0.1 Regional New Keynesian Model

The model is populated by four agents: (1) a representative household, (2) a continuum of firms producing intermediate-goods, (3) a firm producing a final-goods, and (4) the monetary authority.

The representative household maximizes utility based on consumption and labor, subject to a budget constraint composed of wages, capital rental rates, and firm profits.

The final-goods firm produces the final-good consumed by households: it aggregates all intermediate-goods produced by intermediate firms, operates under perfect competition and seeks to maximize profit subject to the bundle technology.

Each intermediate-goods firm produces a single intermediate-good, all exhibiting imperfect substitution, thus operating in monopolistic competition. Intermediate-goods firms have two problems to solve: minimize costs subject to the production technology available and choose an optimal price to maximize the intertemporal profit flow.

Periodically, a portion of intermediate-goods firms have the opportunity to adjust prices, while others miss this chance, following a **calvo_staggered_1983** rule. This mechanism generates nominal price rigidities, altering equilibrium relationships in the system. These rigidities lead to the non-neutrality of money in the short term, as explained by **costa_junior_understanding_2016**.

The monetary authority determines the nominal interest rate in response to fluctuations in previous period's inflation and production, aiming to control price levels and growth, following a **taylor_discretion_1993** rule.

Stochastic shocks will be present in the intermediate-goods firms' productivity and in the monetary policy.

These elements define a canonical NK DSGE model, as presented by (**solis-garcia_ucb_2022**). The model will be adapted to accommodate two distinct regions: the main region and the rest of the country, replacing the single aggregated region. To achieve this, an index will differentiate the studied region from the rest of the country, resulting in separate households, intermediate- and final-goods firms for each region. Households lack mobility between regions. The link connecting the two regions is established through the final-goods, allowing households to consume from both regions.

Then, equilibrium conditions of the system will be determined. Assuming the system tends toward long-term equilibrium, a steady state will be reached where variables cease to change. Thus, for a given $t \to \infty$, there is a $\boldsymbol{X}_t = \boldsymbol{X}_{t+1} = \boldsymbol{X}_{ss} \implies \dot{\boldsymbol{X}} = 0$, where $\boldsymbol{X}$ denotes the vector of system variables, the subscript ss indicates the steady state and $\dot{\boldsymbol{X}} = {}^{\partial X}/_{\partial t}$.

After that, the log-linearization method proposed by **uhlig_toolkit_1999** will be employed to convert the system of equations into a linear system, so that this linear system can be solved by the program Dynare, which computes the solution and produces impulse-response graphs based on the stochastic shocks.

Regions

Regions will have an index $\eta \in \{1, 2, \ldots, n\}$ representing the variables of each region. Whenever necessary, a second region index $\nu \in \{1, 2, \ldots, n\}$ will be used. For example, the variable C_t represents the total consumption (the grand total of all regions), $C_{\eta t}$ represents the consumption composition of region η and $C_{\eta \nu t}$ represents the consumption of final-good of region ν by region η (the first index indicating the destination and the second one indicating the origin of the goods). Without loss of generality, the model will have two regions: the main region 1 and the remaining of the country 2, so that $\eta, \nu \in \{1, 2\}$.

Model Diagram

Figure (1) illustrates the model's mechanics. In this diagram, black arrows depict the real economy, while green arrows represent the nominal economy. The representative household supplies labor and capital to intermediate-goods firms in exchange for wages and capital rent, respectively. Using these resources, intermediate-goods firms produce goods, which are then sold to the final-goods firm. The final-goods firm aggregates all intermediate-goods into a final product, sold back to the household. Operating under a monetary rule, the monetary authority determines the nominal interest rate to achieve output growth and price stability.

Figure 1: Model Diagram, created by the author.

In the next section, the mathematical structure of the model is presented.

0.1.1 Household

The household problem can be divided into two steps: first, minimizing consumption costs, and then maximizing utility subject to a budget constraint.

Cost Minimization Problem

Considering that the representative household must decide to consume goods from both regions, there must be a consumption bundle index $C_{\eta t}$ and a consumption price index $Q_{\eta t}$ the minimizes the total consumption cost $Q_{\eta t}C_{\eta t}$, as demonstrated by **walsh_monetary_2017**:

$$\min_{C_{\eta 1t}, C_{\eta 2t}} : \quad Q_{\eta t}C_{\eta t} = P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} \tag{0.1}$$

$$\text{s. t.} : \quad C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \tag{0.2}$$

$$C_{\eta t} > 0$$

where P_{1t} and P_{2t} are the prices of goods 1 and 2, respectively, $C_{\eta 1t}$ and $C_{\eta 2t}$ are the goods produced in region 1 and 2, respectively, and consumed in region η. In the consumption aggregation, $\omega_{\eta 1}$ and $(1 - \omega_{\eta 1})$ are the weights of goods $C_{\eta 1t}$ and $C_{\eta 2t}$, respectively, in the consumption bundle $C_{\eta t}$.

Lagrangian

The minimization problem with a constraint can be reformulated into one without a constraint by applying the Lagrangian function:

$$\mathscr{L} = P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} - Q_{\eta t}(C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} - C_{\eta t}) \tag{0.3}$$

First Order Conditions

The first order conditions are:

$$
\begin{aligned}
C_{\eta 1t}: \quad & P_{1t} - Q_{\eta t}\omega_{\eta 1} C_{\eta 1t}^{\omega_{\eta 1}-1} C_{\eta 2t}^{1-\omega_{\eta 1}} = 0 \implies \\
& C_{\eta 1t} = \frac{\omega_{\eta 1} Q_{\eta t} C_{\eta t}}{P_{1t}} && (0.4) \\
C_{\eta 2t}: \quad & P_{2t} - Q_{\eta t}(1-\omega_{\eta 1}) C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{-\omega_{\eta 1}} = 0 \implies \\
& C_{\eta 2t} = \frac{(1-\omega_{\eta 1}) Q_{\eta t} C_{\eta t}}{P_{2t}} && (0.5) \\
Q_{\eta t}: \quad & C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} && (0.2)
\end{aligned}
$$

Solutions

Divide 0.5 by 0.4:

$$
\begin{aligned}
\frac{C_{\eta 2t}}{C_{\eta 1t}} &= \frac{(1-\omega_{\eta 1}) Q_{\eta t} C_{\eta t}/P_{2t}}{\omega_{\eta 1} Q_{\eta t} C_{\eta t}/P_{1t}} \implies \\
C_{\eta 2t} &= C_{\eta 1t} \frac{(1-\omega_{\eta 1}) P_{1t}}{\omega_{\eta 1} P_{2t}} && (0.6)
\end{aligned}
$$

Substitute 0.6 in 0.2:

$$
\begin{aligned}
C_{\eta t} &= C_{\eta 1t}^{\omega_{\eta 1}} \left[C_{\eta 1t} \frac{(1-\omega_{\eta 1}) P_{1t}}{\omega_{\eta 1} P_{2t}} \right]^{1-\omega_{\eta 1}} \implies \\
C_{\eta 1t} &= C_{\eta t} \left(\frac{P_{2t}\omega_{\eta 1}}{P_{1t}(1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} && (0.7)
\end{aligned}
$$

Substitute 0.4 and 0.5 in 0.2:

$$
\begin{aligned}
C_{\eta t} &= \left(\frac{\omega_{\eta 1} Q_{\eta t} C_{\eta t}}{P_{1t}} \right)^{\omega_{\eta 1}} \left(\frac{(1-\omega_{\eta 1}) Q_{\eta t} C_{\eta t}}{P_{2t}} \right)^{1-\omega_{\eta 1}} \implies \\
Q_{\eta t} &= \left(\frac{P_{1t}}{\omega_{\eta 1}} \right)^{\omega_{\eta 1}} \left(\frac{P_{2t}}{1-\omega_{\eta 1}} \right)^{1-\omega_{\eta 1}} && (0.8)
\end{aligned}
$$

Divide 0.8 of region 1 by region 2:

$$\frac{Q_{1t}}{Q_{2t}} = \frac{\left(\frac{P_{1t}}{\omega_{11}}\right)^{\omega_{11}} \left(\frac{P_{2t}}{1-\omega_{11}}\right)^{1-\omega_{11}}}{\left(\frac{P_{1t}}{\omega_{21}}\right)^{\omega_{21}} \left(\frac{P_{2t}}{1-\omega_{21}}\right)^{1-\omega_{21}}} \implies$$

$$\frac{Q_{1t}}{Q_{2t}} = \frac{\omega_{21}^{\omega_{21}}(1-\omega_{21})^{1-\omega_{21}}}{\omega_{11}^{\omega_{11}}(1-\omega_{11})^{1-\omega_{11}}} \tag{0.9}$$

Therefore, there is a consumption bundle $C_{\eta t}$ and a consumption price index $Q_{\eta t}$ that minimize the total consumption cost $Q_{\eta t}C_{\eta t}$ for the household in region η. Notice that the cost problems of both regions are (must be) related, as the consumption level in one region influences the demand for goods in both regions. Now, this result will be used in the next problem that the household faces.

