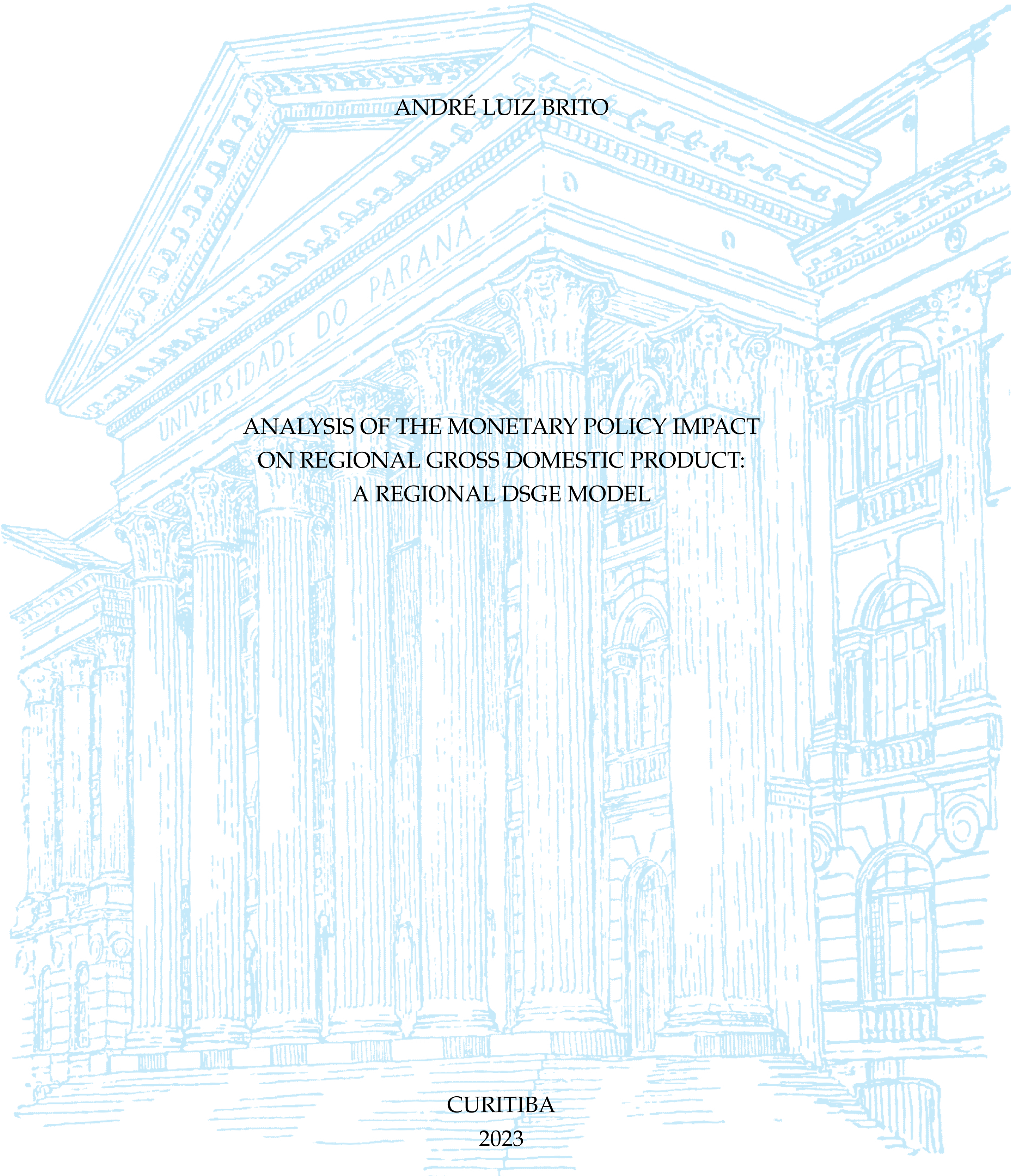
UNIVERSIDADE FEDERAL DO PARANÁ

ANDRÉ LUIZ BRITO



ANALYSIS OF THE MONETARY POLICY IMPACT ON REGIONAL GROSS DOMESTIC PRODUCT: A REGIONAL DSGE MODEL

CURITIBA
2023

ANDRÉ LUIZ BRITO

ANALYSIS OF THE MONETARY POLICY IMPACT ON REGIONAL GROSS DOMESTIC PRODUCT: A REGIONAL DSGE MODEL

Dissertation presented to the Postgraduate Program in Economic Development - PPGDE, Department of Applied Social Sciences, Federal University of Paraná, as a partial requirement for the attainment of the Master's degree in Economic Development.

Advisor: Prof. Dr. Armando Vaz Sampaio

CURITIBA
2023

Brito, André Luiz

Analysis of the Monetary Police Impact on regional Gross Domestic Product: A Regional DSGE Model / André Luiz Brito; orientador Armando Vaz Sampaio – Curitiba, 2023.

100 p.

Dissertação (Mestrado - Programa de Pós-Graduação em Desenvolvimento Econômico) – Setor de Ciências Sociais Aplicadas, Universidade Federal do Paraná, 2023.

1. Macroeconomia, 2. Economia Regional, 3. Política Monetária, 4. DSGE

approval form

To my mother, Diva,
and to my guardian angel, Kellen.

Agradecimentos

Esta dissertação é a reta final de uma longa jornada chamada mestrado. Impossível chegar aqui sozinho: somente com o apoio da família, amigos, professores e instituições é que conseguimos dar este importante passo na vida acadêmica.

À CAPES, pele bolsa de estudos que permite aos alunos terem dedicação exclusiva durante o mestrado. À Universidade Federal do Paraná, professores e servidores, pela excelência do ensino e administração do curso, em especial, professor José Felipe Araujo de Almeida e Laís de Lima Rocha.

A cada um dos professores do PPGDE, pela dedicação incondicional ao ensino, em especial aqueles que tive o privilégio de ser aluno: professores Armando Vaz Sampaio, Marcos Minoru Hasegawa, Maurício Vaz Lobo Bittencourt, Fernando Motta Correia, Victor Rodrigues de Oliveira, Kênia Barreiro de Souza, Terciane Sabadini Carvalho, Vinicius de Almeida Vale. Aos professores que, mesmo longe, não hesitaram em responder os emails de um aluno desconhecido: professores Mario Solis-Garcia e Celso José Costa Junior. Aos professores que não poupam esforços em manter o aplicativo Dynare e o fórum: professores Willi Mutschler e Johannes Pfeifer.

Ao meu orientador Armando Vaz Sampaio, pela atenção, entusiasmo e sugestões que fizeram diferença na construção desta dissertação. Aos integrantes da banca de qualificação e de defesa, professores Celso José Costa Junior e Fernando Motta Correia, pelas importantes sugestões para melhoria desta pesquisa.

Ao meu chefe Greisson Almeida Pereira, por acreditar neste projeto desde a primeira versão e pelas importantes sugestões que enriqueceram a pesquisa.

Aos meus colegas do mestrado, em especial Felipe Duplat e Matheus Canhete, pela companhia, pelo apoio moral e questionamentos que enriqueceram este trabalho.

Ao meu mentor Flávio von der Osten, por todos os conselhos e cobranças.

À minha heroína Kellen, pelo apoio incondicional.

À minha mãe, pelo exemplo de força e perseverança.

A todos que acompanharam esta jornada e me deram apoio e coragem, meu muito obrigado.

Neo: I know kung fu.
Morpheus: Show me.

Resumo

O presente projeto de pesquisa propõe criar um modelo DSGE (*Dynamic and Stochastic General Equilibrium* ou Equilíbrio Geral Dinâmico e Estocástico) para investigar os impactos da taxa de juros nominal sobre o produto interno bruto de uma região brasileira.

Abstract

The present research project aims to develop a Dynamic and Stochastic General Equilibrium (DSGE) model to investigate the effects of the nominal interest rate on the Gross Domestic Product (GDP) of a Brazilian region.

List of Figures

List of Tables

list of abbreviations (glossary)

List fo Variables

Contents

1 Introduction

The importance of macroeconomic modeling as a tool for studying the connections between monetary economy and the outcomes of a country's aggregates is undeniable, as stated by Galí (2015). Considering as well that Brazilian regions possess heterogeneous economic matrices and sectors that respond in different ways to monetary authority decisions, as indicated by Bertanha and Haddad (2008), the need for a structural model capable of relating macroeconomic variables to regional variables becomes evident.

In this context, the present research proposes the development of a macroeconomic model with regional extensions, using the DSGE methodology[1], which can demonstrate the existing relationships among the various considered variables and present impulse response functions that illustrate these relationships. With this model, we aim to investigate the existing relationship between the nominal interest rate of the Brazilian economy and the level of regional gross domestic product.

Problem and Justification

The main issue to be investigated is the impact of monetary authority decisions — especially changes in the nominal interest rate — on regional macroeconomic variables, particularly the Gross Domestic Product (GDP) of a given Brazilian region (such as a state, for example).

Given that Brazilian regions have distinct economic matrices (agriculture, industry, extraction, etc.), and within each of these specializations, some sectors are more labor-intensive while others are capital-intensive, it is plausible to assume that regional diversity allows each region to react differently to changes in the interest rate.

Given the problem, we need to determine how the study will be conducted. As this is a topic that combines knowledge from Macroeconomics and Regional Economics, it will be necessary to address the main concepts from both areas to then determine a methodology capable of integrating all of this content.

Regional Economics investigations often employ tools borrowed from Macroeconomics, as highlighted by Rickman (2010). Examples include the Leontief input-output

[1] Dynamic and Stochastic General Equilibrium.

model, the Walrasian general equilibrium applied model, and the system of macroeconometric equations. These instances demonstrate how models from one field can be adapted and utilized by the other.

In line with this notion, the objective of this work is to utilize a DSGE model (derived from Macroeconomics) to establish relationships between macro variables and regional variables. Subsequently, Brazilian economic data will be employed to ascertain the degree of correlation between these variables.

Numerous studies address the effects of national aggregates on regional variables, and these will be appropriately presented in section 2. However, in these studies, we have not found one that specifically investigates the relationship between the national nominal interest rate and regional GDP.

The significance of this work can be identified by recognizing that, given the diversity of Brazilian regions, it is not plausible that a single macroeconomic variable will have the same effect in each of them (or at least not with the same intensity). Thus, a tool capable of quantifying the regional effect of a macroeconomic variable is an important addition to economic literature, as it investigates the transmission mechanisms of monetary policy to the regional aggregate. Additionally, it also adds to the array of policy evaluation instruments, such that various economic agents can use this tool to determine the conduct of their own internal policies. For example, banks can quantify the credit interest rate for a specific region based on the projected interest rate, considering the needs and potential development of each region separately from the rest of the country.

Objectives

The main objective is to create a structural model capable of relating a macroeconomic variable (the nominal interest rate) to a regional variable (the Gross Domestic Product of a Brazilian region), in order to assess the impact of an expansionary (or contractionary) monetary policy on a specific Brazilian region and the magnitude of that impact.

The specific objectives are (1) elaborate a NK DSGE model with households, firms, monetary authority, price stickiness, productivity and monetary shocks to demonstrate that the nominal interest rate determined by the monetary authority influences the national GDP; (2) determine which variables must be regionalized in order to make a regional environment in order to demonstrate that two regions may have different responses to the monetary policy shocks; (3) produce IRF and analyse the results of both models.

2 Literature Review

Macroeconomics and Regional Economics

The assessment by Rickman (2010) on the importance of the link between Macroeconomics and Regional Economics was made at a time when the use of structural models to investigate regional issues was not yet common. Since then, several studies have addressed this connection.

Initially, we present two works that served as inspiration for the present topic. The first, developed by Costa Junior et al. (2022), investigates the impacts of fiscal policy on the state of Goiás, considering the other states of the nation. In this work, the authors develop a regionalized and open structure, individualizing a Brazilian state from the rest, considering both a national and a state fiscal authority; state expenses and revenues are disaggregated, and thus, the authors seek to identify whether there are differences between the impacts of a tax exemption in the state under study compared to the others. With the model calibrated to data from 2003 to 2019, the authors demonstrate that there is indeed a difference in state performance due to the distinction of the tax exemption occurring in the state or in the rest of the country.

The second work also presents a DSGE model, but with the objective of evaluating whether there are differences in the effects of Foreign Direct Investment (FDI), considering its location. The model developed by Mora and Costa Junior (2019) encompasses an open economy with the main region (Bogotá, 25% of the national GDP) and the rest of the country (Colombia), two types of households[2], habit formation, capital adjustment costs, as well as typical elements of a New Keynesian (NK) model[3]. With the model calibrated to data from 2002 to 2015, the authors demonstrate that there is indeed a difference in the effects of FDI depending on the region where it is applied, such that when applied in the rest of the country, there are growth effects that spread throughout the country through spillovers, including to the main region.

Both works aim to, despite dealing with distinct causes (fiscal policy and FDI), verify whether differences exist when the cause occurs in one of the two different modeled regions. Additionally, they share the same modeling approach, that of a Dynamic and Stochastic General Equilibrium (DSGE). And this was the advancement that Rick-

2 Ricardian and non-Ricardian agents.

3 nominal price rigidity, monopolistic competition, non-neutrality of monetary policy in the short term.

man (2010) wanted to see happen: the use of structural models to address regional questions.

Macroeconomic Modeling

The scientific literature on DSGE modeling is extensive, as it allows for the formulation of various questions and their answers through a general equilibrium model. This includes the aforementioned topics and, also, labor market, as explored by Ribeiro (2023); the real estate market, as studied by Albuquerquemello (2018); and even deforestation, as investigated by Pereira and Góes (2013). As remarked by Solis-Garcia (2022): *if you have a cohesive economic idea, you can put it in terms of a DSGE model*.

The works of Costa Junior (2016), Solis-Garcia (2022), Bergholt (2012), and Galí (2015) are the pillars of macroeconomic modeling theory, as they guide the reader in developing a DSGE model step-by-step. Costa Junior (2016) starts from a Real Business Cycles (RBC) model and chapter by chapter adds elements of New Keynesian (NK) theory to the model. Solis-Garcia (2022) focuses on the mathematical details necessary to develop a DSGE model, beginning with a RBC model and turning it into a canonical NK model. Bergholt (2012) discusses the key elements of a New Keynesian model and also demonstrates the necessary programming to run the model using the `Dynare` software. Galí (2015) shows the evolution from an RBC model to an NK model, adding complexity with each chapter.

Macroeconomic Modeling with Regions

Among the works employing DSGE modeling with regions, there are the study by Tamegawa (2012), which assesses the effects of fiscal policy on two regions using a model featuring two types of households, firms, banks, a national government, and a regional government. Using literature parameters to calibrate the model, the results indicate that indeed there are differences in the effects of fiscal policy depending on which region implements it. It is important to note that the difference between a macroeconomic model and a regional one lies in the fact that in the former, aggregate variables are considered only at the national level, whereas in the latter, both national and regional variables are considered, and depending on the size of the region, the latter might not be able to affect the former, as explained by Tamegawa (2013).

In a similar vein of demonstrating regional relationships, Pytlarczyk (2005) investigates aspects of the European Monetary Union (EMU), focusing on the German economy, using a structural model with two regions; Galí and Monacelli (2005) also evaluates the functioning of the EMU, but with a model where regions form a unitary continuum, such that one region cannot affect the entire economy. Alpanda and Aysun (2014) utilize a two-region model to assess the effects of US financial shocks on the euro area economy.

A framework to assess the economic evolution of a region in Japan is constructed by Okano et al. (2015), with the aim of identifying the causes of stagnation in the Kansai region.

More recently, the article by Croitorov et al. (2020) seeks to identify spillovers between regions, building a model with three regions: the Euro area, the US, and the rest of the world. Similarly investigating spillovers, Corbo and Strid (2020) present a regional model encompassing Sweden and the rest of the world.[4]"

Monetary Policy

DSGE models are widely employed within the macroeconomic literature to examine the effects of monetary policy on macroeconomic aggregates, as pointed by Galí (2015). In this context, it is important to add to the review the papers that develop models describing the monetary policy.

Smets and Wouters (2003) and Smets and Wouters (2007) present models that evaluate various types of shocks in the Eurozone and the United States, respectively. Walque et al. (2010) assess the role of the banking sector in market liquidity recovery, considering the endogenous possibilities of agent default.

Holm et al. (2021) study the transmission of monetary policy to household consumption, estimating the response of consumption, income, and savings. They utilize a heterogeneous agent New Keynesian model (HANK). The results demonstrate that a restrictive monetary policy prompts households with lower liquidity to reduce consumption as disposable income starts to decline, while households with average liquidity save less or borrow more. The study also highlights the differences in consumption changes between savers and borrowers in the face of a monetary policy alteration.

[4] Spillovers: effects that are transmitted from one region to another due to an exogenous factor, such as being neighboring regions.

Capeleti et al. (2022) evaluate the effects of pro-cyclical and counter-cyclical credit expansions by public banks on economic growth. The model implements a banking sector with public and private banks competing in a Cournot oligopoly. The results show that the supply of public credit has a stronger effect when the policy is counter-cyclical.

Soltani et al. (2021) investigate financial and monetary shocks on macroeconomic variables, with special attention to the role of banks. For this analysis, the model considers an economy with a banking sector. The results indicate that banking activity can influence the effects of economic policies.

Vinhado and Divino (2016) employ a model with financial frictions to examine the transmission of monetary policy to the banking sector and economic activity. The results demonstrate that the banking sector plays a significant role in economic activity and impacts the outcomes of monetary policy by having to adjust the bank spread in response to changes in the interest rate or reserve requirements.

3 Methodology

Contents

3.1 Regional New Keynesian Model

The model is populated by four agents: (1) a representative household, (2) a continuum of firms producing intermediate goods, (3) a firm producing a final good, and (4) the monetary authority.

The representative household maximizes utility based on consumption and labor, subject to a budget constraint composed of wages, capital rental rates, and firm profits.

The final-goods firm produces the final-good consumed by households: it aggregates all intermediate-goods produced by intermediate firms, operates under perfect competition and seeks to maximize profit subject to the bundle technology.

Intermediate firms each produce a single intermediate good, all exhibiting imperfect substitution, thus operating in monopolistic competition. Intermediate-goods firms have two problems to solve: minimize costs subject to production level and choose an optimal price to maximize the intertemporal profit flow.

Periodically, a portion of intermediate-goods firms have the opportunity to adjust prices, while others miss this chance, following to a Calvo (1983) rule. This mechanism generates nominal price rigidities, altering equilibrium relationships in the system. These rigidities lead to non-neutrality of money in the short term, as explained by Costa Junior (2016, p.191).

The monetary authority determines the nominal interest rate in response to fluctuations in previous period's inflation and production, aiming to control price levels and growth, following a Taylor (1993) rule.

Stochastic shocks will be present in the intermediate-goods firms' productivity and in the monetary policy.

For regionalization of the model, an index will be used to differentiate the studied region from the rest of Brazil, resulting in separate households, intermediate- and final-goods firms for each region. Households will lack mobility between regions. The link connecting the two regions will be the final-goods.

Then, equilibrium conditions of the system will be determined. Assuming the system tends toward long-term equilibrium, a steady state will be reached where variables cease to change. Thus, for a given $t \longrightarrow \infty$, we will have $\boldsymbol{X}_t = \boldsymbol{X}_{t+1} = \boldsymbol{X}_{ss} \implies \dot{\boldsymbol{X}} = 0$, where $\boldsymbol{X}$ denotes the vector of system variables, ss indicates the steady state and $\dot{\boldsymbol{X}} = \partial X/\partial t$.

After that, the log-linearization method proposed by Uhlig (1999) will be employed to convert the system of equations into a linear system, so that this linear system can be solved by the program Dynare, which computes the solution and produces impulse-response graphs based on the stochastic shocks.

Regions

falta revisar esta parte e agrupar por agentes da economia.

colocar estatística descritiva para justificar as variáveis.

Regions will have an index $\eta \in \{1, 2, \ldots, n\}$ representing the variables of each region. Whenever necessary, a second region index $\nu \in \{1, 2, \ldots, n\}$ will be used. For example, the variable C_t represents the total consumption (the grand total of all regions), $C_{\eta t}$ represents the consumption composition of region η and $C_{\eta 2t}$ represents the consumption of final-good of region ν by region η (the first index indicating the origin and the second the destiny of the good). Without loss of generality, the model will have two regions: the main region 1 and the remaining of the country 2, so that $\eta, \nu \in \{1, 2\}$.

For each region, the variables are:

- Consumption $C_{\eta 2t}$: households from region $\eta \in \{1, 2\}$ consume from both regions $\eta \in \{1, 2\}$.
- Labor $L_{\eta t}$: there is no mobility in the labor market, so that households will work for firms in the same region they live.
- Investment and Capital $I_{\eta t}, K_{\eta t}$: there is no mobility in investments and capital rent: households will invest and rent capital in their own region.
- Final-good production $Y_{\eta t}$: there is one representative final-good firm in each region that aggregates all intermediate-goods of that region.
- Final-good price $P_{\eta t}$ and regional inflation $\pi_{\eta t}$: in each region, there is a final-good price and a regional inflation level.
- Intermediate-goods firms $Y_{\eta j t}$: there is a continuum $j \in [0, 1]$ for each region and these firms will demand labor and capital from within the region.

- Productivity level $Z_{A\eta t}$ and capital weight in production α_η: each region has its own characteristics and because of that has a difference productivity level subject to different shock rule and a different capital weight in production.

Model Diagram

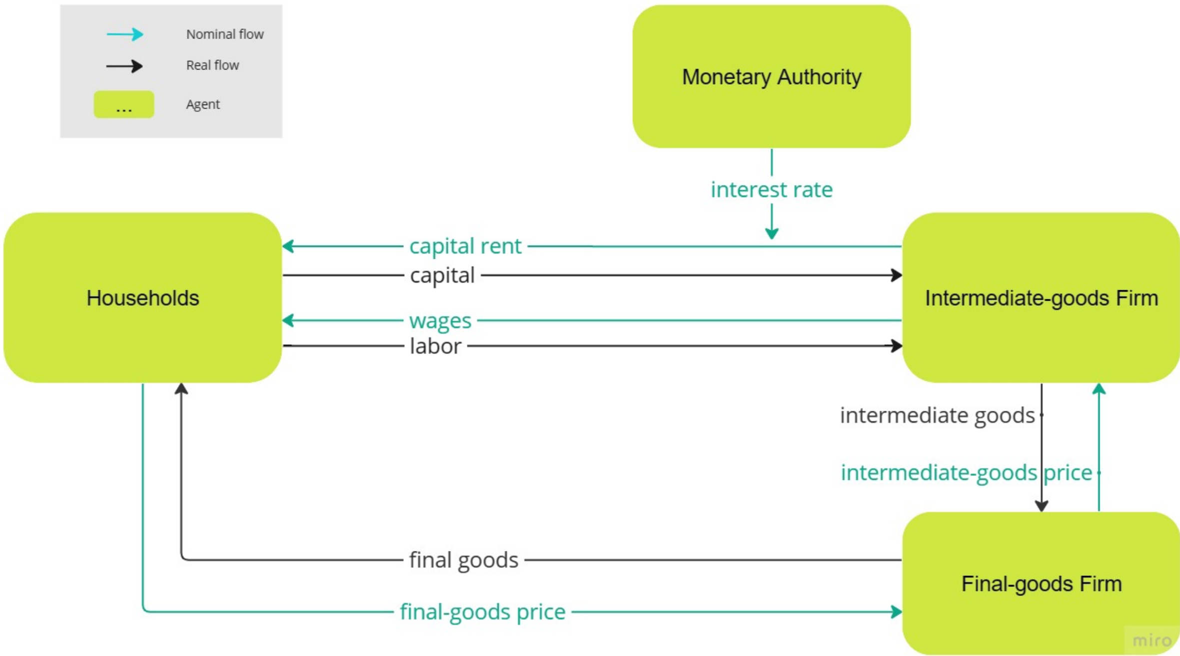


Figure 1: Model Diagram

3.1.1 Household

Utility Maximization Problem

Following the models presented by Costa Junior (2016) and Solis-Garcia (2022), the representative household problem is to maximize an intertemporal utility function U_η with respect to consumption $C_{\eta t}$ and labor $L_{\eta t}$, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_{\eta t}, L_{\eta t}, B_{\eta t}, K_{\eta,t+1}} : \quad U_\eta(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{3.1}$$

$$\begin{aligned} \text{s. t.}: \quad & P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ & = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \tag{3.2}$$

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{3.3}$$

$$C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \tag{3.4}$$

$$C_{\eta t}, L_{\eta t}, K_{\eta,t+1} > 0 \,; K_0 \text{ given.}$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the relative labor weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, $P_{\eta t}$ is the price level of region η, $C_{\eta 1t}$ is the good produced in region η and consumed in region η, $C_{\eta 2t}$ is the good produced in region ν and consumed in region η, $I_{\eta t}$ is the investment, $B_{\eta t}$ are the bonds, W_t is the wage level, $K_{\eta t}$ is the capital stock, R_{Kt} is the return on capital, R_t is the return on bonds (which is also the nominal interest rate of the economy) and $\Pi_{\eta t}$ is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate. In the consumption aggregation, $\omega_{\eta 1}$ is the weight of good $C_{\eta 1t}$ in the consumption bundle $C_{\eta t}$ of region η.

