1 Regional Model

model illustration as in Osterno (2022).

The model is populated by four agents: (1) a representative household, (2) a continuum of firms producing intermediate goods, (3) a firm producing a final good, and (4) the monetary authority.

1.1 Household

Utility Maximization Problem

Following the models presented by Costa Junior (2016) and Solis-Garcia (2022), the representative household problem is to maximize an intertemporal utility function U with respect to consumption $C_{\eta t}$ and labor $L_{\eta t}$, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_{\eta t}, L_{\eta t}, K_{\eta, t+1}} : \quad U(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{1.1}$$

$$\text{s. t.} : \quad P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{1.2}$$

$$K_{\eta, t+1} = (1-\delta) K_{\eta t} + I_{\eta t} \tag{1.3}$$

$$C_{\eta t}, L_{\eta t}, K_{\eta, t+1} \geq 0 \, ; K_0 \text{ given.}$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, P_t is the price level, $I_{\eta t}$ is the investment, W_t is the wage level, $K_{\eta t}$ is the capital stock, R_t is the return on capital, and $\Pi_{\eta t}$ is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate.

Isolate $I_{\eta t}$ in 1.3 and substitute in 1.2:

$$K_{\eta, t+1} = (1-\delta) K_{\eta t} + I_{\eta t} \implies I_{\eta t} = K_{\eta, t+1} - (1-\delta) K_{\eta t} \tag{1.3}$$

$$P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \implies \tag{1.2}$$

$$P_t(C_{\eta t} + K_{\eta, t+1} - (1-\delta) K_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{1.4}$$

Lagrangian

The maximization problem with restriction can be transformed in one without restriction using the Lagrangian function $\mathscr{L}$ with 1.1 and 1.4:

$$\mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) - \right.$$
$$\left. -\mu_t \Big[P_t (C_{\eta t} + K_{\eta,t+1} - (1-\delta) K_{\eta t}) - (W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t}) \Big] \right\} \quad (1.5)$$

First Order Conditions

The first order conditions with respect to $C_{\eta t}$, $L_{\eta t}$, $K_{\eta,t+1}$ and μ_t are:

$$C_{\eta t}: \quad C_{\eta t}^{-\sigma} - \mu_t P_t = 0 \implies \mu_t = \frac{C_{\eta t}^{-\sigma}}{P_t} \quad (1.6)$$

$$L_{\eta t}: \quad -\phi L_{\eta t}^{\varphi} + \mu_t W_t = 0 \implies \mu_t = \frac{\phi L_{\eta t}^{\varphi}}{W_t} \quad (1.7)$$

$$\begin{aligned} K_{\eta,t+1}: \quad & -\mu_t P_t + \beta \mathbb{E}_t \mu_{t+1} [(1-\delta) P_{t+1} + R_{t+1}] = 0 \implies \\ & \mu_t P_t = \beta \mathbb{E}_t \mu_{t+1} [(1-\delta) P_{t+1} + R_{t+1}] \end{aligned} \quad (1.8)$$

$$\mu_t: \quad P_t (C_{\eta t} + K_{\eta,t+1} - (1-\delta) K_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \quad (1.4)$$

Solutions

Match equations 1.6 and 1.7:

$$\frac{C_{\eta t}^{-\sigma}}{P_t} = \frac{\phi L_{\eta t}^{\varphi}}{W_t} \implies \frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \quad (1.9)$$

Equation 1.9 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute μ_t and μ_{t+1} from equation 1.6 in 1.8:

$$\mu_t P_t = \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] \quad \Longrightarrow$$
$$\frac{C_{\eta t}^{-\sigma}}{P_t} P_t = \beta \mathbb{E}_t \frac{C_{\eta,t+1}^{-\sigma}}{P_{t+1}}[(1-\delta)P_{t+1} + R_{t+1}] \Longrightarrow$$
$$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta \left[(1-\delta) + \mathbb{E}_t \left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{1.10}$$

Equation 1.10 is the Household Euler equation.

Firms

Consider two types of firms: (1) a continuum of intermediate-good firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-good firm, which aggregates all the varieties into a final bundle and operates in perfect competition.

1.2 Final-Good Firm

Profit Maximization Problem

The role of the final-good firm is to aggregate all the varieties produced by the intermediate-good firms, so that the representative consumer can buy only one good $Y_{\eta t}$, the bundle good. The final-good firm problem is to maximize its profit, considering that its output is the bundle $Y_{\eta t}$ formed by the continuum of intermediate goods $Y_{\eta j t}$, where $j \in [0, 1]$ and ψ is the elasticity of substitution between intermediate goods:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_t Y_{\eta t} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{1.11}$$

$$\text{s. t.} : \quad Y_{\eta j t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} \tag{1.12}$$

Substitute 1.12 in 1.11:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_t \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{1.13}$$

First Order Condition and Solutions

The first order condition is:

$$
\begin{aligned}
Y_{\eta jt}: \quad & P_t \left(\frac{\psi}{\psi-1}\right) \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \,\mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}-1} \left(\frac{\psi-1}{\psi}\right) Y_{\eta jt}^{\frac{\psi-1}{\psi}-1} - P_{\eta jt} = 0 \implies \\
& Y_{\eta jt} = Y_t \left(\frac{P_t}{P_{\eta jt}}\right)^{\psi} \qquad (1.14)
\end{aligned}
$$

Equation 1.14 shows that the demand for variety j depends on its relative price.

