
\end{aligned}
\tag{0.111}
$$

[6] see definition **??**.

Substitute 0.102 in 0.111:

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{\eta t} \tag{0.112}$$

Equation 0.112 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

Law of Motion for Capital

Log-linearize 0.3:

$$\begin{aligned}
K_{\eta,t+1} &= (1-\delta)K_{\eta t} + I_{\eta t} && \Longrightarrow && (0.3)\\
K_{\eta}(1+\hat{K}_{\eta,t+1}) &= (1-\delta)K_{\eta}(1+\hat{K}_{\eta t}) + I_{\eta}(1+\hat{I}_{\eta t}) && \Longrightarrow &&\\
\hat{K}_{\eta,t+1} &= (1-\delta)\hat{K}_{\eta t} + \delta\hat{I}_{\eta t} && && (0.113)
\end{aligned}$$

Regional Levels of Consumption and Prices

Log-linearize 0.19:

$$\begin{aligned}
C_{\eta 1t} &= \mathcal{E}_{\eta t}\frac{\omega_{\eta 1}}{P_{1t}} && \Longrightarrow && (0.19)\\
C_{\eta 1}(1+\hat{C}_{\eta 1t}) &= \mathcal{E}_{\eta}\frac{\omega_{\eta 1}}{P_1}(1+\hat{\mathcal{E}}_{\eta t} - \hat{P}_{1t}) && \Longrightarrow &&\\
\hat{C}_{\eta 1t} &= \hat{\mathcal{E}}_{\eta t} - \hat{P}_{1t} \iff \hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 1t} + \hat{P}_{1t} && && (0.114)
\end{aligned}$$

Log-linearize 0.18:

$$\begin{aligned}
C_{\eta 2t} &= \mathcal{E}_{\eta t}\frac{1-\omega_{\eta 1}}{P_{2t}} && \Longrightarrow && (0.18)\\
C_{\eta 2}(1+\hat{C}_{\eta 2t}) &= \mathcal{E}_{\eta}\frac{1-\omega_{\eta 1}}{P_2}(1+\hat{\mathcal{E}}_{\eta t} - \hat{P}_{2t}) && \Longrightarrow &&\\
\hat{C}_{\eta 2t} &= \hat{\mathcal{E}}_{\eta t} - \hat{P}_{2t} \iff \hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 2t} + \hat{P}_{2t} && && (0.115)
\end{aligned}$$

Match equations 0.114 and 0.115:

$$\hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 1t} + \hat{P}_{1t} = \hat{C}_{\eta 2t} + \hat{P}_{2t} \implies$$
$$\hat{C}_{\eta 1t} - \hat{C}_{\eta 2t} = \hat{P}_{2t} - \hat{P}_{1t} \tag{0.116}$$

Equation 0.116 shows that the variation of both goods consumption in each region correspond to the distance between the variations of the regional price levels.

Total Expenses

Log-linearize 0.17:

$$\mathcal{E}_{\eta t} = C_{\eta t} \left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \implies \tag{0.17}$$
$$\mathcal{E}_{\eta}(1+\hat{\mathcal{E}}_{\eta t}) = \frac{C_{\eta} P_1^{\omega_{\eta 1}} P_2^{1-\omega_{\eta 1}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{(1-\omega_{\eta 1})}} \cdot$$
$$\cdot (1 + \hat{C}_{\eta t} + \omega_{\eta 1}\hat{P}_{1t} + (1-\omega_{\eta 1})\hat{P}_{2t}) \implies$$
$$\hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta t} + \omega_{\eta 1}\hat{P}_{1t} + (1-\omega_{\eta 1})\hat{P}_{2t} \tag{0.117}$$

Labor Supply

Log-linearize 0.21:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \implies \tag{0.21}$$
$$\varphi \hat{L}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} = \hat{W}_t - \hat{\mathcal{E}}_{\eta t} \tag{0.118}$$

Euler equation for the bonds return

Log-linearize 0.22:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1+R_t)\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}\right\} \implies \tag{0.22}$$

