ANALYSIS OF THE MONETARY POLICY IMPACT ON REGIONAL GROSS DOMESTIC PRODUCT: A REGIONAL DSGE MODEL

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March 9, 2025

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Resumo

Esta pesquisa tem como objetivo criar um modelo DSGE[1] regional para investigar os impactos da taxa de juros nominal sobre o produto interno bruto de uma região de um dado país. Além dos elementos tradicionais da teoria Novo-Keynesiana, como competição monopolística e fricções de preços, o modelo apresenta duas regiões que se comunicam através do consumo do bem final de cada região por ambas. Estas regiões se diferenciam pelo nível de produtividade e pela participação do capital na função de produção da firma produtora de bens intermediários. As funções impulso-resposta demonstram que regiões com estruturas econômicas diferentes reagem de forma diferente a um choque de política monetária. A região mais intensiva em capital é mais sensível ao choque de política monetária, como esperado.

Palavras-Chave: Modelagem Macroeconômica; Modelo DSGE; Teoria Novo-Keynesiana; Política Monetária; Economia Regional; Produto Interno Bruto Regional

[1] *Dynamic and Stochastic General Equilibrium* ou Equilíbrio Geral Dinâmico e Estocástico

Abstract

This research aims to create a regional DSGE[2] model to investigate the impact of the nominal interest rate on the gross domestic product of a region of a given country. In addition to the traditional elements of New Keynesian theory, such as monopolistic competition and price frictions, the model features two regions that communicate through the consumption of the final good from each region by both. These regions differ in productivity levels and the share of capital in the production function of the intermediate-goods firm. The impulse-response functions demonstrate that regions with different economic structures react differently to a monetary policy shock. The more capital-intensive region is more sensitive to the monetary policy shock, as expected.

Keywords: Macroeconomic Modeling; DSGE Model; New-Keynesian Theory; Monetary Policy; Regional Economics; Regional Gross Domestic Product

[2] Dynamic and Stochastic General Equilibrium

1 INTRODUCTION

The importance of macroeconomic modeling as a tool for studying the connections between monetary economy and the outcomes of a country's aggregates is undeniable, as stated by **gali_monetary_2015**. Considering as well that Brazilian regions possess heterogeneous economic matrices and sectors that respond in different ways to monetary authority decisions, as indicated by **bertanha_efeitos_2008**, the need for a structural model capable of relating macroeconomic variables to regional variables becomes evident.

In this context, the present research proposes the development of a macroeconomic model with regional extensions, using the DSGE methodology[3], which can demonstrate the existing relationships among the various considered variables and present impulse response functions that illustrate these relationships. With this model, we aim to investigate the existing relationship between the nominal interest rate of the Brazilian economy and the level of regional gross domestic product.

The main issue to be investigated is the impact of monetary authority decisions — especially changes in the nominal interest rate — on regional macroeconomic variables, particularly the Gross Domestic Product (GDP) of a given Brazilian region (in this context, a region can be anything from a Municipality, a State, an Economic Region or any other composition of the country).

Given that Brazilian regions have distinct economic matrices (agriculture, industry, extraction, etc.), and within each of these specializations, some sectors are more labor-intensive while others are more capital-intensive, it is plausible to assume that regional diversity allows each region to react differently to changes in the interest rate, as demonstrated by **haddad_matriz_2017** and **osterno_uma_2022**.

Regional Economics investigations often employ tools borrowed from Macroeconomics, as highlighted by **rickman_modern_2010**. Examples include the Leontief input-output model, the Walrasian general equilibrium applied model, and the system of macroeconometric equations. These instances demonstrate how models from one field can be adapted and utilized by the other.

In line with this notion, the objective of this work is to utilize a DSGE model (a commonly used tool in Macroeconomics) to establish relationships between macroeconomic variables and regional ones. Subsequently, Brazilian economic data will be employed to calibrate the model, enabling the derivation of impulse response functions that closely resemble the dynamics of the Brazilian economy. DSGE models have already been employed to address regional questions in excellent works such as **tamegawa_two-region_2012**, **tamegawa_constructing_2013**, **mora_fdi_2019**, **costa_junior_dsge_2022**, and **osterno_uma_2022**, to cite a few.

Numerous studies have addressed the effects of national aggregates on regional variables, as those cited above, and these will be appropriately presented in Section (2). Among them, **osterno_uma_2022** investigates the relations between monetary policy and regional variables using a DSGE model. However, the approach taken here differs from theirs: while **osterno_uma_2022** employs a top-down approach, augmenting the SAMBA model from **castro_samba_2015** to include regional variables, our approach is bottom-up. We create a regional DSGE model from scratch to illustrate the rela-

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[3] Dynamic and Stochastic General Equilibrium.

tionships between macroeconomic and regional variables. Consequently, it is reasonable to expect that the results may differ in intensity, as some variables and relationships present in their study are not replicated here. Nevertheless, the direction of the reactions of the existing variables may exhibit some similarity.

The significance of this work can be identified by recognizing that, given the diversity of Brazilian regions, it is not plausible that a single macroeconomic variable will have the same effect in each of them (or at least not with the same intensity). Thus, a tool capable of quantifying the regional effect of a macroeconomic variable is an important addition to economic literature, as it investigates the transmission mechanisms of monetary policy to the regional aggregates. Additionally, it also adds to the array of policy evaluation instruments, such that various economic agents can use this tool to determine the conduct of their own internal policies. For example, banks can quantify the credit interest rate for a specific region based on the projected interest rate of the economy, considering the needs and potential development of each region separately from the rest of the country.

The main objective is to create a DSGE model capable of relating a macroeconomic variable (the nominal interest rate) to a regional variable (the Gross Domestic Product of a Brazilian region), in order to assess the impact of an expansionary (or contractionary) monetary policy on a specific Brazilian region and the magnitude of that impact.

The specific objectives are (1) elaborate a New Keynesian DSGE model with households, firms, monetary authority, price stickiness, productivity and monetary shocks and two regions (the main region and the rest of the country) to verify if the nominal interest rate determined by the monetary authority influences the regional GDP; (2) produce the impulse response functions (IRF) and analyse the results of the regional model.

The other sections are organized as follows. Section (2) summarizes the related literature. Section (3) describes the proposed regional DSGE model. Section (4) presents the results and discussion. Finally, Section (5) provides a summary of what was learned and outlines the next challenges. Additionally, I have included an appendix where some details and results are clarified.

2 LITERATURE REVIEW

This section provides a literature review, exploring the intersection between Regional Economics and Macroeconomics, emphasizing the importance of monetary policy, and delving into the applicability of DSGE models to address diverse economic challenges, including regional and monetary dilemmas. The discussion also underscores the need for a clear definition and methodological framework in utilizing DSGE models.

The assessment by **rickman_modern_2010** on the importance of the link between Macroeconomics and Regional Economics was made at a time when the use of DSGE models to investigate regional issues was not yet common. Since then, several studies have addressed this connection.

Initially, we present two works that served as inspiration for the present research. The first, developed by **costa_junior_dsge_2022**, investigates the impacts of fiscal policy on the state of Goiás, considering the other states of the nation. In this work, the authors develop a regionalized and open structure, individualizing a Brazilian state from the rest, considering both a national and a state fiscal authority; state expenses and revenues are disaggregated, and thus, the authors seek to identify whether there are differences between the impacts of a tax exemption in the state under study compared to the others. With the model calibrated to data from 2003 to 2019, the authors demonstrate that there is indeed a difference in state performance due to the distinction of the tax exemption occurring in the state or in the rest of the country.

The second work also presents a DSGE model, but with the objective of evaluating whether there are differences in the effects of Foreign Direct Investment (FDI), considering its location. The model developed by **mora_fdi_2019** encompasses an open economy with the main region (Bogotá, 25% of the national GDP) and the rest of the country (Colombia), two types of households[4], habit formation, capital adjustment costs, as well as typical elements of a New Keynesian (NK) model[5]. With the model calibrated to data from 2002 to 2015, the authors demonstrate that there is indeed a difference in the effects of FDI depending on the region where it is applied, such that when applied in the rest of the country, there are growth effects that spread throughout the country through spillovers, including to the main region.

Both works aim to, despite dealing with distinct causes (fiscal policy and FDI), verify whether differences exist when the cause occurs in one of the two different modeled regions. Additionally, they share the same modeling approach, that of a Dynamic and Stochastic General Equilibrium (DSGE). And this was the advancement that **rickman_modern_2010** wanted to see happen: the use of DSGE models to address regional questions.

DSGE models are widely employed within the macroeconomic literature to examine the effects of monetary policy on macroeconomic aggregates, as pointed by **gali_monetary_2015**. In this context, it is important to add to the review the papers that develop models describing the monetary policy.

smets_estimated_2003 and **smets_shocks_2007** present models that evaluate various types of

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[4] Ricardian and non-Ricardian agents.

[5] nominal price rigidity, monopolistic competition, non-neutrality of monetary policy in the short term.

shocks in the Eurozone and the United States, respectively. **walque_financial_2010** assess the role of the banking sector in market liquidity recovery, considering the endogenous possibilities of agent default.

vinhado_politica_2016 employ a model with financial frictions to examine the transmission of monetary policy to the banking sector and economic activity. The results demonstrate that the banking sector plays a significant role in economic activity and impacts the outcomes of monetary policy by having to adjust the bank spread in response to changes in the interest rate or reserve requirements.

soltani_investigating_2021 investigate financial and monetary shocks on macroeconomic variables, with special attention to the role of banks. For this analysis, the model considers an economy with a banking sector. The results indicate that banking activity can influence the effects of economic policies.

holm_transmission_2021 study the transmission of monetary policy to household consumption, estimating the response of consumption, income, and savings. They utilize a heterogeneous agent New Keynesian model (HANK). The results demonstrate that a restrictive monetary policy prompts households with lower liquidity to reduce consumption as disposable income starts to decline, while households with average liquidity save less or borrow more. The study also highlights the differences in consumption changes between savers and borrowers in the face of a monetary policy alteration.

capeleti_countercyclical_2022 evaluate the effects of pro-cyclical and counter-cyclical credit expansions by public banks on economic growth. The model implements a banking sector with public and private banks competing in a Cournot oligopoly. The results show that the supply of public credit has a stronger effect when the policy is counter-cyclical.

The literature on DSGE modeling is extensive, as this methodology allows the formulation of various questions and their answers through a general equilibrium model. This includes the aforementioned topics and, also, labor market, as explored by **ribeiro_alongamento_2023**; the real estate market, as studied by **albuquerquemello_mercado_2018**; and even deforestation, as investigated by **pereira_desmatamento_2013**. As remarked by **solis-garcia_ucb_2022**: *if you have a cohesive economic idea, you can put it in terms of a DSGE model*.

The works of **costa_junior_understanding_2016**, **solis-garcia_ucb_2022**, **bergholt_basic_2012**, and **gali_monetary_2015**, between others, are essential materials for macroeconomic modeling theory, as they guide the reader in developing a DSGE model step-by-step. **costa_junior_understanding_2016** starts from a Real Business Cycles (RBC) model and chapter by chapter adds elements of New Keynesian (NK) theory to the model. **solis-garcia_ucb_2022** focuses on the mathematical details necessary to develop a DSGE model, beginning with a RBC model and turning it into a canonical NK model. **bergholt_basic_2012** discusses the key elements of a New Keynesian model and also demonstrates the necessary programming to run the model using the Dynare software, developed by **adjemian_dynare_2023**. **gali_monetary_2015** shows the evolution from an RBC model to an NK model, adding complexity with each chapter.

Among the works employing DSGE modeling with regions, beside the already mentioned before, there is the study by **tamegawa_two-region_2012**, which assesses the effects of fiscal policy on two regions using a model featuring two types of households, firms, banks, a national government, and a regional government. Using literature parameters to calibrate the model, the results indicate that indeed there are differences in the effects of fiscal policy depending on which region implements it. It is important to note that the difference between a macroeconomic model and a regional one lies in the fact that in the former, aggregate variables are considered only at the national level, whereas in the latter, both national and regional variables are considered, and depending on the size of the region, the latter might not be able to affect the former, as explained by **tamegawa_constructing_2013**. A framework to assess the economic evolution of a region in Japan is constructed by **okano_development_2015**, with the aim of identifying the causes of stagnation in the Kansai region.

In a similar vein of demonstrating regional relationships, **pytlarczyk_estimated_2005** investigates aspects of the European Monetary Union (EMU), focusing on the German economy, using a structural model with two regions; **gali_optimal_2005** also evaluates the functioning of the EMU, but with a model where regions form a unitary continuum, such that one region cannot affect the entire economy. **alpanda_international_2014** utilize a two-region model to assess the effects of US financial shocks on the euro area economy.

The article by **croitorov_financial_2020** seeks to identify spillovers between regions, building a model with three regions: the Euro area, the US, and the rest of the world. Similarly investigating spillovers, **corbo_maja_2020** present a regional model encompassing Sweden and the rest of the world.[6]

More recently, a landmark was established by **osterno_uma_2022** in the field of regional models for the Brazilian economy: their endeavor adapted the aggregated Brazilian DSGE model developed by **castro_samba_2015** to include regional disaggregation, enabling the observation of local variable reactions in response to fiscal and monetary shocks. Using a top-down approach, the disaggregation of the main variables allows for the incorporation of regional data. The respective impulse response functions demonstrate that different regions exhibit different reactions to fiscal and monetary shocks.

The papers mentioned here demonstrate the importance of the relationship between Regional Economics and Macroeconomics, and how both areas can benefit from the usage of DSGE models to illustrate the existing relations among regional and national variables. With these concepts in mind, the next section presents the development of a DSGE model using a bottom-up approach: the model is created with built-in regions, ensuring that regional relationships are integral to its development.

[6] Spillovers: effects that are transmitted from one region to another due to an exogenous factor, such as being neighboring regions.

3 METHODOLOGY

The DSGE methodology, as the name implies, involves the utilization of a Dynamic and Stochastic General Equilibrium model. This model outlines the problem to be addressed, requiring the definition of agents, variables, and parameters. In this research, the objective is to assess the impact of monetary policy on regional gross domestic product using the Canonical New Keynesian structure, as proposed by **solis-garcia_ucb_2022**. The structure comprises four representative agents: a household, a retail firm, a continuum of wholesale firms, and a monetary authority. It incorporates key elements of the New Keynesian theory, including monopolistic competition among wholesale firms, the price stickiness they encounter, and the consequential role of monetary policy in the short run.

3.1 REGIONAL NEW KEYNESIAN MODEL

The model is populated by four agents: (1) a representative household, (2) a continuum of firms producing intermediate-goods, (3) a firm producing final-goods, and (4) the monetary authority.

The representative household maximizes utility based on consumption and labor, subject to a budget constraint composed of wages, capital rental rates, and firm profits.

The final-goods firm produces the final-good consumed by households: it aggregates all intermediate-goods produced by intermediate firms, operates under perfect competition and seeks to maximize profit subject to the bundle technology.

Each intermediate-goods firm produces a single intermediate-good, all exhibiting imperfect substitution, thus operating in monopolistic competition. Intermediate-goods firms have two problems to solve: minimize costs subject to the production technology available and choose an optimal price to maximize the intertemporal profit flow.

Periodically, a portion of intermediate-goods firms have the opportunity to adjust prices, while others miss this chance, following a **calvo_staggered_1983** rule. This mechanism generates nominal price rigidities, altering equilibrium relationships in the system. These rigidities lead to the non-neutrality of money in the short term, as explained by **costa_junior_understanding_2016**.

The monetary authority determines the nominal interest rate in response to fluctuations in previous period's inflation and production, aiming to control price levels and growth, following a **taylor_discretion_1993** rule.

Stochastic shocks will be present in the intermediate-goods firms' productivity and in the monetary policy.

These elements define a canonical NK DSGE model, as presented by (**solis-garcia_ucb_2022**). The model will be adapted to accommodate two distinct regions: the main region and the rest of the country, replacing the single aggregated region. To achieve this, an index will differentiate the studied region from the rest of the country, resulting in separate households, intermediate- and final-goods firms for each region. Households lack mobility between regions. The link connecting the two regions is established through the final-goods, allowing households to consume from both regions.

Then, equilibrium conditions of the system will be determined. Assuming the system tends toward long-term equilibrium, a steady state will be reached where variables cease to change. Thus, for a given $t \to \infty$, there is a $\boldsymbol{X}_t = \boldsymbol{X}_{t+1} = \boldsymbol{X}_{ss} \implies \dot{\boldsymbol{X}} = 0$, where $\boldsymbol{X}$ denotes the vector of system variables, the subscript ss indicates the steady state and $\dot{\boldsymbol{X}} = {}^{\partial X}/_{\partial t}$.

After that, the log-linearization method proposed by **uhlig_toolkit_1999** will be employed to convert the system of equations into a linear system, so that this linear system can be solved by the program Dynare, which computes the solution and produces impulse-response graphs based on the stochastic shocks.

3.1.1 Regions

Regions will be indexed by $\eta \in \{1, 2, \dots, n\}$, representing the variables of each region. Whenever necessary, a second region index, $\nu \in \{1, 2, \dots, n\}$, will be used. For example, the variable C_t represents the total consumption (the aggregate of all regions), $C_{\eta t}$ represents the consumption composition of region η, and $C_{\eta \nu t}$ represents the consumption of the final good produced in region ν and consumed in region η (with the first index indicating the destination and the second one indicating the origin of the goods). Without loss of generality, the model will consider two regions: the main region labeled as 1 and the rest of the country as 2, so that $\eta, \nu \in \{1, 2\}$.

Figure (1) illustrates the model's mechanics. In this diagram, black arrows depict the real economy, while green arrows represent the nominal economy. The representative household supplies labor and capital to intermediate-goods firms in exchange for wages and capital rent, respectively. Using these resources, intermediate-goods firms produce goods, which are then sold to the final-goods firm. The final-goods firm aggregates all intermediate-goods into a final product, sold back to the household. Operating under a monetary rule, the monetary authority determines the nominal interest rate to achieve output growth and price stability.

3.1.2 Household

The household problem is divided into two steps: first, the household must minimize the consumption costs, and then maximize the utility, which is subject to a budget constraint.

To solve the cost minimization problem, consider that the representative household must decide on consuming goods from both regions. For this purpose, there must be a consumption bundle index $C_{\eta t}$ and a consumption price index $Q_{\eta t}$ that minimize the total consumption cost $Q_{\eta t} C_{\eta t}$, as demonstrated by **walsh_monetary_2017**:

$$\min_{C_{\eta 1t}, C_{\eta 2t}} : \quad Q_{\eta t} C_{\eta t} = P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} \tag{3.1}$$

$$\text{s. t.} : \quad C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \tag{3.2}$$

$$C_{\eta t} > 0$$

where P_{1t} and P_{2t} are the prices of goods 1 and 2, respectively, $C_{\eta 1t}$ and $C_{\eta 2t}$ are the goods produced in region 1 and 2, respectively, and consumed in region η. In the consumption aggregation, $\omega_{\eta 1}$ and $(1-\omega_{\eta 1})$ are the weights of goods $C_{\eta 1t}$ and $C_{\eta 2t}$, respectively, in the consumption bundle $C_{\eta t}$.