Substitute 1.14 in 1.12:

$$
\begin{aligned}
Y_t &= \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}} \,\mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} \implies \\
Y_t &= \left(\int_0^1 \left[Y_t \left(\frac{P_t}{P_{\eta jt}}\right)^{\psi}\right]^{\frac{\psi-1}{\psi}} \,\mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} \implies \\
P_t &= \left[\int_0^1 P_{\eta jt}^{1-\psi} \,\mathrm{d}\, j\right]^{\frac{1}{1-\psi}} \qquad (1.15)
\end{aligned}
$$

Equation 1.15 is the final-good firm's markup.

1.3 Intermediate-Good Firms

Cost Minimization Problem

The intermediate-good firms, denoted by $j \in [0, 1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose capital $K_{\eta jt}$ and labor N_{jt} to minimize production costs, subject to a technology rule.

$$
\begin{aligned}
\min_{K_{\eta jt}, L_{\eta jt}} : \quad & R_t K_{\eta jt} + W_t L_{\eta jt} \qquad (1.16) \\
\text{s. t.}: \quad & Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \qquad (1.17)
\end{aligned}
$$

where $Y_{\eta jt}$ is the output obtained by the production technology level $Z_{A\eta t}$[1] that transforms capital $K_{\eta jt}$ and labor $L_{\eta jt}$ in proportions α_η and $(1-\alpha_\eta)$, respectively, into intermediate goods.

Lagrangian

Applying the Lagrangian:

$$\mathscr{L} = (R_t K_{\eta jt} + W_t L_{\eta jt}) - \Lambda_{\eta jt}(Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} - Y_{\eta jt}) \tag{1.18}$$

where the Lagrangian multiplier $\Lambda_{\eta jt}$ is the marginal cost[2].

First Order Conditions

The first-order conditions are:

$$\begin{aligned}
K_{\eta jt}: \quad & R_t - \Lambda_{\eta jt} Z_{A\eta t} \alpha_\eta K_{\eta jt}^{\alpha_\eta - 1} L_{\eta jt}^{1-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& K_{\eta jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta jt}}{R_t} && (1.19) \\
L_{\eta jt}: \quad & W_t - \Lambda_{\eta jt} Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} (1-\alpha_\eta) L_{\eta jt}^{-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& L_{\eta jt} = (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta jt}}{W_t} && (1.20) \\
\Lambda_{\eta jt}: \quad & Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} && (1.17)
\end{aligned}$$

Solutions

Divide equation 1.19 by 1.20:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \frac{\alpha_\eta Y_{\eta jt} \Lambda_{\eta jt} / R_t}{(1-\alpha_\eta) Y_{\eta jt} \Lambda_{\eta jt} / W_t} \quad \Longrightarrow \quad \frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \tag{1.21}$$

Equation 1.21 demonstrates the relationship between the technical marginal rate of substitution (TMRS) and the economical marginal rate of substitution (EMRS).

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[1] the production technology level $Z_{A\eta t}$ will be submitted to a productivity shock, detailed in section 1.5.

[2] see Lemma **??**

Substitute $L_{\eta jt}$ from equation 1.21 in 1.17:

$$\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} && \Longrightarrow \\
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} \left[\left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t K_{\eta jt}}{W_t} \right]^{1-\alpha_\eta} && \Longrightarrow \\
K_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{1-\alpha_\eta} && \qquad (1.22)
\end{aligned}$$

Equation 1.22 is the intermediate-good firm demand for capital.

Substitute 1.22 in 1.21:

$$\begin{aligned}
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t K_{\eta jt}}{W_t} && \Longrightarrow \\
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_t}{W_t} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{1-\alpha_\eta} && \Longrightarrow \\
L_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{-\alpha_\eta} && \qquad (1.23)
\end{aligned}$$

Equation 1.23 is the intermediate-good firm demand for labor.

Total and Marginal Costs

Calculate the total cost using 1.22 and 1.23:

$$\begin{aligned}
TC_{\eta jt} &= W_t L_{\eta jt} + R_t K_{\eta jt} && \Longrightarrow \\
TC_{\eta jt} &= W_t \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{-\alpha_\eta} + R_t \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_t} \right]^{1-\alpha_\eta} && \Longrightarrow \\
TC_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta} \right)^{1-\alpha_\eta} && \qquad (1.24)
\end{aligned}$$

Calculate the marginal cost using 1.24:

$$\Lambda_{\eta jt} = \frac{\partial TC_{\eta jt}}{\partial Y_{\eta jt}} \implies \Lambda_{\eta jt} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta} \right)^{1-\alpha_\eta} \qquad (1.25)$$

The marginal cost depends on the technological level $Z_{A\eta t}$, the nominal interest rate R_t and the nominal wage level W_t, which are the same for all intermediate-good firms, and because of that, the index j may be dropped:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{1.26}$$

notice that:

$$\Lambda_{\eta t} = \frac{TC_{\eta jt}}{Y_{\eta jt}} \implies TC_{\eta jt} = \Lambda_{\eta t} Y_{\eta jt} \tag{1.27}$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (CALVO, 1983): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{j,t+1} = P_{j,t})$, and a probability of $(1-\theta)$ of setting a new optimal price $P^*_{j,t}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price.

