Analysis of the Monetary Policy Impact on Regional Gross Domestic Product: A Regional DSGE Model

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Dedication

Acknowledgments

Epigraph

Neo: I know kung fu.
Morpheus: Show me.

Abstract

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Resumo

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List of Variables

MPC model predictive control

TLA Three Letter Acronym

U utility function

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1 Introduction

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2 Literature Review

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3 Model

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model illustration as in Osterno (2022).

The model is populated by four agents: (1) a representative household, (2) a continuum of firms producing intermediate goods, (3) a firm producing a final good, and (4) the monetary authority.

3.1 Household

Utility Maximization Problem

an utility function (U) or U or utility function (U)

Following the models presented by Costa Junior (2016) and Solis-Garcia (2022), the representative household problem is to maximize an intertemporal utility function U with respect to consumption C_t and labor L_t, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_t, L_t, K_{t+1}} : \quad U(C_t, L_t) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \phi \frac{L_t^{1+\varphi}}{1+\varphi} \right) \tag{3.1}$$

$$\text{s. t.}: \quad P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \tag{3.2}$$

$$K_{t+1} = (1-\delta) K_t + I_t \tag{3.3}$$

$$C_t, L_t, K_{t+1} \geq 0 \,; K_0 \text{ given.}$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, P_t is the price level, I_t is the investment, W_t is the wage level, K_t is the capital stock, R_t is the return on capital, and Π_t is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate.

Isolate I_t in 3.3 and substitute in 3.2:

$$K_{t+1} = (1-\delta)K_t + I_t \implies I_t = K_{t+1} - (1-\delta)K_t \tag{3.3}$$

$$P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \implies \tag{3.2}$$

$$P_t(C_t + K_{t+1} - (1-\delta)K_t) = W_t L_t + R_t K_t + \Pi_t \tag{3.4}$$

Lagrangian

The maximization problem with restriction can be transformed in one without restriction using the Lagrangian function $\mathscr{L}$ with 3.1 and 3.4:

$$\begin{aligned} \mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \phi \frac{L_t^{1+\varphi}}{1+\varphi} \right) - \\ -\mu_t \Big[P_t(C_t + K_{t+1} - (1-\delta)K_t) - (W_t L_t + R_t K_t + \Pi_t) \Big] \Bigg\} \end{aligned} \tag{3.5}$$

First Order Conditions

The first order conditions with respect to C_t, L_t, K_{t+1} and μ_t are:

$$C_t: \quad C_t^{-\sigma} - \mu_t P_t = 0 \implies \mu_t = \frac{C_t^{-\sigma}}{P_t} \tag{3.6}$$

$$L_t: \quad -\phi L_t^{\varphi} + \mu_t W_t = 0 \implies \mu_t = \frac{\phi L_t^{\varphi}}{W_t} \tag{3.7}$$

$$\begin{aligned} K_{t+1}: \quad & -\mu_t P_t + \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] = 0 \implies \\ & \mu_t P_t = \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] \end{aligned} \tag{3.8}$$

$$\mu_t: \quad P_t(C_t + K_{t+1} - (1-\delta)K_t) = W_t L_t + R_t K_t + \Pi_t \tag{3.4}$$

Solutions

Match equations 3.6 and 3.7:

$$\frac{C_t^{-\sigma}}{P_t} = \frac{\phi L_t^{\varphi}}{W_t} \implies \frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \tag{3.9}$$

Equation 3.9 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute μ_t and μ_{t+1} from equation 3.6 in 3.8:

$$\begin{aligned}
\mu_t P_t &= \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] \implies \\
\frac{C_t^{-\sigma}}{P_t} P_t &= \beta \mathbb{E}_t \frac{C_{t+1}^{-\sigma}}{P_{t+1}}[(1-\delta)P_{t+1} + R_{t+1}] \implies \\
\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} &= \beta \left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]
\end{aligned} \tag{3.10}$$

Equation 3.10 is the Household Euler equation.

Firms

Consider two types of firms: (1) a continuum of intermediate-good firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-good firm, which aggregates all the varieties into a final bundle and operates in perfect competition.

3.2 Final-Good Firm

Profit Maximization Problem

The role of the final-good firm is to aggregate all the varieties produced by the intermediate-good firms, so that the representative consumer can buy only one good Y_t, the bundle good. The final-good firm problem is to maximize its profit, considering that its output is the bundle Y_t formed by the continuum of intermediate goods Y_{jt}, where $j \in [0, 1]$ and ψ is the elasticity of substitution between intermediate goods:

$$\max_{Y_{jt}} : \quad \Pi_t = P_t Y_t - \int_0^1 P_{jt} Y_{jt} \, \mathrm{d}\, j \tag{3.11}$$

$$\text{s. t.} : \quad Y_t = \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \tag{3.12}$$

Substitute 3.12 in 3.11:

$$\max_{Y_{jt}} : \quad \Pi_t = P_t \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{jt} Y_{jt} \, \mathrm{d}\, j \tag{3.13}$$

First Order Condition and Solutions

The first order condition is:

$$\begin{aligned} Y_{jt} : \quad & P_t \left(\frac{\psi}{\psi - 1} \right) \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1} - 1} \left(\frac{\psi - 1}{\psi} \right) Y_{jt}^{\frac{\psi-1}{\psi} - 1} - P_{jt} = 0 \implies \\ & Y_{jt} = Y_t \left(\frac{P_t}{P_{jt}} \right)^{\psi} \end{aligned} \tag{3.14}$$

Equation 3.14 shows that the demand for variety j depends on its relative price.

Substitute 3.14 in 3.12:

$$\begin{aligned} Y_t &= \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} && \implies \\ Y_t &= \left(\int_0^1 \left[Y_t \left(\frac{P_t}{P_{jt}} \right)^{\psi} \right]^{\frac{\psi-1}{\psi}} \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} && \implies \\ P_t &= \left[\int_0^1 P_{jt}^{1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \end{aligned} \tag{3.15}$$

Equation 3.15 is the final-good firm's markup.

3.3 Intermediate-Good Firms

Cost Minimization Problem

The intermediate-good firms, denoted by $j \in [0,1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose capital K_{jt} and labor N_{jt} to minimize production costs, subject to a technology rule.

$$\min_{K_{jt},L_{jt}} : \quad R_t K_{jt} + W_t L_{jt} \tag{3.16}$$

$$\text{s. t.} : \quad Y_{jt} = Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} \tag{3.17}$$

where Y_{jt} is the output obtained by the production technology level Z_{At}[1] that transforms capital K_{jt} and labor L_{jt} in proportions α and $(1-\alpha)$, respectively, into intermediate goods.

Lagrangian

Applying the Lagrangian:

$$\mathscr{L} = (R_t K_{jt} + W_t L_{jt}) - \Lambda_t (Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} - Y_{jt}) \tag{3.18}$$

where the Lagrangian multiplier Λ_t is the marginal cost[2].

First Order Conditions

The first-order conditions are:

$$K_{jt} : \quad R_t - \Lambda_t Z_{At} \alpha K_{jt}^{\alpha-1} L_{jt}^{1-\alpha} = 0 \qquad \Longrightarrow \quad K_{jt} = \alpha Y_{jt} \frac{\Lambda_t}{R_t} \tag{3.19}$$

$$L_{jt} : \quad W_t - \Lambda_t Z_{At} K_{jt}^{\alpha} (1-\alpha) L_{jt}^{-\alpha} = 0 \qquad \Longrightarrow \quad L_{jt} = (1-\alpha) Y_{jt} \frac{\Lambda_t}{W_t} \tag{3.20}$$

$$\Lambda_t : \quad Y_{jt} = Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} \tag{3.17}$$

[1] the production technology level Z_{At} will be submitted to a productivity shock, detailed in section 4.5.

[2] see Lemma A.1

Solutions

Divide equation 3.19 by 3.20:

$$\frac{K_{jt}}{L_{jt}} = \frac{\alpha Y_{jt}\Lambda_t/R_t}{(1-\alpha)Y_{jt}\Lambda_t/W_t} \implies \frac{K_{jt}}{L_{jt}} = \left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t} \tag{3.21}$$

Equation 3.21 demonstrates the relationship between the technical marginal rate of substitution (TMRS) and the economical marginal rate of substitution (EMRS).

Substitute L_{jt} from equation 3.21 in 3.17:

$$\begin{aligned}
Y_{jt} &= Z_{At}K_{jt}^{\alpha}L_{jt}^{1-\alpha} && \implies \\
Y_{jt} &= Z_{At}K_{jt}^{\alpha}\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{R_tK_{jt}}{W_t}\right]^{1-\alpha} && \implies \\
K_{jt} &= \frac{Y_{jt}}{Z_{At}}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha} && (3.22)
\end{aligned}$$

Equation 3.22 is the intermediate-good firm demand for capital.

Substitute 3.22 in 3.21:

$$\begin{aligned}
L_{jt} &= \left(\frac{1-\alpha}{\alpha}\right)\frac{R_tK_{jt}}{W_t} && \implies \\
L_{jt} &= \left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}\frac{Y_{jt}}{Z_{At}}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha} && \implies \\
L_{jt} &= \frac{Y_{jt}}{Z_{At}}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{-\alpha} && (3.23)
\end{aligned}$$

Equation 3.23 is the intermediate-good firm demand for labor.

Total and Marginal Costs

Calculate the total cost using 3.22 and 3.23:

$$
\begin{aligned}
TC_{jt} &= W_t L_{jt} + R_t K_{jt} && \Longrightarrow \\
TC_{jt} &= W_t \frac{Y_{jt}}{Z_{At}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_t} \right]^{-\alpha} + R_t \frac{Y_{jt}}{Z_{At}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_t} \right]^{1-\alpha} && \Longrightarrow \\
TC_{jt} &= \frac{Y_{jt}}{Z_{At}} \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} && (3.24)
\end{aligned}
$$

Calculate the marginal cost using 3.24:

$$
\Lambda_{jt} = \frac{\partial TC_{jt}}{\partial Y_{jt}} \Longrightarrow \Lambda_{jt} = \frac{1}{Z_{At}} \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \tag{3.25}
$$

The marginal cost depends on the technological level Z_{At}, the nominal interest rate R_t and the nominal wage level W_t, which are the same for all intermediate-good firms, and because of that, the index j may be dropped:

$$
\Lambda_t = \frac{1}{Z_{At}} \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \tag{3.26}
$$

notice that:

$$
\Lambda_t = \frac{TC_{jt}}{Y_{jt}} \Longrightarrow TC_{jt} = \Lambda_t Y_{jt} \tag{3.27}
$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (CALVO, 1983): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{j,t+1} = P_{jt})$, and a probability $(1-\theta)$ of setting a new optimal price P^*_{jt} that maximizes its profits. Each firm selects its optimal price to maximize the present value of the profit flow, taking into account the nominal interest rate R_t, despite the uncertainty regarding its ability to adjust prices in future periods.

Consider an economy with price stickiness, following the Calvo Rule (CALVO, 1983): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{j,t+1} = P_{j,t})$, and a probability of $(1-\theta)$ of setting a new optimal price $P^*_{j,t}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price.

$$\max_{P_{jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt} Y_{j,t+s} - TC_{j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{3.28}$$

$$\text{s. t.}: \quad Y_{jt} = Y_t \left(\frac{P_t}{P_{jt}}\right)^{\psi} \tag{3.14}$$

Substitute 3.27 in 3.28:

$$\max_{P_{jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt} Y_{j,t+s} - \Lambda_{t+s} Y_{j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{3.29}$$

Substitute 3.14 in 3.29 and rearrange the variables:

$$\max_{P_{jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt} Y_{t+s} \left(\frac{P_{t+s}}{P_{jt}}\right)^{\psi} - \Lambda_{t+s} Y_{t+s} \left(\frac{P_{t+s}}{P_{jt}}\right)^{\psi}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$\max_{P_{jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt}^{1-\psi} P_{t+s}^{\psi} Y_{t+s} - P_{jt}^{-\psi} P_{t+s}^{\psi} Y_{t+s} \Lambda_{t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to P_{jt} is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{jt}^{-\psi} P_{t+s}^{\psi} Y_{t+s} - (-\psi) P_{jt}^{-\psi-1} P_{t+s}^{\psi} Y_{t+s} \Lambda_{t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) \left(\frac{P_{t+s}}{P_{jt}}\right)^{\psi} Y_{t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{jt}^{-1} \left(\frac{P_{t+s}}{P_{jt}}\right)^{\psi} Y_{t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{3.30}$$

Substitute 3.14 in 3.30:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) Y_{j,t+s}}{\prod_{k=0}^{s-1}(1 + R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{jt}^{-1} Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1}(1 + R_{t+k})} \right\} \implies$$

$$(\psi - 1)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s}}{\prod_{k=0}^{s-1}(1 + R_{t+k})} \right\} = \psi P_{jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1}(1 + R_{t+k})} \right\} \implies$$

$$P_{jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s}}{\prod_{k=0}^{s-1}(1 + R_{t+k})} \right\} = \frac{\psi}{\psi - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1}(1 + R_{t+k})} \right\} \implies$$

$$P_{jt}^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1}(1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1}(1 + R_{t+k}) \right\}} \tag{3.31}$$

Equation 3.31 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1}(1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1}(1 + R_{t+k}) \right\}} \tag{3.32}$$

3.3.1 Final-Good Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1 - \theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 3.15 to determine the aggregate price level:

$$P_t = \left[\int_0^{\theta} P_{t-1}^{1-\psi} \, \mathrm{d}\, j + \int_{\theta}^{1} P_t^{*1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies$$

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1 - \theta) P_t^{*1-\psi} \right]^{\frac{1}{1-\psi}} \tag{3.33}$$

Equation 3.33 is the aggregate price level.

3.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (TAYLOR, 1993) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \tag{3.34}$$

where π_t is the gross inflation rate, defined by:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{3.35}$$

and R, π, Y are the variables in steady state, γ_R is the smoothing parameter for the interest rate R_t, while γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively and Z_{Mt} is the monetary shock[3].

3.5 Stochastic Shocks

Productivity Shock

The production technology level Z_{At} will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{At} = (1 - \rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \tag{3.36}$$

where $\rho_A \in [0, 1]$ and $\varepsilon_{At} \sim \mathcal{N}(0, \sigma_A)$.