The solutions for the household cost minimization problem are:

$$C_{\eta 2t} = C_{\eta 1t}\frac{(1-\omega_{\eta 1})P_{1t}}{\omega_{\eta 1}P_{2t}} \tag{3.3}$$

$$C_{\eta 1t} = C_{\eta t}\left(\frac{P_{2t}\omega_{\eta 1}}{P_{1t}(1-\omega_{\eta 1})}\right)^{1-\omega_{\eta 1}} \tag{3.4}$$

$$Q_{\eta t} = \left(\frac{P_{1t}}{\omega_{\eta 1}}\right)^{\omega_{\eta 1}}\left(\frac{P_{2t}}{1-\omega_{\eta 1}}\right)^{1-\omega_{\eta 1}} \tag{3.5}$$

Therefore, there is a consumption bundle $C_{\eta t}$ and a consumption price index $Q_{\eta t}$ that minimize the total consumption cost $Q_{\eta t}C_{\eta t}$ for the household in region η. Notice that the cost problems of both regions are (must be) related, as the consumption level in one region influences the demand for goods in both regions. Now, this result will be used in the next problem that the household faces.

Following the models presented by **costa_junior_understanding_2016** and **solis-garcia_ucb_2022**, the representative household next problem is to maximize an intertemporal utility function U_η with respect to consumption $C_{\eta t}$ and labor $L_{\eta t}$, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_{\eta t}, L_{\eta t}, K_{\eta, t+1}} : \quad U_\eta(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi\frac{L_{\eta t}^{1+\varphi}}{1+\varphi}\right) \tag{3.6}$$

$$\text{s. t.}: \quad Q_{\eta t}C_{\eta t} + P_{\eta t}I_{\eta t} = W_{\eta t}L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{3.7}$$

$$K_{\eta, t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{3.8}$$

$$C_{\eta t}, L_{\eta t}, K_{\eta t} > 0$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, $I_{\eta t}$ is the investment, $W_{\eta t}$ is the wage level, $K_{\eta t}$ is the capital, R_t is the return on capital, and $\Pi_{\eta t}$ is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate.

The solutions for the household utility maximization problem are:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_{\eta t}}{Q_{\eta t}} \tag{3.9}$$

Equation 3.9 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

$$\frac{\mathbb{E}_t\{Q_{\eta, t+1}C_{\eta, t+1}^{\sigma}\}}{Q_{\eta t}C_{\eta t}^{\sigma}} = \beta\frac{\mathbb{E}_t\{P_{\eta, t+1}(1-\delta) + R_{t+1}\}}{P_{\eta t}} \tag{3.10}$$

Equation 3.10 is the Euler equation for the return on capital.

3.1.3 Firms

Consider two types of firms: (1) a continuum of intermediate-goods firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-goods firm, which aggregates all these varieties into a final bundle and operates in perfect competition.

3.1.4 Final-Goods Firm

The role of the final-goods firm is to aggregate all the varieties $Y_{\eta jt}$ produced by the intermediate-goods firms in each region $\eta \in \{1, 2\}$, so that the representative consumer can buy only one good $Y_{\eta t}$, the bundle good, from each region.

The final-goods firm problem is to maximize its profit, considering that its output is the bundle $Y_{\eta t}$ formed by a continuum $j \in [0, 1]$ of intermediate-goods $Y_{\eta jt}$, with elasticity of substitution between intermediate-goods ψ:

$$\max_{Y_{\eta jt}} : \quad P_{\eta t} Y_{\eta t} - \int_0^1 P_{\eta jt} Y_{\eta jt} \, \mathrm{d}\, j \tag{3.11}$$

$$\text{s. t.} : \quad Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi - 1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi - 1}} \tag{3.12}$$

The solutions for the final-goods firm problem are:

$$Y_{\eta jt} = Y_t \left(\frac{P_{\eta t}}{P_{\eta jt}} \right)^{\psi} \tag{3.13}$$

Equation 3.13 shows that the demand for variety j depends on its relative price.

$$P_{\eta t} = \left[\int_0^1 P_{\eta jt}^{1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \tag{3.14}$$

Equation 3.14 is the final-goods firm's markup.

3.1.5 Intermediate-Goods Firms

The intermediate-goods firms, denoted by $j \in [0, 1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose labor $L_{\eta jt}$ to minimize production costs, subject to a technology rule.

$$\min_{K_{\eta jt}, L_{\eta jt}} : \quad R_t K_{\eta jt} + W_t L_{\eta jt} \tag{3.15}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \tag{3.16}$$

where $Y_{\eta jt}$ is the output obtained by the production technology level $Z_{A\eta t}$ that transforms capital $K_{\eta jt}$ and labor $L_{\eta jt}$ in proportions α_η and $(1-\alpha_\eta)$, respectively, into intermediate goods.[7]

The solutions for the intermediate-goods firm problem are:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right) \frac{W_{\eta t}}{R_t} \tag{3.17}$$

Equation 3.17 demonstrates the relationship between the technical marginal rate of substitution (TMRS) and the economic marginal rate of substitution (EMRS).

$$K_{\eta jt} = \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right) \frac{W_{\eta t}}{R_t}\right]^{1-\alpha_\eta} \tag{3.18}$$

Equation 3.18 is the intermediate-goods firm demand for capital.

$$L_{\eta jt} = \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right) \frac{W_{\eta t}}{R_t}\right]^{-\alpha_\eta} \tag{3.19}$$

Equation 3.19 is the intermediate-goods firm demand for labor.

$$TC_{\eta jt} = \frac{Y_{\eta jt}}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_{\eta t}}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{3.20}$$

Equation 3.20 is the intermediate-goods firm total cost.

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_{\eta t}}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{3.21}$$

Equation 3.21 is the intermediate-goods firm marginal cost.

[7] the production technology level $Z_{A\eta t}$ will be submitted to a productivity shock, detailed in section A.1.5.

Consider an economy with price stickiness, following a Calvo Rule (**calvo_staggered_1983**): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{\eta j,t+1} = P_{\eta jt})$, and a probability $(1-\theta)$ of setting a new optimal price $P^*_{\eta jt}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{3.22}$$

$$\text{s. t.}: \quad Y_{\eta jt} = Y_{\eta t} \left(\frac{P_{\eta t}}{P_{\eta jt}} \right)^{\psi} \tag{3.13}$$

where s is the period in time when the decision must be made; t is the last period in time when the price was updated and k is the period in the future when the interest rate applies.

The solution to the optimal price problem is equation 3.23, which represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price and for that the index j can be omitted.

$$P^*_{\eta t} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}} \tag{3.23}$$

For the sake of closure, the intermediate-goods firm profit must be defined:

$$\Pi_{\eta t} = P_{\eta t} \int_0^1 Y_{\eta jt} \, \mathrm{d}\, j - W_{\eta t} L_{\eta t} \tag{3.24}$$

And the household constraint in terms of the intermediate-goods firm profit is:

$$Q_{\eta t} C_{\eta t} + P_{\eta t} I_{\eta t} = R_t K_{\eta t} + P_{\eta t} \int_0^1 Y_{\eta jt} \, \mathrm{d}\, j \tag{3.25}$$

3.1.6 Final-Goods Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1-\theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 3.14 to determine the aggregate price level:

$$P_{\eta t} = \left[\theta P_{\eta,t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \tag{3.26}$$

Equation 3.26 is the aggregate price level.

3.1.7 Regional Inflation

In each region, the price level $P_{\eta t}$ generates a regional inflation rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{3.27}$$

3.1.8 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (**taylor_discretion_1993**) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \tag{3.28}$$

where R, π, Y are the nominal interest rate, gross inflation rate and the production level in steady state, respectively; γ_R is the smoothing parameter for the interest rate R_t, γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively, Z_{Mt} is the monetary shock and π_t is the gross inflation rate, defined by:[8]

$$\pi_t = \pi_{1t}^{\theta_\pi} \pi_{2t}^{1-\theta_\pi} \tag{3.29}$$

$$\text{where: } \theta_\pi = \frac{P_{1t}Y_{1t}}{P_{1t}Y_{1t} + P_{2t}Y_{2t}} \tag{3.30}$$

3.1.9 Stochastic Shocks

The production technology level $Z_{A\eta t}$ will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{3.31}$$

where $\rho_{A\eta} \in [0, 1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{3.32}$$

where $\rho_M \in [0, 1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

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[8] for the monetary shock definition, see section A.1.5.

3.1.10 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P^*_{\eta t}, R^*_t, W^*_{\eta t}\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C^*_{\eta 1t}, C^*_{\eta 2t}, L^*_{\eta t}, I^*_{\eta t}, K^*_{\eta,t+1}\}$ and allocations for firms $\boldsymbol{\mathcal{A}}_F := \{K^*_{\eta jt}, L^*_{\eta jt}, Y^*_{\eta jt}, Y^*_{\eta t}\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms' problems, and the markets for goods and labor clear:

$$Y_t = Y_{1t} + Y_{2t} \tag{3.33}$$

$$\text{where:} \quad Y_{\eta t} = C_{\eta t} + I_{\eta t} \tag{3.34}$$

$$L_{\eta t} = \int_0^1 L_{\eta jt} \, \mathrm{d}\, j \tag{3.35}$$

3.2 STEADY STATE

The steady state of a variable is defined by its constancy over time. For any given variable X_t, it is in a steady state if $t \to \infty \implies \mathbb{E}tXt + 1 = X_t = X_{t-1} = X_{ss}$ (**costa_junior_understanding_2016**). For conciseness, the ss index representing the steady state will be omitted, so X is defined as X_{ss}.[9]

3.2.1 Steady State Solution

For the steady-state solution, all endogenous variables are determined with respect to the parameters. It is assumed that the price level P and the productivity level Z_A of region 1 are equal to one. For region 2, it is assumed that these levels are in proportion to the corresponding values of region 1 by factors $\langle \theta_P \quad \theta_Z \rangle$. The solution of the model in steady state is:[10]

$$\vec{\mathbf{1}} = \langle P_1 \quad Z_{A1} \rangle \tag{3.36}$$

$$\langle P_2 \quad Z_{A2} \rangle = \langle P_1 \quad \theta_Z Z_{A1} \rangle \tag{3.37}$$

$$\vec{\mathbf{1}} = \langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle \tag{3.38}$$

$$\vec{\mathbf{0}} = \langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle \tag{3.39}$$

$$R = P_1 \left[\frac{1}{\beta} - (1 - \delta) \right] \tag{3.40}$$

$$P^*_\eta = P_\eta \tag{3.41}$$

$$Q_\eta = \frac{P_1}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{3.42}$$

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9 The structural model in the steady state can be consulted in Section (A.1.8).

10 where $\vec{\mathbf{1}}$ is the unit vector.

$$\Lambda_\eta = P_\eta \frac{\psi - 1}{\psi} \tag{3.43}$$

$$W_\eta = (1 - \alpha_\eta)\left[\Lambda_\eta Z_{A\eta}\left(\frac{\alpha_\eta}{R}\right)^{\alpha_\eta}\right]^{\frac{1}{1-\alpha_\eta}} \tag{3.44}$$

$$a_\eta = \left[\frac{W_\eta}{\phi Q_\eta}\left[Z_{A\eta}\left(\frac{\alpha_\eta W_\eta}{(1-\alpha_\eta)R}\right)^{\alpha_\eta}\right]^{\varphi}\right]^{\frac{1}{\sigma}} \tag{3.45}$$

$$b_\eta = \frac{\delta}{Z_{A\eta}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}\right]^{1-\alpha_\eta} \tag{3.46}$$

$$Y_\eta = \left(\frac{a_\eta}{1-b_\eta}\right)^{\frac{\sigma}{\sigma+\varphi}} \tag{3.47}$$

$$Y = Y_1 + Y_2 \tag{3.48}$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{3.49}$$

$$I_\eta = b_\eta Y_\eta \tag{3.50}$$

$$K_\eta = \frac{I_\eta}{\delta} \tag{3.51}$$

$$C_{\eta 1} = C_\eta\left(\frac{P_2\omega_{\eta 1}}{P_1(1-\omega_{\eta 1})}\right)^{1-\omega_{\eta 1}} \tag{3.52}$$

$$C_{\eta 2} = C_{\eta 1}\frac{(1-\omega_{\eta 1})P_1}{\omega_{\eta 1}P_2} \tag{3.53}$$

$$L_\eta = \frac{Y_\eta}{Z_{A\eta}}\left[\left(\frac{1-\alpha_\eta}{\alpha_\eta}\right)\frac{R}{W_\eta}\right]^{\alpha_\eta} \tag{3.54}$$

3.3 LOG-LINEARIZATION

Due to the number of variables and equations to be solved, computational brute force will be necessary. `Dynare` is a specialized software for macroeconomic modeling, commonly used for solving DSGE models. Before the model can be processed by the software, it must undergo linearization to eliminate the infinite sum in Equation 3.23. For this purpose, Uhlig's rules of log-linearization (**uhlig_toolkit_1999**) will be applied to all equations in the model. For any given variable X_t, its deviation will be represented with a hat, $\hat{X}_t$.[11] The log-linear model is a square system of 30 variables and equations, summarized as follows:

– Real Variables: $\langle \hat{C}_\eta \quad \hat{L}_\eta \quad \hat{K}_\eta \quad \hat{I}_\eta \quad \hat{C}_{\eta 1} \quad \hat{C}_{\eta 2} \quad \hat{Y}_\eta \quad \hat{Y} \quad \hat{Z}_{A\eta} \quad \hat{Z}_M\rangle$;

– Nominal Variables: $\langle \hat{Q}_\eta \quad \hat{P}_\eta \quad \hat{R} \quad \hat{\pi} \quad \hat{W}_\eta \quad \hat{\lambda}_\eta \quad \hat{\pi}_\eta\rangle$.

– Regional Gross Inflation Rate

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{3.55}$$

[11] see Lemma A.3 for details.

– New Keynesian Phillips Curve

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{\eta t} \tag{3.56}$$

– Regional Consumption Weight

$$\hat{C}_{\eta 2t} - \hat{C}_{\eta 1t} = \hat{P}_{1t} - \hat{P}_{2t} \tag{3.57}$$

– Regional Consumption of Good 1

$$\hat{C}_{\eta t} - \hat{C}_{\eta 1t} = (1-\omega_{\eta 1})(\hat{P}_{1t} - \hat{P}_{2t}) \tag{3.58}$$

– Regional Price Index

$$\hat{Q}_{\eta t} = \omega_{\eta 1} \hat{P}_{1t} + (1-\omega_{\eta 1}) \hat{P}_{2t} \tag{3.59}$$

– Labor Supply

$$\varphi \hat{L}_{\eta t} + \sigma \hat{C}_{\eta t} = \hat{W}_{\eta t} - \hat{Q}_{\eta t} \tag{3.60}$$

– Law of Motion for Capital

$$\hat{K}_{\eta,t+1} = (1-\delta)\hat{K}_{\eta t} + \delta \hat{I}_{\eta t} \tag{3.61}$$

– Euler equation for capital return

$$\begin{aligned}(\hat{Q}_{\eta,t+1} - \hat{Q}_{\eta t}) + \sigma(\hat{C}_{\eta,t+1} - \hat{C}_{\eta t}) - (\hat{P}_{\eta,t+1} - \hat{P}_{\eta,t}) = \\ = \beta r (\hat{R}_{\eta,t+1} - \hat{P}_{\eta,t+1})\end{aligned} \tag{3.62}$$

– Production Function

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{\eta t} + (1-\alpha_\eta)\hat{L}_{\eta t} \tag{3.63}$$

– Technical and Economic Marginal Rates of Substitution

$$\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{W}_{\eta t} - \hat{R}_t \tag{3.64}$$

– Marginal Cost

$$\hat{\lambda}_{\eta t} = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_{\eta t} - \hat{Z}_{A\eta t} - \hat{P}_{\eta t} \tag{3.65}$$

– Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{3.66}$$

– National Gross Inflation Rate

$$\hat{\pi}_t = \theta_\pi \hat{\pi}_{1t} + (1-\theta_\pi)\hat{\pi}_{2t} \tag{3.67}$$

– Productivity Shock

$$\hat{Z}_{A\eta t} = \rho_{A\eta} \hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} \tag{3.68}$$

– Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{3.69}$$

– Goods-Market Clearing Condition

$$\hat{Y}_t = \theta_Y \hat{Y}_{1t} + (1 - \theta_Y)\hat{Y}_{2t} \tag{3.70}$$

– Regional Goods-Market Clearing Condition

$$\hat{Y}_{\eta t} = \theta_{C\eta} \hat{C}_{\eta t} + (1 - \theta_{C\eta})\hat{I}_{\eta t} \tag{3.71}$$

3.3.1 Eigenvalues and Forward Looking Variables

As it stands, the model has more forward-looking variables than eigenvalues greater than one, indicating that the model is indeterminate. To transform the model into one with a single solution, the number of eigenvalues and forward-looking variables must be equal. To address this, **farmer_solving_2015** employs a method where excess forward-looking variables are substituted with an expectational variable at time t, along with a expectational shock $sunspot_\eta$, representing the deviation between the expected and the realized values. For the present model, the variables created are the expected regional gross inflation rates $\pi^X_{\eta t}$ and the expected capital deviation $K^X_{\eta t}$:

$$\pi^X_{\eta t} = \mathbb{E}_t \hat{\pi}_{\eta,t+1} \tag{3.72}$$

$$sunspot_\eta = \hat{\pi}_{\eta t} - \pi^X_{\eta,t-1} \tag{3.73}$$

$$K^X_{\eta t} = \hat{K}_{\eta,t+1} \tag{3.74}$$

$$sunspot_{K\eta} = \hat{K}_{\eta t} - K^X_{\eta,t-1} \tag{3.75}$$

3.4 CALIBRATION

In this section, the calibration process to replicate Brazilian economic characteristics is presented, utilizing both literature and datasets. First, a region must be selected: in this study, region 1 corresponds to the State of São Paulo, which contributes 30% to the national GDP, as reported by **ibge_GDP_2023**. While other regions or groupings are viable options, such as groups of States or Cities, the limitation is that the smallest region cannot be less than 0.03 of the total, ensuring model stability, as emphasized by **konopkova_pitfalls_2019**. Second, a year must be chosen: to avoid the pandemic shock that occurred in 2020, the year 2019 will be set as the baseline for establishing the parameters.

To designate Region 1 as more capital-intensive, the capital elasticity of production for Region 1 must be such that $\alpha_1 > \alpha_2$, while Region 2 is assigned the commonly accepted value of 0.30. Various parameters, such as the intertemporal discount factor β; interest-rate smoothing parameter γ_R; interest-rate sensitivity in relation to inflation γ_π; interest-rate sensitivity in relation to product γ_Y; capital depreciation rate δ; price stickness parameter θ; autoregressive parameter of productivity in region 1 ρ_{A1}; autoregressive parameter of productivity in region 2 ρ_{A2}; autoregressive parameter of monetary policy ρ_M; relative risk aversion coefficient σ; relative labor weight in utility ϕ; marginal disutility of labor supply φ; elasticity of substitution between intermediate goods ψ; standard deviation of productivity shock $\sigma_{A\eta}$; standard deviation of monetary shock σ_M , are drawn from existing literature, as documented in **costa_junior_understanding_2016** and **pereira_rbc_2021**.

The weight of region 1 in total production, θ_Y, is determined by the São Paulo GDP to Brazil GDP ratio, as presented in Table (1).