Utility Maximization Problem

Following the models presented by **costa_junior_understanding_2016** and **solis-garcia_ucb_2022**, the representative household next problem is to maximize an intertemporal utility function U_η with respect to consumption $C_{\eta t}$ and labor $L_{\eta t}$, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_{\eta t}, L_{\eta t}, B_{\eta t}} : \quad U_\eta(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{0.10}$$

$$\text{s. t.} : \quad Q_{\eta t}C_{\eta t} + B_{\eta t} = W_{\eta t}L_{\eta t} + (1+R_{t-1})B_{\eta, t-1} + \Pi_{\eta t} \tag{0.11}$$

$$C_{\eta t}, L_{\eta t} > 0$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, P_{1t} and P_{2t} are the prices of goods 1 and 2, respectively, $C_{\eta 1t}$ and $C_{\eta 2t}$ are the goods produced in region 1 and 2, respectively, and consumed in region η, $B_{\eta t}$ are the bonds, $W_{\eta t}$ is the wage level, R_t is the return on bonds (which is also the nominal interest rate of the economy) and $\Pi_{\eta t}$ is the firm profit.

Lagrangian

The maximization problem with restrictions can be transformed into one without restriction using the Lagrangian function $\mathscr{L}$ formed by 0.10 and 0.11:

$$\begin{aligned}\mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) - \\ - \mu_{\eta t} \Big[Q_{\eta t} C_{\eta t} + B_{\eta t} - (W_{\eta t} L_{\eta t} + (1+R_{t-1}) B_{\eta, t-1} + \Pi_{\eta t}) \Big] \Bigg\}\end{aligned} \tag{0.12}$$

First Order Conditions

The first order conditions are:

$$C_{\eta t}: \quad \beta^t \left\{ \frac{(1-\sigma) C_{\eta t}^{-\sigma}}{1-\sigma} - \mu_{\eta t} \left[Q_{\eta t} \right] \right\} = 0 \implies$$

$$\mu_{\eta t} = \frac{C_{\eta t}^{-\sigma}}{Q_{\eta t}} \tag{0.13}$$

$$L_{\eta t}: \quad \beta^t \left\{ -\phi \frac{(1+\varphi) L_{\eta t}^{1+\varphi}}{1+\varphi} - \mu_{\eta t} \left[-W_{\eta t} \right] \right\} = 0 \implies$$

$$\mu_{\eta t} = \frac{\phi L_{\eta t}^{\varphi}}{W_{\eta t}} \tag{0.14}$$

$$B_{\eta t}: \quad \beta^t \{ -\mu_{\eta t} \} + \mathbb{E}_t \beta^{t+1} \{ -\mu_{\eta, t+1} [-(1+R_t)] \} = 0 \implies$$

$$\mu_{\eta t} = \beta (1+R_t) \mathbb{E}_t \mu_{\eta, t+1} \tag{0.15}$$

$$\mu_{\eta t}: \quad Q_{\eta t} C_{\eta t} + B_{\eta t} = W_{\eta t} L_{\eta t} + (1+R_{t-1}) B_{\eta, t-1} + \Pi_{\eta t} \tag{0.11}$$

Solutions

Match 0.13 and 0.14:

$$\mu_{\eta t} = \frac{C_{\eta t}^{-\sigma}}{Q_{\eta t}} = \frac{\phi L_{\eta t}^{\varphi}}{W_{\eta t}} \implies$$

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_{\eta t}}{Q_{\eta t}} \tag{0.16}$$

Equation 0.16 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute $\mu_{\eta t}$ and $\mu_{\eta,t+1}$ from equation 0.13 in 0.15:

$$\frac{C_{\eta t}^{-\sigma}}{Q_{\eta t}} = \beta(1+R_t)\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{-\sigma}}{Q_{\eta,t+1}}\right\} \implies$$

$$\frac{\mathbb{E}_t\left\{Q_{\eta,t+1}C_{\eta,t+1}^{\sigma}\right\}}{Q_{\eta t}C_{\eta t}^{\sigma}} = \beta(1+R_t) \tag{0.17}$$

Equation 0.17 is the Euler equation for the bonds return.

Divide 0.17 of region one by region two:

$$\frac{\mathbb{E}_t\left\{Q_{1,t+1}C_{1,t+1}^{\sigma}\right\}}{\mathbb{E}_t\left\{Q_{2,t+1}C_{2,t+1}^{\sigma}\right\}} = \frac{\beta(1+R_t)Q_{1t}C_{1t}^{\sigma}}{\beta(1+R_t)Q_{2t}C_{2t}^{\sigma}} \implies$$

$$\frac{\mathbb{E}_t\left\{Q_{1,t+1}C_{1,t+1}^{\sigma}\right\}}{Q_{1t}C_{1t}^{\sigma}} = \frac{\mathbb{E}_t\left\{Q_{2,t+1}C_{2,t+1}^{\sigma}\right\}}{Q_{2t}C_{2t}^{\sigma}} \tag{0.18}$$

Firms

Consider two types of firms: (1) a continuum of intermediate-goods firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-goods firm, which aggregates all these varieties into a final bundle and operates in perfect competition.

0.1.2 Final-Goods Firm

Profit Maximization Problem

The role of the final-goods firm is to aggregate all the varieties $Y_{\eta jt}$ produced by the intermediate-goods firms in each region $\eta \in \{1,2\}$, so that the representative consumer can buy only one good $Y_{\eta t}$, the bundle good, from each region.

The final-goods firm problem is to maximize its profit, considering that its output is the bundle $Y_{\eta t}$ formed by a continuum $j \in [0,1]$ of intermediate-goods $Y_{\eta jt}$, with elasticity of substitution between intermediate-goods ψ:

$$\max_{Y_{\eta jt}}: \quad P_{\eta t}Y_{\eta t} - \int_0^1 P_{\eta jt}Y_{\eta jt}\,\mathrm{d}\,j \tag{0.19}$$

$$\text{s. t.}: \quad Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \tag{0.20}$$

Substitute 0.20 in 0.19:

$$\max_{Y_{\eta jt}}: \quad \Pi_{\eta t} = P_{\eta t}\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{\eta jt}Y_{\eta jt}\,\mathrm{d}\,j \tag{0.21}$$

First Order Condition and Solutions

The first order condition is:

$$Y_{\eta jt}: \quad P_{\eta t}\left(\frac{\psi}{\psi-1}\right)\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}-1}\left(\frac{\psi-1}{\psi}\right)Y_{\eta jt}^{\frac{\psi-1}{\psi}-1} - P_{\eta jt} = 0 \implies$$

$$Y_{\eta jt} = Y_t\left(\frac{P_{\eta t}}{P_{\eta jt}}\right)^{\psi} \tag{0.22}$$

Equation 0.22 shows that the demand for variety j depends on its relative price.

Substitute 0.22 in 0.20:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \implies$$

$$Y_{\eta t} = \left(\int_0^1 \left[Y_{\eta t}\left(\frac{P_{\eta t}}{P_{\eta jt}}\right)^{\psi}\right]^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \implies$$

$$P_{\eta t} = \left[\int_0^1 P_{\eta jt}^{1-\psi}\,\mathrm{d}\,j\right]^{\frac{1}{1-\psi}} \tag{0.23}$$

Equation 0.23 is the final-goods firm's markup.