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} \cdot \frac{\mathbb{E}_t\mathcal{E}_{\eta t+1}}{\mathbb{E}_t C_{\eta,t+1}^{1-\sigma}} = \beta + \beta R_t \implies$$

$$\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \cdot \frac{\mathcal{E}_{\eta}}{C_{\eta}^{1-\sigma}}(1+(1-\sigma)\hat{C}_{\eta t} - \hat{\mathcal{E}}_{\eta t} + \mathbb{E}_t\{\hat{\mathcal{E}}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1}\}) =$$

$$= \beta + \beta(R(1+\hat{R}_t)) \implies$$

$$\mathbb{E}_t\left\{\hat{\mathcal{E}}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1}\right\} - [\hat{\mathcal{E}}_{\eta t} - (1-\sigma)\hat{C}_{\eta t}] = (1-\beta)\hat{R}_t \tag{0.119}$$

Euler equation for the capital return

Log-linearize 0.23:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}}P_{\eta t} = \beta\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\right\} \implies \tag{0.23}$$

$$\frac{P_{\eta t}C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}}\mathbb{E}_t\left\{\frac{\mathcal{E}_{\eta t+1}}{P_{\eta,t+1}C_{\eta,t+1}^{1-\sigma}}\right\} = \beta\left[(1-\delta) + \mathbb{E}_t\left\{\frac{R_{K,t+1}}{P_{\eta,t+1}}\right\}\right] \implies$$

$$\frac{P_{\eta}C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \cdot \frac{\mathcal{E}_{\eta}}{P_{\eta}C_{\eta}^{1-\sigma}}(1+\hat{P}_{\eta t} + (1-\sigma)\hat{C}_{\eta t} - \hat{\mathcal{E}}_{\eta t} +$$

$$+ \mathbb{E}_t\{\hat{\mathcal{E}}_{\eta,t+1} - \hat{P}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1}\}) =$$

$$= \beta\left[(1-\delta) + \frac{R_K}{P_\eta}\mathbb{E}_t\left\{1 + \hat{R}_{K,t+1} - \hat{P}_{\eta,t+1}\right\}\right] \implies$$

$$\mathbb{E}_t\left\{\hat{\mathcal{E}}_{\eta,t+1} - \hat{P}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1}\right\} - (\hat{\mathcal{E}}_{\eta t} - \hat{P}_{\eta t} - (1-\sigma)\hat{C}_{\eta t}) =$$

$$= \beta r_K \mathbb{E}_t\left\{\hat{R}_{K,t+1} - \hat{P}_{\eta,t+1}\right\} \tag{0.120}$$

$$\text{where: } r_K = \frac{R_K}{P} \tag{0.121}$$

Bundle Technology

Apply the natural logarithm to 0.25:

$$
\begin{aligned}
Y_{\eta t} &= \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \implies && (0.25) \\
\ln Y_{\eta t} &= \frac{\psi}{\psi-1} \ln \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)
\end{aligned}
$$

Log-linearize using corollary **??**:

$$
\begin{aligned}
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \frac{\psi}{\psi-1} \left[\ln \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right) + \frac{\psi-1}{\psi} \int_0^1 \hat{Y}_{\eta j t} \, \mathrm{d}\, j \right] \implies \\
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \frac{\psi}{\psi-1} \left[\ln \left(Y_{\eta j}^{\frac{\psi-1}{\psi}} \int_0^1 \mathrm{d}\, j \right) + \frac{\psi-1}{\psi} \int_0^1 \hat{Y}_{\eta j t} \, \mathrm{d}\, j \right] \implies \\
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \cancel{\frac{\psi}{\psi-1}} \left[\cancel{\frac{\psi-1}{\psi}} \ln Y_{\eta j} + \cancel{\ln 1} + \cancel{\frac{\psi-1}{\psi}} \int_0^1 \hat{Y}_{\eta j t} \, \mathrm{d}\, j \right] \implies \\
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta j t} \, \mathrm{d}\, j
\end{aligned}
$$

Apply corollary **??**:

$$
\begin{aligned}
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta j t} \, \mathrm{d}\, j \implies \\
\hat{Y}_{\eta t} &= \int_0^1 \hat{Y}_{\eta j t} \, \mathrm{d}\, j && (0.122)
\end{aligned}
$$