The productivity ratio θ_Z quantifies the relative productivity of Region 2 compared to Region 1, using GDP per total hours worked as a measure, as discussed by **krugman_defining_1997**. Table (1) presents the regional GDP and total worked hours. The total worked hours are given by the product of the number of the employed population and the average worked hour. Productivity is determined by the regional GDP to the total worked hours ratio. Therefore, the productivity Z_η of each region and the productivity ratio θ_Z are:

$$Z_\eta := \frac{Y_\eta}{n_\eta L_\eta} \tag{3.76}$$

$$\theta_Z := \frac{Z_2}{Z_1} \approx \frac{469}{662} \approx 0.7076 \tag{3.77}$$

The weight of good 1 in the consumption composition $\omega_{\eta 1}$ for both regions is sourced from **haddad_matriz_2017**. Specifically, ω_{11} corresponds to item $a_{SP \times SP}$. Additionally, ω_{21} is calculated as the weighted mean of all state production (except São Paulo) with São Paulo as the final demand (column SP), taking into account the total production Y_i of each state:

$$\omega_{21} = \frac{\sum_{i=1}^{26} \omega_{iSP} Y_i}{\sum_{i=1}^{26} Y_i} \approx 0.095 \tag{3.78}$$

The parameters are summarized in Table (2) and the steady state variables are presented in Table (3).

Table 1: Brazilian GDP, Worked Hours and Productivity in 2019

State	GDP (R$)	GDP (%)	Total Worked Hours (h)	Productivity (R$/hour)
São Paulo	2.348.338.000	31,8	3.545.301	662
Rio de Janeiro	779.927.917	10,6	1.211.037	644
Minas Gerais	651.872.684	8,8	1.514.471	430
Rio Grande do Sul	482.464.177	6,5	893.768	540
Paraná	466.377.036	6,3	867.265	538
Santa Catarina	323.263.857	4,4	592.856	545
Bahia	293.240.504	4,0	832.538	352
Distrito Federal	273.613.711	3,7	222.054	1.232
Goiás	208.672.492	2,8	531.835	392
Pernambuco	197.853.378	2,7	538.199	368
Pará	178.376.984	2,4	497.051	359
Ceará	163.575.327	2,2	542.370	302
Mato Grosso	142.122.028	1,9	265.546	535
Espírito Santo	137.345.595	1,9	299.144	459
Amazonas	108.181.091	1,5	232.169	466
Mato Grosso do Sul	106.943.246	1,4	197.764	541
Maranhão	97.339.938	1,3	317.848	306
Rio Grande do Norte	71.336.780	1,0	191.595	372
Paraíba	67.986.074	0,9	219.648	310
Alagoas	58.963.729	0,8	153.311	385
Piauí	52.780.785	0,7	167.634	315
Rondônia	47.091.336	0,6	121.275	388
Sergipe	44.689.483	0,6	131.390	340
Tocantins	39.355.941	0,5	96.137	409
Amapá	17.496.661	0,2	46.468	377
Acre	15.630.017	0,2	44.427	352
Roraima	14.292.227	0,2	32.173	444
Brazil	7.389.131.000	100,0	14.300.643	517
Rest of Brazil	5.040.793.000	68,2	10.755.342	469

Source: **ibge_workers_2023**, **ibge_GDP_2023**, **ibge_hours_2023**

Table 2: Parameter Calibration

Parameter	Definition	Calibration
α_1	capital elasticity of production in region 1	0.4
α_2	capital elasticity of production in region 2	0.3
β	intertemporal discount factor	0.985
γ_R	interest-rate smoothing parameter	0.79
γ_π	interest-rate sensitivity in relation to inflation	2.43
γ_Y	interest-rate sensitivity in relation to product	0.16
δ	capital depreciation rate	0.025
θ	price stickness parameter	0.8
θ_{C1}	weight of consumption on production of region 1	0.65
θ_{C2}	weight of consumption on production of region 2	0.65
θ_Y	weight of region 1 in total production	0.318
θ_P	region 2 to 1 price level ratio	1
θ_Z	productivity proportion between regions	0.7076
ρ_{A1}	autoregressive parameter of productivity in region 1	0.95
ρ_{A2}	autoregressive parameter of productivity in region 2	0.95
ρ_M	autoregressive parameter of monetary policy	0.9
σ	relative risk aversion coefficient	2
ϕ	relative labor weight in utility	1
φ	marginal disutility of labor supply	1.5
ψ	elasticity of substitution between intermediate goods	8
$\sigma_{A\eta}$	standard deviation of productivity shock	0.01
σ_M	standard deviation of monetary shock	0.01
ω_{11}	weight of good 1 in consumption composition of region 1	0.528
ω_{21}	weight of good 1 in consumption composition of region 2	0.095

Sources: The Author and **costa_junior_understanding_2016**

Table 3: Variables at Steady State

Variable	**Steady State Value**
$\langle P_1 \quad Z_{A1} \rangle$	$\vec{\mathbf{1}}$
$\langle P_2 \quad Z_{A2} \rangle$	$\langle 1 \quad .7076 \rangle$
$\langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle$	$\vec{\mathbf{1}}$
R	.0402
$\langle \Lambda_1 \quad \Lambda_2 \rangle$	$\langle .875 \quad .875 \rangle$
$\langle W_1 \quad W_2 \rangle$	$\langle 2.2208 \quad .8349 \rangle$
$\langle a_1 \quad a_2 \rangle$	$\langle 4.3957 \quad 1.3703 \rangle$
$\langle b_1 \quad b_2 \rangle$	$\langle .2175 \quad .1631 \rangle$
$\langle Y_1 \quad Y_2 \rangle$	$\langle 2.6811 \quad 1.3255 \rangle$
$\langle I_1 \quad I_2 \rangle$	$\langle .5832 \quad .2162 \rangle$
$\langle K_1 \quad K_2 \rangle$	$\langle 23.3263 \quad 8.6491 \rangle$
$\langle C_1 \quad C_2 \rangle$	$\langle 2.0979 \quad 1.1093 \rangle$
$\langle Q_1 \quad Q_2 \rangle$	$\langle 1,9969 \quad 1,3688 \rangle$
$\langle C_{11} \quad C_{12} \rangle$	$\langle 2.2119 \quad 1.9773 \rangle$
$\langle C_{21} \quad C_{22} \rangle$	$\langle .1442 \quad 1.3741 \rangle$
$\langle L_1 \quad L_2 \rangle$	$\langle .6338 \quad .9724 \rangle$

Source: The Author.

4 RESULTS

In this section, the Impulse Response Functions of each variable of the model is presented and their reaction to the monetary policy shock is discussed.

Figures (2) and (3) depict the reaction of each variable of the model to a 1% monetary policy shock, initiating a response in the economy by elevating the interest rate by the same percentage. The revised interest rate, also representing the price of capital rent, prompts households to reevaluate their consumption decisions. It becomes more advantageous to save now through investment and consume more in the future. This choice leads to an increase in investment and a decrease in demand.

However, despite the higher capital rent, firms to decide to produce more, driven by a higher demand for final products (which will become investment for the next period) and a higher supply of capital. While firms now require more labor, households also supply more labor, seeking higher income for additional investments. As wages decrease, it is reasonable to assume that supply exceeds demand. Wages will rise, but less than the price of capital. As households demand more production to transform into investments, production increases. Although capital rent is higher, lower wages lead to a decline in marginal costs.

Consumer and firm price levels initially rise and then fall as the effects of the monetary shock progress towards stabilizing the economy. In the new equilibrium, there is a decrease in nominal levels, indicating a successful monetary shock. Investment, capital, labor, and production are now at higher levels, while consumption is at a lower level, reflecting a positive outcome of the monetary policy shock. The outcome described can be synthesized in the flow on (4.1). It is important to note that price variations are small in magnitude due to the fact that the capital price corresponds to the interest rate of the economy, denoted as $R \in (0, 1)$.

$$
\begin{array}{lllllllll}
\hat{Z}_M \uparrow & \Longrightarrow & \hat{R} \uparrow & \Longrightarrow & \hat{I}^D_\eta \uparrow & \Longrightarrow & \hat{C}_\eta \downarrow & \Longrightarrow & \\
\hat{K}_\eta \downarrow & \Longrightarrow & \hat{L}^S_\eta \uparrow > \hat{L}^D_\eta \uparrow & \Longrightarrow & \hat{Y}_\eta \uparrow & \Longrightarrow & \hat{\lambda}_\eta \downarrow & \Longrightarrow & \\
\hat{W}_\eta \downarrow & \Longrightarrow & \Delta\hat{Q}_\eta ; \Delta\hat{P}_\eta \to 0 & & & & & &
\end{array}
\tag{4.1}
$$

From a regional perspective, Figures (4) and (5) show the variables of each region in the same plot, indicating that the monetary policy shock will elicit diverse reactions across structurally distinct regions, aligning with the initial hypothesis. The production growth in region 1 surpasses that in region 2. This difference can be attributed to the fact that region 1 is more capital-intensive. As households opt for investment in the present and consumption in the future, the demand for production and subsequent supply of capital benefit firms in region 1 more than in region 2, which is more labor-intensive. This leads to a comparatively lower impact from changes in interest rates in region 2, resulting in a less pronounced reaction in its production. While it may initially seem counter-intuitive that higher nominal interest rates result in increased capital rental costs, a crucial aspect of this model should be noted. In this framework, the savings channel accessible to households is represented by investments, subsequently transformed into capital. The augmentation of investment and subsequent capital supply, in turn, contributes to an elevation in production.

Similarly, Figures (6), (7), (8) and (9) show a negative monetary policy shock, also set at 1%. Note that all the reactions described earlier occur in the opposite direction in this scenario. Therefore, the negative monetary shock yields a mirrored result if compared to the positive one, demonstrating the effectiveness of the interest rate as an instrument of monetary policy in both directions.

The results presented in this study are in line with the existing literature on regional economic dynamics. **bertanha_efeitos_2008**, although using a different method, explored the effects of monetary policy on regional economies and found that regions with diverse economic structures exhibit responses with different intensities to policy shocks. Similarly, **osterno_uma_2022**, using the DSGE methodology, also emphasize the importance of considering regional variables in monetary policy formulation. Overall, the convergence of our findings with previous researchs underscores the robustness of our analysis and contributes to a deeper understanding of regional economic dynamics.

* CPL: Consumer Price Level.

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5 FINAL REMARKS

The primary objective of this thesis was to construct a DSGE model for evaluating the influence of monetary policy on regions within the same country that possess distinct productive structures. The impulse response functions revealed discernible differences in the intensity of reactions, attributable to variations in capital elasticity and technological levels among firms in different regions, while the direction of the reactions remained consistent.

From this information, one can infer that a national monetary policy may not uniformly exhibit effectiveness across all regions and may not be universally optimal and will have different impacts on different regions. For one region, the monetary policy decision may be just what it needed, while to the other it may be too harsh or too soft. Regardless of the decision, it is evident that a country with diverse regions should consider that the interest rate, among many available monetary policy mechanisms, is not universally optimal and will have different impacts on different regions.

Consequently, contemplating the implementation of alternative policies in parallel may be necessary to ensure the desired effects are achieved. Alternatively, the formulation of monetary policy should consider regional variables, allowing the intensity of the policy to be weighted according to the characteristics of each region.

Considering the potential benefits of coordinated fiscal and monetary policies, it becomes imperative to explore avenues for collaboration between fiscal and monetary authorities. Given the existence of regional fiscal authorities, one plausible approach is to introduce regional fiscal policies that complement monetary policy measures. In this context, a historical example comes to mind: the issuance of State Bonds by Brazilian regional governments. Such regional mechanisms could be strategically leveraged in coordination with monetary policies to achieve the desired economic effects. But this is another story, for another thesis.

While the model presented here lays the groundwork for understanding the dynamics of regional economies, future studies could further enrich our understanding by incorporating additional elements to this framework, such as: (1) non-Ricardian households, given that a significant portion of the Brazilian population lacks access to credit; (2) habit formation, as this feature provides a more accurate description of household behavior; (3) labor market, as rigidities within it contribute to a better alignment of the model with reality; (4) a fiscal authority, recognizing the significance of government decisions on private agents; (5) adjustment costs on investment, acknowledging that higher investments can increase its overall expense; (6) inclusion of bonds and other assets, as there are various financial products within the economy; (7) consideration of the foreign market, recognizing the influence of other global economies on internal decisions.

Besides these elements, future implementations should also consider the following constraints the model presented: (1) An important addition to the model will be the inclusion of a credit channel to demonstrate the transmission of monetary policy via credit; (2) The model considers only one representative family in each region, meaning that in this model, as Region 1 has higher productivity than Region 2, it is expected that the production of the former will be greater than that of the latter. In the hypothesis of adding the population element to the model, this would be reflected in production: Region 1, represented by the state of São Paulo, even with higher productivity, would have lower production due to having a smaller population; (3) The model takes more than 40 periods to converge to the steady state, indicating a convergence problem; (4) Despite the initial increase in investment resulting from the monetary policy shock, there is an initial decrease in the capital stock, which should not exist. ■

A APPENDIX

Contents

A.1 EXTENDED MODEL DEVELOPMENT

In this section, we present the extended and detailed development of the regional model presented in Section (3).

A.1.1 Household

The household problem is divided into two steps: first, the household must minimize the consumption costs, and then maximize the utility, which is subject to a budget constraint.

Cost Minimization Problem

Considering that the representative household must decide to consume goods from both regions, there must be a consumption bundle index $C_{\eta t}$ and a consumption price index $Q_{\eta t}$ that minimize the total consumption cost $Q_{\eta t} C_{\eta t}$, as demonstrated by **walsh_monetary_2017**:

$$\min_{C_{\eta 1t}, C_{\eta 2t}} : \quad Q_{\eta t} C_{\eta t} = P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} \tag{A.1}$$

$$\text{s. t.} : \quad C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \tag{A.2}$$

$$C_{\eta t} > 0$$

where P_{1t} and P_{2t} are the prices of goods 1 and 2, respectively, $C_{\eta 1t}$ and $C_{\eta 2t}$ are the goods produced in region 1 and 2, respectively, and consumed in region η. In the consumption aggregation, $\omega_{\eta 1}$ and $(1 - \omega_{\eta 1})$ are the weights of goods $C_{\eta 1t}$ and $C_{\eta 2t}$, respectively, in the consumption bundle $C_{\eta t}$.

Lagrangian

The minimization problem with a constraint can be reformulated into one without a constraint by applying the Lagrangian function:

$$\mathscr{L} = P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} - Q_{\eta t} (C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} - C_{\eta t}) \tag{A.3}$$

First Order Conditions

The first order conditions are:

$$C_{\eta 1t}: \quad P_{1t} - Q_{\eta t}\omega_{\eta 1} C_{\eta 1t}^{\omega_{\eta 1}-1} C_{\eta 2t}^{1-\omega_{\eta 1}} = 0 \implies$$

$$C_{\eta 1t} = \frac{\omega_{\eta 1} Q_{\eta t} C_{\eta t}}{P_{1t}} \tag{A.4}$$

$$C_{\eta 2t}: \quad P_{2t} - Q_{\eta t}(1-\omega_{\eta 1}) C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{-\omega_{\eta 1}} = 0 \implies$$

$$C_{\eta 2t} = \frac{(1-\omega_{\eta 1}) Q_{\eta t} C_{\eta t}}{P_{2t}} \tag{A.5}$$

$$Q_{\eta t}: \quad C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \tag{A.2}$$

Solutions

Divide A.5 by A.4:

$$\frac{C_{\eta 2t}}{C_{\eta 1t}} = \frac{(1-\omega_{\eta 1}) Q_{\eta t} C_{\eta t}/P_{2t}}{\omega_{\eta 1} Q_{\eta t} C_{\eta t}/P_{1t}} \implies$$

$$C_{\eta 2t} = C_{\eta 1t} \frac{(1-\omega_{\eta 1}) P_{1t}}{\omega_{\eta 1} P_{2t}} \tag{A.6}$$

Substitute A.6 in A.2:

$$C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} \left[C_{\eta 1t} \frac{(1-\omega_{\eta 1}) P_{1t}}{\omega_{\eta 1} P_{2t}} \right]^{1-\omega_{\eta 1}} \implies$$

$$C_{\eta 1t} = C_{\eta t} \left(\frac{P_{2t}\omega_{\eta 1}}{P_{1t}(1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \tag{A.7}$$

Substitute A.4 and A.5 in A.2:

$$C_{\eta t} = \left(\frac{\omega_{\eta 1} Q_{\eta t} C_{\eta t}}{P_{1t}} \right)^{\omega_{\eta 1}} \left(\frac{(1-\omega_{\eta 1}) Q_{\eta t} C_{\eta t}}{P_{2t}} \right)^{1-\omega_{\eta 1}} \implies$$

$$Q_{\eta t} = \left(\frac{P_{1t}}{\omega_{\eta 1}} \right)^{\omega_{\eta 1}} \left(\frac{P_{2t}}{1-\omega_{\eta 1}} \right)^{1-\omega_{\eta 1}} \tag{A.8}$$

Therefore, there is a consumption bundle $C_{\eta t}$ and a consumption price index $Q_{\eta t}$ that minimize the total consumption cost $Q_{\eta t} C_{\eta t}$ for the household in region η. Notice that the cost problems of both regions are (must be) related, as the consumption level in one region influences the demand for goods in both regions. Now, this result will be used in the next problem that the household faces.

Utility Maximization Problem

Following the models presented by **costa_junior_understanding_2016** and **solis-garcia_ucb_2022**, the representative household next problem is to maximize an intertemporal utility function U_η with respect to consumption $C_{\eta t}$ and labor $L_{\eta t}$, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\begin{aligned}
\max_{C_{\eta t}, L_{\eta t}, K_{\eta, t+1}} \quad : \quad & U_\eta(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) && \text{(A.9)} \\
\text{s. t.}: \quad & Q_{\eta t} C_{\eta t} + P_{\eta t} I_{\eta t} = W_{\eta t} L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} && \text{(A.10)} \\
& K_{\eta, t+1} = (1-\delta) K_{\eta t} + I_{\eta t} && \text{(A.11)} \\
& C_{\eta t}, L_{\eta t}, K_{\eta t} > 0
\end{aligned}$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, $I_{\eta t}$ is the investment, $W_{\eta t}$ is the wage level, $K_{\eta t}$ is the capital, R_t is the return on capital, and $\Pi_{\eta t}$ is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate.

