0.1 Household

The utility maximization problem is the same for the representative household in each region, so is the solution:

Utility Maximization Problem

$$\max_{C_{\eta t}, L_{\eta t}, K_{\eta, t+1}} : \quad U(C_{\eta t}, L_{\eta t}) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{\eta t}^{1-\sigma}}{1-\sigma} - \phi \frac{L_{\eta t}^{1+\varphi}}{1+\varphi} \right) \tag{0.1}$$

$$\text{s. t.} : \quad P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{0.2}$$

$$K_{\eta, t+1} = (1-\delta) K_{\eta t} + I_{\eta t} \tag{0.3}$$

$$C_{\eta t}, L_{\eta t}, K_{\eta, t+1} \geq 0 \; ; K_0 \text{ given.}$$

Solutions

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \tag{0.4}$$

$$\left(\frac{\mathbb{E}_t C_{\eta, t+1}}{C_{\eta t}} \right)^{\sigma} = \beta \left[(1-\delta) + \mathbb{E}_t \left(\frac{R_{t+1}}{P_{t+1}} \right) \right] \tag{0.5}$$

Firms

0.2 Final-Good Firm

As the final-good firms operate in perfect competition, the price level P_t is the same for both.

The profit maximization problem is the same for the representative final-good firm in each region, so is the solution:

Profit Maximization Problem

$$\max_{Y_{\eta jt}} : \quad \Pi_{\eta t} = P_t Y_{\eta t} - \int_0^1 P_{\eta jt} Y_{\eta jt} \, \mathrm{d}\, j \tag{0.6}$$

$$\text{s. t.} : \quad Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi - 1}{\psi}} \, \mathrm{d}\, j \right)^{\frac{\psi}{\psi - 1}} \tag{0.7}$$

Intermediary Solutions

$$Y_{\eta jt} = Y_{\eta t} \left(\frac{P_t}{P_{\eta jt}} \right)^{\psi} \tag{0.8}$$

$$P_t = \left[\int_0^1 P_{\eta jt}^{1-\psi} \, \mathrm{d}\, j \right]^{\frac{1}{1-\psi}} \tag{0.9}$$

0.3 Intermediate-Good Firms

Cost Minimization Problem

The minimization cost problem is the same for the intermediate-good firms in each region, so is the solution:

$$\min_{K_{\eta jt}, L_{\eta jt}} : \quad R_t K_{\eta jt} + W_t L_{\eta jt} \tag{0.10}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Z_{A\eta t} K_{\eta jt}^{\alpha_\eta} L_{\eta jt}^{1-\alpha_\eta} \tag{0.11}$$

$$K_{jt} = \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_t} \tag{0.12}$$

$$L_{jt} = (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} \tag{0.13}$$

$$\tag{0.14}$$

Solutions

$$\frac{K_{\eta jt}}{L_{\eta jt}} = \left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_t}{R_t} \tag{0.15}$$

$$K_{\eta jt} = \frac{Y_{\eta jt}}{Z_{A\eta t}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_t}{R_t}\right]^{1-\alpha_\eta} \tag{0.16}$$

$$L_{jt} = \frac{Y_{\eta jt}}{Z_{A\eta t}}\left[\left(\frac{\alpha_\eta}{1-\alpha_\eta}\right)\frac{W_t}{R_t}\right]^{-\alpha_\eta} \tag{0.17}$$

Total and Marginal Costs

$$TC_{\eta jt} = \frac{Y_{\eta jt}}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{0.18}$$

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}}\left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{0.19}$$

Optimal Price Problem

$$\max_{P_{\eta jt}} : \quad \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s \left[P_{\eta jt} Y_{\eta j,t+s} - TC_{\eta j,t+s} \right]}{\prod_{k=0}^{s-1} (1 + R_{t+k})} \right\} \tag{0.20}$$

$$\text{s. t.} : \quad Y_{\eta jt} = Y_t \left(\frac{P_t}{P_{\eta jt}} \right)^{\psi} \tag{0.8}$$

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1 + R_{t+k}) \right\}} \tag{0.21}$$

0.3.1 Final-Good Firm, part II

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1 - \theta) P_t^{*1-\psi} \right]^{\frac{1}{1-\psi}} \tag{0.22}$$

0.4 Monetary Authority

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \right]^{1-\gamma_R} Z_{Mt} \tag{0.23}$$

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{0.24}$$

0.5 Stochastic Shocks

Productivity Shock

$$\ln Z_{A\eta t} = (1 - \rho_{A\eta}) \ln Z_{A\eta} + \rho_{A\eta} \ln Z_{A\eta,t-1} + \varepsilon_{A\eta t} \tag{0.25}$$

where $\rho_{A\eta} \in [0,1]$ and $\varepsilon_{A\eta t} \sim \mathcal{N}(0, \sigma_{A\eta})$.

Monetary Shock

$$\ln Z_{Mt} = (1 - \rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{0.26}$$

where $\rho_M \in [0,1]$ and $\varepsilon_{Mt} \sim \mathcal{N}(0, \sigma_M)$.