Production Function

Log-linearize 0.30:

$$
\begin{aligned}
Y_{\eta j t} &= Z_{A \eta t} K_{\eta j t}^{\alpha} L_{\eta j t}^{1-\alpha} && \implies && (0.30) \\
Y_{\eta j}(1 + \hat{Y}_{\eta j t}) &= Z_{A \eta} K_{\eta j}^{\alpha} L_{\eta j}^{1-\alpha} (1 + \hat{Z}_{A \eta t} + \alpha \hat{K}_{\eta j t} + (1-\alpha) \hat{L}_{\eta j t}) && \implies \\
\hat{Y}_{\eta j t} &= \hat{Z}_{A \eta t} + \alpha \hat{K}_{\eta j t} + (1-\alpha) \hat{L}_{\eta j t} && && (0.123)
\end{aligned}
$$

Substitute 0.123 in 0.122:

$$\begin{aligned}
\hat{Y}_{\eta t} &= \int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j && \implies && (0.122)\\
\hat{Y}_{\eta t} &= \int_0^1 \left[\hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta j t} + (1-\alpha)\hat{L}_{\eta j t}\right]\mathrm{d}\,j && \implies &&\\
\hat{Y}_{\eta t} &= \hat{Z}_{A\eta t} + \alpha \int_0^1 \hat{K}_{\eta j t}\,\mathrm{d}\,j + (1-\alpha)\int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j && && (0.124)
\end{aligned}$$

Apply the natural logarithm and then log-linearize 0.55:

$$\begin{aligned}
L_{\eta t} &= \int_0^1 L_{\eta j t}\,\mathrm{d}\,j && \implies && (0.55)\\
\ln L_{\eta t} &= \ln\left[\int_0^1 L_{\eta j t}\,\mathrm{d}\,j\right] && \implies &&\\
\ln L + \hat{L}_{\eta t} &= \ln\left[\int_0^1 L_{\eta j}\,\mathrm{d}\,j\right] + \int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j && \implies &&\\
\ln L + \hat{L}_{\eta t} &= \ln L_{\eta j} + \ln 1 + \int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j && &&
\end{aligned}$$

Apply corollary **??**:

$$\implies \hat{L}_{\eta t} = \int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j \tag{0.125}$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_{\eta t} = \int_0^1 \hat{K}_{\eta j t}\,\mathrm{d}\,j \tag{0.126}$$

Substitute 0.125 and 0.126 in 0.124:

$$\begin{aligned}
\hat{Y}_{\eta t} &= \hat{Z}_{A\eta t} + \alpha \int_0^1 \hat{K}_{\eta j t}\,\mathrm{d}\,j + (1-\alpha)\int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j \implies && (0.124)\\
\hat{Y}_{\eta t} &= \hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta t} + (1-\alpha)\hat{L}_{\eta t} && (0.127)
\end{aligned}$$

Capital Demand

Log-linearize 0.32:

$$\begin{aligned} K_{\eta jt} &= \alpha Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_{Kt}} && \Longrightarrow \\ K_{\eta j}(1+\hat{K}_{\eta jt}) &= \alpha Y_{\eta j} \frac{\Lambda_{\eta}}{R_K}(1+\hat{Y}_{\eta jt}+\hat{\Lambda}_{\eta t}-\hat{R}_{Kt}) && \Longrightarrow \\ \hat{K}_{\eta jt} &= \hat{Y}_{\eta jt}+\hat{\Lambda}_{\eta t}-\hat{R}_{Kt} \end{aligned} \tag{0.32}$$

Integrate both sides and then substitute 0.126 and 0.122:

$$\begin{aligned} \int_0^1 \hat{K}_{\eta jt}\,\mathrm{d}\,j &= \int_0^1 (\hat{Y}_{\eta jt}+\hat{\Lambda}_{\eta t}-\hat{R}_{Kt})\,\mathrm{d}\,j \quad \Longrightarrow \\ \hat{K}_{\eta t} &= \hat{Y}_{\eta t}+\hat{\Lambda}_{\eta t}-\hat{R}_{Kt} \end{aligned} \tag{0.128}$$