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{1.28}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Y_{\eta t} \left(\frac{P_t}{P_{\eta jt}}\right)^{\psi} \tag{1.14}$$

Substitute 1.27 in 1.28:

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - \Lambda_{\eta t+s} Y_{\eta j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{1.29}$$

Substitute 1.14 in 1.29 and rearrange the variables:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta t+s} \left(\frac{P_{t+s}}{P_{\eta jt}}\right)^{\psi} - \Lambda_{\eta t+s} Y_{\eta t+s} \left(\frac{P_{t+s}}{P_{\eta jt}}\right)^{\psi} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt}^{1-\psi} P_{t+s}^{\psi} Y_{\eta t+s} - P_{\eta jt}^{-\psi} P_{t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to $P_{\eta jt}$ is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{\eta jt}^{-\psi} P_{t+s}^{\psi} Y_{\eta t+s} - (-\psi) P_{\eta jt}^{-\psi-1} P_{t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi-1) \left(\frac{P_{t+s}}{P_{\eta jt}}\right)^{\psi} Y_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} &= \\ = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} \left(\frac{P_{t+s}}{P_{\eta jt}}\right)^{\psi} Y_{\eta t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} & \end{aligned} \tag{1.30}$$

Substitute 1.14 in 1.30:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi-1) Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$(\psi-1)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \psi P_{\eta jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt}^{*} = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}} \tag{1.31}$$

Equation 1.31 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P_{\eta t}^{*} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{1.32}$$

1.3.1 Final-Good Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1-\theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 1.15 to determine the aggregate price level:

$$P_t = \left[\int_0^{\theta} P_{\eta, t-1}^{1-\psi} \, \mathrm{d}\, j + \int_{\theta}^{1} P_{\eta t}^{*1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies$$

$$P_t = \left[\theta P_{\eta, t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \tag{1.33}$$

Equation 1.33 is the aggregate price level.

1.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (TAYLOR, 1993) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{1.34}$$

where π_t is the gross inflation rate, defined by:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{1.35}$$

and R, π, Y are the variables in steady state, γ_R is the smoothing parameter for the interest rate R_t, while γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively and Z_{Mt} is the monetary shock[3].

1.5 Stochastic Shocks

Productivity Shock

The production technology level $Z_{A\eta t}$ will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{1.36}$$

where $\rho_{A\eta} \in [0,1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{1.37}$$

where $\rho_M \in [0,1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

1.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P_t^*, R_t^*, W_t^*\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C_{\eta t}^*, L_{\eta t}^*, K_{\eta,t+1}^*\}$ and for firms $\boldsymbol{\mathcal{A}}_F := \{K_{\eta jt}^*, L_{\eta jt}^*, Y_{\eta jt}^*, Y_{\eta t}^*\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms'

[3] for the monetary shock definition, see section 1.5.

problems, and the markets for goods and labor clear:

$$Y_t = \sum_{\eta=1}^{n} (C_{\eta t} + I_{\eta t}) \tag{1.38}$$

$$L_{\eta t} = \int_0^1 L_{\eta j t} \, \mathrm{d}\, j \tag{1.39}$$

1.6.1 Model Structure

The model is composed of the preview solutions, forming a square system of 27 variables and 27 equations, summarized as follows:

- Variables:
  - from the household problem: $C_{\eta t}, L_{\eta t}, K_{\eta,t+1}$;
  - from the final-good firm problem: $Y_{\eta t}, Y_{\eta jt}, P_t$;
  - from the intermediate-good firm problems: $K_{\eta jt}, L_{\eta jt}, P_t^*$;
  - from the market clearing condition: $Y_t, I_{\eta t}$;
  - prices: $W_t, R_t, \Lambda_{\eta t}, \pi_t$;
  - shocks: $Z_{A\eta t}, Z_{Mt}$.
- Equations:
  1. Labor Supply:
  $$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \tag{1.9}$$
  2. Household Euler Equation:
  $$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{1.10}$$
  3. Budget Constraint:
  $$P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{1.2}$$
  4. Law of Motion for Capital:
  $$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{1.3}$$
  5. Bundle Technology:
  $$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{1.12}$$
  6. General Price Level:
  $$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{1.33}$$

7. Capital Demand:

$$K_{\eta jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_t} \tag{1.19}$$

8. Labor Demand:

$$L_{\eta jt} = (1 - \alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \tag{1.20}$$

9. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1 - \alpha_\eta} \right)^{1-\alpha_\eta} \tag{1.26}$$

10. Production Function:

$$Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \tag{1.17}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{1.32}$$

12. Market Clearing Condition:

$$Y_t = \sum_{\eta=1}^{n} (C_{\eta t} + I_{\eta t}) \tag{1.38}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{1.34}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{1.35}$$

15. Productivity Shock:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{1.36}$$

16. Monetary Shock:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{1.37}$$

1.7 Steady State

The steady state is defined by the constancy of the variables through time. For any given endogenous variable X_t, it is in steady state if $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (COSTA JUNIOR, 2016, p.41). For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The steady state of each equation of the model is:

1. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \implies \frac{\phi L_{\eta}^{\varphi}}{C_{\eta}^{-\sigma}} = \frac{W}{P} \tag{1.40}$$

2. Household Euler Equation:

$$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies$$
$$1 = \beta\left[(1-\delta) + \frac{R}{P}\right] \tag{1.41}$$

3. Budget Constraint:

$$P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \implies$$
$$P(C_{\eta} + I_{\eta}) = W L_{\eta} + R K_{\eta} + \Pi_{\eta} \tag{1.42}$$

4. Law of Motion for Capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies K_{\eta} = (1-\delta)K_{\eta} + I_{\eta} \implies$$
$$I_{\eta} = \delta K_{\eta} \tag{1.43}$$

5. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \implies Y_{\eta} = \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{1.44}$$

6. General Price Level:

$$\begin{aligned}
P_t &= \left[\theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} && \Longrightarrow \\
P^{1-\psi} &= \theta P^{1-\psi} + (1-\theta) P^{*1-\psi} && \Longrightarrow \\
(1-\theta) P^{1-\psi} &= (1-\theta) P^{*1-\psi} && \Longrightarrow \quad P = P^* \qquad (1.45)
\end{aligned}$$

7. Capital Demand:

$$K_{\eta jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_t} \implies K_j = \alpha_\eta Y j \frac{\Lambda}{R} \qquad (1.46)$$

8. Labor Demand:

$$L_{\eta jt} = (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \implies L_j = (1-\alpha_\eta) Y j \frac{\Lambda}{W} \qquad (1.47)$$

9. Marginal Cost:

$$\begin{aligned}
\Lambda_{\eta t} &= \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies \\
\Lambda_\eta &= \frac{1}{Z_{A\eta}} \left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \qquad (1.48)
\end{aligned}$$

10. Production Technology:

$$Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \implies Y_{\eta j} = Z_{A\eta} K_{\eta j}^{\alpha_\eta} L_{\eta j}^{1-\alpha_\eta} \qquad (1.49)$$

11. Optimal Price:

$$\begin{aligned}
P_t^* &= \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \implies \qquad (1.32) \\
P^* &= \frac{\psi}{\psi-1} \cdot \frac{Y_{\eta j} \Lambda_\eta / [1-\theta(1-R)]}{Y_{\eta j} / [1-\theta(1-R)]} \implies \\
P^* &= \frac{\psi}{\psi-1} \Lambda_\eta \qquad (1.50)
\end{aligned}$$

12. Market Clearing Condition:

$$Y_t = \sum_{\eta=1}^{n} (C_{\eta t} + I_{\eta t}) \implies Y = \sum_{\eta=1}^{n} (C_\eta + I_\eta) \tag{1.51}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \tag{1.52}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \pi = 1 \tag{1.53}$$

15. Productivity Shock:

$$\begin{aligned} \ln Z_{A\eta t} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} && \implies \\ \ln Z_{A\eta} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta} + \varepsilon_{A\eta} && \implies \\ \varepsilon_{A\eta} &= 0 \end{aligned} \tag{1.54}$$

16. Monetary Shock:

$$\begin{aligned} \ln Z_{Mt} &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \implies \\ \ln Z_M &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_M + \varepsilon_M && \implies \\ \varepsilon_M &= 0 \end{aligned} \tag{1.55}$$

1.7.1 Variables in Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It's assumed that the productivity and the price level are normalized to one: $\left[\, P \;\; Z_{A\eta} \,\right] = \vec{\mathbf{1}}$ [4].

From 1.45, the optimal price P^* is:

$$P^* = P \tag{1.56}$$

[4] where $\vec{\mathbf{1}}$ is the unit vector.