0.1.3 Intermediate-Goods Firms

Cost Minimization Problem

The intermediate-goods firms, denoted by $j \in [0, 1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose labor $L_{\eta jt}$ to minimize production costs, subject to a technology rule.

$$\min_{L_{\eta jt}} : \quad W_{\eta t} L_{\eta jt} \tag{0.24}$$

$$\text{s.t.} : \quad Y_{\eta jt} = Z_{A\eta t} L_{\eta jt} \tag{0.25}$$

where $Y_{\eta jt}$ is the output obtained by the technology level $Z_{A\eta t}$ that transforms labor $L_{\eta jt}$ into intermediate-goods, with constants returns to scale.[1]

Lagrangian

Transform the minimization problem with restriction into one without restriction applying the Lagrangian function $\mathscr{L}$:

$$\mathscr{L} = (W_{\eta t} L_{\eta jt}) - \Lambda_{\eta jt}(Z_{A\eta t} L_{\eta jt} - Y_{\eta jt}) \tag{0.26}$$

where the Lagrangian multiplier $\Lambda_{\eta jt}$ is the marginal cost.[2]

First Order Condition

The first-order condition is:

$$L_{\eta jt} : \quad W_{\eta t} - \Lambda_{\eta jt} Z_{A\eta t} = 0 \quad \Longrightarrow$$

$$\Lambda_{\eta jt} = \frac{W_{\eta t}}{Z_{A\eta t}} \tag{0.27}$$

$$\Lambda_{\eta jt} : \quad Y_{\eta jt} = Z_{A\eta t} L_{\eta jt} \tag{0.25}$$

---

1 the production technology level $Z_{A\eta t}$ will be submitted to a productivity shock, detailed in section 0.1.5.

2 see Lemma **??**

As salaries and technology are the same for all firms in region η, the j index can be dropped from the marginal cost Λ:

$$\Lambda_{\eta t} = \frac{W_{\eta t}}{Z_{A\eta t}} \tag{0.28}$$

Total and Marginal Costs

Notice that:

$$\begin{aligned} TC_{\eta jt} &= W_{\eta t} L_{\eta jt} = \Lambda_{\eta t} Y_{\eta jt} \\ MC_{\eta jt} &= \frac{\partial TC_{\eta jt}}{\partial Y_{\eta jt}} = \Lambda_{\eta t} \end{aligned} \tag{0.29}$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (**calvo_staggered_1983**): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{\eta j,t+1} = P_{\eta jt})$, and a probability $(1 - \theta)$ of setting a new optimal price $P^*_{\eta jt}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price:

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \tag{0.30}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Y_{\eta t} \left(\frac{P_{\eta t}}{P_{\eta jt}} \right)^{\psi} \tag{0.22}$$

where s is the period in time when the decision must be made; t is the last period in time when the price was updated and k is the period in the future when the interest rate applies.

Substitute 0.29 in 0.30:

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - \Lambda_{\eta,t+s} Y_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \tag{0.31}$$

Substitute 0.22 in 0.31 and rearrange the variables:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} - \Lambda_{\eta,t+s} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt}^{1-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to $P_{\eta jt}$ is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - (-\psi) P_{\eta jt}^{-\psi-1} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi-1) \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} Y_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \\ = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}}\right)^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \end{aligned} \tag{0.32}$$

Substitute 0.22 in 0.32:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi-1) Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$(\psi-1)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \psi P_{\eta jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt}\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \Longrightarrow$$

$$P_{\eta jt}^{*} = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}} \tag{0.33}$$

Equation 0.33 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P^*_{\eta t} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1 + R_{t+k}) \right\}} \tag{0.34}$$

Final-Goods Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1-\theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 0.23 to determine the aggregate price level:

$$P_{\eta t} = \left[\int_0^\theta P^{1-\psi}_{\eta,t-1} \, \mathrm{d}\, j + \int_\theta^1 P^{*1-\psi}_{\eta t} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies$$

$$P_{\eta t} = \left[\theta P^{1-\psi}_{\eta,t-1} + (1-\theta) P^{*1-\psi}_{\eta t} \right]^{\frac{1}{1-\psi}} \tag{0.35}$$

Equation 0.35 is the aggregate price level.

0.1.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (**taylor_discretion_1993**) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{0.36}$$

where R, π, Y are the nominal interest rate, gross inflation rate and the production level in steady state, respectively; γ_R is the smoothing parameter for the interest rate R_{Kt}, γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively, Z_{Mt} is the monetary shock and π_t is the gross inflation rate, defined by:[3]

$$\pi_t = \pi_{1t}^{\theta_\pi} \pi_{2t}^{1-\theta_\pi} \tag{0.37}$$

$$\text{where: } \theta_\pi = \frac{P_{1t}Y_{1t}}{P_{1t}Y_{1t} + P_{2t}Y_{2t}} \tag{0.38}$$

Regional Inflation

There is one price level $P_{\eta t}$ in each region, generating a regional inflation rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{0.39}$$

0.1.5 Stochastic Shocks

Productivity Shock

The production technology level $Z_{A\eta t}$ will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{0.40}$$

where $\rho_{A\eta} \in [0, 1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{0.41}$$

where $\rho_M \in [0, 1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

---

[3] for the monetary shock definition, see section 0.1.5.

0.1.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P^*_{\eta t}, R^*_t, R^*_{Kt}, W^*_{\eta t}\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C^*_{\eta 1t}, C^*_{\eta 2t}, L^*_{\eta t}, B^*_{\eta t}, K^*_{\eta,t+1}\}$ and allocations for firms $\boldsymbol{\mathcal{A}}_F := \{K^*_{\eta jt}, L^*_{\eta jt}, Y^*_{\eta jt}, Y^*_{\eta t}\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms' problems, and the markets for goods, labor and bonds clear:

$$Y_t = Y_{1t} + Y_{2t} \tag{0.42}$$

$$\text{where:} \quad P_{\eta t} Y_{\eta t} = Q_{\eta t} C_{\eta t} + B_{\eta t} \tag{0.43}$$

$$L_{\eta t} = \int_0^1 L_{\eta jt} \, \mathrm{d}\, j \tag{0.44}$$

$$B_{1t} + B_{2t} = 0 \tag{0.45}$$

Intermediate-Goods Firm Profit

For the sake of closure, the intermediate-goods firm profit must be defined:

$$\Pi_{\eta t} = \int_0^1 \Pi_{\eta jt} \, \mathrm{d}\, j \tag{0.46}$$

$$\Pi_{\eta jt} = P_{\eta t} Y_{\eta jt} - W_{\eta t} L_{\eta jt} \tag{0.47}$$

Substitute 0.47 and 0.44 in 0.46:

$$\Pi_{\eta t} = P_{\eta t} \int_0^1 Y_{\eta jt} \, \mathrm{d}\, j - W_{\eta t} L_{\eta t} \tag{0.48}$$

Substitute 0.48 in 0.11:

$$Q_{\eta t} C_{\eta t} + B_{\eta t} = W_{\eta t} L_{\eta t} + (1 + R_{t-1}) B_{\eta, t-1} + P_{\eta t} \int_0^1 Y_{\eta jt} \, \mathrm{d}\, j - W_{\eta t} L_{\eta t} \implies$$

$$Q_{\eta t} C_{\eta t} + B_{\eta t} = (1 + R_{t-1}) B_{\eta, t-1} + P_{\eta t} \int_0^1 Y_{\eta jt} \, \mathrm{d}\, j \tag{0.49}$$

0.1.7 Model Structure

The model is composed of the preview solutions, forming a square system of 34 variables and equations, summarized as follows:

- Variables:
  - from the household problem: $\langle C_\eta \quad L_\eta \quad B_\eta \quad C_{\eta 1} \quad C_{\eta 2} \quad Q_\eta \rangle$;
  - from the final-goods firm problem: $\langle Y_{\eta j} \quad Y_\eta \quad P_\eta \rangle$;
  - from the intermediate-goods firm problems: $\langle L_{\eta j} \quad P^*_\eta \rangle$;
  - from the monetary policy: $\langle R \quad \pi \quad Y \rangle$;
  - prices: $\langle W_\eta \quad \Lambda_\eta \quad \pi_\eta \rangle$;
  - shocks: $\langle Z_{A\eta} \quad Z_M \rangle$.
- Equations:
  1. Regional Consumption Weight:
$$C_{\eta 2t} = C_{\eta 1t} \frac{(1-\omega_{\eta 1})P_{1t}}{\omega_{\eta 1} P_{2t}} \tag{0.6}$$
  2. Regional Consumption of Good 1:
$$C_{\eta 1t} = C_{\eta t} \left(\frac{P_{2t}\omega_{\eta 1}}{P_{1t}(1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \tag{0.7}$$
  3. Region 1 Price Index:
$$Q_{1t} = \left(\frac{P_{1t}}{\omega_{11}} \right)^{\omega_{11}} \left(\frac{P_{2t}}{1-\omega_{11}} \right)^{1-\omega_{11}} \tag{0.8}$$
  4. Regional Terms of Trade:
$$\frac{Q_{1t}}{Q_{2t}} = \frac{\omega_{21}^{\omega_{21}}(1-\omega_{21})^{1-\omega_{21}}}{\omega_{11}^{\omega_{11}}(1-\omega_{11})^{1-\omega_{11}}} \tag{0.9}$$
  5. Labor Supply:
$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_{\eta t}}{Q_{\eta t}} \tag{0.16}$$
  6. Region 1 Euler equation for the bonds return:
$$\frac{\mathbb{E}_t \left\{ Q_{1,t+1} C_{1,t+1}^{\sigma} \right\}}{Q_{1t} C_{1t}^{\sigma}} = \beta(1+R_t) \tag{0.17}$$