Labor Demand

Log-linearize 0.33:

$$\begin{aligned} L_{\eta jt} &= (1-\alpha) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} && \Longrightarrow \\ L_{\eta j}(1+\hat{L}_{\eta jt}) &= (1-\alpha) Y_{\eta j} \frac{\Lambda_{\eta}}{W}(1+\hat{Y}_{\eta jt}+\hat{\Lambda}_{\eta t}-\hat{W}_t) && \Longrightarrow \\ \hat{L}_{\eta jt} &= \hat{Y}_{\eta jt}+\hat{\Lambda}_{\eta t}-\hat{W}_t \end{aligned} \tag{0.33}$$

Integrate both sides and then substitute 0.125 and 0.122:

$$\begin{aligned} \int_0^1 \hat{L}_{\eta jt}\,\mathrm{d}\,j &= \int_0^1 \hat{Y}_{\eta jt}+\hat{\Lambda}_{\eta t}-\hat{W}_t\,\mathrm{d}\,j \quad \Longrightarrow \\ \hat{L}_{\eta t} &= \hat{Y}_{\eta t}+\hat{\Lambda}_{\eta t}-\hat{W}_t \end{aligned} \tag{0.129}$$

Subtract 0.129 from 0.128:

$$\begin{aligned} &\hat{K}_{\eta t}-\hat{L}_{\eta t} = \hat{Y}_{\eta t}+\hat{\Lambda}_{\eta t}-\hat{R}_{Kt}-(\hat{Y}_{\eta t}+\hat{\Lambda}_{\eta t}-\hat{W}_t) \quad \Longrightarrow \\ &\hat{K}_{\eta t}-\hat{L}_{\eta t} = \hat{W}_t-\hat{R}_{Kt} \end{aligned} \tag{0.130}$$

Equation 0.130 is the log-linearized version of 0.34.

Marginal Cost

Log-linearize 0.39:

$$\Lambda_{\eta t} = Z_{A\eta t}^{-1} \frac{R_{Kt}^{\alpha} W_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \implies \tag{0.39}$$

$$\Lambda(1+\hat{\Lambda}_{\eta t}) =$$

$$= \frac{1}{Z_{A\eta}} \left(\frac{R_K}{\alpha}\right)^{\alpha} \left(\frac{W}{1-\alpha}\right)^{1-\alpha} (1 - \hat{Z}_{A\eta t} + \alpha \hat{R}_{Kt} + (1-\alpha)\hat{W}_t) \implies$$

$$\hat{\Lambda}_{\eta t} = \alpha \hat{R}_{Kt} + (1-\alpha)\hat{W}_t - \hat{Z}_{A\eta t} \tag{0.131}$$

Substitute 0.107 in 0.131:

$$\begin{aligned}
\hat{\Lambda}_{\eta t} &= \alpha \hat{R}_{Kt} + (1-\alpha)\hat{W}_t - \hat{Z}_{A\eta t} && \implies \\
\hat{P}_{\eta t} + \hat{\lambda}_{\eta t} &= \alpha \hat{R}_{Kt} + (1-\alpha)\hat{W}_t - \hat{Z}_{A\eta t} && \implies \\
\hat{\lambda}_{\eta t} &= \alpha \hat{R}_{Kt} + (1-\alpha)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_{\eta t}
\end{aligned} \tag{0.132}$$

Monetary Policy

Log-linearize 0.47:

$$\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi} Y_t^{\gamma_Y})^{(1-\gamma_R)} Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \implies \tag{0.47}$$

$$\frac{R(1+\hat{R}_t)}{R} = \frac{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)} Z_M}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \cdot$$

$$\cdot [1 + \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt}] \implies$$

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{0.133}$$

National Gross Inflation Rate

Log-linearize 0.48 and define the level deviation of gross inflation rate $\hat{\pi}_t$:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \tag{0.48}$$

$$\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{0.134}$$

Market Clearing Condition

Log-linearize 0.53:

$$\begin{aligned} Y_t &= Y_{1t} + Y_{2t} && (0.53)\\ Y(1+\hat{Y}_t) &= Y_1(1+\hat{Y}_{1t}) + Y_2(1+\hat{Y}_{2t}) \implies \\ \hat{Y}_t &= \frac{Y_1}{Y}\hat{Y}_{1t} + \frac{Y_2}{Y}\hat{Y}_{2t} && (0.135) \end{aligned}$$

