Intelligent Systems

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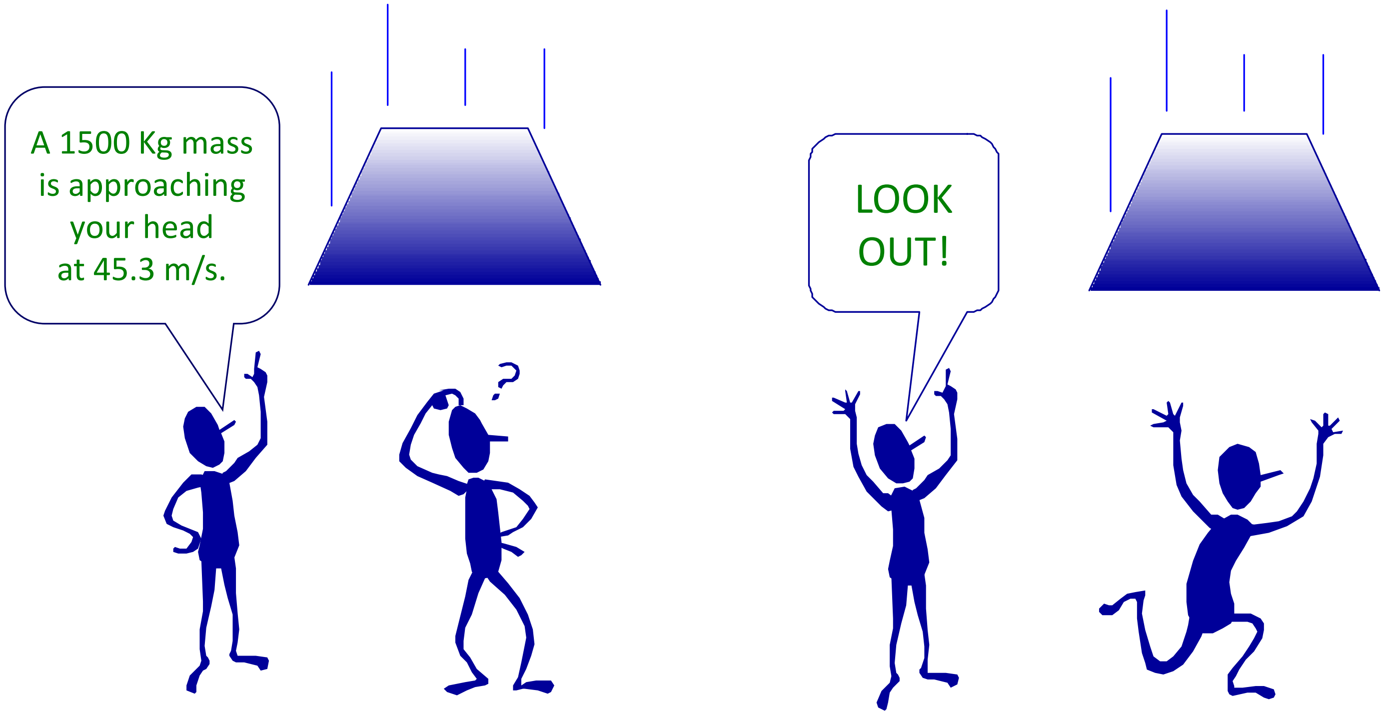
FUZZY SETS

SI2 – Introduction to Fuzzy Sets

Reading: Part I Fuzzy Set Theory: Chapter 2 Fuzzy Sets

J.-S. Jang, C.-T. Sun and E. Mizutani. ***Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence.*** Prentice Hall, New Jersey, 1997.

Precision vs. Relevancy



Introduction

- How to simplify very complex systems?
  - *Allow some degree of imprecision in their description!*
- How to deal mathematically with uncertainty?
  - Using probabilistic theory (*stochastic*).
  - Using the **theory of fuzzy sets** (*non-stochastic*).
- Proposed in 1965 by Lotfi Zadeh (Fuzzy Sets, *Information Control*, 8, pp. 338-353).
- Imprecision or vagueness in natural language **does not** imply a loss of accuracy or meaningfulness!

Examples

- *Give travel directions in terms of city blocks* **OR** *in meters?*
- *The day is sunny* **OR** *the sky is covered by 5% of clouds?*
  - If the sky is covered by 10% of clouds is still *sunny*?
  - And 25%?
  - And 50%?
  - **Where to draw the line from** *sunny* **to** *not sunny*?
  - Member and not member or **membership degree**?

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Probability vs. Possibility

- **Event u**: Hans ate X eggs for breakfast.
- Probability distribution: $P_X(u)$
- Possibility distribution: $\pi_X(u)$

u	1	2	3	4	5	6	7	8
$P_X(u)$	0.1	0.8	0.1	0	0	0	0	0
$\pi_X(u)$	1	1	1					

Probability vs. Fuzzy membership

- **You’re lost in the outback; dying of thirst**

"Honestly, the water hole is back
that way... Why would I lie?"

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Probability vs. Fuzzy membership

- **You come upon two bottles containing liquid**

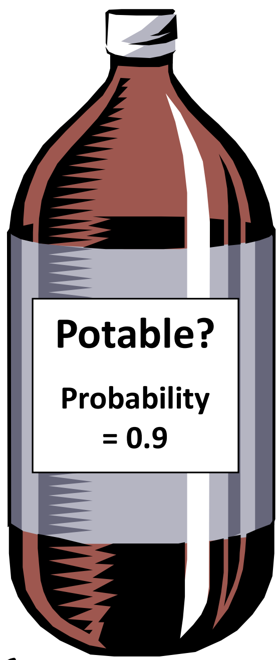
Probability vs. Fuzzy membership

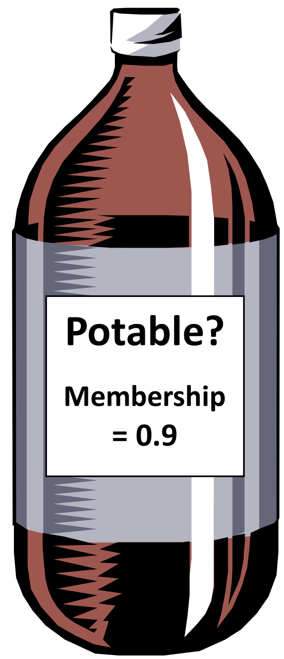
- **Which one will you choose?**

How will you process the information?

Probability vs. Fuzzy membership

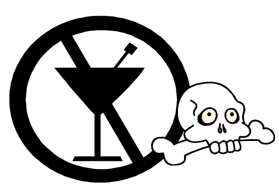
0.9 probability of belonging to the **set of non-poisonous liquid**





0.9 degree of membership to the **set of non-poisonous liquid**

OR



1 out of 10

Might taste funky, but shouldn't kill you

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Applications of fuzzy sets

- Fuzzy sets belong to "conventional" mathematics (measures, relations, topology, etc.)
- **Fuzzy logic** and **AI** (approximate reasoning, expert systems, etc.)
- **Fuzzy systems**
  - Fuzzy modeling
  - Fuzzy control, etc.
- **Fuzzy decision making**
  - Multi-criteria optimization
  - Optimization techniques
- …

Classical set theory

- **Set**: collection of objects with a common property.
- **Examples**:
  - Set of basic colors:

$$A = \{\text{red, green, blue}\}$$

  - Set of positive integers:

$$A = \{x \in \mathbf{Z} \mid x \geq 0\}$$

  - A line in $\mathbf{R}^3$:

$$A = \{(x,y,z) \mid ax + by + cz + d = 0\}$$

Representation of sets

- Enumeration of elements: $A = \{x_1, x_2,\ldots, x_n\}$
- Definition by property P: $A = \{x \in X \mid P(x)\}$
- ***Characteristic function*** $\mu_{A(x)}: X \longrightarrow \{0,1\}$

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \text{ is member of } A \\ 0, & \text{if } x \text{ is not member of } A \end{cases}$$

  - **Example:**
    - Set of odd numbers: $\mu_A(x) = x \bmod 2$

Set operations

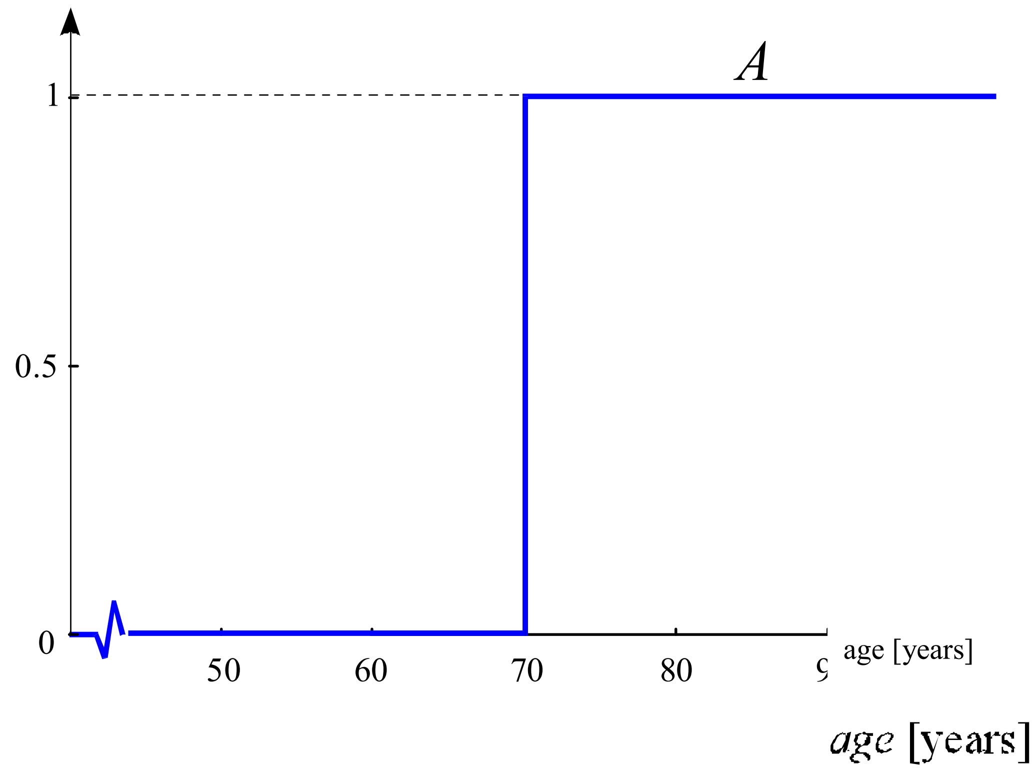
- **Intersection**: $C = A \cap B$
  - *C* contains elements that belong to A and B
  - Characteristic function: $\mu_C = \min(\mu_A\,,\, \mu_B) = \mu_A \cdot \mu_B$
- **Union**: $C = A \cup B$
  - *C* contains elements that belong to A or to B
  - Characteristic function: $\mu_C = \max(\mu_A\,,\, \mu_B)$
- **Complement**: $C = \bar{A}$
  - C contains elements that do not belong to A
  - Characteristic function: $\mu_C = 1 - \mu_A$

Fuzzy sets

- Represent ***uncertain*** (vague, ambiguous, etc.) knowledge in the form of propositions, rules, etc.
- Propositions:
  - expensive cars,
  - cloudy sky,...
- Rules (decisions):
  - Want to buy a big and new house for a low price.
  - **If** the temperature is ***low***, **then** ***increase*** the heating.
  - …