From 1.53, the gross inflation rate is:

$$\pi = 1 \tag{1.57}$$

From 1.52, the monetary shock is:

$$Z_M = 1 \tag{1.58}$$

From 1.54 and 1.55, the productivity and monetary shocks are:

$$\begin{bmatrix} \varepsilon_{A\eta} & \varepsilon_M \end{bmatrix} = \vec{\mathbf{0}} \tag{1.59}$$

From 1.41, the return on capital R is:

$$1 = \beta \left[(1-\delta) + \frac{R}{P} \right] \implies R = P \left[\frac{1}{\beta} - (1-\delta) \right] \tag{1.60}$$

From 1.50 and 1.45, the marginal cost Λ_η is:

$$P^* = \frac{\psi}{\psi - 1} \Lambda_\eta \implies \Lambda_\eta = P \frac{\psi - 1}{\psi} \tag{1.61}$$

From equation 1.48, the nominal wage W is:

$$\Lambda_\eta = \frac{1}{Z_{A\eta}} \left(\frac{R}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta} \right)^{1-\alpha_\eta} \implies$$
$$W = (1-\alpha_\eta) \left[\Lambda_\eta Z_{A\eta} \left(\frac{\alpha_\eta}{R} \right)^{\alpha_\eta} \right]^{\frac{1}{1-\alpha_\eta}} \tag{1.62}$$

@@@ CONTINUAR DAQUI @@@

In steady state, prices are the same ($P = P^*$), resulting in a gross inflation level of one ($\pi = 1$), and all firms producing the same output level ($Y_j = Y$) due to the price parity (SOLIS-GARCIA, 2022, Lecture 13, p.12). For this reason, they all demand the same amount of factors (K, L), and equations 1.46, 1.47, and 1.49 become:

$$Y = Z_{A\eta} K^{\alpha_\eta} L^{1-\alpha_\eta} \tag{1.63}$$

$$K = \alpha_\eta Y \frac{\Lambda}{R} \tag{1.64}$$

$$L = (1 - \alpha_\eta) Y \frac{\Lambda}{W} \tag{1.65}$$

Substitute 1.64 in 1.43:

$$I = \delta K \implies I = \delta \alpha_\eta Y \frac{\Lambda}{R} \tag{1.66}$$

Substitute 1.65 in 1.40:

$$\frac{\phi L^{\varphi}}{C^{-\sigma}} = \frac{W}{P} \implies C = \left[L^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} \implies C = \left[\left((1 - \alpha_\eta) Y \frac{\Lambda}{W} \right)^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} \tag{1.67}$$

Substitute 1.66 and 1.67 in 1.51:

$$\begin{aligned}
Y &= C + I && \implies \\
Y &= \left[\left((1 - \alpha_\eta) Y \frac{\Lambda}{W} \right)^{-\varphi} \frac{W}{\phi P} \right]^{\frac{1}{\sigma}} + \left[\delta \alpha_\eta Y \frac{\Lambda}{R} \right] && \implies \\
Y &= \left[\left(\frac{W}{\phi P} \right) \left(\frac{W}{(1 - \alpha_\eta)\Lambda} \right)^{\varphi} \left(\frac{R}{R - \delta \alpha_\eta \Lambda} \right)^{\sigma} \right]^{\frac{1}{\varphi + \sigma}} && (1.68)
\end{aligned}$$

For C, K, L, I, use the result from 1.68 in 1.67, 1.64, 1.65 and 1.43, respectively.

1.7.2 Steady State Solution

$$\begin{bmatrix} P & P^* & \pi & Z_{A\eta} & Z_M \end{bmatrix} = \vec{1} \tag{1.69}$$

$$\begin{bmatrix} \varepsilon_A & \varepsilon_M \end{bmatrix} = \vec{0} \tag{1.70}$$

$$R = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{1.60}$$

$$\Lambda = P\frac{\psi - 1}{\psi} \tag{1.61}$$

$$W = (1-\alpha_\eta)\left[\Lambda Z_{A\eta}\left(\frac{\alpha_\eta}{R}\right)^{\alpha_\eta}\right]^{\frac{1}{1-\alpha_\eta}} \tag{1.62}$$

$$Y = \left[\left(\frac{W}{\phi P}\right)\left(\frac{W}{(1-\alpha_\eta)\Lambda}\right)^{\varphi}\left(\frac{R}{R-\delta\alpha_\eta\Lambda}\right)^{\sigma}\right]^{\frac{1}{\varphi+\sigma}} \tag{1.68}$$

$$C = \left[\left((1-\alpha_\eta)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \tag{1.67}$$

$$K = \alpha_\eta Y\frac{\Lambda}{R} \tag{1.64}$$

$$L = (1-\alpha_\eta)Y\frac{\Lambda}{W} \tag{1.65}$$

$$I = \delta K \tag{1.43}$$

1.8 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. Dynare is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation 1.32. For this purpose, Uhlig's rules of log-linearization (UHLIG, 1999) will be applied to all equations in the model[5].

[5] see lemma **??** for details.

