Intelligent Systems

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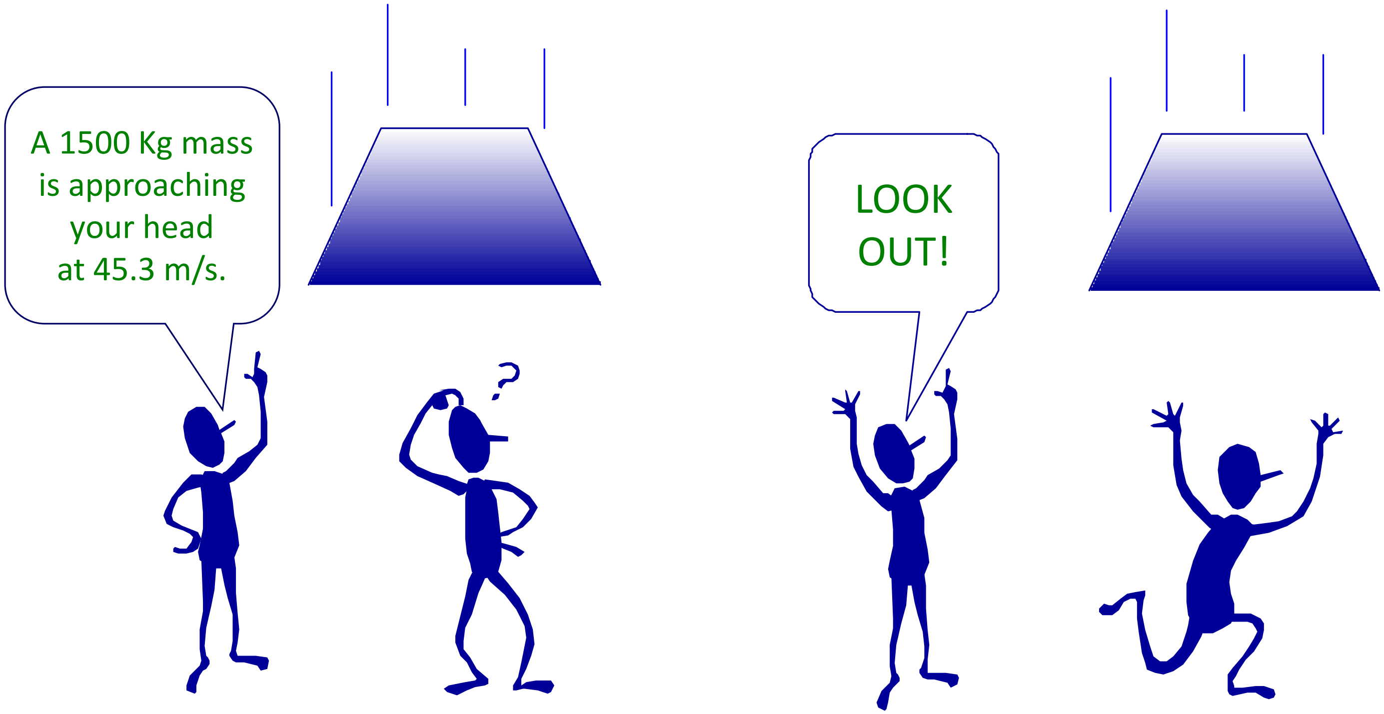
FUZZY SETS

SI2 – Introduction to Fuzzy Sets

Reading: Part I Fuzzy Set Theory: Chapter 2 Fuzzy Sets

J.-S. Jang, C.-T. Sun and E. Mizutani. ***Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence.*** Prentice Hall, New Jersey, 1997.

Precision vs. Relevancy



Introduction

- How to simplify very complex systems?
  - *Allow some degree of imprecision in their description!*
- How to deal mathematically with uncertainty?
  - Using probabilistic theory (*stochastic*).
  - Using the **theory of fuzzy sets** (*non-stochastic*).
- Proposed in 1965 by Lotfi Zadeh (Fuzzy Sets, *Information Control*, 8, pp. 338-353).
- Imprecision or vagueness in natural language **does not** imply a loss of accuracy or meaningfulness!

Examples

- *Give travel directions in terms of city blocks* **OR** *in meters?*
- *The day is sunny* **OR** *the sky is covered by 5% of clouds?*
  - If the sky is covered by 10% of clouds is still *sunny*?
  - And 25%?
  - And 50%?
  - **Where to draw the line from** *sunny* **to** *not sunny*?
  - Member and not member or **membership degree**?

Probability vs. Possibility

- **Event u**: Hans ate X eggs for breakfast.
- Probability distribution: $P_X(u)$
- Possibility distribution: $\pi_X(u)$

u	1	2	3	4	5	6	7	8
$P_X(u)$	0.1	0.8	0.1	0	0	0	0	0
$\pi_X(u)$	1	1	1					

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Probability vs. Fuzzy membership

- **You’re lost in the outback; dying of thirst**

"Honestly, the water hole is back that way... Why would I lie?"

Probability vs. Fuzzy membership

- **You come upon two bottles containing liquid**

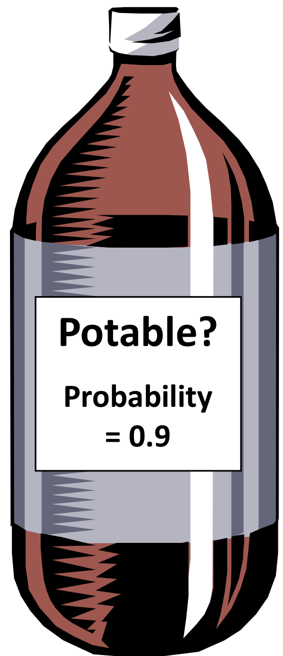
Probability vs. Fuzzy membership

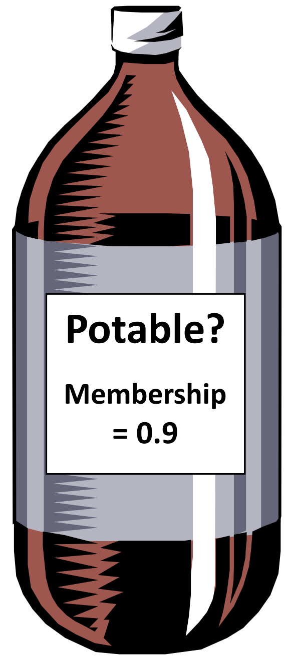
- **Which one will you choose?**

  How will you process the information?

Probability vs. Fuzzy membership

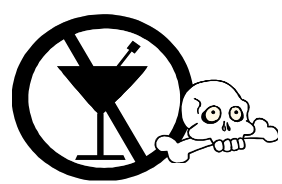
0.9 probability of belonging to the **set of non-poisonous liquid**





0.9 degree of membership to the **set of non-poisonous liquid**

OR



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Might taste funky, but shouldn't kill you

Why would anyone want to *fuzzify* logic?

- Isn't "fuzzy logic" an inherent contradiction? Why would anyone want to fuzzify logic?
- Fuzzy sets and logic must be viewed as a formal mathematical theory for the representation of uncertainty. Uncertainty is crucial for the management of real systems: if you had to park your car PRECISELY in one place, it would not be possible. Instead, you work within, say, 10 cm tolerances. The presence of uncertainty is the price you pay for handling a complex system.

Why would anyone want to *fuzzify* logic?

- Isn't "fuzzy logic" an inherent contradiction? Why would anyone want to fuzzify logic?
- Nevertheless, fuzzy logic is a mathematical formalism, and a membership grade is a precise number. What's crucial to realize is that fuzzy logic is a logic OF fuzziness, not a logic which is ITSELF fuzzy. But that's OK: just as the laws of probability are not random, so the laws of fuzziness are not vague.

Applications of fuzzy sets

- Fuzzy sets belong to “conventional” mathematics (measures, relations, topology, etc.)
- **Fuzzy logic** and **AI** (approximate reasoning, expert systems, etc.)
- **Fuzzy systems**
  - Fuzzy modeling
  - Fuzzy control, etc.
- **Fuzzy decision making**
  - Multi-criteria optimization
  - Optimization techniques
- …

Classical set theory

- **Set**: collection of objects with a common property.
- **Examples**:
  - Set of basic colors:

$$A = \{\text{red, green, blue}\}$$

  - Set of positive integers:

$$A = \{x \in \mathbf{Z} \mid x \geq 0\}$$

  - A line in $\mathbf{R}^3$:

$$A = \{(x,y,z) \mid ax + by + cz + d = 0\}$$

Representation of sets

- Enumeration of elements: $A = \{x_1, x_2, \ldots, x_n\}$
- Definition by property P: $A = \{x \in X \mid P(x)\}$
- ***Characteristic function*** $\mu_{A(x)}: X \rightarrow \{0,1\}$

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \text{ is member of } A \\ 0, & \text{if } x \text{ is not member of } A \end{cases}$$

  - **Example:**
    - Set of odd numbers: $\mu_A(x) = x \bmod 2$

Set operations

- **Intersection**: $C = A \cap B$
  - *C* contains elements that belong to A and B
  - Characteristic function: $\mu_C = \min(\mu_A, \mu_B) = \mu_A \cdot \mu_B$
- **Union**: $C = A \cup B$
  - *C* contains elements that belong to A or to B
  - Characteristic function: $\mu_C = \max(\mu_A, \mu_B)$
- **Complement**: $C = \bar{A}$
  - C contains elements that do not belong to A
  - Characteristic function: $\mu_C = 1 - \mu_A$

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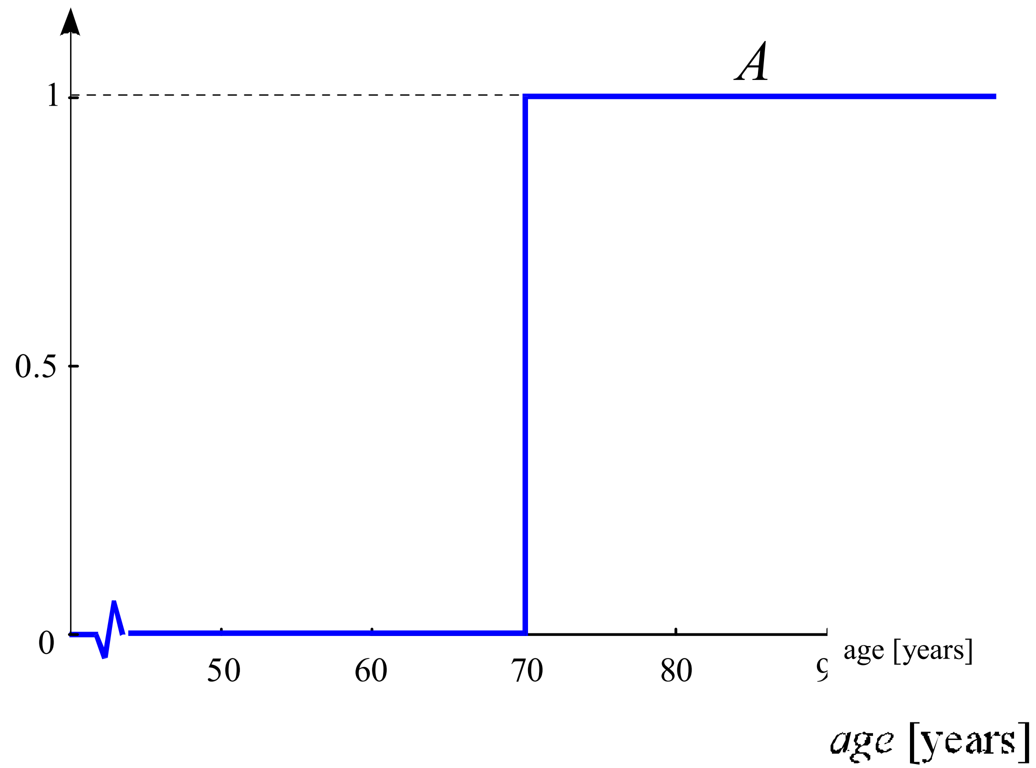
Fuzzy sets

- Represent ***uncertain*** (vague, ambiguous, etc.) knowledge in the form of propositions, rules, etc.
- Propositions:
  - expensive cars,
  - cloudy sky,...
- Rules (decisions):
  - Want to buy a big and new house for a low price.
  - **If** the temperature is ***low***, **then** ***increase*** the heating.
  - …

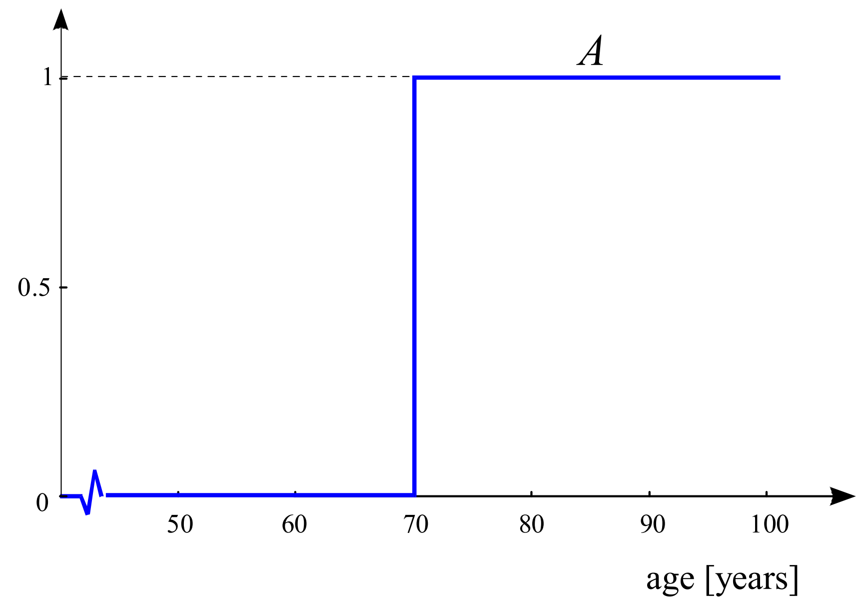
Classical set

- **Example**: set of *old people* $A = \{age \mid age \geq 70\}$



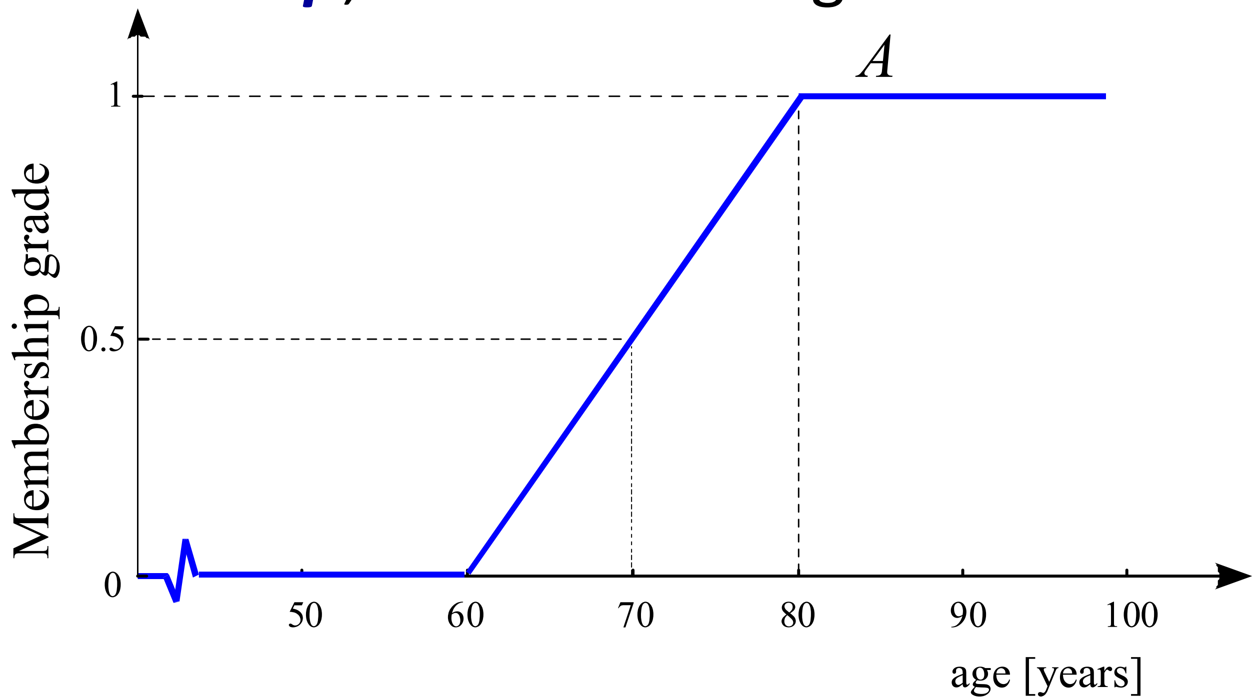
Logic propositions

- "Nick is old" ... true or false
- Nick's age:
  - $age_{\text{Nick}} = 70$, $\mu_A(70) = 1$ (true)
  - $age_{\text{Nick}} = 69.9$, $\mu_A(69.9) = 0$ (false)



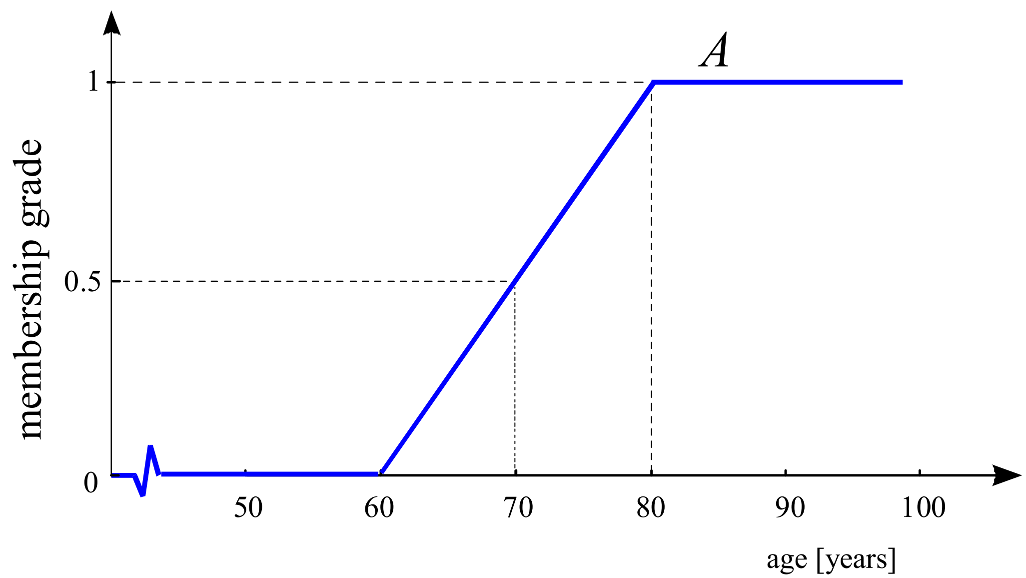
Fuzzy set

- ***Graded membership***, element belongs to a set to a certain degree.