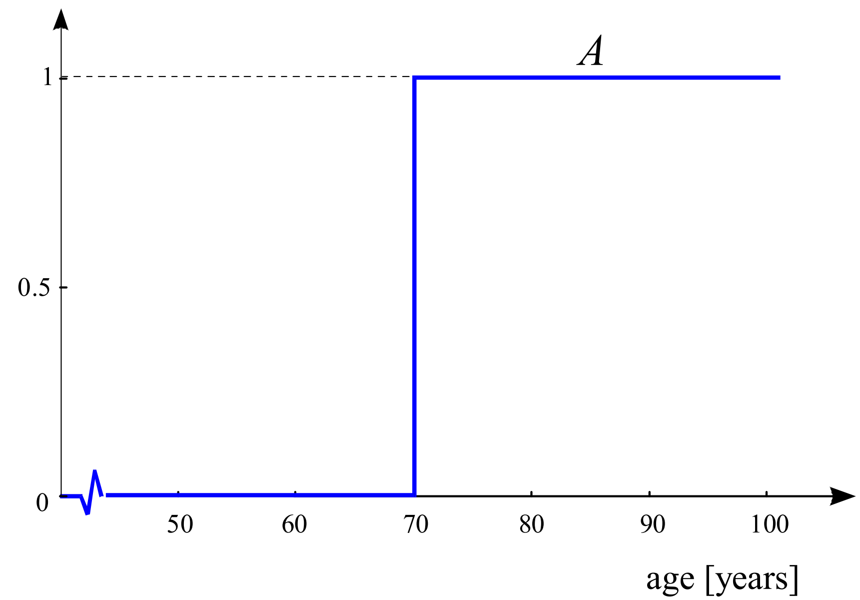
Classical set

- **Example**: set of *old people* $A = \{age \mid age \geq 70\}$



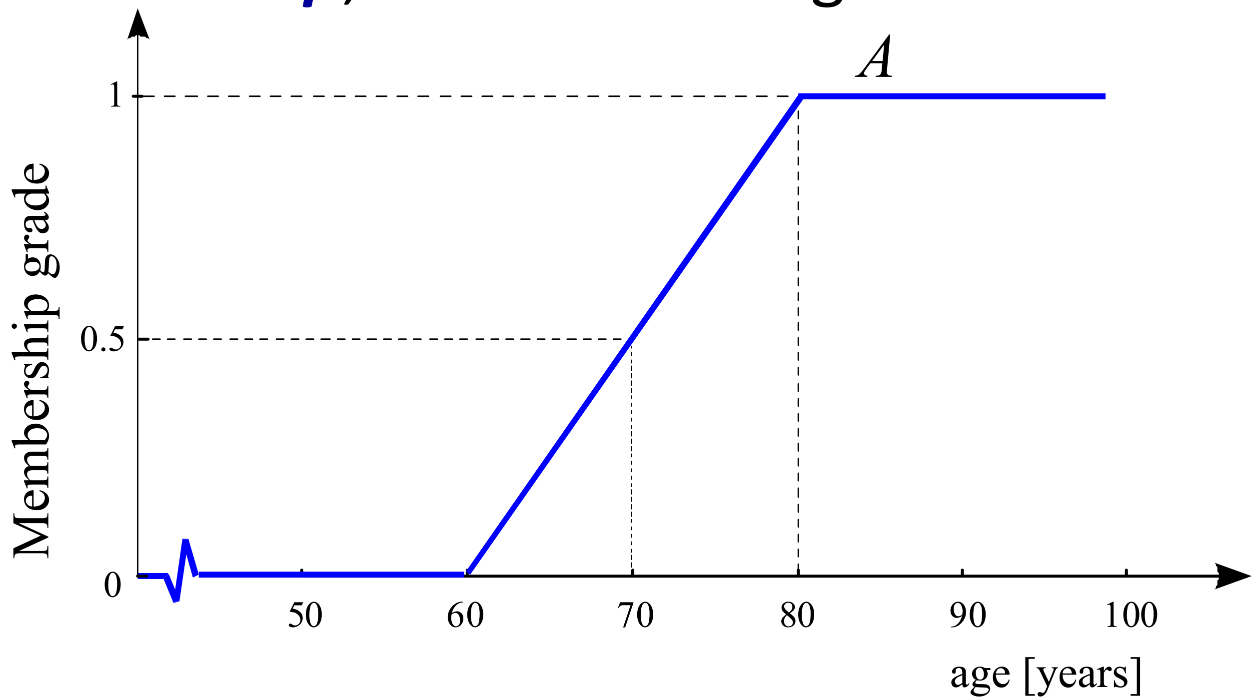
Logic propositions

- "Nick is old" ... true or false
- Nick's age:
  - $age_{\text{Nick}} = 70, \quad \mu_A(70) = 1$ (true)
  - $age_{\text{Nick}} = 69.9, \quad \mu_A(69.9) = 0$ (false)



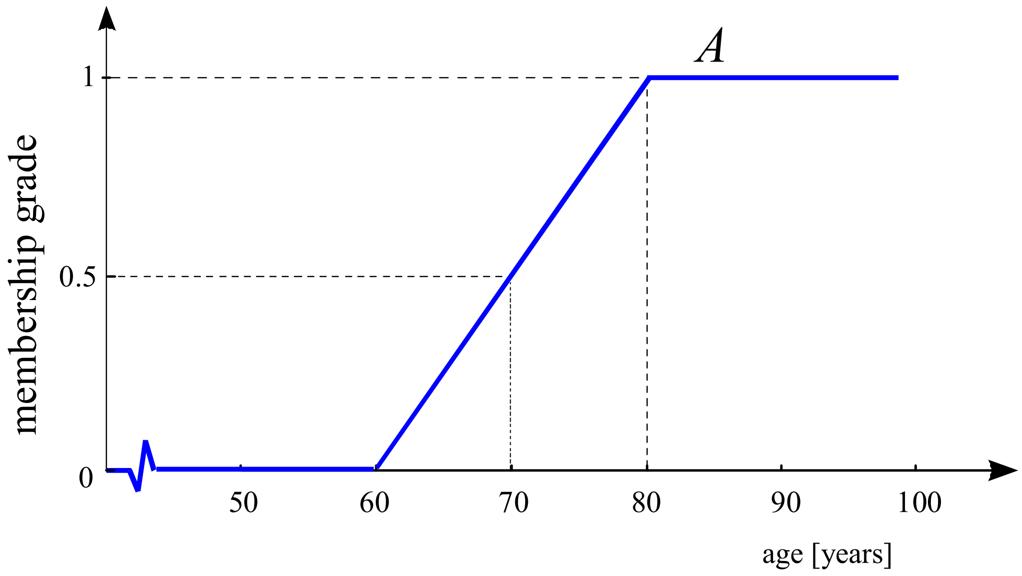
Fuzzy set

- ***Graded membership***, element belongs to a set to a certain degree.

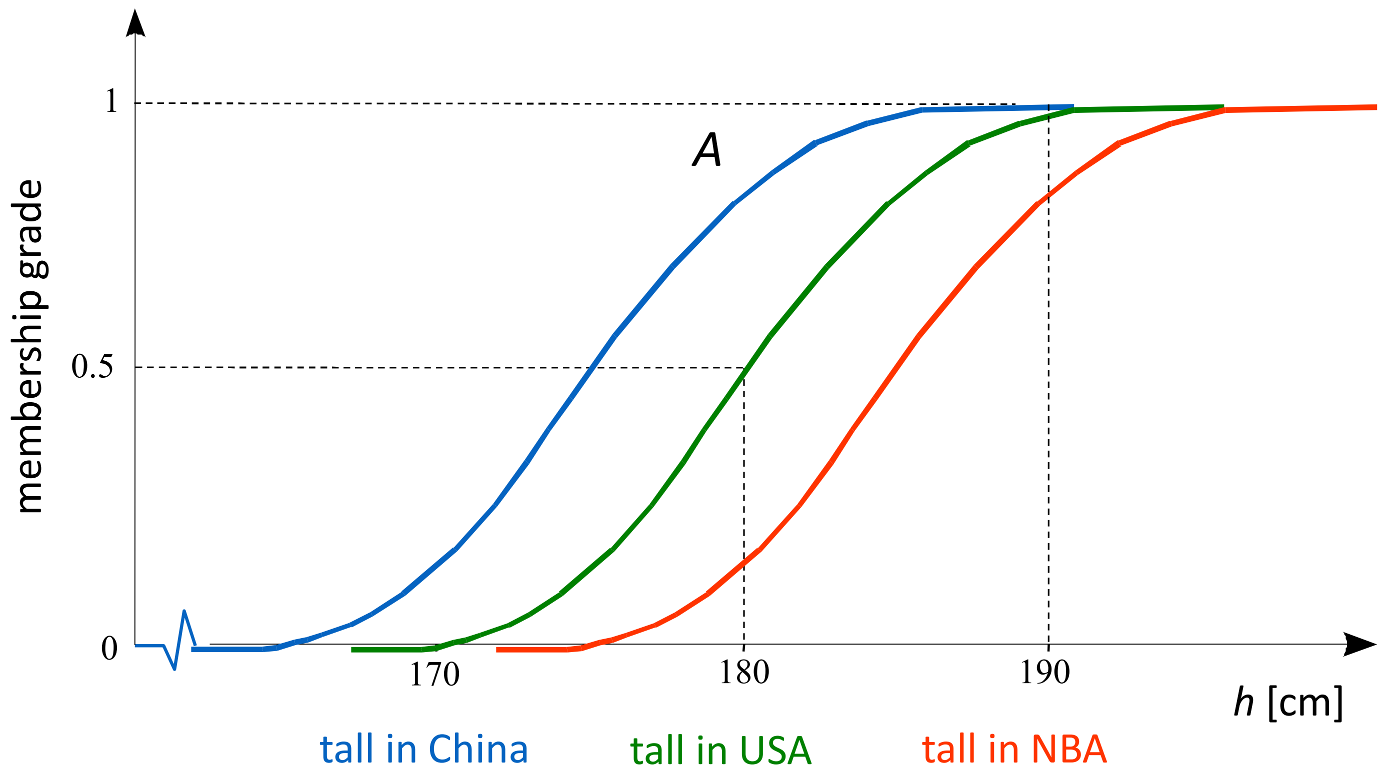


Fuzzy proposition

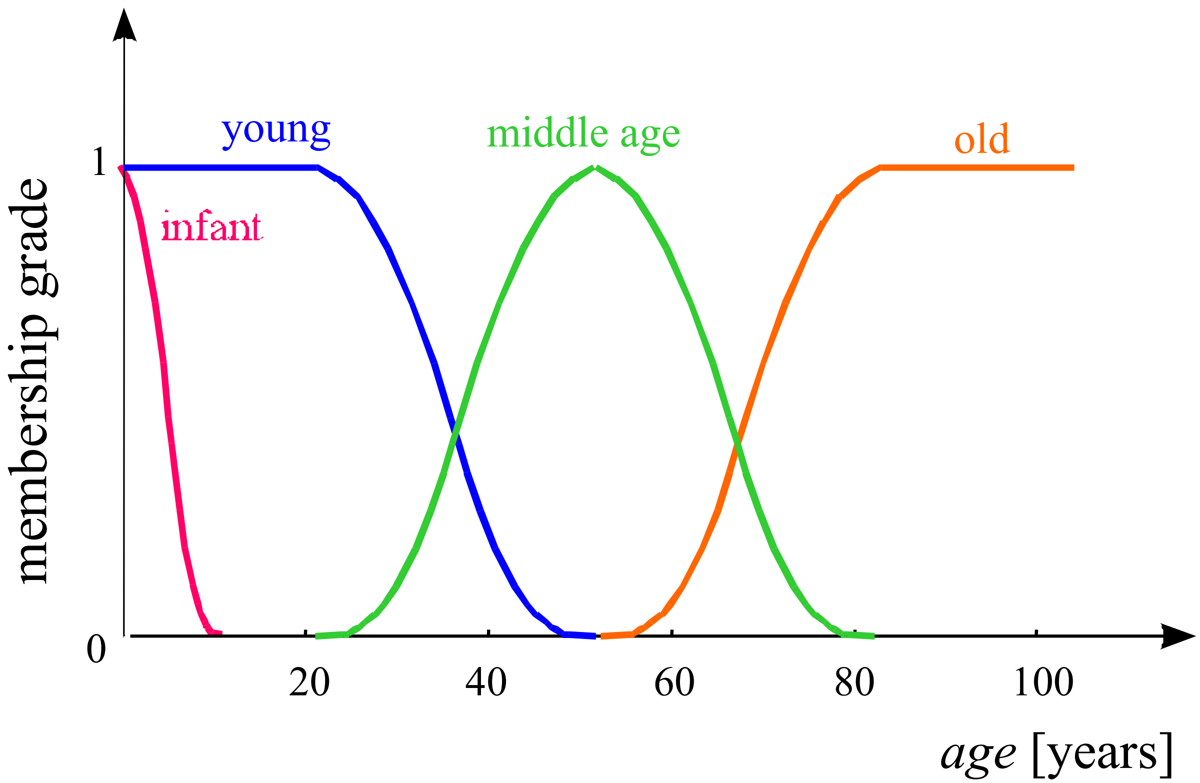
- "Nick is old"... degree of truth
  - $age_{\text{Nick}} = 70$, $\mu_A(70) = 0.5$
  - $age_{\text{Nick}} = 69.9$, $\mu_A(69.9) = 0.49$
  - $age_{\text{Nick}} = 90$, $\mu_A(90) = 1$