Isolate $I_{\eta t}$ in A.11 and substitute in A.10:

$$\begin{aligned}
& I_{\eta t} = K_{\eta, t+1} - (1-\delta) K_{\eta t} && \text{(A.12)} \\
& Q_{\eta t} C_{\eta t} + P_{\eta t} (K_{\eta, t+1} - (1-\delta) K_{\eta t}) = W_{\eta t} L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} && \text{(A.13)}
\end{aligned}$$

Lagrangian

The maximization problem with restrictions can be transformed into one without restriction using the Lagrangian function $\mathscr{L}$ formed by A.9 and A.13:

$$\begin{aligned}
\mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) - \\
& - \mu_{\eta t} \Big[Q_{\eta t} C_{\eta t} + P_{\eta t} (K_{\eta, t+1} - (1-\delta) K_{\eta t}) - (W_{\eta t} L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t}) \Big] \Bigg\}
\end{aligned} \tag{A.14}$$

First Order Conditions

The first order conditions are:

$$C_{\eta t}: \quad \beta^t \left\{ \frac{(1-\sigma)C_{\eta t}^{-\sigma}}{1-\sigma} - \mu_{\eta t}\left[Q_{\eta t}\right] \right\} = 0 \implies$$

$$\mu_{\eta t} = \frac{C_{\eta t}^{-\sigma}}{Q_{\eta t}} \tag{A.15}$$

$$L_{\eta t}: \quad \beta^t \left\{ -\phi \frac{(1+\varphi)L_{\eta t}^{1+\varphi}}{1+\varphi} - \mu_{\eta t}\left[-W_{\eta t}\right] \right\} = 0 \implies$$

$$\mu_{\eta t} = \frac{\phi L_{\eta t}^{\varphi}}{W_{\eta t}} \tag{A.16}$$

$$K_{\eta,t+1}: \quad \beta^t\{-\mu_{\eta t}[P_{\eta t}]\} + \mathbb{E}_t\beta^{t+1}\{-\mu_{\eta,t+1}[-(P_{\eta,t+1}(1-\delta)+R_{t+1})]\} = 0 \implies$$

$$\mu_{\eta t}P_{\eta t} = \beta\mathbb{E}_t\{\mu_{\eta,t+1}[P_{\eta,t+1}(1-\delta)+R_{t+1}]\} \tag{A.17}$$

$$\mu_{\eta t}: \quad Q_{\eta t}C_{\eta t} + P_{\eta t}(K_{\eta,t+1} - (1-\delta)K_{\eta t}) = W_{\eta t}L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{A.13}$$

Solutions

Match A.15 and A.16:

$$\mu_{\eta t} = \frac{C_{\eta t}^{-\sigma}}{Q_{\eta t}} = \frac{\phi L_{\eta t}^{\varphi}}{W_{\eta t}} \implies$$

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_{\eta t}}{Q_{\eta t}} \tag{A.18}$$

Equation A.18 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute $\mu_{\eta t}$ and $\mu_{\eta,t+1}$ from equation A.15 in A.17:

$$\frac{C_{\eta t}^{-\sigma}}{Q_{\eta t}}P_{\eta t} = \beta\mathbb{E}_t\left\{ \frac{C_{\eta t}^{-\sigma}}{Q_{\eta t}}[P_{\eta,t+1}(1-\delta)+R_{t+1}] \right\} \implies$$

$$\frac{\mathbb{E}_t\{Q_{\eta,t+1}C_{\eta,t+1}^{\sigma}\}}{Q_{\eta t}C_{\eta t}^{\sigma}} = \beta\frac{\mathbb{E}_t\{P_{\eta,t+1}(1-\delta)+R_{t+1}\}}{P_{\eta t}} \tag{A.19}$$

Equation A.19 is the Euler equation for the return on capital.

Firms

Consider two types of firms: (1) a continuum of intermediate-goods firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-goods firm, which aggregates all these varieties into a final bundle and operates in perfect competition.

A.1.2 Final-Goods Firm

Profit Maximization Problem

The role of the final-goods firm is to aggregate all the varieties $Y_{\eta jt}$ produced by the intermediate-goods firms in each region $\eta \in \{1, 2\}$, so that the representative consumer can buy only one good $Y_{\eta t}$, the bundle good, from each region.

The final-goods firm problem is to maximize its profit, considering that its output is the bundle $Y_{\eta t}$ formed by a continuum $j \in [0, 1]$ of intermediate-goods $Y_{\eta jt}$, with elasticity of substitution between intermediate-goods ψ:

$$\max_{Y_{\eta jt}} : \quad P_{\eta t} Y_{\eta t} - \int_0^1 P_{\eta jt} Y_{\eta jt} \, \mathrm{d}\, j \tag{A.20}$$

$$\text{s. t.} : \quad Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \tag{A.21}$$

Substitute A.21 in A.20:

$$\max_{Y_{\eta jt}} : \quad \Pi_{\eta t} = P_{\eta t} \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{\eta jt} Y_{\eta jt} \, \mathrm{d}\, j \tag{A.22}$$

First Order Condition and Solutions

The first order condition is:

$$Y_{\eta jt} : \quad P_{\eta t} \left(\frac{\psi}{\psi - 1} \right) \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1} - 1} \left(\frac{\psi - 1}{\psi} \right) Y_{\eta jt}^{\frac{\psi-1}{\psi} - 1} - P_{\eta jt} = 0 \implies$$

$$Y_{\eta jt} = Y_t \left(\frac{P_{\eta t}}{P_{\eta jt}} \right)^{\psi} \tag{A.23}$$

Equation A.23 shows that the demand for variety j depends on its relative price.

Substitute A.23 in A.21:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \qquad \Longrightarrow$$

$$Y_{\eta t} = \left(\int_0^1 \left[Y_{\eta t} \left(\frac{P_{\eta t}}{P_{\eta jt}} \right)^{\psi} \right]^{\frac{\psi-1}{\psi}} \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \qquad \Longrightarrow$$

$$P_{\eta t} = \left[\int_0^1 P_{\eta jt}^{1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \tag{A.24}$$

Equation A.24 is the final-goods firm's markup.

A.1.3 Intermediate-Goods Firms

Cost Minimization Problem

The intermediate-goods firms, denoted by $j \in [0,1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose labor $L_{\eta jt}$ to minimize production costs, subject to a technology rule.

$$\min_{K_{\eta jt}, L_{\eta jt}} : \quad R_t K_{\eta jt} + W_t L_{\eta jt} \tag{A.25}$$

$$\text{s. t.}: \quad Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \tag{A.26}$$

where $Y_{\eta jt}$ is the output obtained by the production technology level $Z_{A\eta t}$ that transforms capital $K_{\eta jt}$ and labor $L_{\eta jt}$ in proportions α_η and $(1-\alpha_\eta)$, respectively, into intermediate goods.[12]

Lagrangian

Transform the minimization problem with restriction into one without restriction applying the Lagrangian function $\mathscr{L}$:

$$\mathscr{L} = (R_t K_{\eta jt} + W_t L_{\eta jt}) - \Lambda_{\eta jt}(Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} - Y_{\eta jt}) \tag{A.27}$$

where the Lagrangian multiplier $\Lambda_{\eta jt}$ is the marginal cost.[13]

---

[12] the production technology level $Z_{A\eta t}$ will be submitted to a productivity shock, detailed in section A.1.5.

[13] see Lemma A.1

First Order Condition

The first-order conditions are:

$$
\begin{aligned}
K_{\eta jt}: \quad & R_t - \Lambda_{\eta jt} Z_{A\eta t} \alpha_\eta K_{\eta jt}^{\alpha_\eta - 1} L_{\eta jt}^{1-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& K_{\eta jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta jt}}{R_t} && \text{(A.28)} \\
L_{\eta jt}: \quad & W_t - \Lambda_{\eta jt} Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} (1-\alpha_\eta) L_{\eta jt}^{-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& L_{\eta jt} = (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta jt}}{W_t} && \text{(A.29)} \\
\Lambda_{\eta jt}: \quad & Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} && \text{(A.26)}
\end{aligned}
$$

Solutions

Divide equation A.28 by A.29:

$$
\frac{K_{\eta jt}}{L_{\eta jt}} = \frac{\alpha_\eta Y_{\eta jt} \Lambda_{\eta jt} / R_t}{(1-\alpha_\eta) Y_{\eta jt} \Lambda_{\eta jt} / W_{\eta t}} \quad \Longrightarrow \quad \frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_{\eta t}}{R_t} \tag{A.30}
$$

Equation A.30 demonstrates the relationship between the technical marginal rate of substitution (TMRS) and the economic marginal rate of substitution (EMRS).

Substitute $L_{\eta jt}$ from equation A.30 in A.26:

$$
\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \qquad \Longrightarrow \\
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} \left[\left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t K_{\eta jt}}{W_{\eta t}} \right]^{1-\alpha_\eta} \qquad \Longrightarrow \\
K_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_{\eta t}}{R_t} \right]^{1-\alpha_\eta} && \text{(A.31)}
\end{aligned}
$$

Equation A.31 is the intermediate-goods firm demand for capital.

Substitute A.31 in A.30:

$$
\begin{aligned}
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t K_{\eta jt}}{W_{\eta t}} \qquad \Longrightarrow \\
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t}{W_{\eta t}} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_{\eta t}}{R_t} \right]^{1-\alpha_\eta} \qquad \Longrightarrow \\
L_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_{\eta t}}{R_t} \right]^{-\alpha_\eta} && \text{(A.32)}
\end{aligned}
$$

Equation A.32 is the intermediate-goods firm demand for labor.

Total and Marginal Costs

Calculate the total cost TC using A.31 and A.32:

$$
\begin{aligned}
TC_{\eta jt} &= W_{\eta t}L_{\eta jt} + R_t K_{\eta jt} && \Longrightarrow \\
TC_{\eta jt} &= W_{\eta t}\frac{Y_{\eta jt}}{Z_{A\eta t}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_{\eta t}}{R_t}\right]^{-\alpha_\eta} + R_t\frac{Y_{\eta jt}}{Z_{A\eta t}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_{\eta t}}{R_t}\right]^{1-\alpha_\eta} && \Longrightarrow \\
TC_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_{\eta t}}{1-\alpha_\eta}\right)^{1-\alpha_\eta} && \text{(A.33)}
\end{aligned}
$$

Calculate the marginal cost Λ using A.33:

$$
\Lambda_{\eta jt} = \frac{\partial TC_{\eta jt}}{\partial Y_{\eta jt}} \Longrightarrow \Lambda_{\eta jt} = \frac{1}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_{\eta t}}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{A.34}
$$

The marginal cost depends on the technological level $Z_{A\eta t}$, the nominal interest rate R_t and the nominal wage level $W_{\eta t}$, which are the same for all intermediate-goods firms, and because of that, the index j may be dropped:

$$
\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_{\eta t}}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{A.35}
$$

notice that:

$$
\Lambda_{\eta t} = \frac{TC_{\eta jt}}{Y_{\eta jt}} \Longrightarrow TC_{\eta jt} = \Lambda_{\eta t}Y_{\eta jt} \tag{A.36}
$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (**calvo_staggered_1983**): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{\eta j,t+1} = P_{\eta jt})$, and a probability $(1-\theta)$ of setting a new optimal price $P^*_{\eta jt}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price:

$$
\begin{aligned}
\max_{P_{\eta jt}}: &\quad \mathbb{E}_t\sum_{s=0}^{\infty}\left\{\frac{\theta^s\left[P_{\eta jt}Y_{\eta j,t+s} - TC_{\eta j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})}\right\} && \text{(A.37)} \\
\text{s. t.}: &\quad Y_{\eta jt} = Y_{\eta t}\left(\frac{P_{\eta t}}{P_{\eta jt}}\right)^{\psi} && \text{(A.23)}
\end{aligned}
$$

where s is the period in time when the decision must be made; t is the last period in time when the price was updated and k is the period in the future when the interest rate applies.

Substitute A.36 in A.37:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - \Lambda_{\eta,t+s} Y_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{A.38}$$

Substitute A.23 in A.38 and rearrange the variables:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} - \Lambda_{\eta,t+s} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt}^{1-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to $P_{\eta jt}$ is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - (-\psi) P_{\eta jt}^{-\psi-1} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} Y_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \\ = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \end{aligned} \tag{A.39}$$

Substitute A.23 in A.39:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi-1) Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$(\psi-1)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \psi P_{\eta jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt}^{*} = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}} \tag{A.40}$$

Equation A.40 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P^*_{\eta t} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{A.41}$$

Final-Goods Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1 - \theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in A.24 to determine the aggregate price level:

$$P_{\eta t} = \left[\int_0^{\theta} P_{\eta,t-1}^{1-\psi} \, \mathrm{d}\, j + \int_{\theta}^{1} P_{\eta t}^{*1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies$$

$$P_{\eta t} = \left[\theta P_{\eta,t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \tag{A.42}$$

Equation A.42 is the aggregate price level.

Regional Inflation

In each region, the price level $P_{\eta t}$ generates a regional inflation rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{A.43}$$

A.1.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (**taylor_discretion_1993**) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{A.44}$$

where R, π, Y are the nominal interest rate, gross inflation rate and the production level in steady state, respectively; γ_R is the smoothing parameter for the interest rate R_t, γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively, Z_{Mt} is the monetary shock and π_t is the gross inflation rate, defined by:[14]

$$\pi_t = \pi_{1t}^{\theta_\pi} \pi_{2t}^{1-\theta_\pi} \tag{A.45}$$

$$\text{where: } \theta_\pi = \frac{P_{1t}Y_{1t}}{P_{1t}Y_{1t} + P_{2t}Y_{2t}} \tag{A.46}$$

A.1.5 Stochastic Shocks

Productivity Shock

The production technology level $Z_{A\eta t}$ will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{A.47}$$

where $\rho_{A\eta} \in [0,1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{A.48}$$

where $\rho_M \in [0,1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

A.1.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P_{\eta t}^*, R_t^*, W_{\eta t}^*\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C_{\eta 1t}^*, C_{\eta 2t}^*, L_{\eta t}^*, I_{\eta t}^*, K_{\eta,t+1}^*\}$ and allocations for firms $\boldsymbol{\mathcal{A}}_F := \{K_{\eta jt}^*, L_{\eta jt}^*, Y_{\eta jt}^*, Y_{\eta t}^*\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms' problems, and the markets for goods and labor clear:

$$Y_t = Y_{1t} + Y_{2t} \tag{A.49}$$

$$\text{where: } \quad Y_{\eta t} = C_{\eta t} + I_{\eta t} \tag{A.50}$$

$$L_{\eta t} = \int_0^1 L_{\eta jt} \, \mathrm{d}\, j \tag{A.51}$$

[14] for the monetary shock definition, see section A.1.5.

Intermediate-Goods Firm Profit

For the sake of closure, the intermediate-goods firm profit must be defined:

$$\Pi_{\eta t} = \int_0^1 \Pi_{\eta j t} \, \mathrm{d}\, j \tag{A.52}$$

$$\Pi_{\eta j t} = P_{\eta t} Y_{\eta j t} - W_{\eta t} L_{\eta j t} \tag{A.53}$$

Substitute A.53 and A.51 in A.52:

$$\Pi_{\eta t} = P_{\eta t} \int_0^1 Y_{\eta j t} \, \mathrm{d}\, j - W_{\eta t} L_{\eta t} \tag{A.54}$$

Substitute A.54 in A.10:

$$Q_{\eta t} C_{\eta t} + P_{\eta t} I_{\eta t} = W_{\eta t} L_{\eta t} + R_t K_{\eta t} + P_{\eta t} \int_0^1 Y_{\eta j t} \, \mathrm{d}\, j - W_{\eta t} L_{\eta t} \implies$$

$$Q_{\eta t} C_{\eta t} + P_{\eta t} I_{\eta t} = R_t K_{\eta t} + P_{\eta t} \int_0^1 Y_{\eta j t} \, \mathrm{d}\, j \tag{A.55}$$

A.1.7 Model Structure

The model is composed of the preview solutions, forming a square system of 38 variables and equations, summarized as follows:

- Variables:
  - from the household problem: $\langle C_\eta \quad L_\eta \quad K_\eta \quad I_\eta \quad C_{\eta 1} \quad C_{\eta 2} \quad Q_\eta \rangle$;
  - from the final-goods firm problem: $\langle Y_{\eta j} \quad Y_\eta \quad P_\eta \rangle$;
  - from the intermediate-goods firm problems: $\langle L_{\eta j} \quad K_{\eta j} \quad P^*_\eta \rangle$;
  - from the monetary policy: $\langle R \quad \pi \quad Y \rangle$;
  - prices: $\langle W_\eta \quad \Lambda_\eta \quad \pi_\eta \rangle$;
  - shocks: $\langle Z_{A\eta} \quad Z_M \rangle$.
- Equations:

1. Regional Consumption Weight:

$$C_{\eta 2t} = C_{\eta 1t} \frac{(1-\omega_{\eta 1})P_{1t}}{\omega_{\eta 1} P_{2t}} \tag{A.6}$$

2. Regional Consumption of Good 1:

$$C_{\eta 1t} = C_{\eta t} \left(\frac{P_{2t}\omega_{\eta 1}}{P_{1t}(1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \tag{A.7}$$

3. Regional Price Index:

$$Q_{\eta t} = \left(\frac{P_{1t}}{\omega_{\eta 1}} \right)^{\omega_{\eta 1}} \left(\frac{P_{2t}}{1-\omega_{\eta 1}} \right)^{1-\omega_{\eta 1}} \tag{A.8}$$

4. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_{\eta t}}{Q_{\eta t}} \tag{A.18}$$

5. Law of motion for capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{A.11}$$

6. Euler equation for the return on capital:

$$\frac{\mathbb{E}_t\{Q_{\eta,t+1}C_{\eta,t+1}^{\sigma}\}}{Q_{\eta t}C_{\eta t}^{\sigma}} = \beta \frac{\mathbb{E}_t\{P_{\eta,t+1}(1-\delta) + R_{t+1}\}}{P_{\eta t}} \tag{A.19}$$

7. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}j \right)^{\frac{\psi}{\psi-1}} \tag{A.21}$$

8. Production Function:

$$Y_{\eta j t} = Z_{A\eta t} K_{\eta j t}^{\alpha_\eta} L_{\eta j t}^{1-\alpha_\eta} \tag{A.26}$$

9. Technical and Economic Marginal Rates of Substitution:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_{\eta t}}{R_t} \tag{A.30}$$

10. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_{\eta t}}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{A.35}$$

11. Optimal Price:

$$P^*_{\eta t} = \frac{\psi}{\psi-1}\cdot\frac{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j,t+s}\Lambda_{\eta,t+s}/\prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j,t+s}/\prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \tag{A.41}$$

12. Regional Price Level:

$$P_{\eta t} = \left[\theta P_{\eta,t-1}^{1-\psi} + (1-\theta)P_{\eta t}^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{A.42}$$

13. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{A.43}$$

14. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi}\left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \tag{A.44}$$

15. National Gross Inflation Rate:

$$\pi_t = \pi_{1t}^{\theta_\pi}\pi_{2t}^{1-\theta_\pi} \tag{A.45}$$

16. Productivity Shock:

$$\ln Z_{A\eta t} = (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}\ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{A.47}$$

17. Monetary Shock:

$$\ln Z_{Mt} = (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{A.48}$$

18. Goods-Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \tag{A.49}$$

19. Regional Goods-Market Clearing Condition:

$$Y_{\eta t} = C_{\eta t} + I_{\eta t} \tag{A.50}$$

20. Regional Labor-Market Clearing Condition:

$$L_{\eta t} = \int_0^1 L_{\eta jt}\,\mathrm{d}j \tag{A.51}$$

21. Budget Constraint:

$$Q_{\eta t} C_{\eta t} + P_{\eta t} I_{\eta t} = R_t K_{\eta t} + P_{\eta t} \int_0^1 Y_{\eta j t} \, \mathrm{d}\, j \tag{A.55}$$

A.1.8 Steady State

The steady state of a variable is defined by its constancy through time. For any given variable X_t, it is in steady state if $t \to \infty \implies \mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (**costa_junior_understanding_2016**). For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The model in steady state is:

1. Regional Consumption Weight:

$$C_{\eta 2} = C_{\eta 1} \frac{(1-\omega_{\eta 1}) P_1}{\omega_{\eta 1} P_2} \tag{A.56}$$

2. Regional Consumption of Good 1:

$$C_{\eta 1} = C_\eta \left(\frac{P_2 \omega_{\eta 1}}{P_1 (1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \tag{A.57}$$

3. Regional Price Index:

$$Q_\eta = \left(\frac{P_1}{\omega_{\eta 1}} \right)^{\omega_{\eta 1}} \left(\frac{P_2}{1-\omega_{\eta 1}} \right)^{1-\omega_{\eta 1}} \tag{A.58}$$