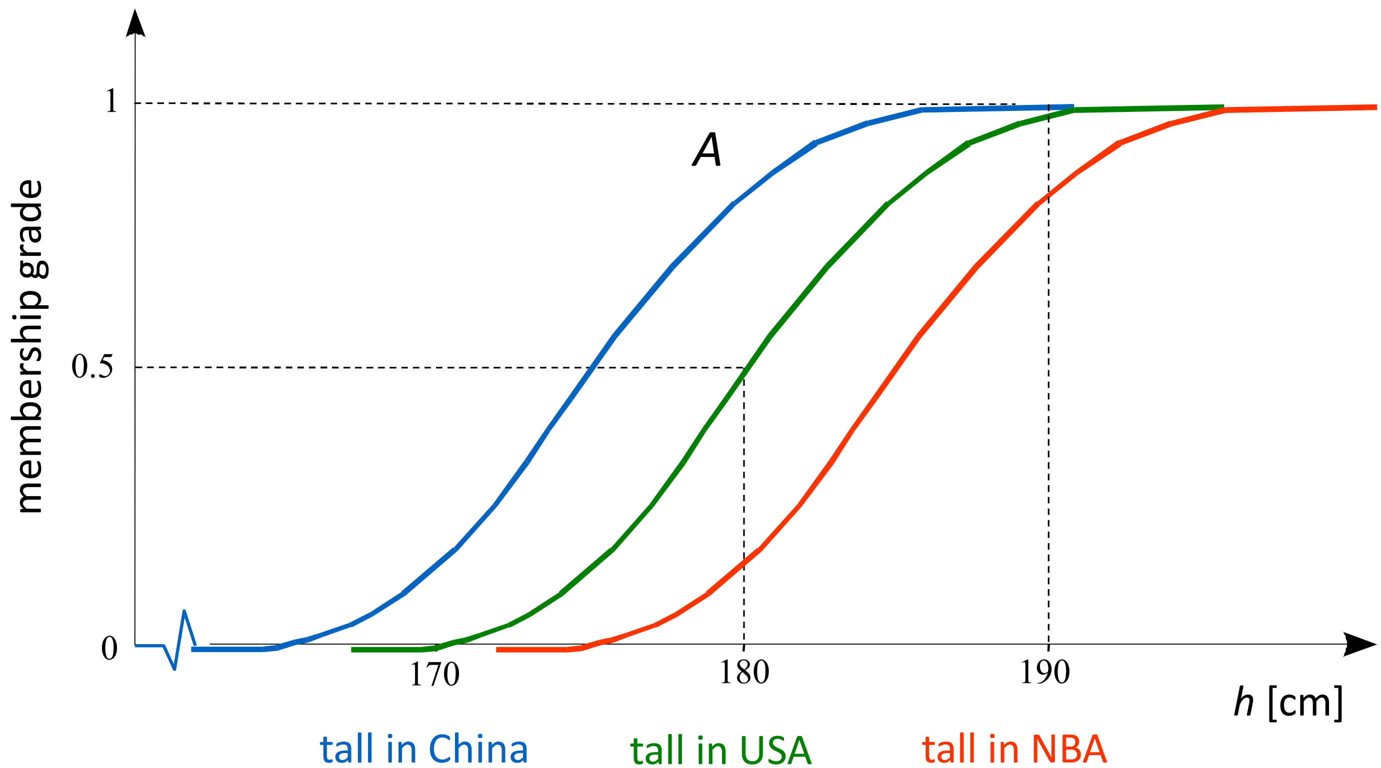


Fuzzy proposition

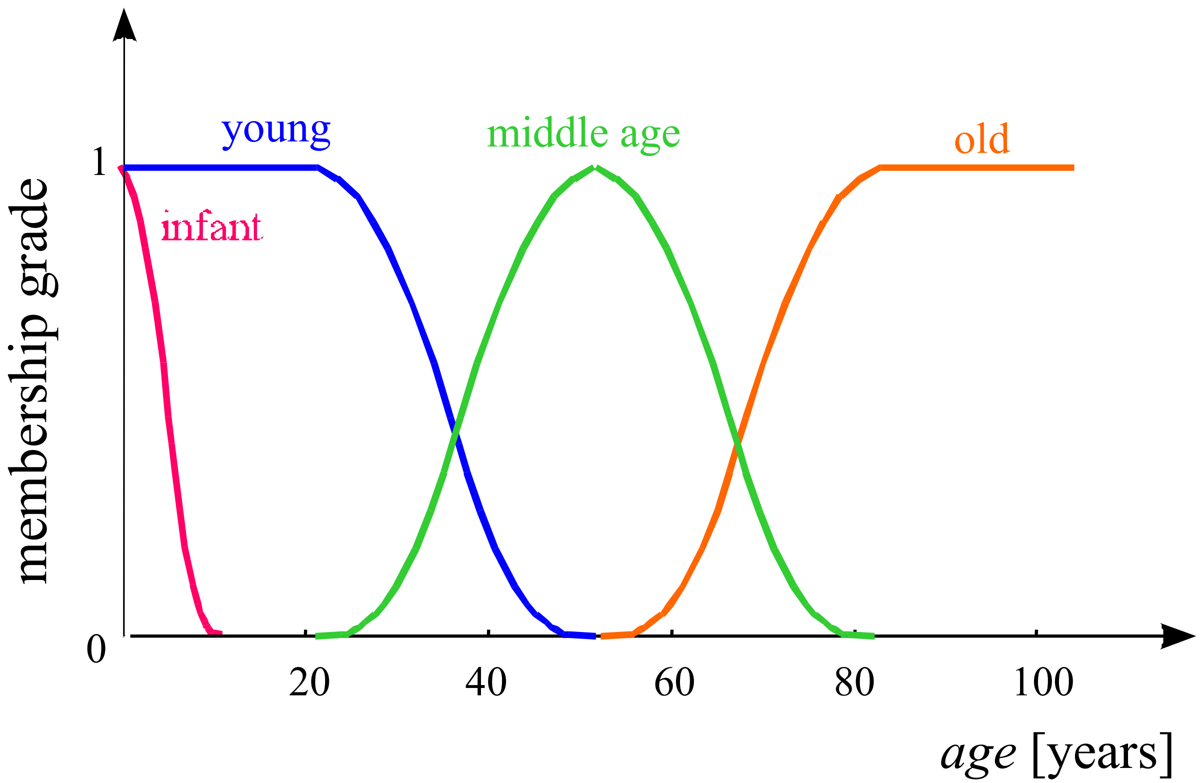
- "Nick is old"... degree of truth
  - $age_{\text{Nick}} = 70$, $\mu_A(70) = 0.5$
  - $age_{\text{Nick}} = 69.9$, $\mu_A(69.9) = 0.49$
  - $age_{\text{Nick}} = 90$, $\mu_A(90) = 1$



Context dependent

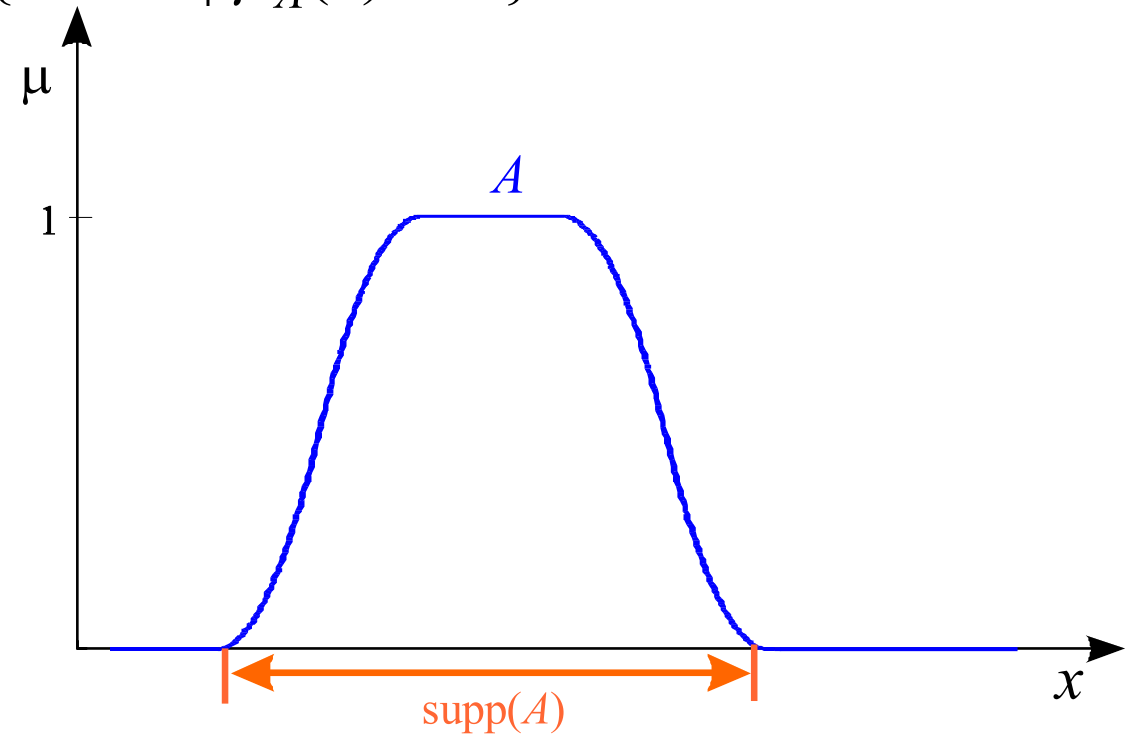


Typical linguistic values



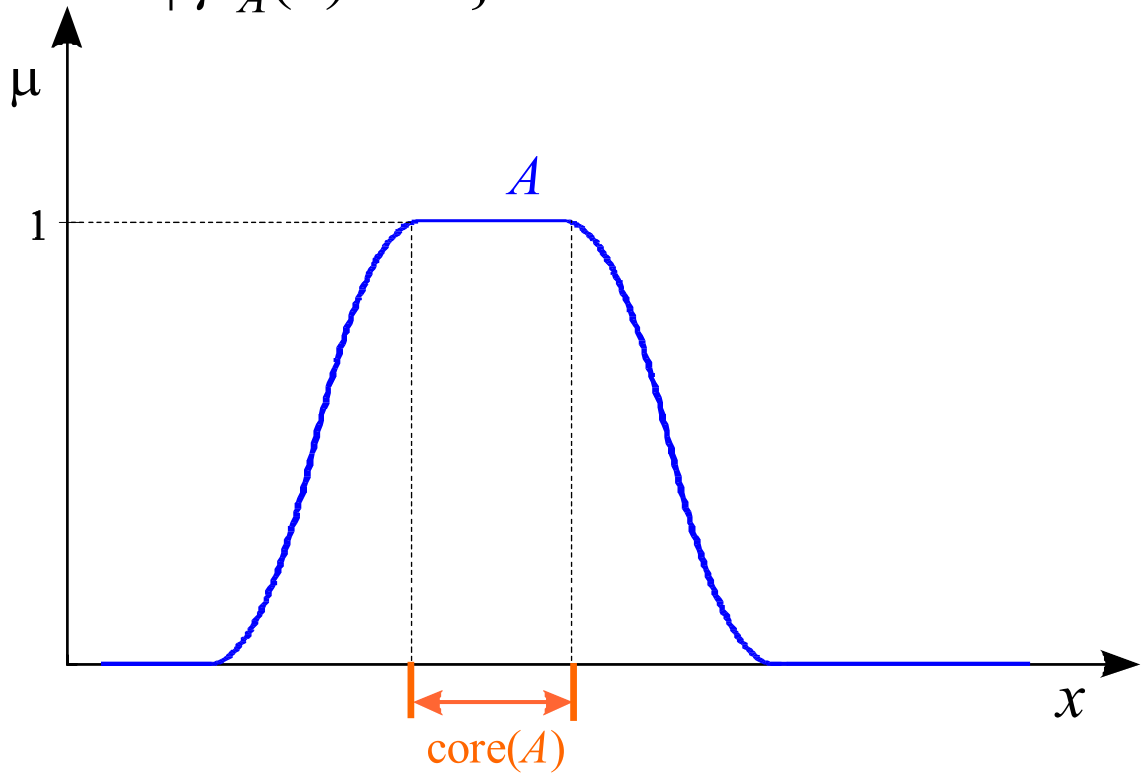
Support of a fuzzy set

- $\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$



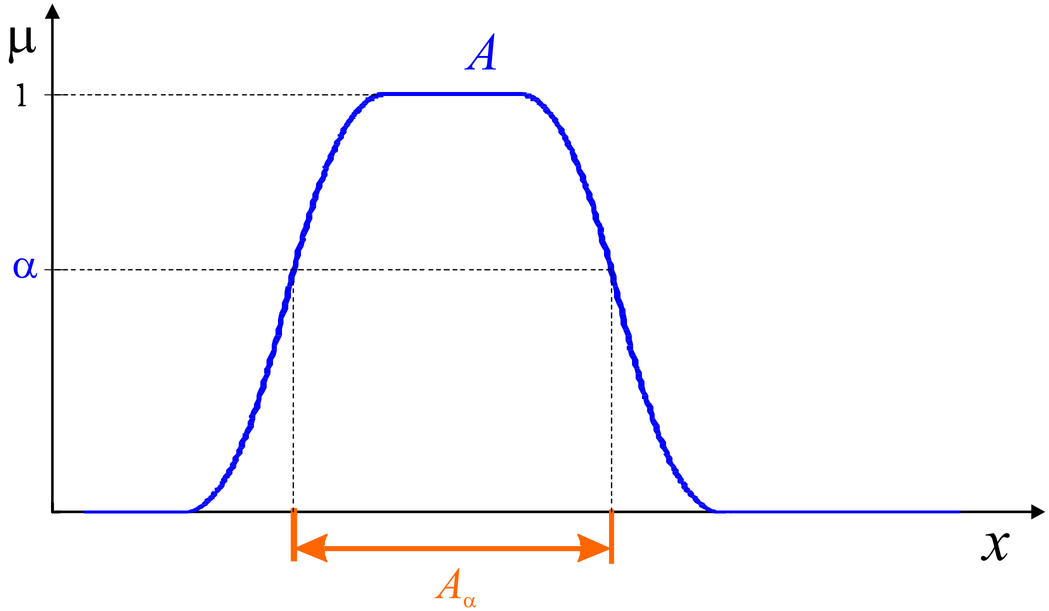
Core (nucleous, kernel)

- $\text{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$



α-cut of a fuzzy set

- α-cut is the crisp set: $A_\alpha = \{\, x \in X \mid \mu_A(x) \geq \alpha\}$
- ***Strong*** α-cut: $A_\alpha = \{\, x \in X \mid \mu_A(x) > \alpha\}$



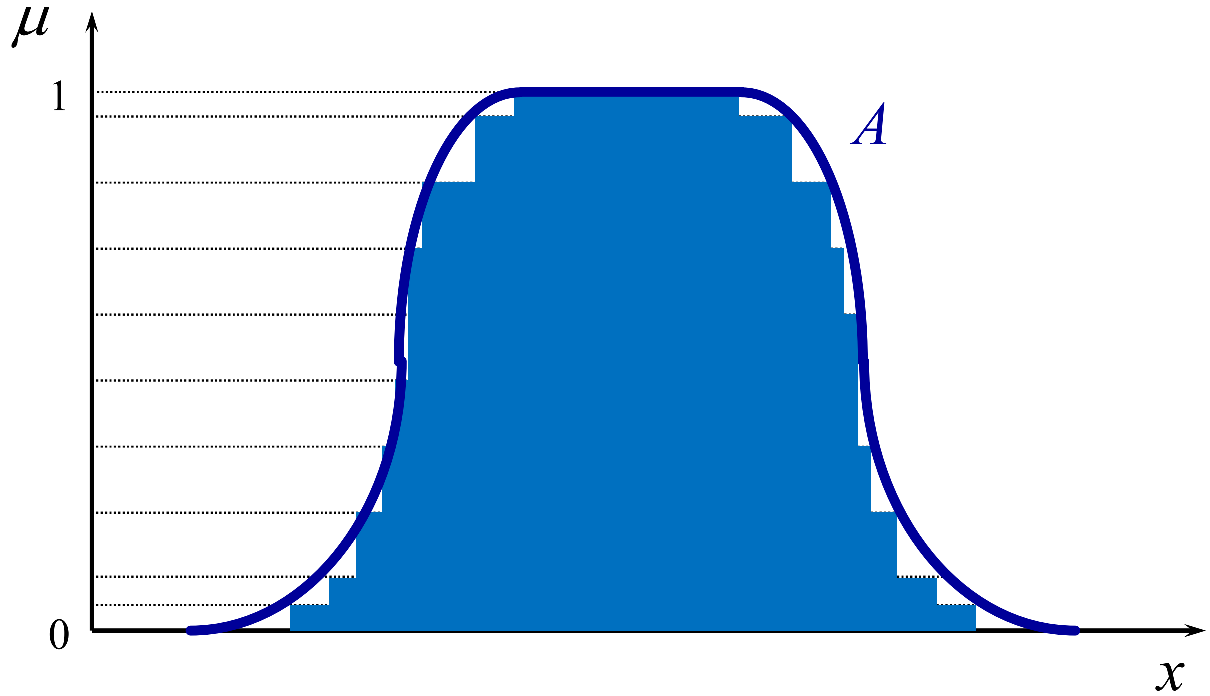
Resolution principle

- Every fuzzy set A can be uniquely represented as a collection of α-level sets according to

$$\mu_A(x) = \sup_{\alpha \in [0,1]} \{\alpha \in [0,1]\ x \in A_\alpha\}$$

- **Resolution principle** implies that fuzzy set theory is a generalization of classical set theory, and that its results can be represented in terms of classical set theory.