0.6 Equilibrium Conditions

A Competitive Equilibrium consists of sequences of prices $\boldsymbol{\mathcal{P}} := \{P_t^*, R_t^*, W_t^*\}$, and for each region, allocations for households $\boldsymbol{\mathcal{A}}_{H\eta} := \{C_{\eta t}^*, L_{\eta t}^*, K_{\eta,t+1}^*\}$ and for firms $\boldsymbol{\mathcal{A}}_{F\eta} := \{K_{\eta jt}^*, L_{\eta jt}^*, Y_{\eta jt}^*, Y_{\eta t}^*\}$. In such an equilibrium, given the set of exogenous variables $\{K_0, Z_{A\eta t}, Z_{Mt}\}$, the elements in $\boldsymbol{\mathcal{A}}_{H\eta}$ solve the household problem, while the elements in $\boldsymbol{\mathcal{A}}_{F\eta}$ solve the firms' problems, and the markets for goods and labor clear:

$$Y_t = \sum_{\eta=1}^{n} \{C_{\eta t} + I_{\eta t}\} \tag{0.27}$$

$$L_{\eta t} = \int_0^1 L_{\eta jt} \, \mathrm{d}\, j \tag{0.28}$$

0.6.1 Model Structure

The model is composed of the preview solutions, forming a square system of 25 variables and 25 equations, summarized as follows:

- Variables:

- from the household problem: $C_{\eta t}, L_{\eta t}, K_{\eta,t+1}$;
- from the final-good firm problem: $Y_{\eta jt}, P_t$;
- from the intermediate-good firm problems: $K_{\eta jt}, L_{\eta jt}, P_t^*$;
- from the market clearing condition: $Y_t, I_{\eta t}$;
- prices: $W_t, R_t, \Lambda_{\eta t}, \pi_t$;
- shocks: $Z_{A\eta t}, Z_{Mt}$.

- Equations:

1. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \tag{0.4}$$

2. Household Euler Equation:

$$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{0.5}$$

3. Budget Constraint:

$$P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \Pi_{\eta t} \tag{0.2}$$

4. Law of Motion for Capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \tag{0.3}$$

5. Bundle Technology:

$$Y_{\eta t} = \left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\, \mathrm{d}j\right)^{\frac{\psi}{\psi-1}} \tag{0.7}$$

6. General Price Level:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} \tag{0.22}$$

7. Capital Demand:

$$K_{jt} = \alpha_\eta Y_{\eta jt}\frac{\Lambda_{\eta t}}{R_t} \tag{0.12}$$

8. Labor Demand:

$$L_{jt} = (1-\alpha_\eta)Y_{\eta jt}\frac{\Lambda_{\eta t}}{W_t} \tag{0.13}$$

9. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{0.19}$$

10. Production Function:

$$Y_{\eta j t} = Z_{A\eta t} K_{jt}^{\alpha_\eta} L_{jt}^{1-\alpha_\eta} \tag{0.11}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi - 1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1}(1+R_{t+k})\right\}} \tag{0.21}$$

12. Market Clearing Condition:

$$Y_t = \sum_{\eta=1}^{n} \{C_{\eta t} + I_{\eta t}\} \tag{0.27}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_{\eta t}}{\pi}\right)^{\gamma_\Pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \tag{0.23}$$

14. Gross Inflation Rate:

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{0.24}$$

15. Productivity Shock:

$$\ln Z_{A\eta t} = (1-\rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \tag{0.25}$$

16. Monetary Shock:

$$\ln Z_{Mt} = (1-\rho_M) \ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} \tag{0.26}$$

0.7 Steady State

The steady state is defined by the constancy of the variables through time. For any given endogenous variable X_t, it is in steady state if $\mathbb{E}_t X_{t+1} = X_t = X_{t-1} = X_{ss}$ (COSTA JUNIOR, 2016, p.41). For conciseness, the ss index representing the steady state will be omitted, so that $X := X_{ss}$. The steady state of each equation of the model is:

1. Labor Supply:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \implies \frac{\phi L^{\varphi}}{C^{-\sigma}} = \frac{W}{P} \tag{0.29}$$

2. Household Euler Equation:

$$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta) + \mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies 1 = \beta\left[(1-\delta) + \frac{R}{P}\right] \tag{0.30}$$

3. Budget Constraint:

$$P_t(C_{\eta t} + I_{\eta t}) = W_t L_{\eta t} + R_t K_{\eta t} + \pi_{\eta t} \implies P(C+I) = WL + RK + \Pi \tag{0.31}$$

4. Law of Motion for Capital:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies K = (1-\delta)K + I \implies I = \delta K \tag{0.32}$$

5. Bundle Technology:

$$Y_t = \left(\int_0^1 Y_{\eta j t}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \implies Y = \left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)^{\frac{\psi}{\psi-1}} \tag{0.33}$$

6. General Price Level:

$$\begin{aligned} P_t &= \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}} && \implies \\ P^{1-\psi} &= \theta P^{1-\psi} + (1-\theta)P^{*1-\psi} && \implies \\ (1-\theta)P^{1-\psi} &= (1-\theta)P^{*1-\psi} && \implies P = P^* \end{aligned} \tag{0.34}$$

7. Capital Demand:

$$K_{jt} = \alpha_\eta Y_{\eta j t}\frac{\Lambda_{\eta t}}{R_t} \implies K_j = \alpha_\eta Y j \frac{\Lambda}{R} \tag{0.35}$$