4. Labor Supply:

$$\frac{\phi L_\eta^\varphi}{C_\eta^{-\sigma}} = \frac{W_\eta}{Q_\eta} \tag{A.59}$$

5. Law of motion for capital:

$$I_\eta = \delta K_\eta \tag{A.60}$$

6. Euler equation for the return on capital:

$$1 = \beta \left[(1-\delta) + \frac{R}{P_\eta} \right] \tag{A.61}$$

7. Bundle Technology:

$$Y_\eta = \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}} \, \mathrm{d}j \right)^{\frac{\psi}{\psi-1}} \tag{A.62}$$

8. Production Function:

$$Y_{\eta j} = Z_{A\eta} K_{\eta j}^{\alpha_\eta} L_{\eta j}^{1-\alpha_\eta} \tag{A.63}$$

9. Technical and Economic Marginal Rates of Substitution:

$$\frac{K_{\eta j}}{L_{\eta j}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R} \tag{A.64}$$

10. Marginal Cost:

$$\Lambda_\eta = \frac{1}{Z_{A\eta}}\left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_\eta}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{A.65}$$

11. Optimal Price:

$$\begin{aligned} P_\eta^* &= \frac{\psi}{\psi-1}\cdot\frac{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j}\Lambda_\eta / \prod_{k=0}^{s-1}(1+R)\right\}}{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j} / \prod_{k=0}^{s-1}(1+R)\right\}} \implies \\ P_\eta^* &= \frac{\psi}{\psi-1}\Lambda_\eta \end{aligned} \tag{A.66}$$

12. Regional Price Level:

$$\begin{aligned} P_\eta &= \left[\theta P_\eta^{1-\psi} + (1-\theta)P_\eta^{*1-\psi}\right]^{\frac{1}{1-\psi}} \implies \\ P_\eta &= P_\eta^* \end{aligned} \tag{A.67}$$

13. Regional Gross Inflation Rate:

$$\pi_\eta = \frac{P_\eta}{P_\eta} = 1 \tag{A.68}$$

14. Monetary Policy:

$$\begin{aligned} \frac{R}{R} &= \left(\frac{R}{R}\right)^{\gamma_R}\left[\left(\frac{\pi}{\pi}\right)^{\gamma_\pi}\left(\frac{Y}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_M \implies \\ Z_M &= 1 \end{aligned} \tag{A.69}$$

15. National Gross Inflation Rate:

$$\pi = \pi_1^{\theta_\pi}\pi_2^{1-\theta_\pi} = 1 \tag{A.70}$$

16. Productivity Shock:

$$\begin{aligned} &\ln Z_{A\eta} = (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}\ln Z_{A\eta} + \varepsilon_{A\eta} \implies \\ &\quad \varepsilon_{A\eta} = 0 \end{aligned} \tag{A.71}$$

17. Monetary Shock:

$$\begin{aligned} &\ln Z_M = (1-\rho_M)\ln Z_M + \rho_M \ln Z_M + \varepsilon_M \implies \\ &\quad \varepsilon_M = 0 \end{aligned} \tag{A.72}$$

18. Goods-Market Clearing Condition:

$$Y = Y_1 + Y_2 \tag{A.73}$$

19. Regional Goods-Market Clearing Condition:

$$Y_\eta = C_\eta + I_\eta \tag{A.74}$$

20. Regional Labor-Market Clearing Condition:

$$L_\eta = \int_0^1 L_{\eta j} \, \mathrm{d}\, j \tag{A.75}$$

21. Budget Constraint:

$$Q_\eta C_\eta + P_\eta I_\eta = RK_\eta + P_\eta \int_0^1 Y_{\eta j} \, \mathrm{d}\, j \tag{A.76}$$

A.1.9 Variables at Steady State

For the steady-state solution, all endogenous variables will be determined with respect to the parameters. It is assumed that the price level and the productivity level of region 1 are equal to one. For region 2, it is assumed that these levels are in proportion to the corresponding values in the first region by factors $\langle \theta_P \quad \theta_Z \rangle$:[15]

$$\langle P_1 \quad Z_{A1} \rangle = \vec{\mathbf{1}} \tag{A.77}$$

$$\langle P_2 \quad Z_{A2} \rangle = \langle \theta_P P_1 \quad \theta_Z Z_{A1} \rangle \tag{A.78}$$

From A.68, A.69 and A.70, the monetary shock, the national and regional gross inflation rates are:

$$\langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle = \vec{\mathbf{1}} \tag{A.79}$$

From A.71 and A.72, the productivity and monetary shocks are:

$$\langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle = \vec{\mathbf{0}} \tag{A.80}$$

From A.61, the return on capital is:

$$1 = \beta \left[(1-\delta) + \frac{R}{P} \right] \implies \tag{A.61}$$

$$R = P_\eta \left[\frac{1}{\beta} - (1-\delta) \right] \tag{A.81}$$

Divide A.81 for one region by the other region:

$$\frac{R}{R} = \frac{P_1 \left[\frac{1}{\beta} - (1-\delta) \right]}{P_2 \left[\frac{1}{\beta} - (1-\delta) \right]} \implies$$

$$P_1 = P_2 \tag{A.82}$$

Substitute A.82 in A.78:

$$\langle P_2 \quad Z_{A2} \rangle = \langle P_1 \quad \theta_Z Z_{A1} \rangle \tag{A.83}$$

From A.67, A.77 and A.82, the regional optimal price P_η^* is:

$$P_\eta^* = P_\eta \implies \langle P_1^* \quad P_2^* \rangle = \langle P_1 \quad P_2 \rangle = \langle P_1 \quad P_1 \rangle \tag{A.84}$$

[15] where $\vec{\mathbf{1}}$ is the unit vector.

Substitute A.82 in A.58 for the price composition of consumption bundle Q_η:

$$Q_\eta = \left(\frac{P_1}{\omega_{\eta 1}}\right)^{\omega_{\eta 1}} \left(\frac{P_2}{1-\omega_{\eta 1}}\right)^{1-\omega_{\eta 1}} \implies \tag{A.58}$$

$$Q_\eta = \frac{P_1}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{A.85}$$

Substitute A.84 in A.66 for the marginal cost Λ_η:

$$P_\eta^* = \frac{\psi}{\psi-1}\Lambda_\eta \implies \tag{A.66}$$

$$\Lambda_\eta = P_\eta \frac{\psi-1}{\psi} \tag{A.86}$$

From A.65, the nominal wage W_η is:

$$\Lambda_\eta = \frac{1}{Z_{A\eta}}\left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_\eta}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies \tag{A.65}$$

$$W_\eta = (1-\alpha_\eta)\left[\Lambda_\eta Z_{A\eta}\left(\frac{\alpha_\eta}{R}\right)^{\alpha_\eta}\right]^{\frac{1}{1-\alpha_\eta}} \tag{A.87}$$

Due to price parity in steady state, where prices are identical ($P_\eta = P_\eta^*$) and resulting in a gross inflation level of one ($\pi_\eta = 1$), all firms produce the same output level ($\forall i, j \in [0,1], Y_{\eta j} = Y_{\eta i}, i \neq j$) (**solis-garcia_ucb_2022**). As a consequence, they uniformly demand the same amount of factors ($\forall j \in [0,1], L_{\eta j} = L_{\eta i}, j \neq i$), and A.62, A.63, A.64, A.75, and A.76 become:

$$Y_\eta = Y_{\eta j} \tag{A.88}$$

$$Y_\eta = Z_{A\eta} K_\eta^{\alpha_\eta} L_\eta^{1-\alpha_\eta} \tag{A.89}$$

$$\frac{K_\eta}{L_\eta} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R} \tag{A.90}$$

$$L_\eta = L_{\eta j} \tag{A.91}$$

$$Q_\eta C_\eta + P_\eta I_\eta = R K_\eta + P_\eta Y_\eta \tag{A.92}$$

Isolate K_η in A.90 and substitute in A.89:

$$K_\eta = L_\eta \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}$$

$$Y_\eta = Z_{A\eta}\left[L_\eta\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}\right]^{\alpha_\eta} L_\eta^{1-\alpha_\eta} \implies$$

$$L_\eta = \frac{Y_\eta}{Z_{A\eta}}\left[\left(\frac{1-\alpha_\eta}{\alpha_\eta}\right)\frac{R}{W_\eta}\right]^{\alpha_\eta} \iff \frac{1}{L_\eta} = \frac{Z_{A\eta}}{Y_\eta}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}\right]^{\alpha_\eta} \tag{A.93}$$

Substitute A.93 in A.90:

$$\frac{K_\eta}{L_\eta} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R} \implies \tag{A.90}$$

$$K_\eta \frac{Z_{A\eta}}{Y_\eta}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}\right]^{\alpha_\eta} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R} \implies$$

$$K_\eta = \frac{Y_\eta}{Z_{A\eta}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}\right]^{1-\alpha_\eta} \tag{A.94}$$

Substitute A.94 in A.60:

$$I_\eta = \delta K_\eta \implies \tag{A.60}$$

$$I_\eta = \delta\frac{Y_\eta}{Z_{A\eta}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}\right]^{1-\alpha_\eta} \implies \tag{A.95}$$

$$I_\eta = b_\eta Y_\eta \tag{A.96}$$

$$\text{where:}\quad b_\eta = \frac{\delta}{Z_{A\eta}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_\eta}{R}\right]^{1-\alpha_\eta} \tag{A.97}$$

Isolate C_η in A.59 and then substitute L_η from A.93:

$$\frac{\phi L_\eta^{\varphi}}{C_\eta^{-\sigma}} = \frac{W_\eta}{Q_\eta} \implies C_\eta^{\sigma} = \frac{W_\eta}{\phi Q_\eta}\cdot\frac{1}{L_\eta^{\varphi}} \implies$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{A.98}$$

$$\text{where:}\quad a_\eta = \left[\frac{W_\eta}{\phi Q_\eta}\left[Z_{A\eta}\left(\frac{\alpha_\eta W_\eta}{(1-\alpha_\eta)R}\right)^{\alpha_\eta}\right]^{\varphi}\right]^{\frac{1}{\sigma}} \tag{A.99}$$

Substitute A.98 and A.96 in A.74:

$$Y_\eta = C_\eta + I_\eta \implies \tag{A.74}$$

$$Y_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} + b_\eta Y_\eta \implies$$

$$Y_\eta = \left(\frac{a_\eta}{1-b_\eta}\right)^{\frac{\sigma}{\sigma+\varphi}} \tag{A.100}$$

The result of A.100 determines $Y, C_\eta, I_\eta, K_\eta, L_\eta, C_{\eta 1}, C_{\eta 2}$ in A.73, A.98, A.60, A.94, A.57, A.56, A.93, respectively.

A.1.10 Steady State Solution

$$\vec{\mathbf{1}} = \langle P_1 \quad Z_{A1} \rangle \tag{A.77}$$

$$\langle P_2 \quad Z_{A2} \rangle = \langle P_1 \quad \theta_Z Z_{A1} \rangle \tag{A.83}$$

$$\vec{\mathbf{1}} = \langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle \tag{A.79}$$

$$\vec{\mathbf{0}} = \langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle \tag{A.80}$$

$$R = P_\eta \left[\frac{1}{\beta} - (1-\delta) \right] \tag{A.81}$$

$$P_\eta^* = P_\eta \tag{A.84}$$

$$Q_\eta = \frac{P_1}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{A.85}$$

$$\Lambda_\eta = P_\eta \frac{\psi - 1}{\psi} \tag{A.86}$$

$$W_\eta = (1-\alpha_\eta) \left[\Lambda_\eta Z_{A\eta} \left(\frac{\alpha_\eta}{R} \right)^{\alpha_\eta} \right]^{\frac{1}{1-\alpha_\eta}} \tag{A.87}$$

$$a_\eta = \left[\frac{W_\eta}{\phi Q_\eta} \left[Z_{A\eta} \left(\frac{\alpha_\eta W_\eta}{(1-\alpha_\eta) R} \right)^{\alpha_\eta} \right]^{\varphi} \right]^{\frac{1}{\sigma}} \tag{A.99}$$

$$b_\eta = \frac{\delta}{Z_{A\eta}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_\eta}{R} \right]^{1-\alpha_\eta} \tag{A.97}$$

$$Y_\eta = \left(\frac{a_\eta}{1-b_\eta} \right)^{\frac{\sigma}{\sigma+\varphi}} \tag{A.100}$$

$$Y = Y_1 + Y_2 \tag{A.73}$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{A.98}$$

$$I_\eta = b_\eta Y_\eta \tag{A.96}$$

$$K_\eta = \frac{I_\eta}{\delta} \tag{A.60}$$

$$C_{\eta 1} = C_\eta \left(\frac{P_2 \omega_{\eta 1}}{P_1 (1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \tag{A.57}$$

$$C_{\eta 2} = C_{\eta 1} \frac{(1-\omega_{\eta 1}) P_1}{\omega_{\eta 1} P_2} \tag{A.56}$$

$$L_\eta = \frac{Y_\eta}{Z_{A\eta}} \left[\left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R}{W_\eta} \right]^{\alpha_\eta} \tag{A.93}$$

A.1.11 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. `Dynare` is specialized software for macroeconomic modeling, commonly used for solving DSGE models. Before the model can be processed by the software, it must undergo linearization to eliminate the infinite sum in Equation A.41. For this purpose, Uhlig's rules of log-linearization (**uhlig_toolkit_1999**) will be applied to all equations in the model. For any given variable X_t, its deviation will be represented with a hat, $\hat{X}_t$.[16]

Regional Gross Inflation Rate

Log-linearize A.43 and define the level deviation of regional inflation rate $\hat{\pi}_{\eta t}$:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{A.43}$$

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{A.101}$$

Regional Price Level

Log-linearize equation A.42:

$$P_{\eta t}^{1-\psi} = \theta P_{\eta,t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi} \quad \Longrightarrow \tag{A.42}$$

$$\begin{aligned} P^{1-\psi}(1+(1-\psi)\hat{P}_{\eta t}) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{\eta,t-1}) + \\ &+ (1-\theta)P^{1-\psi}(1+(1-\psi)\hat{P}^{*}_{\eta t}) \quad \Longrightarrow \end{aligned}$$

$$\hat{P}_{\eta t} = \theta \hat{P}_{\eta,t-1} + (1-\theta)\hat{P}^{*}_{\eta t} \tag{A.102}$$

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[16] see Lemma A.3 for details.

New Keynesian Phillips Curve

In order to log-linearize equation A.41, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply Lemma A.5:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{A.41}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j,t+s}}{(1+R)^s \left(1+\frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\qquad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{(1+R)^s \left(1+\frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{A.103}$$

First, log-linearize the left hand side of equation A.103 with respect to $P_{\eta t}^*, Y_{\eta j t}, \widetilde{R}_t$:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j,t+s}}{(1+R)^s \left(1+\frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies$$

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P_{\eta}^* Y_{\eta j}\left(1+\hat{P}_{\eta t}^* + \hat{Y}_{\eta j,t+s}\right)}{1+\frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies$$

$$P_{\eta}^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1+\hat{P}_{\eta t}^* + \hat{Y}_{\eta j,t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\}$$

Separate the terms not dependent on s:

$$\begin{aligned} &P_{\eta}^* Y_{\eta j}(1+\hat{P}_{\eta t}^*) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \right\} + \\ &\qquad + P_{\eta}^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \implies \end{aligned}$$

Apply definition A.3 on the first term:

$$\frac{P_{\eta}^* Y_{\eta j}(1+\hat{P}_{\eta t}^*)}{1-\theta/(1+R)} + P_{\eta}^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\}$$

Second, log-linearize the left hand side of A.103 with respect to $\Lambda^*_{\eta t}, Y_{\eta j t}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\frac{\theta^s Y_{\eta j,t+s}\Lambda_{\eta,t+s}}{(1+R)^s\left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)}\right\} \Longrightarrow$$

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\frac{Y_{\eta j}\Lambda_\eta(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s})}{1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right\} \Longrightarrow$$

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\right\}+ \\ +\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Apply definition A.3 on the first term:

$$\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+ \\ +\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Join both sides of the equation again:

$$\frac{P^*_\eta Y_{\eta j}(1+\hat{P}^*_{\eta t})}{1-\theta/(1+R)}+P^*_\eta Y_{\eta j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}= \\ =\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+ \\ +\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} \quad \text{(A.104)}$$

Define a discount rate ϱ:

$$\varrho=\frac{1}{(1+R)} \quad \text{(A.105)}$$

Substitute A.105 in A.104:

$$\frac{P_\eta^* Y_{\eta j}(1+\hat{P}_{\eta t}^*)}{1-\theta\varrho} + P_\eta^* Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\varrho)^s\left(\hat{Y}_{\eta j,t+s} - \varrho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} =$$

$$= \frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta\varrho} + \tag{A.106}$$

$$+\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\varrho)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\varrho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Substitute A.86 in A.106 and simplify all common terms:

$$\cancel{\frac{P_\eta^* Y_{\eta j}}{1-\theta\varrho}} + \frac{\cancel{P_\eta^* Y_{\eta j}}\hat{P}_{\eta t}^*}{1-\theta\varrho} + \cancel{P_\eta^* Y_{\eta j}}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\varrho)^s\left(\cancel{\hat{Y}_{\eta j,t+s}-\varrho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right)\right\} =$$

$$= \cancel{\frac{P_\eta^* Y_{\eta j}}{1-\theta\varrho}} + \cancel{P_\eta^* Y_{\eta j}}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\varrho)^s\left(\cancel{\hat{Y}_{\eta j,t+s}-\varrho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}+\hat{\Lambda}_{\eta,t+s}\right)\right\} \Longrightarrow$$

$$\frac{\hat{P}_{\eta t}^*}{1-\theta\varrho} = \mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\varrho)^s\left(\hat{\Lambda}_{\eta,t+s}\right)\right\} \tag{A.107}$$

Define the real marginal cost $\lambda_{\eta t}$ and log-linearize it:

$$\lambda_{\eta t} = \frac{\Lambda_{\eta t}}{P_{\eta t}} \Longrightarrow \Lambda_{\eta t} = P_{\eta t}\lambda_{\eta t} \Longrightarrow \tag{A.108}$$

$$\hat{\Lambda}_{\eta t} = \hat{P}_{\eta t} + \hat{\lambda}_{\eta t} \tag{A.109}$$

Substitute A.109 in A.107:

$$\hat{P}_{\eta t}^* = (1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) \tag{A.110}$$

Substitute A.110 in A.102:

$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)\hat{P}_{\eta t}^* \tag{A.102}$$

$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) \tag{A.111}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\varrho\mathbb{L}^{-1})$ in A.111:[17]

$$
\begin{aligned}
(1-\theta\varrho\mathbb{L}^{-1})\hat{P}_{\eta t} &= (1-\theta\varrho\mathbb{L}^{-1})\Big[\theta\hat{P}_{\eta,t-1}+\\
&\quad +(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right)\Big] \implies\\
\hat{P}_{\eta t}-\theta\varrho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1}-\theta\varrho\theta\hat{P}_{\eta t}+\\
&\quad (1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right)-\\
&\quad -\theta\varrho(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)
\end{aligned}
\tag{A.112}
$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\varrho$ within the operator. Then, cancel the summations and rearrange the terms:

$$
\begin{aligned}
\hat{P}_{\eta t}-\theta\varrho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1}-\theta\varrho\theta\hat{P}_{\eta t}+\\
&\quad (1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right)-\\
&\quad -\theta\varrho(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right) \implies\\
\hat{P}_{\eta t}-\theta\varrho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1}-\theta\varrho\theta\hat{P}_{\eta t}+(1-\theta)(1-\theta\varrho)(\hat{P}_{\eta t}+\hat{\lambda}_{\eta t})+\\
&\quad \cancel{+(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)}-\\
&\quad \cancel{-(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)} \implies\\
\hat{P}_{\eta t}-\theta\varrho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1}-\theta^2\varrho\hat{P}_{\eta t}+\\
&\quad +(1-\theta-\theta\varrho+\theta^2\varrho)\hat{P}_{\eta t}+(1-\theta)(1-\theta\varrho)\hat{\lambda}_{\eta t} \implies\\
(\hat{P}_{\eta t}-\hat{P}_{\eta,t-1}) &= \varrho(\mathbb{E}_t\hat{P}_{t+1}-\hat{P}_{\eta t})+\frac{(1-\theta)(1-\theta\varrho)}{\theta}\hat{\lambda}_{\eta t}
\end{aligned}
\tag{A.113}
$$

Substitute A.101 in A.113:

$$
\hat{\pi}_{\eta t}=\varrho\mathbb{E}_t\hat{\pi}_{\eta,t+1}+\frac{(1-\theta)(1-\theta\varrho)}{\theta}\hat{\lambda}_{\eta t}
\tag{A.114}
$$

Equation A.114 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

[17] see Corollary A.5.1.