Context dependent

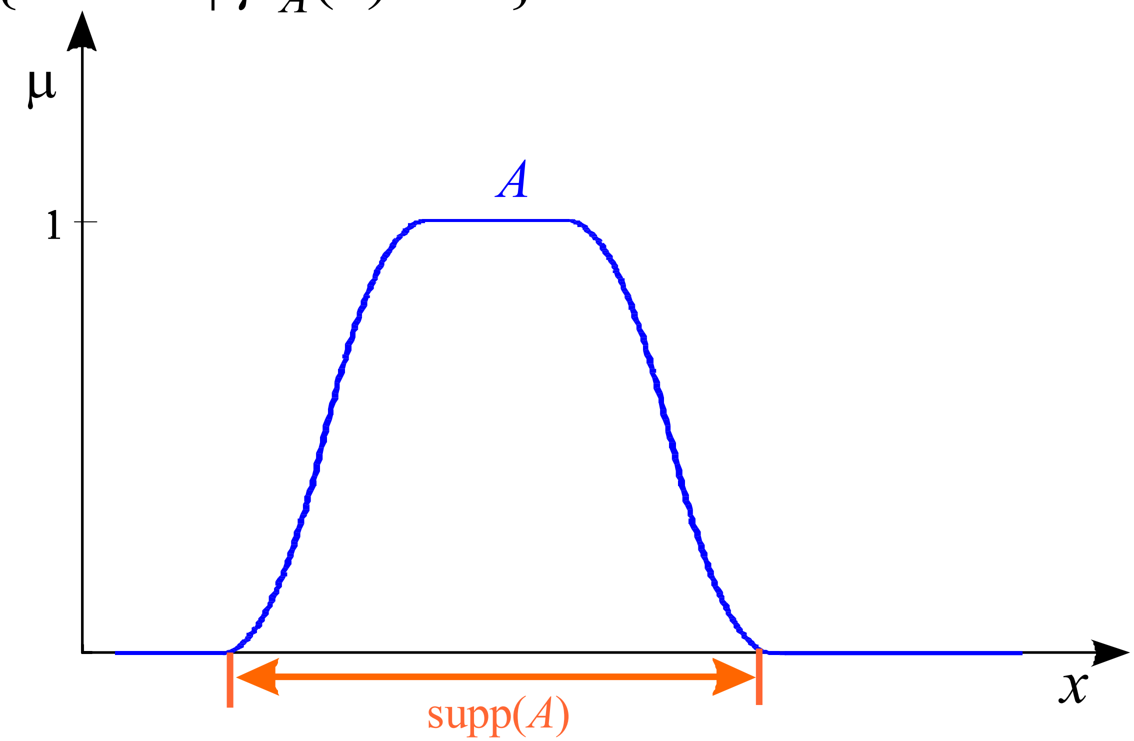


Typical linguistic values



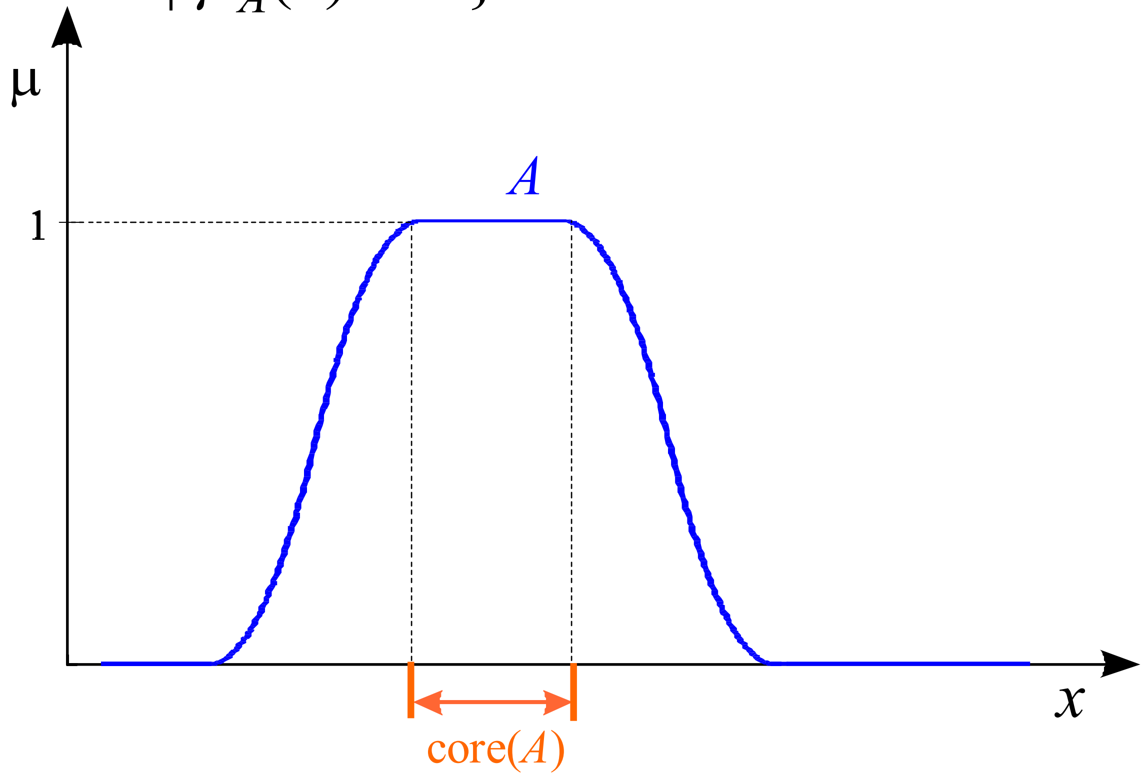
Support of a fuzzy set

- $\mathrm{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$



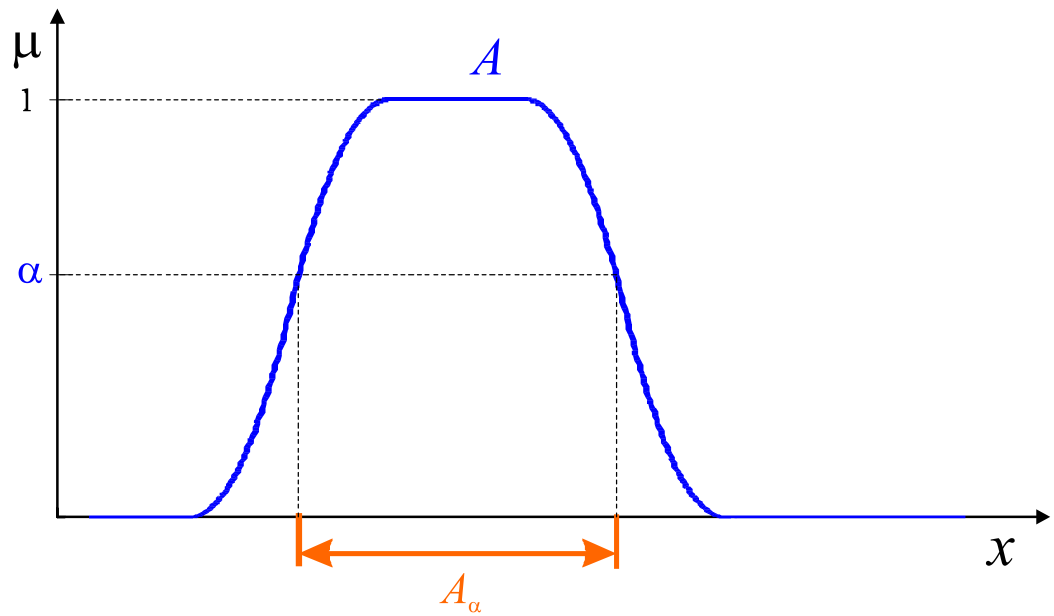
Core (nucleous, kernel)

- $\mathrm{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$



α-cut of a fuzzy set

- α-cut is the crisp set: $A_\alpha = \{\, x \in X \mid \mu_A(x) \geq \alpha \}$
- ***Strong*** α-cut: $A_\alpha = \{\, x \in X \mid \mu_A(x) > \alpha \}$



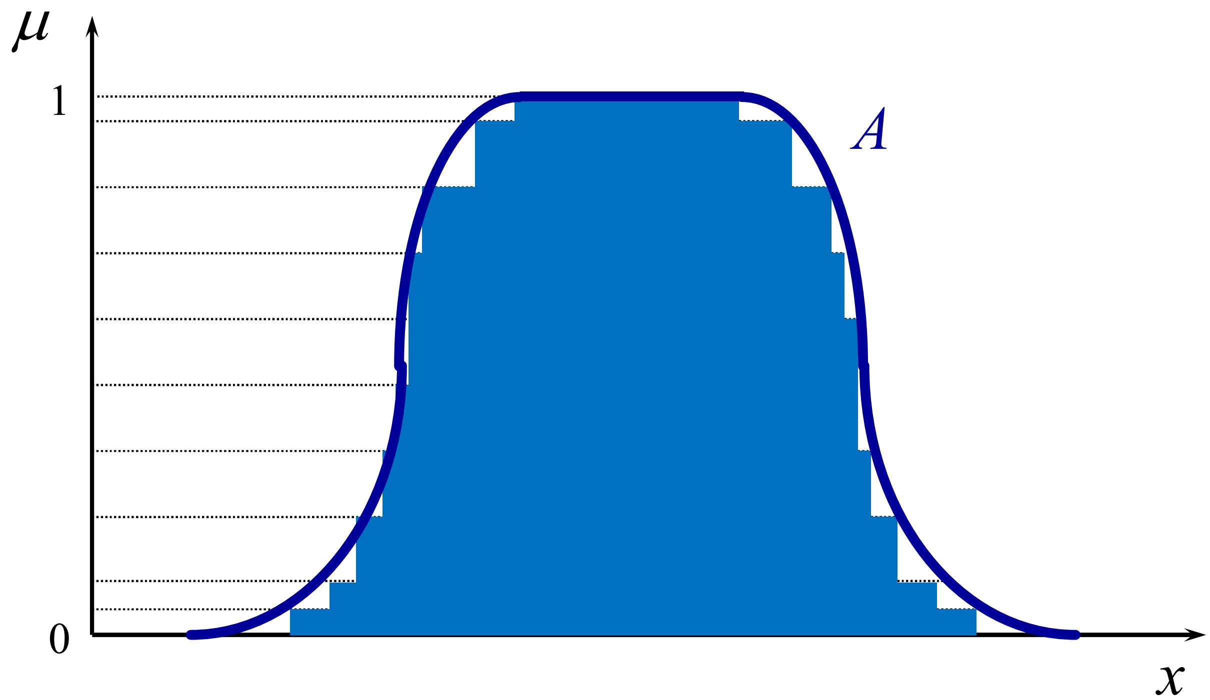
Resolution principle

- Every fuzzy set A can be uniquely represented as a collection of α-level sets according to

$$\mu_A(x) = \sup_{\alpha \in [0,1]} \{\alpha \in [0,1]\ x \in A_\alpha\}$$

- **Resolution principle** implies that fuzzy set theory is a generalization of classical set theory, and that its results can be represented in terms of classical set theory.

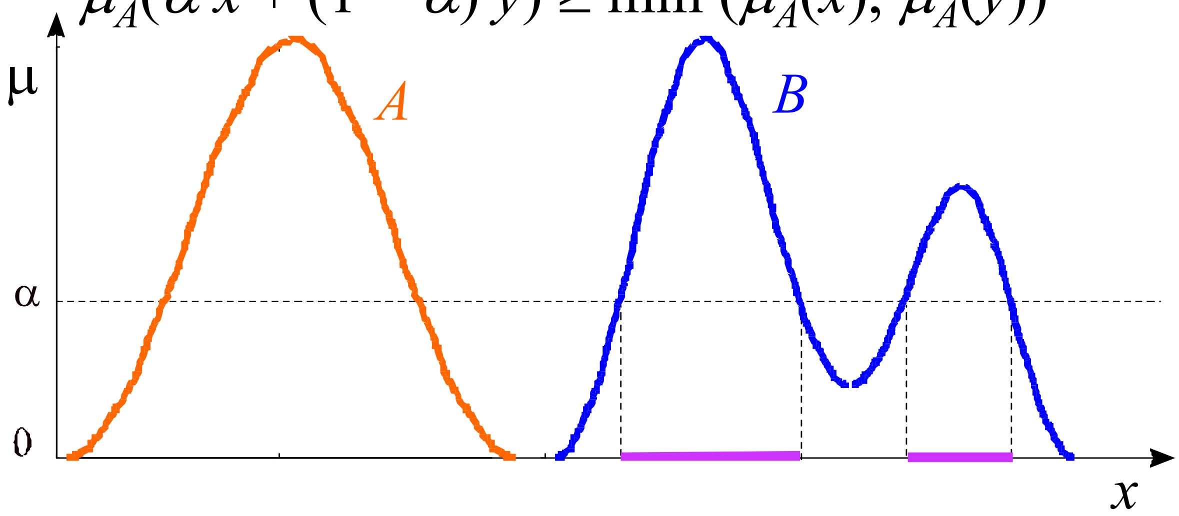
Resolution principle



Other properties

- **Height** of a fuzzy set: $\mathrm{hgt}\,(A) = \sup \mu_A(x),\ x \in X$
- Fuzzy set is **normal(ized)** when $\mathrm{hgt}\,(A) = 1$.
- A fuzzy set A is **convex** iff $\forall\ x,y \in X$ and $\alpha \in [0,1]$:

$$\mu_A(\alpha\, x + (1 - \alpha)\, y) \geq \min\,(\mu_A(x),\ \mu_A(y))$$



Other properties (2)

- **Fuzzy singleton**: single point $x \in X$ where $\mu_A(x) = 1$.
- **Fuzzy number**: fuzzy set in R that is ***normal*** and ***convex***.
- Two fuzzy sets are ***equal*** $(A = B)$ iff:

$$\forall x \in X,\ \mu_A(x) = \mu_B(x)$$

- A is a ***subset*** of B iff:

$$\forall x \in X,\ \mu_A(x) \leq \mu_B(x)$$

Other properties (3)