Define the regional weights $\left\langle \theta_{Y1} \;\; (1-\theta_{Y1}) \right\rangle$ in the production total:

$$\left\langle \theta_{Y1} \quad (1-\theta_{Y1}) \right\rangle := \left\langle \tfrac{Y_1}{Y} \quad \tfrac{Y_2}{Y} \right\rangle \tag{0.136}$$

Substitute 0.136 in 0.135:

$$\hat{Y}_t = \theta_{Y1}\hat{Y}_{1t} + (1-\theta_{Y1})\hat{Y}_{2t} \tag{0.137}$$

National Price Level

Log-linearize 0.49:

$$\begin{aligned} P_tY_t &= P_{1t}Y_{1t} + P_{2t}Y_{2t} \implies && (0.49)\\ PY(1+\hat{P}_t+\hat{Y}_t) &= P_1Y_1(1+\hat{P}_{1t}+\hat{Y}_{1t}) + P_2Y_2(1+\hat{P}_{2t}+\hat{Y}_{2t}) \implies \\ \hat{P}_t+\hat{Y}_t &= \frac{Y_1}{Y}(\hat{P}_{1t}+\hat{Y}_{1t}) + \frac{Y_2}{Y}(\hat{P}_{2t}+\hat{Y}_{2t}) && (0.138) \end{aligned}$$

Substitute 0.136 in 0.138:

$$\hat{P}_t+\hat{Y}_t = \theta_{Y1}(\hat{P}_{1t}+\hat{Y}_{1t}) + (1-\theta_{Y1})(\hat{P}_{2t}+\hat{Y}_{2t}) \tag{0.139}$$

Subtract 0.137 from 0.139:

$$\begin{aligned} \hat{P}_t+\hat{Y}_t-\hat{Y}_t &= \theta_{Y1}(\hat{P}_{1t}+\hat{Y}_{1t}) + (1-\theta_{Y1})(\hat{P}_{2t}+\hat{Y}_{2t}) - \\ &\quad - (\theta_{Y1}\hat{Y}_{1t} + (1-\theta_{Y1})\hat{Y}_{2t}) \implies \\ \hat{P}_t &= \theta_{Y1}\hat{P}_{1t} + (1-\theta_{Y1})\hat{P}_{2t} && (0.140) \end{aligned}$$

Productivity Shock

Log-linearize 0.51:

$$\begin{aligned}
\ln Z_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}\ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} && \implies && (0.51)\\
\ln Z_{A\eta} + \hat{Z}_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}(\ln Z_{A\eta} + \hat{Z}_{A\eta,t-1}) + \varepsilon_{A\eta} && \implies && \\
\hat{Z}_{A\eta t} &= \rho_{A\eta}\hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} && && (0.141)
\end{aligned}$$

Monetary Shock

Log-linearize 0.52:

$$\begin{aligned}
\ln Z_{Mt} &= (1-\rho_{M})\ln Z_{M} + \rho_{M}\ln Z_{M,t-1} + \varepsilon_{Mt} && \implies && (0.52)\\
\ln Z_{M} + \hat{Z}_{Mt} &= (1-\rho_{M})\ln Z_{M} + \rho_{M}(\ln Z_{M} + \hat{Z}_{M,t-1}) + \varepsilon_{M} && \implies && \\
\hat{Z}_{Mt} &= \rho_{M}\hat{Z}_{M,t-1} + \varepsilon_{M} && && (0.142)
\end{aligned}$$

Regional Market Clearing Condition

Log-linearize 0.54:

$$\begin{aligned}
Y_{\eta t} &= C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \implies && (0.54)\\
Y_{\eta}(1+\hat{Y}_{\eta t}) &= C_{\eta 1}(1+\hat{C}_{\eta 1t}) + C_{\eta 2}(1+\hat{C}_{\eta 2t}) + I_{\eta}(1+\hat{I}_{\eta t}) \implies && \\
\hat{Y}_{\eta t} &= \frac{C_{\eta 1}}{Y_{\eta}}\hat{C}_{\eta 1t} + \frac{C_{\eta 2}}{Y_{\eta}}\hat{C}_{\eta 2t} + \frac{I_{\eta}}{Y_{\eta}}\hat{I}_{\eta t} && (0.143)
\end{aligned}$$