[3] for the monetary shock definition, see section 4.5.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{3.37}$$

where $\rho_M \in [0, 1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

3.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P_t^*, R_t^*, W_t^*\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C_t^*, L_t^*, K_{t+1}^*\}$ and for firms $\boldsymbol{\mathcal{A}}_F := \{K_{jt}^*, L_{jt}^*, Y_{jt}^*, Y_t^*\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{At}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms' problems, and the markets for goods and labor clear:

$$Y_t = C_t + I_t \tag{3.38}$$

$$L_t = \int_0^1 L_{jt} \, \mathrm{d}\, j \tag{3.39}$$

3.6.1 Model Structure

The model is composed of the preview solutions, forming a square system of 16 variables and 16 equations, summarized as follows:

- Variables (16):
  - from the household problem: C_t, L_t, K_{t+1};
  - from the final-good firm problem: Y_{jt}, P_t;
  - from the intermediate-good firm problems: K_{jt}, L_{jt}, P_t^*;
  - from the market clearing condition: Y_t, I_t;
  - prices: $W_t, R_t, \Lambda_t, \pi_t$;
  - shocks: Z_{At}, Z_{Mt}.
- Equations (16):

1. Labor Supply:

$$\frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \tag{3.9}$$

2. Household Euler Equation:

$$\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{3.10}$$

3. Budget Constraint:

$$P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \tag{3.2}$$

4. Law of Motion for Capital:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{3.3}$$

5. Bundle Technology:

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} \tag{3.12}$$

6. General Price Level:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{3.33}$$

7. Capital Demand:

$$K_{jt} = \alpha Y_{jt}\frac{\Lambda_t}{R_t} \tag{3.19}$$

8. Labor Demand:

$$L_{jt} = (1-\alpha)Y_{jt}\frac{\Lambda_t}{W_t} \tag{3.20}$$

9. Marginal Cost:

$$\Lambda_t = \frac{1}{Z_{At}}\left(\frac{R_t}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \tag{3.26}$$

10. Production Function:

$$Y_{jt} = Z_{At}K_{jt}^{\alpha}L_{jt}^{1-\alpha} \tag{3.17}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{3.32}$$

12. Market Clearing Condition:

$$Y_t = C_t + I_t \tag{3.38}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{3.34}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{3.35}$$

15. Productivity Shock:

$$\ln Z_{At} = (1 - \rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \tag{3.36}$$

16. Monetary Shock:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{3.37}$$

3.7 Steady State

The steady state is defined by the constancy of the variables through time. For any given endogenous variable X_t, it is in steady state if $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (COSTA JUNIOR, 2016, p.41). For conciseness, the *ss* index representing the steady state will be omitted, so that $X := X_{ss}$. The steady state of each equation of the model is:

1. Labor Supply:

$$\frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \implies \frac{\phi L^{\varphi}}{C^{-\sigma}} = \frac{W}{P} \tag{3.40}$$

2. Household Euler Equation:

$$\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies 1 = \beta\left[(1-\delta) + \frac{R}{P}\right] \tag{3.41}$$

3. Budget Constraint:

$$P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \implies P(C+I) = WL + RK + \Pi \tag{3.42}$$

4. Law of Motion for Capital:

$$K_{t+1} = (1-\delta)K_t + I_t \implies K = (1-\delta)K + I \implies I = \delta K \tag{3.43}$$

5. Bundle Technology:

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \implies Y = \left(\int_0^1 Y_j^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \tag{3.44}$$

6. General Price Level:

$$\begin{aligned} P_t &= \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} &&\implies \\ P^{1-\psi} &= \theta P^{1-\psi} + (1-\theta)P^{*1-\psi} &&\implies \\ (1-\theta)P^{1-\psi} &= (1-\theta)P^{*1-\psi} &&\implies P = P^* \end{aligned} \tag{3.45}$$

7. Capital Demand:

$$K_{jt} = \alpha Y_{jt}\frac{\Lambda_t}{R_t} \implies K_j = \alpha Y j\frac{\Lambda}{R} \tag{3.46}$$

8. Labor Demand:

$$L_{jt} = (1-\alpha)Y_{jt}\frac{\Lambda_t}{W_t} \implies L_j = (1-\alpha)Y j\frac{\Lambda}{W} \tag{3.47}$$

9. Marginal Cost:

$$\Lambda_t = \frac{1}{Z_{At}}\left(\frac{R_t}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \implies \Lambda = \frac{1}{Z_A}\left(\frac{R}{\alpha}\right)^{\alpha}\left(\frac{W}{1-\alpha}\right)^{1-\alpha} \tag{3.48}$$

10. Production Technology:

$$Y_{jt} = Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} \implies Y_j = Z_A K_j^{\alpha} L_j^{1-\alpha} \tag{3.49}$$

11. Optimal Price:

$$\begin{aligned}
P_t^* &= \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}} \implies && (3.32) \\
P^* &= \frac{\psi}{\psi-1} \cdot \frac{Y_j \Lambda / [1-\theta(1-R)]}{Y_j / [1-\theta(1-R)]} \implies && \\
P^* &= \frac{\psi}{\psi-1} \Lambda && (3.50)
\end{aligned}$$

12. Market Clearing Condition:

$$Y_t = C_t + I_t \implies Y = C + I \tag{3.51}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \tag{3.52}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \pi = 1 \tag{3.53}$$

15. Productivity Shock:

$$\begin{aligned}
\ln Z_{At} &= (1-\rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \implies \\
\ln Z_A &= (1-\rho_A) \ln Z_A + \rho_A \ln Z_A + \varepsilon_A \implies \\
\varepsilon_A &= 0 && (3.54)
\end{aligned}$$

16. Monetary Shock:

$$\begin{aligned}
\ln Z_{Mt} &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \implies \\
\ln Z_M &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_M + \varepsilon_M \implies \\
\varepsilon_M &= 0 && (3.55)
\end{aligned}$$

3.7.1 Variables in Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It's assumed that the productivity and the price level are normalized to one: $\left[\, P \; Z_A \,\right] = \vec{\mathbf{1}}$ [4].

From 3.45, the optimal price P^* is:

$$P^* = P \tag{3.56}$$

From 3.53, the gross inflation rate is:

$$\pi = 1 \tag{3.57}$$

From 3.52, the monetary shock is:

$$Z_M = 1 \tag{3.58}$$

From 3.54 and 3.55, the productivity and monetary shocks are:

$$\varepsilon_A = \varepsilon_M = 0 \tag{3.59}$$

From 3.41, the return on capital R is:

$$1 = \beta \left[(1-\delta) + \frac{R}{P} \right] \implies R = P \left[\frac{1}{\beta} - (1-\delta) \right] \tag{3.60}$$

From 3.50 and 3.45, the marginal cost Λ is:

$$P^* = \frac{\psi}{\psi - 1} \Lambda \implies \Lambda = P \frac{\psi - 1}{\psi} \tag{3.61}$$

From equation 3.48, the nominal wage W is:

$$\Lambda = \frac{1}{Z_A} \left(\frac{R}{\alpha} \right)^{\alpha} \left(\frac{W}{1-\alpha} \right)^{1-\alpha} \implies W = (1-\alpha) \left[\Lambda Z_A \left(\frac{\alpha}{R} \right)^{\alpha} \right]^{\frac{1}{1-\alpha}} \tag{3.62}$$

---

[4] where $\vec{\mathbf{1}}$ is the unit vector.

In steady state, prices are the same ($P = P^*$), resulting in a gross inflation level of one ($\pi = 1$), and all firms producing the same output level ($Y_j = Y$) due to the price parity (SOLIS-GARCIA, 2022, Lecture 13, p.12). For this reason, they all demand the same amount of factors (K, L), and equations 3.46, 3.47, and 3.49 become:

$$Y = Z_A K^{\alpha} L^{1-\alpha} \tag{3.63}$$

$$K = \alpha Y \frac{\Lambda}{R} \tag{3.64}$$

$$L = (1-\alpha) Y \frac{\Lambda}{W} \tag{3.65}$$

Substitute 3.64 in 3.43:

$$I = \delta K \implies I = \delta \alpha Y \frac{\Lambda}{R} \tag{3.66}$$

Substitute 3.65 in 3.40:

$$\frac{\phi L^{\varphi}}{C^{-\sigma}} = \frac{W}{P} \implies C = \left[L^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} \implies C = \left[\left((1-\alpha) Y \frac{\Lambda}{W} \right)^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} \tag{3.67}$$

Substitute 3.66 and 3.67 in 3.51:

$$\begin{aligned}
Y &= C + I && \implies \\
Y &= \left[\left((1-\alpha) Y \frac{\Lambda}{W} \right)^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} + \left[\delta \alpha Y \frac{\Lambda}{R} \right] && \implies \\
Y &= \left[\left(\frac{W}{\phi P} \right) \left(\frac{W}{(1-\alpha)\Lambda} \right)^{\varphi} \left(\frac{R}{R - \delta \alpha \Lambda} \right)^{\sigma} \right]^{\frac{1}{\varphi + \sigma}} && (3.68)
\end{aligned}$$

For C, K, L, I, use the result from 3.68 in 3.67, 3.64, 3.65 and 3.43, respectively.

3.7.2 Steady State Solution

$$\begin{bmatrix} P & P^* & \pi & Z_A & Z_M \end{bmatrix} = \vec{\mathbf{1}} \tag{3.69}$$

$$\begin{bmatrix} \varepsilon_A & \varepsilon_M \end{bmatrix} = \vec{\mathbf{0}} \tag{3.70}$$

$$R = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{3.60}$$

$$\Lambda = P\frac{\psi - 1}{\psi} \tag{3.61}$$

$$W = (1-\alpha)\left[\Lambda Z_A \left(\frac{\alpha}{R}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} \tag{3.62}$$

$$Y = \left[\left(\frac{W}{\phi P}\right)\left(\frac{W}{(1-\alpha)\Lambda}\right)^{\varphi}\left(\frac{R}{R-\delta\alpha\Lambda}\right)^{\sigma}\right]^{\frac{1}{\varphi+\sigma}} \tag{3.68}$$

$$C = \left[\left((1-\alpha)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \tag{3.67}$$

$$K = \alpha Y\frac{\Lambda}{R} \tag{3.64}$$

$$L = (1-\alpha)Y\frac{\Lambda}{W} \tag{3.65}$$

$$I = \delta K \tag{3.43}$$

3.8 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. Dynare is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation 3.32. For this purpose, Uhlig's rules of log-linearization (UHLIG, 1999) will be applied to all equations in the model[5].

[5] see lemma A.3 for details.

3.8.1 Gross Inflation Rate

Log-linearize 3.35 and define the level deviation of gross inflation rate $\widetilde{\pi}_t$:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \tag{3.35}$$

$$\widetilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{3.71}$$

3.8.2 New Keynesian Phillips Curve

In order to log-linearize equation 3.32, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma A.5:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{3.32}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\quad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{3.72}$$

First, log-linearize the left hand side of equation 3.72 with respect to $P_t^*, Y_{j,t}, \widetilde{R}_t$:

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies \\ &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P^* Y_j \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies \\ &P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \end{aligned}$$

Separate the terms not dependent on s:

$$P^* Y_j (1+\hat{P}_t^*)\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\right\}+$$

$$+P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{j,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} \Longrightarrow$$

Apply definition A.10 on the first term:

$$\frac{P^* Y_j (1+\hat{P}_t^*)}{1-\theta/(1+R)}+P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{j,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Second, log-linearize the left hand side of equation 3.72 with respect to $\Lambda_t^*, Y_{j,t}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\frac{\theta^s Y_{j,t+s}\Lambda_{t+s}}{(1+R)^s\left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)}\right\} \Longrightarrow$$

$$\frac{\psi}{\psi-1}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s \frac{Y_j\Lambda(1+\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s})}{1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right\} \Longrightarrow$$

$$\frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(1+\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\right\}+$$

$$+\frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Apply definition A.10 on the first term:

$$\frac{\psi}{\psi-1}\cdot\frac{Y_j\Lambda}{1-\theta/(1+R)}+$$

$$+\frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Join both sides of the equation again:

$$\frac{P^* Y_j (1 + \hat{P}_t^*)}{1 - \theta/(1+R)} + P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} =$$
$$= \frac{\psi}{\psi - 1} \cdot \frac{Y_j \Lambda}{1 - \theta/(1+R)} +$$
$$+ \frac{\psi}{\psi - 1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \tag{3.73}$$

Define a nominal discount rate ρ in steady state:

$$1 = \rho(1+R) \implies \rho = \frac{1}{1+R} \tag{3.74}$$

Substitute 3.74 in 3.73:

$$\frac{P^* Y_j (1 + \hat{P}_t^*)}{1 - \theta\rho} + P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\rho)^s \left(\hat{Y}_{j,t+s} - \rho \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} = \frac{\psi}{\psi - 1} \cdot \frac{Y_j \Lambda}{1 - \theta\rho} +$$
$$+ \frac{\psi}{\psi - 1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\rho)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \rho \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \tag{3.75}$$

Substitute 3.61 in 3.75 and simplify all common terms:

$$\cancel{\frac{P^* Y_j}{1 - \theta\rho}} + \frac{\cancel{P^* Y_j} \hat{P}_t^*}{1 - \theta\rho} + \cancel{P^* Y_j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\rho)^s \left(\cancel{\hat{Y}_{j,t+s} - \rho \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right) \right\} =$$
$$= \cancel{\frac{P^* Y_j}{1 - \theta\rho}} + \cancel{P^* Y_j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\rho)^s \left(\cancel{\hat{Y}_{j,t+s} - \rho \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} + \hat{\Lambda}_{t+s} \right) \right\} \implies$$
$$\frac{\hat{P}_t^*}{1 - \theta\rho} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\rho)^s \left(\hat{\Lambda}_{t+s} \right) \right\} \tag{3.76}$$

Define the real marginal cost λ_t:

$$\lambda_t = \frac{\Lambda_t}{P_t} \implies \Lambda_t = P_t \lambda_t \implies$$
$$\hat{\Lambda}_t = \hat{P}_t + \hat{\lambda}_t \tag{3.77}$$

