

### 3.1.1 Household

#### Utility Maximization Problem

Following the models presented by Costa Junior (2016) and Solis-Garcia (2022), the representative household problem is to maximize an intertemporal utility function  $U_\eta$  with respect to consumption  $C_{\eta t}$  and labor  $L_{\eta t}$ , subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_{\eta t}, L_{\eta t}, B_{\eta t}, K_{\eta,t+1}} : U_\eta(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left( \frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \quad (3.1)$$

$$\begin{aligned} \text{s. t. : } & P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} + P_{\eta t} I_{\eta t} + B_{\eta t} = \\ & = W_t L_{\eta t} + R_{Kt} K_{\eta t} + (1 + R_{t-1}) B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \quad (3.2)$$

$$K_{\eta,t+1} = (1 - \delta) K_{\eta t} + I_{\eta t} \quad (3.3)$$

$$C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \quad (3.4)$$

$$C_{\eta t}, L_{\eta t}, K_{\eta,t+1} > 0 ; K_0 \text{ given.}$$

where  $\mathbb{E}_t$  is the expectation operator,  $\beta$  is the intertemporal discount factor,  $\sigma$  is the relative risk aversion coefficient,  $\phi$  is the relative labor weight in utility,  $\varphi$  is the marginal disutility of labor supply. In the budget constraint,  $P_{1t}$  and  $P_{2t}$  are the prices of goods 1 and 2, respectively,  $C_{\eta 1t}$  and  $C_{\eta 2t}$  are the goods produced in region 1 and 2, respectively, and consumed in region  $\eta$ ,  $I_{\eta t}$  is the investment,  $B_{\eta t}$  are the bonds,  $W_t$  is the wage level,  $K_{\eta t}$  is the capital,  $R_{Kt}$  is the return on capital,  $R_t$  is the return on bonds (which is also the nominal interest rate of the economy) and  $\Pi_{\eta t}$  is the firm profit. In the capital accumulation rule,  $\delta$  is the capital depreciation rate. In the consumption aggregation,  $\omega_{\eta 1}$  and  $1 - \omega_{\eta 1}$  are the weights of goods  $C_{\eta 1t}$  and  $C_{\eta 2t}$ , respectively, in the consumption bundle  $C_{\eta t}$ .

To solve the household problem, first substitute 3.4 in 3.1:

$$U_\eta(C_{\eta 1t}, C_{\eta 2t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left( \frac{\left[ C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \quad (3.5)$$

Isolate  $I_{\eta t}$  in 3.3 and substitute in 3.2:

$$K_{\eta,t+1} = (1 - \delta)K_{\eta t} + I_{\eta t} \implies I_{\eta t} = K_{\eta,t+1} - (1 - \delta)K_{\eta t} \quad (3.3)$$

$$\begin{aligned} P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} &= \\ &= W_tL_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \implies \end{aligned} \quad (3.2)$$

$$\begin{aligned} P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1 - \delta)K_{\eta t}] + B_{\eta t} &= \\ &= W_tL_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \quad (3.6)$$

## Lagrangian

The maximization problem with restrictions can be transformed into one without restriction using the Lagrangian function  $\mathcal{L}$  formed by 3.5 and 3.6:

$$\begin{aligned} \mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \left( \frac{\left[ C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) - \right. \\ \left. - \mu_t \left[ P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1 - \delta)K_{\eta t}] + B_{\eta t} - \right. \right. \\ \left. \left. - (W_tL_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t}) \right] \right\} \end{aligned} \quad (3.7)$$

## First Order Conditions

The first order conditions are:

$$C_{\eta 1t} : \mu_t = \frac{\omega_{\eta 1} C_{\eta 1t}^{\omega_{\eta 1}(1-\sigma)-1} C_{\eta 2t}^{(1-\omega_{\eta 1})(1-\sigma)}}{P_{1t}} = \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} C_{\eta t}^{1-\sigma} \quad (3.8)$$

$$C_{\eta 2t} : \mu_t = \frac{(1 - \omega_{\eta 1}) C_{\eta 1t}^{\omega_{\eta 1}(1-\sigma)} C_{\eta 2t}^{(1-\omega_{\eta 1})(1-\sigma)-1}}{P_{2t}} = \frac{(1 - \omega_{\eta 1})}{P_{2t} C_{\eta 2t}} C_{\eta t}^{1-\sigma} \quad (3.9)$$

$$L_{\eta t} : -\phi L_{\eta t}^\varphi + \mu_t W_t = 0 \implies \mu_t = \frac{\phi L_{\eta t}^\varphi}{W_t} \quad (3.10)$$

$$B_{\eta t} : \beta^t \{-\mu_t\} + \mathbb{E}_t \beta^{t+1} \{-\mu_{t+1}[-(1 + R_t)]\} = 0 \implies \mu_t = \beta(1 + R_t) \mathbb{E}_t \mu_{t+1} \quad (3.11)$$

$$K_{\eta, t+1} : -\mu_t P_{\eta t} + \mathbb{E}_t \beta \{\mu_{t+1}[(1 - \delta) P_{\eta, t+1} + R_{K, t+1}]\} = 0 \implies \mu_t P_{\eta t} = \beta \mathbb{E}_t \{\mu_{t+1}[(1 - \delta) P_{\eta, t+1} + R_{K, t+1}]\} \quad (3.12)$$

$$\begin{aligned} \mu_t : & P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} + P_{\eta t} [K_{\eta, t+1} - (1 - \delta) K_{\eta t}] + B_{\eta t} = \\ & = W_t L_{\eta t} + R_{Kt} K_{\eta t} + (1 + R_{t-1}) B_{\eta, t-1} + \Pi_{\eta t} \end{aligned} \quad (3.6)$$

## Solutions

Match 3.8 and 3.9:

$$\begin{aligned} \mu_t &= \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} C_{\eta t}^{1-\sigma} = \frac{(1 - \omega_{\eta 1})}{P_{2t} C_{\eta 2t}} C_{\eta t}^{1-\sigma} \implies \\ \frac{C_{\eta 1t}}{C_{\eta 2t}} &= \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1 - \omega_{\eta 1}} \iff C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1 - \omega_{\eta 1}} \cdot C_{\eta 2t} \end{aligned} \quad (3.13)$$

Equation 3.13 is the relative consumption of regional goods in region  $\eta$ .

Substitute 3.13 in 3.4:

$$C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \implies \quad (3.4)$$

$$\begin{aligned} C_{\eta t} &= \left[ \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1 - \omega_{\eta 1}} \cdot C_{\eta 2t} \right]^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \implies \\ C_{\eta 2t} &= C_{\eta t} \left[ \frac{P_{1t}}{P_{2t}} \cdot \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \end{aligned} \quad (3.14)$$

Substitute 3.14 in 3.13:

$$C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1 - \omega_{\eta 1}} \cdot C_{\eta 2t} \implies \quad (3.13)$$

$$\begin{aligned} C_{\eta 1t} &= \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1 - \omega_{\eta 1}} \cdot C_{\eta t} \left[ \frac{P_{1t}}{P_{2t}} \cdot \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \implies \\ C_{\eta 1t} &= C_{\eta t} \left[ \frac{P_{1t}}{P_{2t}} \cdot \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}-1} \end{aligned} \quad (3.15)$$

Define the total goods expense  $\mathcal{E}_{\eta t}$  of household  $\eta$ :

$$\mathcal{E}_{\eta t} = P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} \quad (3.16)$$

Substitute 3.14 and 3.15 in 3.16:

$$\begin{aligned} \mathcal{E}_{\eta t} &= P_{1t}C_{\eta t} \left[ \frac{P_{1t}}{P_{2t}} \cdot \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}-1} + P_{2t}C_{\eta t} \left[ \frac{P_{1t}}{P_{2t}} \cdot \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \implies \\ \mathcal{E}_{\eta t} &= C_{\eta t} \left[ \frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_{2t}}{1 - \omega_{\eta 1}} \right]^{1-\omega_{\eta 1}} \end{aligned} \quad (3.17)$$

Equation 3.17 shows that the total expense with goods is proportional to the goods' prices  $P_{1t}$  and  $P_{2t}$ .

Rewrite 3.14 in terms of 3.17:

$$C_{\eta 2t} = C_{\eta t} \left[ \frac{P_{1t}}{P_{2t}} \cdot \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \implies \quad (3.14)$$

$$\begin{aligned} C_{\eta 2t} &= C_{\eta t} \left[ \frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_{2t}}{1 - \omega_{\eta 1}} \right]^{-\omega_{\eta 1}} \left[ \frac{P_{2t}}{1 - \omega_{\eta 1}} \cdot \frac{1 - \omega_{\eta 1}}{P_{2t}} \right] \implies \\ C_{\eta 2t} &= \mathcal{E}_{\eta t} \frac{1 - \omega_{\eta 1}}{P_{2t}} \end{aligned} \quad (3.18)$$

Rewrite 3.15 in terms of 3.17 and isolate  $(\omega_{\eta 1} / P_{1t} C_{\eta 1t})$ :

$$C_{\eta 1t} = C_{\eta t} \left[ \frac{P_{1t}}{P_{2t}} \cdot \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1} - 1} \implies \quad (3.15)$$

$$\begin{aligned} C_{\eta 1t} &= C_{\eta t} \left[ \frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_{2t}}{1 - \omega_{\eta 1}} \right]^{1 - \omega_{\eta 1}} \left[ \frac{\omega_{\eta 1}}{P_{1t}} \right] \implies \\ C_{\eta 1t} &= \mathcal{E}_{\eta t} \frac{\omega_{\eta 1}}{P_{1t}} \iff \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} = \frac{1}{\mathcal{E}_{\eta t}} \end{aligned} \quad (3.19)$$

Substitute 3.19 in 3.8:

$$\mu_t = \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} C_{\eta t}^{1-\sigma} \implies \quad (3.8)$$

$$\mu_t = \frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} \quad (3.20)$$

Match 3.20 and 3.10:

$$\begin{aligned} \mu_t &= \frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \frac{\phi L_{\eta t}^\varphi}{W_t} \implies \\ \frac{\phi L_{\eta t}^\varphi}{C_{\eta t}^{1-\sigma}} &= \frac{W_t}{\mathcal{E}_{\eta t}} \end{aligned} \quad (3.21)$$

Equation 3.21 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute  $\mu_t$  and  $\mu_{t+1}$  from equation 3.20 in 3.11:

$$\mu_t = \beta(1 + R_t) \mathbb{E}_t \mu_{t+1} \implies \quad (3.11)$$

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1 + R_t) \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta,t+1}} \right\} \quad (3.22)$$

Equation 3.22 is the Euler equation for the bonds return.

Substitute  $\mu_t$  and  $\mu_{t+1}$  from equation 3.20 in 3.12:

$$\begin{aligned} \mu_t P_{\eta t} &= \beta \mathbb{E}_t \{ \mu_{t+1} [(1 - \delta) P_{\eta, t+1} + R_{K, t+1}] \} \implies (3.12) \\ \frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} &= \beta \mathbb{E}_t \left\{ \frac{C_{\eta, t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} [(1 - \delta) P_{\eta, t+1} + R_{K, t+1}] \right\} \end{aligned} \quad (3.23)$$

Equation 3.23 is the Euler equation for the capital return.

## Firms

Consider two types of firms: (1) a continuum of intermediate-goods firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-goods firm, which aggregates all the varieties into a final bundle and operates in perfect competition.