- **Bandwith (**or **width)**: of ***normal*** and ***convex*** fuzzy sets is defined as the distance between two unique crossover points:

$$\text{width}(A) = |x_2 - x_1|, \text{ where } \mu_A(x_1) = \mu_A(x_2) = 0.5.$$

- **Symmetry**: a fuzzy set A is symmetric if its μ_A is symmetric around a certain point $x =$ c:

$$\mu_A(\mathrm{c} + x) = \mu_A(x + c),\ \forall x \in X,$$

- **Open left, open right, closed:**

$$\lim x \to -\infty\ \mu_A(x) = 1 \text{ and } \lim x \to +\infty\ \mu_A(x) = 0$$

$$\lim x \to -\infty\ \mu_A(x) = 0 \text{ and } \lim x \to +\infty\ \mu_A(x) = 1$$

$$\lim x \to -\infty\ \mu_A(x) = \lim x \to +\infty\ \mu_A(x) = 0$$

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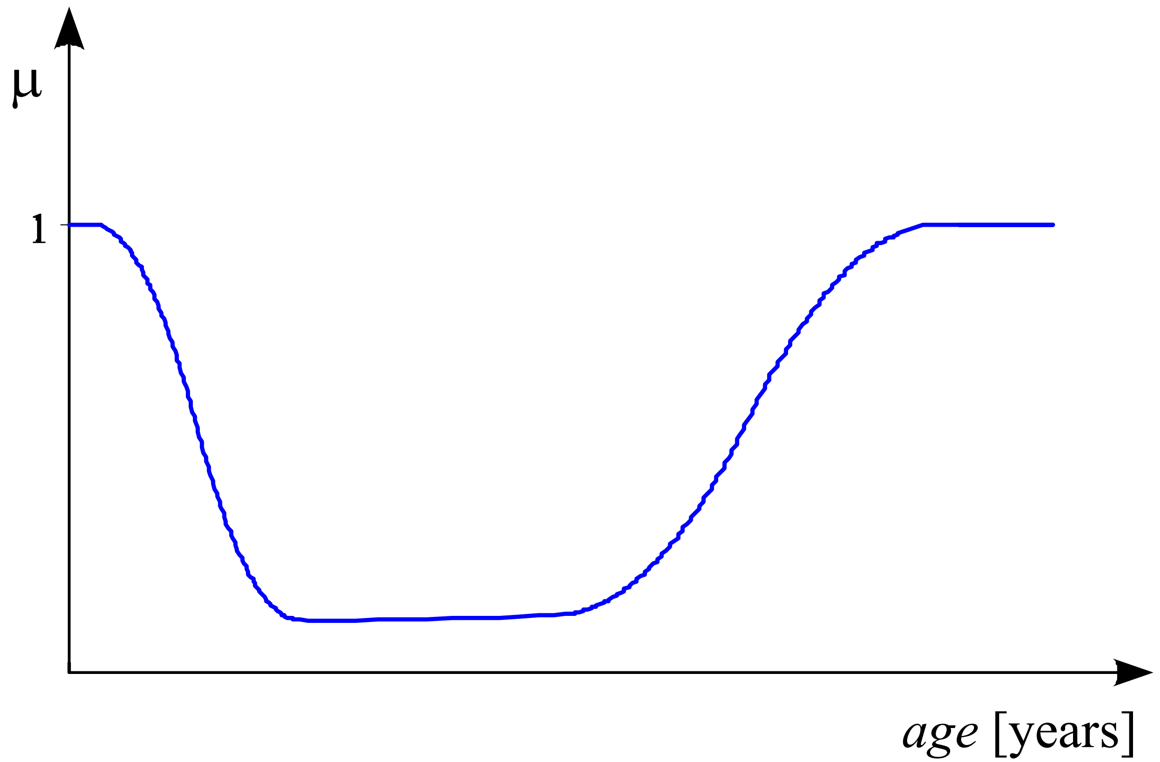
Non-convex fuzzy sets

- **Example**: car insurance risk



Representation of fuzzy sets

Discrete Universe of Discourse:

- Point-wise as a list of membership/element pairs:
  - $A = \mu_A(x_1)/x_1 + \ldots + \mu_A(x_n)/x_n = \Sigma_i\, \mu_A(x_i)/x_i$
  - $A = \{\mu_A(x_1)/x_1, \ldots, \mu_A(x_n)/x_n\} = \{\mu_A(x_i)/x_i \mid x_i \in X\}$

- As a list of α-level/ α-cut pairs:
  - $A = \{\alpha_1/A_{\alpha_1}, \ldots, \alpha_n/A_{\alpha_n}\} = \{\alpha_i/A_{\alpha_i} \mid \alpha_i \in [0,1]\}$

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Representation of fuzzy sets

Continuous Universe of Discourse:

- $A = \int_X \mu_A(x)/x$
- Analytical formula: $\mu_A(x) = \dfrac{1}{1+x^2}, \quad x \in \mathbf{R}$
- Various possible notations:
  - $\mu_A(x)$, $A(x)$, A, a, etc.

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Examples

Discrete universe

- Fuzzy set A =“sensible number of children”.
  - number of children: $X = \{0, 1, 2, 3, 4, 5, 6\}$
  - $A = 0.1/0 + 0.3/1 + 0.7/2 + 1/3 + 0.6/4 + 0.2/5 + 0.1/6$

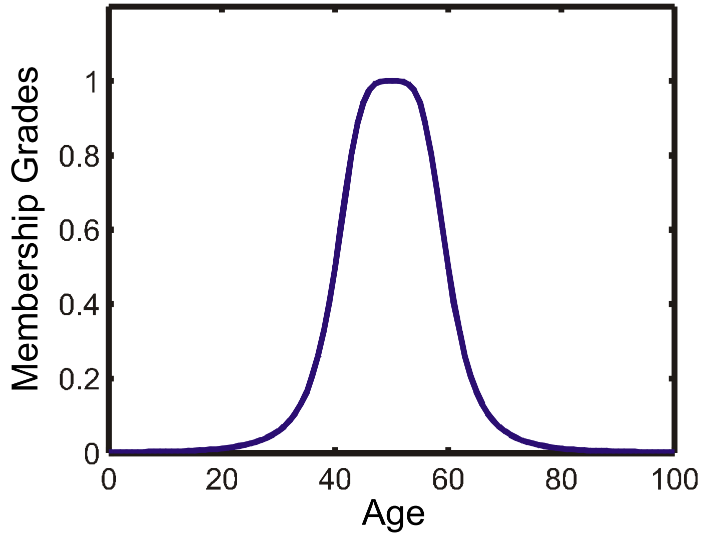
- Fuzzy set C = “desirable city to live in”
  - $X = \{\text{SF, Boston, LA}\}$ (discrete and non-ordered)
  - $C = \{(\text{SF}, 0.9), (\text{Boston}, 0.8), (\text{LA}, 0.6)\}$

Examples

Continuous universe

- Fuzzy set B = "about 50 years old"
  - X = R⁺ (set of positive real numbers)
  - $B = \{(x, \mu_B(x)) \mid x \in X\}$

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}$$



Complement of a fuzzy set

$$c: [0,1] \longrightarrow [0,1]; \qquad \mu_A(x) \longrightarrow c(\mu_A(x))$$

- **Fundamental axioms**

1. ***Boundary conditions*** **-** c behaves as the ordinary complement

$$c(0) = 1; \qquad c(1) = 0$$

2. ***Monotonic non-increasing***

$$\forall\ a,b \in [0,1], \text{ if } a < b, \text{ then } c(a) \geq c(b)$$

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Complement of a fuzzy set

Other axioms:

- c is a ***continuous*** function.
- c is ***involutive***, which means that

$$c(c(a)) = a, \ \ \forall a \in [0,1]$$

Complement of a fuzzy set

Equilibrium point

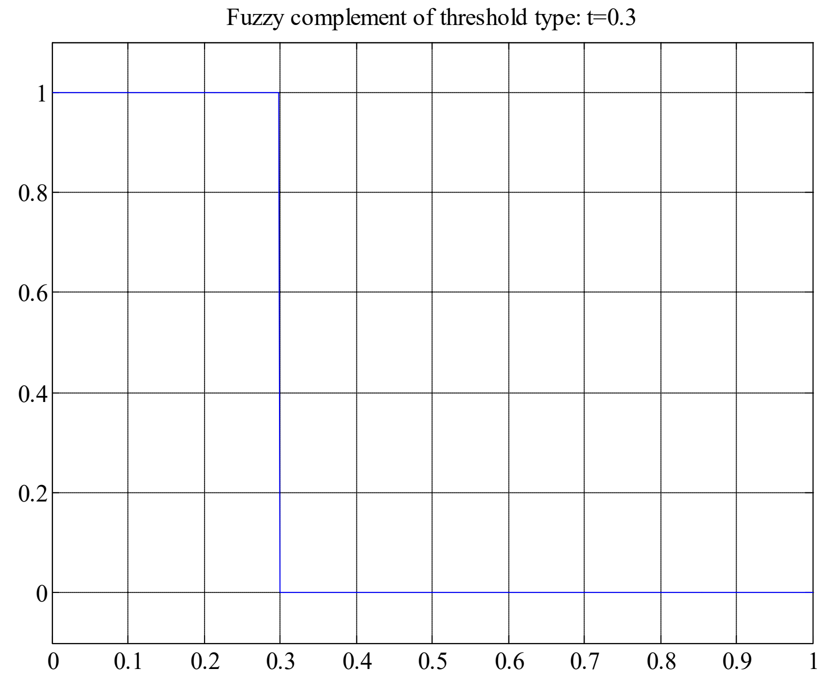
$$c(a) = a = e_c, \ \ \forall a \in [0,1]$$

- Each complement has at most one equilibrium.
- If c is a continuous fuzzy complement, it has one equilibrium point.

Examples of fuzzy complements

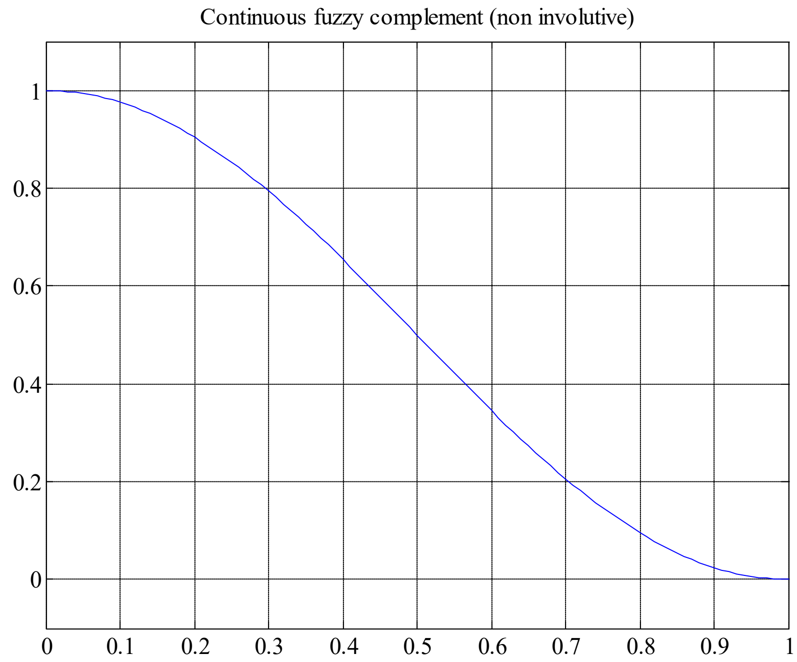
- **Standard complement:**
  - Satisfying only fundamental axioms:

$$c(a)=\begin{cases}1, & \text{if } a \le t\\ 0, & \text{if } a > t\end{cases}$$



Examples of fuzzy complements

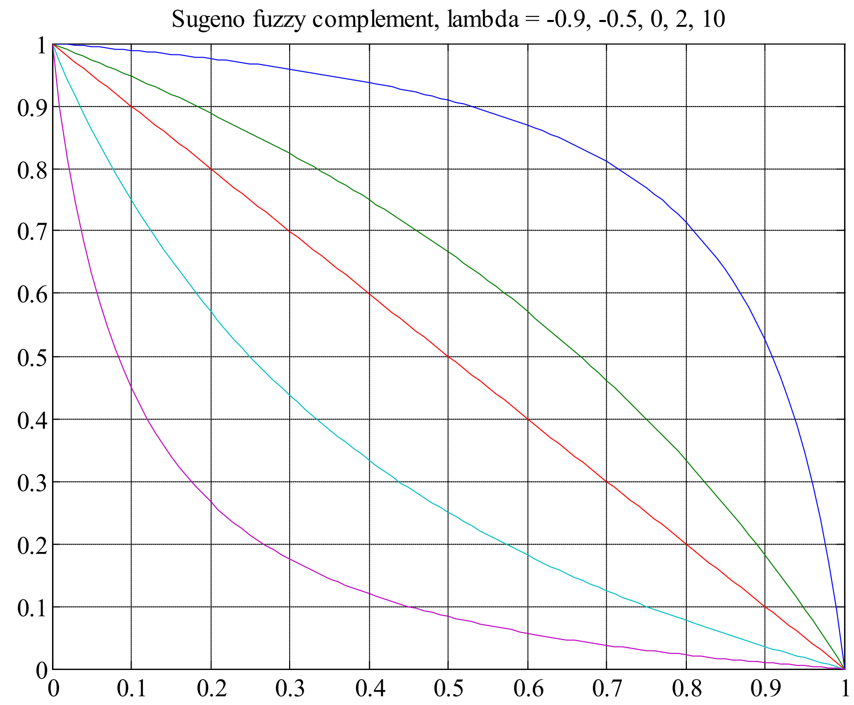
- **Cosine complement:** $$c(a) = \frac{1}{2}(1 + \cos \pi a)$$
  - Satisfying fundamental axioms and continuity:



Examples of fuzzy complements

- **Sugeno complement:**

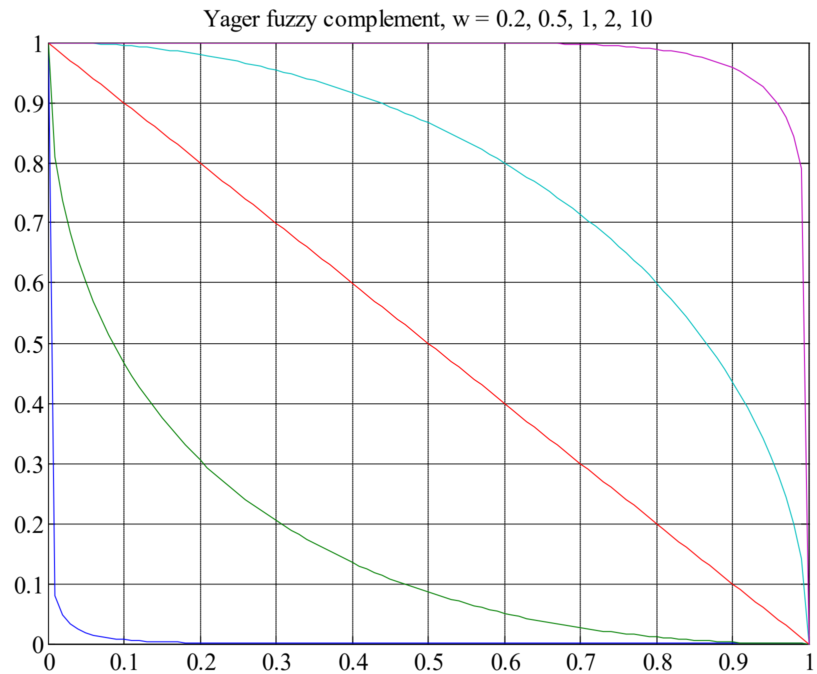
$$c_\lambda(a) = \frac{1-a}{1+\lambda a},\ \ \lambda \in]-1, \infty]$$



Examples of fuzzy complement

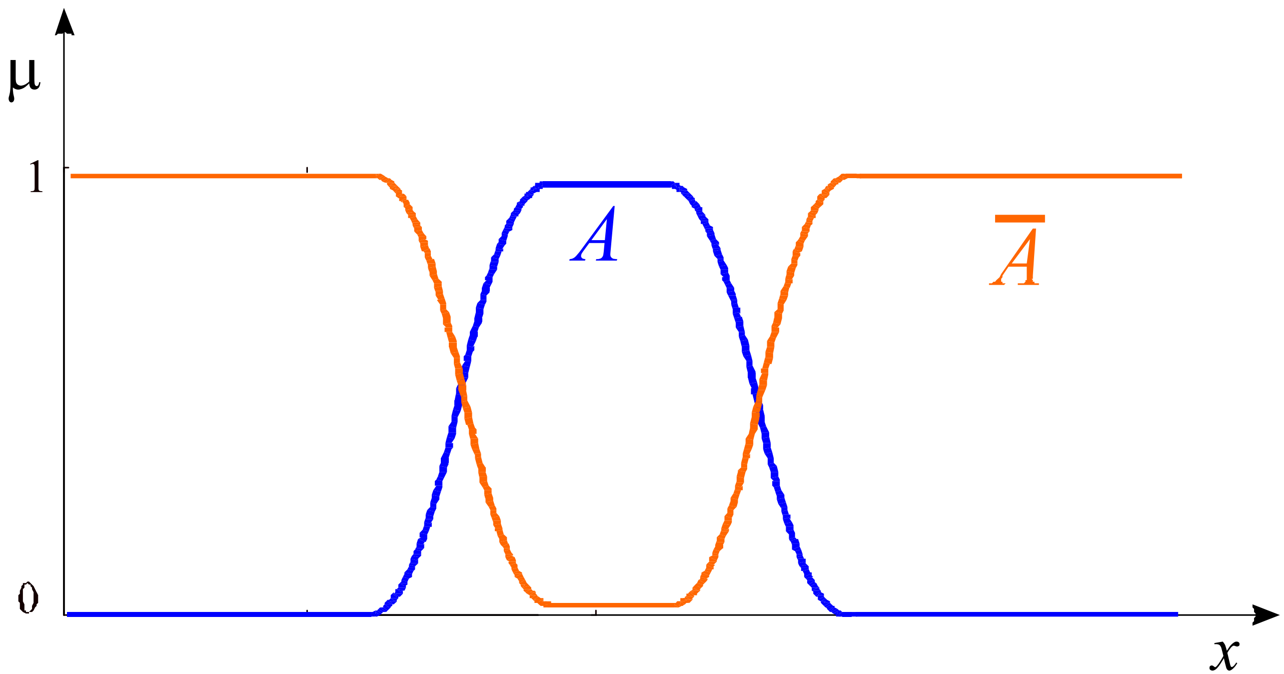
- **Yager complement:**

$$c_w(a) = \left(1 - a^w\right)^{1/w}, \quad w \in]0, \infty]$$



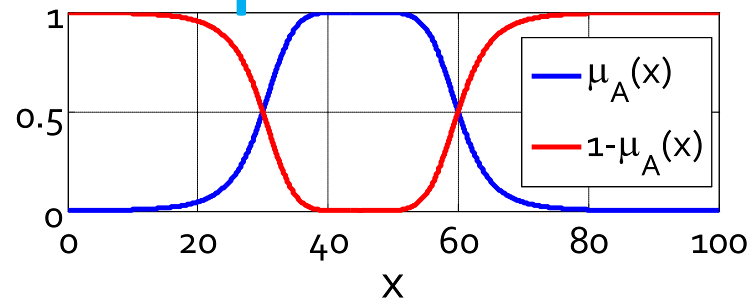
Representation of complement

- $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$

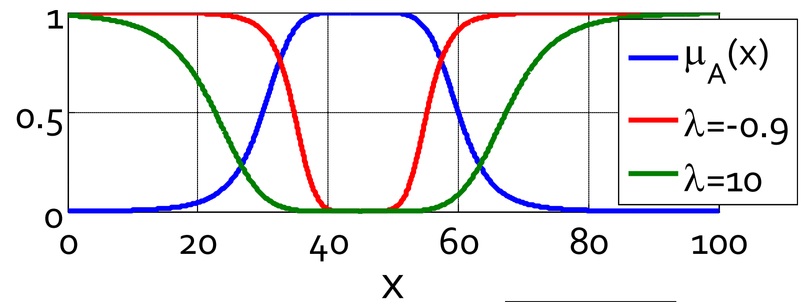


Representation of complement

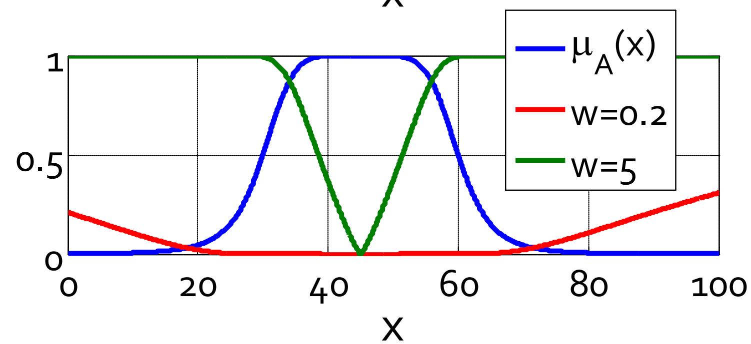
Standard complement



Sugeno complement



Yager complement



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Intersection of fuzzy sets

$$i: [0,1] \times [0,1] \rightarrow [0,1];$$
$$\mu_{A\cap B}(x) \rightarrow i(\mu_A(x),\, \mu_B(x))$$

- **Fundamental axioms:** *triangular norm* or *t*-norm

1. ***Boundary conditions*** **-** i behaves as the classical intersection

$$i(1,1) = 1;$$
$$i(0,1) = i(1,0) = i(0,0) = 0$$

2. ***Commutativity***

$$i(a,b) = i(b,a)$$

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Intersection of fuzzy sets

3. Monotonicity

If $a \leq a'$ and $b \leq b'$, then $i(a,b) \leq i(a',b')$

4. Associativity

$i(i(a,b),c) = i(a,i(b,c))$

- **Other axioms:**
  - i is a ***continuous*** function.
  - $i(a,a) = a$ (idempotent).

Examples of fuzzy conjunctions

- **Zadeh**

$$\mu_{A\cap B}(x) = \min(\mu_A(x),\ \mu_B(x))$$

- **Probabilistic (or algebraic product)**

$$\mu_{A\cap B}(x) = \mu_A(x)\ .\ \mu_B(x)$$

- **Łukaziewicz**

$$\mu_{A\cap B}(x) = \max(0,\ \mu_A(x) + \mu_B(x) - 1)$$

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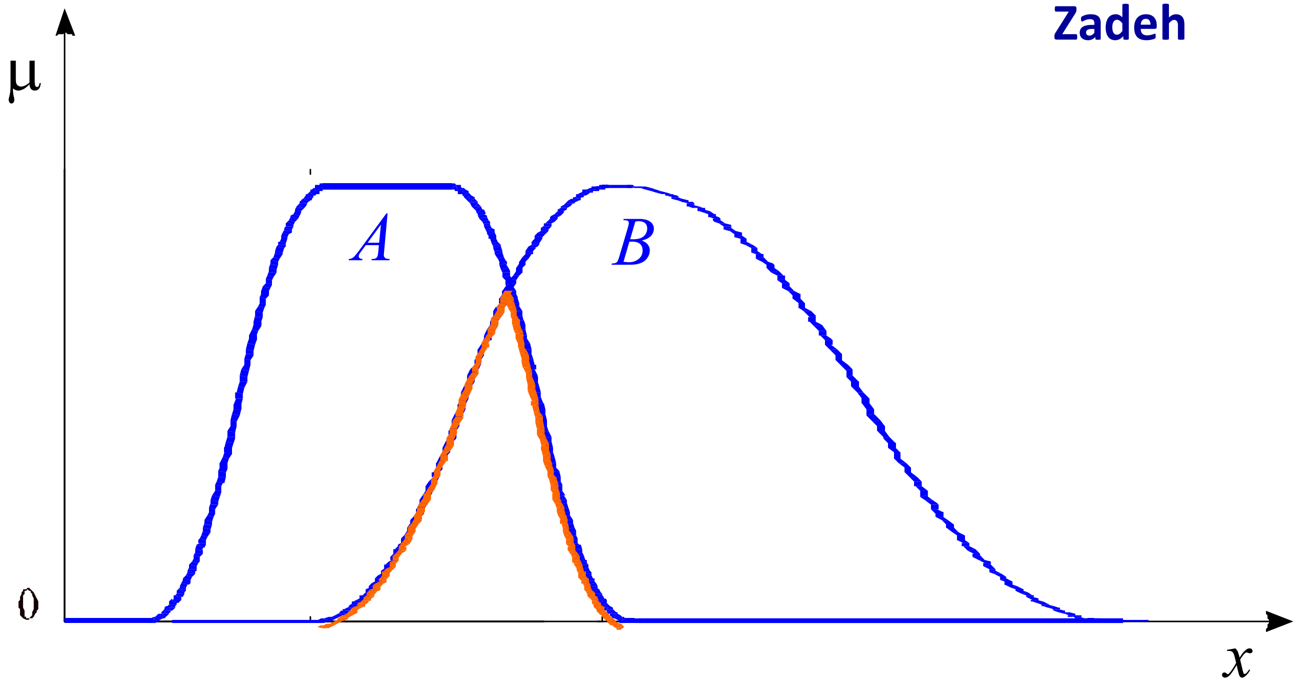
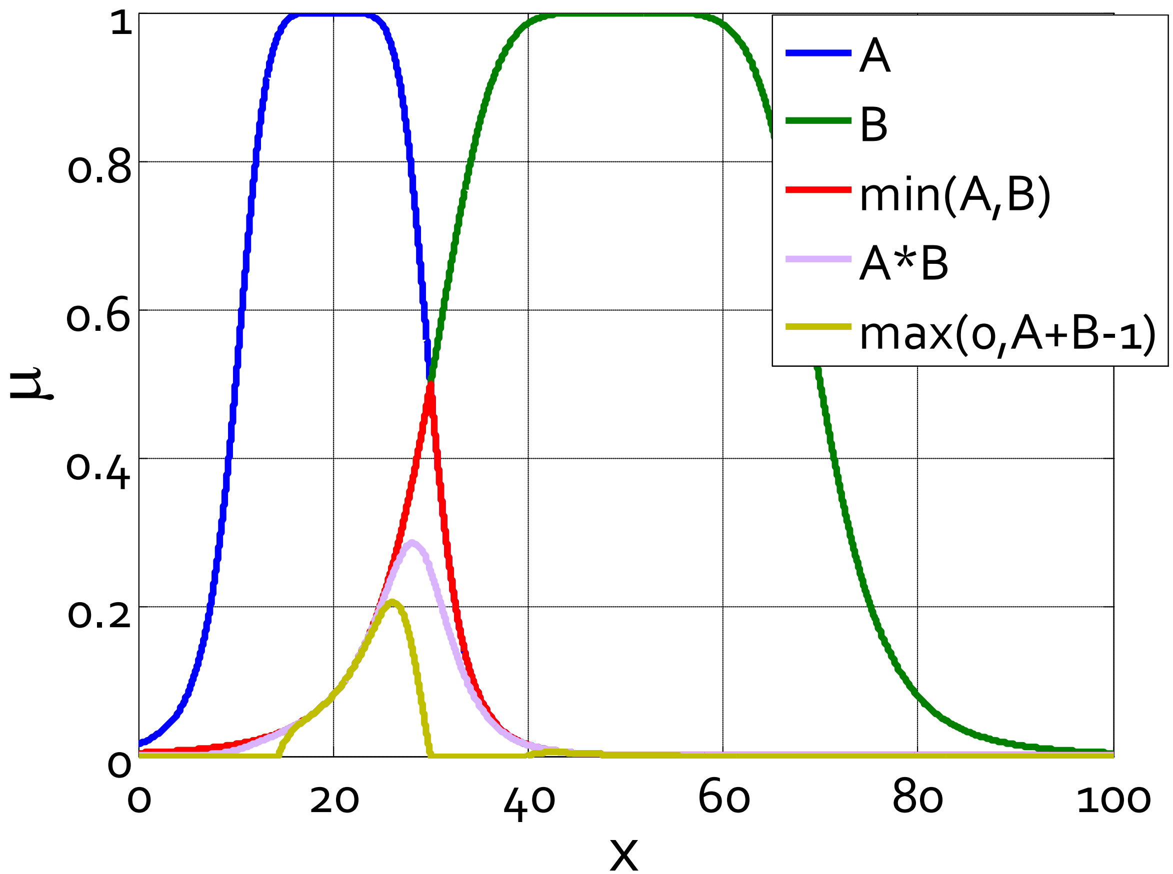
Intersection of fuzzy sets