Substitute 3.77 in 3.76:

$$\hat{P}_t^* = (1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{3.78}$$

Log-linearize equation 3.33:

$$\begin{aligned} P_t^{1-\psi} &= \theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi} \qquad \Longrightarrow & (3.33)\\ P^{1-\psi}(1+(1-\psi)\hat{P}_t) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{t-1}) + \\ &+ (1-\theta)P^{1-\psi}(1+(1-\psi)\hat{P}_t^*) \implies \\ \hat{P}_t &= \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* & (3.79) \end{aligned}$$

Substitute 3.78 in 3.79:

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* \tag{3.79}$$

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{3.80}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\rho\mathbb{L}^{-1})$[6] in 3.80:

$$\begin{aligned} (1-\theta\rho\mathbb{L}^{-1})\hat{P}_t &= (1-\theta\rho\mathbb{L}^{-1}) \Big[\theta\hat{P}_{t-1} + \\ &+(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right)\Big] \implies \\ \hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + \\ &(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) - \\ &- \theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right) \end{aligned} \tag{3.81}$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\rho$ within the operator. Then, cancel the summations and rearrange

[6] see definition A.11.

the terms:

$$\begin{aligned}
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + \\
&\quad (1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^s\left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) - \\
&\quad - \theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^s\left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right) \implies \\
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + (1-\theta)(1-\theta\rho)(\hat{P}_t + \hat{\lambda}_t) + \\
&\quad + \cancel{(1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^{s+1}\left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right)} - \\
&\quad - \cancel{(1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^{s+1}\left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right)} \implies \\
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta^2\rho\hat{P}_t + (1-\theta-\theta\rho+\theta^2\rho)\hat{P}_t + (1-\theta)(1-\theta\rho)\hat{\lambda}_t \implies \\
(\hat{P}_t - \hat{P}_{t-1}) &= \rho(\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_t) + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \qquad (3.82)
\end{aligned}$$

Substitute 3.71 in 3.82:

$$\tilde{\pi}_t = \rho\mathbb{E}_t\tilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \qquad (3.83)$$

Equation 3.83 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

3.8.3 Labor Supply

Log-linearize 3.9:

$$\frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \implies \qquad (3.9)$$

$$\varphi\hat{L}_t + \sigma\hat{C}_t = \hat{W}_t + \hat{P}_t \qquad (3.84)$$

3.8.4 Household Euler Equation

Log-linearize 3.10:

$$\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies \tag{3.10}$$

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{3.85}$$

3.8.5 Law of Motion for Capital

Log-linearize 3.3:

$$K_{t+1} = (1-\delta)K_t + I_t \implies \tag{3.3}$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{3.86}$$

3.8.6 Bundle Technology

Apply the natural logarithm to 3.12:

$$\ln Y_t = \frac{\psi}{\psi-1}\ln\left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)$$

Log-linearize using corollary A.3.1:

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(\int_0^1 Y_j^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(Y_j^{\frac{\psi-1}{\psi}}\int_0^1 \mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \cancel{\frac{\psi}{\psi-1}}\left[\cancel{\frac{\psi-1}{\psi}}\ln Y_j + \cancel{\ln 1} + \cancel{\frac{\psi-1}{\psi}}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j$$

Apply corollary A.2.1:

$$\begin{aligned}\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j \implies \\ \hat{Y}_t = \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\end{aligned} \tag{3.87}$$

3.8.7 Marginal Cost

Log-linearize 3.26:

$$\Lambda_t = Z_{At}^{-1}\frac{R_t^\alpha W_t^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} \implies \tag{3.26}$$

$$\Lambda(1+\hat{\Lambda}_t) = \frac{1}{Z_A}\left(\frac{R}{\alpha}\right)^\alpha\left(\frac{W}{1-\alpha}\right)^{1-\alpha}(1-\hat{Z}_{At}+\alpha\hat{R}_t+(1-\alpha)\hat{W}_t) \implies$$

$$\hat{\Lambda}_t = \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} \tag{3.88}$$

Substitute 3.77 in 3.88:

$$\begin{aligned}\hat{\Lambda}_t &= \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} \implies \\ \hat{P}_t + \hat{\lambda}_t &= \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} \implies \\ \hat{\lambda}_t &= \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} - \hat{P}_t\end{aligned} \tag{3.89}$$

3.8.8 Production Function

Log-linearize 3.17:

$$Y_{jt} = Z_{At}K_{jt}^\alpha L_{jt}^{1-\alpha} \implies \tag{3.17}$$

$$\begin{aligned}Y_j(1+\hat{Y}_{jt}) &= Z_A K_j^\alpha L_j^{1-\alpha}(1+\hat{Z}_{At}+\alpha\hat{K}_{jt}+(1-\alpha)\hat{L}_{jt}) \implies \\ \hat{Y}_{jt} &= \hat{Z}_{At} + \alpha\hat{K}_{jt} + (1-\alpha)\hat{L}_{jt}\end{aligned} \tag{3.90}$$

Substitute 3.90 in 3.87:

$$\hat{Y}_t = \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j \qquad \Longrightarrow \tag{3.87}$$

$$\hat{Y}_t = \int_0^1 \left[\hat{Z}_{At} + \alpha\hat{K}_{jt} + (1-\alpha)\hat{L}_{jt}\right]\mathrm{d}\,j \qquad \Longrightarrow$$

$$\hat{Y}_t = \hat{Z}_{At} + \alpha\int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \tag{3.91}$$

Apply the natural logarithm and then log-linearize 3.39:

$$L_t = \int_0^1 L_{jt}\,\mathrm{d}\,j \qquad \Longrightarrow \tag{3.39}$$

$$\ln L_t = \ln\left[\int_0^1 L_{jt}\,\mathrm{d}\,j\right] \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln\left[\int_0^1 L_j\,\mathrm{d}\,j\right] + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln L_j + \ln 1 + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j$$

Apply corollary A.2.1:

$$\Longrightarrow\ \hat{L}_t = \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \tag{3.92}$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_t = \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j \tag{3.93}$$

Substitute 3.92 and 3.93 in 3.91:

$$\hat{Y}_t = \hat{Z}_{At} + \alpha\int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \ \Longrightarrow \tag{3.91}$$

$$\hat{Y}_t = \hat{Z}_{At} + \alpha\hat{K}_t + (1-\alpha)\hat{L}_t \tag{3.94}$$

3.8.9 Capital Demand

Log-linearize 3.19:

$$\begin{aligned} K_{jt} &= \alpha Y_{jt}\frac{\Lambda_t}{R_t} && \implies && (3.19)\\ K_j(1+\hat{K}_{jt}) &= \alpha Y_j\frac{\Lambda}{R}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t) && \implies \\ \hat{K}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t \end{aligned}$$

Integrate both sides and then substitute 3.93 and 3.87:

$$\begin{aligned} \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j &= \int_0^1 (\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t)\,\mathrm{d}\,j \implies \\ \hat{K}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t \end{aligned} \tag{3.95}$$

3.8.10 Labor Demand

Log-linearize 3.20:

$$\begin{aligned} L_{jt} &= (1-\alpha)Y_{jt}\frac{\Lambda_t}{W_t} && \implies && (3.20)\\ L_j(1+\hat{L}_{jt}) &= (1-\alpha)Y_j\frac{\Lambda}{W}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t) && \implies \\ \hat{L}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t \end{aligned}$$

Integrate both sides and then substitute 3.92 and 3.87:

$$\begin{aligned} \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j &= \int_0^1 \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t\,\mathrm{d}\,j \implies \\ \hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t \end{aligned} \tag{3.96}$$

Subtract 3.96 from 3.95:

$$\begin{aligned} \hat{K}_t-\hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t-(\hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t) \implies \\ \hat{K}_t-\hat{L}_t &= \hat{W}_t-\hat{R}_t \end{aligned} \tag{3.97}$$

Equation 3.97 is the log-linearized version of 3.21.

3.8.11 Market Clearing Condition

Log-linearize 3.38:

$$
\begin{aligned}
Y_t &= C_t + I_t && \implies && (3.38)\\
Y(1+\hat{Y}_t) &= C(1+\hat{C}_t) + I(1+\hat{I}_t) && \implies\\
Y + Y\hat{Y}_t &= C + C\hat{C}_t + I + I\hat{I}_t && \implies\\
Y\hat{Y}_t &= C\hat{C}_t + I\hat{I}_t && \implies\\
\hat{Y}_t &= \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t && && (3.98)
\end{aligned}
$$

Define the consumption and investment weights $\begin{bmatrix}\theta_C & \theta_I\end{bmatrix}$ in the production total:

$$
\begin{bmatrix}\theta_C & \theta_I\end{bmatrix} := \begin{bmatrix}\frac{C}{Y} & \frac{I}{Y}\end{bmatrix} \tag{3.99}
$$

Substitute 3.99 in 3.98:

$$
\begin{aligned}
&\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t \implies\\
&\hat{Y}_t = \theta_C\hat{C}_t + \theta_I\hat{I}_t && (3.100)
\end{aligned}
$$

3.8.12 Monetary Policy

Log-linearize 3.34:

$$
\begin{aligned}
&\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi} Y_t^{\gamma_Y})^{(1-\gamma_R)} Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \implies && (3.34)\\
&\frac{R(1+\hat{R}_t)}{R} =\\
&= \frac{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)} Z_M[1 + \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt}]}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \implies\\
&\hat{R}_t = \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt} && (3.101)
\end{aligned}
$$

3.8.13 Productivity Shock

Log-linearize 3.36:

$$\ln Z_{At} = (1-\rho_A)\ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \quad \Longrightarrow \tag{3.36}$$
$$\ln Z_A + \hat{Z}_{At} = (1-\rho_A)\ln Z_A + \rho_A(\ln Z_A + \hat{Z}_{A,t-1}) + \varepsilon_A \quad \Longrightarrow$$
$$\hat{Z}_{At} = \rho_A \hat{Z}_{A,t-1} + \varepsilon_A \tag{3.102}$$

3.8.14 Monetary Shock

Log-linearize 3.37:

$$\ln Z_{Mt} = (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \quad \Longrightarrow \tag{3.37}$$
$$\ln Z_M + \hat{Z}_{Mt} = (1-\rho_M)\ln Z_M + \rho_M(\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M \quad \Longrightarrow$$
$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{3.103}$$

3.8.15 Log-linear Model Structure

The log-linear model is a square system of 12 variables and 12 equations, summarized as follows:

- Variables: $(\tilde{\pi} \quad \hat{P} \quad \tilde{\lambda} \quad \hat{C} \quad \hat{L} \quad \hat{R} \quad \hat{K} \quad \hat{I} \quad \hat{W} \quad \hat{Z}_A \quad \hat{Y} \quad \hat{Z}_M)$
- Equations:

1. Gross Inflation Rate:

$$\widetilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{3.71}$$

2. New Keynesian Phillips Curve:

$$\widetilde{\pi}_t = \rho \mathbb{E}_t \widetilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \tag{3.83}$$

3. Labor Supply:

$$\varphi \hat{L}_t + \sigma \hat{C}_t = \hat{W}_t + \hat{P}_t \tag{3.84}$$

4. Household Euler Equation:

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{3.85}$$

5. Law of Motion for Capital:

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{3.86}$$

6. Real Marginal Cost:

$$\hat{\lambda}_t = \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} - \hat{P}_t \tag{3.89}$$

7. Production Function:

$$\hat{Y}_t = \hat{Z}_{At} + \alpha\hat{K}_t + (1-\alpha)\hat{L}_t \tag{3.94}$$

8. Marginal Rates of Substitution of Factors:

$$\hat{K}_t - \hat{L}_t = \hat{W}_t - \hat{R}_t \tag{3.97}$$

9. Market Clearing Condition:

$$\hat{Y}_t = \theta_C\hat{C}_t + \theta_I\hat{I}_t \tag{3.100}$$

10. Monetary Policy:

$$\hat{R}_t = \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt} \tag{3.101}$$

11. Productivity Shock:

$$\hat{Z}_{At} = \rho_A\hat{Z}_{A,t-1} + \varepsilon_A \tag{3.102}$$

12. Monetary Shock:

$$\hat{Z}_{Mt} = \rho_M\hat{Z}_{M,t-1} + \varepsilon_M \tag{3.103}$$

4 Regions

Regions will be identified by the index $\eta \in \{A, B, \ldots, n\}$. For example, the variable C_t represents the total consumption, while $C_{\eta t}$ represents the consumption of region η. Without loss of generality, the model will have two regions: the main region A and the remaining of the country B, so that $\eta \in \{A, B\}$.

Determining whether the variable (or parameter) should be region-specific (or not) requires justification:

falta revisar esta parte e agrupar por agente da economia

colocar estatística descritiva para justificar as variáveis

- C_{it} and I_{it}: Consumption from region to region should vary based on the abundance of natural resources and the available technology in that region: each region will specialize in producing goods that are resource-intensive, considering the resources that are abundant in that specific region. This will increase the supply, decreasing their relative price and making them more demanded. Investment is decided based on the household maximization problem, in which consumption level must be decided regionally.

- σ_i: Consumer preference should be somehow tied to cultural aspects, such as food choices (coastal regions will have a higher emphasis on seafood) or climate characteristics (warmer regions require air conditioning, while colder regions need heaters).

- L_{it} and φ_{it}: The same reasoning applies to the supply of labor and the marginal disutility of labor: the cultural and climatic aspects of each region should influence these two factors.

- Y_t and P_t: Final-good production and price levels should be both unique for the whole country, considering that there is only one final-good representative firm. This firm works in perfect competition so that the price is given.

- W_{it}: There is no mobility for families, just for goods, so that each region will have its own wage level based on its closed labor market. The same applies to the profits: intermediate-good firms will operate in monopolist competition in each region, generating different return levels.

- R_t: Nominal rate level is a macroeconomic variable and the instrument of the monetary authority, which is a central government entity.

- Y_{ijt}, P_{ijt}, Π_{ijt} and θ_{it}: Each region i has j intermediate-good firms, each operating with market power enabling a differentiated price (and profits) for its variety and also submitted to a different possibility of updating the its price each period.

- A_{it} and α_i: The assumption is that each region has unique characteristics: exogenous (geographic and cultural) and endogenous (technological level, capital and labor supply). Because capital and labor supply levels are different, the intensity of each in the production function should be also different.