8. Labor Demand:

$$L_{jt} = (1-\alpha_\eta) Y_{\eta j t}\frac{\Lambda_{\eta t}}{W_t} \implies L_j = (1-\alpha_\eta) Y j \frac{\Lambda}{W} \tag{0.36}$$

9. Marginal Cost:

$$\Lambda_{\eta t} = \frac{1}{Z_{A\eta t}} \left(\frac{R_t}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W_t}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies \Lambda = \frac{1}{Z_A} \left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta} \left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \tag{0.37}$$

10. Production Technology:

$$Y_{\eta jt} = Z_{A\eta t} K_{jt}^{\alpha_\eta} L_{jt}^{1-\alpha_\eta} \implies Y_{\eta j} = Z_A K_j^{\alpha_\eta} L_j^{1-\alpha_\eta} \tag{0.38}$$

11. Optimal Price:

$$P_t^* = \frac{\psi}{\psi-1} \cdot \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s} / \prod_{k=0}^{s-1} (1+R_{t+k})\right\}}{\mathbb{E}_t \sum_{s=0}^{\infty} \left\{\theta^s Y_{\eta j,t+s} / \prod_{k=0}^{s-1} (1+R_{t+k})\right\}} \implies \tag{0.21}$$

$$P^* = \frac{\psi}{\psi-1} \cdot \frac{Y_j \Lambda / [1-\theta(1-R)]}{Y_j / [1-\theta(1-R)]} \implies$$

$$P^* = \frac{\psi}{\psi-1} \Lambda \tag{0.39}$$

12. Market Clearing Condition:

$$Y_t = C_{\eta t} + I_{\eta t} \implies Y = C + I \tag{0.40}$$

13. Monetary Policy:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left[\left(\frac{\pi_{\eta t}}{\Pi}\right)^{\gamma_\Pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y}\right]^{1-\gamma_R} Z_{Mt} \implies Z_M = 1 \tag{0.41}$$

14. Gross Inflation Rate:

$$\pi_{\eta t} = \frac{P_t}{P_{t-1}} \implies \Pi = 1 \tag{0.42}$$

15. Productivity Shock:

$$\ln Z_{A\eta t} = (1-\rho_A) \ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} \implies$$

$$\ln Z_A = (1-\rho_A) \ln Z_A + \rho_A \ln Z_A + \varepsilon_A \implies$$

$$\varepsilon_A = 0 \tag{0.43}$$

16. Monetary Shock:

$$\begin{aligned}
\ln Z_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \Longrightarrow \\
\ln Z_M &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_M + \varepsilon_M && \Longrightarrow \\
\varepsilon_M &= 0 &&
\end{aligned} \tag{0.44}$$

0.7.1 Variables in Steady State

For the steady state solution, all endogenous variables will be determined with respect to the parameters. It's assumed that the productivity and the price level are normalized to one: $[\,P \; Z_A\,] = \vec{\mathbf{1}}$ [1].

From 0.34, the optimal price P^* is:

$$P^* = P \tag{0.45}$$

From 0.42, the gross inflation rate is:

$$\Pi = 1 \tag{0.46}$$

From 0.41, the monetary shock is:

$$Z_M = 1 \tag{0.47}$$

From 0.43 and 0.44, the productivity and monetary shocks are:

$$\varepsilon_A = \varepsilon_M = 0 \tag{0.48}$$

From 0.30, the return on capital R is:

$$1 = \beta\left[(1-\delta) + \frac{R}{P}\right] \Longrightarrow R = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{0.49}$$

[1] where $\vec{\mathbf{1}}$ is the unit vector.

From 0.39 and 0.34, the marginal cost Λ is:

$$P^* = \frac{\psi}{\psi - 1}\Lambda \implies \Lambda = P\frac{\psi - 1}{\psi} \tag{0.50}$$

From equation 0.37, the nominal wage W is:

$$\Lambda = \frac{1}{Z_A}\left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta} \implies W = (1-\alpha_\eta)\left[\Lambda Z_A \left(\frac{\alpha_\eta}{R}\right)^{\alpha_\eta}\right]^{\frac{1}{1-\alpha_\eta}} \tag{0.51}$$

In steady state, prices are the same ($P = P^*$), resulting in a gross inflation level of one ($\Pi = 1$), and all firms producing the same output level ($Y_j = Y$) due to the price parity (SOLIS-GARCIA, 2022, Lecture 13, p.12). For this reason, they all demand the same amount of factors (K, L), and equations 0.35, 0.36, and 0.38 become:

$$Y = Z_A K^{\alpha_\eta} L^{1-\alpha_\eta} \tag{0.52}$$

$$K = \alpha_\eta Y \frac{\Lambda}{R} \tag{0.53}$$

$$L = (1-\alpha_\eta) Y \frac{\Lambda}{W} \tag{0.54}$$

Substitute 0.53 in 0.32:

$$I = \delta K \implies I = \delta \alpha_\eta Y \frac{\Lambda}{R} \tag{0.55}$$

Substitute 0.54 in 0.29:

$$\frac{\phi L^\varphi}{C^{-\sigma}} = \frac{W}{P} \implies C = \left[L^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \implies C = \left[\left((1-\alpha_\eta)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \tag{0.56}$$

Substitute 0.55 and 0.56 in 0.40:

$$Y = C + I \implies$$

$$Y = \left[\left((1-\alpha_\eta)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} + \left[\delta\alpha_\eta Y\frac{\Lambda}{R}\right] \implies$$

$$Y = \left[\left(\frac{W}{\phi P}\right)\left(\frac{W}{(1-\alpha_\eta)\Lambda}\right)^{\varphi}\left(\frac{R}{R-\delta\alpha_\eta\Lambda}\right)^{\sigma}\right]^{\frac{1}{\varphi+\sigma}} \tag{0.57}$$

For C, K, L, I, use the result from 0.57 in 0.56, 0.53, 0.54 and 0.32, respectively.