7. Euler equation for regional consumption:

$$\frac{\mathbb{E}_t\left\{Q_{1,t+1}C_{1,t+1}^{\sigma}\right\}}{Q_{1t}C_{1t}^{\sigma}} = \frac{\mathbb{E}_t\left\{Q_{2,t+1}C_{2,t+1}^{\sigma}\right\}}{Q_{2t}C_{2t}^{\sigma}} \tag{0.18}$$

8. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \tag{0.20}$$

9. Production Function:

$$Y_{\eta jt} = Z_{A\eta t}L_{\eta jt} \tag{0.25}$$

10. Marginal Cost:

$$\Lambda_{\eta t} = \frac{W_{\eta t}}{Z_{A\eta t}} \tag{0.28}$$

11. Optimal Price:

$$P_{\eta t}^{*} = \frac{\psi}{\psi-1}\cdot\frac{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j,t+s}\Lambda_{\eta,t+s}/\prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j,t+s}/\prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \tag{0.34}$$

12. Regional Price Level:

$$P_{\eta t} = \left[\theta P_{\eta,t-1}^{1-\psi} + (1-\theta)P_{\eta t}^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{0.35}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi}\left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \tag{0.36}$$

14. National Gross Inflation Rate:

$$\pi_t = \pi_{1t}^{\theta_\pi}\pi_{2t}^{1-\theta_\pi} \tag{0.37}$$

15. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{0.39}$$

16. Productivity Shock:

$$\ln Z_{A\eta t} = (1-\rho_{A\eta})\ln Z_{A\eta} + \rho_{A\eta}\ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{0.40}$$

17. Monetary Shock:

$$\ln Z_{Mt} = (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{0.41}$$

18. Goods-Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \tag{0.42}$$

19. Regional Goods-Market Clearing Condition:

$$P_{\eta t} Y_{\eta t} = Q_{\eta t} C_{\eta t} + B_{\eta t} \tag{0.43}$$

20. Regional Labor-Market Clearing Condition:

$$L_{\eta t} = \int_0^1 L_{\eta j t}\, \mathrm{d}\, j \tag{0.44}$$

21. Budget Constraint:

$$Q_{\eta t} C_{\eta t} + B_{\eta t} = (1 + R_{t-1}) B_{\eta, t-1} + P_{\eta t} \int_0^1 Y_{\eta j t}\, \mathrm{d}\, j \tag{0.49}$$

0.2 Steady State

The steady state of a variable is defined by its constancy through time. For any given variable X_t, it is in steady state if $t \to \infty \implies \mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (**costa_junior_understanding_2016**). For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The model in steady state is:

1. Regional Consumption Weight:

$$C_{\eta 2} = C_{\eta 1} \frac{(1-\omega_{\eta 1})P_1}{\omega_{\eta 1} P_2} \tag{0.50}$$

2. Regional Consumption of Good 1:

$$C_{\eta 1} = C_{\eta} \left(\frac{P_2 \omega_{\eta 1}}{P_1(1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \tag{0.51}$$

3. Region 1 Price Index:

$$Q_1 = \left(\frac{P_1}{\omega_{11}} \right)^{\omega_{11}} \left(\frac{P_2}{1-\omega_{11}} \right)^{1-\omega_{11}} \tag{0.52}$$

4. Regional Terms of Trade:

$$\frac{Q_1}{Q_2} = \frac{\omega_{21}^{\omega_{21}}(1-\omega_{21})^{1-\omega_{21}}}{\omega_{11}^{\omega_{11}}(1-\omega_{11})^{1-\omega_{11}}} \tag{0.53}$$

5. Labor Supply:

$$\frac{\phi L_{\eta}^{\varphi}}{C_{\eta}^{-\sigma}} = \frac{W_{\eta}}{Q_{\eta}} \tag{0.54}$$

6. Region 1 Euler equation for the bonds return:

$$\frac{\mathbb{E}_t \left\{ Q_1 C_1^{\sigma} \right\}}{Q_1 C_1^{\sigma}} = \beta(1+R) \implies$$

$$1 = \beta(1+R) \tag{0.55}$$

7. Euler equation for regional consumption:

$$\frac{\mathbb{E}_t\left\{Q_1 C_1^{\sigma}\right\}}{Q_1 C_1^{\sigma}} = \frac{\mathbb{E}_t\left\{Q_2 C_2^{\sigma}\right\}}{Q_2 C_2^{\sigma}} = 1 \tag{0.56}$$

8. Bundle Technology:

$$Y_{\eta} = \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \tag{0.57}$$

9. Production Function:

$$Y_{\eta j} = Z_{A\eta} L_{\eta j} \tag{0.58}$$

10. Marginal Cost:

$$\Lambda_{\eta} = \frac{W_{\eta}}{Z_{A\eta}} \tag{0.59}$$

11. Optimal Price:

$$\begin{aligned} P_{\eta}^{*} &= \frac{\psi}{\psi-1}\cdot\frac{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j}\Lambda_{\eta}\,/\,\prod_{k=0}^{s-1}(1+R)\right\}}{\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\theta^s Y_{\eta j}\,/\,\prod_{k=0}^{s-1}(1+R)\right\}} \implies \\ P_{\eta}^{*} &= \frac{\psi}{\psi-1}\Lambda_{\eta} \end{aligned} \tag{0.60}$$

12. Regional Price Level:

$$\begin{aligned} P_{\eta} &= \left[\theta P_{\eta}^{1-\psi} + (1-\theta)P_{\eta}^{*1-\psi}\right]^{\frac{1}{1-\psi}} \implies \\ P_{\eta} &= P_{\eta}^{*} \end{aligned} \tag{0.61}$$

13. Monetary Policy:

$$\begin{aligned} \frac{R}{R} &= \left(\frac{R}{R}\right)^{\gamma_R}\left[\left(\frac{\pi}{\pi}\right)^{\gamma_{\pi}}\left(\frac{Y}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_M \implies \\ Z_M &= 1 \end{aligned} \tag{0.62}$$

14. National Gross Inflation Rate:

$$\pi = \pi_1^{\theta_\pi} \pi_2^{1-\theta_\pi} \tag{0.63}$$

15. Regional Gross Inflation Rate:

$$\pi_\eta = \frac{P_\eta}{P_\eta} = 1 \tag{0.64}$$

16. Productivity Shock:

$$\begin{aligned} \ln Z_{A\eta} &= (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta} + \varepsilon_{A\eta} \implies \\ \varepsilon_{A\eta} &= 0 \end{aligned} \tag{0.65}$$

17. Monetary Shock:

$$\begin{aligned} \ln Z_M &= (1 - \rho_M) \ln Z_M + \rho_M \ln Z_M + \varepsilon_M \implies \\ \varepsilon_M &= 0 \end{aligned} \tag{0.66}$$

18. Goods-Market Clearing Condition:

$$Y = Y_1 + Y_2 \tag{0.67}$$

19. Regional Goods-Market Clearing Condition:

$$P_\eta Y_\eta = Q_\eta C_\eta + B_\eta \tag{0.68}$$

20. Regional Labor-Market Clearing Condition:

$$L_\eta = \int_0^1 L_{\eta j} \, \mathrm{d}\, j \tag{0.69}$$

21. Budget Constraint:

$$Q_\eta C_\eta = RB_\eta + P_\eta \int_0^1 Y_{\eta j} \, \mathrm{d}\, j \tag{0.70}$$

0.2.1 Variables at Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It is assumed that one regional price level and one productivity level are equal to one. The other productivity level is a multiple of the first:[4]

$$\langle P_1 \quad Z_{A1} \rangle = \vec{\mathbf{1}} \tag{0.71}$$

$$\langle P_2 \quad Z_{A2} \rangle = \langle \theta_P P_1 \quad \theta_Z Z_{A1} \rangle \tag{0.72}$$