Substitute 3.4 in 3.1:

$$U_\eta(C_{\eta 1t}, C_{\eta 2t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{\left[C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{3.5}$$

Isolate $I_{\eta t}$ in 3.3 and substitute in 3.2:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies I_{\eta t} = K_{\eta,t+1} - (1-\delta)K_{\eta t} \tag{3.3}$$

$$\begin{aligned} &P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ &\quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \implies \end{aligned} \tag{3.2}$$

$$\begin{aligned} &P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1-\delta)K_{\eta t}] + B_{\eta t} = \\ &\quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \tag{3.6}$$

Lagrangian

The maximization problem with restriction can be transformed in one without restriction using the Lagrangian function $\mathscr{L}$ with 3.5 and 3.6:

$$\begin{aligned} \mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \left(\frac{\left[C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}}\right]^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) - \\ & - \mu_t \Big[P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1-\delta)K_{\eta t}] + B_{\eta t} - \\ & \quad - (W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1+R_{t-1})B_{\eta,t-1} + \Pi_{\eta t}) \Big] \Bigg\} \end{aligned} \tag{3.7}$$

First Order Conditions

The first order conditions are:

$$C_{\eta 1t}: \quad \mu_t = \frac{\omega_{\eta 1} C_{\eta 1t}^{\omega_{\eta 1}(1-\sigma)-1} C_{\eta 2t}^{(1-\omega_{\eta 1})(1-\sigma)}}{P_{1t}} = \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} C_{\eta t}^{1-\sigma} \tag{3.8}$$

$$C_{\eta 2t}: \quad \mu_t = \frac{(1-\omega_{\eta 1}) C_{\eta 1t}^{\omega_{\eta 1}(1-\sigma)} C_{\eta 2t}^{(1-\omega_{\eta 1})(1-\sigma)-1}}{P_{2t}} = \frac{(1-\omega_{\eta 1})}{P_{2t} C_{\eta 2t}} C_{\eta t}^{1-\sigma} \tag{3.9}$$

$$L_{\eta t}: \quad -\phi L_{\eta t}^{\varphi} + \mu_t W_t = 0 \implies \mu_t = \frac{\phi L_{\eta t}^{\varphi}}{W_t} \tag{3.10}$$

$$\begin{aligned} B_{\eta t}: \quad & \beta^t\{-\mu_t\} + \mathbb{E}_t \beta^{t+1}\{-\mu_{t+1}[-(1+R_t)]\} = 0 \implies \\ & \mu_t = \beta(1+R_t)\mathbb{E}_t \mu_{t+1} \end{aligned} \tag{3.11}$$

$$\begin{aligned} K_{\eta,t+1}: \quad & -\mu_t P_{\eta t} + \mathbb{E}_t \beta\{\mu_{t+1}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\} = 0 \implies \\ & \mu_t P_{\eta t} = \beta \mathbb{E}_t\{\mu_{t+1}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \mu_t: \quad & P_{1t} C_{\eta 1t} + P_{2t} C_{\eta 2t} + P_{\eta t}[K_{\eta,t+1} - (1-\delta)K_{\eta t}] + B_{\eta t} = \\ & = W_t L_{\eta t} + R_{Kt} K_{\eta t} + (1+R_{t-1}) B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \tag{3.6}$$

Solutions

Match 3.8 and 3.9:

$$\mu_t = \frac{\omega_{\eta 1}}{P_{1t} C_{\eta 1t}} C_{\eta t}^{1-\sigma} = \frac{(1-\omega_{\eta 1})}{P_{2t} C_{\eta 2t}} C_{\eta t}^{1-\sigma} \implies$$

$$\frac{C_{\eta 1t}}{C_{\eta 2t}} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \iff C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta 2t} \tag{3.13}$$

Equation 3.13 is the relative consumption of regional goods in region η.

Substitute 3.13 in 3.4:

$$C_{\eta t} = C_{\eta 1t}^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \implies \tag{3.4}$$

$$C_{\eta t} = \left[\frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta 2t}\right]^{\omega_{\eta 1}} C_{\eta 2t}^{1-\omega_{\eta 1}} \implies$$

$$C_{\eta 2t} = C_{\eta t}\left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \tag{3.14}$$

Substitute 3.14 in 3.13:

$$C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta 2t} \qquad \Longrightarrow \tag{3.13}$$

$$C_{\eta 1t} = \frac{P_{2t}}{P_{1t}} \cdot \frac{\omega_{\eta 1}}{1-\omega_{\eta 1}} \cdot C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \Longrightarrow$$

$$C_{\eta 1t} = C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}-1} \tag{3.15}$$

Define the total goods expense $\mathcal{E}_{\eta t}$ of household η:

$$\mathcal{E}_{\eta t} = P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} \tag{3.16}$$

Substitute 3.14 and 3.15 in 3.16:

$$\mathcal{E}_{\eta t} = P_{1t}C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}-1} + P_{2t}C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \Longrightarrow$$

$$\mathcal{E}_{\eta t} = C_{\eta t} \left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \tag{3.17}$$

Equation 3.17 shows that the total expense with goods is proportional to the goods' prices P_{1t} and P_{2t}.

Rewrite 3.14 in terms of 3.17:

$$C_{\eta 2t} = C_{\eta t} \left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \qquad \Longrightarrow \tag{3.14}$$

$$C_{\eta 2t} = C_{\eta t} \left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{-\omega_{\eta 1}} \left[\frac{P_{2t}}{1-\omega_{\eta 1}} \cdot \frac{1-\omega_{\eta 1}}{P_{2t}}\right] \Longrightarrow$$

$$C_{\eta 2t} = \mathcal{E}_{\eta t} \frac{1-\omega_{\eta 1}}{P_{2t}} \tag{3.18}$$

Rewrite 3.15 in terms of 3.17 and isolate $(\omega_{\eta 1}/P_{1t}C_{\eta 1t})$:

$$C_{\eta 1t} = C_{\eta t}\left[\frac{P_{1t}}{P_{2t}} \cdot \frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}-1} \implies \tag{3.15}$$

$$C_{\eta 1t} = C_{\eta t}\left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}}\left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}}\left[\frac{\omega_{\eta 1}}{P_{1t}}\right] \implies$$

$$C_{\eta 1t} = \mathcal{E}_{\eta t}\frac{\omega_{\eta 1}}{P_{1t}} \iff \frac{\omega_{\eta 1}}{P_{1t}C_{\eta 1t}} = \frac{1}{\mathcal{E}_{\eta t}} \tag{3.19}$$

Substitute 3.19 in 3.8:

$$\mu_t = \frac{\omega_{\eta 1}}{P_{1t}C_{\eta 1t}}C_{\eta t}^{1-\sigma} \implies \tag{3.8}$$

$$\mu_t = \frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} \tag{3.20}$$

Match 3.20 and 3.10:

$$\mu_t = \frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \frac{\phi L_{\eta t}^{\varphi}}{W_t} \implies$$

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \tag{3.21}$$

Equation 3.21 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute μ_t and μ_{t+1} from equation 3.20 in 3.11:

$$\mu_t = \beta(1+R_t)\mathbb{E}_t\mu_{t+1} \implies \tag{3.11}$$

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1+R_t)\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}\right\} \tag{3.22}$$

Equation 3.22 is the Euler equation for the bonds return.

Substitute μ_t and μ_{t+1} from equation 3.20 in 3.12:

$$\mu_t P_{\eta t} = \beta \mathbb{E}_t\{\mu_{t+1}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\} \qquad \Longrightarrow \tag{3.12}$$

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} = \beta \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} [(1-\delta)P_{\eta,t+1} + R_{K,t+1}] \right\} \tag{3.23}$$

Equation 3.23 is the Euler equation for the capital return.

Firms

Consider two types of firms: (1) a continuum of intermediate-goods firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-goods firm, which aggregates all the varieties into a final bundle and operates in perfect competition.

3.1.2 Final-Goods Firm

Profit Maximization Problem

The role of the final-goods firm is to aggregate all the varieties $Y_{\eta j t}$ produced by the intermediate-goods firms in each region $\eta \in \{1, 2\}$, so that the representative consumer can buy only one good $Y_{\eta t}$, the bundle good.

The final-goods firm problem is to maximize its profit, considering that its output is the bundle $Y_{\eta t}$ formed by a continuum $j \in [0, 1]$ of intermediate goods $Y_{\eta j t}$, with elasticity of substitution between intermediate goods ψ:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_{\eta t} Y_{\eta t} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{3.24}$$

$$\text{s. t.} : \quad Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} \tag{3.25}$$

Substitute 3.25 in 3.24:

$$\max_{Y_{\eta j t}} : \quad \Pi_{\eta t} = P_{\eta t} \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{\eta j t} Y_{\eta j t} \, \mathrm{d}\, j \tag{3.26}$$

First Order Condition and Solutions

The first order condition is:

$$Y_{\eta jt}: \quad P_{\eta t}\left(\frac{\psi}{\psi-1}\right)\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}-1}\left(\frac{\psi-1}{\psi}\right)Y_{\eta jt}^{\frac{\psi-1}{\psi}-1}-P_{\eta jt}=0 \implies$$

$$Y_{\eta jt}=Y_t\left(\frac{P_{\eta t}}{P_{\eta jt}}\right)^{\psi} \tag{3.27}$$

Equation 3.27 shows that the demand for variety j depends on its relative price.

Substitute 3.27 in 3.25:

$$Y_{\eta t}=\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \implies$$

$$Y_{\eta t}=\left(\int_0^1\left[Y_{\eta t}\left(\frac{P_{\eta t}}{P_{\eta jt}}\right)^{\psi}\right]^{\frac{\psi-1}{\psi}}\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \implies$$

$$P_{\eta t}=\left[\int_0^1 P_{\eta jt}^{1-\psi}\,\mathrm{d}\,j\right]^{\frac{1}{1-\psi}} \tag{3.28}$$

Equation 3.28 is the final-goods firm's markup.

3.1.3 Intermediate-Goods Firms

Cost Minimization Problem

The intermediate-goods firms, denoted by $j\in[0,1]$, produce varieties of a representative good with a certain level of substitutability. Each of these firms has to choose capital $K_{\eta jt}$ and labor $L_{\eta jt}$ to minimize production costs, subject to a technology rule.

$$\min_{K_{\eta jt},L_{\eta jt}} : \quad R_{Kt}K_{\eta jt}+W_tL_{\eta jt} \tag{3.29}$$

$$\text{s. t.}: \quad Y_{\eta jt}=Z_{A\eta t}K_{\eta jt}^{\alpha_\eta}L_{\eta jt}^{1-\alpha_\eta} \tag{3.30}$$

where $Y_{\eta jt}$ is the output obtained by the production technology level $Z_{A\eta t}$ that transforms capital $K_{\eta jt}$ and labor $L_{\eta jt}$ in proportions α_η and $(1-\alpha_\eta)$, respectively, into intermediate goods.[5]

Lagrangian

Applying the Lagrangian:

$$\mathcal{L} = (R_{Kt}K_{\eta jt} + W_t L_{\eta jt}) - \Lambda_{\eta jt}(Z_{A\eta t}K_{\eta jt}^{\alpha_\eta}L_{\eta jt}^{1-\alpha_\eta} - Y_{\eta jt}) \tag{3.31}$$

where the Lagrangian multiplier $\Lambda_{\eta jt}$ is the marginal cost.[6]

First Order Conditions

The first-order conditions are:

$$\begin{aligned}
K_{\eta jt}: \quad & R_{Kt} - \Lambda_{\eta jt}Z_{A\eta t}\alpha_\eta K_{\eta jt}^{\alpha_\eta - 1}L_{\eta jt}^{1-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& K_{\eta jt} = \alpha_\eta Y_{\eta jt}\frac{\Lambda_{\eta jt}}{R_{Kt}} && (3.32)\\
L_{\eta jt}: \quad & W_t - \Lambda_{\eta jt}Z_{A\eta t}K_{\eta jt}^{\alpha_\eta}(1-\alpha_\eta)L_{\eta jt}^{-\alpha_\eta} = 0 \qquad \Longrightarrow \\
& L_{\eta jt} = (1-\alpha_\eta)Y_{\eta jt}\frac{\Lambda_{\eta jt}}{W_t} && (3.33)\\
\Lambda_{\eta jt}: \quad & Y_{\eta jt} = Z_{A\eta t}K_{\eta jt}^{\alpha_\eta}L_{\eta jt}^{1-\alpha_\eta} && (3.30)
\end{aligned}$$

Solutions

Divide equation 3.32 by 3.33:

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \frac{\alpha_\eta Y_{\eta jt}\Lambda_{\eta jt}/R_{Kt}}{(1-\alpha_\eta)Y_{\eta jt}\Lambda_{\eta jt}/W_t} \Longrightarrow \frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_t}{R_{Kt}} \tag{3.34}$$

Equation 3.34 demonstrates the relationship between the technical marginal rate of substitution (TMRS) and the economical marginal rate of substitution (EMRS).

---

[5] the production technology level $Z_{A\eta t}$ will be submitted to a productivity shock, detailed in section 3.1.5.

[6] see Lemma A.1

Substitute $L_{\eta jt}$ from equation 3.34 in 3.30:

$$
\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} && \Longrightarrow \\
Y_{\eta jt} &= Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} \left[\left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_{Kt} K_{\eta jt}}{W_t} \right]^{1-\alpha_\eta} && \Longrightarrow \\
K_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha_\eta} && (3.35)
\end{aligned}
$$

Equation 3.35 is the intermediate-goods firm demand for capital.

Substitute 3.35 in 3.34:

$$
\begin{aligned}
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_{Kt} K_{\eta jt}}{W_t} && \Longrightarrow \\
L_{\eta jt} &= \left(\frac{1-\alpha_\eta}{\alpha_\eta} \right) \frac{R_{Kt}}{W_t} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha_\eta} && \Longrightarrow \\
L_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_{Kt}} \right]^{-\alpha_\eta} && (3.36)
\end{aligned}
$$

Equation 3.36 is the intermediate-goods firm demand for labor.

Total and Marginal Costs

Calculate the total cost TC using 3.35 and 3.36:

$$
\begin{aligned}
TC_{\eta jt} &= W_t L_{\eta jt} + R_{Kt} K_{\eta jt} && \Longrightarrow \\
TC_{\eta jt} &= W_t \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_{Kt}} \right]^{-\alpha_\eta} + R_{Kt} \frac{Y_{\eta jt}}{Z_{A\eta t}} \left[\left(\frac{\alpha_\eta}{1-\alpha_\eta} \right) \frac{W_t}{R_{Kt}} \right]^{1-\alpha_\eta} && \Longrightarrow \\
TC_{\eta jt} &= \frac{Y_{\eta jt}}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta} \right)^{1-\alpha_\eta} && (3.37)
\end{aligned}
$$

Calculate the marginal cost Λ using 3.37:

$$
\Lambda_{\eta jt} = \frac{\partial TC_{\eta jt}}{\partial Y_{\eta jt}} \Longrightarrow \Lambda_{\eta jt} = \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta} \right)^{1-\alpha_\eta} \qquad (3.38)
$$

The marginal cost depends on the technological level $Z_{A\eta t}$, the nominal interest rate R_{Kt} and the nominal wage level W_t, which are the same for all intermediate-goods firms, and because of that, the index j may be dropped:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta} \right)^{1-\alpha_\eta} \tag{3.39}$$

notice that:

$$\Lambda_{\eta t} = \frac{TC_{\eta jt}}{Y_{\eta jt}} \implies TC_{\eta jt} = \Lambda_{\eta t} Y_{\eta jt} \tag{3.40}$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (CALVO, 1983): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period ($P_{\eta j,t+1} = P_{\eta jt}$), and a probability of $(1-\theta)$ of setting a new optimal price $P^*_{\eta jt}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price.

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \tag{3.41}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Y_{\eta t} \left(\frac{P_{\eta t}}{P_{\eta jt}} \right)^{\psi} \tag{3.27}$$

Substitute 3.40 in 3.41:

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - \Lambda_{\eta,t+s} Y_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \tag{3.42}$$

Substitute 3.27 in 3.42 and rearrange the variables:

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} - \Lambda_{\eta,t+s} Y_{\eta t+s} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies$$

$$\max_{P_{\eta jt}}: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt}^{1-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to $P_{\eta jt}$ is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{\eta jt}^{-\psi} P_{\eta,t+s}^{\psi} Y_{\eta t+s} - (-\psi) P_{\eta jt}^{-\psi-1} P_{\eta,t+s}^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s} \right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\begin{aligned} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} Y_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \\ = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} \left(\frac{P_{\eta,t+s}}{P_{\eta jt}} \right)^{\psi} Y_{\eta t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \end{aligned} \tag{3.43}$$

Substitute 3.27 in 3.43:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi-1) Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{\eta jt}^{-1} Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies$$

$$(\psi-1)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \psi P_{\eta jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies$$

$$P_{\eta jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies$$

$$P_{\eta jt}^{*} = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k}) \right\}} \tag{3.44}$$

Equation 3.44 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P_{\eta t}^{*} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} \Lambda_{\eta, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j, t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{3.45}$$

Final-Goods Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1 - \theta)$ firms will choose a new optimal price. The price level for each period will be a composition of these two prices. Use this information in 3.28 to determine the aggregate price level:

$$\begin{aligned} P_{\eta t} &= \left[\int_0^{\theta} P_{\eta, t-1}^{1-\psi} \, \mathrm{d}\, j + \int_{\theta}^{1} P_{\eta t}^{*1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies \\ P_{\eta t} &= \left[\theta P_{\eta, t-1}^{1-\psi} + (1 - \theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \end{aligned} \tag{3.46}$$

Equation 3.46 is the aggregate price level.

3.1.4 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (TAYLOR, 1993) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{3.47}$$

where R, π, Y are the variables in steady state, γ_R is the smoothing parameter for the interest rate R_{Kt}, while γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively and Z_{Mt} is the monetary shock.[7]

[7] for the monetary shock definition, see section 3.1.5.

and π_t is the gross inflation rate, defined by:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{3.48}$$

where P_t is the national price level, defined by:

$$\begin{aligned} P_t Y_t &= P_{1t} Y_{1t} + P_{2t} Y_{2t} \implies \\ P_t &= \frac{P_{1t} Y_{1t} + P_{2t} Y_{2t}}{Y_t} \end{aligned} \tag{3.49}$$

Regional Inflation

There is one price level $P_{\eta t}$ in each region, generating a regional inflation rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{3.50}$$

3.1.5 Stochastic Shocks

Productivity Shock

The production technology level $Z_{A\eta t}$ will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{3.51}$$

where $\rho_{A\eta} \in [0, 1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{3.52}$$

where $\rho_M \in [0, 1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

3.1.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P^*_{\eta t}, R^*_t, R^*_{Kt}, W^*_t\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C^*_{\eta 1t}, C^*_{\eta 2t}, L^*_{\eta t}, B^*_{\eta t}, K^*_{\eta,t+1}\}$ and allocations for firms $\boldsymbol{\mathcal{A}}_F := \{K^*_{\eta jt}, L^*_{\eta jt}, Y^*_{\eta jt}, Y^*_{\eta t}\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms' problems, and the markets for goods and labor clear.

$$Y_t = Y_{1t} + Y_{2t} \tag{3.53}$$

$$\text{where:} \quad Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \tag{3.54}$$

$$L_{\eta t} = \int_0^1 L_{\eta jt} \, \mathrm{d}\, j \tag{3.55}$$

3.1.7 Model Structure

The model is composed of the preview solutions, forming a square system of 41 variables and 41 equations, summarized as follows:

- Variables:
  - from the household problem: $\langle\, C_{\eta t}\ L_{\eta t}\ B_{\eta t}\ K_{\eta,t+1}\ C_{\eta 1t}\ C_{\eta 2t}\ \mathcal{E}_{\eta t} \,\rangle$;
  - from the final-goods firm problem: $\langle\, Y_{\eta jt}\ Y_{\eta t}\ P_{\eta t} \,\rangle$;
  - from the intermediate-goods firm problems: $\langle\, K_{\eta jt}\ L_{\eta jt}\ P^*_{\eta t} \,\rangle$;
  - from the monetary policy: $\langle\, R_t\ \pi_t\ Y_t \,\rangle$;
  - from the market clearing condition: $\langle\, I_{\eta t} \,\rangle$;
  - prices: $\langle\, W_t\ R_{Kt}\ \Lambda_{\eta t}\ P_t\ \pi_{\eta t} \,\rangle$;
  - shocks: $\langle\, Z_{A\eta t}\ Z_{Mt} \,\rangle$.
- Equations:
  1. Budget Constraint:

$$\begin{aligned} &P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ &\quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \end{aligned} \tag{3.2}$$

  2. Law of Motion for Capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{3.3}$$

  3. Regional Consumption of good 1:

$$C_{\eta 1t} = \mathcal{E}_{\eta t}\frac{\omega_{\eta 1}}{P_{1t}} \tag{3.19}$$

4. Regional Consumption of good 2:

$$C_{\eta 2t} = \mathcal{E}_{\eta t} \frac{1 - \omega_{\eta 1}}{P_{2t}} \tag{3.18}$$

5. Price Composition of Consumption Bundle:

$$\mathcal{E}_{\eta t} = C_{\eta t} \left[\frac{P_{1t}}{\omega_{\eta 1}} \right]^{\omega_{\eta 1}} \left[\frac{P_{2t}}{1 - \omega_{\eta 1}} \right]^{1-\omega_{\eta 1}} \tag{3.17}$$

6. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \tag{3.21}$$

7. Euler equation for the bonds return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1 + R_t)\mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} \right\} \tag{3.22}$$

8. Euler equation for the capital return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} P_{\eta t} = \beta \mathbb{E}_t \left\{ \frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}} [(1 - \delta) P_{\eta,t+1} + R_{K,t+1}] \right\} \tag{3.23}$$

9. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi - 1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi - 1}} \tag{3.25}$$

10. Production Function:

$$Y_{\eta j t} = Z_{A\eta t} K_{\eta j t}^{\alpha_\eta} L_{\eta j t}^{1-\alpha_\eta} \tag{3.30}$$

11. Capital Demand:

$$K_{\eta j t} = \alpha_\eta Y_{\eta j t} \frac{\Lambda_{\eta t}}{R_{Kt}} \tag{3.32}$$

12. Labor Demand:

$$L_{\eta j t} = (1 - \alpha_\eta) Y_{\eta j t} \frac{\Lambda_{\eta t}}{W_t} \tag{3.33}$$

13. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha_\eta} \right)^{\alpha_\eta} \left(\frac{W_t}{1 - \alpha_\eta} \right)^{1-\alpha_\eta} \tag{3.39}$$

14. Optimal Price:

$$P^*_{\eta t} = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{3.45}$$

15. Regional Price Level:

$$P_{\eta t} = \left[\theta P_{\eta,t-1}^{1-\psi} + (1 - \theta) P_{\eta t}^{*1-\psi} \right]^{\frac{1}{1-\psi}} \tag{3.46}$$

16. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{3.47}$$

17. National Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{3.48}$$

18. National Price Level:

$$P_t = \frac{P_{1t} Y_{1t} + P_{2t} Y_{2t}}{Y_t} \tag{3.49}$$

19. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{3.50}$$

20. Productivity Shock:

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{3.51}$$

21. Monetary Shock:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{3.52}$$

22. Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \tag{3.53}$$

23. Regional Market Clearing Condition:

$$Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \tag{3.54}$$

3.2 Steady State

The steady state of a variable is defined by its constancy through time. For any given variable X_t, it is in steady state if $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (COSTA JUNIOR, 2016, p.41). For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The model in steady state is:

1. Budget Constraint:

$$\begin{aligned} &P_{1t}C_{\eta 1t} + P_{2t}C_{\eta 2t} + P_{\eta t}I_{\eta t} + B_{\eta t} = \\ &\quad = W_t L_{\eta t} + R_{Kt}K_{\eta t} + (1 + R_{t-1})B_{\eta,t-1} + \Pi_{\eta t} \implies \end{aligned} \tag{3.2}$$

$$P_1 C_{\eta 1} + P_2 C_{\eta 2} + P_\eta I_\eta = W L_\eta + R_K K_\eta + R B_\eta + \Pi_\eta \tag{3.56}$$

2. Law of Motion for Capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies \tag{3.3}$$

$$K_\eta = (1-\delta)K_\eta + I_\eta \implies$$

$$I_\eta = \delta K_\eta \tag{3.57}$$

3. Regional Consumption of good 1:

$$C_{\eta 1t} = \mathcal{E}_{\eta t}\frac{\omega_{\eta 1}}{P_{1t}} \implies \tag{3.19}$$

$$C_{\eta 1} = \mathcal{E}_{\eta}\frac{\omega_{\eta 1}}{P_{1}} \tag{3.58}$$

4. Regional Consumption of good 2:

$$C_{\eta 2t} = \mathcal{E}_{\eta t}\frac{1-\omega_{\eta 1}}{P_{2t}} \implies \tag{3.18}$$

$$C_{\eta 2} = \mathcal{E}_{\eta}\frac{1-\omega_{\eta 1}}{P_{2}} \tag{3.59}$$

5. Price Composition of Consumption Bundle:

$$\mathcal{E}_{\eta t} = C_{\eta t}\left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}}\left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \implies \tag{3.17}$$

$$\mathcal{E}_{\eta} = C_{\eta}\left[\frac{P_{1}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}}\left[\frac{P_{2}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \tag{3.60}$$

6. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \implies \tag{3.21}$$

$$\frac{\phi L_{\eta}^{\varphi}}{C_{\eta}^{1-\sigma}} = \frac{W}{\mathcal{E}_{\eta}} \tag{3.61}$$

7. Euler equation for the bonds return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1+R_t)\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}\right\} \implies \tag{3.22}$$

$$\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} = \beta(1+R)\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \implies$$

$$\beta = \frac{1}{(1+R)} \tag{3.62}$$

8. Euler equation for the capital return:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}}P_{\eta t} = \beta\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}[(1-\delta)P_{\eta,t+1}+R_{K,t+1}]\right\} \implies \tag{3.23}$$

$$\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}}P_{\eta} = \beta\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}}P_{\eta}\left[(1-\delta)+\frac{R_K}{P_\eta}\right] \implies$$

$$1 = \beta\left[(1-\delta)+\frac{R_K}{P_\eta}\right] \tag{3.63}$$

9. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \implies Y_{\eta} = \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{3.64}$$

10. Production Function:

$$Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \implies Y_{\eta j} = Z_{A\eta} K_{\eta j}^{\alpha_\eta} L_{\eta j}^{1-\alpha_\eta} \tag{3.65}$$

11. Capital Demand:

$$K_{\eta jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_{Kt}} \implies K_{\eta j} = \alpha_\eta Y_{\eta j} \frac{\Lambda_\eta}{R_K} \tag{3.66}$$

12. Labor Demand:

$$L_{\eta jt} = (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \implies L_{\eta j} = (1-\alpha_\eta) Y_{\eta j} \frac{\Lambda_\eta}{W} \tag{3.67}$$

13. Marginal Cost:

$$\begin{aligned} \Lambda_{\eta t} &= \frac{1}{Z_{A\eta t}} \left(\frac{R_{Kt}}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies \\ \Lambda_\eta &= \frac{1}{Z_{A\eta}} \left(\frac{R_K}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \end{aligned} \tag{3.68}$$

14. Optimal Price:

$$P_{\eta t}^* = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^\infty \left\{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t \sum_{s=0}^\infty \left\{\theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \tag{3.45}$$

$$\begin{aligned} P_\eta^* &= \frac{\psi}{\psi-1} \cdot \frac{Y_{\eta j}\Lambda_\eta / [1-\theta(1-R)]}{Y_{\eta j}/[1-\theta(1-R)]} \implies \\ P_\eta^* &= \frac{\psi}{\psi-1} \Lambda_\eta \end{aligned} \tag{3.69}$$

15. Regional Price Level:

$$\begin{aligned} P_{\eta t} &= \left[\theta P_{\eta t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi}\right]^{\frac{1}{1-\psi}} && \implies \\ P_\eta^{1-\psi} &= \theta P_\eta^{1-\psi} + (1-\theta) P_\eta^{*1-\psi} && \implies \\ (1-\theta) P_\eta^{1-\psi} &= (1-\theta) P_\eta^{*1-\psi} && \implies P_\eta = P_\eta^* \end{aligned} \tag{3.70}$$

16. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \tag{3.71}$$

17. National Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \pi = 1 \tag{3.72}$$

18. National Price Level:

$$P_t = \frac{P_{1t}Y_{1t} + P_{2t}Y_{2t}}{Y_t} \implies P = \frac{P_1 Y_1 + P_2 Y_2}{Y} \tag{3.73}$$

19. Regional Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta, t-1}} \implies \pi_\eta = 1 \tag{3.74}$$

20. Productivity Shock:

$$\begin{aligned} \ln Z_{A\eta t} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} &&\implies \\ \ln Z_{A\eta} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta} + \varepsilon_{A\eta} &&\implies \\ \varepsilon_{A\eta} &= 0 \end{aligned} \tag{3.75}$$

21. Monetary Shock:

$$\begin{aligned} \ln Z_{Mt} &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} &&\implies \\ \ln Z_M &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_M + \varepsilon_M &&\implies \\ \varepsilon_M &= 0 \end{aligned} \tag{3.76}$$

22. Market Clearing Condition:

$$Y_t = Y_{1t} + Y_{2t} \implies \tag{3.53}$$

$$Y = Y_1 + Y_2 \tag{3.77}$$

23. Regional Market Clearing Condition:

$$Y_{\eta t} = C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \implies \tag{3.54}$$
$$Y_{\eta} = C_{\eta 1} + C_{\eta 2} + I_{\eta} \tag{3.78}$$

3.2.1 Variables at Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It is assumed that one regional price level and both productivity levels are normalized to one:[8]

$$\langle P_1 \quad Z_{A1} \quad Z_{A2} \rangle = \vec{\mathbf{1}} \tag{3.79}$$

From 3.71, 3.72 and 3.74, the monetary shock, the national and regional gross inflation rates are:

$$\langle Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle = \vec{\mathbf{1}} \tag{3.80}$$

From 3.75 and 3.76, the productivity and monetary shocks are:

$$\langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle = \vec{\mathbf{0}} \tag{3.81}$$

From 3.62, the return on bonds is:

$$\beta = \frac{1}{(1+R)} \implies \tag{3.62}$$
$$R = \frac{1}{\beta} - 1 \tag{3.82}$$

From 3.63, the return on capital R_K for $\eta = 1$ is:

$$1 = \beta \left[(1-\delta) + \frac{R_K}{P_1} \right] \implies \tag{3.63}$$
$$R_K = P_1 \left[\frac{1}{\beta} - (1-\delta) \right] \tag{3.83}$$

[8] where $\vec{\mathbf{1}}$ is the unit vector.

Divide 3.83 for $\eta = 1$ by 3.83 for $\eta = 2$:

$$\frac{R_K}{R_K} = \frac{P_1\left[\frac{1}{\beta} - (1-\delta)\right]}{P_2\left[\frac{1}{\beta} - (1-\delta)\right]} \implies$$

$$1 = \frac{P_1}{P_2} \implies P_1 = P_2 \tag{3.84}$$

From 3.70 and 3.84, the regional optimal price P^*_η is:

$$P^*_\eta = P_\eta \implies \langle P^*_1 \quad P^*_2 \rangle = \langle P_1 \quad P_2 \rangle = \langle P_1 \quad P_1 \rangle \tag{3.70}$$

Substitute 3.84 and then 3.78 in 3.73 for the national price level:

$$P = \frac{P_1 Y_1 + P_2 Y_2}{Y} = \frac{P_1 Y_1 + P_1 Y_2}{Y} \implies \tag{3.73}$$

$$P = P_1 = P_2 \tag{3.85}$$

Substitute 3.85 in 3.60 for the price composition of consumption bundle $\mathcal{E}_\eta$:

$$\mathcal{E}_\eta = C_\eta \left[\frac{P_1}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \left[\frac{P_2}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} = C_\eta \left[\frac{P}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}} \left[\frac{P}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \implies \tag{3.60}$$

$$\mathcal{E}_\eta = \frac{P C_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{3.86}$$

Substitute 3.70 and 3.85 in 3.69 for the marginal cost Λ_η:

$$P^*_\eta = \frac{\psi}{\psi - 1}\Lambda_\eta \implies \tag{3.69}$$

$$\Lambda_\eta = P\frac{\psi - 1}{\psi} = \Lambda \tag{3.87}$$

Substitute 3.79 and 3.87 in 3.68 for the nominal wage W in terms of region 1:

$$\Lambda_\eta = \frac{1}{Z_{A\eta}} \left(\frac{R_K}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies \tag{3.68}$$

$$W = (1-\alpha_1)\left[\Lambda Z_{A1} \left(\frac{\alpha_1}{R_K}\right)^{\alpha_1}\right]^{\frac{1}{1-\alpha_1}} \tag{3.88}$$

In steady state, prices are the same ($P_\eta = P_\eta^*$), resulting in a gross inflation level of one ($\pi_\eta = 1$), and all firms producing the same output level ($Y_{\eta j} = Y_\eta$) due to the price parity (SOLIS-GARCIA, 2022, Lecture 13, p.12). For this reason, they all demand the same amount of factors (K_η, L_η), and equations 3.65, 3.66 and 3.67 become:

$$Y_\eta = Z_{A\eta} K_\eta^{\alpha_\eta} L_\eta^{1-\alpha_\eta} \tag{3.89}$$

$$K_\eta = \alpha_\eta Y_\eta \frac{\Lambda}{R_K} \tag{3.90}$$

$$L_\eta = (1-\alpha_\eta) Y_\eta \frac{\Lambda}{W} \tag{3.91}$$

Substitute 3.90 in 3.57:

$$I_\eta = \delta K_\eta \implies I_\eta = \delta \alpha_\eta \frac{\Lambda}{R_K} Y_\eta \implies \tag{3.57}$$

$$I_\eta = b_\eta Y_\eta \tag{3.92}$$

$$\text{where:} \quad b_\eta = \delta \alpha_\eta \frac{\Lambda}{R_K} \tag{3.93}$$

Isolate C_η in 3.61 and then substitute 3.86 and 3.91:

$$\frac{\phi L_\eta^\varphi}{C_\eta^{1-\sigma}} = \frac{W}{\mathcal{E}_\eta} \implies C_\eta^{\sigma-1} = \frac{W}{\mathcal{E}_\eta \phi L_\eta^\varphi} \implies$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{3.94}$$

$$\text{where:} \quad a_\eta = \left[\frac{W^{1+\varphi} \omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}{\phi P \left[(1-\alpha_\eta)\Lambda\right]^\varphi} \right]^{\frac{1}{\sigma}} \tag{3.95}$$

Substitute 3.94 in 3.86:

$$\mathcal{E}_\eta = \frac{P C_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \implies \tag{3.86}$$

$$\mathcal{E}_\eta = \frac{P a_\eta Y_\eta^{\frac{-\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{3.96}$$

Substitute 3.96 in 3.58:

$$C_{\eta 1} = \mathcal{E}_{\eta} \frac{\omega_{\eta 1}}{P_1} \Longrightarrow \tag{3.58}$$

$$C_{\eta 1} = \frac{P a_{\eta} Y_{\eta}^{\frac{-\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \cdot \frac{\omega_{\eta 1}}{P} \Longrightarrow$$

$$C_{\eta 1} = \left(\frac{\omega_{\eta 1}}{1-\omega_{\eta 1}}\right)^{1-\omega_{\eta 1}} a_{\eta} Y_{\eta}^{\frac{-\varphi}{\sigma}} \tag{3.97}$$

Substitute 3.96 in 3.59:

$$C_{\eta 2} = \mathcal{E}_{\eta} \frac{1-\omega_{\eta 1}}{P_2} \Longrightarrow \tag{3.59}$$

$$C_{\eta 2} = \frac{P a_{\eta} Y_{\eta}^{\frac{-\varphi}{\sigma}}}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \cdot \frac{1-\omega_{\eta 1}}{P} \Longrightarrow$$

$$C_{\eta 2} = \left(\frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right)^{\omega_{\eta 1}} a_{\eta} Y_{\eta}^{\frac{-\varphi}{\sigma}} \tag{3.98}$$

Substitute 3.92, 3.97 and 3.98 in 3.78:

$$Y_{\eta} = C_{\eta 1} + C_{\eta 2} + I_{\eta} \Longrightarrow \tag{3.78}$$

$$Y_{\eta} = \left(\frac{\omega_{\eta 1}}{1-\omega_{\eta 1}}\right)^{1-\omega_{\eta 1}} a_{\eta} Y_{\eta}^{\frac{-\varphi}{\sigma}} + \left(\frac{1-\omega_{\eta 1}}{\omega_{\eta 1}}\right)^{\omega_{\eta 1}} a_{\eta} Y_{\eta}^{\frac{-\varphi}{\sigma}} + b_{\eta} Y_{\eta} \Longrightarrow$$

$$Y_{\eta} = \left[\left(\frac{a_1}{1-b_1}\right)\left(\frac{1}{\omega_{\eta 1}^{\omega_{\eta 1}} (1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}\right)\right]^{\frac{\sigma}{\sigma+\varphi}} \tag{3.99}$$

The result of 3.99 determines $I_{\eta}, C_{\eta}, \mathcal{E}_{\eta}, C_{\eta 1}, C_{\eta 2}, K_{\eta}, L_{\eta}$ in 3.92, 3.94, 3.86, 3.58, 3.59, 3.90 and 3.91, respectively.

3.2.2 Steady State Solution

$$\vec{\mathbf{1}} = \langle P \quad P_1 \quad P_2 \quad Z_{A1} \quad Z_{A2} \quad Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle \tag{3.100}$$

$$\vec{\mathbf{0}} = \langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle \tag{3.81}$$

$$R = \frac{1}{\beta} - 1 \tag{3.82}$$

$$R_K = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{3.83}$$

$$\Lambda = P\frac{\psi - 1}{\psi} \tag{3.87}$$

$$W = (1-\alpha_1)\left[\Lambda Z_{A1}\left(\frac{\alpha_1}{R_K}\right)^{\alpha_1}\right]^{\frac{1}{1-\alpha_1}} \tag{3.88}$$

$$a_\eta = \left[\frac{W^{1+\varphi}\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}{\phi P\left[(1-\alpha_\eta)\Lambda\right]^{\varphi}}\right]^{\frac{1}{\sigma}} \tag{3.95}$$

$$b_\eta = \delta\alpha_\eta\frac{\Lambda}{R_K} \tag{3.93}$$

$$Y_\eta = \left[\left(\frac{a_1}{1-b_1}\right)\left(\frac{1}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}}\right)\right]^{\frac{\sigma}{\sigma+\varphi}} \tag{3.99}$$

$$I_\eta = b_\eta Y_\eta \tag{3.92}$$

$$C_\eta = a_\eta Y_\eta^{\frac{-\varphi}{\sigma}} \tag{3.94}$$

$$\mathcal{E}_\eta = \frac{PC_\eta}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{1-\omega_{\eta 1}}} \tag{3.86}$$

$$C_{\eta 1} = \mathcal{E}_\eta\frac{\omega_{\eta 1}}{P} \tag{3.58}$$

$$C_{\eta 2} = \mathcal{E}_\eta\frac{1-\omega_{\eta 1}}{P} \tag{3.59}$$

$$K_\eta = \alpha_\eta Y_\eta\frac{\Lambda}{R_K} \tag{3.90}$$

$$L_\eta = (1-\alpha_\eta)Y_\eta\frac{\Lambda}{W} \tag{3.91}$$

3.3 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. Dynare is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation 3.45. For this purpose, Uhlig's rules of log-linearization (UHLIG, 1999) will be applied to all equations in the model.[9]

Regional Price Level

Log-linearize equation 3.46:

$$
\begin{aligned}
P_{\eta t}^{1-\psi} &= \theta P_{\eta,t-1}^{1-\psi} + (1-\theta) P_{\eta t}^{*1-\psi} && \Longrightarrow && (3.46) \\
P^{1-\psi}(1+(1-\psi)\hat{P}_{\eta t}) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{\eta,t-1}) + \\
&+ (1-\theta) P^{1-\psi}(1+(1-\psi)\hat{P}_{\eta t}^{*}) && \Longrightarrow \\
\hat{P}_{\eta t} &= \theta \hat{P}_{\eta,t-1} + (1-\theta)\hat{P}_{\eta t}^{*} && && (3.101)
\end{aligned}
$$

Regional Gross Inflation Rate

Log-linearize 3.50 and define the level deviation of regional inflation rate $\hat{\pi}_{\eta t}$:

$$\pi_{\eta t} = \frac{P_{\eta t}}{P_{\eta,t-1}} \tag{3.50}$$

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{3.102}$$

[9] see lemma A.3 for details.

New Keynesian Phillips Curve

In order to log-linearize equation 3.45, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma A.5:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P^*_{\eta t} Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{3.45}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P^*_{\eta t} Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\qquad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{3.103}$$

First, log-linearize the left hand side of equation 3.103 with respect to $P^*_{\eta t}, Y_{\eta jt}, \widetilde{R}_t$:

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P^*_{\eta t} Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies \\ &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P^*_{\eta} Y_{\eta j} \left(1 + \hat{P}^*_{\eta t} + \hat{Y}_{\eta j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies \\ &P^*_{\eta} Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}^*_{\eta t} + \hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \end{aligned}$$

Separate the terms not dependent on s:

$$\begin{aligned} &P^*_{\eta} Y_{\eta j}(1 + \hat{P}^*_{\eta t}) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \right\} + \\ &\qquad + P^*_{\eta} Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \implies \end{aligned}$$

Apply definition A.9 on the first term:

$$\frac{P^*_{\eta} Y_{\eta j}(1 + \hat{P}^*_{\eta t})}{1 - \theta/(1+R)} + P^*_{\eta} Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{\eta j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\}$$

Second, log-linearize the left hand side of 3.103 with respect to $\Lambda^*_{\eta t}, Y_{\eta jt}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\frac{\theta^s Y_{\eta j,t+s}\Lambda_{\eta,t+s}}{(1+R)^s\left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)}\right\} \Longrightarrow$$

$$\frac{\psi}{\psi-1}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\frac{Y_{\eta j}\Lambda_\eta(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s})}{1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right\} \Longrightarrow$$

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(1+\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\right\}+$$
$$+\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Apply definition A.9 on the first term:

$$\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+$$
$$+\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}$$

Join both sides of the equation again:

$$\frac{P^*_\eta Y_{\eta j}(1+\hat{P}^*_{\eta t})}{1-\theta/(1+R)}+P^*_\eta Y_{\eta j}\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}=$$
$$=\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta/(1+R)}+$$
$$+\frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t\sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s\left(\hat{Y}_{\eta j,t+s}+\hat{\Lambda}_{\eta,t+s}-\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} \quad (3.104)$$

Substitute 3.62 in 3.104:

$$\beta = \frac{1}{(1+R)} \tag{3.62}$$

$$\begin{aligned}
&\frac{P^*_\eta Y_{\eta j}(1+\hat{P}^*_{\eta t})}{1-\theta\beta} + P^*_\eta Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} = \\
&\quad = \frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda_\eta}{1-\theta\beta} + \\
&\quad + \frac{\psi}{\psi-1}Y_{\eta j}\Lambda_\eta\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{Y}_{\eta j,t+s} + \hat{\Lambda}_{\eta,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}
\end{aligned} \tag{3.105}$$

Substitute 3.87 in 3.105 and simplify all common terms:

$$\begin{aligned}
&\cancel{\frac{P^*_\eta Y_{\eta j}}{1-\theta\beta}} + \frac{\cancel{P^*_\eta Y_{\eta j}}\hat{P}^*_{\eta t}}{1-\theta\beta} + \cancel{P^*_\eta Y_{\eta j}}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\cancel{\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right)\right\} = \\
&= \cancel{\frac{P^*_\eta Y_{\eta j}}{1-\theta\beta}} + \cancel{P^*_\eta Y_{\eta j}}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\cancel{\hat{Y}_{\eta j,t+s} - \beta\sum_{k=0}^{s-1}\widetilde{R}_{t+k}} + \hat{\Lambda}_{\eta,t+s}\right)\right\} \implies \\
&\frac{\hat{P}^*_{\eta t}}{1-\theta\beta} = \mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\beta)^s\left(\hat{\Lambda}_{\eta,t+s}\right)\right\}
\end{aligned} \tag{3.106}$$

Define the real marginal cost $\lambda_{\eta t}$:

$$\begin{aligned}
&\lambda_{\eta t} = \frac{\Lambda_{\eta t}}{P_{\eta t}} \implies \Lambda_{\eta t} = P_{\eta t}\lambda_{\eta t} \implies \\
&\hat{\Lambda}_{\eta t} = \hat{P}_{\eta t} + \hat{\lambda}_{\eta t}
\end{aligned} \tag{3.107}$$

Substitute 3.107 in 3.106:

$$\hat{P}^*_{\eta t} = (1-\theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}\right) \tag{3.108}$$

Substitute 3.108 in 3.101:

$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)\hat{P}^*_{\eta t} \tag{3.101}$$

$$\hat{P}_{\eta t} = \theta\hat{P}_{\eta,t-1} + (1-\theta)(1-\theta\beta)\mathbb{E}_t \sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s} + \hat{\lambda}_{\eta,t+s}\right) \tag{3.109}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\beta\mathbb{L}^{-1})$ in 3.109:[10]

$$
\begin{aligned}
(1-\theta\beta\mathbb{L}^{-1})\hat{P}_{\eta t} = {} & (1-\theta\beta\mathbb{L}^{-1})\left[\theta\hat{P}_{\eta,t-1} + \right. \\
& \left. +(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right)\right] \implies \\
\hat{P}_{\eta t}-\theta\beta\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{\eta,t-1}-\theta\beta\theta\hat{P}_{\eta t} + \\
& (1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) - \\
& -\theta\beta(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)
\end{aligned}
\tag{3.110}
$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\beta$ within the operator. Then, cancel the summations and rearrange the terms:

$$
\begin{aligned}
\hat{P}_{\eta t}-\theta\beta\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{\eta,t-1}-\theta\beta\theta\hat{P}_{\eta t} + \\
& (1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{\eta,t+s}+\hat{\lambda}_{\eta,t+s}\right) - \\
& -\theta\beta(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right) \implies \\
\hat{P}_{\eta t}-\theta\beta\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{\eta,t-1}-\theta\beta\theta\hat{P}_{\eta t}+(1-\theta)(1-\theta\beta)(\hat{P}_{\eta t}+\hat{\lambda}_{\eta t}) + \\
& +\cancel{(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)} - \\
& -\cancel{(1-\theta)(1-\theta\beta)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\beta)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{\eta,t+s+1}\right)} \implies \\
\hat{P}_{\eta t}-\theta\beta\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{\eta,t-1}-\theta^2\beta\hat{P}_{\eta t}+ \\
& +(1-\theta-\theta\beta+\theta^2\beta)\hat{P}_{\eta t}+(1-\theta)(1-\theta\beta)\hat{\lambda}_{\eta t} \implies \\
(\hat{P}_{\eta t}-\hat{P}_{\eta,t-1}) = {} & \beta(\mathbb{E}_t\hat{P}_{t+1}-\hat{P}_{\eta t})+\frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{\eta t}
\end{aligned}
\tag{3.111}
$$

[10] see definition A.10.