1.8.1 Gross Inflation Rate

Log-linearize 1.35 and define the level deviation of gross inflation rate $\widetilde{\pi}_t$:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \tag{1.35}$$

$$\widetilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{1.71}$$

1.8.2 New Keynesian Phillips Curve

In order to log-linearize equation 1.32, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma **??**:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{1.32}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\qquad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{1.72}$$

First, log-linearize the left hand side of equation 1.72 with respect to $P_t^*, Y_{j,t}, \widetilde{R}_t$:

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies \\ &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P^* Y_j \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies \\ &P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \end{aligned}$$

Separate the terms not dependent on s:

$$P^* Y_j (1+\hat{P}_t^*) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \right\} +$$
$$+ P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \implies$$

Apply definition **??** on the first term:

$$\frac{P^* Y_j (1+\hat{P}_t^*)}{1-\theta/(1+R)} + P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Second, log-linearize the left hand side of equation 1.72 with respect to $\Lambda_{\eta t}^*, Y_{j,t}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right)} \right\} \implies$$
$$\frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \frac{Y_j \Lambda (1 + \hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s})}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies$$
$$\frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(1 + \hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \right\} +$$
$$+ \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Apply definition **??** on the first term:

$$\frac{\psi}{\psi-1} \cdot \frac{Y_j \Lambda}{1-\theta/(1+R)} +$$
$$+ \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Join both sides of the equation again:

$$\frac{P^*Y_j(1+\hat{P}_t^*)}{1-\theta/(1+R)} + P^*Y_j\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{j,t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} =$$
$$= \frac{\psi}{\psi-1}\cdot\frac{Y_j\Lambda}{1-\theta/(1+R)} +$$
$$+\frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} \quad (1.73)$$

Define a nominal discount rate ρ in steady state:

$$1 = \rho(1+R) \implies \rho = \frac{1}{1+R} \quad (1.74)$$

Substitute 1.74 in 1.73:

$$\frac{P^*Y_j(1+\hat{P}_t^*)}{1-\theta\rho} + P^*Y_j\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^s\left(\hat{Y}_{j,t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} = \frac{\psi}{\psi-1}\cdot\frac{Y_j\Lambda}{1-\theta\rho} +$$
$$+\frac{\psi}{\psi-1}Y_j\Lambda\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^s\left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} \quad (1.75)$$

Substitute 1.61 in 1.75 and simplify all common terms:

$$\cancel{\frac{P^*Y_j}{1-\theta\rho}} + \frac{\cancel{P^*Y_j}\hat{P}_t^*}{1-\theta\rho} + \cancel{P^*Y_j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^s\left(\cancel{\hat{Y}_{j,t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right)\right\} =$$
$$= \cancel{\frac{P^*Y_j}{1-\theta\rho}} + \cancel{P^*Y_j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^s\left(\cancel{\hat{Y}_{j,t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}} + \hat{\Lambda}_{t+s}\right)\right\} \implies$$
$$\frac{\hat{P}_t^*}{1-\theta\rho} = \mathbb{E}_t\sum_{s=0}^{\infty}\left\{(\theta\rho)^s\left(\hat{\Lambda}_{t+s}\right)\right\} \quad (1.76)$$

Define the real marginal cost λ_t:

$$\lambda_t = \frac{\Lambda_{\eta t}}{P_t} \implies \Lambda_{\eta t} = P_t\lambda_t \implies$$
$$\hat{\Lambda}_t = \hat{P}_t + \hat{\lambda}_t \quad (1.77)$$

Substitute 1.77 in 1.76:

$$\hat{P}_t^* = (1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{1.78}$$

Log-linearize equation 1.33:

$$\begin{aligned}
P_t^{1-\psi} &= \theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi} & \Longrightarrow \quad (1.33)\\
P^{1-\psi}(1+(1-\psi)\hat{P}_t) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{t-1}) + \\
&\quad + (1-\theta)P^{1-\psi}(1+(1-\psi)\hat{P}_t^*) & \Longrightarrow \\
\hat{P}_t &= \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* & (1.79)
\end{aligned}$$

Substitute 1.78 in 1.79:

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* \tag{1.79}$$

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{1.80}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\rho\mathbb{L}^{-1})$[6] in 1.80:

$$\begin{aligned}
(1-\theta\rho\mathbb{L}^{-1})\hat{P}_t &= (1-\theta\rho\mathbb{L}^{-1})\Big[\theta\hat{P}_{t-1} + \\
&\quad +(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right)\Big] \Longrightarrow \\
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + \\
&\quad (1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) - \\
&\quad - \theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right)
\end{aligned} \tag{1.81}$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\rho$ within the operator. Then, cancel the summations and rearrange

[6] see definition ??.

the terms:

$$
\begin{aligned}
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + \\
& (1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^s\left(\hat{P}_{t+s}+\hat{\lambda}_{t+s}\right) - \\
& -\theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right) \implies \\
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + (1-\theta)(1-\theta\rho)(\hat{P}_t+\hat{\lambda}_t) + \\
& +\cancel{(1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right)} - \\
& -\cancel{(1-\theta)(1-\theta\rho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\rho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right)} \implies \\
\hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{t-1} - \theta^2\rho\hat{P}_t + (1-\theta-\theta\rho+\theta^2\rho)\hat{P}_t + (1-\theta)(1-\theta\rho)\hat{\lambda}_t \implies
\end{aligned}
$$

$$
(\hat{P}_t - \hat{P}_{t-1}) = \rho(\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_t) + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \tag{1.82}
$$