- $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

Zadeh



Intersection of fuzzy sets



Zadeh

Probabilistic

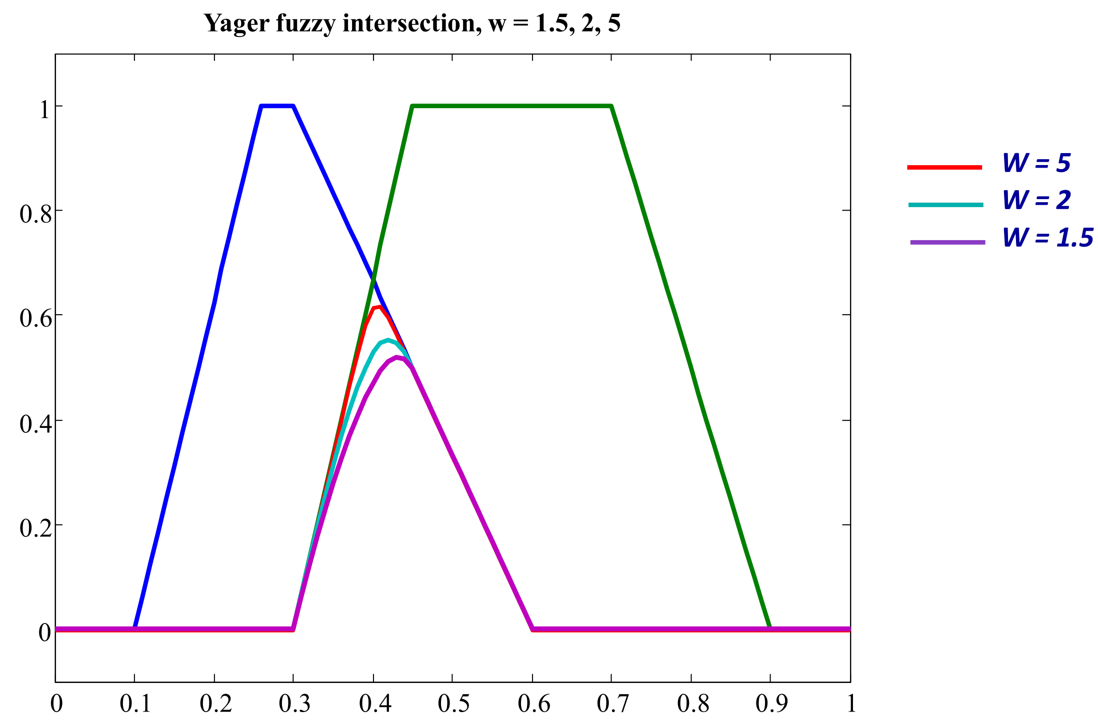
Łukaziewicz

Yager *t*-norm

$$i_w(a,b) = 1 - \min\left[1, \left((1-a)^w + (1-b)^w\right)^{1/w}\right], \; w \in]0, \infty]$$

- Example of ***weak*** and ***strong*** intersections:

Parametric t-norm



Union of fuzzy sets

$$u: [0,1] \times [0,1] \to [0,1];$$
$$\mu_{A \cup B}(x) \to u(\mu_A(x), \mu_B(x))$$

- **Fundamental axioms:** *triangular co-norm* or *s*-norm

1. ***Boundary conditions*** **-** u behaves as the classical union

$$u(0,0) = 0;$$
$$u(0,1) = u(1,0) = u(1,1) = 1$$

2. ***Commutativity***

$$u(a,b) = u(b,a)$$

Union of fuzzy sets

3. Monotonicity

If $a \leq a'$ and $b \leq b'$, then $u(a,b) \leq u(a',b')$

4. Associativity

$$u(u(a,b),c) = u(a,u(b,c))$$

- **Other axioms:**
  - u is a ***continuous*** function.
  - $u(a,a) = a$ (idempotent).

Examples of fuzzy disjunctions

- **Zadeh**

$$\mu_{A\cup B}(x) = \max(\mu_A(x),\mu_B(x))$$

- **Probabilistic (algebraic sum)**

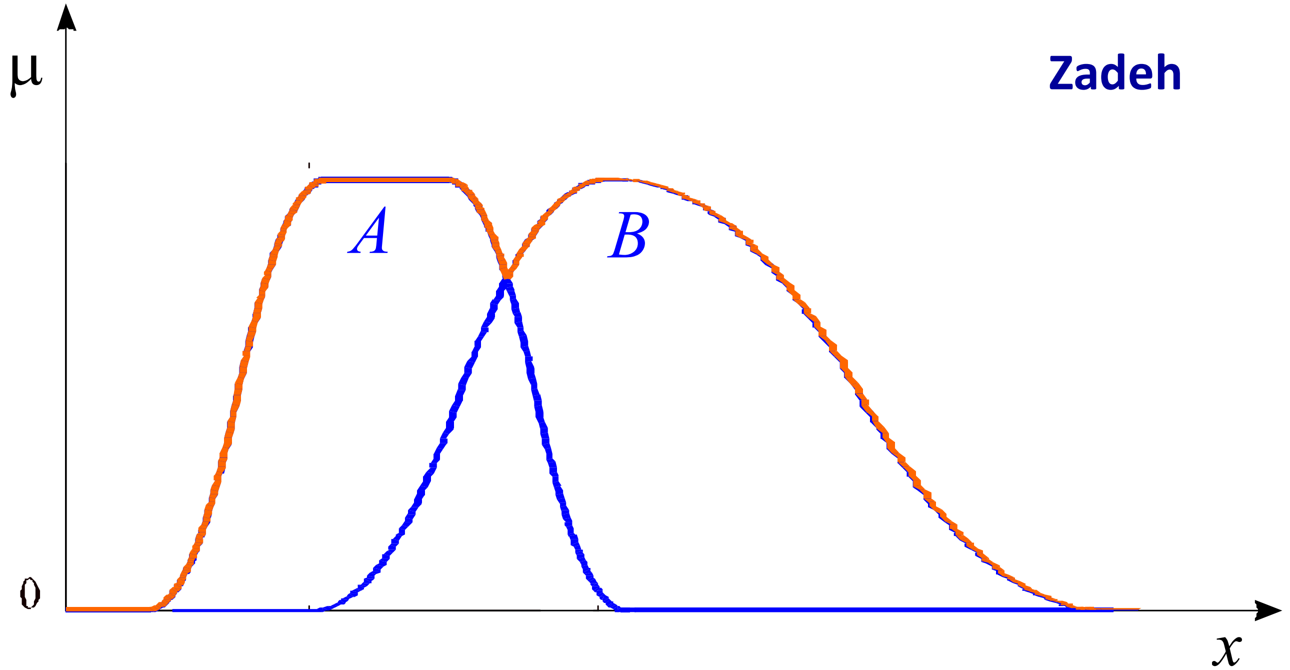
$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\ .\ \mu_B(x)$$

- **Łukasiewicz**

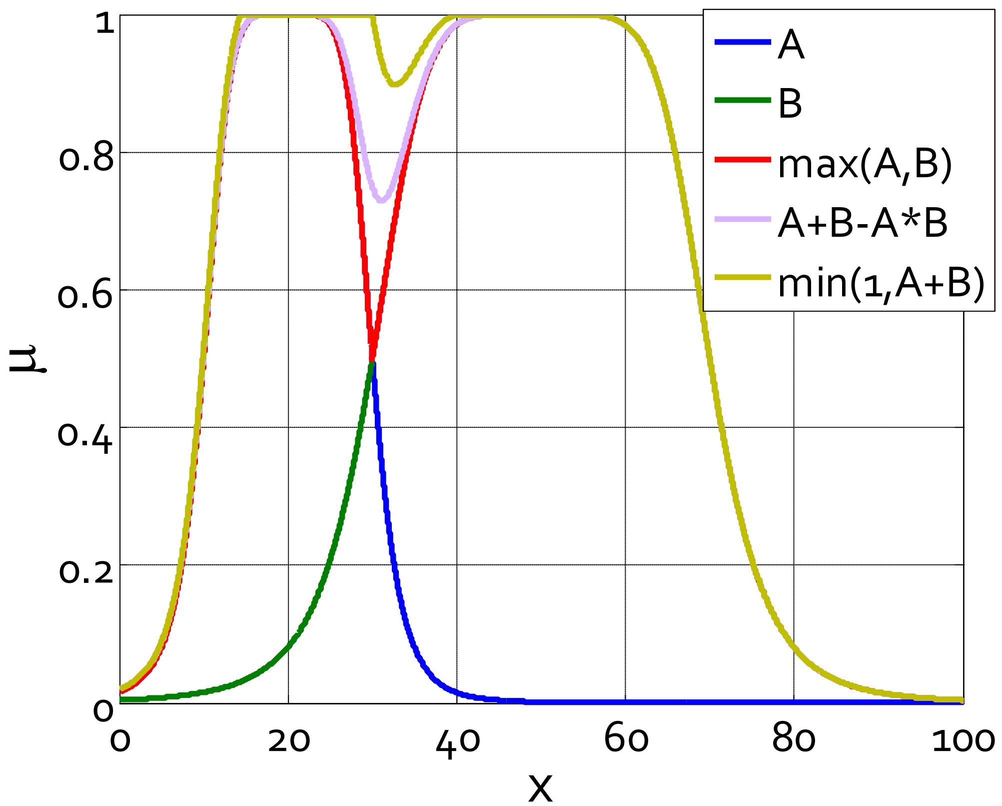
$$\mu_{A\cup B}(x) = \min(1,\ \mu_A(x) + \mu_B(x))$$