Substitute 3.102 in 3.111:

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\lambda}_{\eta t} \tag{3.112}$$

Equation 3.112 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

Law of Motion for Capital

Log-linearize 3.3:

$$\begin{aligned} K_{\eta,t+1} &= (1-\delta)K_{\eta t} + I_{\eta t} && \Longrightarrow && (3.3) \\ K_\eta(1+\hat{K}_{\eta,t+1}) &= (1-\delta)K_\eta(1+\hat{K}_{\eta t}) + I_\eta(1+\hat{I}_{\eta t}) && \Longrightarrow \\ \hat{K}_{\eta,t+1} &= (1-\delta)\hat{K}_{\eta t} + \delta \hat{I}_{\eta t} && && (3.113) \end{aligned}$$

Regional Levels of Consumption and Prices

Log-linearize 3.19:

$$\begin{aligned} C_{\eta 1t} &= \mathcal{E}_{\eta t}\frac{\omega_{\eta 1}}{P_{1t}} && \Longrightarrow && (3.19) \\ C_{\eta 1}(1+\hat{C}_{\eta 1t}) &= \mathcal{E}_\eta \frac{\omega_{\eta 1}}{P_1}(1+\hat{\mathcal{E}}_{\eta t} - \hat{P}_{1t}) && \Longrightarrow \\ \hat{C}_{\eta 1t} &= \hat{\mathcal{E}}_{\eta t} - \hat{P}_{1t} \iff \hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 1t} + \hat{P}_{1t} && && (3.114) \end{aligned}$$

Log-linearize 3.18:

$$\begin{aligned} C_{\eta 2t} &= \mathcal{E}_{\eta t}\frac{1-\omega_{\eta 1}}{P_{2t}} && \Longrightarrow && (3.18) \\ C_{\eta 2}(1+\hat{C}_{\eta 2t}) &= \mathcal{E}_\eta \frac{1-\omega_{\eta 1}}{P_2}(1+\hat{\mathcal{E}}_{\eta t} - \hat{P}_{2t}) && \Longrightarrow \\ \hat{C}_{\eta 2t} &= \hat{\mathcal{E}}_{\eta t} - \hat{P}_{2t} \iff \hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 2t} + \hat{P}_{2t} && && (3.115) \end{aligned}$$

Match equations 3.114 and 3.115:

$$\hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta 1t} + \hat{P}_{1t} = \hat{C}_{\eta 2t} + \hat{P}_{2t} \implies$$
$$\hat{C}_{\eta 1t} - \hat{C}_{\eta 2t} = \hat{P}_{2t} - \hat{P}_{1t} \tag{3.116}$$

Equation 3.116 shows that the variation of both goods consumption in each region correspond to the distance between the variations of the regional price levels.

Total Expenses

Log-linearize 3.17:

$$\mathcal{E}_{\eta t} = C_{\eta t}\left[\frac{P_{1t}}{\omega_{\eta 1}}\right]^{\omega_{\eta 1}}\left[\frac{P_{2t}}{1-\omega_{\eta 1}}\right]^{1-\omega_{\eta 1}} \implies \tag{3.17}$$
$$\mathcal{E}_{\eta}(1+\hat{\mathcal{E}}_{\eta t}) = \frac{C_{\eta}P_1^{\omega_{\eta 1}}P_2^{1-\omega_{\eta 1}}}{\omega_{\eta 1}^{\omega_{\eta 1}}(1-\omega_{\eta 1})^{(1-\omega_{\eta 1})}}\cdot$$
$$\cdot\,(1+\hat{C}_{\eta t}+\omega_{\eta 1}\hat{P}_{1t}+(1-\omega_{\eta 1})\hat{P}_{2t}) \implies$$
$$\hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta t}+\omega_{\eta 1}\hat{P}_{1t}+(1-\omega_{\eta 1})\hat{P}_{2t} \tag{3.117}$$

Labor Supply

Log-linearize 3.21:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{1-\sigma}} = \frac{W_t}{\mathcal{E}_{\eta t}} \implies \tag{3.21}$$
$$\varphi\hat{L}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} = \hat{W}_t - \hat{\mathcal{E}}_{\eta t} \tag{3.118}$$

Euler equation for the bonds return

Log-linearize 3.22:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} = \beta(1+R_t)\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}\right\} \qquad \Longrightarrow \tag{3.22}$$

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}} \cdot \frac{\mathbb{E}_t\mathcal{E}_{\eta t+1}}{\mathbb{E}_t C_{\eta,t+1}^{1-\sigma}} = \beta + \beta R_t \qquad \Longrightarrow$$

$$\frac{C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \cdot \frac{\mathcal{E}_{\eta}}{C_{\eta}^{1-\sigma}}(1+(1-\sigma)\hat{C}_{\eta t} - \hat{\mathcal{E}}_{\eta t} + \mathbb{E}_t\{\hat{\mathcal{E}}_{\eta t+1} - \hat{C}_{\eta t+1}\}) =$$

$$= \beta + \beta(R(1+R_t)) \implies$$

$$\mathbb{E}_t\left\{\hat{\mathcal{E}}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1}\right\} - [\hat{\mathcal{E}}_{\eta t} - (1-\sigma)\hat{C}_{\eta t}] = (1-\beta)\hat{R}_t \tag{3.119}$$

Euler equation for the capital return

Log-linearize 3.23:

$$\frac{C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}}P_{\eta t} = \beta\mathbb{E}_t\left\{\frac{C_{\eta,t+1}^{1-\sigma}}{\mathcal{E}_{\eta t+1}}[(1-\delta)P_{\eta,t+1} + R_{K,t+1}]\right\} \Longrightarrow \tag{3.23}$$

$$\frac{P_{\eta t}C_{\eta t}^{1-\sigma}}{\mathcal{E}_{\eta t}}\mathbb{E}_t\left\{\frac{\mathcal{E}_{\eta t+1}}{P_{\eta,t+1}C_{\eta,t+1}^{1-\sigma}}\right\} = \beta\left[(1-\delta) + \mathbb{E}_t\left\{\frac{R_{K,t+1}}{P_{\eta,t+1}}\right\}\right] \Longrightarrow$$

$$\frac{P_{\eta}C_{\eta}^{1-\sigma}}{\mathcal{E}_{\eta}} \cdot \frac{\mathcal{E}_{\eta}}{P_{\eta}C_{\eta}^{1-\sigma}}(1+\hat{P}_{\eta t} + (1-\sigma)\hat{C}_{\eta t} - \hat{\mathcal{E}}_{\eta t} +$$

$$+ \mathbb{E}_t\{\hat{\mathcal{E}}_{\eta,t+1} - \hat{P}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1}\}) =$$

$$= \beta\left[(1-\delta) + \frac{R_K}{P_\eta}\mathbb{E}_t\left\{1 + \hat{R}_{K,t+1} - \hat{P}_{\eta,t+1}\right\}\right] \Longrightarrow$$

$$\mathbb{E}_t\left\{\hat{\mathcal{E}}_{\eta,t+1} - \hat{P}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1}\right\} - (\hat{\mathcal{E}}_{\eta t} - \hat{P}_{\eta t} - (1-\sigma)\hat{C}_{\eta t}) =$$

$$= \beta\frac{R_K}{P_\eta}\mathbb{E}_t\left\{\hat{R}_{K,t+1} - \hat{P}_{\eta,t+1}\right\} \tag{3.120}$$

Bundle Technology

Apply the natural logarithm to 3.25:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \implies \tag{3.25}$$

$$\ln Y_{\eta t} = \frac{\psi}{\psi-1}\ln\left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)$$

Log-linearize using corollary A.3.1:

$$\ln Y_\eta + \hat{Y}_{\eta t} = \frac{\psi}{\psi-1}\left[\ln\left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y_\eta + \hat{Y}_{\eta t} = \frac{\psi}{\psi-1}\left[\ln\left(Y_{\eta j}^{\frac{\psi-1}{\psi}}\int_0^1 \mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y_\eta + \hat{Y}_{\eta t} = \frac{\cancel{\psi}}{\cancel{\psi}-1}\left[\frac{\psi-\cancel{1}}{\cancel{\psi}}\ln Y_{\eta j} + \cancel{\ln 1} + \frac{\psi-\cancel{1}}{\cancel{\psi}}\int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y_\eta + \hat{Y}_{\eta t} = \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j$$

Apply corollary A.2.1:

$$\ln Y_\eta + \hat{Y}_{\eta t} = \ln Y_{\eta j} + \int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j \implies$$

$$\hat{Y}_{\eta t} = \int_0^1 \hat{Y}_{\eta j t}\,\mathrm{d}\,j \tag{3.121}$$

Production Function

Log-linearize 3.30:

$$Y_{\eta j t} = Z_{A\eta t} K_{\eta j t}^{\alpha_\eta} L_{\eta j t}^{1-\alpha_\eta} \implies \tag{3.30}$$

$$Y_{\eta j}(1+\hat{Y}_{\eta j t}) = Z_{A\eta} K_{\eta j}^{\alpha_\eta} L_{\eta j}^{1-\alpha_\eta}(1+\hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{\eta j t} + (1-\alpha_\eta)\hat{L}_{\eta j t}) \implies$$

$$\hat{Y}_{\eta j t} = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{\eta j t} + (1-\alpha_\eta)\hat{L}_{\eta j t} \tag{3.122}$$

Substitute 3.122 in 3.121:

$$\hat{Y}_{\eta t} = \int_0^1 \hat{Y}_{\eta j t} \, \mathrm{d}\, j \quad \Longrightarrow \tag{3.121}$$

$$\hat{Y}_{\eta t} = \int_0^1 \left[\hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{\eta j t} + (1-\alpha_\eta)\hat{L}_{\eta j t}\right] \mathrm{d}\, j \quad \Longrightarrow$$

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha_\eta \int_0^1 \hat{K}_{\eta j t} \, \mathrm{d}\, j + (1-\alpha_\eta) \int_0^1 \hat{L}_{\eta j t} \, \mathrm{d}\, j \tag{3.123}$$

Apply the natural logarithm and then log-linearize 3.55:

$$L_{\eta t} = \int_0^1 L_{\eta j t} \, \mathrm{d}\, j \quad \Longrightarrow \tag{3.55}$$

$$\ln L_{\eta t} = \ln \left[\int_0^1 L_{\eta j t} \, \mathrm{d}\, j\right] \quad \Longrightarrow$$

$$\ln L + \hat{L}_{\eta t} = \ln \left[\int_0^1 L_{\eta j} \, \mathrm{d}\, j\right] + \int_0^1 \hat{L}_{\eta j t} \, \mathrm{d}\, j \quad \Longrightarrow$$

$$\ln L + \hat{L}_{\eta t} = \ln L_{\eta j} + \ln 1 + \int_0^1 \hat{L}_{\eta j t} \, \mathrm{d}\, j$$

Apply corollary A.2.1:

$$\Longrightarrow \hat{L}_{\eta t} = \int_0^1 \hat{L}_{\eta j t} \, \mathrm{d}\, j \tag{3.124}$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_{\eta t} = \int_0^1 \hat{K}_{\eta j t} \, \mathrm{d}\, j \tag{3.125}$$

Substitute 3.124 and 3.125 in 3.123:

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha_\eta \int_0^1 \hat{K}_{\eta j t} \, \mathrm{d}\, j + (1-\alpha_\eta) \int_0^1 \hat{L}_{\eta j t} \, \mathrm{d}\, j \quad \Longrightarrow \tag{3.123}$$

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{\eta t} + (1-\alpha_\eta)\hat{L}_{\eta t} \tag{3.126}$$

Capital Demand

Log-linearize 3.32:

$$\begin{aligned} K_{\eta jt} &= \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_{Kt}} && \implies && (3.32) \\ K_{\eta j}(1 + \hat{K}_{\eta jt}) &= \alpha_\eta Y_{\eta j} \frac{\Lambda_\eta}{R_K}(1 + \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt}) && \implies \\ \hat{K}_{\eta jt} &= \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt} \end{aligned}$$

Integrate both sides and then substitute 3.125 and 3.121:

$$\begin{aligned} \int_0^1 \hat{K}_{\eta jt} \, \mathrm{d}\, j &= \int_0^1 (\hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt}) \, \mathrm{d}\, j \quad \implies \\ \hat{K}_{\eta t} &= \hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt} \end{aligned} \tag{3.127}$$

Labor Demand

Log-linearize 3.33:

$$\begin{aligned} L_{\eta jt} &= (1 - \alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} && \implies && (3.33) \\ L_{\eta j}(1 + \hat{L}_{\eta jt}) &= (1 - \alpha_\eta) Y_{\eta j} \frac{\Lambda_\eta}{W}(1 + \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{W}_t) && \implies \\ \hat{L}_{\eta jt} &= \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{W}_t \end{aligned}$$

Integrate both sides and then substitute 3.124 and 3.121:

$$\begin{aligned} \int_0^1 \hat{L}_{\eta jt} \, \mathrm{d}\, j &= \int_0^1 \hat{Y}_{\eta jt} + \hat{\Lambda}_{\eta t} - \hat{W}_t \, \mathrm{d}\, j \implies \\ \hat{L}_{\eta t} &= \hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{W}_t \end{aligned} \tag{3.128}$$

Subtract 3.128 from 3.127:

$$\begin{aligned} &\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{R}_{Kt} - (\hat{Y}_{\eta t} + \hat{\Lambda}_{\eta t} - \hat{W}_t) \implies \\ &\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{W}_t - \hat{R}_{Kt} \end{aligned} \tag{3.129}$$

Equation 3.129 is the log-linearized version of 3.34.

Marginal Cost

Log-linearize 3.39:

$$\Lambda_{\eta t} = Z_{A\eta t}^{-1} \frac{R_{Kt}^{\alpha_\eta} W_t^{1-\alpha_\eta}}{\alpha_\eta{}^{\alpha_\eta}(1-\alpha_\eta)^{1-\alpha_\eta}} \implies \tag{3.39}$$

$$\Lambda(1+\hat{\Lambda}_{\eta t}) =$$

$$= \frac{1}{Z_{A\eta}} \left(\frac{R_K}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} (1 - \hat{Z}_{A\eta t} + \alpha_\eta \hat{R}_{Kt} + (1-\alpha_\eta)\hat{W}_t) \implies$$

$$\hat{\Lambda}_{\eta t} = \alpha_\eta \hat{R}_{Kt} + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \tag{3.130}$$

Substitute 3.107 in 3.130:

$$\hat{\Lambda}_{\eta t} = \alpha_\eta \hat{R}_{Kt} + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \implies$$

$$\hat{P}_{\eta t} + \hat{\lambda}_{\eta t} = \alpha_\eta \hat{R}_{Kt} + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} \implies$$

$$\hat{\lambda}_{\eta t} = \alpha_\eta \hat{R}_{Kt} + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_{\eta t} \tag{3.131}$$

Monetary Policy

Log-linearize 3.47:

$$\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi} Y_t^{\gamma_Y})^{(1-\gamma_R)} Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \implies \tag{3.47}$$

$$\frac{R(1+\hat{R}_t)}{R} = \frac{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)} Z_M}{R^{\gamma_R}(\pi^{\gamma_\pi} Y^{\gamma_Y})^{(1-\gamma_R)}} \cdot$$

$$\cdot [1 + \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt}] \implies$$

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{3.132}$$

National Gross Inflation Rate

Log-linearize 3.48 and define the level deviation of gross inflation rate $\hat{\pi}_t$:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \tag{3.48}$$

$$\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{3.133}$$

National Price Level

Log-linearize 3.49:

$$\begin{aligned} P_t Y_t &= P_{1t} Y_{1t} + P_{2t} Y_{2t} && \Longrightarrow && (3.49) \\ PY(1+\hat{P}_t+\hat{Y}_t) &= P_1 Y_1 (1+\hat{P}_{1t}+\hat{Y}_{1t}) + P_2 Y_2 (1+\hat{P}_{2t}+\hat{Y}_{2t}) && \Longrightarrow && \\ \hat{P}_t+\hat{Y}_t &= \frac{P_1 Y_1}{PY}(\hat{P}_{1t}+\hat{Y}_{1t}) + \frac{P_2 Y_2}{PY}(\hat{P}_{2t}+\hat{Y}_{2t}) && && (3.134) \end{aligned}$$

Define the regional weights $\left\langle \theta_{PY1} \quad (1-\theta_{PY1}) \right\rangle$ in the gross domestic production:

$$\left\langle \theta_{PY1} \quad (1-\theta_{PY1}) \right\rangle := \left\langle \tfrac{P_1 Y_1}{PY} \quad \tfrac{P_2 Y_2}{PY} \right\rangle \tag{3.135}$$

Substitute 3.135 in 3.134:

$$\begin{aligned} \hat{P}_t+\hat{Y}_t &= \frac{P_1 Y_1}{PY}(\hat{P}_{1t}+\hat{Y}_{1t}) + \frac{P_2 Y_2}{PY}(\hat{P}_{2t}+\hat{Y}_{2t}) \quad \Longrightarrow && (3.134) \\ \hat{P}_t+\hat{Y}_t &= \theta_{PY1}(\hat{P}_{1t}+\hat{Y}_{1t}) + (1-\theta_{PY1})(\hat{P}_{2t}+\hat{Y}_{2t}) && (3.136) \end{aligned}$$

Productivity Shock

Log-linearize 3.51:

$$\begin{aligned} \ln Z_{A\eta t} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} && \Longrightarrow && (3.51) \\ \ln Z_{A\eta} + \hat{Z}_{A\eta t} &= (1-\rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} (\ln Z_{A\eta} + \hat{Z}_{A,t-1}) + \varepsilon_{A\eta} && \Longrightarrow && \\ \hat{Z}_{A\eta t} &= \rho_{A\eta} \hat{Z}_{A,t-1} + \varepsilon_{A\eta} && && (3.137) \end{aligned}$$

Monetary Shock

Log-linearize 3.52:

$$\begin{aligned} \ln Z_{Mt} &= (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \Longrightarrow && (3.52) \\ \ln Z_M + \hat{Z}_{Mt} &= (1-\rho_M) \ln Z_M + \rho_M (\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M && \Longrightarrow && \\ \hat{Z}_{Mt} &= \rho_M \hat{Z}_{M,t-1} + \varepsilon_M && && (3.138) \end{aligned}$$

Market Clearing Condition

Log-linearize 3.53:

$$
\begin{aligned}
Y_t &= Y_{1t} + Y_{2t} && (3.53) \\
Y(1+\hat{Y}_t) &= Y_1(1+\hat{Y}_{1t}) + Y_2(1+\hat{Y}_{2t}) \implies \\
\hat{Y}_t &= \frac{Y_1}{Y}\hat{Y}_{1t} + \frac{Y_2}{Y}\hat{Y}_{2t} && (3.139)
\end{aligned}
$$

Define the regional weights $\left\langle \theta_{Y1} \quad (1-\theta_{Y1}) \right\rangle$ in the production total:

$$
\left\langle \theta_{Y1} \quad (1-\theta_{Y1}) \right\rangle := \left\langle \tfrac{Y_1}{Y} \quad \tfrac{Y_2}{Y} \right\rangle \tag{3.140}
$$

Substitute 3.140 in 3.139:

$$
\hat{Y}_t = \theta_{Y1}\hat{Y}_{1t} + (1-\theta_{Y1})\hat{Y}_{2t} \tag{3.141}
$$

Regional Market Clearing Condition

Log-linearize 3.54:

$$
\begin{aligned}
Y_{\eta t} &= C_{\eta 1t} + C_{\eta 2t} + I_{\eta t} \implies && (3.54) \\
Y_\eta(1+\hat{Y}_{\eta t}) &= C_{\eta 1}(1+\hat{C}_{\eta 1t}) + C_{\eta 2}(1+\hat{C}_{\eta 2t}) + I_\eta(1+\hat{I}_{\eta t}) \implies \\
\hat{Y}_{\eta t} &= \frac{C_{\eta 1}}{Y_\eta}\hat{C}_{\eta 1t} + \frac{C_{\eta 2}}{Y_\eta}\hat{C}_{\eta 2t} + \frac{I_\eta}{Y_\eta}\hat{I}_{\eta t} && (3.142)
\end{aligned}
$$

Define the consumption and investment weights $\left\langle \theta_{C\eta 1} \; \theta_{C\eta 2} \; (1-\theta_{C\eta 1}-\theta_{C\eta 2}) \right\rangle$ in the regional production:

$$
\left\langle \theta_{C\eta 1} \quad \theta_{C\eta 2} \quad (1-\theta_{C\eta 1}-\theta_{C\eta 2}) \right\rangle := \left\langle \tfrac{C_{\eta 1}}{Y_\eta} \quad \tfrac{C_{\eta 2}}{Y_\eta} \quad \tfrac{I_\eta}{Y_\eta} \right\rangle \tag{3.143}
$$

Substitute 3.143 in 3.142:

$$
\hat{Y}_{\eta t} = \theta_{C\eta 1}\hat{C}_{\eta 1t} + \theta_{C\eta 2}\hat{C}_{\eta 2t} + (1-\theta_{C\eta 1}-\theta_{C\eta 2})\hat{I}_{\eta t} \tag{3.144}
$$

3.3.1 Log-linear Model Structure

The log-linear model is a square system of 31 variables and 31 equations, summarized as follows:

- Variables:
  Real Variables: $\langle\, \hat{C}_\eta\ \hat{C}_{\eta 1}\ \hat{C}_{\eta 2}\ \hat{L}_\eta\ \hat{K}_\eta\ \hat{Y}\ \hat{Y}_\eta\ \hat{I}_\eta\ \hat{Z}_{A\eta} \,\rangle$;
  Nominal Variables: $\langle\, \hat{\mathcal{E}}_\eta\ \hat{P}\ \hat{P}_\eta\ \hat{R}\ \hat{R}_K\ \hat{W}\ \hat{\pi}\ \hat{\pi}_\eta\ \hat{\lambda}_\eta\ \hat{Z}_M \,\rangle$.
- Equations:

1. Regional Gross Inflation Rate

$$\hat{\pi}_{\eta t} = \hat{P}_{\eta t} - \hat{P}_{\eta,t-1} \tag{3.102}$$

2. New Keynesian Phillips Curve

$$\hat{\pi}_{\eta t} = \beta \mathbb{E}_t \hat{\pi}_{\eta,t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\hat{\lambda}_{\eta t} \tag{3.112}$$

3. Law of Motion for Capital

$$\hat{K}_{\eta,t+1} = (1-\delta)\hat{K}_{\eta t} + \delta \hat{I}_{\eta t} \tag{3.113}$$

4. Regional Levels of Consumption and Prices

$$\hat{C}_{\eta 1t} - \hat{C}_{\eta 2t} = \hat{P}_{2t} - \hat{P}_{1t} \tag{3.116}$$

5. Total Expenses

$$\hat{\mathcal{E}}_{\eta t} = \hat{C}_{\eta t} + \omega_{\eta 1}\hat{P}_{1t} + (1-\omega_{\eta 1})\hat{P}_{2t} \tag{3.117}$$

6. Labor Supply

$$\varphi \hat{L}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} = \hat{W}_t - \hat{\mathcal{E}}_{\eta t} \tag{3.118}$$

7. Euler equation for the bonds return

$$\begin{aligned} &\mathbb{E}_t \left\{ \hat{\mathcal{E}}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1} \right\} - \\ &\quad - \left[\hat{\mathcal{E}}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} \right] = (1-\beta)\hat{R}_t \end{aligned} \tag{3.119}$$

8. Euler equation for the capital return

$$\begin{aligned} &\mathbb{E}_t \left\{ \hat{\mathcal{E}}_{\eta,t+1} - \hat{P}_{\eta,t+1} - (1-\sigma)\hat{C}_{\eta,t+1} \right\} - \left(\hat{\mathcal{E}}_{\eta t} - \hat{P}_{\eta t} - (1-\sigma)\hat{C}_{\eta t} \right) = \\ &\quad = \beta \frac{R_K}{P_\eta} \mathbb{E}_t \left\{ \hat{R}_{K,t+1} - \hat{P}_{\eta,t+1} \right\} \end{aligned} \tag{3.120}$$

9. Production Function

$$\hat{Y}_{\eta t} = \hat{Z}_{A\eta t} + \alpha_\eta \hat{K}_{\eta t} + (1-\alpha_\eta)\hat{L}_{\eta t} \tag{3.126}$$

10. Marginal Rates of Substitution of Factors

$$\hat{K}_{\eta t} - \hat{L}_{\eta t} = \hat{W}_t - \hat{R}_{Kt} \tag{3.129}$$

11. Marginal Cost

$$\hat{\lambda}_{\eta t} = \alpha_\eta \hat{R}_{Kt} + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{A\eta t} - \hat{P}_{\eta t} \tag{3.131}$$

12. Monetary Policy

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi \hat{\pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{3.132}$$

13. National Gross Inflation Rate

$$\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{3.133}$$

14. National Price Level

$$\hat{P}_t + \hat{Y}_t = \theta_{PY1}(\hat{P}_{1t} + \hat{Y}_{1t}) + (1-\theta_{PY1})(\hat{P}_{2t} + \hat{Y}_{2t}) \tag{3.134}$$

15. Productivity Shock

$$\hat{Z}_{A\eta t} = \rho_{A\eta}\hat{Z}_{A,t-1} + \varepsilon_{A\eta} \tag{3.137}$$

16. Monetary Shock

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{3.138}$$

17. Market Clearing Condition

$$\hat{Y}_t = \theta_{Y1}\hat{Y}_{1t} + (1-\theta_{Y1})\hat{Y}_{2t} \tag{3.141}$$

18. Regional Market Clearing Condition

$$\hat{Y}_{\eta t} = \theta_{C\eta 1}\hat{C}_{\eta 1t} + \theta_{C\eta 2}\hat{C}_{\eta 2t} + (1-\theta_{C\eta 1} - \theta_{C\eta 2})\hat{I}_{\eta t} \tag{3.144}$$

3.4 Data

In this section, the data necessary to estimate the model parameters will be discussed using descriptive statistics. The intention is to demonstrate, through graphics and tables, a visual correspondence between the nominal interest rate and the gross domestic product of select Brazilian states, particularly those specialized in agriculture and industries. This emphasis aims to show that regional differences play an important role in how a region will react to monetary policy.

3.4.1 Parameter Calibration

Table 1: Parameter Calibration

Parameter	Definition	Calibration
α_1	capital elasticity of production in region 1	0.4
α_2	capital elasticity of production in region 2	0.2
β	intertemporal discount factor	0.985
γ_R	interest-rate smoothing parameter	0.79
γ_π	interest-rate sensitivity in relation to inflation	2.43
γ_Y	interest-rate sensitivity in relation to product	0.16
δ	capital depreciation rate	0.025
θ	price stickness parameter	0.8
θ_{C11}	weight of good 1 in demand of region 1	0.4
θ_{C12}	weight of good 2 in demand of region 1	0.4
θ_{C21}	weight of good 1 in demand of region 2	0.4
θ_{C22}	weight of good 2 in demand of region 2	0.4
θ_{PY1}	weight of region 1 in gross domestic product	0.3
θ_{Y1}	weight of region 1 in total production	0.3
ρ_{A1}	autoregressive parameter of productivity in region 1	0.95
ρ_{A2}	autoregressive parameter of productivity in region 2	0.95
ρ_M	autoregressive parameter of monetary policy	0.9
σ	relative risk aversion coefficient	2
ϕ	relative labor weight in utility	1
φ	marginal disutility of labor supply	1.5
ψ	elasticity of substitution between intermediate goods	8
ω_{11}	weight of good 1 in consumption composition of region 1	0.5
ω_{21}	weight of good 1 in consumption composition of region 2	0.5

Sources: The Author and Costa Junior (2016)

3.4.2 Variables at Steady State

Table 2: Variables at Steady State

Variable	Steady State Value
$\langle P \quad P_1 \quad P_2 \quad Z_{A1} \quad Z_{A2} \quad Z_M \quad \pi \quad \pi_1 \quad \pi_2 \rangle$	$\vec{\mathbf{1}}$
$\langle \varepsilon_{A1} \quad \varepsilon_{A2} \quad \varepsilon_M \rangle$	$\vec{\mathbf{0}}$
R	0.0402
R_K	0.0402
Λ	0.8750
W	1.6967
$\langle a_1 \quad a_2 \rangle$	$\langle a_1 \quad a_2 \rangle$
$\langle b_1 \quad b_2 \rangle$	$\langle b_1 \quad b_2 \rangle$
$\langle Y_1 \quad Y_2 \rangle$	$\langle Y_1 \quad Y_2 \rangle$
$\langle I_1 \quad I_2 \rangle$	$\langle I_1 \quad I_2 \rangle$
$\langle C_1 \quad C_2 \rangle$	$\langle C_1 \quad C_2 \rangle$
$\langle \mathcal{E}_1 \quad \mathcal{E}_2 \rangle$	$\langle \mathcal{E}_1 \quad \mathcal{E}_2 \rangle$
$\langle C_{11} \quad C_{12} \rangle$	$\langle C_{11} \quad C_{12} \rangle$
$\langle C_{21} \quad C_{22} \rangle$	$\langle C_{21} \quad C_{22} \rangle$
$\langle K_1 \quad K_2 \rangle$	$\langle K_1 \quad K_2 \rangle$
$\langle L_1 \quad L_2 \rangle$	$\langle L_1 \quad L_2 \rangle$

Source: The Author.

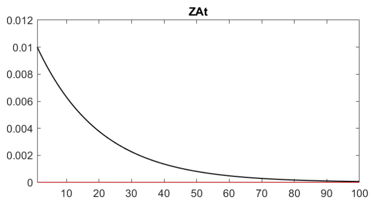
4 Results

Following the data, a Bayesian estimation will be performed to estimate the model parameters.

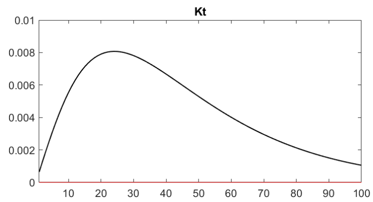
In due time, a thorough analysis of the results will be conducted.

4.1 Impulse Response Functions

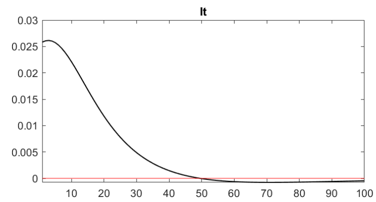
4.1.1 Productivity Shock



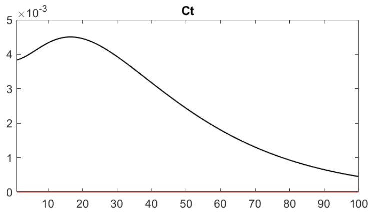
(a) Productivity Shock



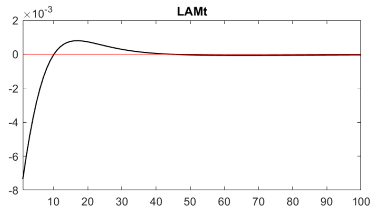
(b) Capital



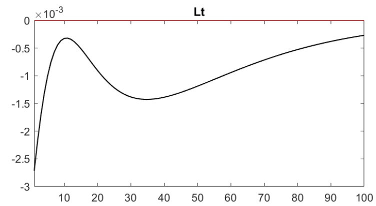
(c) Investment



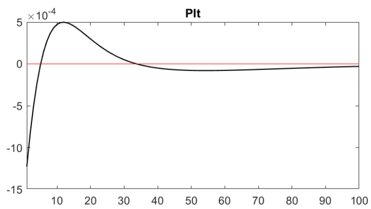
(d) Consumption



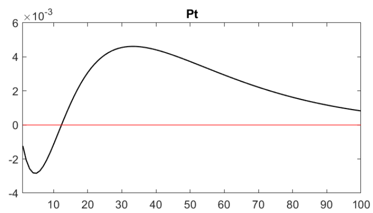
(e) Marginal Cost



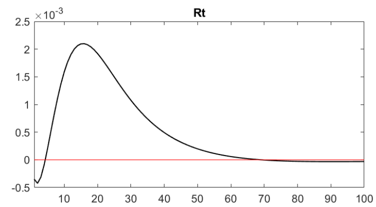
(f) Labor



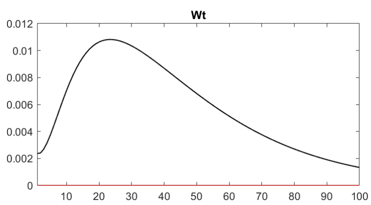
(g) Inflation



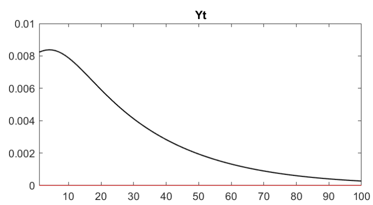
(h) Price Level



(i) Interest Rate



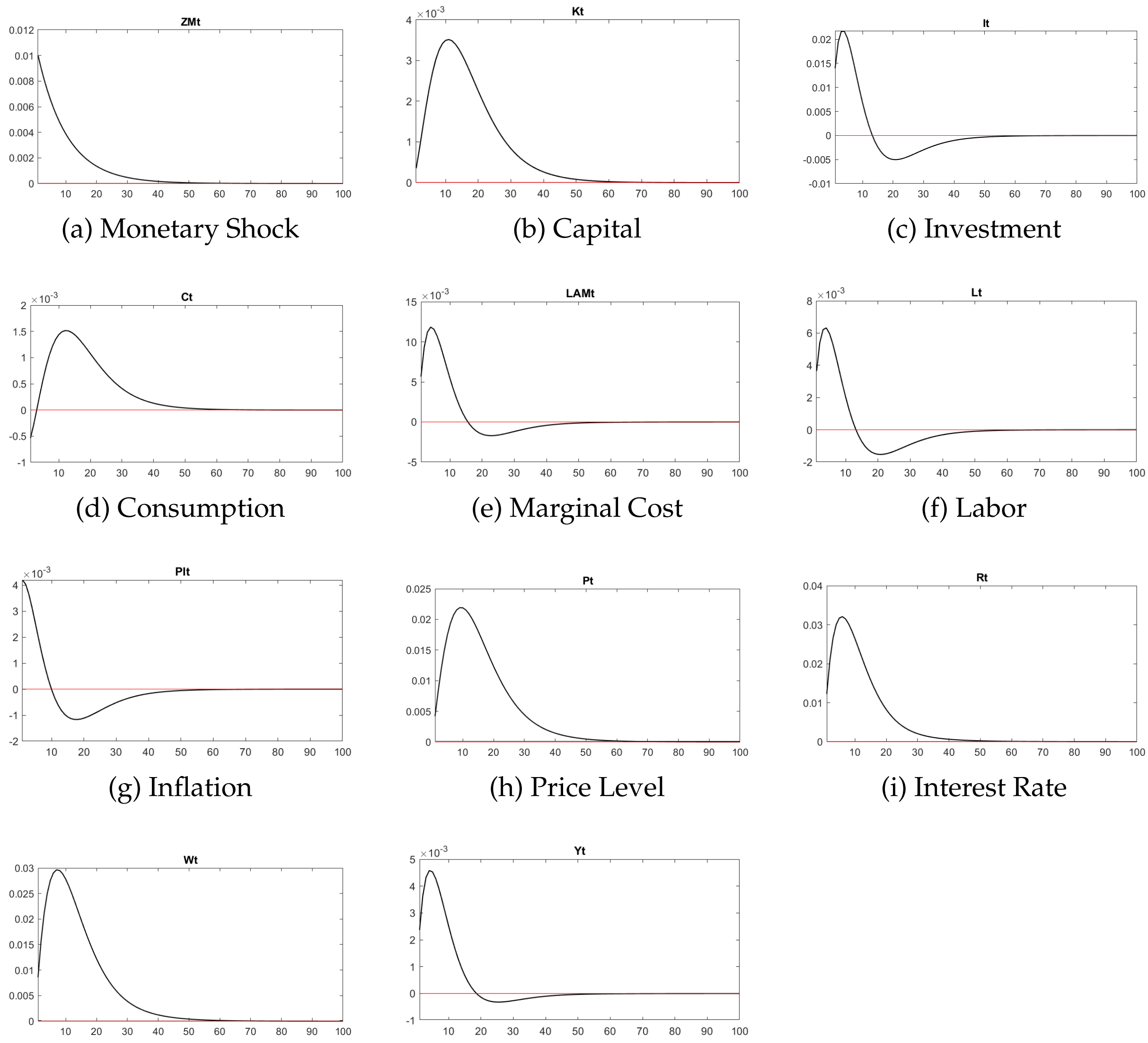
(j) Wage



(k) Production

Figure 2: Productivity Shock Impulse Response Functions

4.1.2 Monetary Shock



(a) Monetary Shock (b) Capital (c) Investment (d) Consumption (e) Marginal Cost (f) Labor (g) Inflation (h) Price Level (i) Interest Rate (j) Wage (k) Production

Figure 3: Monetary Shock Impulse Response Functions

4.2 Parametrization

To be done.

This section will summarize and discuss the main findings, implications, and potential future work related to your research.

References

ALBUQUERQUEMELLO, V. P. d. **Mercado imobiliário, crédito e o Ciclo Real de Negócios: evidências a partir de um modelo DSGE para a economia estadunidense**. 2018. PhD thesis. Cit. on p. 19.

ALPANDA, S.; AYSUN, U. International transmission of financial shocks in an estimated DSGE model. **Journal of International Money and Finance**, v. 47, Oct. 2014. DOI: 10.1016/j.jimonfin.2014.04.007. Cit. on p. 20.

BERGHOLT, D. The Basic New Keynesian Model. [S.l.], 2012. Available from: <https://bergholt.weebly.com/uploads/1/1/8/4/11843961/the_basic_new_keynesian_model_-_drago_bergholt.pdf>. Cit. on p. 19.

BERTANHA, M.; HADDAD, E. A. Efeitos regionais da política monetária no Brasil: impactos e transbordamentos espaciais. **Revista Brasileira de Economia**, Mar. 2008. DOI: 10.1590/S0034-71402008000100001. Cit. on p. 15.

CALVO, G. A. Staggered Prices In a Utility-maximizing Framework. **Journal of Monetary Economics**, Sept. 1983. DOI: 10.1016/0304-3932(83)90060-0. Cit. on pp. 23, 35, 103.

CAPELETI, P.; GARCIA, M.; MIESSI SANCHES, F. Countercyclical credit policies and banking concentration: Evidence from Brazil. **Journal of Banking & Finance**, Oct. 2022. DOI: 10.1016/j.jbankfin.2022.106589. Cit. on p. 21.

CORBO, V.; STRID, I. **MAJA A two-region DSGE model for Sweden and its main trading partners**. [S.l.]: Sveriges Riksbank Working Paper Series, 2020. Available from: <https://www.econstor.eu/handle/10419/232594>. Cit. on p. 20.

COSTA JUNIOR, C. J. **Understanding DSGE**. Wilmington, Delaware: Vernon Press, 2016. (Vernon series in economic methodology). Cit. on pp. 19, 23, 26, 42, 68, 97, 108.

COSTA JUNIOR, C. J.; TEIXEIRA, A. M.; SILVA, M. F. d. DSGE para Macroeconomia Regional: Uma Aplicação para o Estado de Goiás. In: ENCONTRO ANPEC. Cit. on p. 18.

CROITOROV, O.; GIOVANNINI, M.; HOHBERGER, S.; RATTO, M.; VOGEL, L. Financial spillover and global risk in a multi-region model of the world economy. **Journal of Economic Behavior & Organization**, Sept. 2020. DOI: 10.1016/j.jebo.2020.05.024. Cit. on p. 20.

GALÍ, J. **Monetary Policy, Inflation, And The Business Cycle: An Introduction To The New Keynesian Framework And Its Applications**. Second edition. Princeton ; Oxford: Princeton University Press, 2015. Cit. on pp. 15, 19, 20, 84.

GALÍ, J.; MONACELLI, T. Optimal Fiscal Policy in a Monetary Union. **Proceedings**, 2005. Publisher: Federal Reserve Bank of San Francisco. Cit. on p. 20.

HOLM, M. B.; PAUL, P.; TISCHBIREK, A. The Transmission of Monetary Policy under the Microscope. **Journal of Political Economy**, v. 129, n. 10, p. 2861–2904, Oct. 2021. DOI: 10.1086/715416. Visited on: 21 Dec. 2022. Cit. on p. 20.

INADA, K.-i. On a Two-Sector Model of Economic Growth: Comments and a Generalization. **The Review of Economic Studies**, v. 30, 1 June 1963. DOI: 10.2307/2295809. Cit. on p. 81.

MORA, J. U.; COSTA JUNIOR, C. J. FDI Asymmetries in Emerging Economies: The Case of Colombia. **International Journal of Economics and Finance**, v. 11, n. 8, 25 June 2019. DOI: 10.5539/ijef.v11n8p35. Cit. on p. 18.

OKANO, M.; IDA, D.; KITANO, S.; MATSUBAYASHI, Y. Development of a Regional DSGE Model in Japan: Empirical Evidence of Economic Stagnation in the Kansai Economy. **APIR Discussion Paper Series**, Apr. 2015. Cit. on p. 20.

PEREIRA, R. M.; GÓES, G. S. O Desmatamento amazônico e o ciclo econômico no Brasil. **http://www.ipea.gov.br**, Instituto de Pesquisa Econômica Aplicada (Ipea), June 2013. Cit. on p. 19.

PYTLARCZYK, E. An Estimated DSGE Model for the German Economy within the Euro Area. **SSRN Electronic Journal**, 2005. DOI: 10.2139/ssrn.2785216. Cit. on p. 19.

RIBEIRO, G. M. **Alongamento dos Ciclos Econômicos - O Preço da Estabilidade**. 2023. Dissertação – UFPR, Curitiba. Cit. on p. 19.

RICKMAN, D. S. Modern Macroeconomics and Regional Economic Modeling. **Journal of Regional Science**, 2010. DOI: 10.1111/j.1467-9787.2009.00647.x. Cit. on pp. 15, 18.

SIMON, C. P.; BLUME, L. E. **Mathematics for Economists**. First edition, international student edition. New York London: W. W. Norton, 1994. Cit. on p. 81.

SMETS, F.; WOUTERS, R. An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area. **Journal of the European Economic Association**, 1 Sept. 2003. DOI: 10.1162/154247603770383415. Cit. on p. 20.

SMETS, F.; WOUTERS, R. Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach. **American Economic Review**, v. 97, June 2007. DOI: 10.1257/aer.97.3.586. Cit. on p. 20.

SOLIS-GARCIA, M. **UCB Macro Modeling Course**. 2022. Available from: <https://sites.google.com/a/macalester.edu/solis-garcia/home/teaching/ucb-macro-modeling-course>. Visited on: 13 Jan. 2023. Cit. on pp. 19, 26, 48, 81–84, 97, 112.

SOLTANI, S.; FALIHI, N.; MEHRABIYAN, A.; AMIRI, H. Investigating the Effects of Monetary and Financial Shocks on the Key Macroeconomic Variables, Focusing on the Intermediary Role of Banks Using DSGE Models. **Journal of Money and Economy**, v. 16, 1 Dec. 2021. DOI: 10.52547/jme.16.4.477. Available from: <http://jme.mbri.ac.ir/article-1-541-en.html>. Cit. on p. 21.

TAMEGAWA, K. Two-Region DSGE Analysis of Regionally Targeted Fiscal Policy. **Review of Regional Studies**, v. 42, 13 Dec. 2012. DOI: 10.52324/001c.8103. Cit. on p. 19.

________. Constructing a Small-Region DSGE Model. **ISRN Economics**, 11 Mar. 2013. DOI: 10.1155/2013/825862. Cit. on p. 19.

TAYLOR, J. B. Discretion Versus Policy Rules In Practice. **Carnegie-Rochester Conference Series on Public Policy**, Dec. 1993. DOI: 10.1016/0167-2231(93)90009-L. Cit. on pp. 23, 37, 105.

UHLIG, H. A Toolkit For Analysing Nonlinear Dynamic Stochastic Models Easily. In: COMPUTATIONAL Methods for the Study of Dynamic Economies. Oxford: Oxford University Press, 1999. P. 30–61. Cit. on pp. 24, 51, 113.

VINHADO, F. d. S.; DIVINO, J. A. Política Monetária, Macroprudencial E Bancos: Análise Da Transmissão Por Meio De Um Modelo DSGE. In: 44ř ENCONTRO NACIONAL DE ECONOMIA. Available from: <https://en.anpec.org.br/previous-editions.php?r=encontro-2016>. Cit. on p. 21.

WALQUE, G. D.; PIERRARD, O.; ROUABAH, A. Financial (In)Stability, Supervision and Liquidity Injections: A Dynamic General Equilibrium Approach. **The Economic Journal**, 1 Dec. 2010. DOI: 10.1111/j.1468-0297.2010.02383.x. Cit. on p. 20.