Regional Model

model illustration as in Osterno (2022).

The model is populated by four agents: (1) a representative household, (2) a continuum of firms producing intermediate goods, (3) a firm producing a final good, and (4) the monetary authority.

4.1 Household

Utility Maximization Problem

Following the models presented by Costa Junior (2016) and Solis-Garcia (2022), the representative household problem is to maximize an intertemporal utility function U_η with respect to consumption $C_{\eta t}$ and labor $L_{\eta t}$, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_{1\eta t}, C_{2\eta t}, L_{\eta t}, K_{\eta, t+1}} : \quad U_\eta(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{4.1}$$

$$\text{s. t.}: \quad P_{1t}C_{1\eta t} + P_{2t}C_{2\eta t} + P_{\eta t} I_{\eta t} = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{4.2}$$

$$K_{\eta, t+1} = (1-\delta) K_{\eta t} + I_{\eta t} \tag{4.3}$$

$$C_{\eta t} = C_{1\eta t}^{\theta_1} C_{2\eta t}^{1-\theta_1} \tag{4.4}$$

$$C_{m\eta t}, L_{\eta t}, K_{\eta, t+1} > 0 \,;\, K_0 \text{ given.}$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, P_t is the price level, $I_{\eta t}$ is the investment, W_t is the wage level, $K_{\eta t}$ is the capital stock, R_t is the return on capital, and $\Pi_{\eta t}$ is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate. In the consumption aggregation, $C_{m\eta t}$ is the good produced in region $m \in \{1, 2\}$ with weight θ_m in the household basket.

Isolate $I_{\eta t}$ in 4.3 and substitute in 4.2:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies I_{\eta t} = K_{\eta,t+1} - (1-\delta)K_{\eta t} \tag{4.3}$$

$$P_{1t}C_{1\eta t} + P_{2t}C_{2\eta t} + P_{\eta t}I_{\eta t} = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \implies \tag{4.2}$$

$$P_{1t}C_{1\eta t} + P_{2t}C_{2\eta t} + P_{\eta t}(K_{\eta,t+1} - (1-\delta)K_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{4.5}$$

Substitute 4.4 in 4.1:

$$U_\eta(C_{1\eta t}, C_{2\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{\left[C_{1\eta t}^{\theta_1} C_{2\eta t}^{1-\theta_1}\right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{4.6}$$

Lagrangian

The maximization problem with restriction can be transformed in one without restriction using the Lagrangian function $\mathscr{L}$ with 4.6 and 4.5:

$$\begin{aligned} \mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\frac{\left[C_{1\eta t}^{\theta_1} C_{2\eta t}^{1-\theta_1}\right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) - \mu_t \Big[P_{1t}C_{1\eta t} + \\ + P_{2t}C_{2\eta t} + P_{\eta t}(K_{\eta,t+1} - (1-\delta)K_{\eta t}) - (W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t}) \Big] \Bigg\} \end{aligned} \tag{4.7}$$

First Order Conditions

The first order conditions are:

$$C_{1\eta t}: \quad \mu_t = \frac{\theta_1 C_{1\eta t}^{\theta_1(1-\sigma)-1} C_{2\eta t}^{(1-\theta_1)(1-\sigma)}}{P_{1t}} \tag{4.8}$$

$$C_{2\eta t}: \quad \mu_t = \frac{(1-\theta_1) C_{1\eta t}^{\theta_1(1-\sigma)} C_{2\eta t}^{(1-\theta_1)(1-\sigma)-1}}{P_{2t}} \tag{4.9}$$

$$L_{\eta t}: \quad -\phi L_{\eta t}^{\varphi} + \mu_t W_t = 0 \implies \mu_t = \frac{\phi L_{\eta t}^{\varphi}}{W_t} \tag{4.10}$$

$$\begin{aligned} K_{\eta,t+1}: \quad & -\mu_t P_t + \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] = 0 \implies \\ & \mu_t P_t = \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mu_t: \quad & P_{1t}C_{1\eta t} + P_{2t}C_{2\eta t} + P_{\eta t}(K_{\eta,t+1} - (1-\delta)K_{\eta t}) = \\ & = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \end{aligned} \tag{4.5}$$

Solutions

Match 4.8 and 4.9:

$$\mu_t = \frac{\theta_1 C_{1\eta t}^{\theta_1(1-\sigma)-1} C_{2\eta t}^{(1-\theta_1)(1-\sigma)}}{P_{1t}} = \frac{(1-\theta_1) C_{1\eta t}^{\theta_1(1-\sigma)} C_{2\eta t}^{(1-\theta_1)(1-\sigma)-1}}{P_{2t}} \implies \tag{4.12}$$

Substitute **??** in 4.8:

$$\mu_t = \frac{C_{\eta t}^{-\sigma}}{P_t} \tag{4.13}$$

Match 4.14 and 4.10:

$$\frac{C_{\eta t}^{-\sigma}}{P_t} = \frac{\phi L_{\eta t}^{\varphi}}{W_t} \implies \frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \tag{4.14}$$

Equation 4.15 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute μ_t and μ_{t+1} from equation 4.14 in 4.11:

$$\begin{aligned}
\mu_t P_t &= \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] \quad \Longrightarrow \\
\frac{C_{\eta t}^{-\sigma}}{P_t} P_t &= \beta \mathbb{E}_t \frac{C_{\eta,t+1}^{-\sigma}}{P_{t+1}}[(1-\delta)P_{t+1} + R_{t+1}] \Longrightarrow \\
\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} &= \beta \left[(1-\delta) + \mathbb{E}_t \left(\frac{R_{t+1}}{P_{t+1}}\right)\right]
\end{aligned} \tag{4.15}$$

Equation 4.16 is the Household Euler equation.

Firms

Consider two types of firms: (1) a continuum of intermediate-good firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-good firm, which aggregates all the varieties into a final bundle and operates in perfect competition.

4.2 Final-Good Firm

Profit Maximization Problem

The role of the final-good firm is to aggregate all the varieties produced by the intermediate-good firms, so that the representative consumer can buy only one good $Y_{\eta t}$, the bundle good. The final-good firm problem is to maximize its profit, considering that its output is the bundle $Y_{\eta t}$ formed by the continuum of intermediate goods $Y_{\eta j t}$, where $j \in [0,1]$ and ψ is the elasticity of substitution between intermediate goods:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_t Y_{\eta t} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{4.16}$$

$$\text{s.t.}: \quad Y_{\eta j t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} \tag{4.17}$$

Substitute 4.18 in 4.17:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_t \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{4.18}$$

First Order Condition and Solutions

The first order condition is:

$$Y_{\eta jt}: \quad P_t\left(\frac{\psi}{\psi-1}\right)\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}-1}\left(\frac{\psi-1}{\psi}\right)Y_{\eta jt}^{\frac{\psi-1}{\psi}-1} - P_{\eta jt} = 0 \implies$$

$$Y_{\eta jt} = Y_t\left(\frac{P_t}{P_{\eta jt}}\right)^{\psi} \tag{4.19}$$

Equation 4.20 shows that the demand for variety j depends on its relative price.

Substitute 4.20 in 4.18:

$$Y_t = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \qquad \implies$$

$$Y_t = \left(\int_0^1 \left[Y_t\left(\frac{P_t}{P_{\eta jt}}\right)^{\psi}\right]^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \qquad \implies$$

$$P_t = \left[\int_0^1 P_{\eta jt}^{1-\psi}\,\mathrm{d}\,j\right]^{\frac{1}{1-\psi}} \tag{4.20}$$

Equation 4.21 is the final-good firm's markup.

4.3 Intermediate-Good Firms

Cost Minimization Problem

The intermediate-good firms, denoted by $j \in [0,1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose capital $K_{\eta jt}$ and labor N_{jt} to minimize production costs, subject to a technology rule.

$$\min_{K_{\eta jt},L_{\eta jt}} : \quad R_t K_{\eta jt} + W_t L_{\eta jt} \tag{4.21}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \tag{4.22}$$

where $Y_{\eta jt}$ is the output obtained by the production technology level $Z_{A\eta t}$[7] that transforms capital $K_{\eta jt}$ and labor $L_{\eta jt}$ in proportions α_η and $(1-\alpha_\eta)$, respectively, into intermediate goods.

Lagrangian

Applying the Lagrangian:

$$\mathcal{L} = (R_t K_{\eta jt} + W_t L_{\eta jt}) - \Lambda_{\eta jt}(Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} - Y_{\eta jt}) \tag{4.23}$$

where the Lagrangian multiplier $\Lambda_{\eta jt}$ is the marginal cost[8].

First Order Conditions

The first-order conditions are:

$$\begin{aligned}
K_{\eta jt}: \quad & R_t - \Lambda_{\eta jt} Z_{A\eta t} \alpha_\eta K_{\eta jt}^{\alpha_\eta - 1} L_{\eta jt}^{1-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& K_{\eta jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta jt}}{R_t} && (4.24) \\
L_{\eta jt}: \quad & W_t - \Lambda_{\eta jt} Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} (1-\alpha_\eta) L_{\eta jt}^{-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& L_{\eta jt} = (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta jt}}{W_t} && (4.25) \\
\Lambda_{\eta jt}: \quad & Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} && (4.23)
\end{aligned}$$

Solutions

Divide equation 4.25 by 4.26:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \frac{\alpha_\eta Y_{\eta jt} \Lambda_{\eta jt} / R_t}{(1-\alpha_\eta) Y_{\eta jt} \Lambda_{\eta jt} / W_t} \quad \Longrightarrow \quad \frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \tag{4.26}$$

Equation 4.27 demonstrates the relationship between the technical marginal rate of substitution (TMRS) and the economical marginal rate of substitution (EMRS).

---

[7] the production technology level $Z_{A\eta t}$ will be submitted to a productivity shock, detailed in section 4.5.

[8] see Lemma A.1

Substitute $L_{\eta jt}$ from equation 4.27 in 4.23:

$$
\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} && \Longrightarrow \\
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} \left[\left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t K_{\eta jt}}{W_t} \right]^{1-\alpha_\eta} && \Longrightarrow \\
K_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{1-\alpha_\eta}
\end{aligned}
\tag{4.27}
$$

Equation 4.28 is the intermediate-good firm demand for capital.

Substitute 4.28 in 4.27:

$$
\begin{aligned}
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t K_{\eta jt}}{W_t} && \Longrightarrow \\
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t}{W_t} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{1-\alpha_\eta} && \Longrightarrow \\
L_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{-\alpha_\eta}
\end{aligned}
\tag{4.28}
$$

Equation 4.29 is the intermediate-good firm demand for labor.

Total and Marginal Costs

Calculate the total cost using 4.28 and 4.29:

$$
\begin{aligned}
TC_{\eta jt} &= W_t L_{\eta jt} + R_t K_{\eta jt} && \Longrightarrow \\
TC_{\eta jt} &= W_t \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{-\alpha_\eta} + R_t \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{1-\alpha_\eta} && \Longrightarrow \\
TC_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta} \right)^{1-\alpha_\eta}
\end{aligned}
\tag{4.29}
$$

Calculate the marginal cost using 4.30:

$$
\Lambda_{\eta jt} = \frac{\partial TC_{\eta jt}}{\partial Y_{\eta jt}} \Longrightarrow \Lambda_{\eta jt} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta} \right)^{1-\alpha_\eta}
\tag{4.30}
$$

The marginal cost depends on the technological level $Z_{A\eta t}$, the nominal interest rate R_t and the nominal wage level W_t, which are the same for all intermediate-good firms, and because of that, the index j may be dropped:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{4.31}$$

notice that:

$$\Lambda_{\eta t} = \frac{TC_{\eta jt}}{Y_{\eta jt}} \implies TC_{\eta jt} = \Lambda_{\eta t} Y_{\eta jt} \tag{4.32}$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (CALVO, 1983): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{j,t+1} = P_{j,t})$, and a probability of $(1-\theta)$ of setting a new optimal price $P^*_{j,t}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price.

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{4.33}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Y_{\eta t} \left(\frac{P_t}{P_{\eta jt}}\right)^{\psi} \tag{4.20}$$

Substitute 4.33 in 4.34:

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - \Lambda_{\eta t+s} Y_{\eta j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{4.34}$$

Substitute 4.20 in 4.35 and rearrange the variables:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta t+s} \left(\frac{P_{t+s}}{P_{\eta jt}} \right)^{\psi} - \Lambda_{\eta t+s} Y_{\eta t+s} \left(\frac{P_{t+s}}{P_{\eta jt}} \right)^{\psi} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt}^{1-\psi} P_{t+s}^{\psi} Y_{\eta t+s} - P_{\eta jt}^{-\psi} P_{t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to $P_{\eta jt}$ is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{\eta jt}^{-\psi} P_{t+s}^{\psi} Y_{\eta t+s} - (-\psi) P_{\eta jt}^{-\psi-1} P_{t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) \left(\frac{P_{t+s}}{P_{\eta jt}} \right)^{\psi} Y_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \\ = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} \left(\frac{P_{t+s}}{P_{\eta jt}} \right)^{\psi} Y_{\eta t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{4.35}$$

Substitute 4.20 in 4.36:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$(\psi - 1) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \psi P_{\eta jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt}^{*} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}} \tag{4.36}$$

Equation 4.37 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P^*_{\eta t} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{4.37}$$

4.3.1 Final-Good Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1-\theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 4.21 to determine the aggregate price level:

$$\begin{aligned} P_t &= \left[\int_0^\theta P^{1-\psi}_{\eta,t-1} \, \mathrm{d}\, j + \int_\theta^1 P^{*1-\psi}_{\eta t} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies \\ P_t &= \left[\theta P^{1-\psi}_{\eta,t-1} + (1-\theta) P^{*1-\psi}_{\eta t} \right]^{\frac{1}{1-\psi}} \end{aligned} \tag{4.38}$$

Equation 4.39 is the aggregate price level.

4.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (TAYLOR, 1993) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{4.39}$$

where π_t is the gross inflation rate, defined by:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{4.40}$$

and R, π, Y are the variables in steady state, γ_R is the smoothing parameter for the interest rate R_t, while γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively and Z_{Mt} is the monetary shock[9].