### 3.1.2 Final-Goods Firm

#### Profit Maximization Problem

The role of the final-goods firm is to aggregate all the varieties  $Y_{\eta jt}$  produced by the intermediate-goods firms in each region  $\eta \in \{1, 2\}$ , so that the representative consumer can buy only one good  $Y_{\eta t}$ , the bundle good, from each region.

The final-goods firm problem is to maximize its profit, considering that its output is the bundle  $Y_{\eta t}$  formed by a continuum  $j \in [0, 1]$  of intermediate goods  $Y_{\eta jt}$ , with elasticity of substitution between intermediate goods  $\psi$ :

$$\max_{Y_{\eta jt}} : \Pi_{\eta t} = P_{\eta t} Y_{\eta t} - \int_0^1 P_{\eta jt} Y_{\eta jt} \, dj \quad (3.24)$$

$$\text{s. t. : } Y_{\eta t} = \left( \int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \, dj \right)^{\frac{\psi}{\psi-1}} \quad (3.25)$$

Substitute 3.25 in 3.24:

$$\max_{Y_{\eta jt}} : \Pi_{\eta t} = P_{\eta t} \left( \int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \, dj \right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{\eta jt} Y_{\eta jt} \, dj \quad (3.26)$$

## First Order Condition and Solutions

The first order condition is:

$$Y_{\eta jt} : P_{\eta t} \left( \frac{\psi}{\psi - 1} \right) \left( \int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}-1} \left( \frac{\psi-1}{\psi} \right) Y_{\eta jt}^{\frac{\psi-1}{\psi}-1} - P_{\eta jt} = 0 \implies \\ Y_{\eta jt} = Y_t \left( \frac{P_{\eta t}}{P_{\eta jt}} \right)^\psi \quad (3.27)$$

Equation 3.27 shows that the demand for variety  $j$  depends on its relative price.

Substitute 3.27 in 3.25:

$$Y_{\eta t} = \left( \int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}} \implies \\ Y_{\eta t} = \left( \int_0^1 \left[ Y_{\eta t} \left( \frac{P_{\eta t}}{P_{\eta jt}} \right)^\psi \right]^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}} \implies \\ P_{\eta t} = \left[ \int_0^1 P_{\eta jt}^{1-\psi} dj \right]^{\frac{1}{1-\psi}} \quad (3.28)$$

Equation 3.28 is the final-goods firm's markup.

### 3.1.3 Intermediate-Goods Firms

#### Cost Minimization Problem

The intermediate-goods firms, denoted by  $j \in [0, 1]$ , produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose capital  $K_{\eta jt}$  and labor  $L_{\eta jt}$  to minimize production costs, subject to a technology rule.

$$\min_{K_{\eta jt}, L_{\eta jt}} : R_{Kt} K_{\eta jt} + W_t L_{\eta jt} \quad (3.29)$$

$$\text{s. t.} : Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^\alpha L_{\eta jt}^{1-\alpha} \quad (3.30)$$

where  $Y_{\eta jt}$  is the output obtained by the production technology level  $Z_{A\eta t}$  that transforms capital  $K_{\eta jt}$  and labor  $L_{\eta jt}$  in proportions  $\alpha$  and  $(1 - \alpha)$ , respectively, into

intermediate goods.<sup>5</sup>

## Lagrangian

Transform the minimization problem with restriction into one without restriction applying the Lagrangian function  $\mathcal{L}$ :

$$\mathcal{L} = (R_{Kt}K_{\eta jt} + W_tL_{\eta jt}) - \Lambda_{\eta jt}(Z_{A\eta t}K_{\eta jt}^\alpha L_{\eta jt}^{1-\alpha} - Y_{\eta jt}) \quad (3.31)$$

where the Lagrangian multiplier  $\Lambda_{\eta jt}$  is the marginal cost.<sup>6</sup>

## First Order Conditions

The first-order conditions are:

$$K_{\eta jt} : R_{Kt} - \Lambda_{\eta jt} Z_{A\eta t} \alpha K_{\eta jt}^{\alpha-1} L_{\eta jt}^{1-\alpha} = 0 \implies K_{\eta jt} = \alpha Y_{\eta jt} \frac{\Lambda_{\eta jt}}{R_{Kt}} \quad (3.32)$$

$$L_{\eta jt} : W_t - \Lambda_{\eta jt} Z_{A\eta t} K_{\eta jt}^\alpha (1-\alpha) L_{\eta jt}^{-\alpha} = 0 \implies L_{\eta jt} = (1-\alpha) Y_{\eta jt} \frac{\Lambda_{\eta jt}}{W_t} \quad (3.33)$$

$$\Lambda_{\eta jt} : Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^\alpha L_{\eta jt}^{1-\alpha} \quad (3.30)$$

## Solutions

Divide equation 3.32 by 3.33:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \frac{\alpha Y_{\eta jt} \Lambda_{\eta jt} / R_{Kt}}{(1-\alpha) Y_{\eta jt} \Lambda_{\eta jt} / W_t} \implies \frac{K_{\eta jt}}{L_{\eta jt}} = \left( \frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \quad (3.34)$$

Equation 3.34 demonstrates the relationship between the technical marginal rate

<sup>5</sup> the production technology level  $Z_{A\eta t}$  will be submitted to a productivity shock, detailed in section 3.1.5.

<sup>6</sup> see Lemma A.1

of substitution (TMRS) and the economical marginal rate of substitution (EMRS).

Substitute  $L_{\eta jt}$  from equation 3.34 in 3.30:

$$\begin{aligned} Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^\alpha L_{\eta jt}^{1-\alpha} \implies \\ Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^\alpha \left[ \left( \frac{1-\alpha}{\alpha} \right) \frac{R_{Kt} K_{\eta jt}}{W_t} \right]^{1-\alpha} \implies \\ K_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[ \left( \frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha} \end{aligned} \quad (3.35)$$

Equation 3.35 is the intermediate-goods firm demand for capital.

Substitute 3.35 in 3.34:

$$\begin{aligned} L_{\eta jt} &= \left( \frac{1-\alpha}{\alpha} \right) \frac{R_{Kt} K_{\eta jt}}{W_t} \implies \\ L_{\eta jt} &= \left( \frac{1-\alpha}{\alpha} \right) \frac{R_{Kt}}{W_t} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[ \left( \frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha} \implies \\ L_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[ \left( \frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{-\alpha} \end{aligned} \quad (3.36)$$

Equation 3.36 is the intermediate-goods firm demand for labor.

## Total and Marginal Costs

Calculate the total cost  $TC$  using 3.35 and 3.36:

$$\begin{aligned} TC_{\eta jt} &= W_t L_{\eta jt} + R_{Kt} K_{\eta jt} \implies \\ TC_{\eta jt} &= W_t \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[ \left( \frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{-\alpha} + R_{Kt} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[ \left( \frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha} \implies \\ TC_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left( \frac{R_{Kt}}{\alpha} \right)^\alpha \left( \frac{W_t}{1-\alpha} \right)^{1-\alpha} \end{aligned} \quad (3.37)$$

Calculate the marginal cost  $\Lambda$  using 3.37:

$$\Lambda_{\eta jt} = \frac{\partial TC_{\eta jt}}{\partial Y_{\eta jt}} \implies \Lambda_{\eta jt} = \frac{1}{Z_{A\eta t}} \left( \frac{R_{Kt}}{\alpha} \right)^\alpha \left( \frac{W_t}{1-\alpha} \right)^{1-\alpha} \quad (3.38)$$

The marginal cost depends on the technological level  $Z_{A\eta t}$ , the nominal interest rate  $R_{Kt}$  and the nominal wage level  $W_t$ , which are the same for all intermediate-goods firms, and because of that, the index  $j$  may be dropped:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left( \frac{R_{Kt}}{\alpha} \right)^\alpha \left( \frac{W_t}{1-\alpha} \right)^{1-\alpha} \quad (3.39)$$

notice that:

$$\Lambda_{\eta t} = \frac{TC_{\eta jt}}{Y_{\eta jt}} \implies TC_{\eta jt} = \Lambda_{\eta t} Y_{\eta jt} \quad (3.40)$$

## Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule ([CALVO, 1983](#)): each firm has a probability ( $0 < \theta < 1$ ) of keeping its price in the next period ( $P_{\eta j,t+1} = P_{\eta jt}$ ), and a probability of ( $1 - \theta$ ) of setting a new optimal price  $P_{\eta jt}^*$  that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate  $R_t$  of each period, is calculated considering the probability  $\theta$  of keeping the previous price.

$$\max_{P_{\eta jt}} : \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s [P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s}]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \quad (3.41)$$

$$\text{s. t. : } Y_{\eta jt} = Y_{\eta t} \left( \frac{P_{\eta t}}{P_{\eta jt}} \right)^\psi \quad (3.27)$$

Substitute [3.40](#) in [3.41](#):

$$\max_{P_{\eta jt}} : \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s [P_{\eta jt} Y_{\eta j,t+s} - \Lambda_{\eta,t+s} Y_{\eta j,t+s}]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \quad (3.42)$$

Substitute 3.27 in 3.42 and rearrange the variables:

$$\begin{aligned} \max_{P_{\eta jt}} : \quad & \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[ P_{\eta jt} Y_{\eta t+s} \left( \frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^\psi - \Lambda_{\eta,t+s} Y_{\eta t+s} \left( \frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^\psi \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \Rightarrow \\ \max_{P_{\eta jt}} : \quad & \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[ P_{\eta jt}^{1-\psi} P_{\eta,t+s}^\psi Y_{\eta t+s} - P_{\eta jt}^{-\psi} P_{\eta,t+s}^\psi Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \end{aligned}$$

### First Order Condition

The first order condition with respect to  $P_{\eta jt}$  is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[ (1 - \psi) P_{\eta jt}^{-\psi} P_{\eta,t+s}^\psi Y_{\eta t+s} - (-\psi) P_{\eta jt}^{-\psi-1} P_{\eta,t+s}^\psi Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) \left( \frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^\psi Y_{\eta t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} &= \\ &= \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} \left( \frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^\psi Y_{\eta t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \end{aligned} \tag{3.43}$$

Substitute 3.27 in 3.43:

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) Y_{\eta j,t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} &= \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \Rightarrow \\ (\psi - 1) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} &= \psi P_{\eta jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \Rightarrow \\ P_{\eta jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} &= \frac{\psi}{\psi - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \Rightarrow \\ P_{\eta jt}^* &= \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \end{aligned} \tag{3.44}$$

Equation 3.44 represents the optimal price that firm  $j$  will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index  $j$  can be omitted:

$$P_{\eta t}^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} \Lambda_{\eta, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \quad (3.45)$$

## Final-Goods Firm, part II

The process of fixing prices is random: in each period,  $\theta$  firms will maintain the price from the previous period, while  $(1 - \theta)$  firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 3.28 to determine the aggregate price level:

$$\begin{aligned} P_{\eta t} &= \left[ \int_0^\theta P_{\eta, t-1}^{1-\psi} dj + \int_\theta^1 P_{\eta t}^{*1-\psi} dj \right]^{\frac{1}{1-\psi}} \implies \\ P_{\eta t} &= \left[ \theta P_{\eta, t-1}^{1-\psi} + (1 - \theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \end{aligned} \quad (3.46)$$

Equation 3.46 is the aggregate price level.