A Appendix

A.1 Table of the Literature Review

A table of the literature review will be presented here, in order to compare the elements of each DSGE model discussed in the text.

A.2 Greek Letters

Table 3: Greek Letters

Parameter	Command	Definition
α	`\alpha`	factor relative weight in production
β	`\beta`	intertemporal discount
γ	`\gamma`	interest rate sensitivity
δ	`\delta`	depreciation rate
ϵ	`\epsilon`	
ε	`\varepsilon`	stochastic error
ζ	`\zeta`	
η	`\eta`	household region (destination)
θ	`\theta`	price rigidity level
ϑ	`\vartheta`	regional inflation relative weight in total inflation
ι	`\iota`	
κ	`\kappa`	
$\varkappa$	`\varkappa`	
λ	`\lambda`	real marginal cost
Λ	`\Lambda`	nominal marginal cost
μ	`\mu`	household Lagrangian multiplier
ν	`\nu`	firm and goods region (origin)
ξ	`\xi`	
o	`\omicron`	
π	`\pi`	inflation
ϖ	`\varpi`	

Continued on next page

Table 3: Greek Letters (Continued)

Parameter	Command	Definition
ρ	`\rho`	autoregressive parameter
ϱ	`\varrho`	nominal discount rate in steady state
σ	`\sigma`	relative risk aversion
ς	`\varsigma`	
τ	`\tau`	
υ	`\upsilon`	
ϕ	`\phi`	labor relative weight in utility
φ	`\varphi`	marginal disutility of labor supply
χ	`\chi`	
ψ	`\psi`	elasticity of substitution between intermediate goods
ω	`\omega`	consumption relative weight in consumption bundle

Source: The Author.

A.3 Definitions and Lemmas

The objective of this appendix is to present the definitions and lemmas used throughout the text.

Household

Definition A.1 (Discount Factor β)**.** Other things the same, a unit of consumption enjoyed tomorrow is less valuable (yields less utility) than a unit of consumption enjoyed today (SOLIS-GARCIA, 2022, Lecture 2, p.1).

Definition A.2 (Inada Condition)**.** The Inada conditions (INADA, 1963) avoid corner solutions. For this purpose, it is assumed that the partial derivatives u_C and u_L of the function $u(C, L)$ satisfy the following rules:

$$\begin{aligned} &\lim_{C \to 0} u_C(C, L^*) = \infty \quad \text{and} \quad \lim_{C \to \infty} u_C(C, L^*) = 0 \\ &\lim_{L \to 0} u_C(C^*, L) = \infty \quad \text{and} \quad \lim_{L \to \infty} u_C(C^*, L) = 0 \end{aligned} \tag{A.1}$$

where $C^*, L^* \in \mathbb{R}_{++}$ and u_j is the partial derivative of the utility function with respect to $j = C, L$ (SOLIS-GARCIA, 2022, Lecture 1, p.2)

Definition A.3 (Transversality Condition)**.** (SOLIS-GARCIA, 2022, Lecture 4, p.4)

Firms

Lemma A.1 (Marginal Cost)**.** *The Lagrangian multiplier Λ_t is the nominal marginal cost of the intermediate-good firm:*

$$MC_t := \frac{\partial TC_t}{\partial Y_t} = \Lambda_t \tag{A.2}$$

Proof. Simon and Blume (1994, p.449). ■

Definition A.4 (Constant Returns to Scale)**.** (SOLIS-GARCIA, 2022, Lecture 1, p.5)

Definition A.5 (Homogeneous Function of Degree k)**.** (SOLIS-GARCIA, 2022, Lecture 1, p.5)

Monetary Authority

Shocks

Equilibrium Conditions

Definition A.6 (Competitive Equilibrium)**.** (SOLIS-GARCIA, 2022, Lecture 1, p.6)

Steady State

Lemma A.2 (Steady State Inflation)**.** *In steady state, prices are stable $P_t = P_{t-1} = P$ and the gross inflation rate is one.*

Proof. Equation A.60. ■

Corollary A.2.1. *In steady state, all firms have the same level of production Y and therefore demand the same amount of factors, capital K and labor L.*

$$P_t = P_{t-1} = P \implies \begin{pmatrix} Y_j & K_j & L_j \end{pmatrix} = \begin{pmatrix} Y & K & L \end{pmatrix}$$

Log-linearization

Definition A.7 (PERCENTAGE DEVIATION)**.** The percentage deviation of a variable x_t from its steady state is given by (SOLIS-GARCIA, 2022, Lecture 6, p.2):

$$\hat{x}_t := \frac{x_t - x}{x} \tag{A.3}$$

Lemma A.3 (UHLIG'S RULES)**.** *The Uhlig's rules are a set of approximations used to log-linearize equations (SOLIS-GARCIA, 2022, Lecture 6, p.2).*

- *Rule 1:*

  $x_t = x(1 + \hat{x}_t)$

- *Rule 2 (Product):*

- *Rule 3 (Exponential):*

Corollary A.3.1 (Logarithm Rule)**.**

$$\ln x_t \approx \ln x + \hat{x}_t$$

Definition A.8 (LEVEL DEVIATION)**.** The level deviation of a variable u_t from its steady state is given by: (SOLIS-GARCIA, 2022, Lecture 9, p.9)

$$\widetilde{u}_t := u_t - u \tag{A.4}$$

Lemma A.4 (UHLIG'S RULES FOR LEVEL DEVIATIONS)**.** *Uhlig's rules can be applied to level deviations in order to log-linearize equations (SOLIS-GARCIA, 2022, Lecture 9, p.9).*

- *Rule 1:*

$$u_t = u + \widetilde{u}_t \tag{A.5}$$

$$u_t = u\left(1 + \frac{\widetilde{u}_t}{u}\right) \tag{A.6}$$

- *Rule 2 (Product):*
- *Rule 3 (Exponential):*
- *Rule 4 (Logarithm):*
- *Rule 5 (Percentage and Level Deviations)*

Lemma A.5 (LEVEL DEVIATION OF THE PRESENT VALUE DISCOUNT FACTOR)**.** *The level deviation of the present value discount factor is equivalent to (SOLIS-GARCIA, 2022, Lecture 13, p.6):*

$$\prod_{k=0}^{s-1}(1+R_{t+k}) = (1+R)^s\left(1 + \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right) \tag{A.7}$$

Proof. Substitute the interest rate by the gross interest rate $GR_t = 1 + R_t$ and apply rule A.6:

$$\prod_{k=0}^{s-1}(1+R_{t+k}) = \prod_{k=0}^{s-1}(GR_{t+k}) \qquad \Longrightarrow$$

$$GR \times \cdots \times GR\left(1 + \frac{1}{GR}\widetilde{GR}_t + \frac{1}{GR}\widetilde{GR}_{t+1} + \cdots + \frac{1}{GR}\widetilde{GR}_{t+s-1}\right) \qquad \Longrightarrow$$

$$GR^s\left(1 + \frac{1}{GR}\sum_{k=0}^{s-1}\widetilde{GR}_{t+k}\right) \qquad \Longrightarrow$$

$$(1+R)^s\left(1 + \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)$$

■

Definition A.9 (Geometric Series)**.** A geometric series is the sum of the terms of a geometric sequence.

$$S_\infty = \sum_{i=0}^{\infty} ar^i \implies S_\infty = \frac{a}{1-r}\,,\ |r| < 1$$

Definition A.10 (LAG AND LEAD OPERATORS)**.** The lag operator $\mathbb{L}$ is a mathematical operator that represents the backshift or lag of a time series (SOLIS-GARCIA, 2022, Lecture 13, p.9):

$$\begin{aligned} \mathbb{L}x_t &= x_{t-1} \\ (1 + a\mathbb{L})y_{t+2} &= y_{t+2} + ay_{t+1} \end{aligned}$$

Analogously, the lead operator $\mathbb{L}^{-1}$ (or inverse lag operator) yields a variable's lead (SOLIS-GARCIA, 2022, Lecture 13, p.9):

$$\begin{aligned} \mathbb{L}^{-1}x_t &= x_{t+1} \\ (1 + a\mathbb{L}^{-1})y_{t+2} &= y_{t+2} + ay_{t+3} \end{aligned}$$

Canonical NK Model

Definition A.11 (Medium Scale DSGE Model)**.** A Medium Scale DSGE Model has habit formation, capital accumulation, indexation, etc. (GALÍ, 2015, p.208).

See Galí, Smets, and Wouters (2012) for an analysis of the sources of unemployment fluctuations in an estimated medium-scale version of the present model.

Definition A.12 (Stochastic Process)**.** (SOLIS-GARCIA, 2022, Lecture 5, p.3).

Definition A.13 (Markov Process)**.** (SOLIS-GARCIA, 2022, Lecture 5, p.4).

Definition A.14 (first-order autoregressive process $AR(1)$)**.** the first-order autoregressive process $AR(1)$ (SOLIS-GARCIA, 2022, Lecture 5, p.4).

Definition A.15 (Blanchard-Kahn Conditions)**.** (SOLIS-GARCIA, 2022, Hands on 5, p.14).

A.4 Dynare Program

This section presents the mod file used in Dynare to solve the model in section 3.1.

```
% command to run dynare and write
% a new file with all the choices:
% dynare NK_Inv_MonPol savemacro=NK_Inv_MonPol_FINAL.mod

% -------------------------------------------------- %
% MODEL OPTIONS                                      %
% -------------------------------------------------- %

% Productivity Shock ON/OFF
@#define ZA_SHOCK    = 1
% Productivity Shock sign: +/-
@#define ZA_POSITIVE = 1
% Monetary Shock ON/OFF
@#define ZM_SHOCK    = 1
% Monetary Shock sign: +/-
@#define ZM_POSITIVE = 1

% -------------------------------------------------- %
% ENDOGENOUS VARIABLES                               %
% -------------------------------------------------- %

var
PIt       ${\tilde{\pi}}$     (long_name='Inflation Rate')
Pt        ${\hat{P}}$         (long_name='Price Level')
LAMt      ${\tilde{\lambda}}$ (long_name='Real Marginal Cost')
Ct        ${\hat{C}}$         (long_name='Consumption')
Lt        ${\hat{L}}$         (long_name='Labor')
Rt        ${\hat{R}}$         (long_name='Interest Rate')
Kt        ${\hat{K}}$         (long_name='Capital')
It        ${\hat{I}}$         (long_name='Investment')
Wt        ${\hat{W}}$         (long_name='Wage')
ZAt       ${\hat{Z}^A}$       (long_name='Productivity')
Yt        ${\hat{Y}}$         (long_name='Production')
ZMt       ${\hat{Z}^M}$       (long_name='Monetary Policy')
;

% -------------------------------------------------- %
% LOCAL VARIABLES                                    %
% -------------------------------------------------- %

% the steady state variables are used as local
variables for the linear model.
```

```
model_local_variable

% steady state variables used as locals:
P
PI
ZA
ZM
R
LAM
W
Y
C
K
L
I

% local variables:
RHO % Steady State Discount Rate
;

% ------------------------------------------------- %
% EXOGENOUS VARIABLES                               %
% ------------------------------------------------- %

varexo
epsilonA ${\varepsilon_A}$   (long_name='productivity shock')
epsilonM ${\varepsilon_M}$   (long_name='monetary shock')
;

% ------------------------------------------------- %
% PARAMETERS                                        %
% ------------------------------------------------- %

parameters
SIGMA   ${\sigma}$     (long_name='Relative Risk Aversion')
PHI     ${\phi}$       (long_name='Labor Disutility Weight')
VARPHI  ${\varphi}$    (long_name='Marginal Disutility of Labor Supply')
BETA    ${\beta}$      (long_name='Intertemporal Discount Factor')
DELTA   ${\delta}$     (long_name='Depreciation Rate')
ALPHA   ${\alpha}$     (long_name='Output Elasticity of Capital')
PSI     ${\psi}$       (long_name='Elasticity of
Substitution between Intermediate Goods')
THETA   ${\theta}$     (long_name='Price Stickness Parameter')
gammaR  ${\gamma_R}$   (long_name='Interest-Rate Smoothing Parameter')
gammaPI ${\gamma_\pi}$ (long_name='Interest-Rate
Sensitivity to Inflation')
gammaY  ${\gamma_Y}$   (long_name='Interest-Rate Sensitivity to Product')
```

```
% maybe it's a local var, right? RHO ${\rho}$
(long_name='Steady State Discount Rate')
rhoA    ${\rho_A}$     (long_name='Autoregressive
Parameter of Productivity Shock')
rhoM    ${\rho_M}$     (long_name='Autoregressive
Parameter of Monetary Policy Shock')
thetaC  ${\theta_C}$   (long_name='Consumption weight
in Output')
thetaI  ${\theta_I}$   (long_name='Investment weight
in Output')

% -------------------------------------------------- %
% standard errors of stochastic shocks               %
% -------------------------------------------------- %

sigmaA ${\sigma_A}$   (long_name='Productivity-Shock
Standard Error')
sigmaM ${\sigma_M}$   (long_name='Monetary-Shock
Standard Error')
;

% -------------------------------------------------- %
% parameters values                                  %
% -------------------------------------------------- %

SIGMA   = 2       ; % Relative Risk Aversion
PHI     = 1       ; % Labor Disutility Weight
VARPHI  = 1.5     ; % Marginal Disutility of Labor
Supply
BETA    = 0.985   ; % Intertemporal Discount Factor
DELTA   = 0.025   ; % Depreciation Rate
ALPHA   = 0.35    ; % Output Elasticity of Capital
PSI     = 8       ; % Elasticity of Substitution
between Intermediate Goods
THETA   = 0.8     ; % Price Stickness Parameter
gammaR  = 0.79    ; % Interest-Rate Smoothing Parameter
gammaPI = 2.43    ; % Interest-Rate Sensitivity
to Inflation
gammaY  = 0.16    ; % Interest-Rate Sensitivity to
Product
% maybe it's a local var, right? RHO = 1/(1+Rs);
% Steady State Discount Rate
rhoA    = 0.95    ; % Autoregressive Parameter of
Productivity Shock
rhoM    = 0.9     ; % Autoregressive Parameter of
Monetary Policy Shock
thetaC  = 0.8     ; % Consumption weight in Output
```

```
thetaI  = 0.2     ; % Investment weight in Output

% ---------------------------------------------------- %
% standard errors values                               %
% ---------------------------------------------------- %

sigmaA = 0.01   ; % Productivity-Shock Standard Error
sigmaM = 0.01   ; % Monetary-Shock Standard Error

% ---------------------------------------------------- %
% MODEL                                                %
% ---------------------------------------------------- %

model(linear);

% First, the steady state variables as local varibles,
for the log-linear use:

#Ps   = 1 ;
#PIs  = 1 ;
#ZAs  = 1 ;
#ZMs  = 1 ;
#Rs   = Ps*(1/BETA-(1-DELTA)) ;
#LAMs = Ps*(PSI-1)/PSI ;
#Ws   = (1-ALPHA)*(LAMs*ZAs*(ALPHA/Rs)^ALPHA)^
(1/(1-ALPHA)) ;
#Ys   = ((Ws/(PHI*Ps))*((Ws/((1-ALPHA)*LAMs))^PSI)*(Rs/
(Rs-DELTA*ALPHA*LAMs))^SIGMA)^(1/(PSI+SIGMA)) ;
#Cs   = ((Ws/(PHI*Ps))*((1-ALPHA)*Ys*LAMs/Ws)^
(-PSI))^(1/SIGMA) ;
#Ks   = ALPHA*Ys*LAMs/Rs ;
#Ls   = (1-ALPHA)*Ys*LAMs/Ws ;
#Is   = DELTA*Ks ;
#RHO  = 1/(1+Rs) ;

% ---------------------------------------------------- %
% MODEL EQUATIONS                                      %
% ---------------------------------------------------- %

% Second, the log-linear model:

% 01 %
[name='Gross Inflation Rate']
PIt = Pt - Pt(-1) ;

% 02 %
[name='New Keynesian Phillips Curve']
```

```
PIt = RHO*PIt(+1)+LAMt*(1-THETA)*(1-THETA*RHO)/THETA ;

% 03 %
[name='Labor Supply']
VARPHI*Lt + SIGMA*Ct = Wt - Pt ;

% 04 %
[name='Household Euler Equation']
Ct(+1) - Ct = (Rt(+1)-Pt(+1))*BETA*Rs/(SIGMA*Ps) ;

% 05 %
[name='Law of Motion for Capital']
Kt = (1-DELTA)*Kt(-1) + DELTA*It ;

% 06 %
[name='Real Marginal Cost']
LAMt = ALPHA*Rt + (1-ALPHA)*Wt - ZAt - Pt ;

% 07 %
[name='Production Function']
Yt = ZAt + ALPHA*Kt(-1) + (1-ALPHA)*Lt ;

% 08 %
[name='Marginal Rates of Substitution of Factors']
Kt(-1) - Lt = Wt - Rt ;

% 09 %
[name='Market Clearing Condition']
Yt = thetaC*Ct + thetaI*It ;

% 10 %
[name='Monetary Policy']
Rt = gammaR*Rt(-1) + (1 - gammaR)*(gammaPI*PIt +
gammaY*Yt) + ZMt ;

% 11 %
[name='Productivity Shock']
@#if ZA_POSITIVE == 1
ZAt = rhoA*ZAt(-1) + epsilonA ;
@#else
ZAt = rhoA*ZAt(-1) - epsilonA ;
@#endif

% 12 %
[name='Monetary Shock']
@#if ZM_POSITIVE == 1
ZMt = rhoM*ZMt(-1) + epsilonM ;
```

```
@#else
ZMt = rhoM*ZMt(-1) - epsilonM ;
@#endif

end;

% ------------------------------------------------- %
% STEADY STATE                                      %
% ------------------------------------------------- %

steady_state_model ;

% in the log-linear model, all steady state variables
are zero (the variation is zero):

PIt  = 0 ;
Pt   = 0 ;
LAMt = 0 ;
Ct   = 0 ;
Lt   = 0 ;
Rt   = 0 ;
Kt   = 0 ;
It   = 0 ;
Wt   = 0 ;
ZAt  = 0 ;
Yt   = 0 ;
ZMt  = 0 ;

end;

% compute the steady state
steady;
check(qz_zero_threshold=1e-20);

% ------------------------------------------------- %
% SHOCKS                                            %
% ------------------------------------------------- %

shocks;

% Productivity Shock
@#if ZA_SHOCK == 1
var    epsilonA;
stderr sigmaA;
@#endif

% Monetary Shock
```

```
@#if ZM_SHOCK == 1
var    epsilonM;
stderr sigmaM;
@#endif

end;

stoch_simul(irf=80, order=1, qz_zero_threshold=1e-20)
ZAt ZMt Yt Pt PIt LAMt Ct Lt Rt Kt It Wt  ;

% -------------------------------------------------- %
% LATEX OUTPUT                                       %
% -------------------------------------------------- %

write_latex_definitions;
write_latex_parameter_table;
write_latex_original_model;
write_latex_dynamic_model;
write_latex_static_model;
write_latex_steady_state_model;
collect_latex_files;
```

A.5 LaTeX

Commands

- checkmark: `\cmark` ✓
- xmark: `\xmark` ✗
- cancel line in equation: `\cancel`
- space before align: `\vspace{-1cm}`
- correct paragraph overfull: `\sloppy`
- indices: i, j, k, ℓ
- hats: $\overline{abc}, \widetilde{abc}, \widehat{abc}, \overrightarrow{abc}, \overleftarrow{abc}, \sqrt[n]{abc}, \xrightarrow{abc}, \xrightarrow{sometext}$
- accents: $\acute{a}, \check{a}, \grave{a}, \widetilde{a}, \hat{a}, \breve{a}, \overline{a}, \bar{a}, \vec{a}, \dot{a}, \ddot{a}, \mathring{a}, \imath, \jmath$
- symbols:

  checkmark: ✓

  dagger: †

  definition symbol: $:=$
- index before the variable:

$$
\begin{aligned}
&+ {}^{NR}C_{t+1}^{\alpha} + {}_{NR}C_{t+1}^{\alpha} + {}_{\mathscr{nr}}C_{t+1}^{\alpha} \\
&+ NRC_{t+1}^{\alpha} + \mathscr{nr}C_{t+1}^{\alpha} + nrC_{t+1}^{\alpha} \\
&+ \mathrm{NR}C_{t+1}^{\alpha} + \mathcal{NR}C_{t+1}^{\alpha} + \mathscr{nr}C_{t+1}^{\alpha} \\
&+ {}_{\mathcal{NR}}C_{t+1}^{\alpha} + {}^{\mathcal{NR}}C_{t+1}^{\alpha} + C_{t+1}^{\mathcal{NR},\alpha} \\
&+ C_{t+1}^{\mathrm{NR},\alpha} + C_{\mathrm{NR},t+1}^{\alpha} + \mathrm{NR}C_{t+1}^{\alpha}
\end{aligned}
$$

- summation and product operator:

$$
\sum_{s=0}^{\infty} \frac{\theta^s}{\prod_{k=0}^{s-1}(1 + R_{t+k})}
$$

$$
\text{Term for } s = 0 : \frac{\theta^0}{\prod_{k=0}^{-1}(1 + R_{t+k})} = \theta^0 = 1
$$

$$\text{Term for } s = 1 : \frac{\theta^1}{\prod_{k=0}^{0}(1 + R_{t+k})} = \frac{\theta^1}{1 + R_{t+0}} = \frac{\theta}{1 + R_t}$$

Font Styles in Math Mode

- San Serif Style: `\mathsf`

  $\mathsf{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathsf{abcdefghijklmnopqrstuvwxyz}$

  $\mathsf{1234567890}$

- Fraktur Style: `\mathfrak`

  $\mathfrak{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathfrak{abcdefghijklmnopqrstuvwxyz}$

  $\mathfrak{1234567890}$

- Fraktur-bold Style: `\mathbffrak`

  $\mathbffrak{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathbffrak{abcdefghijklmnopqrstuvwxyz}$

  $\mathbffrak{1234567890}$

- Calligraphic Style: `\mathcal`

  $\mathcal{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathcal{abcdefghijklmnopqrstuvwxyz}$

- Calligraphic-bold Style: `\mathbfcal`

  $\mathbfcal{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

  $\mathbfcal{abcdefghijklmnopqrstuvwxyz}$

- Script Style: `\mathscr`

  $\mathscr{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

- Script-bold Style: `\mathbfscr`

  $\mathbfscr{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$

- Blackboard-bold Style: `\mathbb`

$$\mathbb{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$$

$$\mathbb{1}$$

Greek Letters

Table 4: Greek Letters

Lower Case	Upper Case	Variation
$\alpha, \boldsymbol{\alpha}$ \alpha	$A, \boldsymbol{A}$	
$\beta, \boldsymbol{\beta}$ \beta	$B, \boldsymbol{B}$	
$\gamma, \boldsymbol{\gamma}$ \gamma	$\Gamma, \boldsymbol{\Gamma}$ \Gamma	
$\delta, \boldsymbol{\delta}$ \delta	$\Delta, \boldsymbol{\Delta}$ \Delta	
$\epsilon, \boldsymbol{\epsilon}$ \epsilon	$E, \boldsymbol{E}$	$\varepsilon, \boldsymbol{\varepsilon}$ \varepsilon
$\zeta, \boldsymbol{\zeta}$ \zeta	$Z, \boldsymbol{Z}$	
$\eta, \boldsymbol{\eta}$ \eta	$H, \boldsymbol{H}$	
$\theta, \boldsymbol{\theta}$ \theta	$\Theta, \boldsymbol{\Theta}$ \Theta	$\vartheta, \boldsymbol{\vartheta}$ \vartheta
$\iota, \boldsymbol{\iota}$ \iota	$I, \boldsymbol{I}$	
$\kappa, \boldsymbol{\kappa}$ \kappa	$K, \boldsymbol{K}$	$\varkappa, \boldsymbol{\varkappa}$ \varkappa
$\lambda, \boldsymbol{\lambda}$ \lambda	$\Lambda, \boldsymbol{\Lambda}$ \Lambda	
$\mu, \boldsymbol{\mu}$ \mu	$M, \boldsymbol{M}$	
$\nu, \boldsymbol{\nu}$ \nu	$N, \boldsymbol{N}$	
$\xi, \boldsymbol{\xi}$ `\xi`	$\Xi, \boldsymbol{\Xi}$ `\Xi`	
$o, \boldsymbol{o}$ (omicron)	$O, \boldsymbol{O}$	
$\pi, \boldsymbol{\pi}$ \pi	$\Pi, \boldsymbol{\Pi}$ \Pi	$\varpi, \boldsymbol{\varpi}$ \varpi
$\rho, \boldsymbol{\rho}$ \rho	$P, \boldsymbol{P}$	$\varrho, \boldsymbol{\varrho}$ \varrho
$\sigma, \boldsymbol{\sigma}$ \sigma	$\Sigma, \boldsymbol{\Sigma}$ \Sigma	$\varsigma, \boldsymbol{\varsigma}$ \varsigma
$\tau, \boldsymbol{\tau}$ \tau	$T, \boldsymbol{T}$	
$\upsilon, \boldsymbol{\upsilon}$ \upsilon	$\Upsilon, \boldsymbol{\Upsilon}$ \Upsilon	
$\phi, \boldsymbol{\phi}$ \phi	$\Phi, \boldsymbol{\Phi}$ \Phi	$\varphi, \boldsymbol{\varphi}$ \varphi
$\chi, \boldsymbol{\chi}$ \chi	$X, \boldsymbol{X}$	
$\psi, \boldsymbol{\psi}$ \psi	$\Psi, \boldsymbol{\Psi}$ \Psi	
$\omega, \boldsymbol{\omega}$ \omega	$\Omega, \boldsymbol{\Omega}$ \Omega	

Source: The Author.