From 0.62, 0.63 and 0.64, the monetary shock, the national and regional gross inflation rates are:

$$\langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle = \vec{\mathbf{1}} \tag{0.73}$$

From 0.65 and 0.66, the productivity and monetary shocks are:

$$\langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle = \vec{\mathbf{0}} \tag{0.74}$$

From 0.55, the return on bonds is:

$$1 = \beta(1 + R) \implies \tag{0.55}$$

$$R = \frac{1}{\beta} - 1 \tag{0.75}$$

From 0.61 and 0.71, the regional optimal price P_η^* is:

$$P_\eta^* = P_\eta \implies \langle P_1^* \quad P_2^* \rangle = \langle P_1 \quad P_2 \rangle = \langle P_1 \quad \theta_P P_1 \rangle \tag{0.76}$$

Substitute 0.71 in 0.52 for the price composition of consumption bundle Q_η:

$$Q_1 = \left(\frac{P_1}{\omega_{11}}\right)^{\omega_{11}} \left(\frac{P_2}{1 - \omega_{11}}\right)^{1-\omega_{11}} \implies \tag{0.52}$$

$$Q_1 = \frac{P_1 \theta_P^{1-\omega_{11}}}{\omega_{11}^{\omega_{11}} (1 - \omega_{11})^{1-\omega_{11}}} \tag{0.77}$$

---

[4] where $\vec{\mathbf{1}}$ is the unit vector.

Substitute 0.77 in 0.53:

$$Q_2 = Q_1 \frac{\omega_{11}^{\omega_{11}}(1-\omega_{11})^{1-\omega_{11}}}{\omega_{21}^{\omega_{21}}(1-\omega_{21})^{1-\omega_{21}}} \implies \tag{0.53}$$

$$Q_2 = \frac{P_1 \theta_P^{1-\omega_{11}}}{\omega_{21}^{\omega_{21}}(1-\omega_{21})^{1-\omega_{21}}} \tag{0.78}$$

Substitute 0.76 in 0.60 for the marginal cost Λ_η:

$$P_\eta^* = \frac{\psi}{\psi - 1}\Lambda_\eta \implies \tag{0.60}$$

$$\Lambda_\eta = P_\eta \frac{\psi - 1}{\psi} \tag{0.79}$$

From 0.59, the nominal wage W_η is:

$$\Lambda_\eta = \frac{W_\eta}{Z_{A\eta}} \tag{0.59}$$

$$W_\eta = \Lambda_\eta Z_{A\eta} \tag{0.80}$$

Due to price parity in steady state, where prices are identical ($P_\eta = P_\eta^*$) and resulting in a gross inflation level of one ($\pi_\eta = 1$), all firms produce the same output level ($\forall i, j \in [0,1], Y_{\eta j} = Y_{\eta i}, i \neq j$) (**solis-garcia_ucb_2022**). As a consequence, they uniformly demand the same amount of factors ($\forall j \in [0,1], L_{\eta j} = L_{\eta i}, j \neq i$), and 0.57, 0.69, 0.58 and 0.70 become:

$$Y_\eta = Y_{\eta j} \tag{0.81}$$

$$L_\eta = L_{\eta j} \tag{0.82}$$

$$Y_\eta = Z_{A\eta} L_\eta \tag{0.83}$$

$$B_\eta = 0 \tag{0.84}$$

Isolate C_η in 0.54 and then substitute 0.77 and 0.71:

$$\frac{\phi L_\eta^\varphi}{C_\eta^{-\sigma}} = \frac{W_\eta}{Q_\eta} \implies C_\eta^\sigma = \frac{W_\eta}{\phi L_\eta^\varphi Q_\eta} \implies$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{0.85}$$

$$\text{where:} \quad a_\eta = \left[\frac{W_\eta Z_{A\eta}^\varphi}{\phi Q_\eta}\right]^{\frac{1}{\sigma}} \tag{0.86}$$

Substitute 0.85 and 0.84 in 0.68:

$$P_\eta Y_\eta = Q_\eta C_\eta + B_\eta \implies \tag{0.68}$$

$$P_\eta Y_\eta = Q_\eta a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \implies$$

$$Y_\eta = \left(\frac{Q_\eta a_\eta}{P_\eta}\right)^{\frac{\sigma}{\varphi+\sigma}} \tag{0.87}$$

The result of 0.87 determines $Y, C_\eta, C_{\eta 1}, C_{\eta 2}, L_\eta$ in 0.67, 0.85, 0.51, 0.50, 0.83, respectively.

0.2.2 Steady State Solution

$$\vec{\mathbf{1}} = \langle P_1 \quad Z_{A1} \rangle \tag{0.71}$$

$$\langle P_2 \quad Z_{A2} \rangle = \langle \theta_P P_1 \quad \theta_Z Z_{A1} \rangle \tag{0.72}$$

$$\vec{\mathbf{1}} = \langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle \tag{0.73}$$

$$\vec{\mathbf{0}} = \langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle \tag{0.74}$$

$$R = \frac{1}{\beta} - 1 \tag{0.75}$$

$$P^*_\eta = P_\eta \tag{0.76}$$

$$Q_1 = \frac{P_1 \theta_P^{1-\omega_{11}}}{\omega_{11}^{\omega_{11}} (1-\omega_{11})^{1-\omega_{11}}} \tag{0.77}$$

$$Q_2 = \frac{P_1 \theta_P^{1-\omega_{11}}}{\omega_{21}^{\omega_{21}} (1-\omega_{21})^{1-\omega_{21}}} \tag{0.78}$$

$$\Lambda_\eta = P_\eta \frac{\psi - 1}{\psi} \tag{0.79}$$

$$W_\eta = \Lambda_\eta Z_{A\eta} \tag{0.80}$$

$$B_\eta = 0 \tag{0.84}$$

$$a_\eta = \left[\frac{W_\eta Z_{A\eta}^{\varphi}}{\phi Q_\eta} \right]^{\frac{1}{\sigma}} \tag{0.86}$$

$$Y_\eta = \left(\frac{Q_\eta a_\eta}{P_\eta} \right)^{\frac{\sigma}{\varphi+\sigma}} \tag{0.87}$$

$$Y = Y_1 + Y_2 \tag{0.67}$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{0.85}$$

$$C_{\eta 1} = C_\eta \left(\frac{P_2 \omega_{\eta 1}}{P_1 (1 - \omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \tag{0.51}$$

$$C_{\eta 2} = C_{\eta 1} \frac{(1-\omega_{\eta 1}) P_1}{\omega_{\eta 1} P_2} \tag{0.50}$$

$$L_\eta = \frac{Y_\eta}{Z_{A\eta}} \tag{0.83}$$

0.3 Log-linearization

Due to the number of variables and equations that need to be solved, computational brute force will be necessary. `Dynare` is specialized software for macroeconomic modeling, commonly used for solving DSGE models. Before the model can be processed by the software, it must undergo linearization to eliminate the infinite sum in Equation 0.34. For this purpose, Uhlig's rules of log-linearization (**uhlig_toolkit_1999**) will be applied to all equations in the model. For any given variable X_t, its deviation will be represented with a hat, $\hat{X}_t$.[5]

Regional Gross Inflation Rate

Log-linearize 0.39 and define the level deviation of regional inflation rate $\hat{\pi}_{\eta t}$:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{0.39}$$

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{0.88}$$

Regional Price Level

Log-linearize equation 0.35:

$$P_{\eta t}^{1-\psi} = \theta P_{\eta,t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi} \implies \tag{0.35}$$

$$\begin{aligned} P^{1-\psi}(1 + (1-\psi)\hat{P}_{\eta t}) &= \theta P^{1-\psi}(1 + (1-\psi)\hat{P}_{\eta,t-1}) + \\ &\quad + (1-\theta)P^{1-\psi}(1 + (1-\psi)\hat{P}^*_{\eta t}) \implies \end{aligned}$$

$$\hat{P}_{\eta t} = \theta \hat{P}_{\eta,t-1} + (1-\theta)\hat{P}^*_{\eta t} \tag{0.89}$$

5 see lemma **??** for details.