0.7.2 Steady State Solution

$$\begin{bmatrix} P & P^* & \Pi & Z_A & Z_M \end{bmatrix} = \vec{1} \tag{0.58}$$

$$\begin{bmatrix} \varepsilon_A & \varepsilon_M \end{bmatrix} = \vec{0} \tag{0.59}$$

$$R = P\left[\frac{1}{\beta} - (1-\delta)\right] \tag{0.49}$$

$$\Lambda = P\frac{\psi - 1}{\psi} \tag{0.50}$$

$$W = (1-\alpha_\eta)\left[\Lambda Z_A \left(\frac{\alpha_\eta}{R}\right)^{\alpha_\eta}\right]^{\frac{1}{1-\alpha_\eta}} \tag{0.51}$$

$$Y = \left[\left(\frac{W}{\phi P}\right)\left(\frac{W}{(1-\alpha_\eta)\Lambda}\right)^{\varphi}\left(\frac{R}{R-\delta\alpha_\eta\Lambda}\right)^{\sigma}\right]^{\frac{1}{\varphi+\sigma}} \tag{0.57}$$

$$C = \left[\left((1-\alpha_\eta)Y\frac{\Lambda}{W}\right)^{-\varphi}\frac{W}{\phi P}\right]^{\frac{1}{\sigma}} \tag{0.56}$$

$$K = \alpha_\eta Y\frac{\Lambda}{R} \tag{0.53}$$

$$L = (1-\alpha_\eta)Y\frac{\Lambda}{W} \tag{0.54}$$

$$I = \delta K \tag{0.32}$$

0.8 Log-linearization

Due to the number of variables and equations to be solved, computational brute force will be necessary. Dynare is a software specialized on macroeconomic modeling, used for solving DSGE models. Before the model can be processed by the software, it must be linearized in order to eliminate the infinite sum in equation 0.21. For this purpose, Uhlig's rules of log-linearization (UHLIG, 1999) will be applied to all equations in the model[2].

[2] see lemma **??** for details.

0.8.1 Gross Inflation Rate

Log-linearize 0.24 and define the level deviation of gross inflation rate $\widetilde{\Pi}_t$:

$$\pi_{\eta t} = \frac{P_t}{P_{t-1}} \implies \tag{0.24}$$

$$\widetilde{\Pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{0.60}$$

0.8.2 New Keynesian Phillips Curve

In order to log-linearize equation 0.21, it is necessary to eliminate both the summation and the product operators. To handle the product operator, apply lemma **??**:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{\prod_{k=0}^{s-1}(1+R_{t+k})} \right\} \implies \tag{0.21}$$

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} = \\ &\qquad = \frac{\psi}{\psi-1} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s} \Lambda_{\eta t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \end{aligned} \tag{0.61}$$

First, log-linearize the left hand side of equation 0.61 with respect to $P_t^*, Y_{\eta j,t}, \widetilde{R}_t$:

$$\begin{aligned} &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s P_t^* Y_{\eta j,t+s}}{(1+R)^s \left(1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right)} \right\} \implies \\ &\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{P^* Y_{\eta j} \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s}\right)}{1 + \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}} \right\} \implies \\ &P^* Y_{\eta j} \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1 + \hat{P}_t^* + \hat{Y}_{j,t+s} - \frac{1}{1+R} \sum_{k=0}^{s-1} \widetilde{R}_{t+k}\right) \right\} \end{aligned}$$

Separate the terms not dependent on s:

$$P^* Y_{\eta j}(1+\hat{P}_t^*)\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \right\} + \\ + P^* Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\} \implies$$

Apply definition **??** on the first term:

$$\frac{P^* Y_{\eta j}(1+\hat{P}_t^*)}{1-\theta/(1+R)} + P^* Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Second, log-linearize the left hand side of equation 0.61 with respect to $\Lambda^*_{\eta t}, Y_{\eta j,t}, \widetilde{R}_t$:

$$\frac{\psi}{\psi-1}\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \frac{\theta^s Y_{\eta j,t+s}\Lambda_{\eta t+s}}{(1+R)^s \left(1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)} \right\} \implies$$

$$\frac{\psi}{\psi-1}\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \frac{Y_{\eta j}\Lambda(1+\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s})}{1+\frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}} \right\} \implies$$

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(1+\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Separate the terms not dependent on s:

$$\frac{\psi}{\psi-1}Y_{\eta j}\Lambda\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \right\} + \\ + \frac{\psi}{\psi-1}Y_{\eta j}\Lambda\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Apply definition **??** on the first term:

$$\frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda}{1-\theta/(1+R)} + \\ + \frac{\psi}{\psi-1}Y_{\eta j}\Lambda\mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s}+\hat{\Lambda}_{t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1} \widetilde{R}_{t+k} \right) \right\}$$