A.6 NK Model

A.6.1 Household

Utility Maximization Problem

Following the models presented by Costa Junior (2016) and Solis-Garcia (2022), the representative household problem is to maximize an intertemporal utility function U with respect to consumption C_t and labor L_t, subject to a budget constraint, a capital accumulation rule and the non-negativity of real variables:

$$\max_{C_t,L_t,K_{t+1}} \; : \quad U(C_t, L_t) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \phi \frac{L_t^{1+\varphi}}{1+\varphi} \right) \tag{A.8}$$

$$\text{s. t.}: \quad P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \tag{A.9}$$

$$K_{t+1} = (1-\delta)K_t + I_t \tag{A.10}$$

$$C_t, L_t, K_{t+1} \geq 0 \; ; K_0 \text{ given.}$$

where $\mathbb{E}_t$ is the expectation operator, β is the intertemporal discount factor, σ is the relative risk aversion coefficient, ϕ is the labor relative weight in utility, φ is the marginal disutility of labor supply. In the budget constraint, P_t is the price level, I_t is the investment, W_t is the wage level, K_t is the capital stock, R_t is the return on capital, and Π_t is the firm profit. In the capital accumulation rule, δ is the capital depreciation rate.

Isolate I_t in A.10 and substitute in A.9:

$$K_{t+1} = (1-\delta)K_t + I_t \implies I_t = K_{t+1} - (1-\delta)K_t \tag{A.10}$$

$$P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \implies \tag{A.9}$$

$$P_t(C_t + K_{t+1} - (1-\delta)K_t) = W_t L_t + R_t K_t + \Pi_t \tag{A.11}$$

Lagrangian

The maximization problem with restriction can be transformed in one without restriction using the Lagrangian function $\mathscr{L}$ with A.8 and A.11:

$$\mathscr{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \phi \frac{L_t^{1+\varphi}}{1+\varphi} \right) - \right.$$
$$\left. -\mu_t \Big[P_t (C_t + K_{t+1} - (1-\delta) K_t) - (W_t L_t + R_t K_t + \Pi_t) \Big] \right\} \quad \text{(A.12)}$$

First Order Conditions

The first order conditions with respect to C_t, L_t, K_{t+1} and μ_t are:

$$C_t: \quad C_t^{-\sigma} - \mu_t P_t = 0 \implies \mu_t = \frac{C_t^{-\sigma}}{P_t} \quad \text{(A.13)}$$

$$L_t: \quad -\phi L_t^{\varphi} + \mu_t W_t = 0 \implies \mu_t = \frac{\phi L_t^{\varphi}}{W_t} \quad \text{(A.14)}$$

$$\begin{aligned} K_{t+1}: \quad & -\mu_t P_t + \beta \mathbb{E}_t \mu_{t+1} [(1-\delta) P_{t+1} + R_{t+1}] = 0 \implies \\ & \mu_t P_t = \beta \mathbb{E}_t \mu_{t+1} [(1-\delta) P_{t+1} + R_{t+1}] \end{aligned} \quad \text{(A.15)}$$

$$\mu_t: \quad P_t (C_t + K_{t+1} - (1-\delta) K_t) = W_t L_t + R_t K_t + \Pi_t \quad \text{(A.11)}$$

Solutions

Match equations A.13 and A.14:

$$\frac{C_t^{-\sigma}}{P_t} = \frac{\phi L_t^{\varphi}}{W_t} \implies \frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \quad \text{(A.16)}$$

Equation A.16 is the Household Labor Supply and shows that the marginal rate of substitution (MRS) of labor for consumption is equal to the real wage, which is the relative price between labor and goods.

Substitute μ_t and μ_{t+1} from equation A.13 in A.15:

$$\mu_t P_t = \beta \mathbb{E}_t \mu_{t+1}[(1-\delta)P_{t+1} + R_{t+1}] \quad \Longrightarrow$$
$$\frac{C_t^{-\sigma}}{P_t} P_t = \beta \mathbb{E}_t \frac{C_{t+1}^{-\sigma}}{P_{t+1}}[(1-\delta)P_{t+1} + R_{t+1}] \quad \Longrightarrow$$
$$\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} = \beta \left[(1-\delta) + \mathbb{E}_t \left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{A.17}$$

Equation A.17 is the Household Euler equation.

Firms

Consider two types of firms: (1) a continuum of intermediate-good firms, which operate in monopolistic competition and each produce one variety with imperfect substitution level between each other and (2) the final-good firm, which aggregates all the varieties into a final bundle and operates in perfect competition.

A.6.2 Final-Good Firm

Profit Maximization Problem

The role of the final-good firm is to aggregate all the varieties produced by the intermediate-good firms, so that the representative consumer can buy only one good Y_t, the bundle good. The final-good firm problem is to maximize its profit, considering that its output is the bundle Y_t formed by the continuum of intermediate goods Y_{jt}, where $j \in [0,1]$ and ψ is the elasticity of substitution between intermediate goods:

$$\max_{Y_{jt}} : \quad \Pi_t = P_t Y_t - \int_0^1 P_{jt} Y_{jt} \, \mathrm{d}\, j \tag{A.18}$$

$$\text{s. t. :} \quad Y_t = \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} \tag{A.19}$$

Substitute A.19 in A.18:

$$\max_{Y_{jt}} : \quad \Pi_t = P_t \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}} \, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} - \int_0^1 P_{jt} Y_{jt} \, \mathrm{d}\, j \tag{A.20}$$

First Order Condition and Solutions

The first order condition is:

$$
\begin{aligned}
Y_{jt}: \quad & P_t \left(\frac{\psi}{\psi-1}\right)\left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}-1}\left(\frac{\psi-1}{\psi}\right) Y_{jt}^{\frac{\psi-1}{\psi}-1} - P_{jt} = 0 \implies \\
& Y_{jt} = Y_t \left(\frac{P_t}{P_{jt}}\right)^{\psi} \qquad \text{(A.21)}
\end{aligned}
$$

Equation A.21 shows that the demand for variety j depends on its relative price.

Substitute A.21 in A.19:

$$
\begin{aligned}
Y_t &= \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} && \implies \\
Y_t &= \left(\int_0^1 \left[Y_t\left(\frac{P_t}{P_{jt}}\right)^{\psi}\right]^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} && \implies \\
P_t &= \left[\int_0^1 P_{jt}^{1-\psi}\,\mathrm{d}j\right]^{\frac{1}{1-\psi}} && \text{(A.22)}
\end{aligned}
$$

Equation A.22 is the final-good firm's markup.

A.6.3 Intermediate-Good Firms

Cost Minimization Problem

There is an intermediate-good continuum, where each firm, denoted by $j \in [0,1]$, produces varieties of a representative good with a specific level of substitutability. Each of these firms must choose capital K_{jt} and labor N_{jt} to minimize production costs, subject to a technology rule:

$$
\min_{K_{jt},L_{jt}}: \quad R_t K_{jt} + W_t L_{jt} \qquad \text{(A.23)}
$$

$$
\text{s.t.}: \quad Y_{jt} = Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} \qquad \text{(A.24)}
$$

where Y_{jt} is the output obtained by the production technology level Z_{At}[11] that transforms capital K_{jt} and labor L_{jt} in proportions α and $(1-\alpha)$, respectively, into intermediate goods.

Lagrangian

Applying the Lagrangian:

$$\mathscr{L} = (R_t K_{jt} + W_t L_{jt}) - \Lambda_t (Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} - Y_{jt}) \tag{A.25}$$

where the Lagrangian multiplier Λ_t is the marginal cost[12].

First Order Conditions

The first-order conditions are:

$$K_{jt}: \quad R_t - \Lambda_t Z_{At} \alpha K_{jt}^{\alpha-1} L_{jt}^{1-\alpha} = 0 \qquad \Longrightarrow \quad K_{jt} = \alpha Y_{jt} \frac{\Lambda_t}{R_t} \tag{A.26}$$

$$L_{jt}: \quad W_t - \Lambda_t Z_{At} K_{jt}^{\alpha} (1-\alpha) L_{jt}^{-\alpha} = 0 \qquad \Longrightarrow \quad L_{jt} = (1-\alpha) Y_{jt} \frac{\Lambda_t}{W_t} \tag{A.27}$$

$$\Lambda_t: \quad Y_{jt} = Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} \tag{A.24}$$

Solutions

Divide equation A.26 by A.27:

$$\frac{K_{jt}}{L_{jt}} = \frac{\alpha Y_{jt} \Lambda_t / R_t}{(1-\alpha) Y_{jt} \Lambda_t / W_t} \Longrightarrow \frac{K_{jt}}{L_{jt}} = \left(\frac{\alpha}{1-\alpha}\right) \frac{W_t}{R_t} \tag{A.28}$$

Equation A.28 demonstrates the relationship between the technical marginal rate of substitution (TMRS) and the economical marginal rate of substitution (EMRS).

---

[11] the production technology level Z_{At} will be submitted to a productivity shock, detailed in section A.6.6.

[12] see Lemma A.1

Substitute L_{jt} from equation A.28 in A.24:

$$
\begin{aligned}
Y_{jt} &= Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} && \Longrightarrow \\
Y_{jt} &= Z_{At} K_{jt}^{\alpha} \left[\left(\frac{1-\alpha}{\alpha} \right) \frac{R_t K_{jt}}{W_t} \right]^{1-\alpha} && \Longrightarrow \\
K_{jt} &= \frac{Y_{jt}}{Z_{At}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_t} \right]^{1-\alpha}
\end{aligned} \tag{A.29}
$$

Equation A.29 is the intermediate-good firm demand for capital.

Substitute A.29 in A.28:

$$
\begin{aligned}
L_{jt} &= \left(\frac{1-\alpha}{\alpha} \right) \frac{R_t K_{jt}}{W_t} && \Longrightarrow \\
L_{jt} &= \left(\frac{1-\alpha}{\alpha} \right) \frac{R_t}{W_t} \frac{Y_{jt}}{Z_{At}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_t} \right]^{1-\alpha} && \Longrightarrow \\
L_{jt} &= \frac{Y_{jt}}{Z_{At}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_t} \right]^{-\alpha}
\end{aligned} \tag{A.30}
$$

Equation A.30 is the intermediate-good firm demand for labor.

Total and Marginal Costs

Calculate the total cost using A.29 and A.30:

$$
\begin{aligned}
TC_{jt} &= W_t L_{jt} + R_t K_{jt} && \Longrightarrow \\
TC_{jt} &= W_t \frac{Y_{jt}}{Z_{At}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_t} \right]^{-\alpha} + R_t \frac{Y_{jt}}{Z_{At}} \left[\left(\frac{\alpha}{1-\alpha} \right) \frac{W_t}{R_t} \right]^{1-\alpha} && \Longrightarrow \\
TC_{jt} &= \frac{Y_{jt}}{Z_{At}} \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha}
\end{aligned} \tag{A.31}
$$

Calculate the marginal cost using A.31:

$$
\Lambda_{jt} = \frac{\partial TC_{jt}}{\partial Y_{jt}} \Longrightarrow \Lambda_{jt} = \frac{1}{Z_{At}} \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \tag{A.32}
$$

The marginal cost depends on the technological level Z_{At}, the nominal interest rate R_t and the nominal wage level W_t, which are the same for all intermediate-good firms, and because of that, the index j may be dropped:

$$\Lambda_t = \frac{1}{Z_{At}} \left(\frac{R_t}{\alpha}\right)^{\alpha} \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \tag{A.33}$$

notice that:

$$\Lambda_t = \frac{TC_{jt}}{Y_{jt}} \implies TC_{jt} = \Lambda_t Y_{jt} \tag{A.34}$$

Optimal Price Problem

Consider an economy with price stickiness, following the Calvo Rule (CALVO, 1983): each firm has a probability $(0 < \theta < 1)$ of keeping its price in the next period $(P_{j,t+1} = P_{j,t})$, and a probability of $(1-\theta)$ of setting a new optimal price $P^{*}_{j,t}$ that maximizes its profits. Therefore, each firm must take this uncertainty into account when deciding the optimal price: the intertemporal profit flow, given the nominal interest rate R_t of each period, is calculated considering the probability θ of keeping the previous price.

$$\max_{P_{jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt} Y_{j,t+s} - TC_{j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{A.35}$$

$$\text{s.t.} : \quad Y_{jt} = Y_t \left(\frac{P_t}{P_{jt}}\right)^{\psi} \tag{A.21}$$

Substitute A.34 in A.35:

$$\max_{P_{jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt} Y_{j,t+s} - \Lambda_{t+s} Y_{j,t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \tag{A.36}$$

Substitute A.21 in A.36 and rearrange the variables:

$$\max_{P_{jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt} Y_{t+s} \left(\frac{P_{t+s}}{P_{jt}}\right)^{\psi} - \Lambda_{t+s} Y_{t+s} \left(\frac{P_{t+s}}{P_{jt}}\right)^{\psi}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies$$

$$\max_{P_{jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{jt}^{1-\psi} P_{t+s}^{\psi} Y_{t+s} - P_{jt}^{-\psi} P_{t+s}^{\psi} Y_{t+s} \Lambda_{t+s}\right]}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\}$$

First Order Condition

The first order condition with respect to P_{jt} is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[(1-\psi) P_{jt}^{-\psi} P_{t+s}^{\psi} Y_{t+s} - (-\psi) P_{jt}^{-\psi-1} P_{t+s}^{\psi} Y_{t+s} \Lambda_{t+s} \right]}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} = 0$$

Separate the summations and rearrange the variables:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) \left(\frac{P_{t+s}}{P_{jt}} \right)^{\psi} Y_{t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{jt}^{-1} \left(\frac{P_{t+s}}{P_{jt}} \right)^{\psi} Y_{t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} \tag{A.37}$$

Substitute A.21 in A.37:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s (\psi - 1) Y_{j,t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \psi P_{jt}^{-1} Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} \implies$$

$$(\psi - 1) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} = \psi P_{jt}^{-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} \implies$$

$$P_{jt} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} = \frac{\psi}{\psi - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1} (1+R_{t+k})} \right\} \implies$$

$$P_{jt}^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}} \tag{A.38}$$

Equation A.38 represents the optimal price that firm j will choose. Since all firms that are able to choose will opt for the highest possible price, they will all select the same price. As a result, the index j can be omitted:

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1} (1+R_{t+k}) \right\}} \tag{A.39}$$

A.6.4 Final-Good Firm, part II

The process of fixing prices is random: in each period, θ firms will maintain the price from the previous period, while $(1-\theta)$ firms will choose a new optimal price.

The price level for each period will be a composition of these two prices. Use this information in A.22 to determine the aggregate price level:

$$P_t = \left[\int_0^\theta P_{t-1}^{1-\psi} \, \mathrm{d}\, j + \int_\theta^1 P_t^{*1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \implies$$
$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*1-\psi} \right]^{\frac{1}{1-\psi}} \tag{A.40}$$

Equation A.40 is the aggregate price level.

A.6.5 Monetary Authority

The objective of the monetary authority is to conduct the economy to price stability and economic growth, using a Taylor rule (TAYLOR, 1993) to determine the nominal interest rate:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{A.41}$$

where π_t is the gross inflation rate, defined by:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{A.42}$$

and R, π, Y are the variables in steady state, γ_R is the smoothing parameter for the interest rate R_t, while γ_π and γ_Y are the interest-rate sensitivities in relation to inflation and product, respectively and Z_{Mt} is the monetary shock[13].

---

[13] for the monetary shock definition, see section A.6.6.

A.6.6 Stochastic Shocks

Productivity Shock

The production technology level Z_{At} will be submitted to a productivity shock defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{At} = (1 - \rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \tag{A.43}$$

where $\rho_A \in [0, 1]$ is the autoregressive parameter and $\varepsilon_{At} \sim \mathcal{N}(0, \sigma_A)$.

Monetary Shock

The monetary policy will also be submitted to a shock, through the variable Z_{Mt}, defined by a first-order autoregressive process $AR(1)$:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{A.44}$$

where $\rho_M \in [0, 1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

A.6.7 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\{P_t^*, R_t^*, W_t^*\}$, allocations for households $\boldsymbol{\mathcal{A}}_H := \{C_t^*, L_t^*, K_{t+1}^*\}$ and for firms $\boldsymbol{\mathcal{A}}_F := \{K_{jt}^*, L_{jt}^*, Y_{jt}^*, Y_t^*\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{At}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_H$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_F$ solve the firms' problems, and the markets for goods and labor clear:

$$Y_t = C_t + I_t \tag{A.45}$$

$$L_t = \int_0^1 L_{jt} \, \mathrm{d}\, j \tag{A.46}$$

A.6.8 Model Structure

The model is composed of the preview solutions, forming a square system of 16 variables and 16 equations, summarized as follows:

- Variables (16):
  - from the household problem: C_t, L_t, K_{t+1};
  - from the final-good firm problem: Y_{jt}, P_t;
  - from the intermediate-good firm problems: K_{jt}, L_{jt}, P_t^*;
  - from the market clearing condition: Y_t, I_t;
  - prices: $W_t, R_t, \Lambda_t, \pi_t$;
  - shocks: Z_{At}, Z_{Mt}.
- Equations (16):
  1. Labor Supply:
  $$\frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \tag{A.16}$$
  2. Household Euler Equation:
  $$\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{A.17}$$
  3. Budget Constraint:
  $$P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \tag{A.9}$$
  4. Law of Motion for Capital:
  $$K_{t+1} = (1-\delta)K_t + I_t \tag{A.10}$$
  5. Bundle Technology:
  $$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\, \mathrm{d}\, j\right)^{\frac{\psi}{\psi-1}} \tag{A.19}$$
  6. General Price Level:
  $$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{A.40}$$
  7. Capital Demand:
  $$K_{jt} = \alpha Y_{jt}\frac{\Lambda_t}{R_t} \tag{A.26}$$
  8. Labor Demand:
  $$L_{jt} = (1-\alpha)Y_{jt}\frac{\Lambda_t}{W_t} \tag{A.27}$$

9. Marginal Cost:

$$\Lambda_t = \frac{1}{Z_{At}} \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \tag{A.33}$$

10. Production Function:

$$Y_{jt} = Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} \tag{A.24}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{A.39}$$

12. Market Clearing Condition:

$$Y_t = C_t + I_t \tag{A.45}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{A.41}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{A.42}$$

15. Productivity Shock:

$$\ln Z_{At} = (1 - \rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \tag{A.43}$$

16. Monetary Shock:

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{A.44}$$

A.6.9 Steady State

The steady state is defined by the constancy of the variables through time. For any given endogenous variable X_t, it is in steady state if $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (COSTA JUNIOR, 2016, p.41). For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The steady state of each equation of the model is:

1. Labor Supply:

$$\frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \implies \frac{\phi L^{\varphi}}{C^{-\sigma}} = \frac{W}{P} \tag{A.47}$$

2. Household Euler Equation:

$$\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies 1 = \beta\left[(1-\delta) + \frac{R}{P}\right] \tag{A.48}$$

3. Budget Constraint:

$$P_t(C_t + I_t) = W_t L_t + R_t K_t + \Pi_t \implies P(C + I) = WL + RK + \Pi \tag{A.49}$$

4. Law of Motion for Capital:

$$K_{t+1} = (1-\delta)K_t + I_t \implies K = (1-\delta)K + I \implies I = \delta K \tag{A.50}$$

5. Bundle Technology:

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \implies Y = \left(\int_0^1 Y_j^{\frac{\psi-1}{\psi}}\,\mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{A.51}$$

6. General Price Level:

$$\begin{aligned} P_t &= \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} && \implies \\ P^{1-\psi} &= \theta P^{1-\psi} + (1-\theta)P^{*1-\psi} && \implies \\ (1-\theta)P^{1-\psi} &= (1-\theta)P^{*1-\psi} && \implies P = P^* \end{aligned} \tag{A.52}$$

7. Capital Demand:

$$K_{jt} = \alpha Y_{jt}\frac{\Lambda_t}{R_t} \implies K_j = \alpha Y j\frac{\Lambda}{R} \tag{A.53}$$

8. Labor Demand:

$$L_{jt} = (1-\alpha)Y_{jt}\frac{\Lambda_t}{W_t} \implies L_j = (1-\alpha)Y j\frac{\Lambda}{W} \tag{A.54}$$

9. Marginal Cost:

$$\Lambda_t = \frac{1}{Z_{At}} \left(\frac{R_t}{\alpha}\right)^{\alpha} \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \implies \Lambda = \frac{1}{Z_A} \left(\frac{R}{\alpha}\right)^{\alpha} \left(\frac{W}{1-\alpha}\right)^{1-\alpha} \tag{A.55}$$

10. Production Technology:

$$Y_{jt} = Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} \implies Y_j = Z_A K_j^{\alpha} L_j^{1-\alpha} \tag{A.56}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} \Lambda_{t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \implies \tag{A.39}$$

$$P^* = \frac{\psi}{\psi - 1} \cdot \frac{Y_j \Lambda / [1 - \theta(1 - R)]}{Y_j / [1 - \theta(1 - R)]} \implies$$

$$P^* = \frac{\psi}{\psi - 1} \Lambda \tag{A.57}$$

12. Market Clearing Condition:

$$Y_t = C_t + I_t \implies Y = C + I \tag{A.58}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \tag{A.59}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \pi = 1 \tag{A.60}$$

15. Productivity Shock:

$$\begin{aligned} \ln Z_{At} &= (1 - \rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} &&\implies \\ \ln Z_A &= (1 - \rho_A) \ln Z_A + \rho_A \ln Z_A + \varepsilon_A &&\implies \\ \varepsilon_A &= 0 \end{aligned} \tag{A.61}$$

16. Monetary Shock:

$$
\begin{aligned}
\ln Z_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} &&\implies \\
\ln Z_M &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_M + \varepsilon_M &&\implies \\
\varepsilon_M &= 0
\end{aligned} \tag{A.62}
$$

A.6.10 Variables in Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It's assumed that the productivity and the price level are normalized to one: $[\, P \;\; Z_A \,] = \vec{\mathbf{1}}$ [14].