### 3.1.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (TAYLOR, 1993) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left( \frac{R_{t-1}}{R} \right)^{\gamma_R} \left[ \left( \frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left( \frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \quad (3.47)$$

where  $R, \pi, Y$  are the variables in steady state,  $\gamma_R$  is the smoothing parameter for the interest rate  $R_{Kt}$ ,  $\gamma_\pi$  and  $\gamma_Y$  are the interest-rate sensitivities in relation to inflation and product, respectively,  $Z_{Mt}$  is the monetary shock and  $\pi_t$  is the gross inflation rate,

defined by:<sup>7</sup>

$$\pi_t = \frac{P_t}{P_{t-1}} \quad (3.48)$$

where  $P_t$  is the national price level, defined by:

$$P_t Y_t = P_{1t} Y_{1t} + P_{2t} Y_{2t} \iff P_t = \frac{P_{1t} Y_{1t} + P_{2t} Y_{2t}}{Y_t} \quad (3.49)$$

## Regional Inflation

There is one price level  $P_{\eta t}$  in each region, generating a regional inflation rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \quad (3.50)$$

### 3.1.5 Stochastic Shocks

#### Productivity Shock

The production technology level  $Z_{A\eta t}$  will be submitted to a productivity shock defined by a first-order autoregressive process  $AR(1)$ :

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \quad (3.51)$$

where  $\rho_{A\eta} \in [0, 1]$  and  $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$ .

#### Monetary Shock

The monetary policy will also be submitted to a shock, through the variable  $Z_{Mt}$ , defined by a first-order autoregressive process  $AR(1)$ :

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \quad (3.52)$$

<sup>7</sup> for the monetary shock definition, see section 3.1.5.

where  $\rho_M \in [0, 1]$  and  $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$ .

### 3.1.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices  $\{P_{\eta t}^*, R_t^*, R_{Kt}^*, W_t^*\}$ , allocations for households  $\mathcal{A}_H := \{C_{\eta 1t}^*, C_{\eta 2t}^*, L_{\eta t}^*, B_{\eta t}^*, K_{\eta,t+1}^*\}$  and allocations for firms  $\mathcal{A}_F := \{K_{\eta jt}^*, L_{\eta jt}^*, Y_{\eta jt}^*, Y_{\eta t}^*\}$ . In such an equilibrium, given the set of exogenous variables  $\{K_0, Z_{A\eta t}, Z_{Mt}\}$ , the elements in  $\mathcal{A}_H$  solve the household problem, while the elements in  $\mathcal{A}_F$  solve the firms' problems, and the markets for goods and labor clear:

$$Y_t = Y_{1t} + Y_{2t} \quad (3.53)$$

$$\text{where: } Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \quad (3.54)$$

$$L_{\eta t} = \int_0^1 L_{\eta jt} \, d j \quad (3.55)$$

### 3.1.7 Model Structure

The model is composed of the preview solutions, forming a square system of 41 variables and 41 equations, summarized as follows:

- Variables:

- from the household problem:  $\langle C_{\eta t} \, L_{\eta t} \, B_{\eta t} \, K_{\eta,t+1} \, C_{\eta 1t} \, C_{\eta 2t} \, \varepsilon_{\eta t} \rangle$ ;
- from the final-goods firm problem:  $\langle Y_{\eta jt} \, Y_{\eta t} \, P_{\eta t} \rangle$ ;
- from the intermediate-goods firm problems:  $\langle K_{\eta jt} \, L_{\eta jt} \, P_{\eta t}^* \rangle$ ;
- from the monetary policy:  $\langle R_t \, \pi_t \, Y_t \rangle$ ;
- from the market clearing condition:  $\langle I_{\eta t} \rangle$ ;
- prices:  $\langle W_t \, R_{Kt} \, \Lambda_{\eta t} \, P_t \, \pi_{\eta t} \rangle$ ;
- shocks:  $\langle Z_{A\eta t} \, Z_{Mt} \rangle$ .

- Equations:

1. Budget Constraint:

$$\begin{aligned} P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} + P_{\eta t} I_{\eta t} + B_{\eta t} &= \\ &= W_t L_{\eta t} + R_{Kt} K_{\eta t} + (1 + R_{t-1}) B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \quad (3.2)$$

2. Law of Motion for Capital:

$$K_{\eta,t+1} = (1 - \delta) K_{\eta t} + I_{\eta t} \quad (3.3)$$

3. Regional Consumption of good 1:

$$C_{\eta 1t} = \mathcal{E}_{\eta t} \frac{\omega_{\eta 1}}{P_{1t}} \quad (3.19)$$

4. Regional Consumption of good 2:

$$C_{\eta 2t} = \mathcal{E}_{\eta t} \frac{1 - \omega_{\eta 1}}{P_{2t}} \quad (3.18)$$

5. Price Composition of Consumption Bundle:

$$\mathcal{E}_{\eta t} = C_{\eta t} \left[ \frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_{2t}}{1 - \omega_{\eta 1}} \right]^{1 - \omega_{\eta 1}} \quad (3.17)$$

6. Labor Supply:

$$\frac{\phi L_{\eta t}^\varphi}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \quad (3.21)$$

7. Euler equation for the bonds return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1 + R_t) \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta,t+1}} \right\} \quad (3.22)$$

8. Euler equation for the capital return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} = \beta \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta,t+1}} [(1 - \delta) P_{\eta,t+1} + R_{K,t+1}] \right\} \quad (3.23)$$

9. Bundle Technology:

$$Y_{\eta t} = \left( \int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \, d j \right)^{\frac{\psi}{\psi-1}} \quad (3.25)$$

10. Production Function:

$$Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^\alpha L_{\eta jt}^{1-\alpha} \quad (3.30)$$

11. Capital Demand:

$$K_{\eta jt} = \alpha Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_{Kt}} \quad (3.32)$$

12. Labor Demand:

$$L_{\eta jt} = (1 - \alpha) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \quad (3.33)$$

13. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left( \frac{R_{Kt}}{\alpha} \right)^{\alpha} \left( \frac{W_t}{1 - \alpha} \right)^{1-\alpha} \quad (3.39)$$

14. Optimal Price:

$$P_{\eta t}^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} \Lambda_{\eta, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \quad (3.45)$$

15. Regional Price Level:

$$P_{\eta t} = \left[ \theta P_{\eta, t-1}^{1-\psi} + (1 - \theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \quad (3.46)$$

16. Monetary Policy:

$$\frac{R_t}{R} = \left( \frac{R_{t-1}}{R} \right)^{\gamma_R} \left[ \left( \frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left( \frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \quad (3.47)$$

17. National Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \quad (3.48)$$

18. National Price Level:

$$P_t = \frac{P_{1t} Y_{1t} + P_{2t} Y_{2t}}{Y_t} \quad (3.49)$$

19. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta, t-1}} \quad (3.50)$$

20. Productivity Shock:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta, t-1} + \varepsilon_{A\eta t} \quad (3.51)$$

21. Monetary Shock:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M, t-1} + \varepsilon_{Mt} \quad (3.52)$$

22. Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \quad (3.53)$$

23. Regional Market Clearing Condition:

$$Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \quad (3.54)$$

## 3.2 Steady State

The steady state of a variable is defined by its constancy through time. For any given variable  $X_t$ , it is in steady state if  $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$  ([COSTA JUNIOR, 2016](#), p.41). For conciseness, the  $ss$  index representing the steady state will be omitted, so that  $X := X_{ss}$ . The model in steady state is:

1. Budget Constraint:

$$\begin{aligned} P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} &= \\ &= W_tL_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \implies \end{aligned} \quad (3.2)$$

$$P_1C_{\eta 1} + P_2C_{\eta 2} + P_{\eta}I_{\eta} = WL_{\eta} + R_KK_{\eta} + RB_{\eta} + \Pi_{\eta} \quad (3.56)$$

2. Law of Motion for Capital:

$$K_{\eta,t+1} = (1 - \delta)K_{\eta t} + I_{\eta t} \implies \quad (3.3)$$

$$\begin{aligned} K_{\eta} &= (1 - \delta)K_{\eta} + I_{\eta} \implies \\ I_{\eta} &= \delta K_{\eta} \end{aligned} \quad (3.57)$$

3. Regional Consumption of good 1:

$$C_{\eta 1t} = \mathcal{E}_{\eta t} \frac{\omega_{\eta 1}}{P_{1t}} \implies \quad (3.19)$$

$$C_{\eta 1} = \mathcal{E}_{\eta} \frac{\omega_{\eta 1}}{P_1} \quad (3.58)$$

4. Regional Consumption of good 2:

$$C_{\eta 2t} = \mathcal{E}_{\eta t} \frac{1 - \omega_{\eta 1}}{P_{2t}} \implies \quad (3.18)$$

$$C_{\eta 2} = \mathcal{E}_{\eta} \frac{1 - \omega_{\eta 1}}{P_2} \quad (3.59)$$

5. Price Composition of Consumption Bundle:

$$\mathcal{E}_{\eta t} = C_{\eta t} \left[ \frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_{2t}}{1 - \omega_{\eta 1}} \right]^{1 - \omega_{\eta 1}} \implies \quad (3.17)$$

$$\mathcal{E}_\eta = C_\eta \left[ \frac{P_1}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_2}{1 - \omega_{\eta 1}} \right]^{1 - \omega_{\eta 1}} \quad (3.60)$$

6. Labor Supply:

$$\frac{\phi L_{\eta t}^\varphi}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \implies \quad (3.21)$$

$$\frac{\phi L_\eta^\varphi}{C_\eta^{1-\sigma}} = \frac{W}{\mathcal{E}_\eta} \quad (3.61)$$

7. Euler equation for the bonds return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1 + R_t) \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta,t+1}} \right\} \implies \quad (3.22)$$

$$\frac{C_\eta^{1-\sigma}}{\mathcal{E}_\eta} = \beta(1 + R) \frac{C_\eta^{1-\sigma}}{\mathcal{E}_\eta} \implies \beta = \frac{1}{(1 + R)} \quad (3.62)$$

8. Euler equation for the capital return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} = \beta \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta,t+1}} [(1 - \delta) P_{\eta,t+1} + R_{K,t+1}] \right\} \implies \quad (3.23)$$

$$\begin{aligned} \frac{C_\eta^{1-\sigma}}{\mathcal{E}_\eta} P_\eta &= \beta \frac{C_\eta^{1-\sigma}}{\mathcal{E}_\eta} P_\eta \left[ (1 - \delta) + \frac{R_K}{P_\eta} \right] \\ 1 &= \beta \left[ (1 - \delta) + \frac{R_K}{P_\eta} \right] \end{aligned} \quad (3.63)$$

9. Bundle Technology:

$$Y_{\eta t} = \left( \int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}} \implies Y_\eta = \left( \int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}} \quad (3.64)$$

10. Production Function:

$$Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^\alpha L_{\eta jt}^{1-\alpha} \implies Y_{\eta j} = Z_{A\eta} K_{\eta j}^\alpha L_{\eta j}^{1-\alpha} \quad (3.65)$$

11. Capital Demand:

$$K_{\eta jt} = \alpha Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_{Kt}} \implies K_{\eta j} = \alpha Y_{\eta j} \frac{\Lambda_\eta}{R_K} \quad (3.66)$$