Join both sides of the equation again:

$$
\begin{aligned}
&\frac{P^* Y_{\eta j}(1+\hat{P}_t^*)}{1-\theta/(1+R)} + P^* Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} = \\
&= \frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda}{1-\theta/(1+R)} + \\
&\qquad + \frac{\psi}{\psi-1} Y_{\eta j}\Lambda\mathbb{E}_t \sum_{s=0}^{\infty}\left\{\left(\frac{\theta}{1+R}\right)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \frac{1}{1+R}\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}
\end{aligned}
\tag{0.62}
$$

Define a nominal discount rate ρ in steady state:

$$
1 = \rho(1+R) \implies \rho = \frac{1}{1+R} \tag{0.63}
$$

Substitute 0.63 in 0.62:

$$
\begin{aligned}
&\frac{P^* Y_{\eta j}(1+\hat{P}_t^*)}{1-\theta\rho} + P^* Y_{\eta j}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\rho)^s \left(\hat{Y}_{j,t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\} = \frac{\psi}{\psi-1}\cdot\frac{Y_{\eta j}\Lambda}{1-\theta\rho} + \\
&\qquad + \frac{\psi}{\psi-1} Y_{\eta j}\Lambda\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\rho)^s \left(\hat{Y}_{j,t+s} + \hat{\Lambda}_{t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}\right)\right\}
\end{aligned}
\tag{0.64}
$$

Substitute 0.50 in 0.64 and simplify all common terms:

$$
\begin{aligned}
&\cancel{\frac{P^* Y_{\eta j}}{1-\theta\rho}} + \frac{\cancel{P^* Y_{\eta j}}\hat{P}_t^*}{1-\theta\rho} + \cancel{P^* Y_{\eta j}}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\rho)^s \left(\cancel{\hat{Y}_{j,t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}}\right)\right\} = \\
&= \cancel{\frac{P^* Y_{\eta j}}{1-\theta\rho}} + \cancel{P^* Y_{\eta j}}\mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\rho)^s \left(\cancel{\hat{Y}_{j,t+s} - \rho\sum_{k=0}^{s-1}\widetilde{R}_{t+k}} + \hat{\Lambda}_{t+s}\right)\right\} \implies \\
&\frac{\hat{P}_t^*}{1-\theta\rho} = \mathbb{E}_t \sum_{s=0}^{\infty}\left\{(\theta\rho)^s \left(\hat{\Lambda}_{t+s}\right)\right\}
\end{aligned}
\tag{0.65}
$$

Define the real marginal cost $\Lambda_{\eta t}$:

$$
\begin{aligned}
&\Lambda_{\eta t} = \frac{\Lambda_{\eta t}}{P_t} \implies \Lambda_{\eta t} = P_t\Lambda_{\eta t} \implies \\
&\hat{\Lambda}_t = \hat{P}_t + \hat{\lambda}_t
\end{aligned}
\tag{0.66}
$$

Substitute 0.66 in 0.65:

$$\hat{P}_t^* = (1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{0.67}$$

Log-linearize equation 0.22:

$$\begin{aligned} P_t^{1-\psi} &= \theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi} && \Longrightarrow && (0.22) \\ P^{1-\psi}(1+(1-\psi)\hat{P}_t) &= \theta P^{1-\psi}(1+(1-\psi)\hat{P}_{t-1}) + \\ &\quad + (1-\theta)P^{1-\psi}(1+(1-\psi)\hat{P}_t^*) && \Longrightarrow \\ \hat{P}_t &= \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* && && (0.68) \end{aligned}$$

Substitute 0.67 in 0.68:

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^* \tag{0.68}$$

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) \tag{0.69}$$

Finally, to eliminate the summation, apply the lead operator $(1-\theta\rho\mathbb{L}^{-1})$[3] in 0.69:

$$\begin{aligned} (1-\theta\rho\mathbb{L}^{-1})\hat{P}_t &= (1-\theta\rho\mathbb{L}^{-1}) \Big[\theta\hat{P}_{t-1} + \\ &\quad +(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right)\Big] \Longrightarrow \\ \hat{P}_t - \theta\rho\mathbb{E}_t\hat{P}_{t+1} &= \theta\hat{P}_{t-1} - \theta\rho\theta\hat{P}_t + \\ &\quad (1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) - \\ &\quad - \theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right) \end{aligned} \tag{0.70}$$

In the first summation, factor out the first term and in the second summation, include the term $\theta\rho$ within the operator. Then, cancel the summations and rearrange

[3] see definition **??**.

the terms:

$$\begin{aligned}
\hat{P}_t - \theta\rho \mathbb{E}_t \hat{P}_{t+1} = \theta \hat{P}_{t-1} - \theta\rho\theta \hat{P}_t + \\
(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s} + \hat{\lambda}_{t+s}\right) - \\
- \theta\rho(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^s \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right) \implies \\
\hat{P}_t - \theta\rho \mathbb{E}_t \hat{P}_{t+1} = \theta \hat{P}_{t-1} - \theta\rho\theta \hat{P}_t + (1-\theta)(1-\theta\rho)(\hat{P}_t + \hat{\lambda}_t) + \\
+ \cancel{(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^{s+1} \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right)} - \\
- \cancel{(1-\theta)(1-\theta\rho)\mathbb{E}_t \sum_{s=0}^{\infty} (\theta\rho)^{s+1} \left(\hat{P}_{t+s+1} + \hat{\lambda}_{t+s+1}\right)} \implies \\
\hat{P}_t - \theta\rho \mathbb{E}_t \hat{P}_{t+1} = \theta \hat{P}_{t-1} - \theta^2\rho \hat{P}_t + (1-\theta-\theta\rho+\theta^2\rho)\hat{P}_t + (1-\theta)(1-\theta\rho)\hat{\lambda}_t \implies \\
(\hat{P}_t - \hat{P}_{t-1}) = \rho(\mathbb{E}_t \hat{P}_{t+1} - \hat{P}_t) + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \qquad (0.71)
\end{aligned}$$