From A.52, the optimal price P^* is:

$$P^* = P \tag{A.63}$$

From A.60, the gross inflation rate is:

$$\pi = 1 \tag{A.64}$$

From A.59, the monetary shock is:

$$Z_M = 1 \tag{A.65}$$

From A.61 and A.62, the productivity and monetary shocks are:

$$\varepsilon_A = \varepsilon_M = 0 \tag{A.66}$$

From A.48, the return on capital R is:

$$1 = \beta\left[(1-\delta) + \frac{R}{P}\right] \implies R = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{A.67}$$

[14] where $\vec{\mathbf{1}}$ is the unit vector.

From A.57 and A.52, the marginal cost Λ is:

$$P^* = \frac{\psi}{\psi - 1}\Lambda \implies \Lambda = P\frac{\psi - 1}{\psi} \tag{A.68}$$

From equation A.55, the nominal wage W is:

$$\Lambda = \frac{1}{Z_A}\left(\frac{R}{\alpha}\right)^{\alpha}\left(\frac{W}{1-\alpha}\right)^{1-\alpha} \implies W = (1-\alpha)\left[\Lambda Z_A\left(\frac{\alpha}{R}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} \tag{A.69}$$

In steady state, prices are the same ($P = P^*$), resulting in a gross inflation level of one ($\pi = 1$), and all firms producing the same output level ($Y_j = Y$) due to the price parity (SOLIS-GARCIA, 2022, Lecture 13, p.12). For this reason, they all demand the same amount of factors (K, L), and equations A.53, A.54, and A.56 become:

$$K = \alpha Y\frac{\Lambda}{R} \tag{A.70}$$

$$L = (1-\alpha)Y\frac{\Lambda}{W} \tag{A.71}$$

$$Y = Z_A K^{\alpha} L^{1-\alpha} \tag{A.72}$$

Substitute A.70 in A.50:

$$I = \delta K \implies I = \delta\alpha Y\frac{\Lambda}{R} \tag{A.73}$$

Substitute A.71 in A.47:

$$\frac{\phi L^{\varphi}}{C^{-\sigma}} = \frac{W}{P} \implies C = \left[L^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \implies C = \left[\left((1-\alpha)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \tag{A.74}$$

Substitute A.73 and A.74 in A.58:

$$Y = C + I \implies$$

$$Y = \left[\left((1-\alpha)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} + \left[\delta\alpha Y\frac{\Lambda}{R}\right] \implies$$

$$Y = \left[\left(\frac{W}{\phi P}\right)\left(\frac{W}{(1-\alpha)\Lambda}\right)^{\varphi}\left(\frac{R}{R-\delta\alpha\Lambda}\right)^{\sigma}\right]^{\frac{1}{\varphi+\sigma}} \tag{A.75}$$

For C, K, L, I, use the result from A.75 in A.74, A.70, A.71 and A.50, respectively.

A.6.11 Steady State Solution

$$\begin{bmatrix} P & P^* & \pi & Z_A & Z_M \end{bmatrix} = \vec{\mathbf{1}} \tag{A.76}$$

$$\begin{bmatrix} \varepsilon_A & \varepsilon_M \end{bmatrix} = \vec{\mathbf{0}} \tag{A.77}$$

$$R = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{A.67}$$

$$\Lambda = P\frac{\psi - 1}{\psi} \tag{A.68}$$

$$W = (1-\alpha)\left[\Lambda Z_A \left(\frac{\alpha}{R}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} \tag{A.69}$$

$$Y = \left[\left(\frac{W}{\phi P}\right)\left(\frac{W}{(1-\alpha)\Lambda}\right)^{\varphi}\left(\frac{R}{R-\delta\alpha\Lambda}\right)^{\sigma}\right]^{\frac{1}{\varphi+\sigma}} \tag{A.75}$$

$$C = \left[\left((1-\alpha)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \tag{A.74}$$

$$K = \alpha Y \frac{\Lambda}{R} \tag{A.70}$$

$$L = (1-\alpha)Y\frac{\Lambda}{W} \tag{A.71}$$

$$I = \delta K \tag{A.50}$$

A.6.12 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. `Dynare` is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation A.39. For this purpose, Uhlig's rules of log-linearization (UHLIG, 1999) will be applied to all equations in the model[15].

[15] see lemma A.3 for details.

Gross Inflation Rate

Log-linearize A.42 and define the level deviation of gross inflation rate $\widetilde{\pi}_t$:

$$\pi_t = \frac{P_t}{P_{t-1}} \implies \tag{A.42}$$

$$\widetilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{A.78}$$

New Keynesian Phillips Curve

In order to log-linearize equation A.39, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma A.5:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{A.39}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\qquad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{A.79}$$

First, log-linearize the left hand side of equation A.79 with respect to $P_t^*, Y_{j,t}, \widetilde{R}_t$:

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies \\ &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P^* Y_j \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies \\ &P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \end{aligned}$$

Separate the terms not dependent on s:

$$P^* Y_j (1+\hat{P}_t^*) \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \right\} +$$
$$+ P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \implies$$

Apply definition A.9 on the first term:

$$\frac{P^* Y_j (1+\hat{P}_t^*)}{1-\theta/(1+R)} + P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Second, log-linearize the left hand side of equation A.79 with respect to $\Lambda_t^*, Y_{j,t}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{j,t+s} \Lambda_{t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies$$
$$\frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \frac{Y_j \Lambda (1 + \hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s})}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies$$
$$\frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(1 + \hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \right\} +$$
$$+ \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Apply definition A.9 on the first term:

$$\frac{\psi}{\psi-1} \cdot \frac{Y_j \Lambda}{1-\theta/(1+R)} +$$
$$+ \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R} \right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Join both sides of the equation again:

$$
\begin{aligned}
&\frac{P^* Y_j(1+\hat{P}_t^*)}{1-\theta/(1+R)} + P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} = \\
&= \frac{\psi}{\psi-1} \cdot \frac{Y_j \Lambda}{1-\theta/(1+R)} + \\
&\quad + \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \qquad \text{(A.80)}
\end{aligned}
$$

Define a nominal discount rate ϱ in steady state:

$$
1 = \varrho(1+R) \implies \varrho = \frac{1}{1+R} \qquad \text{(A.81)}
$$

Substitute A.81 in A.80:

$$
\begin{aligned}
&\frac{P^* Y_j(1+\hat{P}_t^*)}{1-\theta\varrho} + P^* Y_j \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\varrho)^s \left(\hat{Y}_{j,t+s} - \varrho \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} = \frac{\psi}{\psi-1} \cdot \frac{Y_j \Lambda}{1-\theta\varrho} + \\
&\quad + \frac{\psi}{\psi-1} Y_j \Lambda \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\varrho)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \varrho \sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \qquad \text{(A.82)}
\end{aligned}
$$

Substitute A.68 in A.82 and simplify all common terms:

$$
\begin{aligned}
&\cancel{\frac{P^* Y_j}{1-\theta\varrho}} + \frac{\cancel{P^* Y_j}\hat{P}_t^*}{1-\theta\varrho} + \cancel{P^* Y_j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\varrho)^s \left(\cancel{\hat{Y}_{j,t+s} - \varrho \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right) \right\} = \\
&= \cancel{\frac{P^* Y_j}{1-\theta\varrho}} + \cancel{P^* Y_j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\varrho)^s \left(\cancel{\hat{Y}_{j,t+s} - \varrho \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} + \hat{\Lambda}_{t+s} \right) \right\} \implies \\
&\frac{\hat{P}_t^*}{1-\theta\varrho} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ (\theta\varrho)^s \left(\hat{\Lambda}_{t+s} \right) \right\} \qquad \text{(A.83)}
\end{aligned}
$$

Define the real marginal cost λ_t:

$$
\begin{aligned}
&\lambda_t = \frac{\Lambda_t}{P_t} \implies \Lambda_t = P_t \lambda_t \implies \\
&\hat{\Lambda}_t = \hat{P}_t + \hat{\lambda}_t \qquad \text{(A.84)}
\end{aligned}
$$

Substitute A.84 in A.83:

$$\hat{P}_t^* = (1-\theta\varrho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\varrho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{A.85}$$

Log-linearize equation A.40:

$$\begin{aligned} P_t^{1-\psi} &= \theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*1-\psi} && \Longrightarrow && \text{(A.40)} \\ P^{1-\psi}(1+(1-\psi)\hat{P}_t) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{t-1}) + \\ &\quad + (1-\theta)P^{1-\psi}(1+(1-\psi)\hat{P}_t^*) && \Longrightarrow \\ \hat{P}_t &= \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* && && \text{(A.86)} \end{aligned}$$

Substitute A.85 in A.86:

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* \tag{A.86}$$

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)(1-\theta\varrho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\varrho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{A.87}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\varrho\mathbb{L}^{-1})$[16] in A.87:

$$\begin{aligned} (1-\theta\varrho\mathbb{L}^{-1})\hat{P}_t &= (1-\theta\varrho\mathbb{L}^{-1}) \Big[\theta\hat{P}_{t-1} + \\ &\quad + (1-\theta)(1-\theta\varrho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\varrho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right)\Big] \Longrightarrow \\ \hat{P}_t - \theta\varrho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\varrho\theta\hat{P}_t + \\ &\quad (1-\theta)(1-\theta\varrho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\varrho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) - \\ &\quad - \theta\varrho(1-\theta)(1-\theta\varrho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\varrho)^s \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right) \end{aligned} \tag{A.88}$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\varrho$ within the operator. Then, cancel the summations and rearrange

[16] see definition A.10.

the terms:

$$
\begin{aligned}
\hat{P}_t - \theta\varrho\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{t-1} - \theta\varrho\theta\hat{P}_t + \\
& (1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{t+s}+\hat{\lambda}_{t+s}\right) - \\
& -\theta\varrho(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^s\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right) \implies \\
\hat{P}_t - \theta\varrho\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{t-1} - \theta\varrho\theta\hat{P}_t + (1-\theta)(1-\theta\varrho)(\hat{P}_t+\hat{\lambda}_t) + \\
& + \cancel{(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right)} - \\
& - \cancel{(1-\theta)(1-\theta\varrho)\mathbb{E}_t\sum_{s=0}^{\infty}(\theta\varrho)^{s+1}\left(\hat{P}_{t+s+1}+\hat{\lambda}_{t+s+1}\right)} \implies \\
\hat{P}_t - \theta\varrho\mathbb{E}_t\hat{P}_{t+1} = {} & \theta\hat{P}_{t-1} - \theta^2\varrho\hat{P}_t + (1-\theta-\theta\varrho+\theta^2\varrho)\hat{P}_t + (1-\theta)(1-\theta\varrho)\hat{\lambda}_t \implies
\end{aligned}
$$

$$
(\hat{P}_t - \hat{P}_{t-1}) = \varrho(\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_t) + \frac{(1-\theta)(1-\theta\varrho)}{\theta}\hat{\lambda}_t \tag{A.89}
$$

Substitute A.78 in A.89:

$$
\widetilde{\pi}_t = \varrho\mathbb{E}_t\widetilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\varrho)}{\theta}\hat{\lambda}_t \tag{A.90}
$$

Equation A.90 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

Labor Supply

Log-linearize A.16:

$$
\frac{\phi L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \qquad \implies \tag{A.16}
$$

$$
\varphi\hat{L}_t + \sigma\hat{C}_t = \hat{W}_t - \hat{P}_t \tag{A.91}
$$

Household Euler Equation

Log-linearize A.17:

$$\left(\frac{\mathbb{E}_t C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies \tag{A.17}$$

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{A.92}$$

Law of Motion for Capital

Log-linearize A.10:

$$K_{t+1} = (1-\delta)K_t + I_t \implies \tag{A.10}$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{A.93}$$

Bundle Technology

Apply the natural logarithm to A.19:

$$\ln Y_t = \frac{\psi}{\psi-1}\ln\left(\int_0^1 Y_{jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)$$

Log-linearize using corollary A.3.1:

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(\int_0^1 Y_j^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(Y_j^{\frac{\psi-1}{\psi}}\int_0^1 \mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \cancel{\frac{\psi}{\psi-1}}\left[\cancel{\frac{\psi-1}{\psi}}\ln Y_j + \cancel{\ln 1} + \cancel{\frac{\psi-1}{\psi}}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j$$

Apply corollary A.2.1:

$$
\begin{aligned}
\ln Y + \hat{Y}_t = \ln Y_j + \int_0^1 \hat{Y}_{jt}\, \mathrm{d}\, j &\implies \\
\hat{Y}_t = \int_0^1 \hat{Y}_{jt}\, \mathrm{d}\, j & \qquad (A.94)
\end{aligned}
$$

Marginal Cost

Log-linearize A.33:

$$
\Lambda_t = Z_{At}^{-1} \frac{R_t^{\alpha} W_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \implies \qquad (A.33)
$$

$$
\Lambda(1+\hat{\Lambda}_t) = \frac{1}{Z_A}\left(\frac{R}{\alpha}\right)^{\alpha}\left(\frac{W}{1-\alpha}\right)^{1-\alpha}(1-\hat{Z}_{At}+\alpha\hat{R}_t+(1-\alpha)\hat{W}_t) \implies
$$

$$
\hat{\Lambda}_t = \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} \qquad (A.95)
$$

Substitute A.84 in A.95:

$$
\begin{aligned}
\hat{\Lambda}_t &= \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} && \implies \\
\hat{P}_t + \hat{\lambda}_t &= \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} && \implies \\
\hat{\lambda}_t &= \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} - \hat{P}_t && \qquad (A.96)
\end{aligned}
$$

Production Function

Log-linearize A.24:

$$
\begin{aligned}
Y_{jt} &= Z_{At} K_{jt}^{\alpha} L_{jt}^{1-\alpha} && \implies \qquad (A.24) \\
Y_j(1+\hat{Y}_{jt}) &= Z_A K_j^{\alpha} L_j^{1-\alpha}(1+\hat{Z}_{At}+\alpha\hat{K}_{jt}+(1-\alpha)\hat{L}_{jt}) && \implies \\
\hat{Y}_{jt} &= \hat{Z}_{At} + \alpha\hat{K}_{jt} + (1-\alpha)\hat{L}_{jt} && \qquad (A.97)
\end{aligned}
$$

Substitute A.97 in A.94:

$$\hat{Y}_t = \int_0^1 \hat{Y}_{jt} \, \mathrm{d}\, j \qquad \Longrightarrow \tag{A.94}$$

$$\hat{Y}_t = \int_0^1 \left[\hat{Z}_{At} + \alpha \hat{K}_{jt} + (1-\alpha)\hat{L}_{jt}\right] \mathrm{d}\, j \qquad \Longrightarrow$$

$$\hat{Y}_t = \hat{Z}_{At} + \alpha \int_0^1 \hat{K}_{jt} \, \mathrm{d}\, j + (1-\alpha) \int_0^1 \hat{L}_{jt} \, \mathrm{d}\, j \tag{A.98}$$

Apply the natural logarithm and then log-linearize A.46:

$$L_t = \int_0^1 L_{jt} \, \mathrm{d}\, j \qquad \Longrightarrow \tag{A.46}$$

$$\ln L_t = \ln \left[\int_0^1 L_{jt} \, \mathrm{d}\, j\right] \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln \left[\int_0^1 L_j \, \mathrm{d}\, j\right] + \int_0^1 \hat{L}_{jt} \, \mathrm{d}\, j \qquad \Longrightarrow$$

$$\ln L + \hat{L}_t = \ln L_j + \ln 1 + \int_0^1 \hat{L}_{jt} \, \mathrm{d}\, j$$

Apply corollary A.2.1:

$$\Longrightarrow \; \hat{L}_t = \int_0^1 \hat{L}_{jt} \, \mathrm{d}\, j \tag{A.99}$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$\hat{K}_t = \int_0^1 \hat{K}_{jt} \, \mathrm{d}\, j \tag{A.100}$$

Substitute A.99 and A.100 in A.98:

$$\hat{Y}_t = \hat{Z}_{At} + \alpha \int_0^1 \hat{K}_{jt} \, \mathrm{d}\, j + (1-\alpha) \int_0^1 \hat{L}_{jt} \, \mathrm{d}\, j \; \Longrightarrow \tag{A.98}$$

$$\hat{Y}_t = \hat{Z}_{At} + \alpha \hat{K}_t + (1-\alpha)\hat{L}_t \tag{A.101}$$

Capital Demand

Log-linearize A.26:

$$\begin{aligned} K_{jt} &= \alpha Y_{jt}\frac{\Lambda_t}{R_t} && \implies && \text{(A.26)} \\ K_j(1+\hat{K}_{jt}) &= \alpha Y_j\frac{\Lambda}{R}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t) && \implies \\ \hat{K}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t \end{aligned}$$

Integrate both sides and then substitute A.100 and A.94:

$$\begin{aligned} \int_0^1 \hat{K}_{jt}\,\mathrm{d}j &= \int_0^1 (\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t)\,\mathrm{d}j \implies \\ \hat{K}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t \end{aligned} \tag{A.102}$$

Labor Demand

Log-linearize A.27:

$$\begin{aligned} L_{jt} &= (1-\alpha)Y_{jt}\frac{\Lambda_t}{W_t} && \implies && \text{(A.27)} \\ L_j(1+\hat{L}_{jt}) &= (1-\alpha)Y_j\frac{\Lambda}{W}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t) && \implies \\ \hat{L}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t \end{aligned}$$

Integrate both sides and then substitute A.99 and A.94:

$$\begin{aligned} \int_0^1 \hat{L}_{jt}\,\mathrm{d}j &= \int_0^1 \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t\,\mathrm{d}j \implies \\ \hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t \end{aligned} \tag{A.103}$$

Subtract A.103 from A.102:

$$\begin{aligned} &\hat{K}_t-\hat{L}_t = \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t-(\hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t) \implies \\ &\hat{K}_t-\hat{L}_t = \hat{W}_t-\hat{R}_t \end{aligned} \tag{A.104}$$

Equation A.104 is the log-linearized version of A.28.

Market Clearing Condition

Log-linearize A.45:

$$
\begin{aligned}
Y_t &= C_t + I_t && \Longrightarrow && \text{(A.45)}\\
Y(1+\hat{Y}_t) &= C(1+\hat{C}_t) + I(1+\hat{I}_t) && \Longrightarrow \\
Y + Y\hat{Y}_t &= C + C\hat{C}_t + I + I\hat{I}_t && \Longrightarrow \\
Y\hat{Y}_t &= C\hat{C}_t + I\hat{I}_t && \Longrightarrow \\
\hat{Y}_t &= \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t && && \text{(A.105)}
\end{aligned}
$$

Define the consumption and investment weights $[\,\theta_C \;\theta_I\,]$ in the production total:

$$
\begin{bmatrix}\theta_C & \theta_I\end{bmatrix} := \begin{bmatrix}\frac{C}{Y} & \frac{I}{Y}\end{bmatrix} \tag{A.106}
$$

Substitute A.106 in A.105:

$$
\begin{aligned}
&\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t \;\Longrightarrow \\
&\hat{Y}_t = \theta_C\hat{C}_t + \theta_I\hat{I}_t && \text{(A.107)}
\end{aligned}
$$

Monetary Policy

Log-linearize A.41:

$$
\begin{aligned}
&\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_t^{\gamma_\pi}Y_t^{\gamma_Y})^{(1-\gamma_R)}Z_{Mt}}{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \;\Longrightarrow && \text{(A.41)}\\
&\frac{R(1+\hat{R}_t)}{R} = \\
&= \frac{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}Z_M[1+\gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt}]}{R^{\gamma_R}(\pi^{\gamma_\pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \;\Longrightarrow \\
&\hat{R}_t = \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt} && \text{(A.108)}
\end{aligned}
$$

Productivity Shock

Log-linearize A.43:

$$\begin{aligned}
\ln Z_{At} &= (1-\rho_A)\ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} && \Longrightarrow && \text{(A.43)}\\
\ln Z_A + \hat{Z}_{At} &= (1-\rho_A)\ln Z_A + \rho_A(\ln Z_A + \hat{Z}_{A,t-1}) + \varepsilon_A && \Longrightarrow \\
\hat{Z}_{At} &= \rho_A \hat{Z}_{A,t-1} + \varepsilon_A && && \text{(A.109)}
\end{aligned}$$

Monetary Shock

Log-linearize A.44:

$$\begin{aligned}
\ln Z_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \Longrightarrow && \text{(A.44)}\\
\ln Z_M + \hat{Z}_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M(\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M && \Longrightarrow \\
\hat{Z}_{Mt} &= \rho_M \hat{Z}_{M,t-1} + \varepsilon_M && && \text{(A.110)}
\end{aligned}$$

A.6.13 Log-linear Model Structure

The log-linear model is a square system of 12 variables and 12 equations, summarized as follows:

- Variables: $(\tilde{\pi} \quad \hat{P} \quad \hat{\lambda} \quad \hat{C} \quad \hat{L} \quad \hat{R} \quad \hat{K} \quad \hat{I} \quad \hat{W} \quad \hat{Z}_A \quad \hat{Y} \quad \hat{Z}_M)$
- Equations:

1. Gross Inflation Rate:

$$\widetilde{\pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{A.78}$$

2. New Keynesian Phillips Curve:

$$\widetilde{\pi}_t = \varrho \mathbb{E}_t \widetilde{\pi}_{t+1} + \frac{(1-\theta)(1-\theta\varrho)}{\theta}\hat{\lambda}_t \tag{A.90}$$

3. Labor Supply:

$$\varphi \hat{L}_t + \sigma \hat{C}_t = \hat{W}_t - \hat{P}_t \tag{A.91}$$

4. Household Euler Equation:

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{A.92}$$

5. Law of Motion for Capital:

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{A.93}$$

6. Real Marginal Cost:

$$\hat{\lambda}_t = \alpha\hat{R}_t + (1-\alpha)\hat{W}_t - \hat{Z}_{At} - \hat{P}_t \tag{A.96}$$

7. Production Function:

$$\hat{Y}_t = \hat{Z}_{At} + \alpha\hat{K}_t + (1-\alpha)\hat{L}_t \tag{A.101}$$

8. Marginal Rates of Substitution of Factors:

$$\hat{K}_t - \hat{L}_t = \hat{W}_t - \hat{R}_t \tag{A.104}$$

9. Market Clearing Condition:

$$\hat{Y}_t = \theta_C\hat{C}_t + \theta_I\hat{I}_t \tag{A.107}$$

10. Monetary Policy:

$$\hat{R}_t = \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\pi\widetilde{\pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt} \tag{A.108}$$

11. Productivity Shock:

$$\hat{Z}_{At} = \rho_A\hat{Z}_{A,t-1} + \varepsilon_A \tag{A.109}$$

12. Monetary Shock:

$$\hat{Z}_{Mt} = \rho_M\hat{Z}_{M,t-1} + \varepsilon_M \tag{A.110}$$

A.7 ToDo List

Todo list