Regional Consumption Weight

Log-linearize A.6:

$$C_{\eta 2t} = C_{\eta 1t}\frac{(1-\omega_{\eta 1})P_{1t}}{\omega_{\eta 1}P_{2t}} \implies \tag{A.6}$$

$$\hat{C}_{\eta 2t} - \hat{C}_{\eta 1t} = \hat{P}_{1t} - \hat{P}_{2t} \tag{A.115}$$

Regional Consumption of Good 1

Log-linearize A.7:

$$C_{\eta 1t} = C_{\eta t}\left(\frac{P_{2t}\omega_{\eta 1}}{P_{1t}(1-\omega_{\eta 1})}\right)^{1-\omega_{\eta 1}} \implies \tag{A.7}$$

$$\hat{C}_{\eta t} - \hat{C}_{\eta 1t} = (1-\omega_{\eta 1})(\hat{P}_{1t} - \hat{P}_{2t}) \tag{A.116}$$

Regional Price Index

Log-linearize A.8:

$$Q_{\eta t} = \left(\frac{P_{1t}}{\omega_{\eta 1}}\right)^{\omega_{\eta 1}}\left(\frac{P_{2t}}{1-\omega_{\eta 1}}\right)^{1-\omega_{\eta 1}} \implies \tag{A.8}$$

$$\hat{Q}_{\eta t} = \omega_{\eta 1}\hat{P}_{1t} + (1-\omega_{\eta 1})\hat{P}_{2t} \tag{A.117}$$

Labor Supply

Log-linearize A.18:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_{\eta t}}{Q_{\eta t}} \implies \tag{A.18}$$

$$\varphi\hat{L}_{\eta t} + \sigma\hat{C}_{\eta t} = \hat{W}_{\eta t} - \hat{Q}_{\eta t} \tag{A.118}$$

Law of Motion for Capital

Log-linearize A.11:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies \tag{A.11}$$

$$K_{\eta}(1+\hat{K}_{\eta,t+1}) = (1-\delta)K_{\eta}(1+\hat{K}_{\eta t}) + I_{\eta}(1+\hat{I}_{\eta t}) \implies$$

$$\hat{K}_{\eta,t+1} = (1-\delta)\hat{K}_{\eta t} + \delta\hat{I}_{\eta t} \tag{A.119}$$

Euler equation for capital return

Log-linearize A.19:

$$\frac{\mathbb{E}_t\{Q_{\eta,t+1}C^{\sigma}_{\eta,t+1}\}}{Q_{\eta t}C^{\sigma}_{\eta t}} = \beta\frac{\mathbb{E}_t\{P_{\eta,t+1}(1-\delta)+R_{t+1}\}}{P_{\eta t}} \iff \tag{A.19}$$

$$\mathbb{E}_t\left\{\frac{Q_{\eta,t+1}C^{\sigma}_{\eta,t+1}}{P_{\eta,t+1}}\right\}\cdot\frac{P_{\eta t}}{Q_{\eta t}C^{\sigma}_{\eta t}} = \beta\mathbb{E}_t\left\{(1-\delta)+\frac{R_{t+1}}{P_{\eta,t+1}}\right\} \implies$$

$$(\hat{Q}_{\eta,t+1}-\hat{Q}_{\eta t})+\sigma(\hat{C}_{\eta,t+1}-\hat{C}_{\eta t})-(\hat{P}_{\eta,t+1}-\hat{P}_{\eta,t}) = \beta r(\hat{R}_{\eta,t+1}-\hat{P}_{\eta,t+1}) \tag{A.120}$$

$$\text{where: } r = \frac{R}{P_\eta} \tag{A.121}$$

Bundle Technology

Apply the natural logarithm to A.21:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \implies \tag{A.21}$$

$$\ln Y_{\eta t} = \frac{\psi}{\psi-1}\ln\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)$$

Log-linearize using corollary A.3.3:

$$\ln Y_\eta + \hat{Y}_{\eta t} = \frac{\psi}{\psi-1}\left[\ln\left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)+\frac{\psi-1}{\psi}\int_0^1\hat{Y}_{\eta jt}\,\mathrm{d}j\right] \implies$$

$$\ln Y_\eta + \hat{Y}_{\eta t} = \frac{\psi}{\psi-1}\left[\ln\left(Y_{\eta j}^{\frac{\psi-1}{\psi}}\int_0^1\mathrm{d}j\right)+\frac{\psi-1}{\psi}\int_0^1\hat{Y}_{\eta jt}\,\mathrm{d}j\right] \implies$$

$$\ln Y_\eta + \hat{Y}_{\eta t} = \cancel{\frac{\psi}{\psi-1}}\left[\cancel{\frac{\psi-1}{\psi}}\ln Y_{\eta j}+\cancel{\ln 1}+\cancel{\frac{\psi-1}{\psi}}\int_0^1\hat{Y}_{\eta jt}\,\mathrm{d}j\right] \implies$$

$$\ln Y_\eta + \hat{Y}_{\eta t} = \ln Y_{\eta j}+\int_0^1\hat{Y}_{\eta jt}\,\mathrm{d}j$$

Apply corollary A.2.1:

$$\ln Y_\eta + \hat{Y}_{\eta t} = \ln Y_{\eta j}+\int_0^1\hat{Y}_{\eta jt}\,\mathrm{d}j \implies$$

$$\hat{Y}_{\eta t} = \int_0^1\hat{Y}_{\eta jt}\,\mathrm{d}j \tag{A.122}$$

Production Function

Log-linearize A.26:

$$\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} && \implies \qquad (A.26) \\
Y_{\eta j}(1+\hat{Y}_{\eta jt}) &= Z_{A\eta} K_{\eta j}^{\alpha_\eta} L_{\eta j}^{1-\alpha_\eta}(1+\hat{Z}_{A\eta t}+\alpha_\eta \hat{K}_{\eta jt}+(1-\alpha_\eta)\hat{L}_{\eta jt}) && \implies \\
\hat{Y}_{\eta jt} &= \hat{Z}_{A\eta t}+\alpha_\eta \hat{K}_{\eta jt}+(1-\alpha_\eta)\hat{L}_{\eta jt} && \qquad (A.123)
\end{aligned}$$

Substitute A.123 in A.122:

$$\begin{aligned}
\hat{Y}_{\eta t} &= \int_0^1 \hat{Y}_{\eta jt}\,\mathrm{d}j && \implies \qquad (A.122) \\
\hat{Y}_{\eta t} &= \int_0^1 \left[\hat{Z}_{A\eta t}+\alpha_\eta \hat{K}_{\eta jt}+(1-\alpha_\eta)\hat{L}_{\eta jt}\right]\mathrm{d}j && \implies \\
\hat{Y}_{\eta t} &= \hat{Z}_{A\eta t}+\alpha_\eta \int_0^1 \hat{K}_{\eta jt}\,\mathrm{d}j+(1-\alpha_\eta)\int_0^1 \hat{L}_{\eta jt}\,\mathrm{d}j && \qquad (A.124)
\end{aligned}$$

Apply the natural logarithm and then log-linearize A.51:

$$\begin{aligned}
L_{\eta t} &= \int_0^1 L_{\eta jt}\,\mathrm{d}j && \implies \qquad (A.51) \\
\ln L_{\eta t} &= \ln\left[\int_0^1 L_{\eta jt}\,\mathrm{d}j\right] && \implies \\
\ln L+\hat{L}_{\eta t} &= \ln\left[\int_0^1 L_{\eta j}\,\mathrm{d}j\right]+\int_0^1 \hat{L}_{\eta jt}\,\mathrm{d}j && \implies \\
\ln L+\hat{L}_{\eta t} &= \ln L_{\eta j}+\ln 1+\int_0^1 \hat{L}_{\eta jt}\,\mathrm{d}j
\end{aligned}$$

Apply corollary A.2.1:

$$\implies \hat{L}_{\eta t} = \int_0^1 \hat{L}_{\eta jt}\,\mathrm{d}j \qquad (A.125)$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_{\eta t} = \int_0^1 \hat{K}_{\eta jt}\,\mathrm{d}j \qquad (A.126)$$

Substitute A.125 and A.126 in A.124:

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t}+\alpha_\eta \hat{K}_{\eta t}+(1-\alpha_\eta)\hat{L}_{\eta t} \qquad (A.127)$$

Technical and Economic Marginal Rates of Substitution (TMRS and EMRS)

Log-linearize A.30 and then apply A.125 and A.126:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_{\eta t}}{R_t} \tag{A.30}$$

$$\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{W}_{\eta t} - \hat{R}_t \tag{A.128}$$

Marginal Cost

Log-linearize A.35:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_{\eta t}}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies \tag{A.35}$$

$$\hat{\Lambda}_{\eta t} = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_{\eta t} - \hat{Z}_{A\eta t} \tag{A.129}$$

Substitute A.109 in A.129:

$$\begin{aligned}
\hat{\Lambda}_{\eta t} &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_{\eta t} - \hat{Z}_{A\eta t} && \implies \\
\hat{P}_{\eta t} + \hat{\lambda}_{\eta t} &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_{\eta t} - \hat{Z}_{A\eta t} && \implies \\
\hat{\lambda}_{\eta t} &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_{\eta t} - \hat{Z}_{A\eta t} - \hat{P}_{\eta t}
\end{aligned} \tag{A.130}$$

Monetary Policy

Log-linearize A.44:

$$\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi} Y_t^{\gamma_Y})^{(1-\gamma_R)} Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \implies \tag{A.44}$$

$$\begin{aligned}
\frac{R(1+\hat{R}_t)}{R} &= \frac{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)} Z_M}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \cdot \\
&\quad \cdot [1 + \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt}] \implies
\end{aligned}$$

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{A.131}$$

National Gross Inflation Rate

Log-linearize A.45:

$$\pi_t = \pi_{1t}^{\theta_\pi} \pi_{2t}^{1-\theta_\pi} \implies \tag{A.45}$$

$$\hat{\pi}_t = \theta_\pi \hat{\pi}_{1t} + (1-\theta_\pi)\hat{\pi}_{2t} \tag{A.132}$$

Productivity Shock

Log-linearize A.47:

$$
\begin{aligned}
\ln Z_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}\ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \implies && \text{(A.47)}\\
\ln Z_{A\eta} + \hat{Z}_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}(\ln Z_{A\eta} + \hat{Z}_{A\eta,t-1}) + \varepsilon_{A\eta} \implies \\
\hat{Z}_{A\eta t} &= \rho_{A\eta}\hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} && \text{(A.133)}
\end{aligned}
$$

Monetary Shock

Log-linearize A.48:

$$
\begin{aligned}
\ln Z_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \implies && \text{(A.48)}\\
\ln Z_M + \hat{Z}_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M(\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M \implies \\
\hat{Z}_{Mt} &= \rho_M \hat{Z}_{M,t-1} + \varepsilon_M && \text{(A.134)}
\end{aligned}
$$

Goods-Market Clearing Condition

Log-linearize A.49:

$$
\begin{aligned}
Y_t &= Y_{1t} + Y_{2t} && \text{(A.49)}\\
Y(1+\hat{Y}_t) &= Y_1(1+\hat{Y}_{1t}) + Y_2(1+\hat{Y}_{2t}) \implies \\
\hat{Y}_t &= \frac{Y_1}{Y}\hat{Y}_{1t} + \frac{Y_2}{Y}\hat{Y}_{2t} && \text{(A.135)}
\end{aligned}
$$

Define the regional weights $\langle \theta_Y \quad (1-\theta_Y)\rangle$ in the production total:

$$
\left\langle \theta_Y \quad (1-\theta_Y) \right\rangle := \left\langle \tfrac{Y_1}{Y} \quad \tfrac{Y_2}{Y} \right\rangle \tag{A.136}
$$

Substitute A.136 in A.135:

$$
\hat{Y}_t = \theta_Y \hat{Y}_{1t} + (1-\theta_Y)\hat{Y}_{2t} \tag{A.137}
$$

Regional Goods-Market Clearing Condition

Log-linearize A.50:

$$Y_{\eta t} = C_{\eta t} + I_{\eta t} \implies \tag{A.50}$$

$$Y_{\eta}(1+\hat{Y}_{\eta t}) = C_{\eta}(1+\hat{C}_{\eta t}) + I_{\eta}(1+\hat{I}_{\eta t}) \implies$$

$$\hat{Y}_{\eta t} = \frac{C_{\eta}}{Y_{\eta}}\hat{C}_{\eta 1t} + \frac{I_{\eta}}{Y_{\eta}}\hat{I}_{\eta t} \tag{A.138}$$

Define the consumption and investment weights $\left\langle \theta_{C\eta} \quad (1-\theta_{C\eta}) \right\rangle$ in the regional production:

$$\left\langle \theta_{C\eta} \quad (1-\theta_{C\eta}) \right\rangle := \left\langle \tfrac{C_{\eta}}{Y_{\eta}} \quad \tfrac{I_{\eta}}{Y_{\eta}} \right\rangle \tag{A.139}$$

Substitute A.139 in A.138:

$$\hat{Y}_{\eta t} = \theta_{C\eta}\hat{C}_{\eta t} + (1-\theta_{C\eta})\hat{I}_{\eta t} \tag{A.140}$$

Budget Constraint

Log-linearize A.55: and then apply A.88 and A.122:

$$Q_{\eta t}C_{\eta t} + P_{\eta t}I_{\eta t} = R_t K_{\eta t} + P_{\eta t}\int_0^1 Y_{\eta jt}\,\mathrm{d}\,j \implies \tag{A.55}$$

$$Q_{\eta}C_{\eta}(1+\hat{Q}_{\eta t}+\hat{C}_{\eta t}) + P_{\eta}I_{\eta}(1+\hat{P}_{\eta t}+\hat{I}_{\eta t}) =$$
$$= RK_{\eta}(1+\hat{R}_t+\hat{K}_{\eta t}) + P_{\eta}\int_0^1 Y_{\eta j}\,\mathrm{d}\,j(1+\hat{P}_{\eta t}+\int_0^1 \hat{Y}_{\eta jt}\,\mathrm{d}\,j) \implies$$

$$\begin{aligned} &Q_{\eta}C_{\eta}(\hat{Q}_{\eta t}+\hat{C}_{\eta t}) + P_{\eta}I_{\eta}(\hat{P}_{\eta t}+\hat{I}_{\eta t}) = \\ &\quad = RK_{\eta}(\hat{R}_t+\hat{K}_{\eta t}) + P_{\eta}Y_{\eta}(\hat{P}_{\eta t}+\hat{Y}_{\eta t}) \end{aligned} \tag{A.141}$$

A.1.12 Log-linear Model Structure

The log-linear model is a square system of 30 variables and equations, summarized as follows:

- Variables:
  - Real Variables: $\langle \hat{C}_\eta \quad \hat{L}_\eta \quad \hat{K}_\eta \quad \hat{I}_\eta \quad \hat{C}_{\eta 1} \quad \hat{C}_{\eta 2} \quad \hat{Y}_\eta \quad \hat{Y} \quad \hat{Z}_{A\eta} \quad \hat{Z}_M \rangle$;
  - Nominal Variables: $\langle \hat{Q}_\eta \quad \hat{P}_\eta \quad \hat{R} \quad \hat{\pi} \quad \hat{W}_\eta \quad \hat{\lambda}_\eta \quad \hat{\pi}_\eta \rangle$.
- Equations:
  1. Regional Gross Inflation Rate
  $$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{A.101}$$
  2. New Keynesian Phillips Curve
  $$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{\eta t} \tag{A.114}$$
  3. Regional Consumption Weight
  $$\hat{C}_{\eta 2t} - \hat{C}_{\eta 1t} = \hat{P}_{1t} - \hat{P}_{2t} \tag{A.115}$$
  4. Regional Consumption of Good 1
  $$\hat{C}_{\eta t} - \hat{C}_{\eta 1t} = (1-\omega_{\eta 1})(\hat{P}_{1t} - \hat{P}_{2t}) \tag{A.116}$$
  5. Regional Price Index
  $$\hat{Q}_{\eta t} = \omega_{\eta 1}\hat{P}_{1t} + (1-\omega_{\eta 1})\hat{P}_{2t} \tag{A.117}$$
  6. Labor Supply
  $$\varphi \hat{L}_{\eta t} + \sigma \hat{C}_{\eta t} = \hat{W}_{\eta t} - \hat{Q}_{\eta t} \tag{A.118}$$
  7. Law of Motion for Capital
  $$\hat{K}_{\eta,t+1} = (1-\delta)\hat{K}_{\eta t} + \delta \hat{I}_{\eta t} \tag{A.119}$$
  8. Euler equation for capital return
  $$\begin{aligned}(\hat{Q}_{\eta,t+1} - \hat{Q}_{\eta t}) + \sigma(\hat{C}_{\eta,t+1} - \hat{C}_{\eta t}) - (\hat{P}_{\eta,t+1} - \hat{P}_{\eta,t}) = \\ = \beta r(\hat{R}_{\eta,t+1} - \hat{P}_{\eta,t+1})\end{aligned} \tag{A.120}$$
  9. Production Function
  $$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{\eta t} + (1-\alpha_\eta)\hat{L}_{\eta t} \tag{A.127}$$
  10. Technical and Economic Marginal Rates of Substitution
  $$\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{W}_{\eta t} - \hat{R}_t \tag{A.128}$$
  11. Marginal Cost
  $$\hat{\lambda}_{\eta t} = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_{\eta t} - \hat{Z}_{A\eta t} - \hat{P}_{\eta t} \tag{A.130}$$

12. Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \quad \text{(A.131)}$$

13. National Gross Inflation Rate

$$\hat{\pi}_t = \theta_\pi \hat{\pi}_{1t} + (1-\theta_\pi)\hat{\pi}_{2t} \quad \text{(A.132)}$$

14. Productivity Shock

$$\hat{Z}_{A\eta t} = \rho_{A\eta} \hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} \quad \text{(A.133)}$$

15. Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \quad \text{(A.134)}$$

16. Goods-Market Clearing Condition

$$\hat{Y}_t = \theta_Y \hat{Y}_{1t} + (1-\theta_Y)\hat{Y}_{2t} \quad \text{(A.137)}$$

17. Regional Goods-Market Clearing Condition

$$\hat{Y}_{\eta t} = \theta_{C\eta} \hat{C}_{\eta t} + (1-\theta_{C\eta})\hat{I}_{\eta t} \quad \text{(A.140)}$$

A.1.13 Eigenvalues and Forward Looking Variables

As it stands, the model has more forward-looking variables than eigenvalues greater than one, indicating that the model is indeterminate. To transform the model into one with a single solution, the number of eigenvalues and forward-looking variables must be equal. To address this, **farmer_solving_2015** employs a method where excess forward-looking variables are substituted with an expectational variable at time t, along with a expectational shock $sunspot_\eta$, representing the deviation between the expected and the realized values. For the present model, the variables created are the expected regional gross inflation rates $\pi^X_{\eta t}$ and the expected capital deviation $K^X_{\eta t}$:

$$\pi^X_{\eta t} = \mathbb{E}_t \hat{\pi}_{\eta,t+1} \tag{A.142}$$

$$sunspot_\eta = \hat{\pi}_{\eta t} - \pi^X_{\eta,t-1} \tag{A.143}$$

$$K^X_{\eta t} = \hat{K}_{\eta,t+1} \tag{A.144}$$

$$sunspot_{K\eta} = \hat{K}_{\eta t} - K^X_{\eta,t-1} \tag{A.145}$$

A.2 DEFINITIONS, LEMMAS AND COROLLARIES

The objective of this appendix is to present the definitions and lemmas used throughout the text.