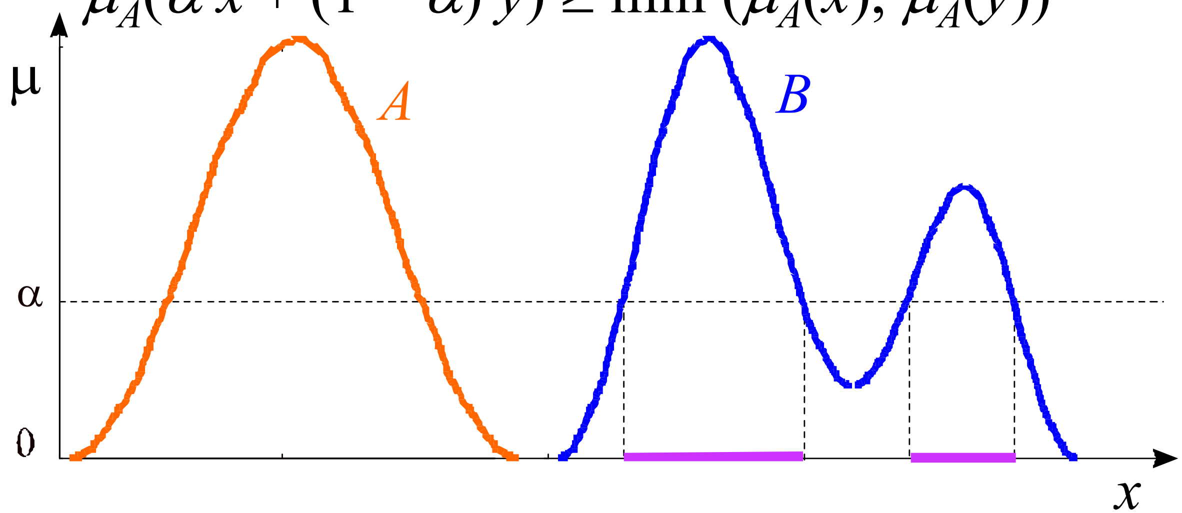
Resolution principle



Other properties

- **Height** of a fuzzy set: $\text{hgt}\,(A) = \sup \mu_A(x),\ x \in X$
- Fuzzy set is **normal(ized)** when $\text{hgt}\,(A) = 1$.
- A fuzzy set A is **convex** iff $\forall\ x,y \in X$ and $\alpha \in [0,1]$:

$$\mu_A(\alpha\, x + (1 - \alpha)\, y) \geq \min\,(\mu_A(x),\ \mu_A(y))$$



Other properties (2)

- **Fuzzy singleton**: single point $x \in X$ where $\mu_A(x) = 1$.
- **Fuzzy number**: fuzzy set in R that is ***normal*** and ***convex***.
- Two fuzzy sets are ***equal*** $(A = B)$ iff:

$$\forall x \in X,\ \mu_A(x) = \mu_B(x)$$

- A is a ***subset*** of B iff:

$$\forall x \in X,\ \mu_A(x) \leq \mu_B(x)$$

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Other properties (3)

- **Bandwith (**or **width)**: of ***normal*** and ***convex*** fuzzy sets is defined as the distance between two unique crossover points:

$$\text{width}(A) = |x_2 - x_1|, \text{ where } \mu_A(x_1) = \mu_A(x_2) = 0.5.$$

- **Symmetry**: a fuzzy set A is symmetric if its μ_A is symmetric around a certain point $x =$ c:

$$\mu_A(\text{c} + x) = \mu_A(x + c), \ \forall x \in X,$$

- **Open left, open right, closed:**

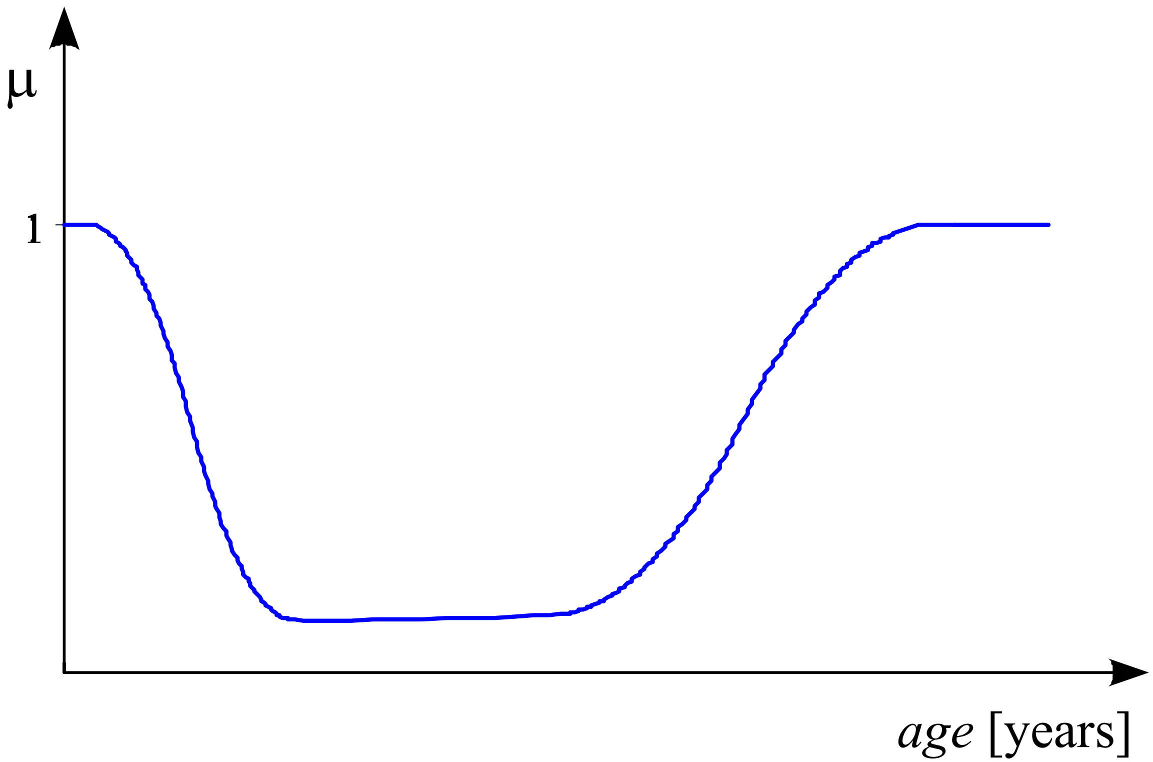
$$\lim x \to -\infty \ \mu_A(x) = 1 \text{ and } \lim x \to +\infty \ \mu_A(x) = 0$$

$$\lim x \to -\infty \ \mu_A(x) = 0 \text{ and } \lim x \to +\infty \ \mu_A(x) = 1$$

$$\lim x \to -\infty \ \mu_A(x) = \lim x \to +\infty \ \mu_A(x) = 0$$

Non-convex fuzzy sets

- **Example**: car insurance risk



Representation of fuzzy sets

Discrete Universe of Discourse:

- Point-wise as a list of membership/element pairs:
  - $A = \mu_A(x_1)/x_1 + \ldots + \mu_A(x_n)/x_n = \Sigma_i \, \mu_A(x_i)/x_i$
  - $A = \{\mu_A(x_1)/x_1, \ldots, \mu_A(x_n)/x_n\} = \{\mu_A(x_i)/x_i \mid x_i \in X\}$

- As a list of α-level/ α-cut pairs:
  - $A = \{\alpha_1/A_{\alpha_1}, \ldots, \alpha_n/A_{\alpha_n}\} = \{\alpha_i/A_{\alpha_i} \mid \alpha_i \in [0,1]\}$

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Representation of fuzzy sets

Continuous Universe of Discourse:

- $A = \int_X \mu_A(x)/x$
- Analytical formula: $\mu_A(x) = \dfrac{1}{1+x^2}, \quad x \in \mathbf{R}$
- Various possible notations:
  - $\mu_A(x)$, $A(x)$, A, a, etc.

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Examples

Discrete universe

- Fuzzy set A ="sensible number of children".
  - number of children: $X = \{0, 1, 2, 3, 4, 5, 6\}$
  - $A = 0.1/0 + 0.3/1 + 0.7/2 + 1/3 + 0.6/4 + 0.2/5 + 0.1/6$

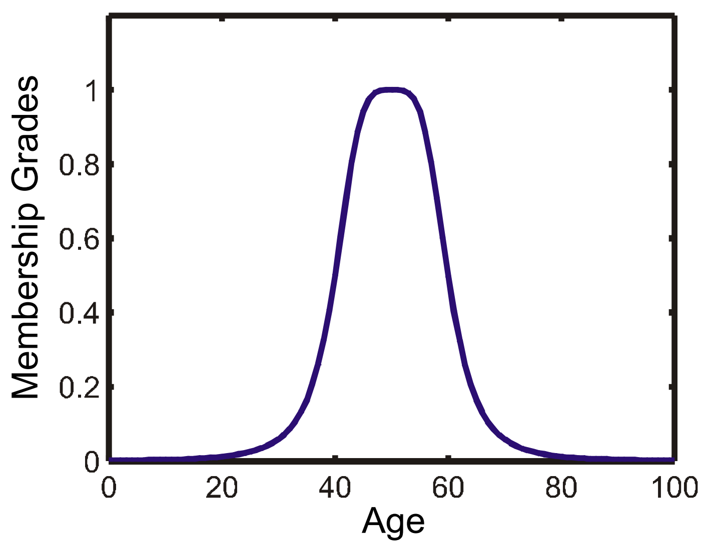
- Fuzzy set C = "desirable city to live in"
  - $X = \{\text{SF, Boston, LA}\}$ (discrete and non-ordered)
  - $C = \{(\text{SF}, 0.9), (\text{Boston}, 0.8), (\text{LA}, 0.6)\}$

Examples

Continuous universe

- Fuzzy set B = "about 50 years old"
  - X = R^+ (set of positive real numbers)
  - $B = \{(x, \mu_B(x)) \mid x \in X\}$

$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$



Complement of a fuzzy set

$$c: [0,1] \rightarrow [0,1]; \qquad \mu_A(x) \rightarrow c(\mu_A(x))$$

- **Fundamental axioms**

1. ***Boundary conditions*** **-** c behaves as the ordinary complement

$$c(0) = 1; \qquad c(1) = 0$$

2. ***Monotonic non-increasing***

$$\forall\ a,b \in [0,1], \text{ if } a < b, \text{ then } c(a) \geq c(b)$$

Complement of a fuzzy set

Other axioms:

- c is a ***continuous*** function.
- c is ***involutive***, which means that

$$c(c(a)) = a, \ \ \forall a \in [0,1]$$

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Complement of a fuzzy set

Equilibrium point

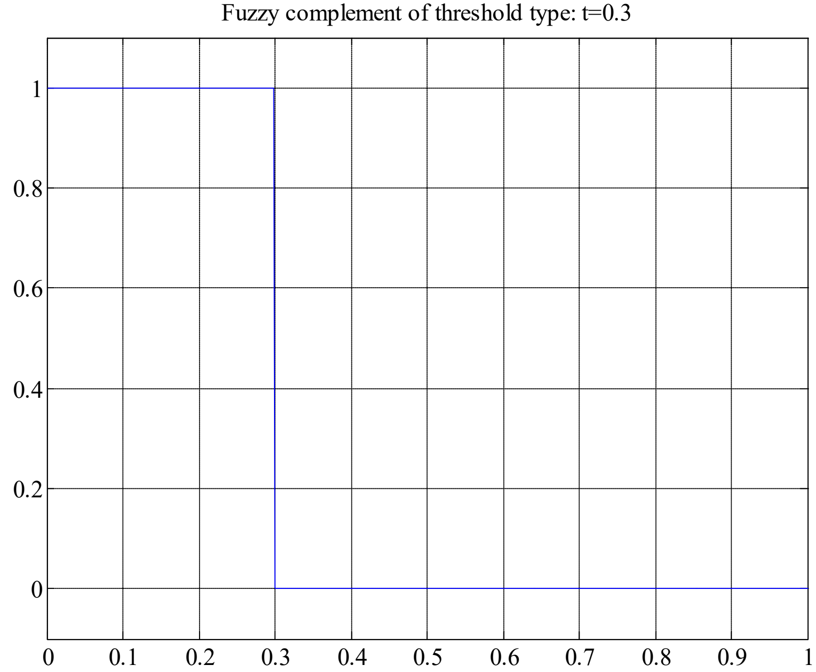
$$c(a) = a = e_c, \ \ \forall a \in [0,1]$$

- Each complement has at most one equilibrium.
- If c is a continuous fuzzy complement, it has one equilibrium point.

Examples of fuzzy complements

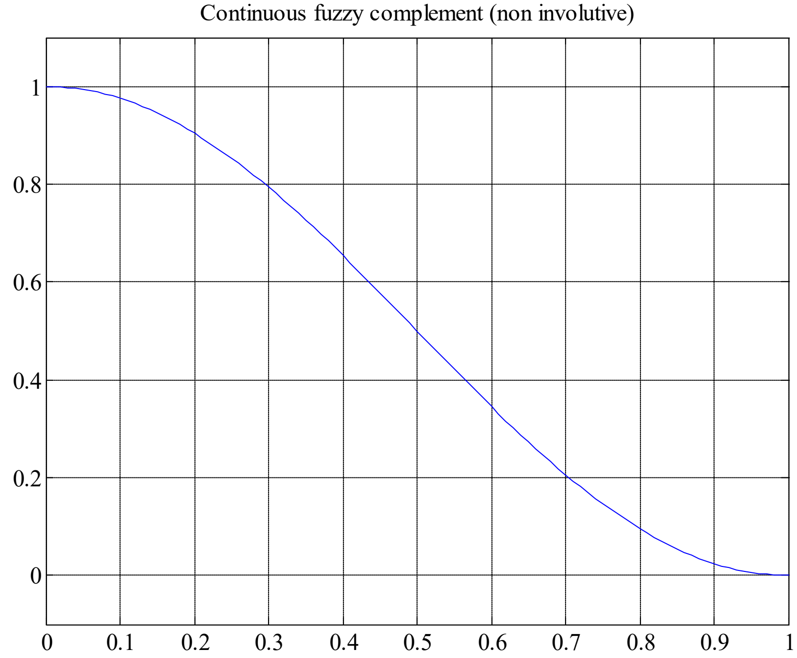
- **Standard complement:**
  - Satisfying only fundamental axioms:

$$c(a)=\begin{cases}1, & \text{if } a \le t\\ 0, & \text{if } a > t\end{cases}$$



Examples of fuzzy complements

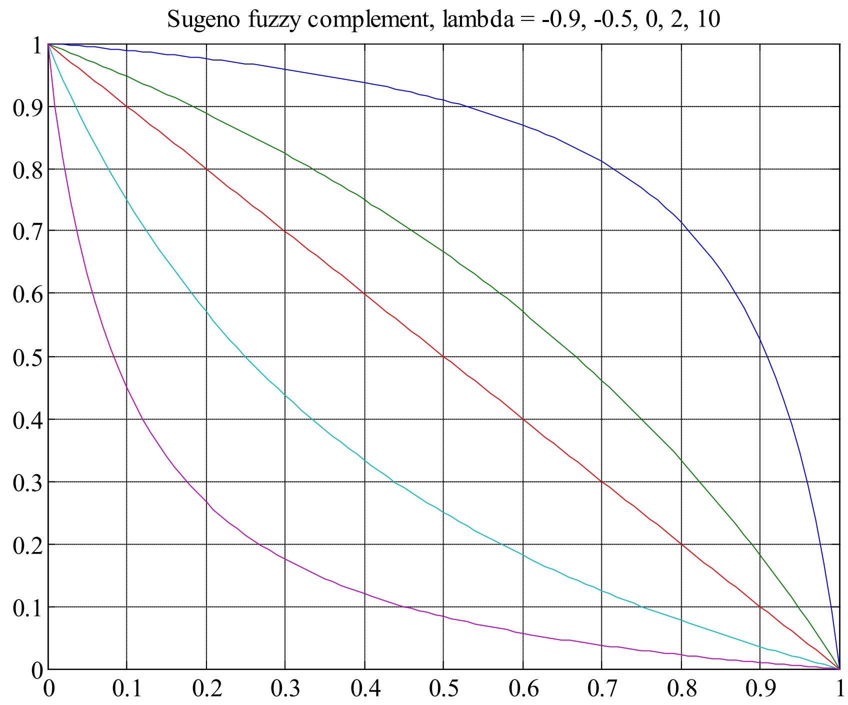
- **Cosine complement:** $$c(a) = \frac{1}{2}(1 + \cos \pi a)$$
  - Satisfying fundamental axioms and continuity:



Examples of fuzzy complements

- **Sugeno complement:**

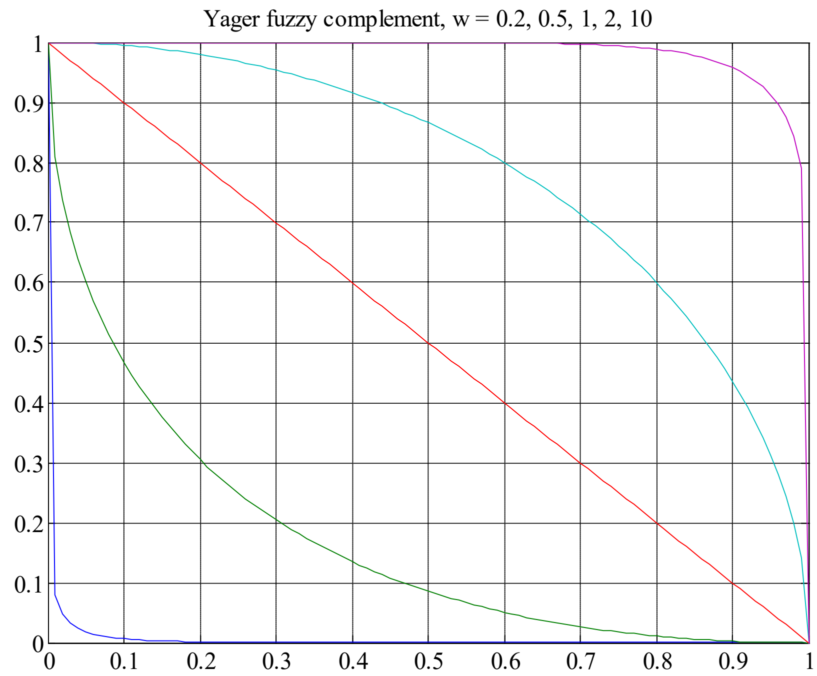
$$c_\lambda(a) = \frac{1-a}{1+\lambda a},\ \lambda \in]-1,\infty]$$



Examples of fuzzy complement

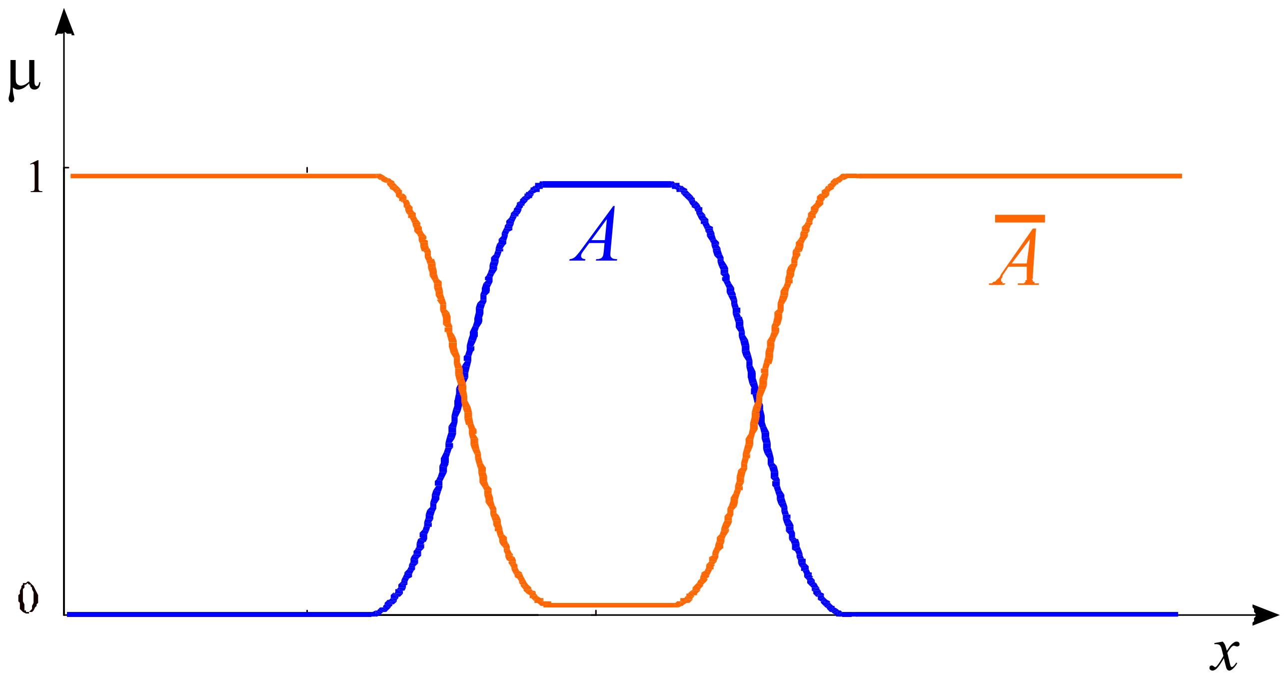
- **Yager complement:**

$$c_w(a) = \left(1 - a^w\right)^{1/w}, \quad w \in]0, \infty]$$



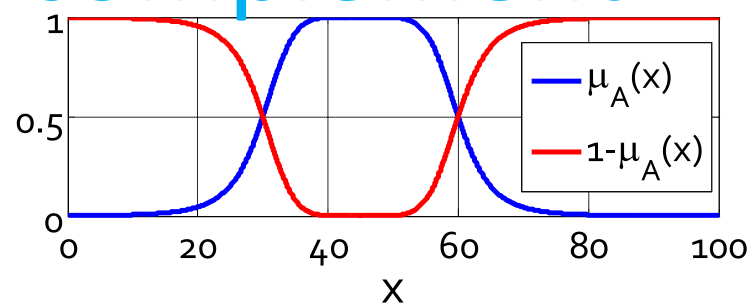
Representation of complement

- $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$

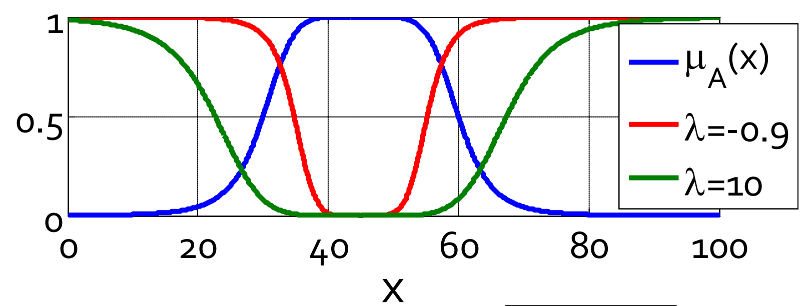


Representation of complement

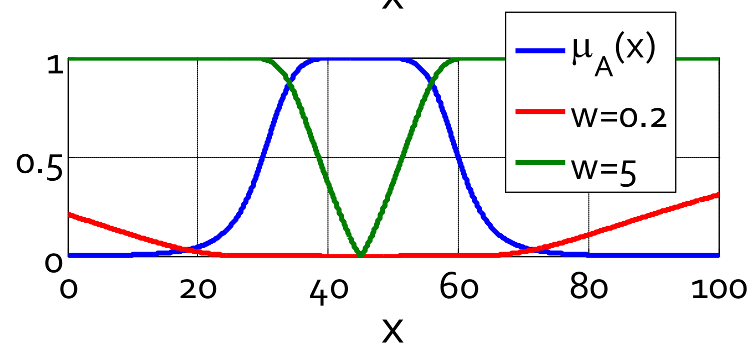
Standard complement



Sugeno complement



Yager complement



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Intersection of fuzzy sets

$$i: [0,1] \times [0,1] \rightarrow [0,1];$$
$$\mu_{A \cap B}(x) \rightarrow i(\mu_A(x),\, \mu_B(x))$$

- **Fundamental axioms:** *triangular norm* or *t*-norm

1. ***Boundary conditions*** **-** i behaves as the classical intersection

$$i(1,1) = 1;$$
$$i(0,1) = i(1,0) = i(0,0) = 0$$

2. ***Commutativity***

$$i(a,b) = i(b,a)$$

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Intersection of fuzzy sets

3. Monotonicity

If $a \leq a'$ and $b \leq b'$, then $i(a,b) \leq i(a',b')$

4. Associativity

$i(i(a,b),c) = i(a,i(b,c))$

- **Other axioms:**
  - i is a ***continuous*** function.
  - $i(a,a) = a$ (idempotent).

Examples of fuzzy conjunctions

- **Zadeh**

$$\mu_{A\cap B}(x) = \min(\mu_A(x),\ \mu_B(x))$$

- **Probabilistic (or algebraic product)**

$$\mu_{A\cap B}(x) = \mu_A(x)\ .\ \mu_B(x)$$

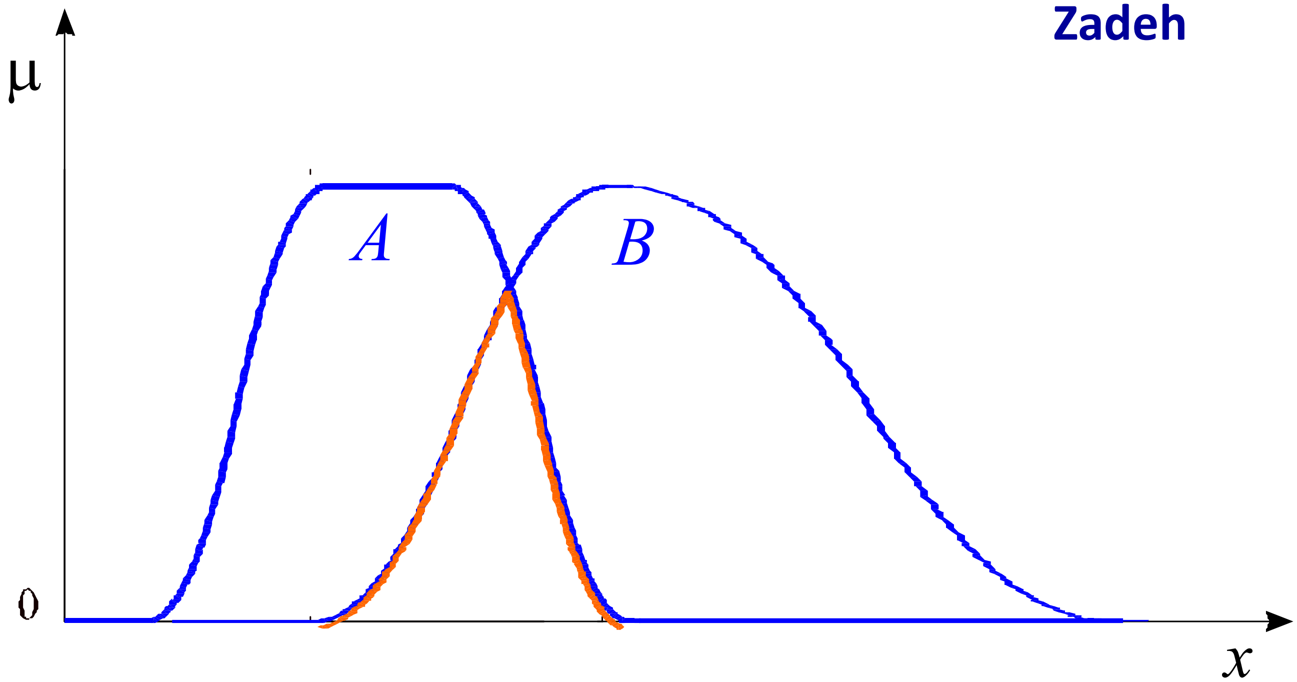
- **Łukaziewicz**

$$\mu_{A\cap B}(x) = \max(0,\ \mu_A(x) + \mu_B(x) - 1)$$

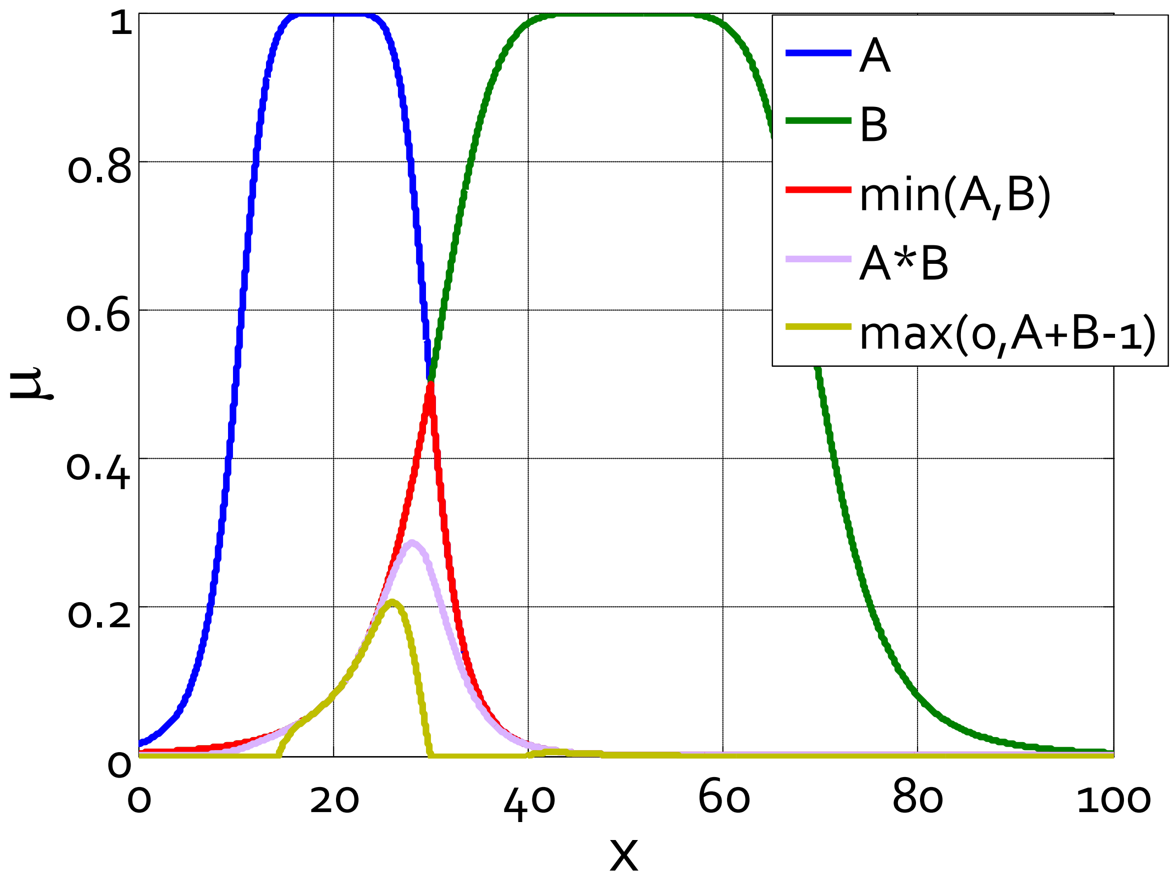
Intersection of fuzzy sets

- $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

Zadeh



Intersection of fuzzy sets



Zadeh

Probabilistic

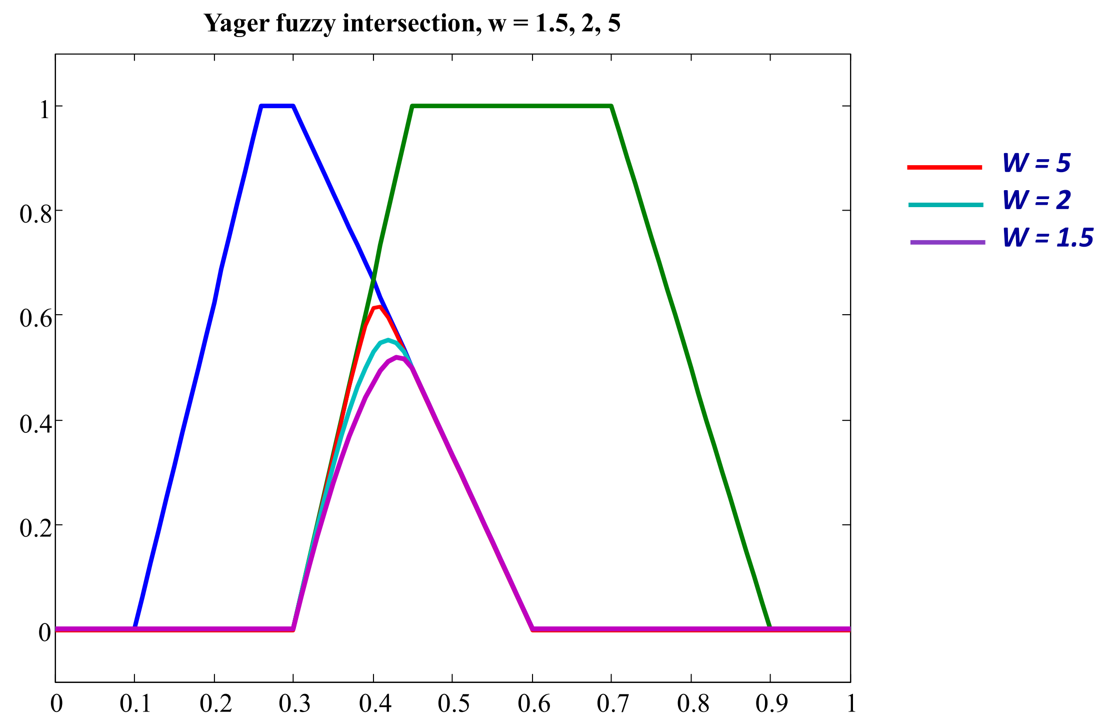
Łukaziewicz

Yager *t*-norm

$$i_w(a,b) = 1 - \min\left[1, \left((1-a)^w + (1-b)^w\right)^{1/w}\right], \ w \in]0,\infty]$$

- Example of ***weak*** and ***strong*** intersections:

Parametric t-norm



Union of fuzzy sets

$$u\colon [0,1] \times [0,1] \to [0,1];$$
$$\mu_{A\cup B}(x) \to u(\mu_A(x), \mu_B(x))$$

- **Fundamental axioms:** *triangular co-norm* or *s*-norm

1. ***Boundary conditions*** **-** u behaves as the classical union

$$u(0,0) = 0;$$
$$u(0,1) = u(1,0) = u(1,1) = 1$$

2. ***Commutativity***

$$u(a,b) = u(b,a)$$

Union of fuzzy sets

3. Monotonicity

If $a \leq a'$ and $b \leq b'$, then $u(a,b) \leq u(a',b')$

4. Associativity

$$u(u(a,b),c) = u(a,u(b,c))$$

- **Other axioms:**
  - u is a ***continuous*** function.
  - $u(a,a) = a$ (idempotent).