Define the consumption and investment weights $\langle\, \theta_{C\eta 1}\ \theta_{C\eta 2}\ (1-\theta_{C\eta 1}-\theta_{C\eta 2})\,\rangle$ in the regional production:

$$\left\langle \theta_{C\eta 1} \quad \theta_{C\eta 2} \quad (1-\theta_{C\eta 1}-\theta_{C\eta 2}) \right\rangle := \left\langle \tfrac{C_{\eta 1}}{Y_{\eta}} \quad \tfrac{C_{\eta 2}}{Y_{\eta}} \quad \tfrac{I_{\eta}}{Y_{\eta}} \right\rangle \tag{0.144}$$

Substitute 0.144 in 0.143:

$$\hat{Y}_{\eta t} = \theta_{C\eta 1}\hat{C}_{\eta 1t} + \theta_{C\eta 2}\hat{C}_{\eta 2t} + (1-\theta_{C\eta 1}-\theta_{C\eta 2})\hat{I}_{\eta t} \tag{0.145}$$

0.3.1 Log-linear Model Structure

The log-linear model is a square system of 31 variables and 31 equations, summarized as follows:

- Variables:
  Real Variables: $\langle\, \hat{C}_\eta\ \hat{C}_{\eta 1}\ \hat{C}_{\eta 2}\ \hat{L}_\eta\ \hat{K}_\eta\ \hat{Y}\ \hat{Y}_\eta\ \hat{I}_\eta\ \hat{Z}_{A\eta} \,\rangle$;
  Nominal Variables: $\langle\, \hat{\mathcal{E}}_\eta\ \hat{P}\ \hat{P}_\eta\ \hat{R}\ \hat{R}_K\ \hat{W}\ \hat{\pi}\ \hat{\pi}_\eta\ \hat{\lambda}_\eta\ \hat{Z}_M \,\rangle$.
- Equations:

1. Regional Gross Inflation Rate

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{0.102}$$

2. New Keynesian Phillips Curve

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{\eta t} \tag{0.112}$$

3. Law of Motion for Capital

$$\hat{K}_{\eta,t+1} = (1-\delta)\hat{K}_{\eta t} + \delta \hat{I}_{\eta t} \tag{0.113}$$

4. Regional Levels of Consumption and Prices

$$\hat{C}_{\eta 1t} - \hat{C}_{\eta 2t} = \hat{P}_{2t} - \hat{P}_{1t} \tag{0.116}$$

5. Total Expenses

$$\hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta t} + \omega_{\eta 1}\hat{P}_{1t} + (1-\omega_{\eta 1})\hat{P}_{2t} \tag{0.117}$$

6. Labor Supply

$$\varphi \hat{L}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} = \hat{W}_t - \hat{\mathcal{E}}_{\eta t} \tag{0.118}$$

7. Euler equation for the bonds return

$$\begin{aligned} &\mathbb{E}_t \left\{ \hat{\mathcal{E}}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1} \right\} - \\ &\quad - \left[\hat{\mathcal{E}}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} \right] = (1-\beta)\hat{R}_t \end{aligned} \tag{0.119}$$

8. Euler equation for the capital return

$$\begin{aligned} &\mathbb{E}_t \left\{ \hat{\mathcal{E}}_{\eta,t+1} - \hat{P}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1} \right\} - \left(\hat{\mathcal{E}}_{\eta t} - \hat{P}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} \right) = \\ &\quad = \beta r_K \mathbb{E}_t \left\{ \hat{R}_{K,t+1} - \hat{P}_{\eta,t+1} \right\} \end{aligned} \tag{0.120}$$

9. Production Function

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta t} + (1-\alpha)\hat{L}_{\eta t} \tag{0.127}$$

10. Marginal Rates of Substitution of Factors

$$\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{W}_t - \hat{R}_{Kt} \tag{0.130}$$

11. Marginal Cost

$$\hat{\lambda}_{\eta t} = \alpha \hat{R}_{Kt} + (1-\alpha)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_{\eta t} \tag{0.132}$$

12. Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{0.133}$$

13. National Gross Inflation Rate

$$\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{0.134}$$

14. National Price Level

$$\hat{P}_t = \theta_{Y1}\hat{P}_{1t} + (1-\theta_{Y1})\hat{P}_{2t} \tag{0.140}$$

15. Productivity Shock

$$\hat{Z}_{A\eta t} = \rho_{A\eta}\hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} \tag{0.141}$$

16. Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{0.142}$$

17. Market Clearing Condition

$$\hat{Y}_t = \theta_{Y1}\hat{Y}_{1t} + (1-\theta_{Y1})\hat{Y}_{2t} \tag{0.137}$$

18. Regional Market Clearing Condition

$$\hat{Y}_{\eta t} = \theta_{C\eta 1}\hat{C}_{\eta 1t} + \theta_{C\eta 2}\hat{C}_{\eta 2t} + (1-\theta_{C\eta 1}-\theta_{C\eta 2})\hat{I}_{\eta t} \tag{0.145}$$

0.3.2 Extended Log-linear Structure

1. Regional Gross Inflation Rate

$$\hat{\pi}_{1t} = \hat{P}_{1t} - \hat{P}_{1,t-1}$$
$$\hat{\pi}_{2t} = \hat{P}_{2t} - \hat{P}_{2,t-1} \tag{0.102}$$

2. New Keynesian Phillips Curve

$$\hat{\pi}_{1t} = \beta \mathbb{E}_t \hat{\pi}_{1,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{1t}$$
$$\hat{\pi}_{2t} = \beta \mathbb{E}_t \hat{\pi}_{2,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{2t} \tag{0.112}$$

3. Law of Motion for Capital

$$\hat{K}_{1,t+1} = (1-\delta)\hat{K}_{1t} + \delta \hat{I}_{1t}$$
$$\hat{K}_{2,t+1} = (1-\delta)\hat{K}_{2t} + \delta \hat{I}_{2t} \tag{0.113}$$

4. Regional Levels of Consumption and Prices

$$\hat{C}_{11t} - \hat{C}_{12t} = \hat{P}_{2t} - \hat{P}_{1t}$$
$$\hat{C}_{21t} - \hat{C}_{22t} = \hat{P}_{2t} - \hat{P}_{1t} \tag{0.116}$$

5. Total Expenses

$$\hat{\mathcal{E}}_{1t} = \hat{C}_{1t} + \omega_{11}\hat{P}_{1t} + (1-\omega_{11})\hat{P}_{2t}$$
$$\hat{\mathcal{E}}_{2t} = \hat{C}_{2t} + \omega_{21}\hat{P}_{1t} + (1-\omega_{21})\hat{P}_{2t} \tag{0.117}$$

6. Labor Supply

$$\varphi \hat{L}_{1t} - (1-\sigma)\hat{C}_{1t} = \hat{W}_t - \hat{\mathcal{E}}_{1t}$$
$$\varphi \hat{L}_{2t} - (1-\sigma)\hat{C}_{2t} = \hat{W}_t - \hat{\mathcal{E}}_{2t} \tag{0.118}$$

7. Euler equation for the bonds return

$$\mathbb{E}_t \left\{ \hat{\mathcal{E}}_{1,t+1} - (1-\sigma)\hat{C}_{1,t+1} \right\} - \left[\hat{\mathcal{E}}_{1t} - (1-\sigma)\hat{C}_{1t} \right] = (1-\beta)\hat{R}_t$$
$$\mathbb{E}_t \left\{ \hat{\mathcal{E}}_{2,t+1} - (1-\sigma)\hat{C}_{2,t+1} \right\} - \left[\hat{\mathcal{E}}_{2t} - (1-\sigma)\hat{C}_{2t} \right] = (1-\beta)\hat{R}_t \tag{0.119}$$