Lemma A.1 (Marginal Cost)**.** *The Lagrangian multiplier Λ_t is the nominal marginal cost of the intermediate-good firm:*

$$MC_t := \frac{\partial TC_t}{\partial Y_t} = \Lambda_t$$

Proof. **simon_mathematics_1994**. ■

Lemma A.2 (Steady State Inflation)**.** *In steady state, prices are stable $P_t = P_{t-1} = P$ and the gross inflation rate is one.*

Proof. Equation A.68. ■

Corollary A.2.1. *In steady state, all firms have the same level of production Y and therefore demand the same amount of factors, capital K and labor L.*

$$P_t = P_{t-1} = P \implies \begin{pmatrix} Y_j & K_j & L_j \end{pmatrix} = \begin{pmatrix} Y & K & L \end{pmatrix}$$

Definition A.1 (Uhlig's Rules)**.** The Uhlig's rules are a set of approximations used to log-linearize equations (**solis-garcia_ucb_2022**).

Lemma A.3 (Rule 1: Percentage Deviation from Steady State)**.**

$$\hat{x}_t := \frac{x_t - x}{x} \iff x_t = x(1 + \hat{x}_t)$$

Corollary A.3.1 (Rule 2: Product)**.**

$$x_t y_t \approx xy(1 + \hat{x}_t + \hat{y}_t)$$

Proof. Apply lemma A.3 to both variables and notice that $\hat{x}_t\hat{y}_t \approx 0$:

$$\begin{aligned} x_t y_t &= x(1 + \hat{x}_t)y(1 + \hat{y}_t) \\ &= xy(1 + \hat{x}_t + \hat{y}_t + \hat{x}_t\hat{y}_t) \\ &\approx xy(1 + \hat{x}_t + \hat{y}_t) \end{aligned}$$

■

Corollary A.3.2 (Rule 3: Exponential)**.**

$$x_t^a \approx x^a(1 + a\hat{x}_t) \quad ; \quad a > 1$$

Proof. Apply corollary A.3.1 $(a - 1)$ times. ■

Corollary A.3.3 (Logarithm Rule)**.**

$$\ln x_t \approx \ln x + \hat{x}_t$$

Proof. Apply corollary A.3 and notice that $\ln(1+x) \approx x$ when $u \to 0_+$. ∎

Definition A.2 (Uhlig's Rules for Level Deviations). Uhlig's rules can be applied to level deviations (when $0 < u_t < 1$) in order to log-linearize equations (**solis-garcia_ucb_2022**).

Lemma A.4 (Level Deviation from Steady State when $0 \leq u \leq 1$).

$$\widetilde{u}_t := u_t - u$$

Corollary A.4.1 (Level Deviation from Steady State when $0 < u < 1$).

$$u_t = u\left(1 + \frac{\widetilde{u}_t}{u}\right)$$

Lemma A.5 (Level Deviation of the Present Value Discount Factor). *The level deviation of the present value discount factor is equivalent to (***solis-garcia_ucb_2022***):*

$$\prod_{k=0}^{s-1}(1+R_{t+k}) = (1+R)^s\left(1 + \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)$$

Proof. Substitute the interest rate by the gross interest rate $GR_t = 1 + R_t$ and apply corollary A.4.1:

$$\begin{aligned}
&\prod_{k=0}^{s-1}(1+R_{t+k}) = \prod_{k=0}^{s-1}(GR_{t+k}) && \Longrightarrow \\
&GR \times \cdots \times GR\left(1 + \frac{1}{GR}\widetilde{GR}_t + \frac{1}{GR}\widetilde{GR}_{t+1} + \cdots + \frac{1}{GR}\widetilde{GR}_{t+s-1}\right) && \Longrightarrow \\
&GR^s\left(1 + \frac{1}{GR}\sum_{k=0}^{s-1}\widetilde{GR}_{t+k}\right) && \Longrightarrow \\
&(1+R)^s\left(1 + \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)
\end{aligned}$$

∎

Definition A.3 (Geometric Series). A geometric series S_∞ is the sum of the terms of a geometric sequence.

$$S_\infty = \sum_{i=0}^{\infty} ar^i \implies S_\infty = \frac{a}{1-r}\,, \ |r| < 1$$

Definition A.4 (Lag Operator). The lag operator $\mathbb{L}$ is a mathematical operator that represents the backshift or lag of a time series (**solis-garcia_ucb_2022**):

$$\begin{aligned}
\mathbb{L}x_t &= x_{t-1} \\
(1 + a\mathbb{L})y_{t+2} &= y_{t+2} + ay_{t+1}
\end{aligned}$$

Corollary A.5.1 (Lead Operator). *Analogously, the lead operator $\mathbb{L}^{-1}$ (or inverse lag operator) yields a vari-*

*able's lead (***solis-garcia_ucb_2022***):*

$$\mathbb{L}^{-1}x_t = x_{t+1}$$
$$(1 + a\mathbb{L}^{-1})y_{t+2} = y_{t+2} + ay_{t+3}$$

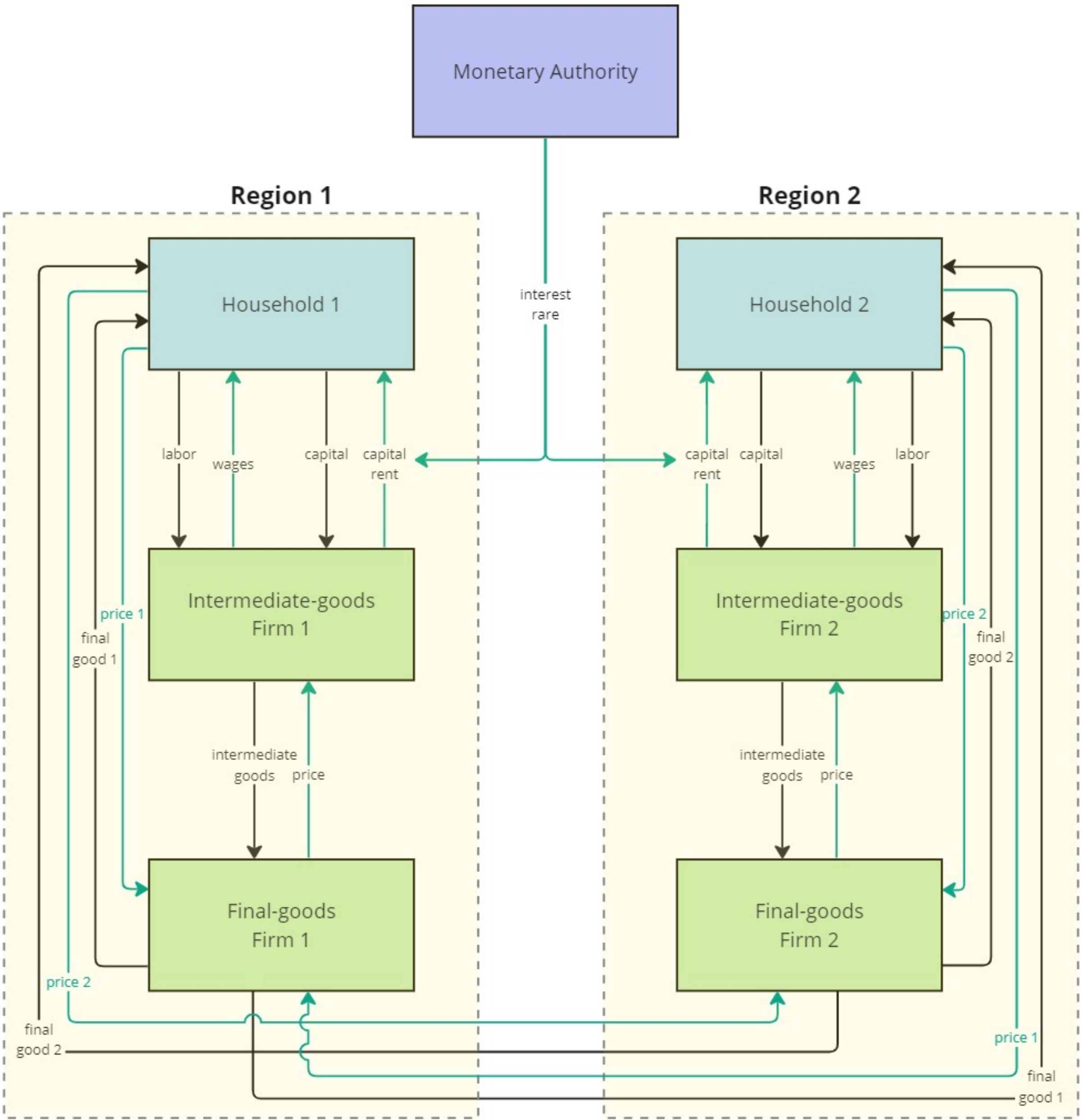
A.3 DYNARE MOD FILE

The Dynare mod file for the model discussed in this thesis is available on my GitHub repository:

`https://github.com/andrlb/mastersthesis`

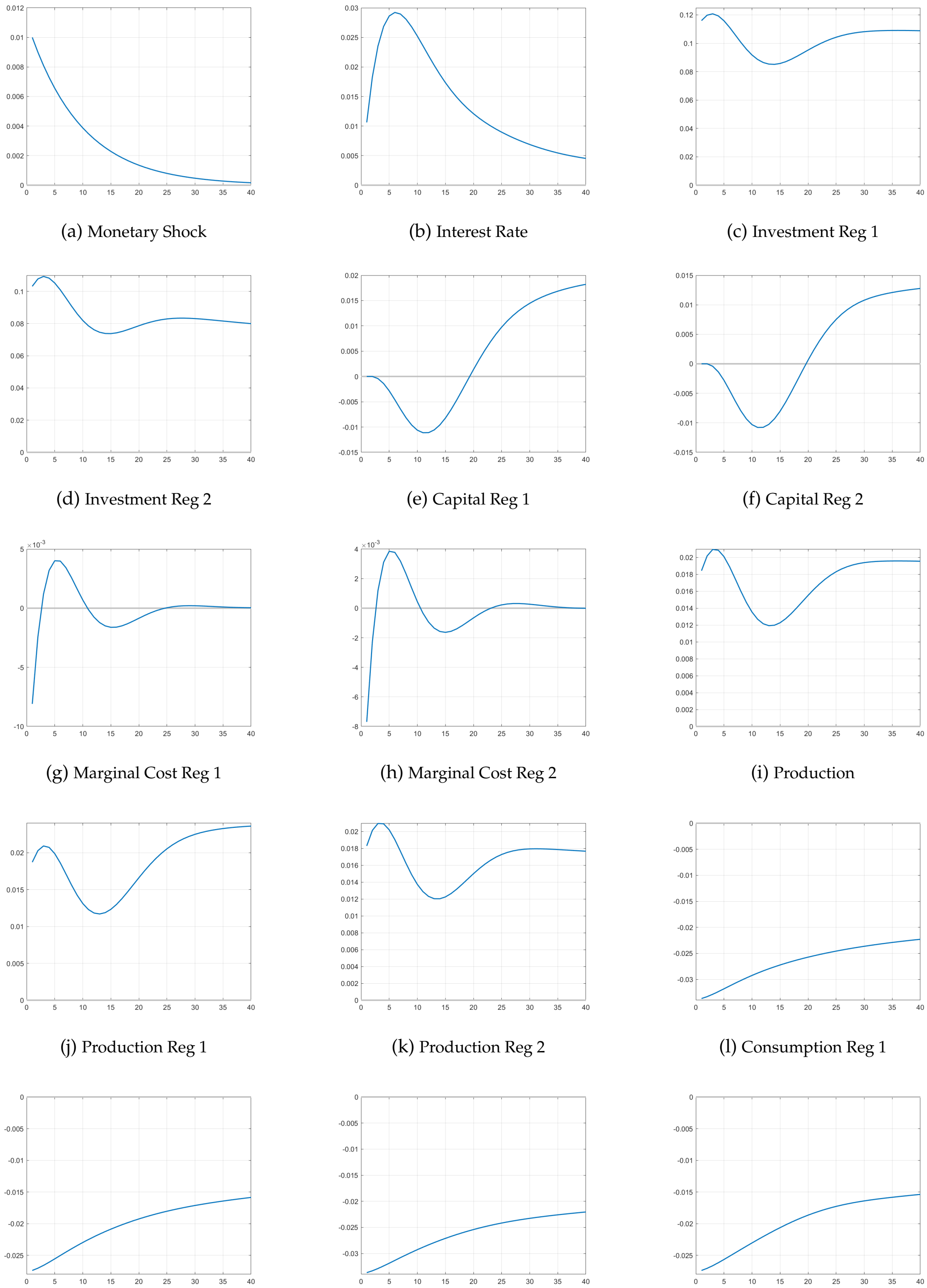
This file is provided to facilitate replication of the model and exploration of its implementation. The code is compatible with Dynare 5.5, and the commit history on GitHub provides details about the evolution of the codebase.

Figure 1: Model Diagram



Source: created by the author.

Figure 2: Positive-Monetary-Shock Impulse Response Functions



(a) Monetary Shock (b) Interest Rate (c) Investment Reg 1

(d) Investment Reg 2 (e) Capital Reg 1 (f) Capital Reg 2

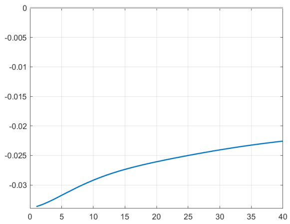
(g) Marginal Cost Reg 1 (h) Marginal Cost Reg 2 (i) Production

(j) Production Reg 1 (k) Production Reg 2 (l) Consumption Reg 1

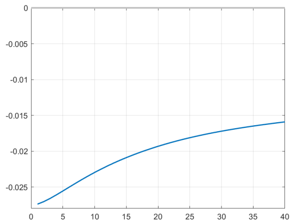
(m) Consumption Reg 2 (n) Region 1, Good 1 (o) Region 2, Good 1

Source: created by the author.

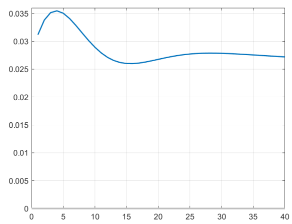
Figure 3: Positive-Monetary-Shock Impulse Response Functions, part 2



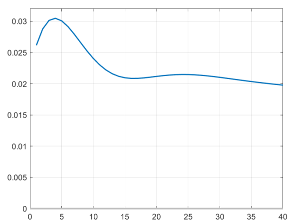
(a) Region 1, Good 2



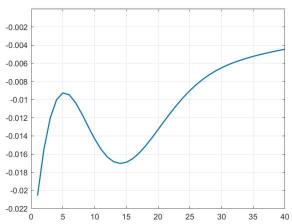
(b) Region 2, Good 2



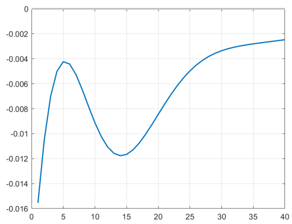
(c) Labor Reg 1



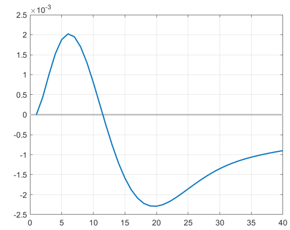
(d) Labor Reg 2



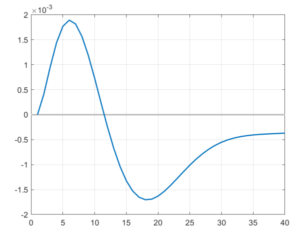
(e) Wages Reg 1



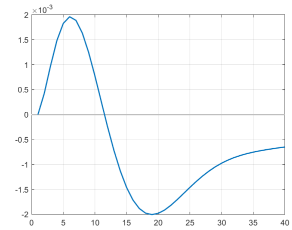
(f) Wages Reg 2



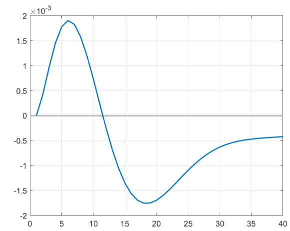
(g) Price Level Reg 1



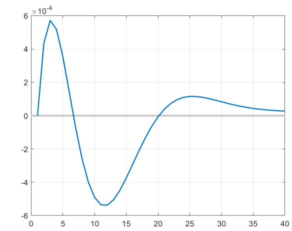
(h) Price Level Reg 2



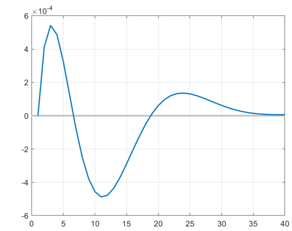
(i) Region 1 CPL*



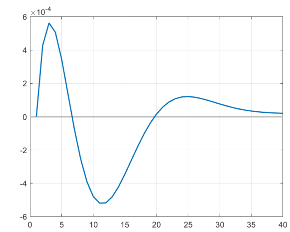
(j) Region 1 CPL*



(k) Inflation Region 1



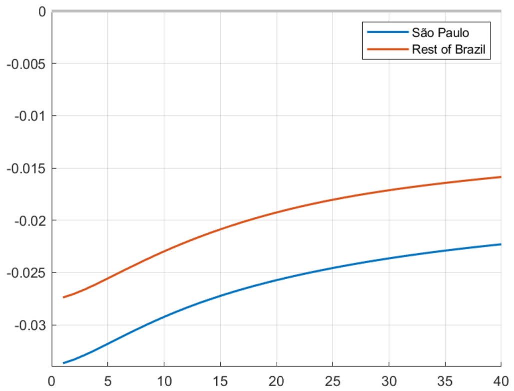
(l) Inflation Region 2



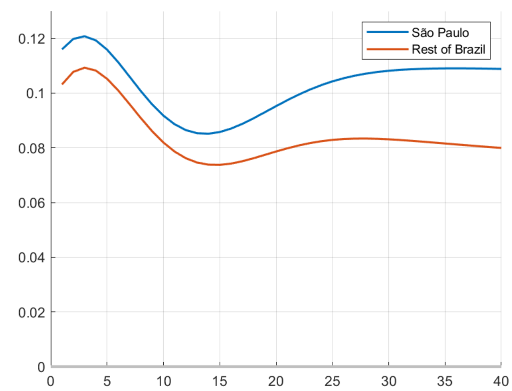
(m) Inflation

Source: created by the author.