4.5 Stochastic Shocks

Productivity Shock

The production technology level $Z_{A\eta t}$ will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{4.41}$$

where $\rho_{A\eta} \in [0, 1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{4.42}$$

where $\rho_M \in [0, 1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

4.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P_t^*, R_t^*, W_t^*\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C_{\eta t}^*, L_{\eta t}^*, K_{\eta,t+1}^*\}$ and for firms $\boldsymbol{\mathcal{A}}_F := \{K_{\eta jt}^*, L_{\eta jt}^*, Y_{\eta jt}^*, Y_{\eta t}^*\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms'

[9] for the monetary shock definition, see section 4.5.

problems, and the markets for goods and labor clear:

$$Y_t = \sum_{\eta=1}^{n} (C_{\eta t} + I_{\eta t}) \tag{4.43}$$

$$L_{\eta t} = \int_0^1 L_{\eta j t} \, \mathrm{d}\, j \tag{4.44}$$

4.6.1 Model Structure

The model is composed of the preview solutions, forming a square system of 27 variables and 27 equations, summarized as follows:

- Variables:
  - from the household problem: $C_{\eta t}, L_{\eta t}, K_{\eta,t+1}$;
  - from the final-good firm problem: $Y_{\eta t}, Y_{\eta jt}, P_t$;
  - from the intermediate-good firm problems: $K_{\eta jt}, L_{\eta jt}, P_t^*$;
  - from the market clearing condition: $Y_t, I_{\eta t}$;
  - prices: $W_t, R_t, \Lambda_{\eta t}, \pi_t$;
  - shocks: $Z_{A\eta t}, Z_{Mt}$.
- Equations:
  1. Labor Supply:
  $$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \tag{4.15}$$
  2. Household Euler Equation:
  $$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{4.16}$$
  3. Budget Constraint:
  $$P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{4.2}$$
  4. Law of Motion for Capital:
  $$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{4.3}$$
  5. Bundle Technology:
  $$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\, \mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{4.18}$$
  6. General Price Level:
  $$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{4.39}$$

7. Capital Demand:

$$K_{\eta jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_t} \tag{4.25}$$

8. Labor Demand:

$$L_{\eta jt} = (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \tag{4.26}$$

9. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{4.32}$$

10. Production Function:

$$Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \tag{4.23}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}} \tag{4.38}$$

12. Market Clearing Condition:

$$Y_t = \sum_{\eta=1}^{n} (C_{\eta t} + I_{\eta t}) \tag{4.44}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{4.40}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{4.41}$$

15. Productivity Shock:

$$\ln Z_{A\eta t} = (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{4.42}$$

16. Monetary Shock:

$$\ln Z_{Mt} = (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{4.43}$$

4.7 Steady State

The steady state is defined by the constancy of the variables through time. For any given endogenous variable X_t, it is in steady state if $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (COSTA JUNIOR, 2016, p.41). For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The steady state of each equation of the model is:

1. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \implies \frac{\phi L_{\eta}^{\varphi}}{C_{\eta}^{-\sigma}} = \frac{W}{P} \tag{4.45}$$

2. Household Euler Equation:

$$\begin{aligned} \left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} &= \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies \\ 1 &= \beta\left[(1-\delta) + \frac{R}{P}\right] \end{aligned} \tag{4.46}$$

3. Budget Constraint:

$$\begin{aligned} P_t(C_{\eta t} + I_{\eta t}) &= W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \implies \\ P(C_\eta + I_\eta) &= W L_\eta + R K_\eta + \Pi_\eta \end{aligned} \tag{4.47}$$

4. Law of Motion for Capital:

$$\begin{aligned} K_{\eta,t+1} &= (1-\delta)K_{\eta t} + I_{\eta t} \implies K_\eta = (1-\delta)K_\eta + I_\eta \implies \\ I_\eta &= \delta K_\eta \end{aligned} \tag{4.48}$$

5. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}}\, \mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \implies Y_\eta = \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\, \mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{4.49}$$

6. General Price Level:

$$\begin{aligned}
P_t &= \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} && \implies \\
P^{1-\psi} &= \theta P^{1-\psi} + (1-\theta)P^{*1-\psi} && \implies \\
(1-\theta)P^{1-\psi} &= (1-\theta)P^{*1-\psi} && \implies P = P^*
\end{aligned} \tag{4.50}$$

7. Capital Demand:

$$K_{\eta jt} = \alpha_\eta Y_{\eta jt}\frac{\Lambda_{\eta t}}{R_t} \implies K_j = \alpha_\eta Yj\frac{\Lambda}{R} \tag{4.51}$$

8. Labor Demand:

$$L_{\eta jt} = (1-\alpha_\eta)Y_{\eta jt}\frac{\Lambda_{\eta t}}{W_t} \implies L_j = (1-\alpha_\eta)Yj\frac{\Lambda}{W} \tag{4.52}$$

9. Marginal Cost:

$$\begin{aligned}
\Lambda_{\eta t} &= \frac{1}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies \\
\Lambda_\eta &= \frac{1}{Z_{A\eta}}\left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta}
\end{aligned} \tag{4.53}$$

10. Production Technology:

$$Y_{\eta jt} = Z_{A\eta t}K_{\eta jt}^{\alpha_\eta}L_{\eta jt}^{1-\alpha_\eta} \implies Y_{\eta j} = Z_{A\eta}K_{\eta j}^{\alpha_\eta}L_{\eta j}^{1-\alpha_\eta} \tag{4.54}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi-1}\cdot\frac{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j,t+s}\Lambda_{\eta t+s}/\prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j,t+s}/\prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \implies \tag{4.38}$$

$$P^* = \frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta/[1-\theta(1-R)]}{Y_{\eta j}/[1-\theta(1-R)]} \implies$$

$$P^* = \frac{\psi}{\psi-1}\Lambda_\eta \tag{4.55}$$

12. Market Clearing Condition:

$$Y_t = \sum_{\eta=1}^{n} (C_{\eta t} + I_{\eta t}) \implies Y = \sum_{\eta=1}^{n} (C_\eta + I_\eta) \tag{4.56}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \tag{4.57}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \pi = 1 \tag{4.58}$$

15. Productivity Shock:

$$\begin{aligned} \ln Z_{A\eta t} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} && \implies \\ \ln Z_{A\eta} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta} + \varepsilon_{A\eta} && \implies \\ \varepsilon_{A\eta} &= 0 \end{aligned} \tag{4.59}$$

16. Monetary Shock:

$$\begin{aligned} \ln Z_{Mt} &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \implies \\ \ln Z_M &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_M + \varepsilon_M && \implies \\ \varepsilon_M &= 0 \end{aligned} \tag{4.60}$$

4.7.1 Variables in Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It's assumed that the productivity and the price level are normalized to one: $[\, P \;\; Z_{A\eta} \,] = \vec{\mathbf{1}}$ [10].

From 4.51, the optimal price P^* is:

$$P^* = P \tag{4.61}$$

[10] where $\vec{\mathbf{1}}$ is the unit vector.

From 4.59, the gross inflation rate is:

$$\pi = 1 \tag{4.62}$$

From 4.58, the monetary shock is:

$$Z_M = 1 \tag{4.63}$$

From 4.60 and 4.61, the productivity and monetary shocks are:

$$\begin{bmatrix} \varepsilon_{A\eta} & \varepsilon_M \end{bmatrix} = \vec{\mathbf{0}} \tag{4.64}$$

From 4.47, the return on capital R is:

$$1 = \beta \left[(1-\delta) + \frac{R}{P} \right] \implies R = P \left[\frac{1}{\beta} - (1-\delta) \right] \tag{4.65}$$

From 4.56 and 4.51, the marginal cost Λ_η is:

$$P^* = \frac{\psi}{\psi - 1} \Lambda_\eta \implies \Lambda_\eta = P \frac{\psi - 1}{\psi} \tag{4.66}$$

From equation 4.54, the nominal wage W is:

$$\Lambda_\eta = \frac{1}{Z_{A\eta}} \left(\frac{R}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta} \right)^{1-\alpha_\eta} \implies$$
$$W = (1-\alpha_\eta) \left[\Lambda_\eta Z_{A\eta} \left(\frac{\alpha_\eta}{R} \right)^{\alpha_\eta} \right]^{\frac{1}{1-\alpha_\eta}} \tag{4.67}$$

@@@ CONTINUAR DAQUI @@@

In steady state, prices are the same ($P = P^*$), resulting in a gross inflation level of one ($\pi = 1$), and all firms producing the same output level ($Y_j = Y$) due to the price parity (SOLIS-GARCIA, 2022, Lecture 13, p.12). For this reason, they all demand the same amount of factors (K, L), and equations 4.52, 4.53, and 4.55 become:

$$Y = Z_{A\eta} K^{\alpha_\eta} L^{1-\alpha_\eta} \tag{4.68}$$

$$K = \alpha_\eta Y \frac{\Lambda}{R} \tag{4.69}$$

$$L = (1 - \alpha_\eta) Y \frac{\Lambda}{W} \tag{4.70}$$

Substitute 4.70 in 4.49:

$$I = \delta K \implies I = \delta \alpha_\eta Y \frac{\Lambda}{R} \tag{4.71}$$

Substitute 4.71 in 4.46:

$$\frac{\phi L^{\varphi}}{C^{-\sigma}} = \frac{W}{P} \implies C = \left[L^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} \implies C = \left[\left((1 - \alpha_\eta) Y \frac{\Lambda}{W} \right)^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} \tag{4.72}$$

Substitute 4.72 and 4.73 in 4.57:

$$\begin{aligned} Y &= C + I && \implies \\ Y &= \left[\left((1 - \alpha_\eta) Y \frac{\Lambda}{W} \right)^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} + \left[\delta \alpha_\eta Y \frac{\Lambda}{R} \right] && \implies \\ Y &= \left[\left(\frac{W}{\phi P} \right) \left(\frac{W}{(1 - \alpha_\eta)\Lambda} \right)^{\varphi} \left(\frac{R}{R - \delta \alpha_\eta \Lambda} \right)^{\sigma} \right]^{\frac{1}{\varphi + \sigma}} \end{aligned} \tag{4.73}$$

For C, K, L, I, use the result from 4.74 in 4.73, 4.70, 4.71 and 4.49, respectively.

4.7.2 Steady State Solution

$$\begin{bmatrix} P & P^* & \pi & Z_{A\eta} & Z_M \end{bmatrix} = \vec{1} \tag{4.74}$$

$$\begin{bmatrix} \varepsilon_A & \varepsilon_M \end{bmatrix} = \vec{0} \tag{4.75}$$

$$R = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{4.66}$$

$$\Lambda = P\frac{\psi - 1}{\psi} \tag{4.67}$$

$$W = (1-\alpha_\eta)\left[\Lambda Z_{A\eta}\left(\frac{\alpha_\eta}{R}\right)^{\alpha_\eta}\right]^{\frac{1}{1-\alpha_\eta}} \tag{4.68}$$

$$Y = \left[\left(\frac{W}{\phi P}\right)\left(\frac{W}{(1-\alpha_\eta)\Lambda}\right)^{\varphi}\left(\frac{R}{R-\delta\alpha_\eta\Lambda}\right)^{\sigma}\right]^{\frac{1}{\varphi+\sigma}} \tag{4.74}$$

$$C = \left[\left((1-\alpha_\eta)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \tag{4.73}$$

$$K = \alpha_\eta Y\frac{\Lambda}{R} \tag{4.70}$$

$$L = (1-\alpha_\eta)Y\frac{\Lambda}{W} \tag{4.71}$$

$$I = \delta K \tag{4.49}$$

4.8 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. Dynare is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation 4.38. For this purpose, Uhlig's rules of log-linearization (UHLIG, 1999) will be applied to all equations in the model[11].

[11]see lemma A.3 for details.

4.8.1 Gross Inflation Rate

Log-linearize 4.41 and define the level deviation of gross inflation rate $\widetilde{\pi}_t$:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \tag{4.41}$$

$$\widetilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{4.76}$$

4.8.2 New Keynesian Phillips Curve

In order to log-linearize equation 4.38, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma A.5:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{4.38}$$

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{4.77}$$

First, log-linearize the left hand side of equation 4.78 with respect to $P_t^*, Y_{j,t}, \widetilde{R}_t$:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies$$

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P^* Y_j \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies$$

$$P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\}$$

Separate the terms not dependent on s:

$$P^* Y_j (1+\hat{P}^*_t)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \right\} +$$
$$+ P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \implies$$

Apply definition A.10 on the first term:

$$\frac{P^* Y_j (1+\hat{P}^*_t)}{1-\theta/(1+R)} + P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Second, log-linearize the left hand side of equation 4.78 with respect to $\Lambda^*_{\eta t}, Y_{j,t}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies$$
$$\frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \frac{Y_j \Lambda (1 + \hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s})}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies$$
$$\frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(1 + \hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \right\} +$$
$$+ \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Apply definition A.10 on the first term:

$$\frac{\psi}{\psi-1} \cdot \frac{Y_j \Lambda}{1-\theta/(1+R)} +$$
$$+ \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Join both sides of the equation again:

$$
\begin{aligned}
&\frac{P^{*}Y_j(1+\hat{P}_t^{*})}{1-\theta/(1+R)} + P^{*}Y_j\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^{s}\left(\hat{Y}_{j,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} = \\
&= \frac{\psi}{\psi-1}\cdot\frac{Y_j\Lambda}{1-\theta/(1+R)} + \\
&\qquad + \frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^{s}\left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}
\end{aligned}
\tag{4.78}
$$

Define a nominal discount rate ρ in steady state:

$$
1=\rho(1+R) \implies \rho=\frac{1}{1+R} \tag{4.79}
$$

Substitute 4.80 in 4.79:

$$
\begin{aligned}
&\frac{P^{*}Y_j(1+\hat{P}_t^{*})}{1-\theta\rho} + P^{*}Y_j\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^{s}\left(\hat{Y}_{j,t+s}-\rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} = \frac{\psi}{\psi-1}\cdot\frac{Y_j\Lambda}{1-\theta\rho} + \\
&\qquad + \frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^{s}\left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s}-\rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}
\end{aligned}
\tag{4.80}
$$