Substitute 1.71 in 1.82:

$$
\widetilde{\pi}_t = \rho\mathbb{E}_t\widetilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \tag{1.83}
$$

Equation 1.83 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

1.8.3 Labor Supply

Log-linearize 1.9:

$$
\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \qquad \implies \tag{1.9}
$$

$$
\varphi\hat{L}_t + \sigma\hat{C}_t = \hat{W}_t + \hat{P}_t \tag{1.84}
$$

1.8.4 Household Euler Equation

Log-linearize 1.10:

$$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies \tag{1.10}$$

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{1.85}$$

1.8.5 Law of Motion for Capital

Log-linearize 1.3:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies \tag{1.3}$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta \hat{I}_t \tag{1.86}$$

1.8.6 Bundle Technology

Apply the natural logarithm to 1.12:

$$\ln Y_t = \frac{\psi}{\psi-1}\ln\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)$$

Log-linearize using corollary **??**:

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(\int_0^1 Y_j^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(Y_j^{\frac{\psi-1}{\psi}}\int_0^1 \mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \cancel{\frac{\psi}{\psi-1}}\left[\cancel{\frac{\psi-1}{\psi}}\ln Y_j + \cancel{\ln 1} + \cancel{\frac{\psi-1}{\psi}}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j$$

Apply corollary **??**:

$$\begin{aligned}\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j \implies \\ \hat{Y}_t = \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j \qquad (1.87)\end{aligned}$$

1.8.7 Marginal Cost

Log-linearize 1.26:

$$\Lambda_{\eta t} = Z_{A\eta t}^{-1} \frac{R_t^{\alpha_\eta} W_t^{1-\alpha_\eta}}{\alpha_\eta{}^{\alpha_\eta}(1-\alpha_\eta)^{1-\alpha_\eta}} \implies \qquad (1.26)$$

$$\Lambda(1+\hat{\Lambda}_t) = \frac{1}{Z_{A\eta}} \left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} (1 - \hat{Z}_{A\eta t} + \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t) \implies$$

$$\hat{\Lambda}_t = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \qquad (1.88)$$

Substitute 1.77 in 1.88:

$$\begin{aligned}\hat{\Lambda}_t &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \implies \\ \hat{P}_t + \hat{\lambda}_t &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \implies \\ \hat{\lambda}_t &= \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_t \qquad (1.89)\end{aligned}$$

1.8.8 Production Function

Log-linearize 1.17:

$$\begin{aligned}Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \implies \qquad (1.17) \\ Y_j(1+\hat{Y}_{jt}) &= Z_{A\eta} K_j^{\alpha_\eta} L_j^{1-\alpha_\eta} (1 + \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt}) \implies \\ \hat{Y}_{jt} &= \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt} \qquad (1.90)\end{aligned}$$

Substitute 1.90 in 1.87:

$$\hat{Y}_t = \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j \qquad \Longrightarrow \tag{1.87}$$

$$\hat{Y}_t = \int_0^1 \left[\hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt}\right]\mathrm{d}\,j \qquad \Longrightarrow$$

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha_\eta)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \tag{1.91}$$

Apply the natural logarithm and then log-linearize 1.39:

$$L_{\eta t} = \int_0^1 L_{\eta jt}\,\mathrm{d}\,j \qquad \Longrightarrow \tag{1.39}$$

$$\ln L_{\eta t} = \ln\left[\int_0^1 L_{\eta jt}\,\mathrm{d}\,j\right] \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln\left[\int_0^1 L_j\,\mathrm{d}\,j\right] + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln L_j + \ln 1 + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j$$

Apply corollary **??**:

$$\Longrightarrow\ \hat{L}_t = \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \tag{1.92}$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_t = \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j \tag{1.93}$$

Substitute 1.92 and 1.93 in 1.91:

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha_\eta)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \ \Longrightarrow \tag{1.91}$$

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_t + (1-\alpha_\eta)\hat{L}_t \tag{1.94}$$

1.8.9 Capital Demand

Log-linearize 1.19:

$$
\begin{aligned}
K_{\eta jt} &= \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_t} && \Longrightarrow && (1.19)\\
K_j(1+\hat{K}_{jt}) &= \alpha_\eta Y_j \frac{\Lambda}{R}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t) && \Longrightarrow\\
\hat{K}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t
\end{aligned}
$$

Integrate both sides and then substitute 1.93 and 1.87:

$$
\begin{aligned}
\int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j &= \int_0^1 (\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t)\,\mathrm{d}\,j \quad \Longrightarrow\\
\hat{K}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t \qquad (1.95)
\end{aligned}
$$

1.8.10 Labor Demand

Log-linearize 1.20:

$$
\begin{aligned}
L_{\eta jt} &= (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} && \Longrightarrow && (1.20)\\
L_j(1+\hat{L}_{jt}) &= (1-\alpha_\eta) Y_j \frac{\Lambda}{W}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t) && \Longrightarrow\\
\hat{L}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t
\end{aligned}
$$

Integrate both sides and then substitute 1.92 and 1.87:

$$
\begin{aligned}
\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j &= \int_0^1 \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t\,\mathrm{d}\,j \quad \Longrightarrow\\
\hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t \qquad (1.96)
\end{aligned}
$$

Subtract 1.96 from 1.95:

$$
\begin{aligned}
\hat{K}_t-\hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t-(\hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t) \quad \Longrightarrow\\
\hat{K}_t-\hat{L}_t &= \hat{W}_t-\hat{R}_t \qquad (1.97)
\end{aligned}
$$

Equation 1.97 is the log-linearized version of 1.21.