Union of fuzzy sets

- $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$



Union of fuzzy sets



Zadeh

Probabilistic

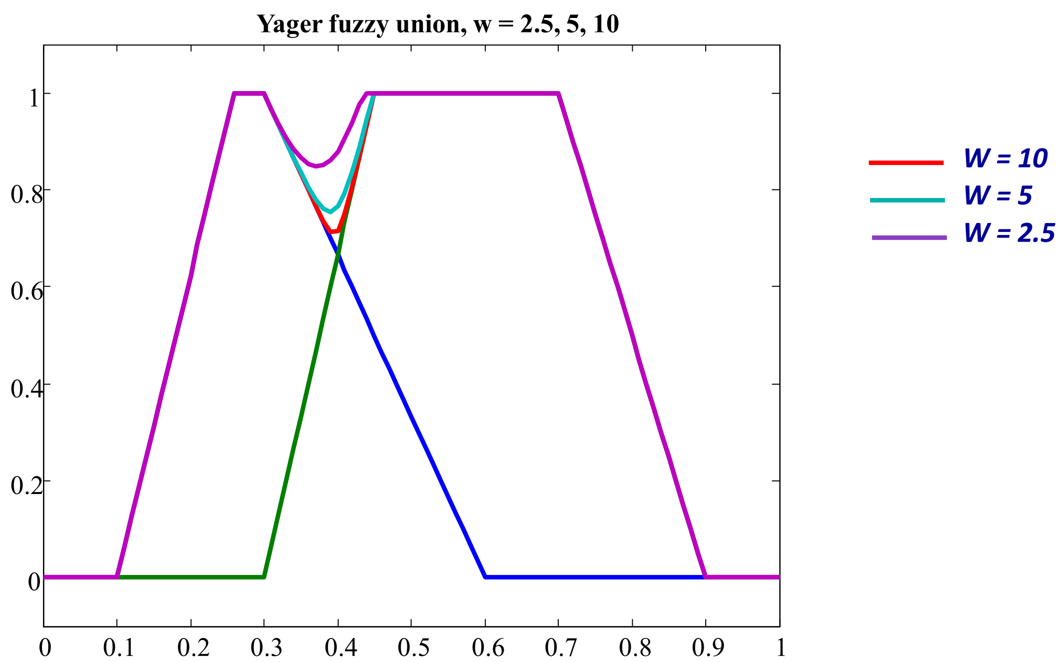
Łukaziewicz

Yager *t*-conorm (s-norm)

$$u_w(a,b) = \min\left[1, \left(a^w + b^w\right)^{1/w}\right], \; w \in]0,\infty]$$

- Example of ***weak*** and ***strong*** disjunctions:

Parametric t-conorm



General aggregation operations

$$h\colon [0,1]^n \rightarrow [0,1];$$
$$\mu_A(x) \rightarrow h(\mu_{A_1}(x),\ldots,\mu_{A_n}(x))$$

- **Axioms**

1. Boundary conditions

$$h(0,\ldots,0) = 0$$
$$h(1,\ldots,1) = 1$$

2. Monotonic non-decreasing

For any pair a_i, $b_i \in [0,1]$, $i \in$ **N**

If $a_i \geq b_i$ then $h(a_i) \geq h(b_i)$

General aggregation operations

- **Other axioms:**
  - *h* is a ***continuous*** function.
  - *h* is a ***symmetric*** function in all its arguments:

$$h(a_i) = h(a_{p(i)})$$

for any permutation p on **N**

Averaging operations

- When all the four axioms hold:

$$\min(a_1,\ldots,a_n) \le h(a_1,\ldots,a_n) \le \max(a_1,\ldots,a_n)$$

- Operator covering this range: **Generalized mean**

$$h_\alpha(a_1,\ldots,a_n) = \left(\frac{\left(a_1^\alpha + \ldots + a_n^\alpha\right)}{n} \right)^{1/\alpha}$$

Generalized mean

- Typical cases:
  - Lower bound: $h_{-\infty} = \min(a_1, \ldots, a_n)$
  - *Geometric mean*: $h_0 = (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{1/n}$
  - *Harmonic mean*: $h_{-1} = \dfrac{n}{\dfrac{1}{a_1} + \ldots + \dfrac{1}{a_n}}$
  - *Arithmetic mean*: $h_1 = \dfrac{a_1 + \ldots + a_n}{n}$
  - Upper bound: $h_{\infty} = \max(a_1, \ldots, a_n)$

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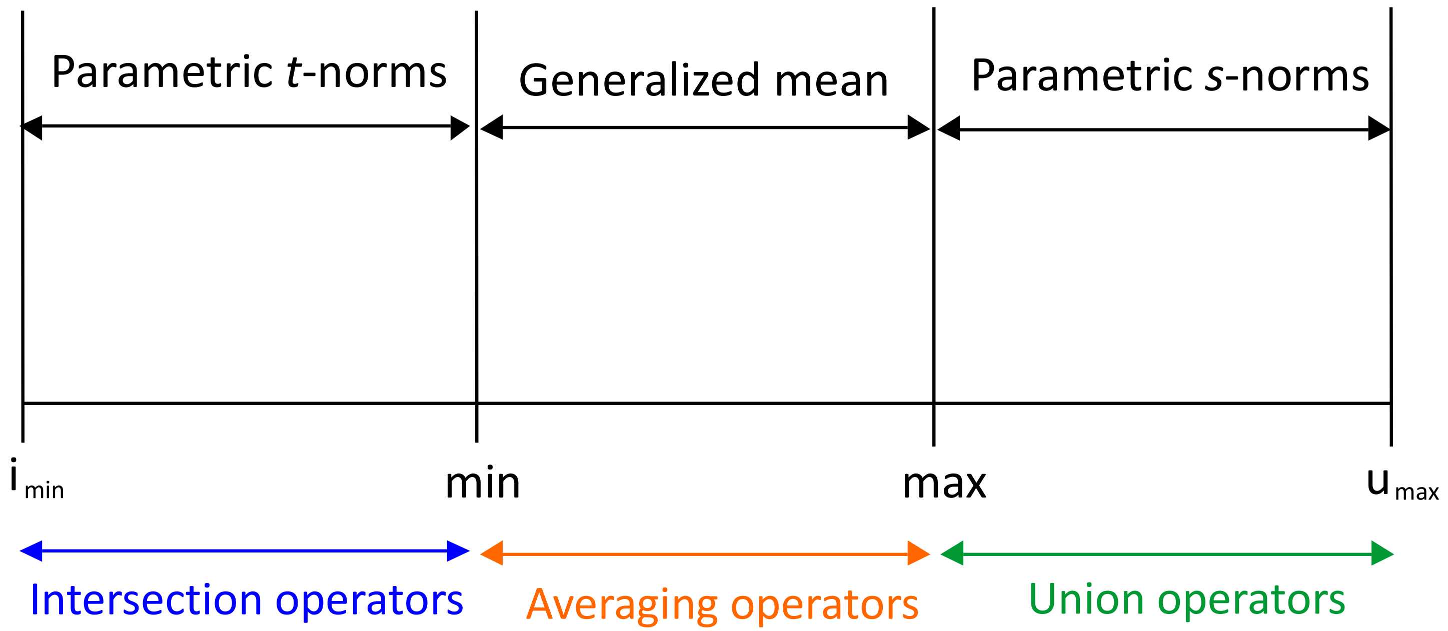
Fuzzy aggregation operations



Membership functions (MF)

- Triangular MF: $$Tr(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$$
- Trapezoidal MF: $$Tp(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),0\right)$$
- Gaussian MF: $$Gs(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$
- Generalized bell MF: $$Bell(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2a}}$$

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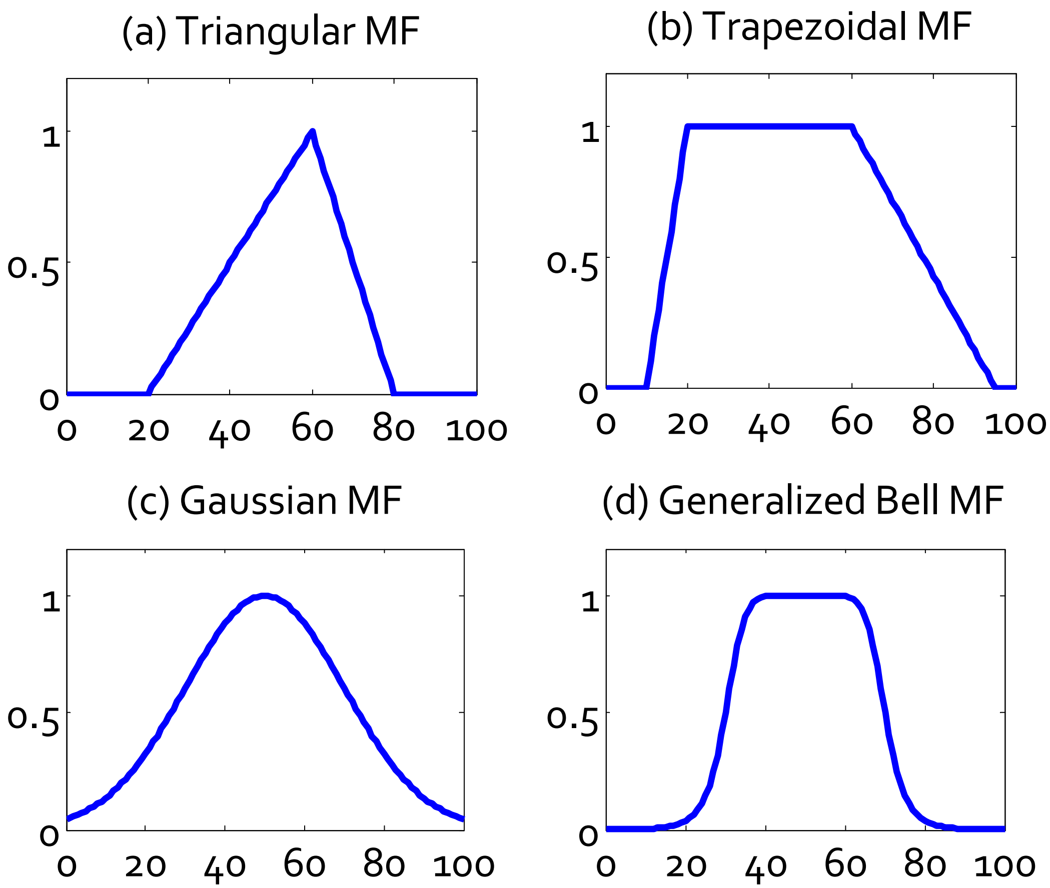
Membership functions



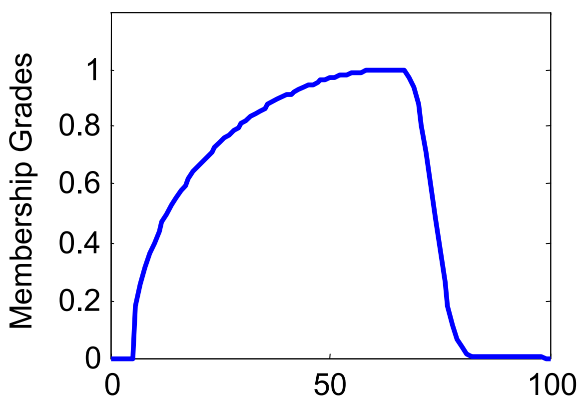
Left-right MF

$$LR(x;c,\alpha,\beta)=\begin{cases} F_L\left(\dfrac{c-x}{\alpha}\right), x<c \\ F_R\left(\dfrac{x-c}{\beta}\right), x\geq c \end{cases}$$