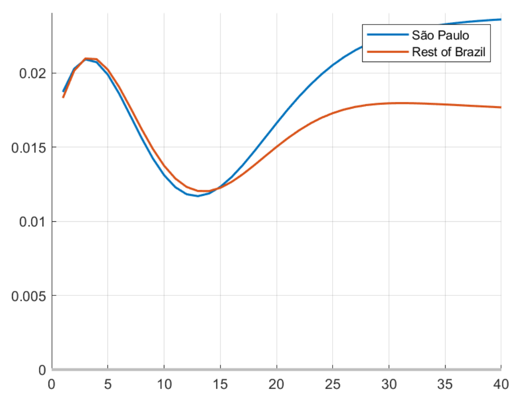
Figure 4: Positive-Monetary-Shock Paired Impulse Response Functions



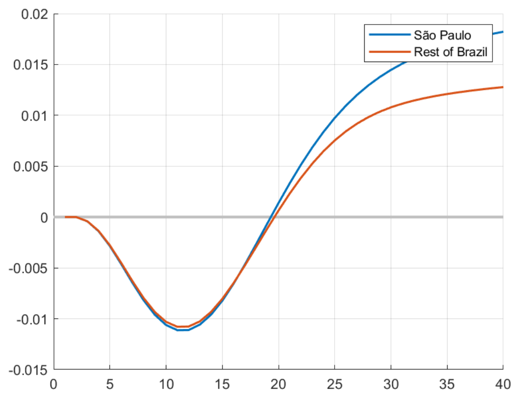
(a) Consumption



(b) Investment



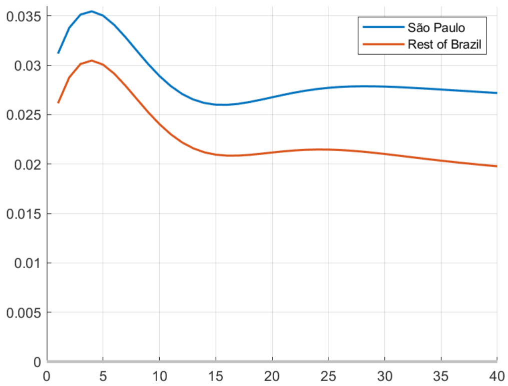
(c) Production



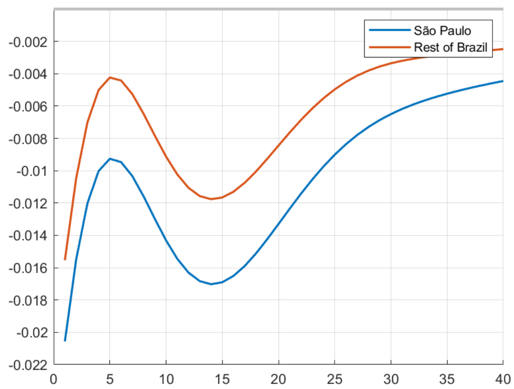
(d) Capital

Source: created by the author.

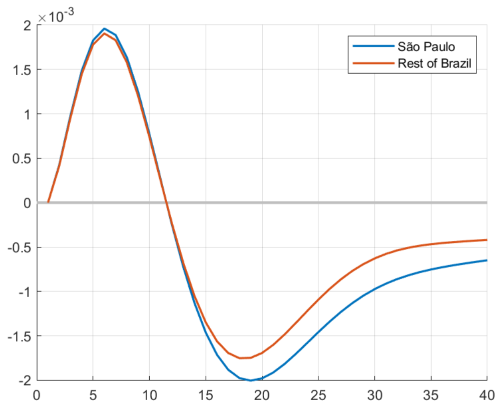
Figure 5: Positive-Monetary-Shock Paired Impulse Response Functions, part 2



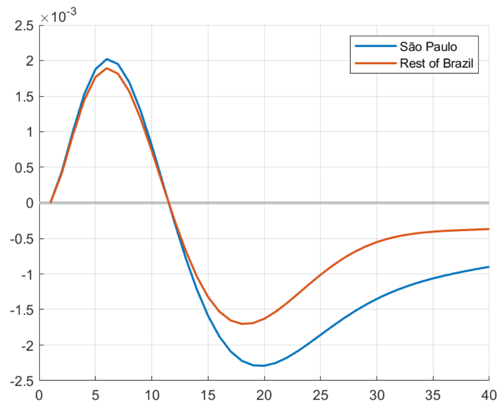
(a) Labor



(b) Wages



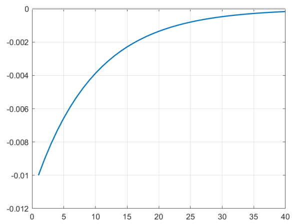
(c) Consumer Price Level



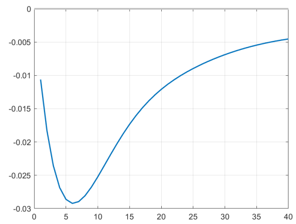
(d) Price Level

Source: created by the author.

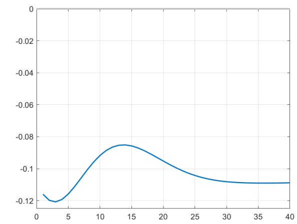
Figure 6: Negative-Monetary-Shock Impulse Response Functions



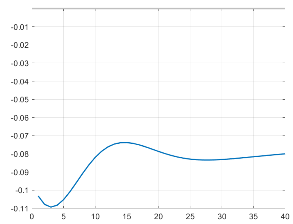
(a) Monetary Shock



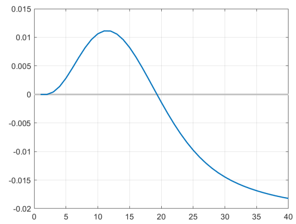
(b) Interest Rate



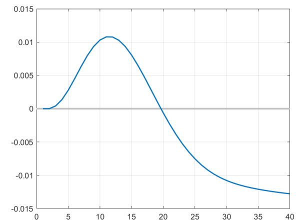
(c) Investment Reg 1



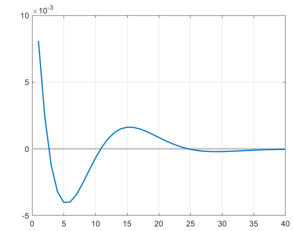
(d) Investment Reg 2



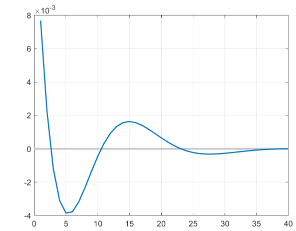
(e) Capital Reg 1



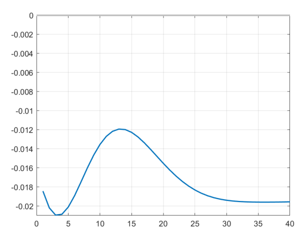
(f) Capital Reg 2



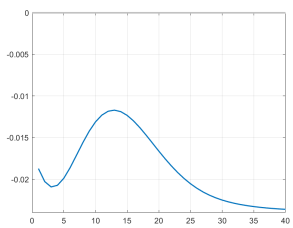
(g) Marginal Cost Reg 1



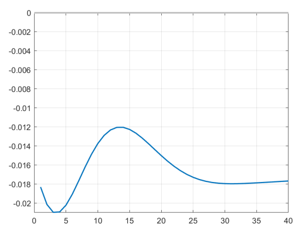
(h) Marginal Cost Reg 2



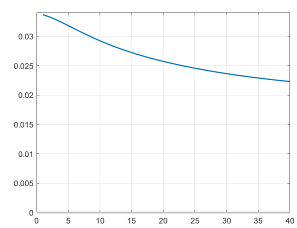
(i) Production



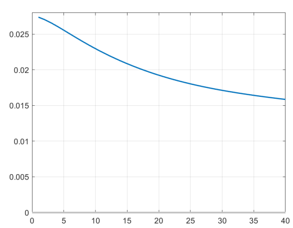
(j) Production Reg 1



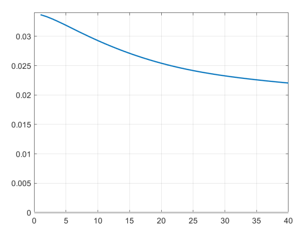
(k) Production Reg 2



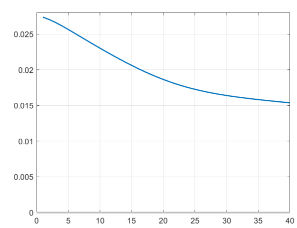
(l) Consumption Reg 1



(m) Consumption Reg 2



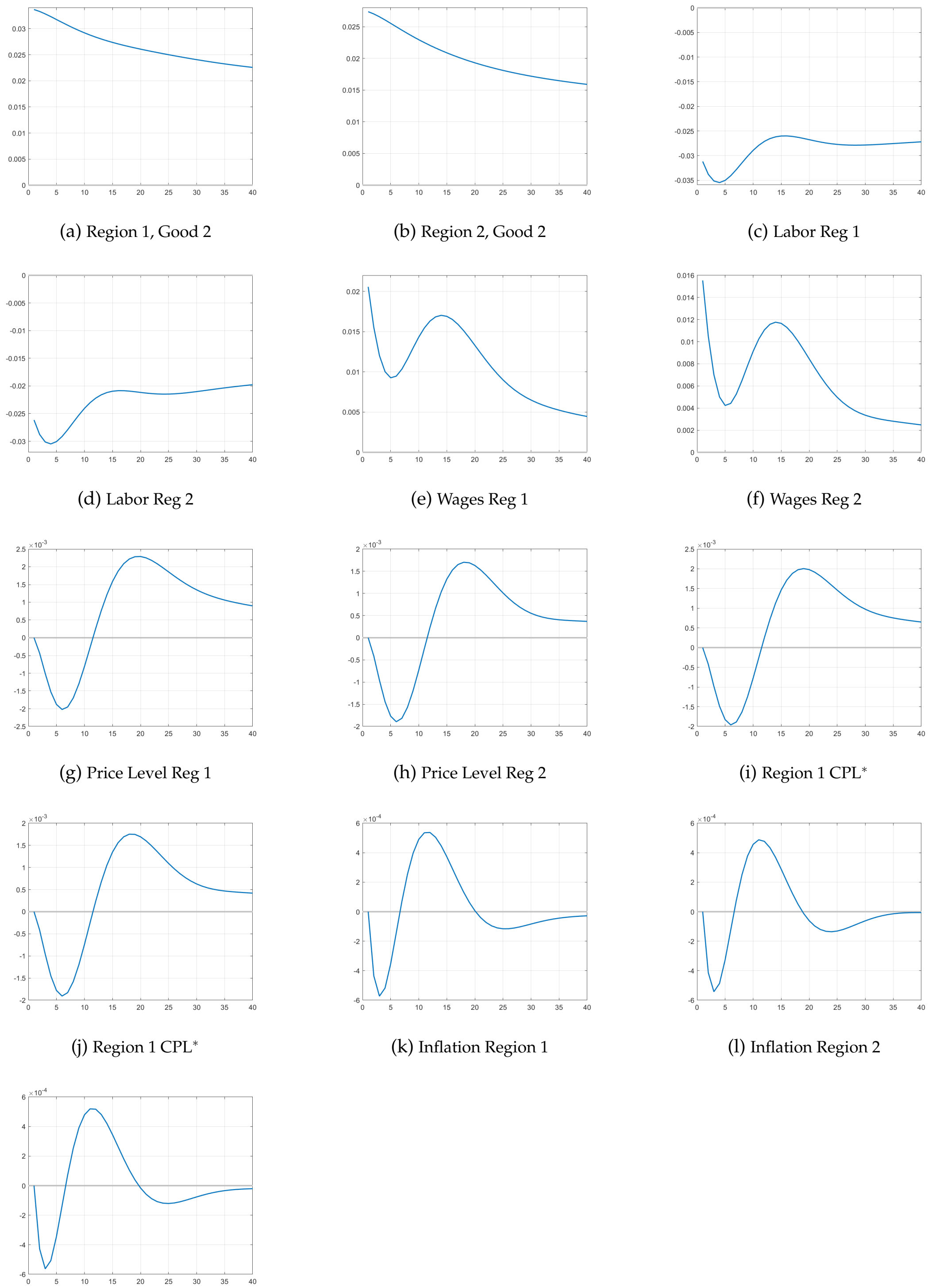
(n) Region 1, Good 1



(o) Region 2, Good 1

Source: created by the author.

Figure 7: Negative-Monetary-Shock Impulse Response Functions, part 2



(a) Region 1, Good 2

(b) Region 2, Good 2

(c) Labor Reg 1

(d) Labor Reg 2

(e) Wages Reg 1

(f) Wages Reg 2

(g) Price Level Reg 1

(h) Price Level Reg 2

(i) Region 1 CPL*

(j) Region 1 CPL*

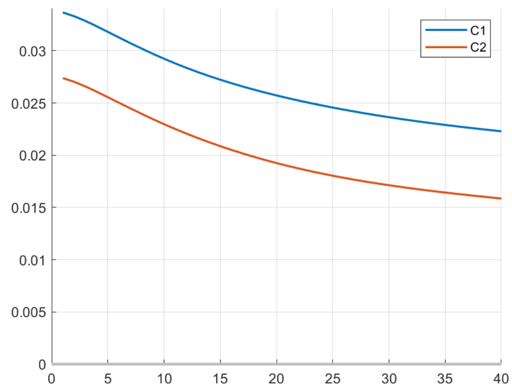
(k) Inflation Region 1

(l) Inflation Region 2

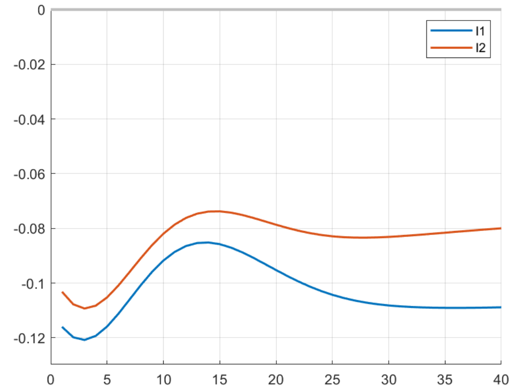
(m) Inflation

Source: created by the author.

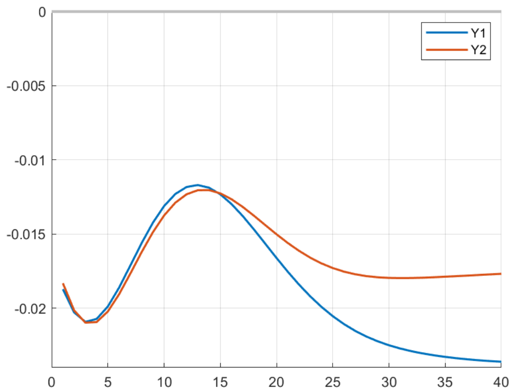
Figure 8: Negative-Monetary-Shock Paired Impulse Response Functions



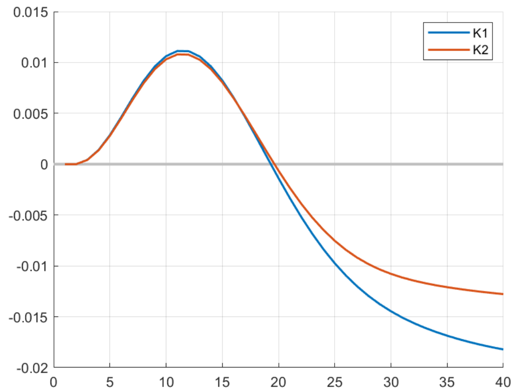
(a) Consumption



(b) Investment



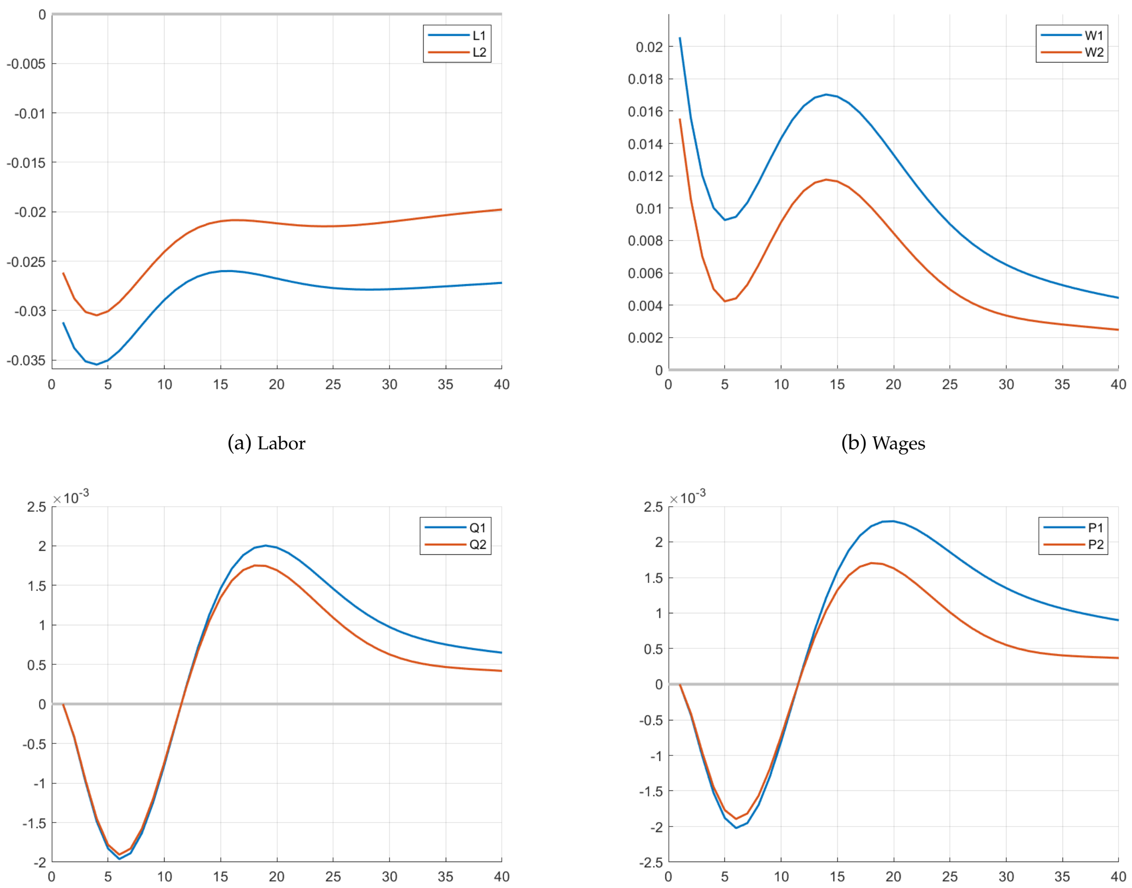
(c) Production



(d) Capital

Source: created by the author.

Figure 9: Negative-Monetary-Shock Paired Impulse Response Functions, part 2



(a) Labor

(b) Wages

(c) Consumer Price Level

(d) Price Level

Source: created by the author.