Examples of fuzzy disjunctions

- **Zadeh**

$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

- **Probabilistic (algebraic sum)**

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \, . \, \mu_B(x)$$

- **Łukasiewicz**

$$\mu_{A\cup B}(x) = \min(1, \, \mu_A(x) + \mu_B(x))$$

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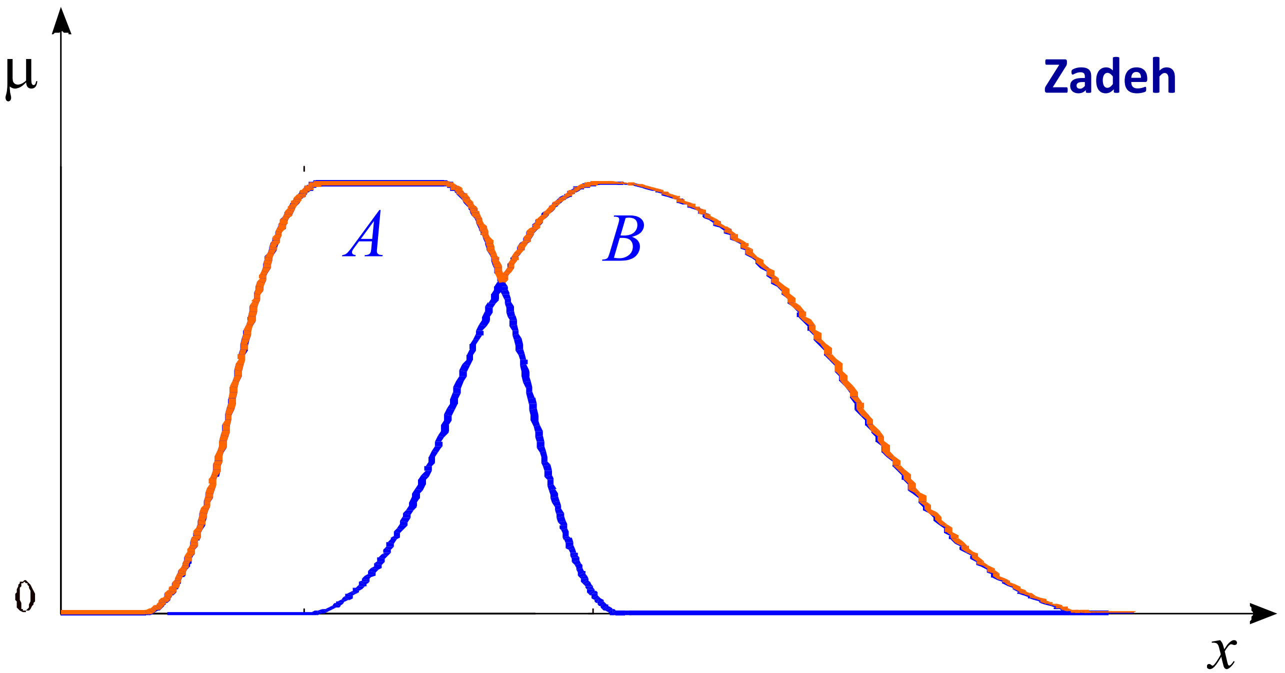
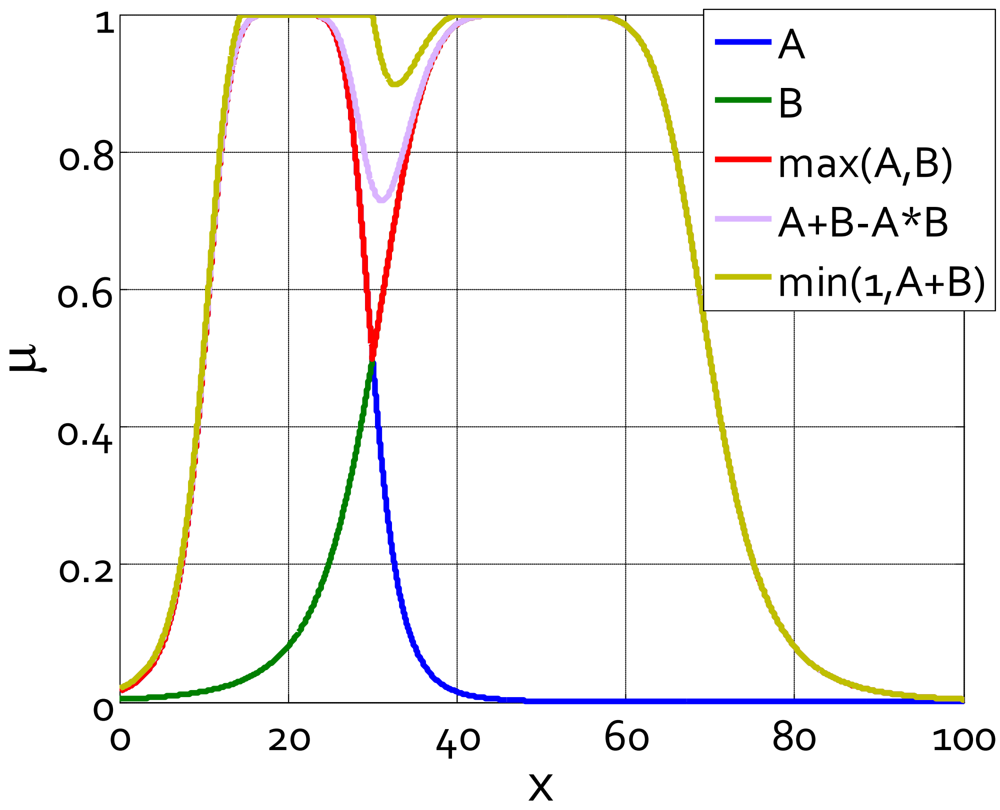
Union of fuzzy sets

- $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$



Union of fuzzy sets



Zadeh

Probabilistic

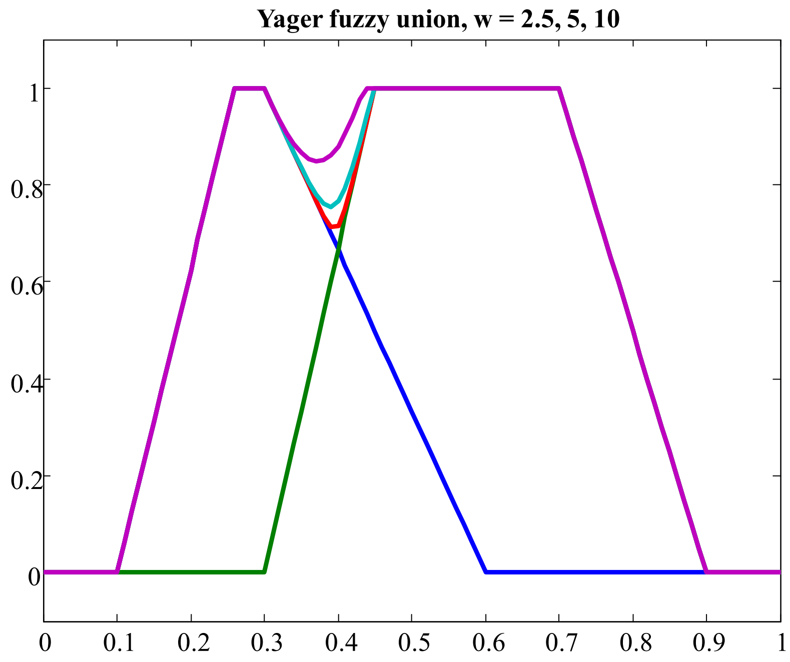
Łukaziewicz

Yager *t*-conorm (s-norm)

$$u_w(a,b) = \min\left[1, \left(a^w + b^w\right)^{1/w}\right], \; w \in]0, \infty]$$

- Example of ***weak*** and ***strong*** disjunctions:

Parametric t-conorm



W = 10
W = 5
W = 2.5

General aggregation operations

$$h: [0,1]^n \rightarrow [0,1];$$
$$\mu_A(x) \rightarrow h(\mu_{A_1}(x),\ldots, \mu_{A_n}(x))$$

- **Axioms**

1. ***Boundary conditions***

   $h(0,\ldots,0) = 0$

   $h(1,\ldots,1) = 1$

2. ***Monotonic non-decreasing***

   For any pair $a_i, b_i \in [0,1]$, $i \in \mathbf{N}$

   If $a_i \geq b_i$ then $h(a_i) \geq h(b_i)$

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General aggregation operations

- **Other axioms:**
  - *h* is a ***continuous*** function.
  - *h* is a ***symmetric*** function in all its arguments:

$$h(a_i) = h(a_{p(i)})$$

for any permutation p on **N**

Averaging operations

- When all the four axioms hold:

$$\min(a_1,\ldots,a_n) \le h(a_1,\ldots,a_n) \le \max(a_1,\ldots,a_n)$$

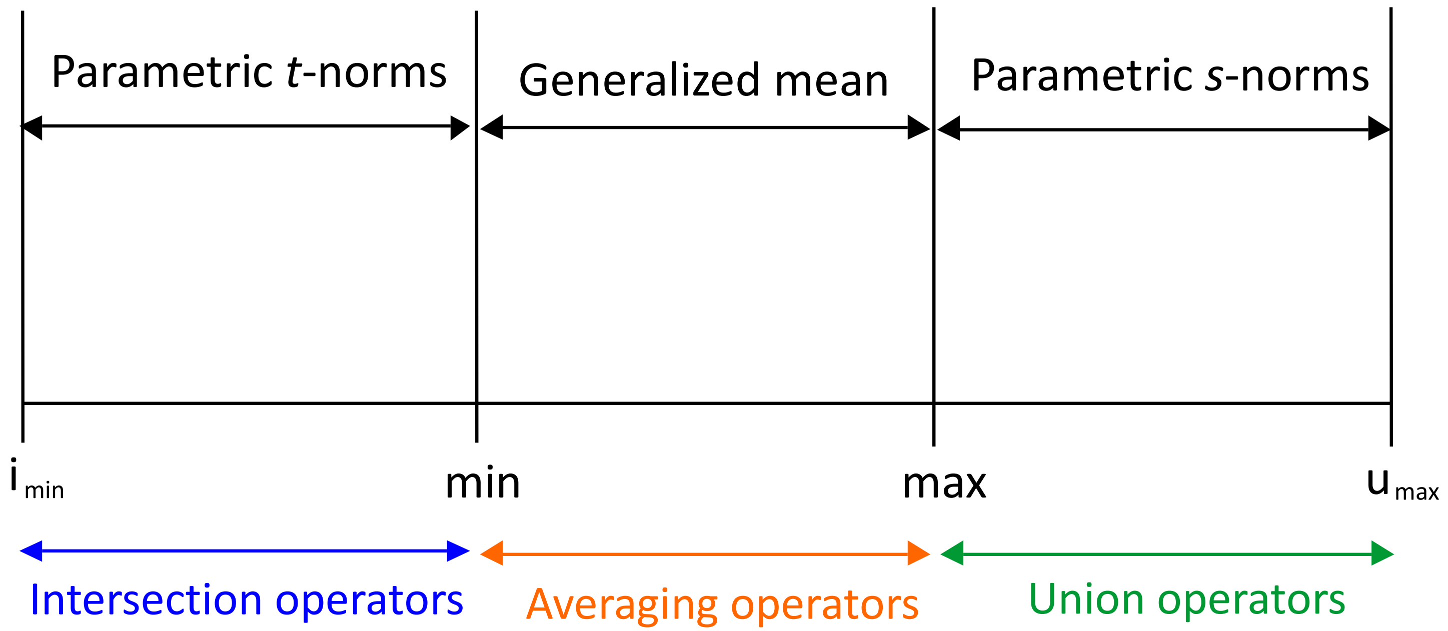
- Operator covering this range: **Generalized mean**

$$h_\alpha(a_1,\ldots,a_n) = \left(\frac{\left(a_1^\alpha + \ldots + a_n^\alpha\right)}{n} \right)^{1/\alpha}$$

Generalized mean

- Typical cases:
  - Lower bound: $h_{-\infty} = \min(a_1, \ldots, a_n)$
  - *Geometric mean*: $h_0 = (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{1/n}$
  - *Harmonic mean*: $h_{-1} = \dfrac{n}{\dfrac{1}{a_1} + \ldots + \dfrac{1}{a_n}}$
  - *Arithmetic mean*: $h_1 = \dfrac{a_1 + \ldots + a_n}{n}$
  - Upper bound: $h_{\infty} = \max(a_1, \ldots, a_n)$

Fuzzy aggregation operations



Membership functions (MF)

- Triangular MF: $$Tr(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$$
- Trapezoidal MF: $$Tp(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),0\right)$$
- Gaussian MF: $$Gs(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$
- Generalized bell MF: $$Bell(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2a}}$$

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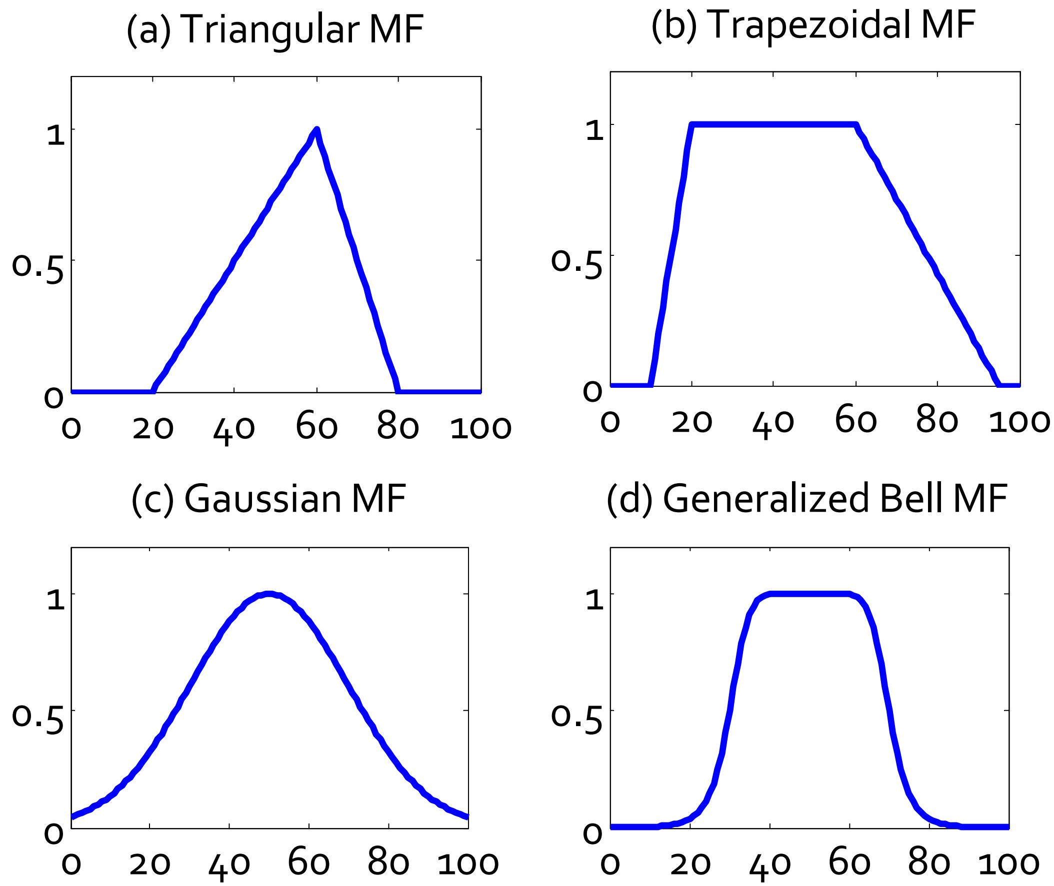
Membership functions