12. Labor Demand:

$$L_{\eta jt} = (1 - \alpha) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \implies L_{\eta j} = (1 - \alpha) Y_{\eta j} \frac{\Lambda_\eta}{W} \quad (3.67)$$

13. Marginal Cost:

$$\begin{aligned} \Lambda_{\eta t} &= \frac{1}{Z_{A\eta t}} \left( \frac{R_{Kt}}{\alpha} \right)^\alpha \left( \frac{W_t}{1 - \alpha} \right)^{1-\alpha} \implies \\ \Lambda_\eta &= \frac{1}{Z_{A\eta}} \left( \frac{R_K}{\alpha} \right)^\alpha \left( \frac{W}{1 - \alpha} \right)^{1-\alpha} \end{aligned} \quad (3.68)$$

14. Optimal Price:

$$\begin{aligned} P_{\eta t}^* &= \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} \Lambda_{\eta, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \quad (3.45) \\ P_\eta^* &= \frac{\psi}{\psi - 1} \cdot \frac{Y_{\eta j} \Lambda_\eta / [1 - \theta(1 - R)]}{Y_{\eta j} / [1 - \theta(1 - R)]} \implies \\ P_\eta^* &= \frac{\psi}{\psi - 1} \Lambda_\eta \end{aligned} \quad (3.69)$$

15. Regional Price Level:

$$\begin{aligned} P_{\eta t} &= \left[ \theta P_{\eta t-1}^{1-\psi} + (1 - \theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \implies \\ P_\eta^{1-\psi} &= \theta P_\eta^{1-\psi} + (1 - \theta) P_\eta^{*1-\psi} \implies \\ (1 - \theta) P_\eta^{1-\psi} &= (1 - \theta) P_\eta^{*1-\psi} \implies P_\eta = P_\eta^* \end{aligned} \quad (3.70)$$

16. Monetary Policy:

$$\frac{R_t}{R} = \left( \frac{R_{t-1}}{R} \right)^{\gamma_R} \left[ \left( \frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left( \frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \quad (3.71)$$

17. National Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \pi = 1 \quad (3.72)$$

18. National Price Level:

$$P_t = \frac{P_{1t} Y_{1t} + P_{2t} Y_{2t}}{Y_t} \implies P = \frac{P_1 Y_1 + P_2 Y_2}{Y} \quad (3.73)$$

19. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \implies \pi_\eta = 1 \quad (3.74)$$

20. Productivity Shock:

$$\begin{aligned} \ln Z_{A\eta t} &= (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \implies \\ \ln Z_{A\eta} &= (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta} + \varepsilon_{A\eta} \implies \\ \varepsilon_{A\eta} &= 0 \end{aligned} \quad (3.75)$$

21. Monetary Shock:

$$\begin{aligned} \ln Z_{Mt} &= (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \implies \\ \ln Z_M &= (1 - \rho_M) \ln Z_M + \rho_M \ln Z_M + \varepsilon_M \implies \\ \varepsilon_M &= 0 \end{aligned} \quad (3.76)$$

22. Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \implies \quad (3.53)$$

$$Y = Y_1 + Y_2 \quad (3.77)$$

23. Regional Market Clearing Condition:

$$Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \implies \quad (3.54)$$

$$Y_{\eta} = C_{\eta 1} + C_{\eta 2} + I_{\eta} \quad (3.78)$$

### 3.2.1 Variables at Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It is assumed that one regional price level and both productivity levels are normalized to one:<sup>8</sup>

$$\langle P_1 \ Z_{A1} \ Z_{A2} \rangle = \vec{1} \quad (3.79)$$

From 3.71, 3.72 and 3.74, the monetary shock, the national and regional gross inflation rates are:

$$\langle Z_M \ \pi \ \pi_1 \ \pi_2 \rangle = \vec{1} \quad (3.80)$$

From 3.75 and 3.76, the productivity and monetary shocks are:

$$\langle \varepsilon_{A1} \ \varepsilon_{A2} \ \varepsilon_M \rangle = \vec{0} \quad (3.81)$$

From 3.62, the return on bonds is:

$$\beta = \frac{1}{(1+R)} \implies \quad (3.62)$$

$$R = \frac{1}{\beta} - 1 \quad (3.82)$$

From 3.63, the return on capital  $R_K$  for  $\eta = 1$  is:

$$1 = \beta \left[ (1 - \delta) + \frac{R_K}{P_1} \right] \implies \quad (3.63)$$

$$R_K = P_1 \left[ \frac{1}{\beta} - (1 - \delta) \right] \quad (3.83)$$

<sup>8</sup> where  $\vec{1}$  is the unit vector.

Divide 3.83 for  $\eta = 1$  by 3.83 for  $\eta = 2$ :

$$\begin{aligned} \frac{R_K}{R_K} &= \frac{P_1 \left[ \frac{1}{\beta} - (1 - \delta) \right]}{P_2 \left[ \frac{1}{\beta} - (1 - \delta) \right]} \implies \\ 1 &= \frac{P_1}{P_2} \implies P_1 = P_2 \end{aligned} \quad (3.84)$$

From 3.70 and 3.84, the regional optimal price  $P_\eta^*$  is:

$$P_\eta^* = P_\eta \implies \langle P_1^* \ P_2^* \rangle = \langle P_1 \ P_2 \rangle = \langle P_1 \ P_1 \rangle \quad (3.70)$$

Substitute 3.84 and then 3.78 in 3.73 for the national price level:

$$\begin{aligned} P &= \frac{P_1 Y_1 + P_2 Y_2}{Y} = \frac{P_1 Y_1 + P_1 Y_2}{Y} \implies \\ P &= P_1 = P_2 \end{aligned} \quad (3.73)$$

Substitute 3.85 in 3.60 for the price composition of consumption bundle  $\mathcal{E}_\eta$ :

$$\mathcal{E}_\eta = C_\eta \left[ \frac{P_1}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_2}{1 - \omega_{\eta 1}} \right]^{1 - \omega_{\eta 1}} = C_\eta \left[ \frac{P}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P}{1 - \omega_{\eta 1}} \right]^{1 - \omega_{\eta 1}} \implies \quad (3.60)$$

$$\mathcal{E}_\eta = \frac{P C_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1 - \omega_{\eta 1}}} \quad (3.86)$$

Substitute 3.70 and 3.85 in 3.69 for the marginal cost  $\Lambda_\eta$ :

$$P_\eta^* = \frac{\psi}{\psi - 1} \Lambda_\eta \implies \quad (3.69)$$

$$\Lambda_\eta = P \frac{\psi - 1}{\psi} = \Lambda \quad (3.87)$$

Substitute 3.79 and 3.87 in 3.68 for the nominal wage  $W$  in terms of region 1:

$$\Lambda_\eta = \frac{1}{Z_{A\eta}} \left( \frac{R_K}{\alpha} \right)^\alpha \left( \frac{W}{1 - \alpha} \right)^{1 - \alpha} \implies \quad (3.68)$$

$$W = (1 - \alpha) \left[ \Lambda Z_{A1} \left( \frac{\alpha}{R_K} \right)^\alpha \right]^{\frac{1}{1 - \alpha}} \quad (3.88)$$

Due to price parity in steady state, where prices are identical ( $P_\eta = P_\eta^*$ ) and resulting in a gross inflation level of one ( $\pi_\eta = 1$ ), all firms produce the same output level ( $Y_{\eta j} = Y_\eta$ ) (SOLIS-GARCIA, 2022, Lecture 13, p.12). As a consequence, they uniformly demand the same amount of factors ( $K_\eta, L_\eta$ ), and equations 3.65, 3.66 and 3.67 become:

$$Y_\eta = Z_{A\eta} K_\eta^\alpha L_\eta^{1-\alpha} \quad (3.89)$$

$$K_\eta = \alpha Y_\eta \frac{\Lambda}{R_K} \quad (3.90)$$

$$L_\eta = (1 - \alpha) Y_\eta \frac{\Lambda}{W} \quad (3.91)$$

Substitute 3.90 in 3.57:

$$I_\eta = \delta K_\eta \implies I_\eta = \delta \alpha \frac{\Lambda}{R_K} Y_\eta \implies \quad (3.57)$$

$$I_\eta = b_\eta Y_\eta \quad (3.92)$$

$$\text{where: } b_\eta = \delta \alpha \frac{\Lambda}{R_K} \quad (3.93)$$

Isolate  $C_\eta$  in 3.61 and then substitute 3.86 and 3.91:

$$\frac{\phi L_\eta^\varphi}{C_\eta^{1-\sigma}} = \frac{W}{\mathcal{E}_\eta} \implies C_\eta^{\sigma-1} = \frac{W}{\mathcal{E}_\eta \phi L_\eta^\varphi} \implies \\ C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \quad (3.94)$$

$$\text{where: } a_\eta = \left[ \frac{W^{1+\varphi} \omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}}{\phi P [(1 - \alpha) \Lambda]^\varphi} \right]^{\frac{1}{\sigma}} \quad (3.95)$$

Substitute 3.94 in 3.86:

$$\mathcal{E}_\eta = \frac{P C_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \implies \quad (3.86)$$

$$\mathcal{E}_\eta = \frac{P a_\eta Y_\eta^{\frac{-\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \quad (3.96)$$

Substitute 3.96 in 3.58:

$$C_{\eta 1} = \mathcal{E}_\eta \frac{\omega_{\eta 1}}{P_1} \implies \quad (3.58)$$

$$\begin{aligned} C_{\eta 1} &= \frac{P a_\eta Y_\eta^{-\frac{\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \cdot \frac{\omega_{\eta 1}}{P} \implies \\ C_{\eta 1} &= \left( \frac{\omega_{\eta 1}}{1 - \omega_{\eta 1}} \right)^{1-\omega_{\eta 1}} a_\eta Y_\eta^{-\frac{\varphi}{\sigma}} \end{aligned} \quad (3.97)$$

Substitute 3.96 in 3.59:

$$C_{\eta 2} = \mathcal{E}_\eta \frac{1 - \omega_{\eta 1}}{P_2} \implies \quad (3.59)$$

$$\begin{aligned} C_{\eta 2} &= \frac{P a_\eta Y_\eta^{-\frac{\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \cdot \frac{1 - \omega_{\eta 1}}{P} \implies \\ C_{\eta 2} &= \left( \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right)^{\omega_{\eta 1}} a_\eta Y_\eta^{-\frac{\varphi}{\sigma}} \end{aligned} \quad (3.98)$$

Substitute 3.92, 3.97 and 3.98 in 3.78:

$$Y_\eta = C_{\eta 1} + C_{\eta 2} + I_\eta \implies \quad (3.78)$$

$$\begin{aligned} Y_\eta &= \left( \frac{\omega_{\eta 1}}{1 - \omega_{\eta 1}} \right)^{1-\omega_{\eta 1}} a_\eta Y_\eta^{-\frac{\varphi}{\sigma}} + \left( \frac{1 - \omega_{\eta 1}}{\omega_{\eta 1}} \right)^{\omega_{\eta 1}} a_\eta Y_\eta^{-\frac{\varphi}{\sigma}} + b_\eta Y_\eta \implies \\ Y_\eta &= \left[ \left( \frac{a_1}{1 - b_1} \right) \left( \frac{1}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \right) \right]^{\frac{\sigma}{\sigma+\varphi}} \end{aligned} \quad (3.99)$$

The result of 3.99 determines  $Y, I_\eta, C_\eta, \mathcal{E}_\eta, C_{\eta 1}, C_{\eta 2}, K_\eta, L_\eta$  in 3.77 3.92, 3.94, 3.86, 3.58, 3.59, 3.90 and 3.91, respectively.