$c=65$

$\alpha=60$

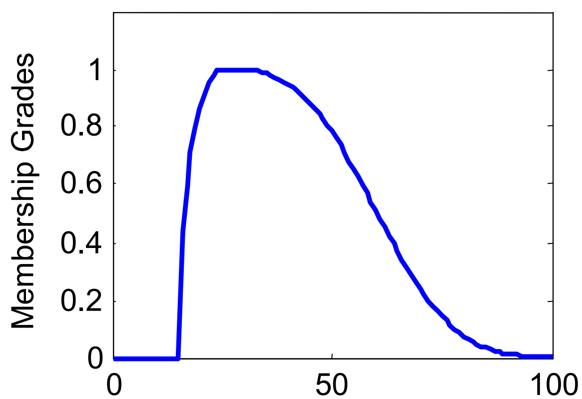
$\beta=10$



$c=25$

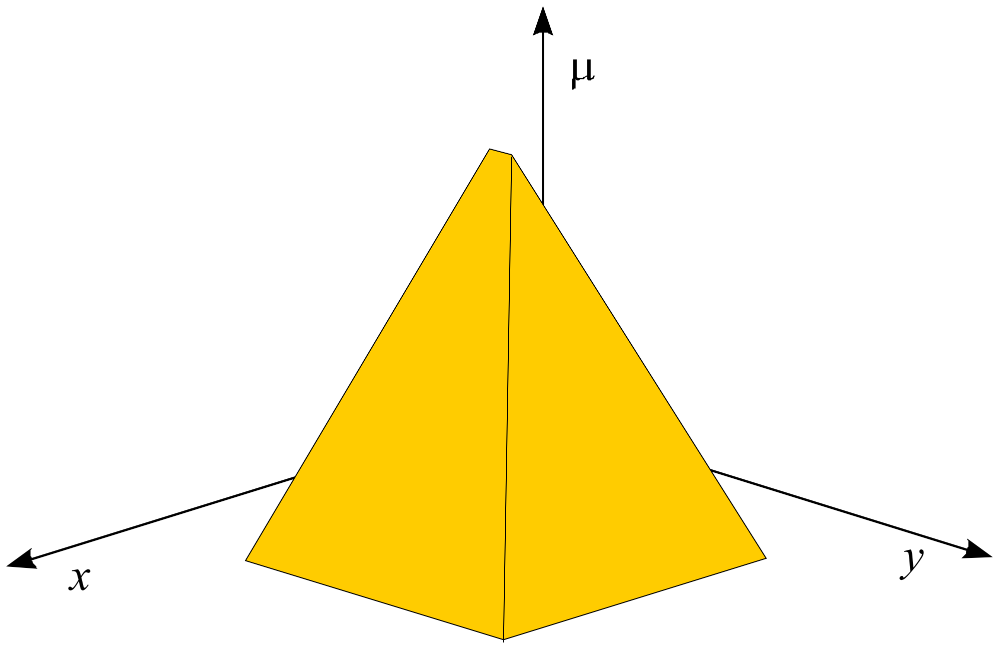
$\alpha=10$

$\beta=40$

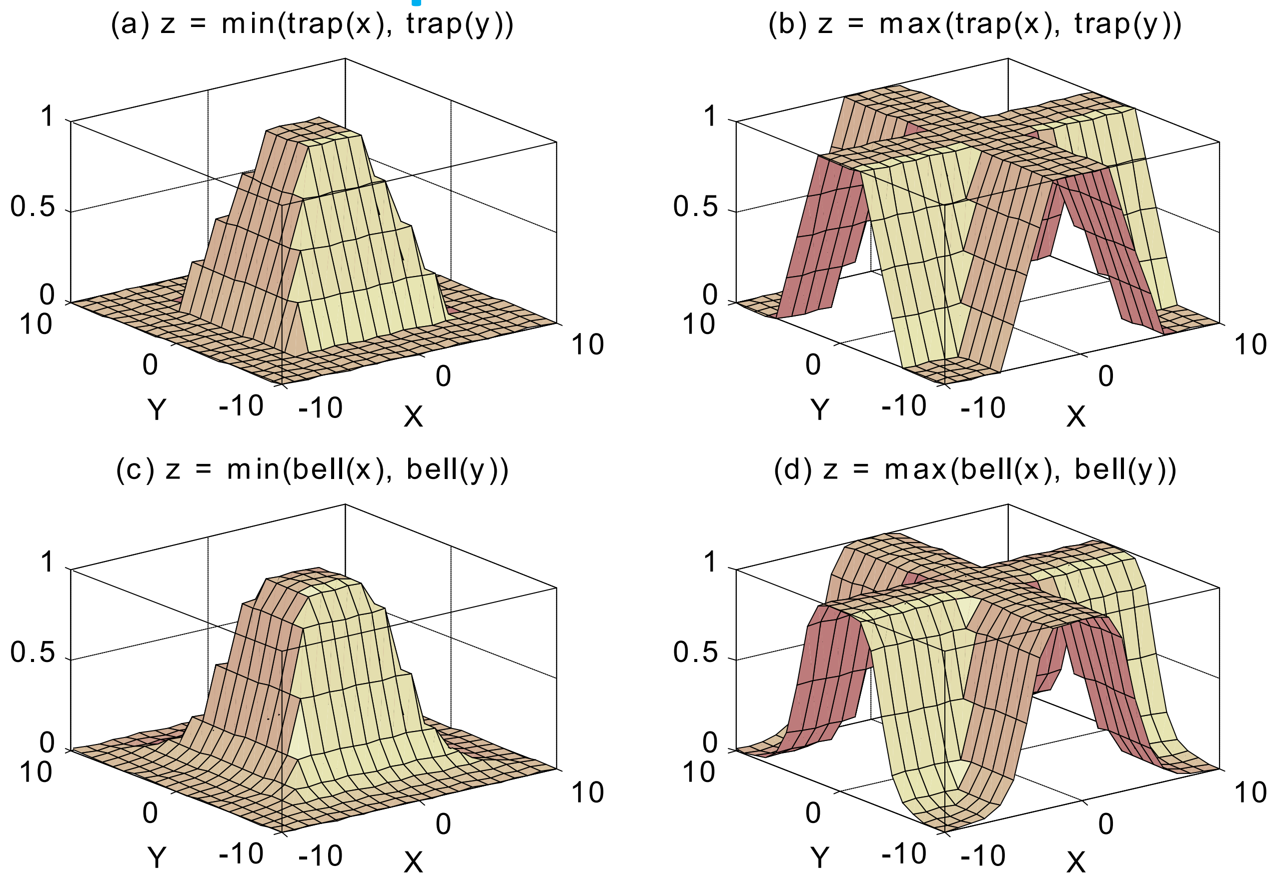


Two-dimensional fuzzy sets

$$A = \int_{X\times Y} \mu_A(x,y) = \left\{ \mu_A(x,y) \middle| (x,y) \in X \times Y \right\}$$

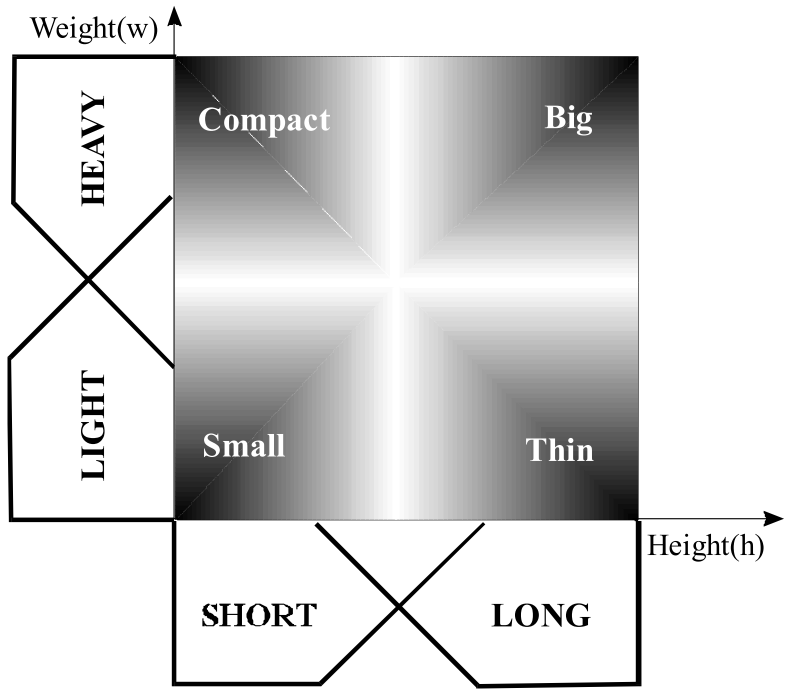


2-D membership functions



Compound fuzzy propositions

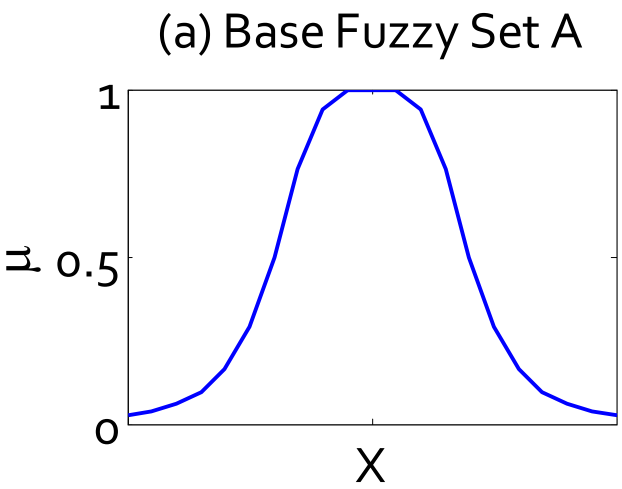
- *Small* = *Short* **and** *Light* (conjunction) $\mu_{Small}(h,w) = \mu_{Short}(h) \cap \mu_{Light}(w)$

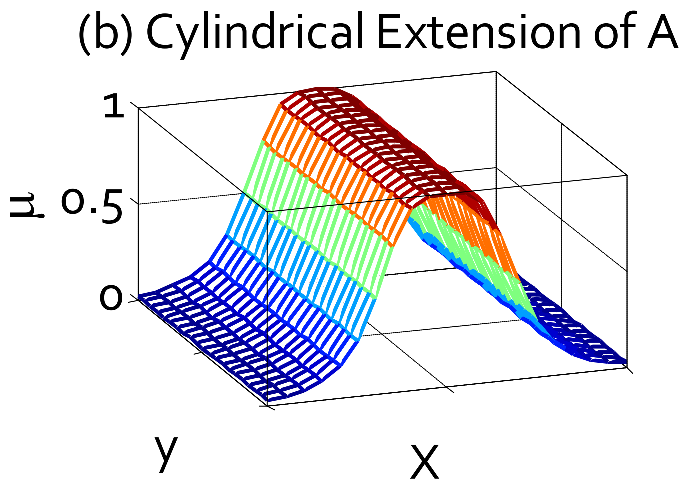


Cylindrical extension

- Cylindrical extension of fuzzy set A in X into Y results in a two-dimensional fuzzy set in $X \times Y$, given by

$$\text{cext}_y(A) = \int_{X \times Y} \mu_A(x)/(x,y) = \left\{ \mu_A(x)/(x,y) \,\middle|\, (x,y) \in X \times Y \right\}$$





Example

- Premise: Fuzzy set $\mu(x)$ that represents '***Young* people'** in the domain *X* representing age:
- X = {18, 20, 22, 25, 30} [years]

μ_{Young}(x)

x	18	20	22	25	30
$\mu_{Young}(x)$	1	1	0.8	0.5	0.2

- Consider the set of **'duration of mobile calls'** Y = {1, 3, 5, 7, 10, 20} [min/call]

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Example

- Compute the cylindrical extension of $\mu(x)$ in to Y:

x	18	20	22	25	30	[years]
$\mu_{Young}(x)$	1	1	0.8	0.5	0.2	

$Y = \{1, 3, 5, 7, 10, 20\}$ [min/call]

$\mu_{Young}(x)$ into X x Y

		y					
x	$\mu_{Young}(x)$	1	3	5	7	10	20
18	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1
22	0.8	0.8	0.8	0.8	0.8	0.8	0.8
25	0.5	0.5	0.5	0.5	0.5	0.5	0.5
30	0.2	0.2	0.2	0.2	0.2	0.2	0.2

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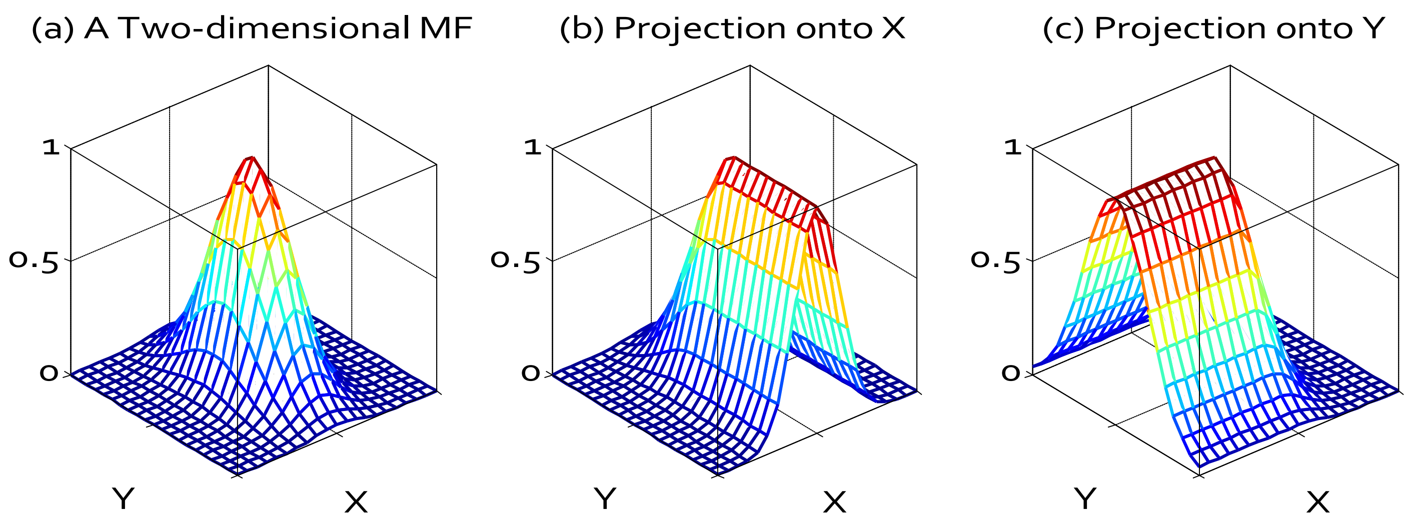
Projection



$$\mu_R(x, y) \qquad \mu_A(x) = \max_y \mu_R(x, y) \qquad \mu_B(y) = \max_x \mu_R(x, y)$$

Cartesian product and co-product

- **Cartesian product** of fuzzy sets A and B is a fuzzy set in the product space $X \times Y$ with membership

$$\mu_{A\times B}(x,y) = \min(\mu_A(x), \mu_B(y))$$

- **Cartesian co-product** of fuzzy sets A and B is a fuzzy set in the product space $X \times Y$ with membership

$$\mu_{A+B}(x,y) = \max(\mu_A(x), \mu_B(y))$$

Cartesian product and co-product