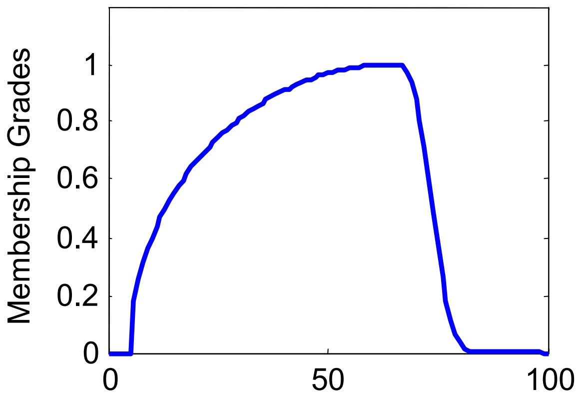
Left-right MF

$$LR(x;c,\alpha,\beta)=\begin{cases}F_L\left(\dfrac{c-x}{\alpha}\right), x<c\\[2ex] F_R\left(\dfrac{x-c}{\beta}\right), x\geq c\end{cases}$$

$c=65$

$\alpha=60$

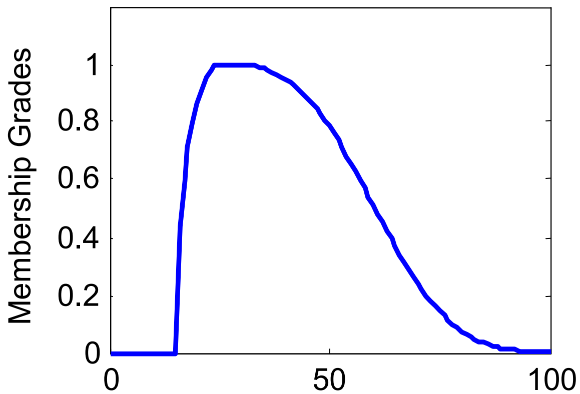
$\beta=10$



$c=25$

$\alpha=10$

$\beta=40$



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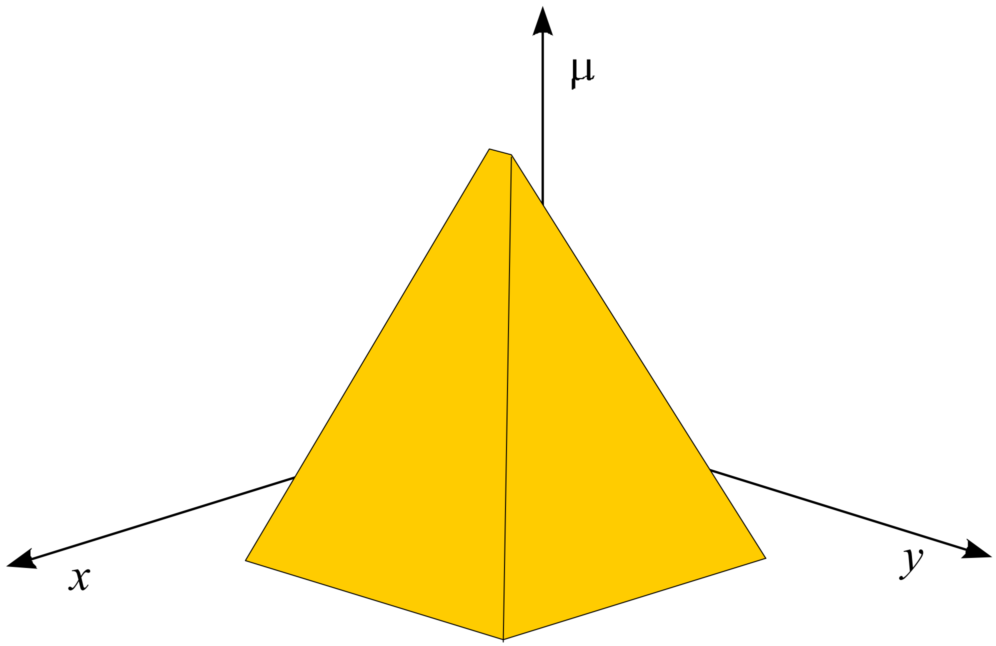
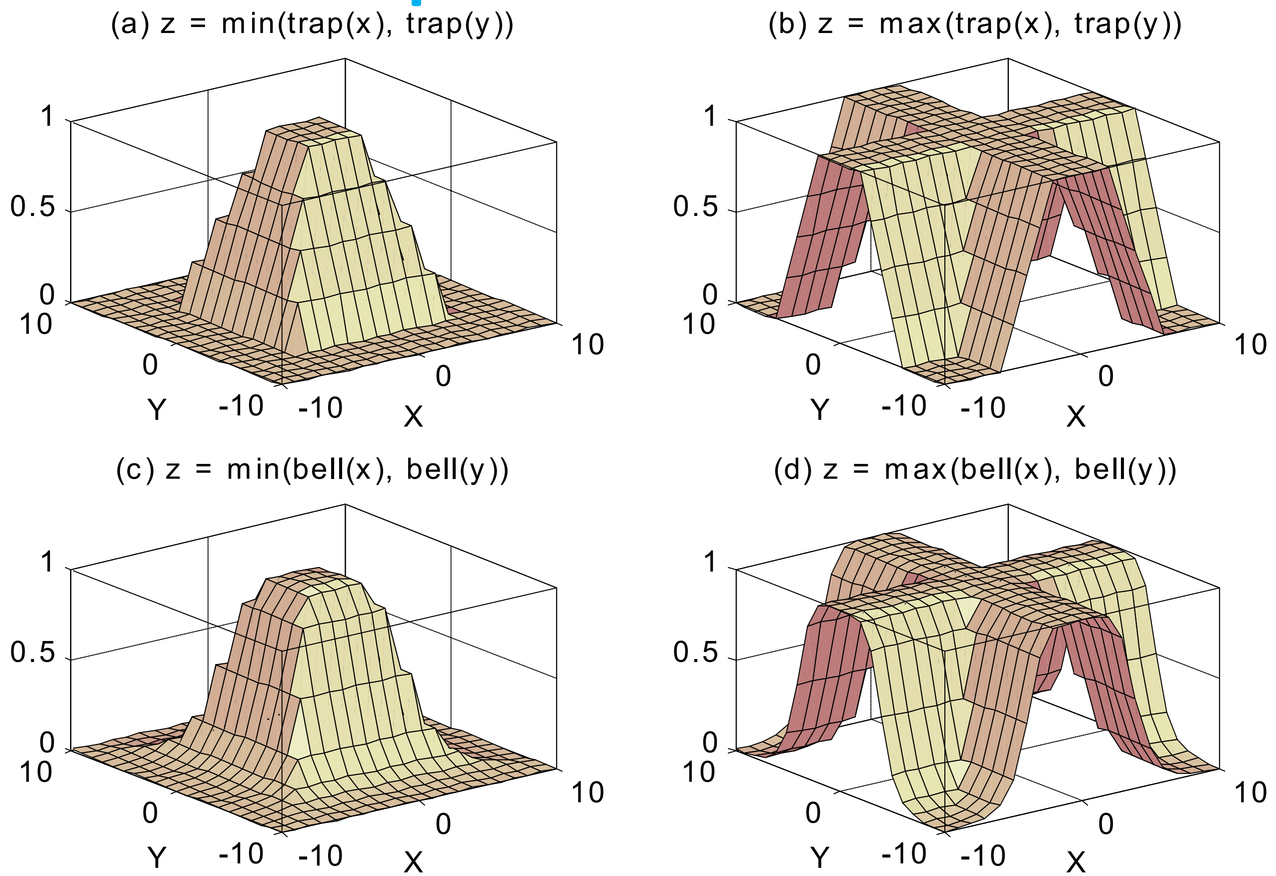
Two-dimensional fuzzy sets

$$A = \int_{X \times Y} \mu_A(x, y) = \left\{ \mu_A(x, y) \middle| (x, y) \in X \times Y \right\}$$

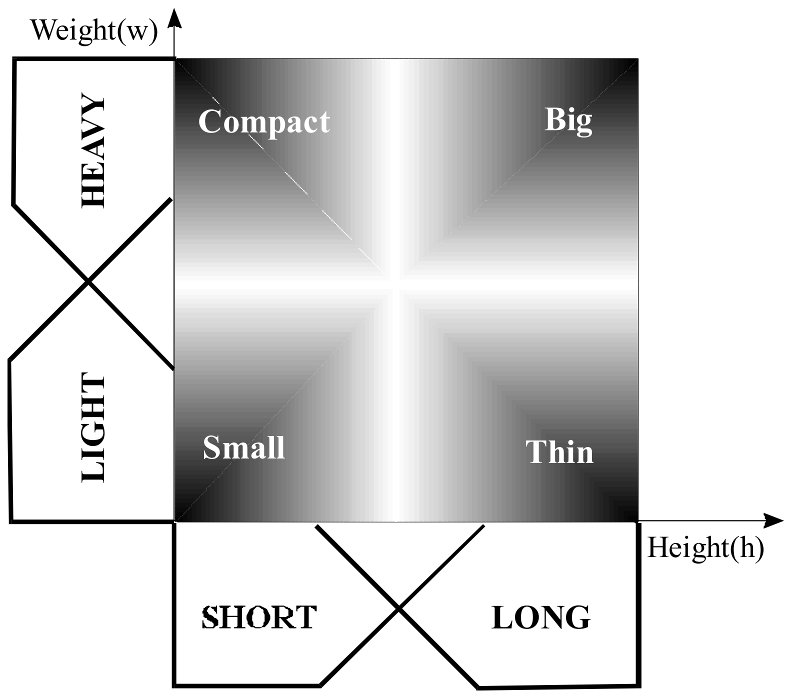


2-D membership functions



Compound fuzzy propositions

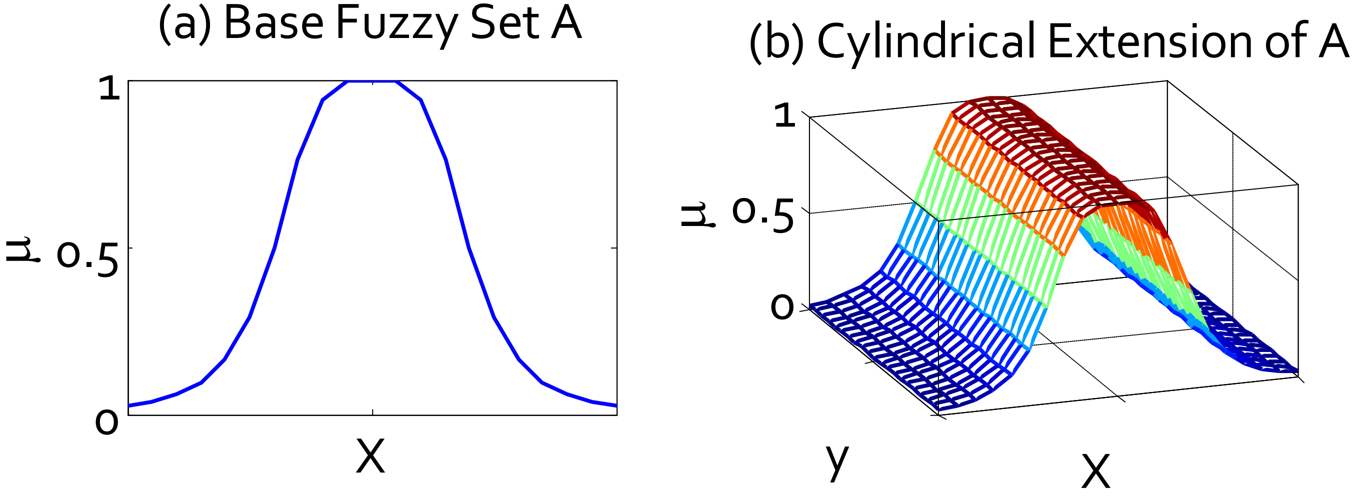
- *Small* = *Short* **and** *Light* (conjunction) $\mu_{Small}(h,w) = \mu_{Short}(h) \cap \mu_{Light}(w)$



Cylindrical extension

- Cylindrical extension of fuzzy set A in X into Y results in a two-dimensional fuzzy set in $X \times Y$, given by

$$\text{cext}_y(A) = \int_{X\times Y} \mu_A(x)/(x,y) = \left\{ \mu_A(x)/(x,y) \middle| (x,y) \in X \times Y \right\}$$



Example

- Premise: Fuzzy set $\mu(x)$ that represents '***Young* people'** in the domain *X* representing age:
- X = {18, 20, 22, 25, 30} [years]

μ_{Young}(x)

x	18	20	22	25	30
$\mu_{Young}(x)$	1	1	0.8	0.5	0.2

- Consider the set of **'duration of mobile calls'** Y = {1, 3, 5, 7, 10, 20} [min/call]

Example

- Compute the cylindrical extension of $\mu(x)$ in to Y:

x	18	20	22	25	30	[years]
$\mu_{Young}(x)$	1	1	0.8	0.5	0.2	

Y = {1, 3, 5, 7, 10, 20} [min/call]

$\mu_{Young}(x)$ into X x Y

x	$\mu_{Young}(x)$	y = 1	3	5	7	10	20
18	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1
22	0.8	0.8	0.8	0.8	0.8	0.8	0.8
25	0.5	0.5	0.5	0.5	0.5	0.5	0.5
30	0.2	0.2	0.2	0.2	0.2	0.2	0.2

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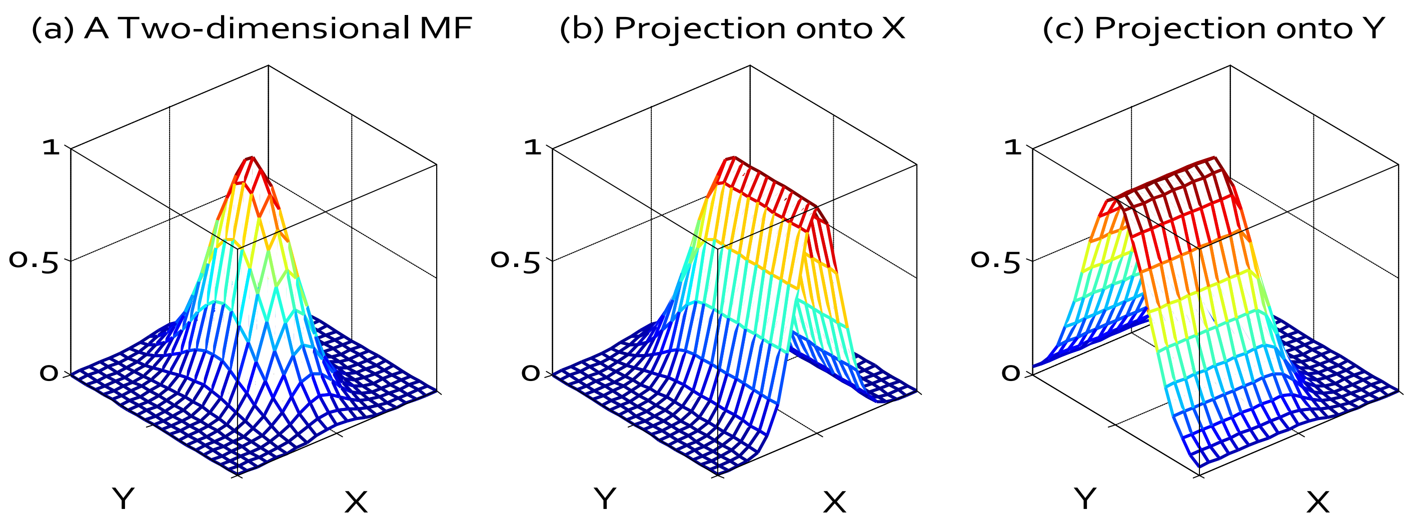
Projection



$$\mu_R(x, y) \qquad \mu_A(x) = \max_y \mu_R(x, y) \qquad \mu_B(y) = \max_x \mu_R(x, y)$$

Cartesian product and co-product

- **Cartesian product** of fuzzy sets A and B is a fuzzy set in the product space $X \times Y$ with membership

$$\mu_{A\times B}(x,y) = \min(\mu_A(x), \mu_B(y))$$

- **Cartesian co-product** of fuzzy sets A and B is a fuzzy set in the product space $X \times Y$ with membership

$$\mu_{A+B}(x,y) = \max(\mu_A(x), \mu_B(y))$$

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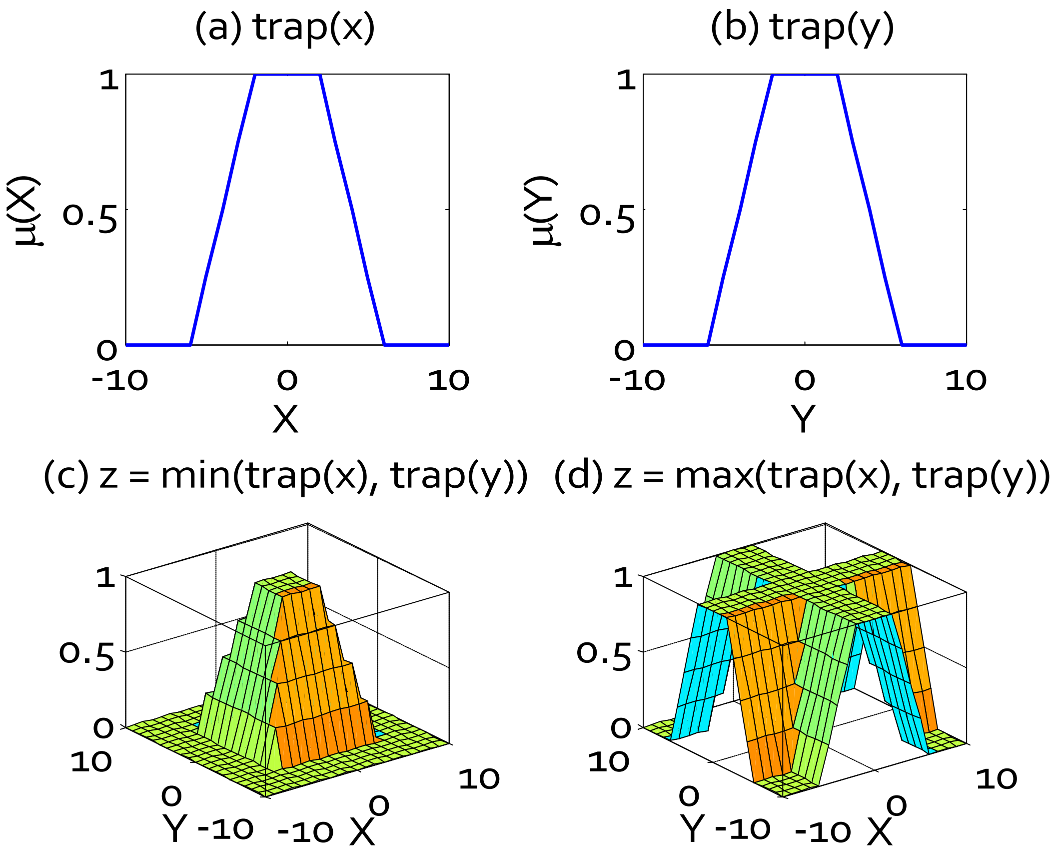
Cartesian product and co-product



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FUZZY RELATIONS

SI3 – Fuzzy Relations

Reading: Part I Fuzzy Set Theory: Chapter 3 Fuzzy Rules and Fuzzy Reasoning

J.-S. Jang, C.-T. Sun and E. Mizutani. ***Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence.*** Prentice Hall, New Jersey, 1997.

