SID #24253445
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Error Analysis Exercise

1) Since radioactive decay is modelled by a Poisson distribution, its error σ is $\sqrt{N}$

$N = 1000 \Rightarrow$ specific activity $= \lambda = \frac{1000}{5} = 200 \frac{\text{decay}}{\text{min}}$

precision of $1\% = 0.01 = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \Rightarrow N = 10^4$

time required $= \frac{N}{\lambda} = \frac{10^4 \text{ decays}}{\left(200 \frac{\text{decay}}{\text{min}}\right)} = \boxed{50 \text{ minutes}}$

assuming that time measurement doesn't contribute to error.

2) a) $\sigma = \boxed{\sqrt{\sigma_A^2 + \sigma_B^2}}$

b) $\sigma = \boxed{\sqrt{\sigma_A^2 + \sigma_B^2}}$ this is basically same as a) ∵ error both ±

c) General function : $f(A,B) = 2A + 2B \rightarrow \frac{\partial f}{\partial A} = 2 \quad \frac{\partial f}{\partial B} = 2$

$$\sigma = \delta f = \sqrt{\left(\frac{\partial f}{\partial A} \delta A\right)^2 + \left(\frac{\partial f}{\partial B} \delta B\right)^2} = \sqrt{(2\sigma_A)^2 + (2\sigma_B)^2} = \boxed{2\sqrt{\sigma_A^2 + \sigma_B^2}}$$

d) for multiplication ∴ $\frac{\delta f}{f} = \sqrt{\left(\frac{\delta A}{A}\right)^2 + \left(\frac{\delta B}{B}\right)^2}$

$f(A,B) = A \times B$

$$\sigma = \delta f = f\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2} = AB\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2} = \sqrt{A^2B^2\frac{\sigma_A^2}{A^2} + A^2B^2\frac{\sigma_B^2}{B^2}}$$

$$\sigma = \sqrt{B^2\sigma_A^2 + A^2\sigma_B^2} = \boxed{\sqrt{(B\sigma_A)^2 + (A\sigma_B)^2}}$$

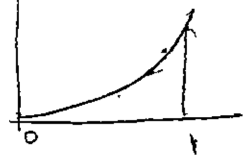
3) $\mu = 0$
① $\sigma = 1$

After sampling 5 times,
the expected mean $= 0$
standard deviation $= 1$
error on the mean $= \sqrt{N} = \sqrt{5} = 2.24$

④ See plot attached

② Using the computer to generate a list of 5 random number
we get $\mu = 0.096$, $\sigma = 0.8418$ but this can vary alot if a pick another set of 5 normally distributed random numbers

③ The histograms centered around 0 which makes sense, because most experiments should yield something close to $\mu = 0$, some don't because there is only 5 number in the list which is relatively small, this explains the wide spread of the histogram...
Counts of experiment within 1, 2, 3 sigma is 675, 955, 998 respectively.
This agree with expected because almost all experiments (99.8%) lie in 3 sigma, less for 1, 2 sigma.



4) ① To generate a exponential distribution use a <u>uniform</u> distribution to draw a random # and put in into $x = -\log(y-1)$

$y = e^{-x}$

normalize

$$f(x) = Ae^{-x} \Rightarrow \int_0^{\infty} A f(x)\,dx = 1 = A\left[e^{-\infty} - e^{0}\right] \Rightarrow A = 1$$

$$\mu = \langle x \rangle = \int_0^{\infty} x e^{-x}\,dx = 1$$

$$\langle x^2 \rangle = \int_0^{\infty} x^2 e^{-x}\,dx = 2$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{2 - 1^2} = \sqrt{1} = 1$$

So both the mean and the standard deviation should be 1

The error is the standard deviation so it should be one. The histogram should be centered around one and with a relatively narrow peak because the standard deviation is relatively small.

The error is $\approx \frac{1}{\sqrt{N}}$ so for the $N=100 \Rightarrow$ Error ≈ 0.1

② For $N = 100$, mean $= 1.1717$, standard deviation $= 1.066$

③ The distribution is centered around 1 and the error is 0.007304 which is close to $\frac{1}{\sqrt{100}} \approx 0.01$ agreeing with what I thought.

④ Yes, the error on the mean scales as $\frac{1}{\sqrt{N}}$ as I thought

5) ① See attached plots

Error $= \sqrt{N}$ ($\because$ Poisson process)

② Mean $= 1.20268 \pm 10$ ← # of datapoints $= 1000$

$\sigma = \sqrt{1000} = 10$

Standard deviation

$= 0.10379 \pm 0.0001$ ← $\because$ peak.dat given with 5 s.f.