Substitute 0.60 in 0.71:

$$\widetilde{\Pi}_t = \rho \mathbb{E}_t \widetilde{\Pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \qquad (0.72)$$

Equation 0.72 is the New Keynesian Phillips Curve in terms of the real marginal cost. It illustrates that the deviation of inflation depends on both the expectation of future inflation deviation and the present marginal cost deviation.

0.8.3 Labor Supply

Log-linearize 0.4:

$$\frac{\phi L_{\eta t}^{\varphi}}{C_{\eta t}^{-\sigma}} = \frac{W_t}{P_t} \qquad \implies \qquad (0.4)$$

$$\varphi \hat{L}_t + \sigma \hat{C}_t = \hat{W}_t + \hat{P}_t \qquad (0.73)$$

0.8.4 Household Euler Equation

Log-linearize 0.5:

$$\left(\frac{\mathbb{E}_t C_{\eta,t+1}}{C_{\eta t}}\right)^{\sigma} = \beta\left[(1-\delta)+\mathbb{E}_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \implies \tag{0.5}$$

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{0.74}$$

0.8.5 Law of Motion for Capital

Log-linearize 0.3:

$$K_{\eta,t+1} = (1-\delta)K_{\eta t} + I_{\eta t} \implies \tag{0.3}$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{0.75}$$

0.8.6 Bundle Technology

Apply the natural logarithm to 0.7:

$$\ln Y_t = \frac{\psi}{\psi-1}\ln\left(\int_0^1 Y_{\eta jt}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right)$$

Log-linearize using corollary **??**:

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(\int_0^1 Y_{\eta j}^{\frac{\psi-1}{\psi}}\,\mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \frac{\psi}{\psi-1}\left[\ln\left(Y_{\eta j}^{\frac{\psi-1}{\psi}}\int_0^1 \mathrm{d}\,j\right) + \frac{\psi-1}{\psi}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \cancel{\frac{\psi}{\psi-1}}\left[\cancel{\frac{\psi-1}{\psi}}\ln Y_{\eta j} + \cancel{\ln 1} + \cancel{\frac{\psi-1}{\psi}}\int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j\right] \implies$$

$$\ln Y + \hat{Y}_t = \ln Y_{\eta j} + \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j$$

Apply corollary **??**:

$$
\begin{aligned}
\ln Y + \hat{Y}_t = \ln Y_{\eta j} + \int_0^1 \hat{Y}_{jt} \,\mathrm{d}\, j \implies \\
\hat{Y}_t = \int_0^1 \hat{Y}_{jt} \,\mathrm{d}\, j \qquad (0.76)
\end{aligned}
$$

0.8.7 Marginal Cost

Log-linearize 0.19:

$$
\Lambda_{\eta t} = Z_{A\eta t}^{-1} \frac{R_t^{\alpha_\eta} W_t^{1-\alpha_\eta}}{\alpha_\eta{}^{\alpha_\eta}(1-\alpha_\eta)^{1-\alpha_\eta}} \implies \qquad (0.19)
$$

$$
\Lambda(1+\hat{\Lambda}_t) = \frac{1}{Z_A}\left(\frac{R}{\alpha_\eta}\right)^{\alpha_\eta}\left(\frac{W}{1-\alpha_\eta}\right)^{1-\alpha_\eta}(1-\hat{Z}_{At}+\alpha_\eta\hat{R}_t+(1-\alpha_\eta)\hat{W}_t) \implies
$$

$$
\hat{\Lambda}_t = \alpha_\eta\hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{At} \qquad (0.77)
$$

Substitute 0.66 in 0.77:

$$
\begin{aligned}
\hat{\Lambda}_t &= \alpha_\eta\hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{At} \implies \\
\hat{P}_t + \hat{\lambda}_t &= \alpha_\eta\hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{At} \implies \\
\hat{\lambda}_t &= \alpha_\eta\hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{At} - \hat{P}_t \qquad (0.78)
\end{aligned}
$$

0.8.8 Production Function

Log-linearize 0.11:

$$
\begin{aligned}
Y_{\eta jt} &= Z_{A\eta t} K_{jt}^{\alpha_\eta} L_{jt}^{1-\alpha_\eta} \implies \qquad (0.11) \\
Y_{\eta j}(1+\hat{Y}_{jt}) &= Z_A K_j^{\alpha_\eta} L_j^{1-\alpha_\eta}(1+\hat{Z}_{At}+\alpha_\eta\hat{K}_{jt}+(1-\alpha_\eta)\hat{L}_{jt}) \implies \\
\hat{Y}_{jt} &= \hat{Z}_{At} + \alpha_\eta\hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt} \qquad (0.79)
\end{aligned}
$$