Substitute 4.67 in 4.81 and simplify all common terms:

$$
\begin{aligned}
&\cancel{\frac{P^{*}Y_j}{1-\theta\rho}} + \frac{\cancel{P^{*}Y_j}\hat{P}_t^{*}}{1-\theta\rho} + \cancel{P^{*}Y_j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^{s}\left(\cancel{\hat{Y}_{j,t+s}-\rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right)\right\} = \\
&= \cancel{\frac{P^{*}Y_j}{1-\theta\rho}} + \cancel{P^{*}Y_j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^{s}\left(\cancel{\hat{Y}_{j,t+s}-\rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}+\hat{\Lambda}_{t+s}\right)\right\} \implies \\
&\frac{\hat{P}_t^{*}}{1-\theta\rho} = \mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^{s}\left(\hat{\Lambda}_{t+s}\right)\right\}
\end{aligned}
\tag{4.81}
$$

Define the real marginal cost λ_t:

$$
\begin{aligned}
&\lambda_t=\frac{\Lambda_{\eta t}}{P_t} \implies \Lambda_{\eta t}=P_t\lambda_t \implies \\
&\hat{\Lambda}_t=\hat{P}_t+\hat{\lambda}_t
\end{aligned}
\tag{4.82}
$$

Substitute 4.83 in 4.82:

$$\hat{P}_t^* = (1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{4.83}$$

Log-linearize equation 4.39:

$$\begin{aligned} P_t^{1-\psi} &= \theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi} && \Longrightarrow && (4.39) \\ P^{1-\psi}(1+(1-\psi)\hat{P}_t) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{t-1}) + \\ &\quad + (1-\theta)P^{1-\psi}(1+(1-\psi)\hat{P}_t^*) && \Longrightarrow \\ \hat{P}_t &= \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* && && (4.84) \end{aligned}$$

Substitute 4.84 in 4.85:

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* \tag{4.85}$$

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{4.85}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\rho\mathbb{L}^{-1})$[12] in 4.86:

$$\begin{aligned} (1-\theta\rho\mathbb{L}^{-1})\hat{P}_t &= (1-\theta\rho\mathbb{L}^{-1})\Big[\theta\hat{P}_{t-1} + \\ &\quad +(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right)\Big] \Longrightarrow \\ \hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + \\ &\quad (1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) - \\ &\quad - \theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right) \end{aligned} \tag{4.86}$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\rho$ within the operator. Then, cancel the summations and rearrange

[12]see definition A.11.

the terms:

$$
\begin{aligned}
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + \\
&\qquad (1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^s\left(\hat{P}_{t+s}+\hat{\lambda}_{t+s}\right) - \\
&\qquad - \theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right) \implies \\
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + (1-\theta)(1-\theta\rho)(\hat{P}_t+\hat{\lambda}_t) + \\
&\qquad \cancel{+ (1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right)} - \\
&\qquad \cancel{- (1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right)} \implies \\
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta^2\rho\hat{P}_t + (1-\theta-\theta\rho+\theta^2\rho)\hat{P}_t + (1-\theta)(1-\theta\rho)\hat{\lambda}_t \implies \\
(\hat{P}_t - \hat{P}_{t-1}) &= \rho(\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_t) + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t
\end{aligned}
\tag{4.87}
$$

Substitute 4.77 in 4.88:

$$
\tilde{\pi}_t = \rho\mathbb{E}_t\tilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \tag{4.88}
$$

Equation 4.89 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

4.8.3 Labor Supply

Log-linearize 4.15:

$$
\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \qquad \implies \tag{4.15}
$$

$$
\varphi\hat{L}_t + \sigma\hat{C}_t = \hat{W}_t + \hat{P}_t \tag{4.89}
$$

4.8.4 Household Euler Equation

Log-linearize 4.16:

$$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies \tag{4.16}$$

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{4.90}$$

4.8.5 Law of Motion for Capital

Log-linearize 4.3:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies \tag{4.3}$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{4.91}$$

4.8.6 Bundle Technology

Apply the natural logarithm to 4.18:

$$\ln Y_t = \frac{\psi}{\psi-1}\ln\left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)$$

Log-linearize using corollary A.3.1:

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(\int_0^1 Y_j^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}j\right] \implies$$

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(Y_j^{\frac{\psi-1}{\psi}}\int_0^1 \mathrm{d}j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}j\right] \implies$$

$$\ln Y + \hat{Y}_t = \cancel{\frac{\psi}{\psi-1}}\left[\cancel{\frac{\psi-1}{\psi}}\ln Y_j + \cancel{\ln 1} + \cancel{\frac{\psi-1}{\psi}}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}j\right] \implies$$

$$\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt}\,\mathrm{d}j$$

Apply corollary A.2.1:

$$
\begin{aligned}
\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt} \, \mathrm{d}\, j \implies& \\
\hat{Y}_t = \int_0^1 \hat{Y}_{jt} \, \mathrm{d}\, j& \qquad (4.92)
\end{aligned}
$$

4.8.7 Marginal Cost

Log-linearize 4.32:

$$
\begin{aligned}
&\Lambda_{\eta t} = Z_{A\eta t}^{-1} \frac{R_t^{\alpha_\eta} W_t^{1-\alpha_\eta}}{\alpha_\eta{}^{\alpha_\eta} (1-\alpha_\eta)^{1-\alpha_\eta}} \implies \qquad (4.32) \\
&\Lambda(1+\hat{\Lambda}_t) = \frac{1}{Z_{A\eta}} \left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} (1 - \hat{Z}_{A\eta t} + \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t) \implies \\
&\hat{\Lambda}_t = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \qquad (4.93)
\end{aligned}
$$

Substitute 4.83 in 4.94:

$$
\begin{aligned}
\hat{\Lambda}_t &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \implies \\
\hat{P}_t + \hat{\lambda}_t &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \implies \\
\hat{\lambda}_t &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_t \qquad (4.94)
\end{aligned}
$$

4.8.8 Production Function

Log-linearize 4.23:

$$
\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \implies \qquad (4.23) \\
Y_j(1+\hat{Y}_{jt}) &= Z_{A\eta} K_j^{\alpha_\eta} L_j^{1-\alpha_\eta} (1 + \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt}) \implies \\
\hat{Y}_{jt} &= \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt} \qquad (4.95)
\end{aligned}
$$

Substitute 4.96 in 4.93:

$$\hat{Y}_t = \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j \qquad \Longrightarrow \tag{4.93}$$

$$\hat{Y}_t = \int_0^1 \left[\hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt}\right]\mathrm{d}\,j \qquad \Longrightarrow$$

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha_\eta)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \tag{4.96}$$

Apply the natural logarithm and then log-linearize 4.45:

$$L_{\eta t} = \int_0^1 L_{\eta jt}\,\mathrm{d}\,j \qquad \Longrightarrow \tag{4.45}$$

$$\ln L_{\eta t} = \ln\left[\int_0^1 L_{\eta jt}\,\mathrm{d}\,j\right] \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln\left[\int_0^1 L_j\,\mathrm{d}\,j\right] + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln L_j + \ln 1 + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j$$

Apply corollary A.2.1:

$$\Longrightarrow \hat{L}_t = \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \tag{4.97}$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_t = \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j \tag{4.98}$$

Substitute 4.98 and 4.99 in 4.97:

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha_\eta)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \Longrightarrow \tag{4.97}$$

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_t + (1-\alpha_\eta)\hat{L}_t \tag{4.99}$$

4.8.9 Capital Demand

Log-linearize 4.25:

$$
\begin{aligned}
K_{\eta jt} &= \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_t} && \Longrightarrow && (4.25)\\
K_j(1+\hat{K}_{jt}) &= \alpha_\eta Y_j \frac{\Lambda}{R}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t) && \Longrightarrow \\
\hat{K}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t
\end{aligned}
$$

Integrate both sides and then substitute 4.99 and 4.93:

$$
\begin{aligned}
\int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j &= \int_0^1 (\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t)\,\mathrm{d}\,j \quad \Longrightarrow \\
\hat{K}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t && (4.100)
\end{aligned}
$$

4.8.10 Labor Demand

Log-linearize 4.26:

$$
\begin{aligned}
L_{\eta jt} &= (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} && \Longrightarrow && (4.26)\\
L_j(1+\hat{L}_{jt}) &= (1-\alpha_\eta) Y_j \frac{\Lambda}{W}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t) && \Longrightarrow \\
\hat{L}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t
\end{aligned}
$$

Integrate both sides and then substitute 4.98 and 4.93:

$$
\begin{aligned}
\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j &= \int_0^1 \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t\,\mathrm{d}\,j \Longrightarrow \\
\hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t && (4.101)
\end{aligned}
$$

Subtract 4.102 from 4.101:

$$
\begin{aligned}
\hat{K}_t-\hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t-(\hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t) \Longrightarrow \\
\hat{K}_t-\hat{L}_t &= \hat{W}_t-\hat{R}_t && (4.102)
\end{aligned}
$$

Equation 4.103 is the log-linearized version of 4.27.

4.8.11 Market Clearing Condition

Log-linearize 4.44:

$$
\begin{aligned}
Y_t &= C_{\eta t} + I_{\eta t} && \Longrightarrow && (4.44)\\
Y(1+\hat{Y}_t) &= C(1+\hat{C}_t) + I(1+\hat{I}_t) && \Longrightarrow \\
Y + Y\hat{Y}_t &= C + C\hat{C}_t + I + I\hat{I}_t && \Longrightarrow \\
Y\hat{Y}_t &= C\hat{C}_t + I\hat{I}_t && \Longrightarrow \\
\hat{Y}_t &= \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t && && (4.103)
\end{aligned}
$$

Define the consumption and investment weights $\left[\,\theta_C \;\; \theta_I\,\right]$ in the production total:

$$
\begin{bmatrix}\theta_C & \theta_I\end{bmatrix} := \begin{bmatrix}\frac{C}{Y} & \frac{I}{Y}\end{bmatrix} \tag{4.104}
$$

Substitute 4.105 in 4.104:

$$
\begin{aligned}
&\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t \;\Longrightarrow \\
&\hat{Y}_t = \theta_C\hat{C}_t + \theta_I\hat{I}_t && (4.105)
\end{aligned}
$$

4.8.12 Monetary Policy

Log-linearize 4.40:

$$
\begin{aligned}
&\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi}Y_t^{\gamma_Y})^{(1-\gamma_R)}Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \;\Longrightarrow && (4.40)\\
&\frac{R(1+\hat{R}_t)}{R} = \\
&= \frac{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}Z_M[1+\gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt}]}{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \;\Longrightarrow \\
&\hat{R}_t = \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt} && (4.106)
\end{aligned}
$$

4.8.13 Productivity Shock

Log-linearize 4.42:

$$\begin{aligned} \ln Z_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}\ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} && \Longrightarrow && (4.42)\\ \ln Z_{A\eta} + \hat{Z}_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}(\ln Z_{A\eta} + \hat{Z}_{A,t-1}) + \varepsilon_{A\eta} && \Longrightarrow \\ \hat{Z}_{A\eta t} &= \rho_{A\eta}\hat{Z}_{A,t-1} + \varepsilon_{A\eta} && && (4.107) \end{aligned}$$

4.8.14 Monetary Shock

Log-linearize 4.43:

$$\begin{aligned} \ln Z_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \Longrightarrow && (4.43)\\ \ln Z_M + \hat{Z}_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M(\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M && \Longrightarrow \\ \hat{Z}_{Mt} &= \rho_M \hat{Z}_{M,t-1} + \varepsilon_M && && (4.108) \end{aligned}$$

4.8.15 Log-linear Model Structure

The log-linear model is a square system of 12 variables and 12 equations, summarized as follows:

- Variables: $(\tilde{\pi} \quad \hat{P} \quad \tilde{\lambda} \quad \hat{C} \quad \hat{L} \quad \hat{R} \quad \hat{K} \quad \hat{I} \quad \hat{W} \quad \hat{Z}_A \quad \hat{Y} \quad \hat{Z}_M)$
- Equations:

1. Gross Inflation Rate:

$$\tilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{4.77}$$

2. New Keynesian Phillips Curve:

$$\tilde{\pi}_t = \rho \mathbb{E}_t \tilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\tilde{\lambda}_t \tag{4.89}$$

3. Labor Supply:

$$\varphi \hat{L}_t + \sigma \hat{C}_t = \hat{W}_t + \hat{P}_t \tag{4.90}$$

4. Household Euler Equation:

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{4.91}$$

5. Law of Motion for Capital:

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{4.92}$$

6. Real Marginal Cost:

$$\hat{\lambda}_t = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_t \tag{4.95}$$

7. Production Function:

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_t + (1-\alpha_\eta)\hat{L}_t \tag{4.100}$$

8. Marginal Rates of Substitution of Factors:

$$\hat{K}_t - \hat{L}_t = \hat{W}_t - \hat{R}_t \tag{4.103}$$

9. Market Clearing Condition:

$$\hat{Y}_t = \theta_C \hat{C}_t + \theta_I \hat{I}_t \tag{4.106}$$

10. Monetary Policy:

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \widetilde{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{4.107}$$

11. Productivity Shock:

$$\hat{Z}_{A\eta t} = \rho_{A\eta} \hat{Z}_{A,t-1} + \varepsilon_{A\eta} \tag{4.108}$$

12. Monetary Shock:

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{4.109}$$

5 Data

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1. Data Sources
2. Data Treatment
3. Descriptive Statistics

6 Results

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6.1 Calibration

6.1.1 Parameter Calibration

$$\begin{bmatrix} \phi \\ \varphi \\ \sigma \\ \beta \\ \delta \\ \psi \\ \theta \\ \alpha \\ \gamma_R \\ \gamma_\pi \\ \gamma_Y \\ \rho_A \\ \rho_M \\ \theta_C \\ \theta_I \end{bmatrix} = \begin{bmatrix} \phi \\ \varphi \\ \sigma \\ \beta \\ \delta \\ \psi \\ \theta \\ \alpha \\ \gamma_R \\ \gamma_\pi \\ \gamma_Y \\ \rho_A \\ \rho_M \\ \theta_C \\ \theta_I \end{bmatrix} \tag{6.1}$$

6.1.2 Variables at the Steady State

$$\begin{bmatrix} P \\ Z_A \\ P^* \\ \pi \\ Z_M \\ R \\ \Lambda \\ W \\ Y \\ C \\ I \\ K \\ L \end{bmatrix} = \begin{bmatrix} P \\ Z_A \\ P^* \\ \pi \\ Z_M \\ R \\ \Lambda \\ W \\ Y \\ C \\ I \\ K \\ L \end{bmatrix} \tag{6.2}$$

6.1.3 Parameter Calibration

Parameter	Definition	Calibration
σ	relative risk aversion coefficient	
ϕ	relative labor weight in utility	
φ	marginal disutility of labor supply	
β	intertemporal discount factor	
δ	capital depreciation rate	
α	production elasticity with respect to capital	
ψ	elasticity of substitution between intermediate goods	
θ	price stickness parameter	
γ_R	interest-rate smoothing parameter	
γ_π	interest-rate sensitivity in relation to inflation	
γ_Y	interest-rate sensitivity in relation to product	
ρ_A	autoregressive parameter of productivity	
ρ_M	autoregressive parameter of monetary policy	
θ_C	consumption weight in production	
θ_I	investment weight in production	

6.1.4 Variables at the Steady State

Variable	Steady State Value
P	
Z_A	
P^*	
π	
Z_M	
R	
Λ	
W	
Y	
C	
I	
K	
L	

6.1.5 Impulse Response Graphics

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6.2 Parametrization

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7 Final Remarks

This section is where you summarize and discuss the main findings, implications, and potential future work related to your research.