8. Euler equation for the capital return

$$
\begin{aligned}
&\mathbb{E}_t\left\{\hat{\mathcal{E}}_{1,t+1}-\hat{P}_{1,t+1}-(1-\sigma)\hat{C}_{1,t+1}\right\}- \\
&\qquad -\left(\hat{\mathcal{E}}_{1t}-\hat{P}_{1t}-(1-\sigma)\hat{C}_{1t}\right)=\beta\frac{R_K}{P_1}\mathbb{E}_t\left\{\hat{R}_{K,t+1}-\hat{P}_{1,t+1}\right\} \\
&\mathbb{E}_t\left\{\hat{\mathcal{E}}_{2,t+1}-\hat{P}_{2,t+1}-(1-\sigma)\hat{C}_{2,t+1}\right\}- \\
&\qquad -\left(\hat{\mathcal{E}}_{2t}-\hat{P}_{2t}-(1-\sigma)\hat{C}_{2t}\right)=\beta\frac{R_K}{P_2}\mathbb{E}_t\left\{\hat{R}_{K,t+1}-\hat{P}_{2,t+1}\right\}
\end{aligned} \tag{0.120}
$$

9. Production Function

$$
\begin{aligned}
\hat{Y}_{1t} &= \hat{Z}_{A1t}+\alpha\hat{K}_{1t}+(1-\alpha)\hat{L}_{1t} \\
\hat{Y}_{2t} &= \hat{Z}_{A2t}+\alpha\hat{K}_{2t}+(1-\alpha)\hat{L}_{2t}
\end{aligned} \tag{0.127}
$$

10. Marginal Rates of Substitution of Factors

$$
\begin{aligned}
\hat{K}_{1t}-\hat{L}_{1t} &= \hat{W}_t-\hat{R}_{Kt} \\
\hat{K}_{2t}-\hat{L}_{2t} &= \hat{W}_t-\hat{R}_{Kt}
\end{aligned} \tag{0.130}
$$

11. Marginal Cost

$$
\begin{aligned}
\hat{\lambda}_{1t} &= \alpha\hat{R}_{Kt}+(1-\alpha)\hat{W}_t-\hat{Z}_{A1t}-\hat{P}_{1t} \\
\hat{\lambda}_{2t} &= \alpha\hat{R}_{Kt}+(1-\alpha)\hat{W}_t-\hat{Z}_{A2t}-\hat{P}_{2t}
\end{aligned} \tag{0.132}
$$

12. Monetary Policy

$$
\hat{R}_t=\gamma_R\hat{R}_{t-1}+(1-\gamma_R)(\gamma_\pi\hat{\pi}_t+\gamma_Y\hat{Y}_t)+\hat{Z}_{Mt} \tag{0.133}
$$

13. National Gross Inflation Rate

$$
\hat{\pi}_t=\hat{P}_t-\hat{P}_{t-1} \tag{0.134}
$$

14. National Price Level

$$
\hat{P}_t+\hat{Y}_t=\theta_{PY1}(\hat{P}_{1t}+\hat{Y}_{1t})+(1-\theta_{PY1})(\hat{P}_{2t}+\hat{Y}_{2t}) \tag{0.140}
$$

15. Productivity Shock

$$
\begin{aligned}
\hat{Z}_{A1t} &= \rho_{A1}\hat{Z}_{A1,t-1}+\varepsilon_{A1} \\
\hat{Z}_{A2t} &= \rho_{A2}\hat{Z}_{A2,t-1}+\varepsilon_{A2}
\end{aligned} \tag{0.141}
$$

16. Monetary Shock

$$
\hat{Z}_{Mt}=\rho_M\hat{Z}_{M,t-1}+\varepsilon_M \tag{0.142}
$$

17. Market Clearing Condition

$$\hat{Y}_t = \theta_{Y1}\hat{Y}_{1t} + (1 - \theta_{Y1})\hat{Y}_{2t} \tag{0.137}$$

18. Regional Market Clearing Condition

$$\hat{Y}_{1t} = \theta_{C11}\hat{C}_{11t} + \theta_{C12}\hat{C}_{12t} + (1 - \theta_{C11} - \theta_{C12})\hat{I}_{1t}$$
$$\hat{Y}_{2t} = \theta_{C21}\hat{C}_{21t} + \theta_{C22}\hat{C}_{22t} + (1 - \theta_{C21} - \theta_{C22})\hat{I}_{2t} \tag{0.145}$$