1.8.11 Market Clearing Condition

Log-linearize 1.38:

$$\begin{aligned}
Y_t &= C_{\eta t} + I_{\eta t} && \Longrightarrow && (1.38)\\
Y(1+\hat{Y}_t) &= C(1+\hat{C}_t) + I(1+\hat{I}_t) && \Longrightarrow\\
Y + Y\hat{Y}_t &= C + C\hat{C}_t + I + I\hat{I}_t && \Longrightarrow\\
Y\hat{Y}_t &= C\hat{C}_t + I\hat{I}_t && \Longrightarrow\\
\hat{Y}_t &= \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t && && (1.98)
\end{aligned}$$

Define the consumption and investment weights $\left[\theta_C \ \theta_I\right]$ in the production total:

$$\begin{bmatrix}\theta_C & \theta_I\end{bmatrix} := \begin{bmatrix}\frac{C}{Y} & \frac{I}{Y}\end{bmatrix} \tag{1.99}$$

Substitute 1.99 in 1.98:

$$\begin{aligned}
&\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t \implies\\
&\hat{Y}_t = \theta_C\hat{C}_t + \theta_I\hat{I}_t && (1.100)
\end{aligned}$$

1.8.12 Monetary Policy

Log-linearize 1.34:

$$\begin{aligned}
&\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi}Y_t^{\gamma_Y})^{(1-\gamma_R)}Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \implies && (1.34)\\
&\frac{R(1+\hat{R}_t)}{R} =\\
&= \frac{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}Z_M[1+\gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt}]}{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \implies\\
&\hat{R}_t = \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt} && (1.101)
\end{aligned}$$

1.8.13 Productivity Shock

Log-linearize 1.36:

$$\begin{aligned}
\ln Z_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}\ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} && \Longrightarrow && (1.36)\\
\ln Z_{A\eta} + \hat{Z}_{A\eta t} &= (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}(\ln Z_{A\eta} + \hat{Z}_{A,t-1}) + \varepsilon_{A\eta} && \Longrightarrow &&\\
\hat{Z}_{A\eta t} &= \rho_{A\eta}\hat{Z}_{A,t-1} + \varepsilon_{A\eta} && && (1.102)
\end{aligned}$$

1.8.14 Monetary Shock

Log-linearize 1.37:

$$\begin{aligned}
\ln Z_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \Longrightarrow && (1.37)\\
\ln Z_M + \hat{Z}_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M(\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M && \Longrightarrow &&\\
\hat{Z}_{Mt} &= \rho_M \hat{Z}_{M,t-1} + \varepsilon_M && && (1.103)
\end{aligned}$$

1.8.15 Log-linear Model Structure

The log-linear model is a square system of 12 variables and 12 equations, summarized as follows:

- Variables: $(\tilde{\pi} \quad \hat{P} \quad \tilde{\lambda} \quad \hat{C} \quad \hat{L} \quad \hat{R} \quad \hat{K} \quad \hat{I} \quad \hat{W} \quad \hat{Z}_A \quad \hat{Y} \quad \hat{Z}_M)$
- Equations:

1. Gross Inflation Rate:

$$\widetilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{1.71}$$

2. New Keynesian Phillips Curve:

$$\widetilde{\pi}_t = \rho \mathbb{E}_t \widetilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \tag{1.83}$$

3. Labor Supply:

$$\varphi \hat{L}_t + \sigma \hat{C}_t = \hat{W}_t + \hat{P}_t \tag{1.84}$$

4. Household Euler Equation:

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{1.85}$$

5. Law of Motion for Capital:

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta \hat{I}_t \tag{1.86}$$

6. Real Marginal Cost:

$$\hat{\lambda}_t = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_t \tag{1.89}$$

7. Production Function:

$$\hat{Y}_t = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_t + (1-\alpha_\eta)\hat{L}_t \tag{1.94}$$

8. Marginal Rates of Substitution of Factors:

$$\hat{K}_t - \hat{L}_t = \hat{W}_t - \hat{R}_t \tag{1.97}$$

9. Market Clearing Condition:

$$\hat{Y}_t = \theta_C \hat{C}_t + \theta_I \hat{I}_t \tag{1.100}$$

10. Monetary Policy:

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \widetilde{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{1.101}$$

11. Productivity Shock:

$$\hat{Z}_{A\eta t} = \rho_{A\eta} \hat{Z}_{A,t-1} + \varepsilon_{A\eta} \tag{1.102}$$

12. Monetary Shock:

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{1.103}$$