New Keynesian Phillips Curve

In order to log-linearize equation 0.34, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma **??**:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{0.34}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\qquad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{0.90}$$

First, log-linearize the left hand side of equation 0.90 with respect to $P_{\eta t}^*, Y_{\eta j t}, \widetilde{R}_t$:

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_{\eta t}^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies \\ &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P_\eta^* Y_{\eta j} \left(1 + \hat{P}_{\eta t}^* + \hat{Y}_{\eta j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies \\ &P_\eta^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}_{\eta t}^* + \hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \end{aligned}$$

Separate the terms not dependent on s:

$$\begin{aligned} &P_\eta^* Y_{\eta j} (1 + \hat{P}_{\eta t}^*) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \right\} + \\ &\qquad + P_\eta^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \implies \end{aligned}$$

Apply definition **??** on the first term:

$$\frac{P_\eta^* Y_{\eta j} (1 + \hat{P}_{\eta t}^*)}{1 - \theta/(1+R)} + P_\eta^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\}$$

Second, log-linearize the left hand side of 0.90 with respect to $\Lambda^*_{\eta t}, Y_{\eta jt}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\frac{\theta^s Y_{\eta j,t+s}\Lambda_{\eta,t+s}}{(1+R)^s\left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)}\right\} \implies$$

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\frac{Y_{\eta j}\Lambda_\eta(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s})}{1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right\} \implies$$

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\right\}+$$
$$+\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Apply definition **??** on the first term:

$$\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+$$
$$+\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Join both sides of the equation again:

$$\frac{P^*_\eta Y_{\eta j}(1+\hat{P}^*_{\eta t})}{1-\theta/(1+R)}+P^*_\eta Y_{\eta j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}=$$
$$=\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+$$
$$+\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} \quad (0.91)$$

Substitute 0.55 in 0.91:

$$\beta = \frac{1}{(1+R)} \tag{0.55}$$

$$\frac{P_\eta^* Y_{\eta j}(1+\hat{P}_{\eta t}^*)}{1-\theta\beta} + P_\eta^* Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} =$$
$$= \frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta\beta} + \tag{0.92}$$
$$+ \frac{\psi}{\psi-1} Y_{\eta j}\Lambda_\eta \mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{Y}_{\eta j,t+s} + \hat{\Lambda}_{\eta,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Substitute 0.79 in 0.92 and simplify all common terms:

$$\cancel{\frac{P_\eta^* Y_{\eta j}}{1-\theta\beta}} + \frac{\cancel{P_\eta^* Y_{\eta j}}\hat{P}_{\eta t}^*}{1-\theta\beta} + \cancel{P_\eta^* Y_{\eta j}}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\cancel{\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right)\right\} =$$
$$= \cancel{\frac{P_\eta^* Y_{\eta j}}{1-\theta\beta}} + \cancel{P_\eta^* Y_{\eta j}}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\cancel{\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}} + \hat{\Lambda}_{\eta,t+s}\right)\right\} \implies$$
$$\frac{\hat{P}_{\eta t}^*}{1-\theta\beta} = \mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{\Lambda}_{\eta,t+s}\right)\right\} \tag{0.93}$$

Define the real marginal cost $\lambda_{\eta t}$ and log-linearize it:

$$\lambda_{\eta t} = \frac{\Lambda_{\eta t}}{P_{\eta t}} \implies \Lambda_{\eta t} = P_{\eta t}\lambda_{\eta t} \implies \tag{0.94}$$
$$\hat{\Lambda}_{\eta t} = \hat{P}_{\eta t} + \hat{\lambda}_{\eta t} \tag{0.95}$$

Substitute 0.95 in 0.93:

$$\hat{P}_{\eta t}^* = (1-\theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}\right) \tag{0.96}$$

Substitute 0.96 in 0.89:

$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)\hat{P}_{\eta t}^* \tag{0.89}$$
$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)(1-\theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}\right) \tag{0.97}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\beta\mathbb{L}^{-1})$ in 0.97:[6]

$$
\begin{aligned}
(1-\theta\beta\mathbb{L}^{-1})\hat{P}_{\eta t} = (1-\theta\beta\mathbb{L}^{-1})\Big[\theta\hat{P}_{\eta,t-1} + \\
+(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right)\Big] \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + \\
(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) - \\
-\theta\beta(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)
\end{aligned}
\tag{0.98}
$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\beta$ within the operator. Then, cancel the summations and rearrange the terms:

$$
\begin{aligned}
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + \\
(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) - \\
-\theta\beta(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right) \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta\beta\theta\hat{P}_{\eta t} + (1-\theta)(1-\theta\beta)(\hat{P}_{\eta t}+\hat{\lambda}_{\eta t}) + \\
+(1-\theta)(1-\theta\beta)\cancel{\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)} - \\
-(1-\theta)(1-\theta\beta)\cancel{\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)} \implies \\
\hat{P}_{\eta t} - \theta\beta\mathbb{E}_t\hat{P}_{t+1} = \theta\hat{P}_{\eta,t-1} - \theta^2\beta\hat{P}_{\eta t} + \\
+(1-\theta-\theta\beta+\theta^2\beta)\hat{P}_{\eta t} + (1-\theta)(1-\theta\beta)\hat{\lambda}_{\eta t} \implies \\
(\hat{P}_{\eta t} - \hat{P}_{\eta,t-1}) = \beta(\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_{\eta t}) + \frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{\eta t}
\end{aligned}
\tag{0.99}
$$

[6] see definition **??**.

Substitute 0.88 in 0.99:

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{\eta t} \tag{0.100}$$

Equation 0.100 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

Regional Consumption Weight

Log-linearize 0.6:

$$C_{\eta 2t} = C_{\eta 1t} \frac{(1-\omega_{\eta 1}) P_{1t}}{\omega_{\eta 1} P_{2t}} \implies \tag{0.6}$$

$$\hat{C}_{\eta 2t} - \hat{C}_{\eta 1t} = \hat{P}_{1t} - \hat{P}_{2t} \tag{0.101}$$

Regional Consumption of Good 1

Log-linearize 0.7:

$$C_{\eta 1t} = C_{\eta t} \left(\frac{P_{2t} \omega_{\eta 1}}{P_{1t}(1-\omega_{\eta 1})} \right)^{1-\omega_{\eta 1}} \implies \tag{0.7}$$

$$\hat{C}_{\eta t} - \hat{C}_{\eta 1t} = (1-\omega_{\eta 1})(\hat{P}_{1t} - \hat{P}_{2t}) \tag{0.102}$$

Region 1 Price Index

Log-linearize 0.8:

$$Q_{1t} = \left(\frac{P_{1t}}{\omega_{11}} \right)^{\omega_{11}} \left(\frac{P_{2t}}{1-\omega_{11}} \right)^{1-\omega_{11}} \implies \tag{0.8}$$

$$\hat{Q}_{1t} = \omega_{11} \hat{P}_{1t} + (1-\omega_{11}) \hat{P}_{2t} \tag{0.103}$$

Regional Terms of Trade

Log-linearize 0.9:

$$\hat{Q}_{1t} = \hat{Q}_{2t} \tag{0.104}$$

Labor Supply

Log-linearize 0.16 and then substitute 0.104:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_{\eta t}}{Q_{\eta t}} \qquad \Longrightarrow \tag{0.16}$$

$$\varphi \hat{L}_{\eta t} + \sigma \hat{C}_{\eta t} = \hat{W}_{\eta t} - \hat{Q}_{1t} \tag{0.105}$$

Region 1 Euler equation for the bonds return

Log-linearize 0.17:

$$\frac{\mathbb{E}_t \left\{ Q_{1,t+1} C_{1,t+1}^{\sigma} \right\}}{Q_{1t} C_{1t}^{\sigma}} = \beta(1 + R_t) \implies \tag{0.17}$$

$$\hat{Q}_{1,t+1} - \hat{Q}_{1t} + \sigma(\hat{C}_{1,t+1} - \hat{C}_{1t}) = (1 - \beta)\hat{R}_t \tag{0.106}$$

Euler equation for regional consumption

Log-linearize 0.17 and then substitute 0.104:

$$\frac{\mathbb{E}_t \left\{ Q_{1,t+1} C_{1,t+1}^{\sigma} \right\}}{Q_{1t} C_{1t}^{\sigma}} = \frac{\mathbb{E}_t \left\{ Q_{2,t+1} C_{2,t+1}^{\sigma} \right\}}{Q_{2t} C_{2t}^{\sigma}} \implies \tag{0.18}$$

$$\hat{Q}_{1,t+1} - \hat{Q}_{1t} + \sigma(\hat{C}_{1,t+1} - \hat{C}_{1t}) =$$

$$= \hat{Q}_{2,t+1} - \hat{Q}_{2t} + \sigma(\hat{C}_{2,t+1} - \hat{C}_{2t}) \implies$$

$$\hat{C}_{1,t+1} - \hat{C}_{1t} = \hat{C}_{2,t+1} - \hat{C}_{2t} \tag{0.107}$$

Bundle Technology

Apply the natural logarithm to 0.20:

$$
\begin{aligned}
Y_{\eta t} &= \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \,\mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \implies \qquad (0.20) \\
\ln Y_{\eta t} &= \frac{\psi}{\psi-1} \ln \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \,\mathrm{d}\, j \right)
\end{aligned}
$$