### 3.2.2 Steady State Solution

$$\vec{1} = \langle P \quad P_1 \quad P_2 \quad Z_{A1} \quad Z_{A2} \quad Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle \quad (3.100)$$

$$\vec{0} = \langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle \quad (3.81)$$

$$R = \frac{1}{\beta} - 1 \quad (3.82)$$

$$R_K = P \left[ \frac{1}{\beta} - (1 - \delta) \right] \quad (3.83)$$

$$\Lambda = P \frac{\psi - 1}{\psi} \quad (3.87)$$

$$W = (1 - \alpha) \left[ \Lambda Z_{A1} \left( \frac{\alpha}{R_K} \right)^\alpha \right]^{\frac{1}{1-\alpha}} \quad (3.88)$$

$$a_\eta = \left[ \frac{W^{1+\varphi} \omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}}{\phi P [(1 - \alpha) \Lambda]^\varphi} \right]^{\frac{1}{\sigma}} \quad (3.95)$$

$$b_\eta = \delta \alpha \frac{\Lambda}{R_K} \quad (3.93)$$

$$Y_\eta = \left[ \left( \frac{a_1}{1 - b_1} \right) \left( \frac{1}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \right) \right]^{\frac{\sigma}{\sigma+\varphi}} \quad (3.99)$$

$$Y = Y_1 + Y_2 \quad (3.77)$$

$$I_\eta = b_\eta Y_\eta \quad (3.92)$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \quad (3.94)$$

$$\mathcal{E}_\eta = \frac{P C_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{1-\omega_{\eta 1}}} \quad (3.86)$$

$$C_{\eta 1} = \mathcal{E}_\eta \frac{\omega_{\eta 1}}{P} \quad (3.58)$$

$$C_{\eta 2} = \mathcal{E}_\eta \frac{1 - \omega_{\eta 1}}{P} \quad (3.59)$$

$$K_\eta = \alpha Y_\eta \frac{\Lambda}{R_K} \quad (3.90)$$

$$L_\eta = (1 - \alpha) Y_\eta \frac{\Lambda}{W} \quad (3.91)$$

### 3.3 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. Dynare is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation 3.45. For this purpose, Uhlig's rules of log-linearization (UHLIG, 1999) will be applied to all equations in the model.<sup>9</sup>

#### Regional Price Level

Log-linearize equation 3.46:

$$\begin{aligned} P_{\eta t}^{1-\psi} &= \theta P_{\eta,t-1}^{1-\psi} + (1 - \theta) P_{\eta t}^{*1-\psi} \implies && (3.46) \\ P^{1-\psi}(1 + (1 - \psi)\hat{P}_{\eta t}) &= \theta P^{1-\psi}(1 + (1 - \psi)\hat{P}_{\eta,t-1}) + \\ &\quad + (1 - \theta)P^{1-\psi}(1 + (1 - \psi)\hat{P}_{\eta t}^*) \implies \\ \hat{P}_{\eta t} &= \theta \hat{P}_{\eta,t-1} + (1 - \theta) \hat{P}_{\eta t}^* && (3.101) \end{aligned}$$

#### Regional Gross Inflation Rate

Log-linearize 3.50 and define the level deviation of regional inflation rate  $\hat{\pi}_{\eta t}$ :

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \quad (3.50)$$

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \quad (3.102)$$

<sup>9</sup> see lemma A.3 for details.

## New Keynesian Phillips Curve

In order to log-linearize equation 3.45, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma A.5:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j, t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} = \frac{\psi}{\psi - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j, t+s} \Lambda_{\eta, t+s}}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \implies \quad (3.45)$$

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j, t+s}}{(1+R)^s \left( 1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right)} \right\} &= \\ &= \frac{\psi}{\psi - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j, t+s} \Lambda_{\eta, t+s}}{(1+R)^s \left( 1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right)} \right\} \end{aligned} \quad (3.103)$$

First, log-linearize the left hand side of equation 3.103 with respect to  $P_{\eta t}^*$ ,  $Y_{\eta j t}$ ,  $\tilde{R}_t$ :

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j, t+s}}{(1+R)^s \left( 1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right)} \right\} &\implies \\ \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \frac{P_{\eta}^* Y_{\eta j} \left( 1 + \hat{P}_{\eta t}^* + \hat{Y}_{\eta j, t+s} \right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k}} \right\} &\implies \\ P_{\eta}^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( 1 + \hat{P}_{\eta t}^* + \hat{Y}_{\eta j, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} \end{aligned}$$

Separate the terms not dependent on  $s$ :

$$\begin{aligned} P_{\eta}^* Y_{\eta j} (1 + \hat{P}_{\eta t}^*) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \right\} + \\ + P_{\eta}^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( \hat{Y}_{\eta j, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} \implies \end{aligned}$$

Apply definition A.9 on the first term:

$$\frac{P_{\eta}^* Y_{\eta j} (1 + \hat{P}_{\eta t}^*)}{1 - \theta / (1+R)} + P_{\eta}^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( \hat{Y}_{\eta j, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\}$$

Second, log-linearize the left hand side of 3.103 with respect to  $\Lambda_{\eta t}^*, Y_{\eta j t}, \tilde{R}_t$ :

$$\begin{aligned} \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j, t+s} \Lambda_{\eta, t+s}}{(1+R)^s \left( 1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right)} \right\} &\implies \\ \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \frac{Y_{\eta j} \Lambda_{\eta} (1 + \hat{Y}_{\eta j, t+s} + \hat{\Lambda}_{\eta, t+s})}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k}} \right\} &\implies \\ \frac{\psi}{\psi-1} Y_{\eta j} \Lambda_{\eta} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( 1 + \hat{Y}_{\eta j, t+s} + \hat{\Lambda}_{\eta, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} \end{aligned}$$

Separate the terms not dependent on  $s$ :

$$\begin{aligned} \frac{\psi}{\psi-1} Y_{\eta j} \Lambda_{\eta} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \right\} + \\ + \frac{\psi}{\psi-1} Y_{\eta j} \Lambda_{\eta} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( \hat{Y}_{\eta j, t+s} + \hat{\Lambda}_{\eta, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} \end{aligned}$$

Apply definition A.9 on the first term:

$$\begin{aligned} \frac{\psi}{\psi-1} \cdot \frac{Y_{\eta j} \Lambda_{\eta}}{1 - \theta/(1+R)} + \\ + \frac{\psi}{\psi-1} Y_{\eta j} \Lambda_{\eta} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( \hat{Y}_{\eta j, t+s} + \hat{\Lambda}_{\eta, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} \end{aligned}$$

Join both sides of the equation again:

$$\begin{aligned} \frac{P_{\eta}^* Y_{\eta j} (1 + \hat{P}_{\eta t}^*)}{1 - \theta/(1+R)} + P_{\eta}^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( \hat{Y}_{\eta j, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} = \\ = \frac{\psi}{\psi-1} \cdot \frac{Y_{\eta j} \Lambda_{\eta}}{1 - \theta/(1+R)} + \\ + \frac{\psi}{\psi-1} Y_{\eta j} \Lambda_{\eta} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left( \frac{\theta}{1+R} \right)^s \left( \hat{Y}_{\eta j, t+s} + \hat{\Lambda}_{\eta, t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} \quad (3.104) \end{aligned}$$

Substitute 3.62 in 3.104:

$$\beta = \frac{1}{(1+R)} \quad (3.62)$$

$$\begin{aligned} & \frac{P_\eta^* Y_{\eta j} (1 + \hat{P}_{\eta t}^*)}{1 - \theta \beta} + P_\eta^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta \beta)^s \left( \hat{Y}_{\eta j, t+s} - \beta \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} = \\ & = \frac{\psi}{\psi - 1} \cdot \frac{Y_{\eta j} \Lambda_\eta}{1 - \theta \beta} + \\ & + \frac{\psi}{\psi - 1} Y_{\eta j} \Lambda_\eta \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta \beta)^s \left( \hat{Y}_{\eta j, t+s} + \hat{\Lambda}_{\eta, t+s} - \beta \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\} \end{aligned} \quad (3.105)$$

Substitute 3.87 in 3.105 and simplify all common terms:

$$\begin{aligned} & \cancel{\frac{P_\eta^* Y_{\eta j}}{1 - \theta \beta}} + \cancel{\frac{P_\eta^* Y_{\eta j} \hat{P}_{\eta t}^*}{1 - \theta \beta}} + \cancel{P_\eta^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta \beta)^s \left( \hat{Y}_{\eta j, t+s} - \beta \sum_{k=0}^{s-1} \tilde{R}_{t+k} \right) \right\}} = \\ & = \cancel{\frac{P_\eta^* Y_{\eta j}}{1 - \theta \beta}} + \cancel{P_\eta^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta \beta)^s \left( \hat{Y}_{\eta j, t+s} - \beta \sum_{k=0}^{s-1} \tilde{R}_{t+k} + \hat{\Lambda}_{\eta, t+s} \right) \right\}} \implies \\ & \frac{\hat{P}_{\eta t}^*}{1 - \theta \beta} = \mathbb{E}_t \sum_{s=0}^{\infty} \{ (\theta \beta)^s (\hat{\Lambda}_{\eta, t+s}) \} \end{aligned} \quad (3.106)$$

Define the real marginal cost  $\lambda_{\eta t}$ :

$$\begin{aligned} \lambda_{\eta t} &= \frac{\Lambda_{\eta t}}{P_{\eta t}} \implies \Lambda_{\eta t} = P_{\eta t} \lambda_{\eta t} \implies \\ \hat{\Lambda}_{\eta t} &= \hat{P}_{\eta t} + \hat{\lambda}_{\eta t} \end{aligned} \quad (3.107)$$

Substitute 3.107 in 3.106:

$$\hat{P}_{\eta t}^* = (1 - \theta \beta) \mathbb{E}_t \sum_{s=0}^{\infty} (\theta \beta)^s (\hat{P}_{\eta, t+s} + \hat{\lambda}_{\eta, t+s}) \quad (3.108)$$

Substitute 3.108 in 3.101:

$$\hat{P}_{\eta t} = \theta \hat{P}_{\eta, t-1} + (1 - \theta) \hat{P}_{\eta t}^* \quad (3.101)$$

$$\hat{P}_{\eta t} = \theta \hat{P}_{\eta, t-1} + (1 - \theta)(1 - \theta \beta) \mathbb{E}_t \sum_{s=0}^{\infty} (\theta \beta)^s (\hat{P}_{\eta, t+s} + \hat{\lambda}_{\eta, t+s}) \quad (3.109)$$

Finally, to eliminate the summation, apply the lead operator  $(1 - \theta\beta\mathbb{L}^{-1})$  in 3.109:<sup>10</sup>

$$\begin{aligned}
(1 - \theta\beta\mathbb{L}^{-1})\hat{P}_{\eta t} &= (1 - \theta\beta\mathbb{L}^{-1}) \left[ \theta\hat{P}_{\eta,t-1} + \right. \\
&\quad \left. + (1 - \theta)(1 - \theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\beta)^s (\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}) \right] \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + \\
&\quad (1 - \theta)(1 - \theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\beta)^s (\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}) - \\
&\quad - \theta\beta(1 - \theta)(1 - \theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\beta)^s (\hat{P}_{t+s+1} + \hat{\lambda}_{\eta,t+s+1}) \tag{3.110}
\end{aligned}$$