③ we fit $y = Ae^{-\frac{(x-\mu)^2}{2\sigma^2}}$

and get $\mu = 1.20117$ and $\sigma = 0.10280$

The difference arise from the fact that the Gaussian assumes a continuous functional form for the fit whereas in ① we computed μ & σ based on its discrete definition.

$$\mu = \frac{\Sigma\, x_1 \ldots x_n}{N} \quad \text{and} \quad \sigma = \sqrt{\frac{\Sigma (x_i - \mu)^2}{N}}$$

④ $\chi^2 = \Sigma \frac{(\text{data}-\text{fit})^2}{\text{data}} = 39.2$

where data is the observed frequency of each bin

The χ^2 is large which means that the Gaussian distribution may not be a very good fit

6) ① see attached

② $\sigma_f = 0.01\,MHz$

Refering to Numerical recipes 15.2, we analytically maximize the log-likelihood

$y = a + bx$ the uncertainty on the estimated parameters are:

$$\sigma_a^2 = \frac{S_{xx}}{\Delta} = \frac{\sum_{i=1}^{N} \frac{x_i^2}{\sigma_i^2}}{S S_{xx} - (S_x)^2} \qquad \sigma_b^2 = \frac{S}{\Delta} = \frac{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}{\Delta}$$

$$S_x = \sum_{i=1}^{N} \frac{x_i}{\sigma_i^2} \qquad S_{xx} = \sum_{i=1}^{N} \frac{x_i^2}{\sigma_i^2}$$

Computing this for $\sigma_f = 0.01\,MHz \Rightarrow \begin{cases} \sigma_a^2 = -1.314 \times 10^{-5} \\ \sigma_b^2 = -6.49 \times 10^{-7} \end{cases}$

for $\sigma_f = 1\,MHz \Rightarrow \begin{cases} \sigma_a^2 = -0.1314 \\ \sigma_b^2 = -0.06494 \end{cases}$

So when the uncertainty on the measured frequency is larger the error propagates to the parameter estimation, yielding a greater uncertainty in the slope and intercept estimated from the fit.

③ Uncertainty measured from $\sigma^2 = \frac{\sum_{N=1}^{N} (y_i - f(x)^{fit})^2}{N-1} = \frac{0.1518}{11} = \boxed{0.014}$

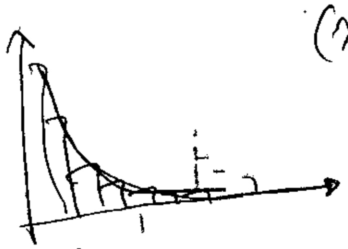
④ Using the weighted least square fit,

I obtained a slope of 3.11287 (rounded up to 3.1) and

and an intercept of −0.04518 (rounded up to 0.05)

See attached plot of the fit

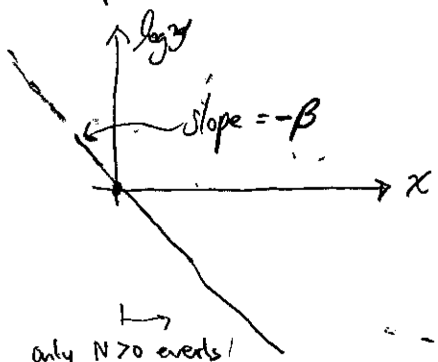
7) ① (x_i, y_i)



$y = e^{-\beta x}$

$\log y = \log e^{-\beta x} = -\beta x$

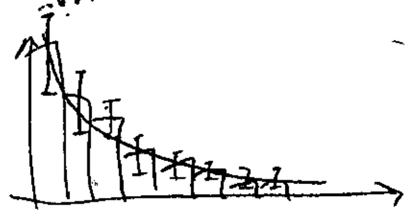
$\underbrace{\log y}_{\text{"y"}} = \underbrace{-\beta x}_{m=-\beta\ x}\ \underbrace{\ }_{b=0}$ linear form

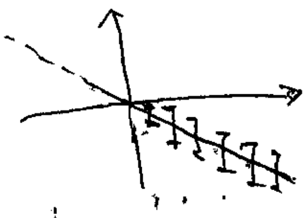


Using the function plt.semilog y this can be easily plotted

Muon lifetime is govered by a Poisson distribution

error $= \frac{1}{\sqrt{N}}$ so as N increases the error bar shrinks!





$\Rightarrow \log y = -\beta x \Rightarrow y = e^{-\beta x} = f(x)$

$\delta f = \sqrt{\left(\frac{\partial f}{\partial x}\delta x\right)^2} = -\beta e^{-\beta N}\frac{1}{\sqrt{N}}$

as N increase errorbar of the semi-logged plot the will depend on $\frac{-\beta e^{-\beta N}}{\sqrt{N}}$

$N^{-\frac{1}{2}}$

② $\log E_0 = 2.1 \pm 0.5$

for error propagation on a general function $y = \log x = f(x)$

$\delta f = \sqrt{\left(\frac{\partial f}{\partial x}\delta x\right)^2} = \left(\frac{1}{x} 0.5\right) = \frac{0.5}{2.1} = \frac{5}{21} = 0.238$

$\boxed{E_0 = 8.2 \pm 0.2}$