0.1 DEFINITIONS, LEMMAS AND COROLLARIES

The objective of this appendix is to present the definitions and lemmas used throughout the text.

Lemma 0.1 (Marginal Cost)**.** *The Lagrangian multiplier Λ_t is the nominal marginal cost of the intermediate-good firm:*

$$MC_t := \frac{\partial TC_t}{\partial Y_t} = \Lambda_t$$

Proof. Simon and Blume (1994, p.449). ∎

Lemma 0.2 (Steady State Inflation)**.** *In steady state, prices are stable $P_t = P_{t-1} = P$ and the gross inflation rate is one.*

Proof. Equation **??**. ∎

Corollary 0.2.1. *In steady state, all firms have the same level of production Y and therefore demand the same amount of factors, capital K and labor L.*

$$P_t = P_{t-1} = P \implies \begin{pmatrix} Y_j & K_j & L_j \end{pmatrix} = \begin{pmatrix} Y & K & L \end{pmatrix}$$

Definition 0.1 (Uhlig's Rules)**.** The Uhlig's rules are a set of approximations used to log-linearize equations (SOLIS-GARCIA, 2022, Lecture 6, p.2).

Lemma 0.3 (Rule 1: Percentage Deviation from Steady State)**.**

$$\hat{x}_t := \frac{x_t - x}{x} \iff x_t = x(1 + \hat{x}_t)$$

Corollary 0.3.1 (Rule 2: Product)**.**

$$x_t y_t \approx xy(1 + \hat{x}_t + \hat{y}_t)$$

Proof. Apply lemma 0.3 to both variables and notice that $\hat{x}_t \hat{y}_t \approx 0$:

$$\begin{aligned} x_t y_t &= x(1 + \hat{x}_t) y(1 + \hat{y}_t) \\ &= xy(1 + \hat{x}_t + \hat{y}_t + \hat{x}_t \hat{y}_t) \\ &\approx xy(1 + \hat{x}_t + \hat{y}_t) \end{aligned}$$

∎

Corollary 0.3.2 (Rule 3: Exponential).

$$x_t^a \approx x^a(1 + a\hat{x}_t) \quad ; \quad a > 1$$

Proof. Apply corollary 0.3.1 $(a - 1)$ times. ∎

Corollary 0.3.3 (Logarithm Rule).

$$\ln x_t \approx \ln x + \hat{x}_t$$

Proof. Apply corollary 0.3 and notice that $\ln(1 + x) \approx x$ when $u \to 0_+$. ∎

Definition 0.2 (Uhlig's Rules for Level Deviations). Uhlig's rules can be applied to level deviations (when $0 < u_t < 1$) in order to log-linearize equations (SOLIS-GARCIA, 2022, Lecture 9, p.9).

Lemma 0.4 (Level Deviation from Steady State when $0 \leq u \leq 1$).

$$\widetilde{u}_t := u_t - u$$

Corollary 0.4.1 (Level Deviation from Steady State when $0 < u < 1$).

$$u_t = u\left(1 + \frac{\widetilde{u}_t}{u}\right)$$

Lemma 0.5 (Level Deviation of the Present Value Discount Factor). *The level deviation of the present value discount factor is equivalent to (SOLIS-GARCIA, 2022, Lecture 13, p.6):*

$$\prod_{k=0}^{s-1}(1 + R_{t+k}) = (1 + R)^s\left(1 + \frac{1}{1 + R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)$$

Proof. Substitute the interest rate by the gross interest rate $GR_t = 1 + R_t$ and apply corollary 0.4.1:

$$\begin{aligned}
&\prod_{k=0}^{s-1}(1 + R_{t+k}) = \prod_{k=0}^{s-1}(GR_{t+k}) && \Longrightarrow \\
&GR \times \cdots \times GR\left(1 + \frac{1}{GR}\widetilde{GR}_t + \frac{1}{GR}\widetilde{GR}_{t+1} + \cdots + \frac{1}{GR}\widetilde{GR}_{t+s-1}\right) && \Longrightarrow \\
&GR^s\left(1 + \frac{1}{GR}\sum_{k=0}^{s-1}\widetilde{GR}_{t+k}\right) && \Longrightarrow \\
&(1 + R)^s\left(1 + \frac{1}{1 + R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)
\end{aligned}$$

$\blacksquare$

Definition 0.3 (Geometric Series)**.** A geometric series S_∞ is the sum of the terms of a geometric sequence.

$$S_\infty = \sum_{i=0}^{\infty} ar^i \implies S_\infty = \frac{a}{1-r}, \ |r| < 1$$

Definition 0.4 (Lag Operator)**.** The lag operator $\mathbb{L}$ is a mathematical operator that represents the backshift or lag of a time series (SOLIS-GARCIA, 2022, Lecture 13, p.9):

$$\begin{aligned} \mathbb{L}x_t &= x_{t-1} \\ (1 + a\mathbb{L})y_{t+2} &= y_{t+2} + ay_{t+1} \end{aligned}$$

Corollary 0.5.1 (Lead Operator)**.** *Analogously, the lead operator $\mathbb{L}^{-1}$ (or inverse lag operator) yields a variable's lead (SOLIS-GARCIA, 2022, Lecture 13, p.9):*

$$\begin{aligned} \mathbb{L}^{-1}x_t &= x_{t+1} \\ (1 + a\mathbb{L}^{-1})y_{t+2} &= y_{t+2} + ay_{t+3} \end{aligned}$$