In the first summation, factor out the first term and in the second summation, include the term  $\theta\beta$  within the operator. Then, cancel the summations and rearrange the terms:

$$\begin{aligned}
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + \\
&\quad (1 - \theta)(1 - \theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\beta)^s (\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}) - \\
&\quad - \theta\beta(1 - \theta)(1 - \theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\beta)^s (\hat{P}_{t+s+1} + \hat{\lambda}_{\eta,t+s+1}) \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + (1 - \theta)(1 - \theta\beta)(\hat{P}_{\eta t} + \hat{\lambda}_{\eta t}) + \\
&\quad + (1 - \theta)(1 - \theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\beta)^{s+1} (\hat{P}_{t+s+1} + \hat{\lambda}_{\eta,t+s+1}) - \\
&\quad - (1 - \theta)(1 - \theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\beta)^{s+1} (\hat{P}_{t+s+1} + \hat{\lambda}_{\eta,t+s+1}) \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{\eta,t-1} - \theta^2\beta\hat{P}_{\eta t} + \\
&\quad + (1 - \theta - \theta\beta + \theta^2\beta)\hat{P}_{\eta t} + (1 - \theta)(1 - \theta\beta)\hat{\lambda}_{\eta t} \implies \\
(\hat{P}_{\eta t} - \hat{P}_{\eta,t-1}) &= \beta(\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_{\eta t}) + \frac{(1 - \theta)(1 - \theta\beta)}{\theta}\hat{\lambda}_{\eta t} \tag{3.111}
\end{aligned}$$

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<sup>10</sup> see definition A.10.

Substitute 3.102 in 3.111:

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{\eta t} \quad (3.112)$$

Equation 3.112 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

## Law of Motion for Capital

Log-linearize 3.3:

$$\begin{aligned} K_{\eta,t+1} &= (1-\delta)K_{\eta t} + I_{\eta t} &\implies (3.3) \\ K_\eta(1+\hat{K}_{\eta,t+1}) &= (1-\delta)K_\eta(1+\hat{K}_{\eta t}) + I_\eta(1+\hat{I}_{\eta t}) \implies \\ \hat{K}_{\eta,t+1} &= (1-\delta)\hat{K}_{\eta t} + \delta\hat{I}_{\eta t} \end{aligned} \quad (3.113)$$

## Regional Levels of Consumption and Prices

Log-linearize 3.19:

$$\begin{aligned} C_{\eta 1t} &= \mathcal{E}_{\eta t} \frac{\omega_{\eta 1}}{P_{1t}} &\implies (3.19) \\ C_{\eta 1}(1+\hat{C}_{\eta 1t}) &= \mathcal{E}_\eta \frac{\omega_{\eta 1}}{P_1} (1+\hat{\mathcal{E}}_{\eta t} - \hat{P}_{1t}) &\implies \\ \hat{C}_{\eta 1t} &= \hat{\mathcal{E}}_{\eta t} - \hat{P}_{1t} \iff \hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 1t} + \hat{P}_{1t} \end{aligned} \quad (3.114)$$

Log-linearize 3.18:

$$\begin{aligned} C_{\eta 2t} &= \mathcal{E}_{\eta t} \frac{1-\omega_{\eta 1}}{P_{2t}} &\implies (3.18) \\ C_{\eta 2}(1+\hat{C}_{\eta 2t}) &= \mathcal{E}_\eta \frac{1-\omega_{\eta 1}}{P_2} (1+\hat{\mathcal{E}}_{\eta t} - \hat{P}_{2t}) &\implies \\ \hat{C}_{\eta 2t} &= \hat{\mathcal{E}}_{\eta t} - \hat{P}_{2t} \iff \hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 2t} + \hat{P}_{2t} \end{aligned} \quad (3.115)$$

Match equations 3.114 and 3.115:

$$\begin{aligned}\hat{\mathcal{E}}_{\eta t} &= \hat{C}_{\eta 1t} + \hat{P}_{1t} = \hat{C}_{\eta 2t} + \hat{P}_{2t} \implies \\ \hat{C}_{\eta 1t} - \hat{C}_{\eta 2t} &= \hat{P}_{2t} - \hat{P}_{1t}\end{aligned}\tag{3.116}$$

Equation 3.116 shows that the variation of both goods consumption in each region correspond to the distance between the variations of the regional price levels.

## Total Expenses

Log-linearize 3.17:

$$\begin{aligned}\mathcal{E}_{\eta t} &= C_{\eta t} \left[ \frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[ \frac{P_{2t}}{1 - \omega_{\eta 1}} \right]^{1 - \omega_{\eta 1}} \implies \\ \mathcal{E}_{\eta}(1 + \hat{\mathcal{E}}_{\eta t}) &= \frac{C_{\eta} P_1^{\omega_{\eta 1}} P_2^{1 - \omega_{\eta 1}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1 - \omega_{\eta 1})^{(1 - \omega_{\eta 1})}} \cdot \\ &\quad \cdot (1 + \hat{C}_{\eta t} + \omega_{\eta 1} \hat{P}_{1t} + (1 - \omega_{\eta 1}) \hat{P}_{2t}) \implies \\ \hat{\mathcal{E}}_{\eta t} &= \hat{C}_{\eta t} + \omega_{\eta 1} \hat{P}_{1t} + (1 - \omega_{\eta 1}) \hat{P}_{2t}\end{aligned}\tag{3.117}$$

## Labor Supply

Log-linearize 3.21:

$$\begin{aligned}\frac{\phi L_{\eta t}^\varphi}{C_{\eta t}^{1-\sigma}} &= \frac{W_t}{\mathcal{E}_{\eta t}} \implies \\ \varphi \hat{L}_{\eta t} - (1 - \sigma) \hat{C}_{\eta t} &= \hat{W}_t - \hat{\mathcal{E}}_{\eta t}\end{aligned}\tag{3.118}$$

## Euler equation for the bonds return

Log-linearize 3.22:

$$\begin{aligned}
\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} &= \beta(1 + R_t) \mathbb{E}_t \left\{ \frac{C_{\eta, t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} \right\} \implies && (3.22) \\
\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} \cdot \frac{\mathbb{E}_t \mathcal{E}_{\eta t+1}}{\mathbb{E}_t C_{\eta, t+1}^{1-\sigma}} &= \beta + \beta R_t \implies \\
\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \cdot \frac{\mathcal{E}_{\eta}}{C_{\eta}^{1-\sigma}} (1 + (1 - \sigma) \hat{C}_{\eta t} - \hat{\mathcal{E}}_{\eta t} + \mathbb{E}_t \{\hat{\mathcal{E}}_{\eta, t+1} - (1 - \sigma) \hat{C}_{\eta, t+1}\}) &= \\
&= \beta + \beta(R(1 + \hat{R}_t)) \implies \\
\mathbb{E}_t \{\hat{\mathcal{E}}_{\eta, t+1} - (1 - \sigma) \hat{C}_{\eta, t+1}\} - [\hat{\mathcal{E}}_{\eta t} - (1 - \sigma) \hat{C}_{\eta t}] &= (1 - \beta) \hat{R}_t && (3.119)
\end{aligned}$$

## Euler equation for the capital return

Log-linearize 3.23:

$$\begin{aligned}
\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} &= \beta \mathbb{E}_t \left\{ \frac{C_{\eta, t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} [(1 - \delta) P_{\eta, t+1} + R_{K, t+1}] \right\} \implies && (3.23) \\
\frac{P_{\eta t} C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} \mathbb{E}_t \left\{ \frac{\mathcal{E}_{\eta t+1}}{P_{\eta, t+1} C_{\eta, t+1}^{1-\sigma}} \right\} &= \beta \left[ (1 - \delta) + \mathbb{E}_t \left\{ \frac{R_{K, t+1}}{P_{\eta, t+1}} \right\} \right] \implies \\
\frac{P_{\eta} C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \cdot \frac{\mathcal{E}_{\eta}}{P_{\eta} C_{\eta}^{1-\sigma}} (1 + \hat{P}_{\eta t} + (1 - \sigma) \hat{C}_{\eta t} - \hat{\mathcal{E}}_{\eta t} + & \\
&+ \mathbb{E}_t \{\hat{\mathcal{E}}_{\eta, t+1} - \hat{P}_{\eta, t+1} - (1 - \sigma) \hat{C}_{\eta, t+1}\}) = \\
&= \beta \left[ (1 - \delta) + \frac{R_K}{P_{\eta}} \mathbb{E}_t \{1 + \hat{R}_{K, t+1} - \hat{P}_{\eta, t+1}\} \right] \implies \\
\mathbb{E}_t \{\hat{\mathcal{E}}_{\eta, t+1} - \hat{P}_{\eta, t+1} - (1 - \sigma) \hat{C}_{\eta, t+1}\} - (\hat{\mathcal{E}}_{\eta t} - \hat{P}_{\eta t} - (1 - \sigma) \hat{C}_{\eta t}) &= \\
&= \beta r_K \mathbb{E}_t \{\hat{R}_{K, t+1} - \hat{P}_{\eta, t+1}\} && (3.120)
\end{aligned}$$

$$\text{where: } r_K = \frac{R_K}{P} \quad (3.121)$$

## Bundle Technology

Apply the natural logarithm to 3.25:

$$Y_{\eta t} = \left( \int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}} \implies \ln Y_{\eta t} = \frac{\psi}{\psi-1} \ln \left( \int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} dj \right) \quad (3.25)$$

Log-linearize using corollary A.3.1:

$$\begin{aligned} \ln Y_\eta + \hat{Y}_{\eta t} &= \frac{\psi}{\psi-1} \left[ \ln \left( \int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}} dj \right) + \frac{\psi-1}{\psi} \int_0^1 \hat{Y}_{\eta jt} dj \right] \implies \\ \ln Y_\eta + \hat{Y}_{\eta t} &= \frac{\psi}{\psi-1} \left[ \ln \left( Y_{\eta j}^{\frac{\psi-1}{\psi}} \int_0^1 dj \right) + \frac{\psi-1}{\psi} \int_0^1 \hat{Y}_{\eta jt} dj \right] \implies \\ \ln Y_\eta + \hat{Y}_{\eta t} &= \cancel{\frac{\psi}{\psi-1}} \left[ \cancel{\frac{\psi-1}{\psi}} \ln Y_{\eta j} + \ln \cancel{1} + \cancel{\frac{\psi-1}{\psi}} \int_0^1 \hat{Y}_{\eta jt} dj \right] \implies \\ \ln Y_\eta + \hat{Y}_{\eta t} &= \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta jt} dj \end{aligned}$$

Apply corollary A.2.1:

$$\begin{aligned} \ln Y_\eta + \hat{Y}_{\eta t} &= \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta jt} dj \implies \\ \hat{Y}_{\eta t} &= \int_0^1 \hat{Y}_{\eta jt} dj \end{aligned} \quad (3.122)$$