Substitute 0.79 in 0.76:

$$
\begin{aligned}
\hat{Y}_t &= \int_0^1 \hat{Y}_{jt}\,\mathrm{d}\,j \implies && (0.76)\\
\hat{Y}_t &= \int_0^1 \left[\hat{Z}_{At} + \alpha_\eta \hat{K}_{jt} + (1-\alpha_\eta)\hat{L}_{jt}\right]\mathrm{d}\,j \implies \\
\hat{Y}_t &= \hat{Z}_{At} + \alpha_\eta \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha_\eta)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j && (0.80)
\end{aligned}
$$

Apply the natural logarithm and then log-linearize 0.28:

$$
\begin{aligned}
L_{\eta t} &= \int_0^1 L_{jt}\,\mathrm{d}\,j \implies && (0.28)\\
\ln L_{\eta t} &= \ln\left[\int_0^1 L_{jt}\,\mathrm{d}\,j\right] \implies \\
\ln L + \hat{L}_t &= \ln\left[\int_0^1 L_j\,\mathrm{d}\,j\right] + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \implies \\
\ln L + \hat{L}_t &= \ln L_j + \ln 1 + \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j
\end{aligned}
$$

Apply corollary **??**:

$$
\implies \hat{L}_t = \int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \tag{0.81}
$$

By analogy, the total capital deviation is the sum of all firm's deviations:

$$
\hat{K}_t = \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j \tag{0.82}
$$

Substitute 0.81 and 0.82 in 0.80:

$$
\begin{aligned}
\hat{Y}_t &= \hat{Z}_{At} + \alpha_\eta \int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j + (1-\alpha_\eta)\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j \implies && (0.80)\\
\hat{Y}_t &= \hat{Z}_{At} + \alpha_\eta \hat{K}_t + (1-\alpha_\eta)\hat{L}_t && (0.83)
\end{aligned}
$$

0.8.9 Capital Demand

Log-linearize 0.12:

$$
\begin{aligned}
K_{jt} &= \alpha_\eta Y_{\eta jt} \frac{\Lambda_{\eta t}}{R_t} && \Longrightarrow && (0.12)\\
K_j(1+\hat{K}_{jt}) &= \alpha_\eta Y_{\eta j} \frac{\Lambda}{R}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t) && \Longrightarrow \\
\hat{K}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t
\end{aligned}
$$

Integrate both sides and then substitute 0.82 and 0.76:

$$
\begin{aligned}
\int_0^1 \hat{K}_{jt}\,\mathrm{d}\,j &= \int_0^1 (\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{R}_t)\,\mathrm{d}\,j \quad \Longrightarrow \\
\hat{K}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t \qquad (0.84)
\end{aligned}
$$

0.8.10 Labor Demand

Log-linearize 0.13:

$$
\begin{aligned}
L_{jt} &= (1-\alpha_\eta) Y_{\eta jt} \frac{\Lambda_{\eta t}}{W_t} && \Longrightarrow && (0.13)\\
L_j(1+\hat{L}_{jt}) &= (1-\alpha_\eta) Y_{\eta j} \frac{\Lambda}{W}(1+\hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t) && \Longrightarrow \\
\hat{L}_{jt} &= \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t
\end{aligned}
$$

Integrate both sides and then substitute 0.81 and 0.76:

$$
\begin{aligned}
\int_0^1 \hat{L}_{jt}\,\mathrm{d}\,j &= \int_0^1 \hat{Y}_{jt}+\hat{\Lambda}_t-\hat{W}_t\,\mathrm{d}\,j \quad \Longrightarrow \\
\hat{L}_t &= \hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t \qquad (0.85)
\end{aligned}
$$

Subtract 0.85 from 0.84:

$$
\begin{aligned}
&\hat{K}_t-\hat{L}_t = \hat{Y}_t+\hat{\Lambda}_t-\hat{R}_t-(\hat{Y}_t+\hat{\Lambda}_t-\hat{W}_t) \quad \Longrightarrow \\
&\hat{K}_t-\hat{L}_t = \hat{W}_t-\hat{R}_t \qquad (0.86)
\end{aligned}
$$

Equation 0.86 is the log-linearized version of 0.15.

0.8.11 Market Clearing Condition

Log-linearize 0.27:

$$
\begin{aligned}
Y_t &= C_{\eta t} + I_{\eta t} && \Longrightarrow && (0.27)\\
Y(1+\hat{Y}_t) &= C(1+\hat{C}_t) + I(1+\hat{I}_t) && \Longrightarrow\\
Y + Y\hat{Y}_t &= C + C\hat{C}_t + I + I\hat{I}_t && \Longrightarrow\\
Y\hat{Y}_t &= C\hat{C}_t + I\hat{I}_t && \Longrightarrow\\
\hat{Y}_t &= \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t && && (0.87)
\end{aligned}
$$

Define the consumption and investment weights $\begin{bmatrix}\theta_C & \theta_I\end{bmatrix}$ in the production total:

$$
\begin{bmatrix}\theta_C & \theta_I\end{bmatrix} := \begin{bmatrix}\frac{C}{Y} & \frac{I}{Y}\end{bmatrix} \tag{0.88}
$$