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A Appendix

A.1 Table of the Literature Review

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A.2 Definitions, Theorems and Lemmas

The objective of this appendix is to present the definitions, theorems, lemmas and proofs used throughout the text.

A.2.1 Model

A.2.2 Household

Definition A.1 (Household Maximization Problem)**.** The utility function is:

- strictly increasing in consumption C;
- strictly increasing in leisure l;
- strictly concave;
- twice continuously differentiable;
- the composite consumption good C is also the numeraire good, so that its price equals one: $p_C = 1$;

- to avoid corner solutions, the Inada conditions[13] hold.

Consider a representative household that maximizes an utility function u that depends on consumption C_t and labor L_t:

$$u \equiv u\left(C_t, L_t\right) \tag{A.1}$$

The utility function is considered to be convex (when a variable increases, the respective marginal utility diminishes)[14]:

$$u_C > 0, \quad u_{CC} < 0, \quad u_L > 0, \quad u_{LL} < 0$$

Definition A.2 (Discount Factor β)**.** other things the same, a unit of consumption enjoyed tomorrow is less valuable (yields less utility) than a unit of consumption enjoyed today (SOLIS-GARCIA, 2022, Lecture 2, p.1).

Definition A.3 (Inada Condition)**.** The Inada conditions (INADA, 1963) avoid corner solutions. For this purpose, it is assumed that the partial derivatives u_C and u_L of the function $u(C, L)$ satisfy the following rules:

$$\begin{aligned} &\lim_{C \to 0} u_C(C, L^*) = \infty \quad \text{and} \quad \lim_{C \to \infty} u_C(C, L^*) = 0 \\ &\lim_{L \to 0} u_C(C^*, L) = \infty \quad \text{and} \quad \lim_{L \to \infty} u_C(C^*, L) = 0 \end{aligned} \tag{A.2}$$

where $C^*, L^* \in \mathbb{R}_{++}$ and u_j is the partial derivative of the utility function with respect to $j = C, L$ (SOLIS-GARCIA, 2022, Lecture 1, p.2)

Definition A.4 (Transversality Condition)**.** (SOLIS-GARCIA, 2022, Lecture 4, p.4)

[13] see definition A.3.

[14] Consider the following notation: given two variables X and Y, the first and second partial derivatives are: $Y_X := \dfrac{\partial Y}{\partial X}$ and $Y_{XX} := \dfrac{\partial^2 Y}{\partial X^2}$.

A.2.3 Firms

Lemma A.1 (Marginal Cost)**.** *The Lagrangian multiplier Λ_t is the nominal marginal cost of the intermediate-good firm:*

$$MC_t := \frac{\partial TC_t}{\partial Y_t} = \Lambda_t \tag{A.3}$$

Proof. Please see Simon and Blume (1994, p.449). ■

Definition A.5 (Constant Returns to Scale)**.** (SOLIS-GARCIA, 2022, Lecture 1, p.5)

Definition A.6 (Homogeneous Function of Degree k)**.** (SOLIS-GARCIA, 2022, Lecture 1, p.5)

A.2.4 Monetary Authority

A.2.5 Shocks

A.2.6 Equilibrium Conditions

Definition A.7 (Competitive Equilibrium)**.** (SOLIS-GARCIA, 2022, Lecture 1, p.6)

A.2.7 Steady State

Lemma A.2 (Steady State Inflation)**.** *In steady state, prices are stable $P_t = P_{t-1} = P$ and the gross inflation rate is one.*

Proof. Equation 3.53. ■

Corollary A.2.1. *In steady state, all firms have the same level of production Y and therefore demand the same amount of factors, capital K and labor L.*

$$P_t = P_{t-1} = P \implies \begin{pmatrix} Y_j & K_j & L_j \end{pmatrix} = \begin{pmatrix} Y & K & L \end{pmatrix}$$

A.2.8 Log-linearization

Definition A.8 (PERCENTAGE DEVIATION)**.** The percentage deviation of a variable x_t from its steady state is given by (SOLIS-GARCIA, 2022, Lecture 6, p.2):

$$\hat{x}_t := \frac{x_t - x}{x} \tag{A.4}$$

Lemma A.3 (UHLIG'S RULES)**.** *The Uhlig's rules are a set of approximations used to log-linearize equations (SOLIS-GARCIA, 2022, Lecture 6, p.2).*

- *Rule 1:*

  $x_t = x(1 + \hat{x}_t)$

- *Rule 2 (Product):*

- *Rule 3 (Exponential):*

Corollary A.3.1 (Logarithm Rule)**.**

$$\ln x_t \approx \ln x + \hat{x}_t$$

Definition A.9 (LEVEL DEVIATION)**.** The level deviation of a variable u_t from its steady state is given by: (SOLIS-GARCIA, 2022, Lecture 9, p.9)

$$\widetilde{u}_t := u_t - u \tag{A.5}$$

Lemma A.4 (UHLIG'S RULES FOR LEVEL DEVIATIONS)**.** *Uhlig's rules can be applied to level deviations in order to log-linearize equations (SOLIS-GARCIA, 2022, Lecture 6, p.2).*

- *Rule 1:*

$$u_t = u + \widetilde{u}_t \tag{A.6}$$

$$u_t = u\left(1 + \frac{\widetilde{u}_t}{u}\right) \tag{A.7}$$

- *Rule 2 (Product):*

- *Rule 3 (Exponential):*

- *Rule 4 (Logarithm):*

- *Rule 5 (Percentage and Level Deviations)*

Lemma A.5 (LEVEL DEVIATION OF THE PRESENT VALUE DISCOUNT FACTOR). *The level deviation of the present value discount factor is equivalent to:*

$$\prod_{k=0}^{s-1}(1+R_{t+k}) = (1+R)^s \left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right) \tag{A.8}$$

Proof. Substitute the interest rate by the gross interest rate $GR_t = 1 + R_t$ and apply rule A.7:

$$\begin{aligned}
&\prod_{k=0}^{s-1}(1+R_{t+k}) = \prod_{k=0}^{s-1}(GR_{t+k}) && \Longrightarrow \\
&GR \times \cdots \times GR \left(1+\frac{1}{GR}\widetilde{GR}_t + \frac{1}{GR}\widetilde{GR}_{t+1} + \cdots + \frac{1}{GR}\widetilde{GR}_{t+s-1}\right) && \Longrightarrow \\
&GR^s \left(1+\frac{1}{GR}\sum_{k=0}^{s-1}\widetilde{GR}_{t+k}\right) && \Longrightarrow \\
&(1+R)^s \left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)
\end{aligned}$$

■

Definition A.10 (Geometric Series). A geometric series is the sum of the terms of a geometric sequence.

$$S_\infty = \sum_{i=0}^{\infty} ar^i \implies S_\infty = \frac{a}{1-r},\ |r| < 1$$

Definition A.11 (LAG AND LEAD OPERATORS). The lag operator $\mathbb{L}$ is a mathematical operator that represents the backshift or lag of a time series (SOLIS-GARCIA, 2022, Lecture 13, p.9):

$$\begin{aligned}
\mathbb{L}x_t &= x_{t-1} \\
(1+a\mathbb{L})y_{t+2} &= y_{t+2} + ay_{t+1}
\end{aligned}$$

Analogously, the lead operator $\mathbb{L}^{-1}$ (or inverse lag operator) yields a variable's lead (SOLIS-GARCIA, 2022, Lecture 13, p.9):

$$\begin{aligned}
\mathbb{L}^{-1}x_t &= x_{t+1} \\
(1+a\mathbb{L}^{-1})y_{t+2} &= y_{t+2} + ay_{t+3}
\end{aligned}$$

A.2.9 Canonical NK Model

Definition A.12 (Canonical NK Model)**.** (SOLIS-GARCIA, 2022, Lecture 13, p.7)

3.1.2 Back to the pricing equation:

log-linearize the left hand equation:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\left(\frac{\theta}{1+R} \right)^s \left(\frac{P_t^* Y_{t+s}(j)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right) \right] \Longrightarrow$$
$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\left(\frac{\theta}{1+R} \right)^s \left(\frac{P^* Y(j)(1 + \widehat{P}_t^* + \widehat{Y}_{t+s}(j))}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right) \right] \Longrightarrow$$
$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\left(\frac{\theta}{1+R} \right)^s \left(\frac{P_t^* Y_{t+s}(j)}{\frac{(1+R) + \sum_{k=0}^{s-1} \widetilde{R}_{t+k}}{1+R}} \right) \right] \Longrightarrow$$
$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\left(\frac{\theta}{1+R} \right)^s \left(\frac{P_t^* Y_{t+s}(j)(1+R)}{(1+R) + \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right) \right] \Longrightarrow$$
$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\left(\frac{\theta}{1+R} \right)^s \left(\frac{P^* Y(j)(1 + \widehat{P}_t^* + \widehat{Y}_{t+s}(j))(1+R)}{(1+R) + \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right) \right]$$

Definition A.13 (Medium Scale DSGE Model)**.** A Medium Scale DSGE Model has habit formation, capital accumulation, indexation, etc. (GALÍ, 2015, p.208).

See Galí, Smets, and Wouters (2012) for an analysis of the sources of unemployment fluctuations in an estimated medium-scale version of the present model.

Definition A.14 (Stochastic Process)**.** (SOLIS-GARCIA, 2022, Lecture 5, p.3).

Definition A.15 (Markov Process)**.** (SOLIS-GARCIA, 2022, Lecture 5, p.4).

Definition A.16 (first-order autoregressive process $AR(1)$)**.** the first-order autoregressive process $AR(1)$ (SOLIS-GARCIA, 2022, Lecture 5, p.4).

Definition A.17 (Blanchard-Kahn Conditions)**.** (SOLIS-GARCIA, 2022, Hands on 5, p.14).