## Production Function

Log-linearize 3.30:

$$\begin{aligned} Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^\alpha L_{\eta jt}^{1-\alpha} \implies \quad (3.30) \\ Y_{\eta j}(1 + \hat{Y}_{\eta jt}) &= Z_{A\eta} K_{\eta j}^\alpha L_{\eta j}^{1-\alpha} (1 + \hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta jt} + (1 - \alpha) \hat{L}_{\eta jt}) \implies \\ \hat{Y}_{\eta jt} &= \hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta jt} + (1 - \alpha) \hat{L}_{\eta jt} \end{aligned} \quad (3.123)$$

Substitute 3.123 in 3.122:

$$\hat{Y}_{\eta t} = \int_0^1 \hat{Y}_{\eta jt} \, dj \implies \quad (3.122)$$

$$\hat{Y}_{\eta t} = \int_0^1 [\hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta jt} + (1 - \alpha) \hat{L}_{\eta jt}] \, dj \implies$$

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha \int_0^1 \hat{K}_{\eta jt} \, dj + (1 - \alpha) \int_0^1 \hat{L}_{\eta jt} \, dj \quad (3.124)$$

Apply the natural logarithm and then log-linearize 3.55:

$$\begin{aligned} L_{\eta t} &= \int_0^1 L_{\eta jt} \, dj \implies \\ \ln L_{\eta t} &= \ln \left[ \int_0^1 L_{\eta jt} \, dj \right] \implies \\ \ln L + \hat{L}_{\eta t} &= \ln \left[ \int_0^1 L_{\eta j} \, dj \right] + \int_0^1 \hat{L}_{\eta jt} \, dj \implies \\ \ln L + \hat{L}_{\eta t} &= \ln L_{\eta j} + \ln 1 + \int_0^1 \hat{L}_{\eta jt} \, dj \end{aligned} \quad (3.55)$$

Apply corollary A.2.1:

$$\implies \hat{L}_{\eta t} = \int_0^1 \hat{L}_{\eta jt} \, dj \quad (3.125)$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_{\eta t} = \int_0^1 \hat{K}_{\eta jt} \, dj \quad (3.126)$$

Substitute 3.125 and 3.126 in 3.124:

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha \int_0^1 \hat{K}_{\eta jt} \, dj + (1 - \alpha) \int_0^1 \hat{L}_{\eta jt} \, dj \implies \quad (3.124)$$

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta t} + (1 - \alpha) \hat{L}_{\eta t} \quad (3.127)$$

## Capital Demand

Log-linearize 3.32:

$$\begin{aligned} K_{\eta jt} &= \alpha Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_{Kt}} \implies \\ K_{\eta j}(1 + \hat{K}_{\eta jt}) &= \alpha Y_{\eta j} \frac{\Lambda_{\eta}}{R_K} (1 + \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt}) \implies \\ \hat{K}_{\eta jt} &= \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt} \end{aligned} \quad (3.32)$$

Integrate both sides and then substitute 3.126 and 3.122:

$$\begin{aligned} \int_0^1 \hat{K}_{\eta jt} \, dj &= \int_0^1 (\hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt}) \, dj \implies \\ \hat{K}_{\eta t} &= \hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt} \end{aligned} \quad (3.128)$$

## Labor Demand

Log-linearize 3.33:

$$\begin{aligned} L_{\eta jt} &= (1 - \alpha) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \implies \\ L_{\eta j}(1 + \hat{L}_{\eta jt}) &= (1 - \alpha) Y_{\eta j} \frac{\Lambda_{\eta}}{W} (1 + \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{W}_t) \implies \\ \hat{L}_{\eta jt} &= \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{W}_t \end{aligned} \quad (3.33)$$

Integrate both sides and then substitute 3.125 and 3.122:

$$\begin{aligned} \int_0^1 \hat{L}_{\eta jt} \, dj &= \int_0^1 \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{W}_t \, dj \implies \\ \hat{L}_{\eta t} &= \hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{W}_t \end{aligned} \quad (3.129)$$

Subtract 3.129 from 3.128:

$$\begin{aligned} \hat{K}_{\eta t} - \hat{L}_{\eta t} &= \hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt} - (\hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{W}_t) \implies \\ \hat{K}_{\eta t} - \hat{L}_{\eta t} &= \hat{W}_t - \hat{R}_{Kt} \end{aligned} \quad (3.130)$$

Equation 3.130 is the log-linearized version of 3.34.

## Marginal Cost

Log-linearize 3.39:

$$\Lambda_{\eta t} = Z_{A\eta t}^{-1} \frac{R_{Kt}^\alpha W_t^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \implies \quad (3.39)$$

$$\begin{aligned} \Lambda(1 + \hat{\Lambda}_{\eta t}) &= \\ &= \frac{1}{Z_{A\eta}} \left( \frac{R_K}{\alpha} \right)^\alpha \left( \frac{W}{1-\alpha} \right)^{1-\alpha} (1 - \hat{Z}_{A\eta t} + \alpha \hat{R}_{Kt} + (1-\alpha) \hat{W}_t) \implies \\ \hat{\Lambda}_{\eta t} &= \alpha \hat{R}_{Kt} + (1-\alpha) \hat{W}_t - \hat{Z}_{A\eta t} \end{aligned} \quad (3.131)$$

Substitute 3.107 in 3.131:

$$\begin{aligned} \hat{\Lambda}_{\eta t} &= \alpha \hat{R}_{Kt} + (1-\alpha) \hat{W}_t - \hat{Z}_{A\eta t} \implies \\ \hat{P}_{\eta t} + \hat{\lambda}_{\eta t} &= \alpha \hat{R}_{Kt} + (1-\alpha) \hat{W}_t - \hat{Z}_{A\eta t} \implies \\ \hat{\lambda}_{\eta t} &= \alpha \hat{R}_{Kt} + (1-\alpha) \hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_{\eta t} \end{aligned} \quad (3.132)$$

## Monetary Policy

Log-linearize 3.47:

$$\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R} (\pi_t^{\gamma_\pi} Y_t^{\gamma_Y})^{(1-\gamma_R)} Z_{Mt}}{R^{\gamma_R} (\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \implies \quad (3.47)$$

$$\begin{aligned} \frac{R(1 + \hat{R}_t)}{R} &= \frac{R^{\gamma_R} (\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)} Z_M}{R^{\gamma_R} (\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \cdot [1 + \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt}] \implies \\ \hat{R}_t &= \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \end{aligned} \quad (3.133)$$

## National Gross Inflation Rate

Log-linearize 3.48 and define the level deviation of gross inflation rate  $\hat{\pi}_t$ :

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \quad (3.48)$$

$$\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \quad (3.134)$$

## Market Clearing Condition

Log-linearize 3.53:

$$\begin{aligned} Y_t &= Y_{1t} + Y_{2t} \\ Y(1 + \hat{Y}_t) &= Y_1(1 + \hat{Y}_{1t}) + Y_2(1 + \hat{Y}_{2t}) \implies \\ \hat{Y}_t &= \frac{Y_1}{Y} \hat{Y}_{1t} + \frac{Y_2}{Y} \hat{Y}_{2t} \end{aligned} \quad (3.135)$$

Define the regional weights  $\langle \theta_{Y1} (1 - \theta_{Y1}) \rangle$  in the production total:

$$\left\langle \theta_{Y1} (1 - \theta_{Y1}) \right\rangle := \left\langle \frac{Y_1}{Y} \quad \frac{Y_2}{Y} \right\rangle \quad (3.136)$$

Substitute 3.136 in 3.135:

$$\hat{Y}_t = \theta_{Y1} \hat{Y}_{1t} + (1 - \theta_{Y1}) \hat{Y}_{2t} \quad (3.137)$$

## National Price Level

Log-linearize 3.49:

$$\begin{aligned} P_t Y_t &= P_{1t} Y_{1t} + P_{2t} Y_{2t} \implies \\ PY(1 + \hat{P}_t + \hat{Y}_t) &= P_1 Y_1(1 + \hat{P}_{1t} + \hat{Y}_{1t}) + P_2 Y_2(1 + \hat{P}_{2t} + \hat{Y}_{2t}) \implies \\ \hat{P}_t + \hat{Y}_t &= \frac{Y_1}{Y} (\hat{P}_{1t} + \hat{Y}_{1t}) + \frac{Y_2}{Y} (\hat{P}_{2t} + \hat{Y}_{2t}) \end{aligned} \quad (3.138)$$

Substitute 3.136 in 3.138:

$$\hat{P}_t + \hat{Y}_t = \theta_{Y1} (\hat{P}_{1t} + \hat{Y}_{1t}) + (1 - \theta_{Y1}) (\hat{P}_{2t} + \hat{Y}_{2t}) \quad (3.139)$$

Subtract 3.137 from 3.139:

$$\begin{aligned} \hat{P}_t + \hat{Y}_t - \hat{Y}_t &= \theta_{Y1} (\hat{P}_{1t} + \hat{Y}_{1t}) + (1 - \theta_{Y1}) (\hat{P}_{2t} + \hat{Y}_{2t}) - \\ &\quad - (\theta_{Y1} \hat{Y}_{1t} + (1 - \theta_{Y1}) \hat{Y}_{2t}) \implies \\ \hat{P}_t &= \theta_{Y1} \hat{P}_{1t} + (1 - \theta_{Y1}) \hat{P}_{2t} \end{aligned} \quad (3.140)$$

## Productivity Shock

Log-linearize 3.51:

$$\begin{aligned}\ln Z_{A\eta t} &= (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \implies (3.51) \\ \ln Z_{A\eta} + \hat{Z}_{A\eta t} &= (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} (\ln Z_{A\eta} + \hat{Z}_{A\eta,t-1}) + \varepsilon_{A\eta} \implies \\ \hat{Z}_{A\eta t} &= \rho_{A\eta} \hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} \end{aligned}\quad (3.141)$$

## Monetary Shock

Log-linearize 3.52:

$$\begin{aligned}\ln Z_{Mt} &= (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \implies (3.52) \\ \ln Z_M + \hat{Z}_{Mt} &= (1 - \rho_M) \ln Z_M + \rho_M (\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M \implies \\ \hat{Z}_{Mt} &= \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \end{aligned}\quad (3.142)$$

## Regional Market Clearing Condition

Log-linearize 3.54:

$$\begin{aligned}Y_{\eta t} &= C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \implies (3.54) \\ Y_{\eta} (1 + \hat{Y}_{\eta t}) &= C_{\eta 1} (1 + \hat{C}_{\eta 1t}) + C_{\eta 2} (1 + \hat{C}_{\eta 2t}) + I_{\eta} (1 + \hat{I}_{\eta t}) \implies \\ \hat{Y}_{\eta t} &= \frac{C_{\eta 1}}{Y_{\eta}} \hat{C}_{\eta 1t} + \frac{C_{\eta 2}}{Y_{\eta}} \hat{C}_{\eta 2t} + \frac{I_{\eta}}{Y_{\eta}} \hat{I}_{\eta t} \end{aligned}\quad (3.143)$$

Define the consumption and investment weights  $\langle \theta_{C\eta 1} \theta_{C\eta 2} (1 - \theta_{C\eta 1} - \theta_{C\eta 2}) \rangle$  in the regional production:

$$\langle \theta_{C\eta 1} \theta_{C\eta 2} (1 - \theta_{C\eta 1} - \theta_{C\eta 2}) \rangle := \left\langle \frac{C_{\eta 1}}{Y_{\eta}} \frac{C_{\eta 2}}{Y_{\eta}} \frac{I_{\eta}}{Y_{\eta}} \right\rangle \quad (3.144)$$

Substitute 3.144 in 3.143:

$$\hat{Y}_{\eta t} = \theta_{C\eta 1} \hat{C}_{\eta 1t} + \theta_{C\eta 2} \hat{C}_{\eta 2t} + (1 - \theta_{C\eta 1} - \theta_{C\eta 2}) \hat{I}_{\eta t} \quad (3.145)$$

### 3.3.1 Log-linear Model Structure

The log-linear model is a square system of 31 variables and 31 equations, summarized as follows:

- Variables:

Real Variables:  $\langle \hat{C}_\eta \hat{C}_{\eta 1} \hat{C}_{\eta 2} \hat{L}_\eta \hat{K}_\eta \hat{Y}_\eta \hat{I}_\eta \hat{Z}_{A\eta} \rangle$ ;

Nominal Variables:  $\langle \hat{\mathcal{E}}_\eta \hat{P} \hat{P}_\eta \hat{R} \hat{R}_K \hat{W} \hat{\pi} \hat{\pi}_\eta \hat{\lambda}_\eta \hat{Z}_M \rangle$ .