Motivation

- A *Crisp relation* represents presence or absence of association, interaction or interconnection between elements of ≥ 2 sets.
  This concept can be generalized to various degrees or strengths of association or interaction between elements.
- A *fuzzy relation* generalizes these degrees to membership grades. So, a crisp relation is a restricted case of a fuzzy relation.

Classical relations

- Classical relation $\mathcal{R}(X_1, X_2, \ldots, X_n)$ is a subset of the Cartesian product:

$$\mathcal{R}\left(X_1, X_2, \ldots, X_n\right) \subset X_1 \times X_2 \times \ldots \times X_n$$

- **Characteristic function:**

$$\mu_{\mathcal{R}}\left(x_1, x_2, \ldots, x_n\right) = \begin{cases} 1, & \text{iff } \left(x_1, x_2, \ldots, x_n\right) \in \mathcal{R} \\ 0, & \text{otherwise} \end{cases}$$

Example

- X = {English, French}
- Y = {dollar, pound, euro}
- Z = {USA, France, Canada, Britain, Germany}
- $\mathcal{R}(X, Y, Z)$ = {(English, dollar, USA),

  (French, euro, France), (English, dollar,Canada),

  (French, dollar, Canada), (English, pound,Britain)}

Matrix representation

$$\begin{array}{c c} & \begin{array}{ccccc} \text{USA} & \text{Fra} & \text{Can} & \text{Brit} & \text{Ger} \end{array} \\ \begin{array}{c} \text{Dollar} \\ \text{Pound} \\ \text{Euro} \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

English

$$\begin{array}{c c} & \begin{array}{ccccc} \text{USA} & \text{Fra} & \text{Can} & \text{Brit} & \text{Ger} \end{array} \\ \begin{array}{c} \text{Dollar} \\ \text{Pound} \\ \text{Euro} \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

French

Fuzzy relation

- **Fuzzy relation:**

$$\mathcal{R}: X_1 \times X_2 \times \ldots \times X_n \to [0,1]$$

- Each tuple $(x_1, x_2, \ldots, x_n)$ has a ***degree of membership.***
- Fuzzy relation can be represented by an *n*-dimensional ***membership function*** (continuous space) or a ***matrix*** (discrete space).
- **Examples:**
  - x is close to y
  - x and y are similar
  - x and y are related (dependent)

Discrete examples

- Relation $\mathcal{R}$ "*very far*" between X = {New York, Lisbon} and Y = {New York, Beijing, London}:

$\mathcal{R}(x,y)$ = 0/(NY,NY) + 1/(NY,Beijing) + 0.6/(NY,London) + 0.5/(Lisbon,NY) + 0.8/(Lisbon,Beijing) + 0.1/(Lisbon,London)

Discrete examples

- Relation: "*is an important trade partner of*"

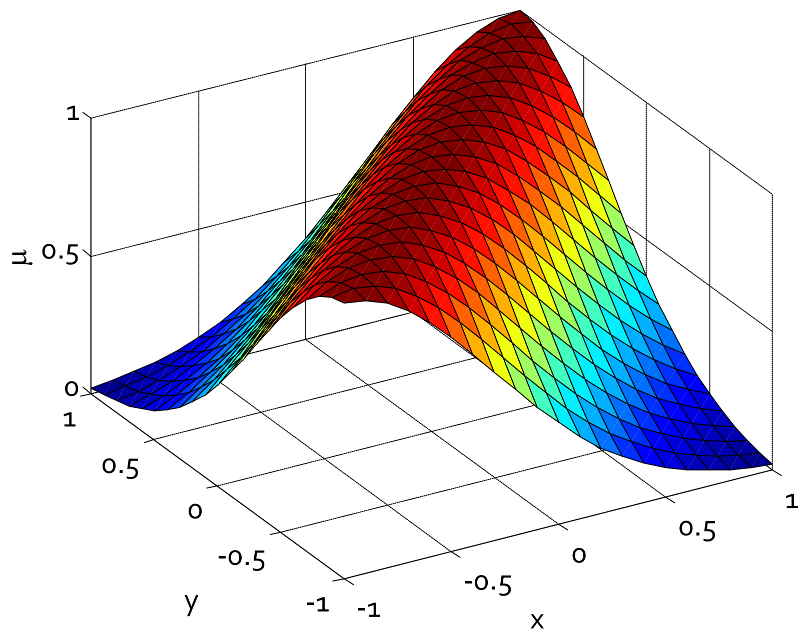
	Holland	Germany	USA	Japan
Holland	1	0,9	0,5	0,2
Germany	0,3	1	0,4	0,2
USA	0,3	0,4	1	0,7
Japan	0,6	0,8	0,9	1

X = {New York, Lisbon}
and Y = {New York, Beijing, London}

Continuous example

- $\mathcal{R}$: $x \approx y$ ("*x* is approximately equal to *y*")

$$\mu_{\Re}(x,y) = e^{-(x-y)^2}$$



Composition of relations

- $\mathscr{R}(X,Z) = \mathscr{P}(X,Y) \circ \mathscr{Q}(Y,Z)$

Conditions:

- $(x,z) \in \mathscr{R}$ iff exists $y \in Y$ such that
- $(x,y) \in \mathscr{P}$ and $(y,z) \in \mathscr{Q}$.

$$\mu_{\mathscr{P}\circ\mathscr{Q}}(x,z) = \max_{y\in Y} \min\left[\mu_{\mathscr{P}}(x,y), \mu_{\mathscr{Q}}(y,z)\right]$$

- **Max-min composition**

Properties

- Associativity: $\mathcal{R} \circ (\mathcal{S} \circ \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \circ \mathcal{T}$
- Distributivity over union: $\mathcal{R} \circ (\mathcal{S} \cup \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \cup (\mathcal{R} \circ \mathcal{T})$
- Weak distributivity over intersection: $\mathcal{R} \circ (\mathcal{S} \cap \mathcal{T}) \subseteq (\mathcal{R} \circ \mathcal{S}) \cap (\mathcal{R} \circ \mathcal{T})$
- Monotonicity: $\mathcal{S} \subseteq \mathcal{T} \Rightarrow (\mathcal{R} \circ \mathcal{S}) \subseteq (\mathcal{R} \circ \mathcal{T})$

Other compositions

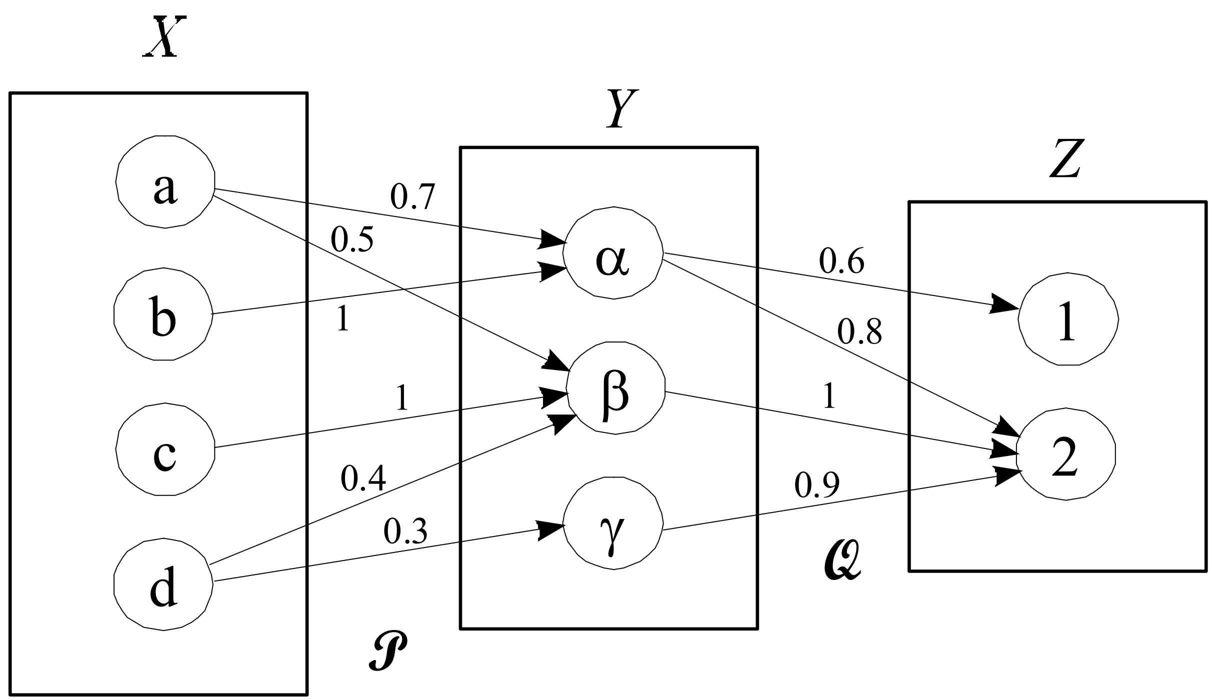
- **Max-prod composition**

$$\mu_{\mathcal{P}\circ\mathcal{Q}}(x,z)=\max_{y\in Y}\left(\mu_{\mathcal{P}}(x,y)\cdot\mu_{\mathcal{Q}}(y,z)\right)$$

- **Max-*t* composition**

$$\mu_{\mathcal{P}\circ\mathcal{Q}}(x,z)=\max_{y\in Y} t\left(\mu_{\mathcal{P}}(x,y),\mu_{\mathcal{Q}}(y,z)\right)$$

Example



Example

- Composition $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$

  Max-min composition

- Composition $\mathcal{R} = \mathcal{P} \otimes \mathcal{Q}$?

  Max-prod composition

x	z	$\mu_R(x,z)$
a	1	0.6
a	2	0.7
b	1	0.6
b	2	0.8
c	2	1
d	2	0.4

Matrix notation examples

Max-min composition: max(min(0.3,0.9),min(0.5,0.3),min(0.8,1.0)) =

=

$$\begin{bmatrix} \boxed{0.3 & 0.5 & 0.8} \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} \boxed{} & & & \\ & & & \\ & & & \end{bmatrix}$$

Matrix notation examples

$$\begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}$$

$$\begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \otimes \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.15 & 0.4 & 0.45 \\ 1 & 0.14 & 0.5 & 0.63 \\ 0.5 & 0.2 & 0.28 & 0.54 \end{bmatrix}$$

Relations on the same universe

- Let $\mathcal{R}$ be a relation defined on $U \times U$, then it is called:

- **Reflexive**, if $\forall\ u \in U$, the pair $(u,u) \in \mathcal{R}$
- **Anti-reflexive**, if $\forall\ u \in U$, $(u,u) \notin \mathcal{R}$
- **Symmetric**, if $\forall\ u,v \in U$, if $(u,v) \in \mathcal{R}$, then $(v,u) \in \mathcal{R}$ too
- **Anti-symmetric**, if $\forall\ u,v \in U$, if (u,v) and $(v,u) \in \mathcal{R}$, then $u = v$
- **Transitive**, if $\forall\ u,v,w \in U$, if (u,v) and $(v,w) \in \mathcal{R}$, then $(u,w) \in \mathcal{R}$ too.

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Examples

- $\mathfrak{R}$ is an *equivalence relation* if it is reflexive, symmetric and transitive.
- $\mathfrak{R}$ is a *partial order relation* if it is reflexive, anti-symmetric and transitive.
- $\mathfrak{R}$ is a *total order relation* if $\mathfrak{R}$ is a partial order relation, and $\forall$ $u, v \in U$, either (u,v) or $(v,u) \in \mathfrak{R}$.

- **Examples**:
  - The subset relation on sets ($\subseteq$) is a partial order relation.
  - The relation $\leq$ on N is a total order relation.

FUZZY SYSTEMS

SI3 – Fuzzy Systems

Reading: Part I Fuzzy Set Theory: Chapter 4 Fuzzy Inference Systems

J.-S. Jang, C.-T. Sun and E. Mizutani. ***Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence.*** Prentice Hall, New Jersey, 1997.

Extension Principle

- General procedure to extend crisp mathematical expressions to fuzzy domains.
- Generalizes a point-to-point mapping into a mapping between fuzzy sets.
- Given a function f mapping points in set X to points in set Y:

$$f: X_1 \times X_2 \times \ldots \times X_n \rightarrow Y, \; y = f(x_1, x_2, \ldots, x_n)$$

- And a fuzzy set A where:

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n,$$

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Extension Principle

- the extension principle states that:

$$f(A) = f(\mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n)$$

$$+ \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \ldots + \mu_A(x_n)/y_n$$

- If f is a many-to-one mapping, then $\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x)$
- Fuzzy set B is the image of A in Y using f, and is given by:

$$\mu_B(y) = \begin{cases} \max_{\mathbf{x}=f^{-1}(y)} \left[\min_i \left(\mu_{A_i}(x_i) \right) \right], & \text{if } f^{-1}(y) \neq \varnothing \\ 0, & \text{if } f^{-1}(y) = \varnothing \end{cases}$$

Example

- Let

$$A = 0.1/{-2} + 0.4/{-1} + 0.8/0 + 0.9/1 + 0.3/2$$

- and

$$f(x) = x^2 - 3$$

- Applying the extension principle:

$$\begin{aligned} B &= \boxed{0.1/1} + \boxed{0.4/{-2}} + 0.8/{-3} + \boxed{0.9/{-2}} + \boxed{0.3/1} \\ &= 0.8/{-3} + (0.4 \vee 0.9)/{-2} + (0.1 \vee 0.3)/1 \\ &= 0.8/{-3} + 0.9/{-2} + 0.3/1 \end{aligned}$$

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Example

$$
\begin{array}{c|cc}
 & x & y \\
\hline
a & p & p \\
b & q & r \\
c & r & p
\end{array}
$$

- $X_1 = \{a,b,c\}$, $X_2 = \{x,y\}$, $Y = \{p,q,r\}$
- $f : X_1 \times X_2 \rightarrow Y$

- $A_1 = .3/a + .9/b + .5/c, \quad A_2 = .5/x + 1/y$
- $\mu_B(p) = \max[\min(.3,.5),\min(.3,1),\min(.5,1)] = .5$

  $\mu_B(q) = \max[\min(.9,.5)] = .5$

  $\mu_B(r) = \max[\min(.5,.5),\min(.9,1)] = .9$
- $f(A_1, A_2) = .5/p + .5/q + .9/r$

Extension Principle Example

- Example for a continuous function

$$[f(A)](y) = \sup_{x|y=f(x)} A(x)$$

for all $A \in \mathcal{F}(X)$ and

$$[f^{-1}(B)](x) = B(f(x))$$

for all $B \in \mathcal{F}(Y)$.