A.3 Dynare Program

```
% command to run dynare and write
% a new file with all the choices:
% dynare NK_Inv_MonPol savemacro=NK_Inv_MonPol_FINAL.mod

% -------------------------------------------------- %
% MODEL OPTIONS                                      %
% -------------------------------------------------- %

% Productivity Shock ON/OFF
@#define ZA_SHOCK    = 1
% Productivity Shock sign: +/-
@#define ZA_POSITIVE = 1
% Monetary Shock ON/OFF
@#define ZM_SHOCK    = 1
% Monetary Shock sign: +/-
@#define ZM_POSITIVE = 1

% -------------------------------------------------- %
% ENDOGENOUS VARIABLES                               %
% -------------------------------------------------- %

var
PIt       ${\tilde{\pi}}$     (long_name='Inflation Rate')
Pt        ${\hat{P}}$         (long_name='Price Level')
LAMt      ${\tilde{\lambda}}$ (long_name='Real Marginal Cost')
Ct        ${\hat{C}}$         (long_name='Consumption')
Lt        ${\hat{L}}$         (long_name='Labor')
Rt        ${\hat{R}}$         (long_name='Interest Rate')
Kt        ${\hat{K}}$         (long_name='Capital')
It        ${\hat{I}}$         (long_name='Investment')
Wt        ${\hat{W}}$         (long_name='Wage')
ZAt       ${\hat{Z}^A}$       (long_name='Productivity')
Yt        ${\hat{Y}}$         (long_name='Production')
ZMt       ${\hat{Z}^M}$       (long_name='Monetary Policy')
;
```

```
% ---------------------------------------------------- %
% LOCAL VARIABLES                                      %
% ---------------------------------------------------- %

% the steady state variables are used as local
variables for the linear model.

model_local_variable

% steady state variables used as locals:
P
PI
ZA
ZM
R
LAM
W
Y
C
K
L
I

% local variables:
RHO % Steady State Discount Rate
;

% ---------------------------------------------------- %
% EXOGENOUS VARIABLES                                  %
% ---------------------------------------------------- %

varexo
epsilonA ${\varepsilon_A}$   (long_name='productivity shock')
epsilonM ${\varepsilon_M}$   (long_name='monetary shock')
;

% ---------------------------------------------------- %
```

```
% PARAMETERS                                          %
% --------------------------------------------------- %

parameters
SIGMA   ${\sigma}$     (long_name='Relative Risk Aversion')
PHI     ${\phi}$       (long_name='Labor Disutility Weight')
VARPHI  ${\varphi}$    (long_name='Marginal Disutility of Labor Supply')
BETA    ${\beta}$      (long_name='Intertemporal Discount Factor')
DELTA   ${\delta}$     (long_name='Depreciation Rate')
ALPHA   ${\alpha}$     (long_name='Output Elasticity of Capital')
PSI     ${\psi}$       (long_name='Elasticity of
Substitution between Intermediate Goods')
THETA   ${\theta}$     (long_name='Price Stickness Parameter')
gammaR  ${\gamma_R}$   (long_name='Interest-Rate Smoothing Parameter')
gammaPI ${\gamma_\pi}$ (long_name='Interest-Rate
Sensitivity to Inflation')
gammaY  ${\gamma_Y}$   (long_name='Interest-Rate Sensitivity to Product')
% maybe it's a local var, right? RHO ${\rho}$
(long_name='Steady State Discount Rate')
rhoA    ${\rho_A}$     (long_name='Autoregressive
Parameter of Productivity Shock')
rhoM    ${\rho_M}$     (long_name='Autoregressive
Parameter of Monetary Policy Shock')
thetaC  ${\theta_C}$   (long_name='Consumption weight
in Output')
thetaI  ${\theta_I}$   (long_name='Investment weight
in Output')

% --------------------------------------------------- %
% standard errors of stochastic shocks                %
% --------------------------------------------------- %

sigmaA ${\sigma_A}$   (long_name='Productivity-Shock
Standard Error')
sigmaM ${\sigma_M}$   (long_name='Monetary-Shock
Standard Error')
;
```

```
% ------------------------------------------------- %
% parameters values                                 %
% ------------------------------------------------- %

SIGMA   = 2       ; % Relative Risk Aversion
PHI     = 1       ; % Labor Disutility Weight
VARPHI  = 1.5     ; % Marginal Disutility of Labor
Supply
BETA    = 0.985   ; % Intertemporal Discount Factor
DELTA   = 0.025   ; % Depreciation Rate
ALPHA   = 0.35    ; % Output Elasticity of Capital
PSI     = 8       ; % Elasticity of Substitution
between Intermediate Goods
THETA   = 0.8     ; % Price Stickness Parameter
gammaR  = 0.79    ; % Interest-Rate Smoothing Parameter
gammaPI = 2.43    ; % Interest-Rate Sensitivity
to Inflation
gammaY  = 0.16    ; % Interest-Rate Sensitivity to
Product
% maybe it's a local var, right? RHO = 1/(1+Rs);
% Steady State Discount Rate
rhoA    = 0.95    ; % Autoregressive Parameter of
Productivity Shock
rhoM    = 0.9     ; % Autoregressive Parameter of
Monetary Policy Shock
thetaC  = 0.8     ; % Consumption weight in Output
thetaI  = 0.2     ; % Investment weight in Output

% ------------------------------------------------- %
% standard errors values                            %
% ------------------------------------------------- %

sigmaA = 0.01   ; % Productivity-Shock Standard Error
sigmaM = 0.01   ; % Monetary-Shock Standard Error

% ------------------------------------------------- %
% MODEL                                             %
% ------------------------------------------------- %
```

```
model(linear);

% First, the steady state variables as local varibles,
for the log-linear use:

#Ps   = 1 ;
#PIs  = 1 ;
#ZAs  = 1 ;
#ZMs  = 1 ;
#Rs   = Ps*(1/BETA-(1-DELTA)) ;
#LAMs = Ps*(PSI-1)/PSI ;
#Ws   = (1-ALPHA)*(LAMs*ZAs*(ALPHA/Rs)^ALPHA)^
(1/(1-ALPHA)) ;
#Ys   = ((Ws/(PHI*Ps))*((Ws/((1-ALPHA)*LAMs))^PSI)*(Rs/
(Rs-DELTA*ALPHA*LAMs))^SIGMA)^(1/(PSI+SIGMA)) ;
#Cs   = ((Ws/(PHI*Ps))*((1-ALPHA)*Ys*LAMs/Ws)^
(-PSI))^(1/SIGMA) ;
#Ks   = ALPHA*Ys*LAMs/Rs ;
#Ls   = (1-ALPHA)*Ys*LAMs/Ws ;
#Is   = DELTA*Ks ;
#RHO  = 1/(1+Rs) ;

% -------------------------------------------------- %
% MODEL EQUATIONS                                    %
% -------------------------------------------------- %

% Second, the log-linear model:

% 01 %
[name='Gross Inflation Rate']
PIt = Pt - Pt(-1) ;

% 02 %
[name='New Keynesian Phillips Curve']
PIt = RHO*PIt(+1)+LAMt*(1-THETA)*(1-THETA*RHO)/THETA ;

% 03 %
```

```
[name='Labor Supply']
VARPHI*Lt + SIGMA*Ct = Wt - Pt ;

% 04 %
[name='Household Euler Equation']
Ct(+1) - Ct = (Rt(+1)-Pt(+1))*BETA*Rs/(SIGMA*Ps) ;

% 05 %
[name='Law of Motion for Capital']
Kt = (1-DELTA)*Kt(-1) + DELTA*It ;

% 06 %
[name='Real Marginal Cost']
LAMt = ALPHA*Rt + (1-ALPHA)*Wt - ZAt - Pt ;

% 07 %
[name='Production Function']
Yt = ZAt + ALPHA*Kt(-1) + (1-ALPHA)*Lt ;

% 08 %
[name='Marginal Rates of Substitution of Factors']
Kt(-1) - Lt = Wt - Rt ;

% 09 %
[name='Market Clearing Condition']
Yt = thetaC*Ct + thetaI*It ;

% 10 %
[name='Monetary Policy']
Rt = gammaR*Rt(-1) + (1 - gammaR)*(gammaPI*PIt +
gammaY*Yt) + ZMt ;

% 11 %
[name='Productivity Shock']
@#if ZA_POSITIVE == 1
ZAt = rhoA*ZAt(-1) + epsilonA ;
@#else
ZAt = rhoA*ZAt(-1) - epsilonA ;
```

```
@#endif

% 12 %
[name='Monetary Shock']
@#if ZM_POSITIVE == 1
ZMt = rhoM*ZMt(-1) + epsilonM ;
@#else
ZMt = rhoM*ZMt(-1) - epsilonM ;
@#endif

end;

% -------------------------------------------------- %
% STEADY STATE                                       %
% -------------------------------------------------- %

steady_state_model ;

% in the log-linear model, all steady state variables
are zero (the variation is zero):

PIt  = 0 ;
Pt   = 0 ;
LAMt = 0 ;
Ct   = 0 ;
Lt   = 0 ;
Rt   = 0 ;
Kt   = 0 ;
It   = 0 ;
Wt   = 0 ;
ZAt  = 0 ;
Yt   = 0 ;
ZMt  = 0 ;

end;

% compute the steady state
steady;
```

```
check(qz_zero_threshold=1e-20);

% ------------------------------------------------- %
% SHOCKS                                            %
% ------------------------------------------------- %

shocks;

% Productivity Shock
@#if ZA_SHOCK == 1
var    epsilonA;
stderr sigmaA;
@#endif

% Monetary Shock
@#if ZM_SHOCK == 1
var    epsilonM;
stderr sigmaM;
@#endif

end;

stoch_simul(irf=80, order=1, qz_zero_threshold=1e-20)
ZAt ZMt Yt Pt PIt LAMt Ct Lt Rt Kt It Wt  ;

% ------------------------------------------------- %
% LATEX OUTPUT                                      %
% ------------------------------------------------- %

write_latex_definitions;
write_latex_parameter_table;
write_latex_original_model;
write_latex_dynamic_model;
write_latex_static_model;
write_latex_steady_state_model;
collect_latex_files;
```

A.4 LaTeX

A.4.1 Commands

- cancel line in equation: `\cancel`
- space before align: `\vspace{-1cm}`
- correct paragraph overfull: `\sloppy`
- indices: i, j, k, ℓ
- hats: $\overline{abc}, \widetilde{abc}, \widehat{abc}, \overrightarrow{abc}, \overleftarrow{abc}, \sqrt[n]{abc}, \xrightarrow{abc}, \xrightarrow{sometext}$
- accents: $\acute{a}, \check{a}, \grave{a}, \tilde{a}, \hat{a}, \breve{a}, \bar{a}, \bar{a}, \vec{a}, \dot{a}, \ddot{a}, \mathring{a}, \imath, \jmath$
- symbols:

  checkmark: ✓

  dagger: †

  definition symbol: $:=$
- index before the variable:

$$
\begin{aligned}
&+ {}^{NR}C^{\alpha}_{t+1} + {}_{NR}C^{\alpha}_{t+1} + {}_{\mathscr{nr}}C^{\alpha}_{t+1} \\
&+ NRC^{\alpha}_{t+1} + \mathscr{nr}C^{\alpha}_{t+1} + nrC^{\alpha}_{t+1} \\
&+ \mathrm{NRC}^{\alpha}_{t+1} + \mathcal{NR}C^{\alpha}_{t+1} + \mathscr{nr}C^{\alpha}_{t+1} \\
&+ {}_{\mathcal{NR}}C^{\alpha}_{t+1} + {}^{\mathcal{NR}}C^{\alpha}_{t+1} + C^{\mathcal{NR},\alpha}_{t+1} \\
&+ C^{\mathrm{NR},\alpha}_{t+1} + C^{\alpha}_{\mathrm{NR},t+1} + \mathrm{NRC}^{\alpha}_{t+1}
\end{aligned}
$$

- summation and product operator:

$$\sum_{s=0}^{\infty} \frac{\theta^s}{\prod_{k=0}^{s-1}(1 + R_{t+k})}$$

$$\text{Term for } s = 0: \frac{\theta^0}{\prod_{k=0}^{-1}(1 + R_{t+k})} = \theta^0 = 1$$

$$\text{Term for } s = 1 : \frac{\theta^1}{\prod_{k=0}^{0}(1 + R_{t+k})} = \frac{\theta^1}{1 + R_{t+0}} = \frac{\theta}{1 + R_t}$$

A.4.2 Font Styles in Math Mode

- San Serif Style: `\mathsf`

  $\mathsf{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathsf{abcdefghijklmnopqrstuvwxyz}$

  $\mathsf{1234567890}$

- Fraktur Style: `\mathfrak`

  $\mathfrak{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathfrak{abcdefghijklmnopqrstuvwxyz}$

  $\mathfrak{1234567890}$

- Fraktur-bold Style: `\mathbffrak`

  $\mathbffrak{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathbffrak{abcdefghijklmnopqrstuvwxyz}$

  $\mathbffrak{1234567890}$

- Calligraphic Style: `\mathcal`

  $\mathcal{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathcal{abcdefghijklmnopqrstuvwxyz}$

- Calligraphic-bold Style: `\mathbfcal`

  $\mathbfcal{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathbfcal{abcdefghijklmnopqrstuvwxyz}$

- Script Style: `\mathscr`

  $\mathscr{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

- Script-bold Style: `\mathbfscr`

  $\mathbfscr{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

- Blackboard-bold Style: `\mathbb`

$$\mathbb{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$$

$$\mathbb{1}$$

A.4.3 Greek Letters

Lower Case	Upper Case	Variation
$\alpha, \boldsymbol{\alpha}$ \alpha	$A, \boldsymbol{A}$	
$\beta, \boldsymbol{\beta}$ \beta	$B, \boldsymbol{B}$	
$\gamma, \boldsymbol{\gamma}$ \gamma	$\Gamma, \boldsymbol{\Gamma}$ \Gamma	
$\delta, \boldsymbol{\delta}$ \delta	$\Delta, \boldsymbol{\Delta}$ \Delta	
$\epsilon, \boldsymbol{\epsilon}$ \epsilon	$E, \boldsymbol{E}$	$\varepsilon, \boldsymbol{\varepsilon}$ \varepsilon
$\zeta, \boldsymbol{\zeta}$ \zeta	$Z, \boldsymbol{Z}$	
$\eta, \boldsymbol{\eta}$ \eta	$H, \boldsymbol{H}$	
$\theta, \boldsymbol{\theta}$ \theta	$\Theta, \boldsymbol{\Theta}$ \Theta	$\vartheta, \boldsymbol{\vartheta}$ \vartheta
$\iota, \boldsymbol{\iota}$ \iota	$I, \boldsymbol{I}$	
$\kappa, \boldsymbol{\kappa}$ \kappa	$K, \boldsymbol{K}$	$\varkappa, \boldsymbol{\varkappa}$ \varkappa
$\lambda, \boldsymbol{\lambda}$ \lambda	$\Lambda, \boldsymbol{\Lambda}$ \Lambda	
$\mu, \boldsymbol{\mu}$ \mu	$M, \boldsymbol{M}$	
$\nu, \boldsymbol{\nu}$ \nu	$N, \boldsymbol{N}$	
$\xi, \boldsymbol{\xi}$ `\xi`	$\Xi, \boldsymbol{\Xi}$ `\Xi`	
$o, \boldsymbol{o}$ (omicron)	$O, \boldsymbol{O}$	
$\pi, \boldsymbol{\pi}$ \pi	$\Pi, \boldsymbol{\Pi}$ \Pi	$\varpi, \boldsymbol{\varpi}$ \varpi
$\rho, \boldsymbol{\rho}$ \rho	$P, \boldsymbol{P}$	$\varrho, \boldsymbol{\varrho}$ \varrho
$\sigma, \boldsymbol{\sigma}$ \sigma	$\Sigma, \boldsymbol{\Sigma}$ \Sigma	$\varsigma, \boldsymbol{\varsigma}$ \varsigma
$\tau, \boldsymbol{\tau}$ \tau	$T, \boldsymbol{T}$	
$\upsilon, \boldsymbol{\upsilon}$ \upsilon	$\Upsilon, \boldsymbol{\Upsilon}$ \Upsilon	
$\phi, \boldsymbol{\phi}$ \phi	$\Phi, \boldsymbol{\Phi}$ \Phi	$\varphi, \boldsymbol{\varphi}$ \varphi
$\chi, \boldsymbol{\chi}$ \chi	$X, \boldsymbol{X}$	
$\psi, \boldsymbol{\psi}$ \psi	$\Psi, \boldsymbol{\Psi}$ \Psi	
$\omega, \boldsymbol{\omega}$ \omega	$\Omega, \boldsymbol{\Omega}$ \Omega	

A.5 ToDo List

Todo list

A.6 Epigraphs

To be yourself in a world that is constantly trying to make you something else is the greatest accomplishment.
— Ralph Waldo Emerson

The reason anyone would do this, if they could, which they can't, would be because they could, which they can't.
— Pickle Rick

THE CIRCLE IS NOW COMPLETE.
— Darth Vader

A.7 Drafts