Log-linearize using corollary **??**:

$$
\begin{aligned}
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \frac{\psi}{\psi-1} \left[\ln \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}} \,\mathrm{d}\, j \right) + \frac{\psi-1}{\psi} \int_0^1 \hat{Y}_{\eta j t} \,\mathrm{d}\, j \right] \implies \\
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \frac{\psi}{\psi-1} \left[\ln \left(Y_{\eta j}^{\frac{\psi-1}{\psi}} \int_0^1 \mathrm{d}\, j \right) + \frac{\psi-1}{\psi} \int_0^1 \hat{Y}_{\eta j t} \,\mathrm{d}\, j \right] \implies \\
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \cancel{\frac{\psi}{\psi-1}} \left[\cancel{\frac{\psi-1}{\psi}} \ln Y_{\eta j} + \cancel{\ln 1} + \cancel{\frac{\psi-1}{\psi}} \int_0^1 \hat{Y}_{\eta j t} \,\mathrm{d}\, j \right] \implies \\
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta j t} \,\mathrm{d}\, j
\end{aligned}
$$

Apply corollary **??**:

$$
\begin{aligned}
\ln Y_{\eta} + \hat{Y}_{\eta t} &= \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta j t} \,\mathrm{d}\, j \implies \\
\hat{Y}_{\eta t} &= \int_0^1 \hat{Y}_{\eta j t} \,\mathrm{d}\, j \qquad (0.108)
\end{aligned}
$$

Production Function

Log-linearize 0.25:

$$
\begin{aligned}
Y_{\eta j t} &= Z_{A\eta t} L_{\eta j t} \implies \qquad (0.25) \\
\hat{Y}_{\eta j t} &= \hat{Z}_{A\eta t} + \hat{L}_{\eta j t} \qquad (0.109)
\end{aligned}
$$

Substitute 0.109 in 0.108:

$$\hat{Y}_{\eta t} = \int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j \qquad \Longrightarrow \tag{0.108}$$

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j \tag{0.110}$$

Apply the natural logarithm and then log-linearize 0.44:

$$\begin{aligned} L_{\eta t} &= \int_0^1 L_{\eta j t}\,\mathrm{d}\,j && \Longrightarrow \\ \ln L_{\eta t} &= \ln\left[\int_0^1 L_{\eta j t}\,\mathrm{d}\,j\right] && \Longrightarrow \\ \ln L + \hat{L}_{\eta t} &= \ln\left[\int_0^1 L_{\eta j}\,\mathrm{d}\,j\right] + \int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j && \Longrightarrow \\ \ln L + \hat{L}_{\eta t} &= \ln L_{\eta j} + \ln 1 + \int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j \end{aligned} \tag{0.44}$$

Apply corollary **??**:

$$\Longrightarrow \hat{L}_{\eta t} = \int_0^1 \hat{L}_{\eta j t}\,\mathrm{d}\,j \tag{0.111}$$

Substitute 0.111 in 0.110:

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \hat{L}_{\eta t} \tag{0.112}$$

Marginal Cost

Log-linearize 0.28 and then substitute 0.95:

$$\begin{aligned} \Lambda_{\eta t} &= \frac{W_{\eta t}}{Z_{A\eta t}} \Longrightarrow \\ \hat{\Lambda}_{\eta t} &= \hat{W}_{\eta t} - \hat{Z}_{A\eta t} \Longrightarrow \end{aligned} \tag{0.28}$$

$$\hat{P}_{\eta t} + \hat{\lambda}_{\eta t} = \hat{W}_{\eta t} - \hat{Z}_{A\eta t} \tag{0.113}$$

Monetary Policy

Log-linearize 0.36:

$$\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi} Y_t^{\gamma_Y})^{(1-\gamma_R)} Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \implies \tag{0.36}$$

$$\frac{R(1+\hat{R}_t)}{R} = \frac{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)} Z_M}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \cdot$$

$$\cdot [1 + \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt}] \implies$$

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{0.114}$$

National Gross Inflation Rate

Log-linearize 0.37:

$$\pi_t = \pi_{1t}^{\theta_\pi} \pi_{2t}^{1-\theta_\pi} \implies \tag{0.37}$$

$$\hat{\pi}_t = \theta_\pi \hat{\pi}_{1t} + (1-\theta_\pi)\hat{\pi}_{2t} \tag{0.115}$$

Productivity Shock

Log-linearize 0.40:

$$\ln Z_{A\eta t} = (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \implies \tag{0.40}$$

$$\ln Z_{A\eta} + \hat{Z}_{A\eta t} = (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta}(\ln Z_{A\eta} + \hat{Z}_{A\eta,t-1}) + \varepsilon_{A\eta} \implies$$

$$\hat{Z}_{A\eta t} = \rho_{A\eta} \hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} \tag{0.116}$$

Monetary Shock

Log-linearize 0.41:

$$\ln Z_{Mt} = (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \implies \tag{0.41}$$

$$\ln Z_M + \hat{Z}_{Mt} = (1-\rho_M) \ln Z_M + \rho_M(\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M \implies$$

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{0.117}$$

Market Clearing Condition

Log-linearize 0.42:

$$Y_t = Y_{1t} + Y_{2t} \tag{0.42}$$

$$Y(1+\hat{Y}_t) = Y_1(1+\hat{Y}_{1t}) + Y_2(1+\hat{Y}_{2t}) \implies$$

$$\hat{Y}_t = \frac{Y_1}{Y}\hat{Y}_{1t} + \frac{Y_2}{Y}\hat{Y}_{2t} \tag{0.118}$$

Define the regional weights $\langle \theta_Y \quad (1-\theta_Y) \rangle$ in the production total:

$$\left\langle \theta_Y \quad (1-\theta_Y) \right\rangle := \left\langle \tfrac{Y_1}{Y} \quad \tfrac{Y_2}{Y} \right\rangle \tag{0.119}$$

Substitute 0.119 in 0.118:

$$\hat{Y}_t = \theta_Y \hat{Y}_{1t} + (1-\theta_Y)\hat{Y}_{2t} \tag{0.120}$$

Regional Market Clearing Condition

Log-linearize 0.43 and then substitute 0.104:

$$P_{\eta t} Y_{\eta t} = Q_{\eta t} C_{\eta t} + B_{\eta t} \implies \tag{0.43}$$

$$\hat{P}_{\eta t} + \hat{Y}_{\eta t} = \frac{Q_\eta C_\eta}{P_\eta Y_\eta}(\hat{Q}_{1t} + \hat{C}_{\eta t}) + \frac{B_\eta}{P_\eta Y_\eta}\hat{B}_{\eta t} \tag{0.121}$$

Define bond and consumption weights on total expenses as $\langle \theta_{B\eta} \quad (1-\theta_{B\eta}) \rangle$:

$$\left\langle \theta_{B\eta} \quad (1-\theta_{B\eta}) \right\rangle := \left\langle \tfrac{B_\eta}{P_\eta Y_\eta} \quad \tfrac{Q_\eta C_\eta}{P_\eta Y_\eta} \right\rangle \tag{0.122}$$

Substitute 0.122 in 0.121:

$$\hat{P}_{\eta t} + \hat{Y}_{\eta t} = \theta_{B\eta}(\hat{Q}_{1t} + \hat{C}_{\eta t}) + (1-\theta_{B\eta})\hat{B}_{\eta t} \tag{0.123}$$

Budget Constraint

Log-linearize 0.49 and then substitute 0.81, 0.84 and 0.104:

$$Q_{\eta t}C_{\eta t} + B_{\eta t} = (1 + R_{t-1})B_{\eta,t-1} + P_{\eta t}\int_0^1 Y_{\eta jt}\,\mathrm{d}\,j \implies \tag{0.49}$$

$$\hat{P}_{\eta t} + \hat{Y}_{\eta t} = \hat{Q}_{1t} + \hat{C}_{\eta t} \tag{0.124}$$

Substitute 0.121 in 0.123:

$$\hat{Q}_{1t} = \hat{B}_{\eta t} - \hat{C}_{\eta t} \tag{0.125}$$

Equation 0.125 demonstrates that the variation on the price level composition will affect the variation of consumption levels.