- Equations:

1. Regional Gross Inflation Rate

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta, t-1} \quad (3.102)$$

2. New Keynesian Phillips Curve

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta, t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{\eta t} \quad (3.112)$$

3. Law of Motion for Capital

$$\hat{K}_{\eta, t+1} = (1-\delta) \hat{K}_{\eta t} + \delta \hat{I}_{\eta t} \quad (3.113)$$

4. Regional Levels of Consumption and Prices

$$\hat{C}_{\eta 1 t} - \hat{C}_{\eta 2 t} = \hat{P}_{2t} - \hat{P}_{1t} \quad (3.116)$$

5. Total Expenses

$$\hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta t} + \omega_{\eta 1} \hat{P}_{1t} + (1 - \omega_{\eta 1}) \hat{P}_{2t} \quad (3.117)$$

6. Labor Supply

$$\varphi \hat{L}_{\eta t} - (1 - \sigma) \hat{C}_{\eta t} = \hat{W}_t - \hat{\mathcal{E}}_{\eta t} \quad (3.118)$$

7. Euler equation for the bonds return

$$\begin{aligned} \mathbb{E}_t \{ \hat{\mathcal{E}}_{\eta, t+1} - (1 - \sigma) \hat{C}_{\eta, t+1} \} - \\ - [\hat{\mathcal{E}}_{\eta t} - (1 - \sigma) \hat{C}_{\eta t}] = (1 - \beta) \hat{R}_t \end{aligned} \quad (3.119)$$

8. Euler equation for the capital return

$$\begin{aligned} \mathbb{E}_t \{ \hat{\mathcal{E}}_{\eta, t+1} - \hat{P}_{\eta, t+1} - (1 - \sigma) \hat{C}_{\eta, t+1} \} - (\hat{\mathcal{E}}_{\eta t} - \hat{P}_{\eta t} - (1 - \sigma) \hat{C}_{\eta t}) = \\ = \beta r_K \mathbb{E}_t \{ \hat{R}_{K, t+1} - \hat{P}_{\eta, t+1} \} \end{aligned} \quad (3.120)$$

9. Production Function

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha \hat{K}_{\eta t} + (1 - \alpha) \hat{L}_{\eta t} \quad (3.127)$$

10. Marginal Rates of Substitution of Factors

$$\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{W}_t - \hat{R}_{Kt} \quad (3.130)$$

11. Marginal Cost

$$\hat{\lambda}_{\eta t} = \alpha \hat{R}_{Kt} + (1 - \alpha) \hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_{\eta t} \quad (3.132)$$

12. Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1 - \gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \quad (3.133)$$

13. National Gross Inflation Rate

$$\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \quad (3.134)$$

14. National Price Level

$$\hat{P}_t = \theta_{Y1} \hat{P}_{1t} + (1 - \theta_{Y1}) \hat{P}_{2t} \quad (3.140)$$

15. Productivity Shock

$$\hat{Z}_{A\eta t} = \rho_{A\eta} \hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} \quad (3.141)$$

16. Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \quad (3.142)$$

17. Market Clearing Condition

$$\hat{Y}_t = \theta_{Y1} \hat{Y}_{1t} + (1 - \theta_{Y1}) \hat{Y}_{2t} \quad (3.137)$$

18. Regional Market Clearing Condition

$$\hat{Y}_{\eta t} = \theta_{C\eta 1} \hat{C}_{\eta 1t} + \theta_{C\eta 2} \hat{C}_{\eta 2t} + (1 - \theta_{C\eta 1} - \theta_{C\eta 2}) \hat{I}_{\eta t} \quad (3.145)$$

### 3.3.2 Extended Log-linear Structure

1. Regional Gross Inflation Rate

$$\begin{aligned}\hat{\pi}_{1t} &= \hat{P}_{1t} - \hat{P}_{1,t-1} \\ \hat{\pi}_{2t} &= \hat{P}_{2t} - \hat{P}_{2,t-1}\end{aligned}\tag{3.102}$$

2. New Keynesian Phillips Curve

$$\begin{aligned}\hat{\pi}_{1t} &= \beta \mathbb{E}_t \hat{\pi}_{1,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{1t} \\ \hat{\pi}_{2t} &= \beta \mathbb{E}_t \hat{\pi}_{2,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{2t}\end{aligned}\tag{3.112}$$

3. Law of Motion for Capital

$$\begin{aligned}\hat{K}_{1,t+1} &= (1-\delta) \hat{K}_{1t} + \delta \hat{I}_{1t} \\ \hat{K}_{2,t+1} &= (1-\delta) \hat{K}_{2t} + \delta \hat{I}_{2t}\end{aligned}\tag{3.113}$$

4. Regional Levels of Consumption and Prices

$$\begin{aligned}\hat{C}_{11t} - \hat{C}_{12t} &= \hat{P}_{2t} - \hat{P}_{1t} \\ \hat{C}_{21t} - \hat{C}_{22t} &= \hat{P}_{2t} - \hat{P}_{1t}\end{aligned}\tag{3.116}$$

5. Total Expenses

$$\begin{aligned}\hat{\mathcal{E}}_{1t} &= \hat{C}_{1t} + \omega_{11} \hat{P}_{1t} + (1-\omega_{11}) \hat{P}_{2t} \\ \hat{\mathcal{E}}_{2t} &= \hat{C}_{2t} + \omega_{21} \hat{P}_{1t} + (1-\omega_{21}) \hat{P}_{2t}\end{aligned}\tag{3.117}$$

6. Labor Supply

$$\begin{aligned}\varphi \hat{L}_{1t} - (1-\sigma) \hat{C}_{1t} &= \hat{W}_t - \hat{\mathcal{E}}_{1t} \\ \varphi \hat{L}_{2t} - (1-\sigma) \hat{C}_{2t} &= \hat{W}_t - \hat{\mathcal{E}}_{2t}\end{aligned}\tag{3.118}$$

7. Euler equation for the bonds return

$$\begin{aligned}\mathbb{E}_t \{ \hat{\mathcal{E}}_{1,t+1} - (1-\sigma) \hat{C}_{1,t+1} \} - [\hat{\mathcal{E}}_{1t} - (1-\sigma) \hat{C}_{1t}] &= (1-\beta) \hat{R}_t \\ \mathbb{E}_t \{ \hat{\mathcal{E}}_{2,t+1} - (1-\sigma) \hat{C}_{2,t+1} \} - [\hat{\mathcal{E}}_{2t} - (1-\sigma) \hat{C}_{2t}] &= (1-\beta) \hat{R}_t\end{aligned}\tag{3.119}$$

8. Euler equation for the capital return

$$\begin{aligned} \mathbb{E}_t \{ \hat{\mathcal{E}}_{1,t+1} - \hat{P}_{1,t+1} - (1 - \sigma) \hat{C}_{1,t+1} \} - \\ - (\hat{\mathcal{E}}_{1t} - \hat{P}_{1t} - (1 - \sigma) \hat{C}_{1t}) = \beta \frac{R_K}{P_1} \mathbb{E}_t \{ \hat{R}_{K,t+1} - \hat{P}_{1,t+1} \} \\ \mathbb{E}_t \{ \hat{\mathcal{E}}_{2,t+1} - \hat{P}_{2,t+1} - (1 - \sigma) \hat{C}_{2,t+1} \} - \\ - (\hat{\mathcal{E}}_{2t} - \hat{P}_{2t} - (1 - \sigma) \hat{C}_{2t}) = \beta \frac{R_K}{P_2} \mathbb{E}_t \{ \hat{R}_{K,t+1} - \hat{P}_{2,t+1} \} \end{aligned} \quad (3.120)$$

9. Production Function

$$\begin{aligned} \hat{Y}_{1t} &= \hat{Z}_{A1t} + \alpha \hat{K}_{1t} + (1 - \alpha) \hat{L}_{1t} \\ \hat{Y}_{2t} &= \hat{Z}_{A2t} + \alpha \hat{K}_{2t} + (1 - \alpha) \hat{L}_{2t} \end{aligned} \quad (3.127)$$

10. Marginal Rates of Substitution of Factors

$$\begin{aligned} \hat{K}_{1t} - \hat{L}_{1t} &= \hat{W}_t - \hat{R}_{Kt} \\ \hat{K}_{2t} - \hat{L}_{2t} &= \hat{W}_t - \hat{R}_{Kt} \end{aligned} \quad (3.130)$$

11. Marginal Cost

$$\begin{aligned} \hat{\lambda}_{1t} &= \alpha \hat{R}_{Kt} + (1 - \alpha) \hat{W}_t - \hat{Z}_{A1t} - \hat{P}_{1t} \\ \hat{\lambda}_{2t} &= \alpha \hat{R}_{Kt} + (1 - \alpha) \hat{W}_t - \hat{Z}_{A2t} - \hat{P}_{2t} \end{aligned} \quad (3.132)$$

12. Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1 - \gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \quad (3.133)$$

13. National Gross Inflation Rate

$$\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \quad (3.134)$$

14. National Price Level

$$\hat{P}_t + \hat{Y}_t = \theta_{PY1}(\hat{P}_{1t} + \hat{Y}_{1t}) + (1 - \theta_{PY1})(\hat{P}_{2t} + \hat{Y}_{2t}) \quad (3.140)$$

15. Productivity Shock

$$\begin{aligned} \hat{Z}_{A1t} &= \rho_{A1} \hat{Z}_{A1,t-1} + \varepsilon_{A1} \\ \hat{Z}_{A2t} &= \rho_{A2} \hat{Z}_{A2,t-1} + \varepsilon_{A2} \end{aligned} \quad (3.141)$$

16. Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \quad (3.142)$$

17. Market Clearing Condition

$$\hat{Y}_t = \theta_{Y1} \hat{Y}_{1t} + (1 - \theta_{Y1}) \hat{Y}_{2t} \quad (3.137)$$

18. Regional Market Clearing Condition

$$\begin{aligned}\hat{Y}_{1t} &= \theta_{C11} \hat{C}_{11t} + \theta_{C12} \hat{C}_{12t} + (1 - \theta_{C11} - \theta_{C12}) \hat{I}_{1t} \\ \hat{Y}_{2t} &= \theta_{C21} \hat{C}_{21t} + \theta_{C22} \hat{C}_{22t} + (1 - \theta_{C21} - \theta_{C22}) \hat{I}_{2t}\end{aligned} \quad (3.145)$$