Substitute 0.88 in 0.87:

$$
\begin{aligned}
&\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t \Longrightarrow\\
&\hat{Y}_t = \theta_C\hat{C}_t + \theta_I\hat{I}_t && (0.89)
\end{aligned}
$$

0.8.12 Monetary Policy

Log-linearize 0.23:

$$
\begin{aligned}
&\frac{R_t}{R} = \frac{R_{t-1}^{\gamma_R}(\pi_{\eta t}^{\gamma_\Pi}Y_t^{\gamma_Y})^{(1-\gamma_R)}Z_{Mt}}{R^{\gamma_R}(\Pi^{\gamma_\Pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \Longrightarrow && (0.23)\\
&\frac{R(1+\hat{R}_t)}{R} =\\
&= \frac{R^{\gamma_R}(\Pi^{\gamma_\Pi}Y^{\gamma_Y})^{(1-\gamma_R)}Z_M[1+\gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\Pi\widetilde{\Pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt}]}{R^{\gamma_R}(\Pi^{\gamma_\Pi}Y^{\gamma_Y})^{(1-\gamma_R)}} \Longrightarrow\\
&\hat{R}_t = \gamma_R\hat{R}_{t-1} + (1-\gamma_R)(\gamma_\Pi\widetilde{\Pi}_t + \gamma_Y\hat{Y}_t) + \hat{Z}_{Mt} && (0.90)
\end{aligned}
$$

0.8.13 Productivity Shock

Log-linearize 0.25:

$$\begin{aligned}
\ln Z_{A\eta t} &= (1-\rho_A)\ln Z_A + \rho_A \ln Z_{A,t-1} + \varepsilon_{At} && \Longrightarrow && (0.25)\\
\ln Z_A + \hat{Z}_{At} &= (1-\rho_A)\ln Z_A + \rho_A(\ln Z_A + \hat{Z}_{A,t-1}) + \varepsilon_A && \Longrightarrow && \\
\hat{Z}_{At} &= \rho_A \hat{Z}_{A,t-1} + \varepsilon_A && && (0.91)
\end{aligned}$$

0.8.14 Monetary Shock

Log-linearize 0.26:

$$\begin{aligned}
\ln Z_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M \ln Z_{M,t-1} + \varepsilon_{Mt} && \Longrightarrow && (0.26)\\
\ln Z_M + \hat{Z}_{Mt} &= (1-\rho_M)\ln Z_M + \rho_M(\ln Z_M + \hat{Z}_{M,t-1}) + \varepsilon_M && \Longrightarrow && \\
\hat{Z}_{Mt} &= \rho_M \hat{Z}_{M,t-1} + \varepsilon_M && && (0.92)
\end{aligned}$$

0.8.15 Log-linear Model Structure

The log-linear model is a square system of 12 variables and 12 equations, summarized as follows:

- Variables: $(\tilde{\Pi} \quad \hat{P} \quad \tilde{\lambda} \quad \hat{C} \quad \hat{L} \quad \hat{R} \quad \hat{K} \quad \hat{I} \quad \hat{W} \quad \hat{Z}_A \quad \hat{Y} \quad \hat{Z}_M)$
- Equations:
  1. Gross Inflation Rate:

$$\widetilde{\Pi}_t = \hat{P}_t - \hat{P}_{t-1} \tag{0.60}$$

  2. New Keynesian Phillips Curve:

$$\widetilde{\Pi}_t = \rho \mathbb{E}_t \widetilde{\Pi}_{t+1} + \frac{(1-\theta)(1-\theta\rho)}{\theta}\hat{\lambda}_t \tag{0.72}$$

  3. Labor Supply:

$$\varphi \hat{L}_t + \sigma \hat{C}_t = \hat{W}_t + \hat{P}_t \tag{0.73}$$

  4. Household Euler Equation:

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{\beta R}{\sigma P}\mathbb{E}_t(\hat{R}_{t+1} - \hat{P}_{t+1}) \tag{0.74}$$

5. Law of Motion for Capital:

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t \tag{0.75}$$

6. Real Marginal Cost:

$$\hat{\lambda}_t = \alpha_\eta \hat{R}_t + (1-\alpha_\eta)\hat{W}_t - \hat{Z}_{At} - \hat{P}_t \tag{0.78}$$

7. Production Function:

$$\hat{Y}_t = \hat{Z}_{At} + \alpha_\eta \hat{K}_t + (1-\alpha_\eta)\hat{L}_t \tag{0.83}$$

8. Marginal Rates of Substitution of Factors:

$$\hat{K}_t - \hat{L}_t = \hat{W}_t - \hat{R}_t \tag{0.86}$$

9. Market Clearing Condition:

$$\hat{Y}_t = \theta_C \hat{C}_t + \theta_I \hat{I}_t \tag{0.89}$$

10. Monetary Policy:

$$\hat{R}_t = \gamma_R \hat{R}_{t-1} + (1-\gamma_R)(\gamma_\Pi \widetilde{\Pi}_t + \gamma_Y \hat{Y}_t) + \hat{Z}_{Mt} \tag{0.90}$$

11. Productivity Shock:

$$\hat{Z}_{At} = \rho_A \hat{Z}_{A,t-1} + \varepsilon_A \tag{0.91}$$

12. Monetary Shock:

$$\hat{Z}_{Mt} = \rho_M \hat{Z}_{M,t-1} + \varepsilon_M \tag{0.92}$$