0.3.1 Log-linear Model Structure

The log-linear model is a square system of 25 variables and equations, summarized as follows:

- Variables:
  - Real Variables: $\langle \hat{C}_\eta \quad \hat{L}_\eta \quad \hat{C}_{\eta 1} \quad \hat{C}_{\eta 2} \quad \hat{Y}_\eta \quad \hat{Y} \quad \hat{Z}_{A\eta} \quad \hat{Z}_M \rangle$;
  - Nominal Variables: $\langle \hat{Q}_1 \quad \hat{P}_\eta \quad \hat{R} \quad \hat{\pi} \quad \hat{W}_\eta \quad \hat{\lambda}_\eta \quad \hat{\pi}_\eta \rangle$.
- Equations:

1. Regional Gross Inflation Rate

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{0.88}$$

2. New Keynesian Phillips Curve

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{\eta t} \tag{0.100}$$

3. Regional Consumption Weight

$$\hat{C}_{\eta 2t} - \hat{C}_{\eta 1t} = \hat{P}_{1t} - \hat{P}_{2t} \tag{0.101}$$

4. Regional Consumption of Good 1

$$\hat{C}_{\eta t} - \hat{C}_{\eta 1t} = (1-\omega_{\eta 1})(\hat{P}_{1t} - \hat{P}_{2t}) \tag{0.102}$$

5. Region 1 Price Index

$$\hat{Q}_{1t} = \omega_{11}\hat{P}_{1t} + (1-\omega_{11})\hat{P}_{2t} \tag{0.103}$$

6. Labor Supply

$$\varphi \hat{L}_{\eta t} + \sigma \hat{C}_{\eta t} = \hat{W}_{\eta t} - \hat{Q}_{1t} \tag{0.105}$$

7. Region 1 Euler equation for the bonds return

$$\hat{Q}_{1,t+1} - \hat{Q}_{1t} + \sigma(\hat{C}_{1,t+1} - \hat{C}_{1t}) = (1-\beta)\hat{R}_t \tag{0.106}$$

8. Euler equation for regional consumption

$$\hat{C}_{1,t+1} - \hat{C}_{1t} = \hat{C}_{2,t+1} - \hat{C}_{2t} \tag{0.107}$$

9. Production Function

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \hat{L}_{\eta t} \tag{0.112}$$

10. Marginal Cost

$$\hat{P}_{\eta t} + \hat{\lambda}_{\eta t} = \hat{W}_{\eta t} - \hat{Z}_{A\eta t} \tag{0.113}$$

11. Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{0.114}$$

12. National Gross Inflation Rate

$$\hat{\pi}_t = \theta_\pi \hat{\pi}_{1t} + (1-\theta_\pi)\hat{\pi}_{2t} \tag{0.115}$$

13. Productivity Shock

$$\hat{Z}_{A\eta t} = \rho_{A\eta} \hat{Z}_{A\eta,t-1} + \varepsilon_{A\eta} \tag{0.116}$$

14. Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{0.117}$$

15. Market Clearing Condition

$$\hat{Y}_t = \theta_Y \hat{Y}_{1t} + (1-\theta_Y)\hat{Y}_{2t} \tag{0.120}$$

16. Regional Market Clearing Condition

$$\hat{P}_{\eta t} + \hat{Y}_{\eta t} = \hat{Q}_{1t} + \hat{C}_{\eta t} \tag{0.121}$$

0.3.2 Extended Log-linear Structure

- Regional Gross Inflation Rate

$$\hat{\pi}_{1t} = \hat{P}_{1t} - \hat{P}_{1,t-1} \tag{0.88a}$$
$$\hat{\pi}_{2t} = \hat{P}_{2t} - \hat{P}_{2,t-1} \tag{0.88b}$$

- New Keynesian Phillips Curve

$$\hat{\pi}_{1t} = \beta \mathbb{E}_t \hat{\pi}_{1,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{1t} \tag{0.100a}$$
$$\hat{\pi}_{2t} = \beta \mathbb{E}_t \hat{\pi}_{2,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{2t} \tag{0.100b}$$

- Regional Consumption Weight

$$\hat{C}_{12t} - \hat{C}_{11t} = \hat{P}_{1t} - \hat{P}_{2t} \tag{0.101a}$$
$$\hat{C}_{22t} - \hat{C}_{21t} = \hat{P}_{1t} - \hat{P}_{2t} \tag{0.101b}$$

- Regional Consumption of Good 1

$$\hat{C}_{1t} - \hat{C}_{11t} = (1-\omega_{11})(\hat{P}_{1t} - \hat{P}_{2t}) \tag{0.102a}$$
$$\hat{C}_{2t} - \hat{C}_{21t} = (1-\omega_{21})(\hat{P}_{1t} - \hat{P}_{2t}) \tag{0.102b}$$

- Region 1 Price Index

$$\hat{Q}_{1t} = \omega_{11}\hat{P}_{1t} + (1-\omega_{11})\hat{P}_{2t} \tag{0.103}$$

- Labor Supply

$$\varphi \hat{L}_{1t} + \sigma \hat{C}_{1t} = \hat{W}_{1t} - \hat{Q}_{1t} \tag{0.105a}$$
$$\varphi \hat{L}_{2t} + \sigma \hat{C}_{2t} = \hat{W}_{2t} - \hat{Q}_{1t} \tag{0.105b}$$

- Region 1 Euler equation for the bonds return

$$\hat{Q}_{1,t+1} - \hat{Q}_{1t} + \sigma(\hat{C}_{1,t+1} - \hat{C}_{1t}) = (1-\beta)\hat{R}_t \tag{0.106}$$

- Euler equation for regional consumption

$$\hat{C}_{1,t+1} - \hat{C}_{1t} = \hat{C}_{2,t+1} - \hat{C}_{2t} \tag{0.107}$$

- Production Function

$$\hat{Y}_{1t} = \hat{Z}_{A1t} + \hat{L}_{1t} \tag{0.112a}$$
$$\hat{Y}_{2t} = \hat{Z}_{A2t} + \hat{L}_{2t} \tag{0.112b}$$

- Marginal Cost

$$\hat{P}_{1t} + \hat{\lambda}_{1t} = \hat{W}_{1t} - \hat{Z}_{A1t} \tag{0.113a}$$
$$\hat{P}_{2t} + \hat{\lambda}_{2t} = \hat{W}_{2t} - \hat{Z}_{A2t} \tag{0.113b}$$

- Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{0.114}$$

- National Gross Inflation Rate

$$\hat{\pi}_t = \theta_\pi \hat{\pi}_{1t} + (1-\theta_\pi)\hat{\pi}_{2t} \tag{0.115}$$

- Productivity Shock

$$\hat{Z}_{A1t} = \rho_{A1}\hat{Z}_{A1,t-1} + \varepsilon_{A1} \tag{0.116a}$$
$$\hat{Z}_{A2t} = \rho_{A2}\hat{Z}_{A2,t-1} + \varepsilon_{A2} \tag{0.116b}$$

- Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{0.117}$$

- Market Clearing Condition

$$\hat{Y}_t = \theta_Y \hat{Y}_{1t} + (1-\theta_Y)\hat{Y}_{2t} \tag{0.120}$$

- Regional Market Clearing Condition

$$\hat{P}_{1t} + \hat{Y}_{1t} = \hat{Q}_{1t} + \hat{C}_{1t} \tag{0.121a}$$
$$\hat{P}_{2t} + \hat{Y}_{2t} = \hat{Q}_{1t} + \hat{C}_{2t} \tag{0.121b}$$

0.3.3 Eigenvalues and Forward Looking Variables

As it stands, the model has more forward-looking variables than eigenvalues greater than one, indicating that the model is indeterminate. To transform the model into one with a single solution, the number of eigenvalues and forward-looking variables must be equal. To address this, **farmer_solving_2015** employs a method where excess forward-looking variables are substituted with an expectational variable at time t, along with a expectational shock $sunspot_\eta$, representing the deviation between the expected and the realized values. For the present model, the variables created are the expected regional gross inflation rates $\pi^X_{\eta t}$:

$$\pi^X_{\eta t} = \mathbb{E}_t \pi_{\eta,t+1} \tag{0.126}$$

$$sunspot_\eta = \pi_{\eta t} - \pi^X_{\eta,t-1} \tag{0.127}$$