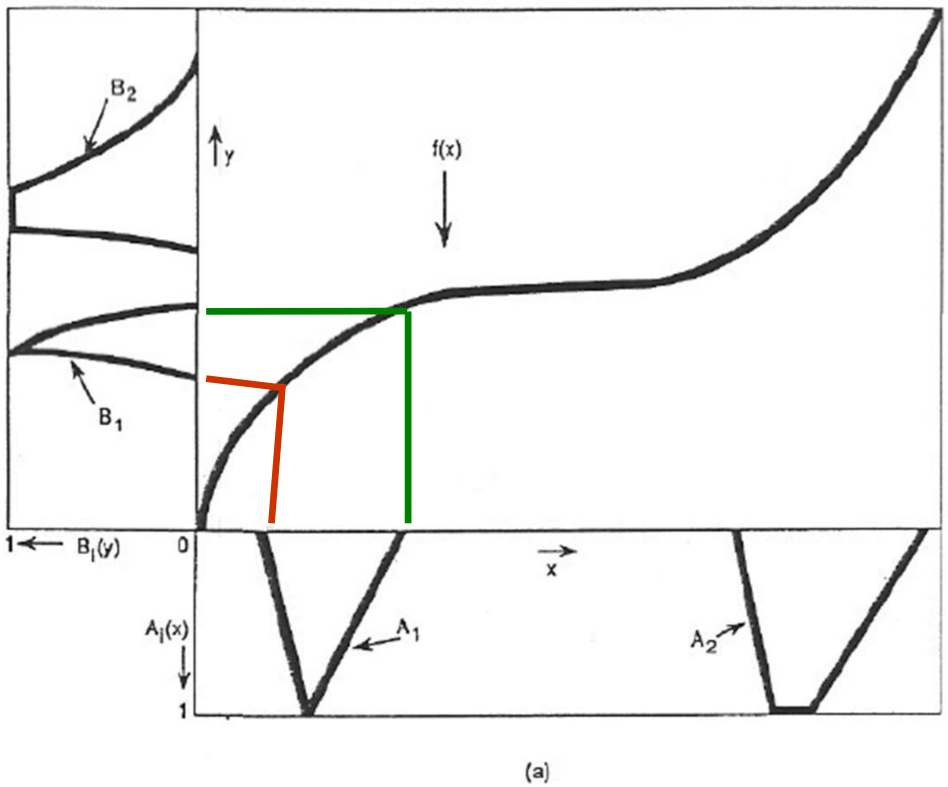


Figure 2.5 Illustration of the extension principle when f is continuous.

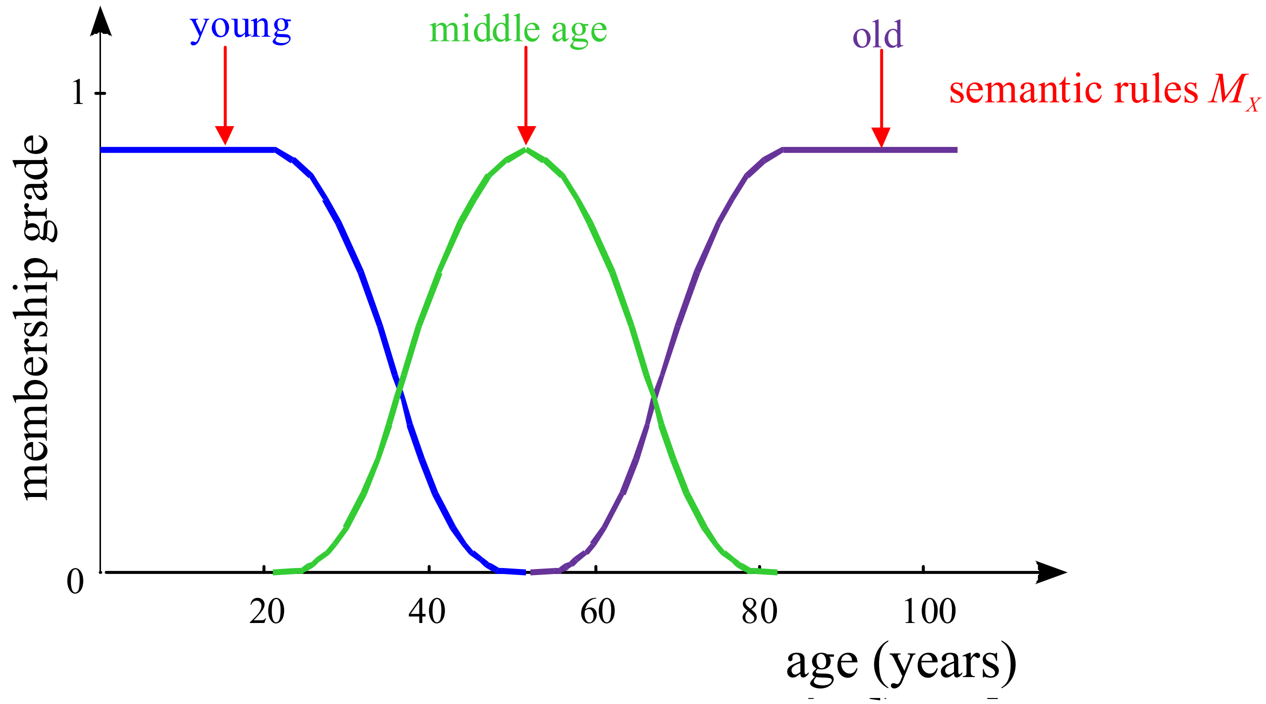
Linguistic variable

$$\{x, \text{LX}, \text{X}, M_X\}$$

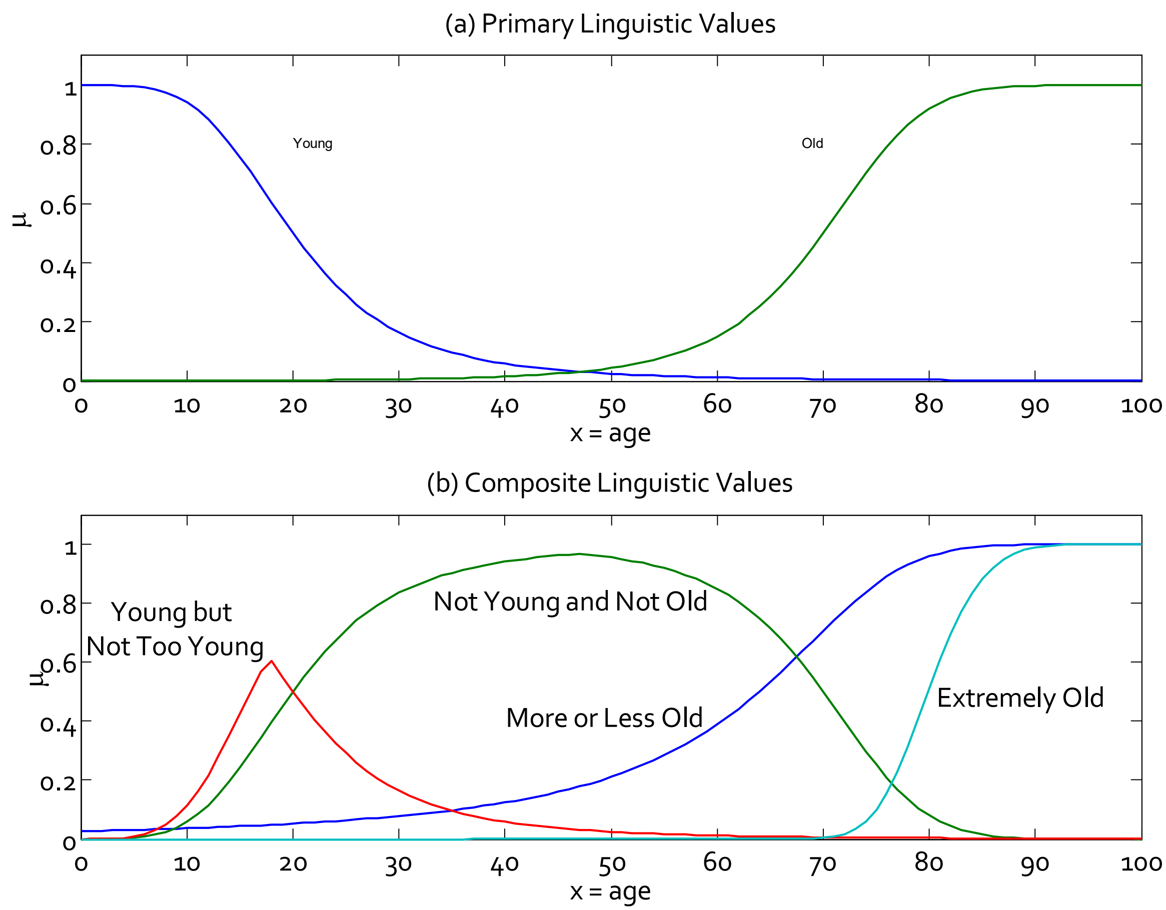
- Where:
  - x – name of the linguistic variable
  - LX – linguistic values (terms)
  - X – Universe of discourse
  - M_X – semantic rule that associates each linguistic value to a membership function.

Example

- x is *age* and LX = {*young*, *middle age*, *old*}



Linguistic values (terms)



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Linguistic hedges (modifiers)

- More common modifiers: ***likely***, ***almost***, ***very***, ***more or less***, ***fairly***, ***rather***, ***too***, ***extremely***, etc.
- Typical **concentration** operator: ***very***

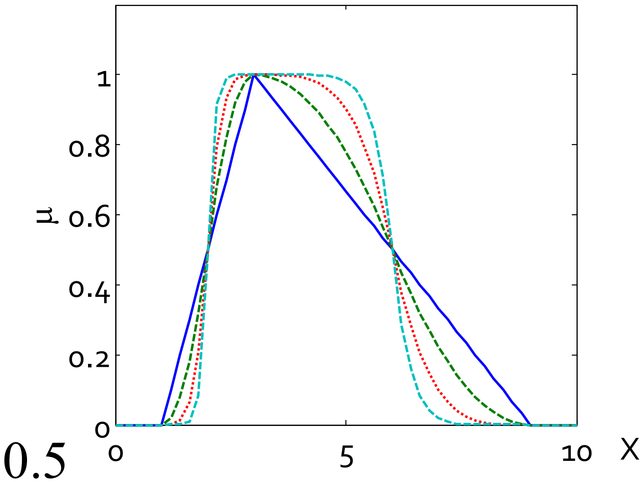
$$\mathrm{CON}(\mu_A(x)) = \mu_A{}^2(x)$$

- Typical **dilation** operator: ***more or less***

$$\mathrm{DIL}(\mu_A(x)) = \mu_A{}^{0.5}(x)$$

- **Contrast intensification** operator:

$$\mathrm{INT}(\mu_A(x)) = \begin{cases} 2\mu_A^2(x), & 0 \le \mu_A(x) \le 0.5 \\ \overline{2}(\overline{\mu}_A^2(x)), & 0.5 \le \mu_A(x) \le 1 \end{cases}$$



Fuzzy systems

- Fuzzy systems manipulate fuzzy sets to model the world.
- Most fuzzy systems are **rule based**.

antecedent

linguistic variables

fuzzy sets

If *age of human person* is *young*
then *risk of heart attack* is *low*

consequent

- Consequent can be a *fuzzy set* or a *crisp function*

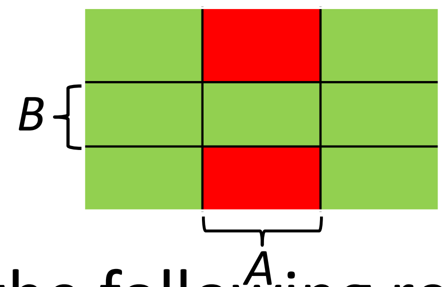
Fuzzy if-then rules

- Linguistic (Mamdani) **fuzzy if-then rule**
  - **If** x is A **then** y is B
- Rule is represented by a ***fuzzy relation*** defined on $X \times Y$
- **Examples:**
  - If the road is slippery then brake softly.
  - If a tomato is red then the tomato is ripe.
  - If the temperature is high then reduce the heat.
  - If the valve is closed then the pressure is high.

Fuzzy inference

- The rule **If** *x* is *A* **then** *y* is *B* can be interpreted as ***A entails B*** (***fuzzy implication***) where

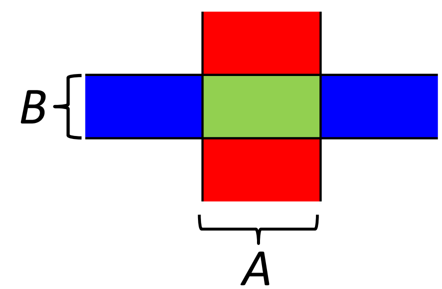
$$\mathscr{R} = \overline{A} \cup B$$



- or that ***A* is coupled with *B***, resulting in the following relation:

$$\mathscr{R} = \operatorname{cext}_Y(A) \cap \operatorname{cext}_X(B)$$

$$\mu_{\mathscr{R}}(x, y) = \min(\mu_A(x), \mu_B(x))$$



Fuzzy implications

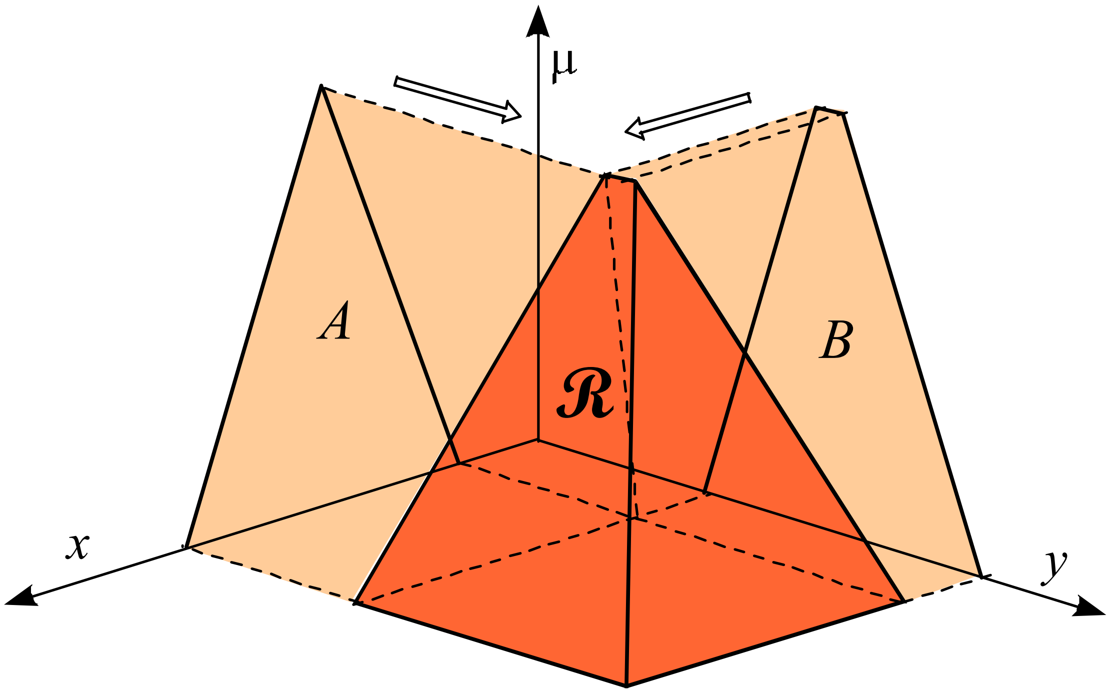
- Fuzzy implications: $\mu_{\mathcal{R}}(x, y) = I(\mu_A(x), \mu_B(y))$

- Examples of implications:

Kleene-Dienes	$I(\mu_A(x), \mu_B(x)) = \max(1 - \mu_A(x), \mu_B(x))$
Lukasiewicz	$I(\mu_A(x), \mu_B(x)) = \min(1, 1 - \mu_A(x) + \mu_B(x))$
Mamdani	$I(\mu_A(x), \mu_B(x)) = \min(\mu_A(x), \mu_B(x))$

Example of Mamdani inference

- **If** *x* is *A* **then** *y* is *B* using the **min**.



Example

- Premise: **Young people make long Mobile calls**
- X = {18, 20, 22, 25, 30} [years]
- Y = {1, 3, 5, 7, 10, 20} [min/call]

young(x)

x	18	20	22	25	30
$\mu(x)$	1	1	0.8	0.5	0.2

long(y)

y	1	3	5	7	10	20
$\mu(y)$	0	0.1	0.2	0.5	0.9	1

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Example

- Compute cylindrical extensions

young(x) into X x Y

x	$\mu(x)$	y = 1	3	5	7	10	20
18	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1
22	0.8	0.8	0.8	0.8	0.8	0.8	0.8
25	0.5	0.5	0.5	0.5	0.5	0.5	0.5
30	0.2	0.2	0.2	0.2	0.2	0.2	0.2

long(y) into X x Y

y	1	3	5	7	10	20
$\mu(y)$	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1

Example

- Compute the aggregation of the two cylindrical extensions, using e.g conjunctive operator minimum.

1	1	1	1	1	1
1	1	1	1	1	1
0.8	0.8	0.8	0.8	0.8	0.8
0.5	0.5	0.5	0.5	0.5	0.5
0.2	0.2	0.2	0.2	0.2	0.2

0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1

min

x \ *y*	1	3	5	7	10	20
18	0	0.1	0.2	0.5	0.9	1
20	0	0.1	0.2	0.5	0.9	1
22	0	0.1	0.2	0.5	0.8	0.8
25	0	0.1	0.2	0.5	0.5	0.5
30	0	0.1	0.2	0.2	0.2	0.2

R

Compositional rule of inference

- Let a fuzzy set A be defined on X, and $\mathcal{R}$ be a fuzzy relation defined in $X \times Y$. The composition of A and $\mathcal{R}$ results in fuzzy set B defined in Y:

$$B = A \circ \mathcal{R} = \mathrm{proj}_X(\mathrm{cext}_Y(A) \cap \mathcal{R})$$

- If intersection is given by the **min** and the projection with the **max**, results in the **max-min composition**:

$$\mu_B(y) = \max_x \min\left[\mu_A(x), \mu_{\mathcal{R}}(x, y)\right]$$

- Returns the image of A transformed through the relation $\mathcal{R}$.

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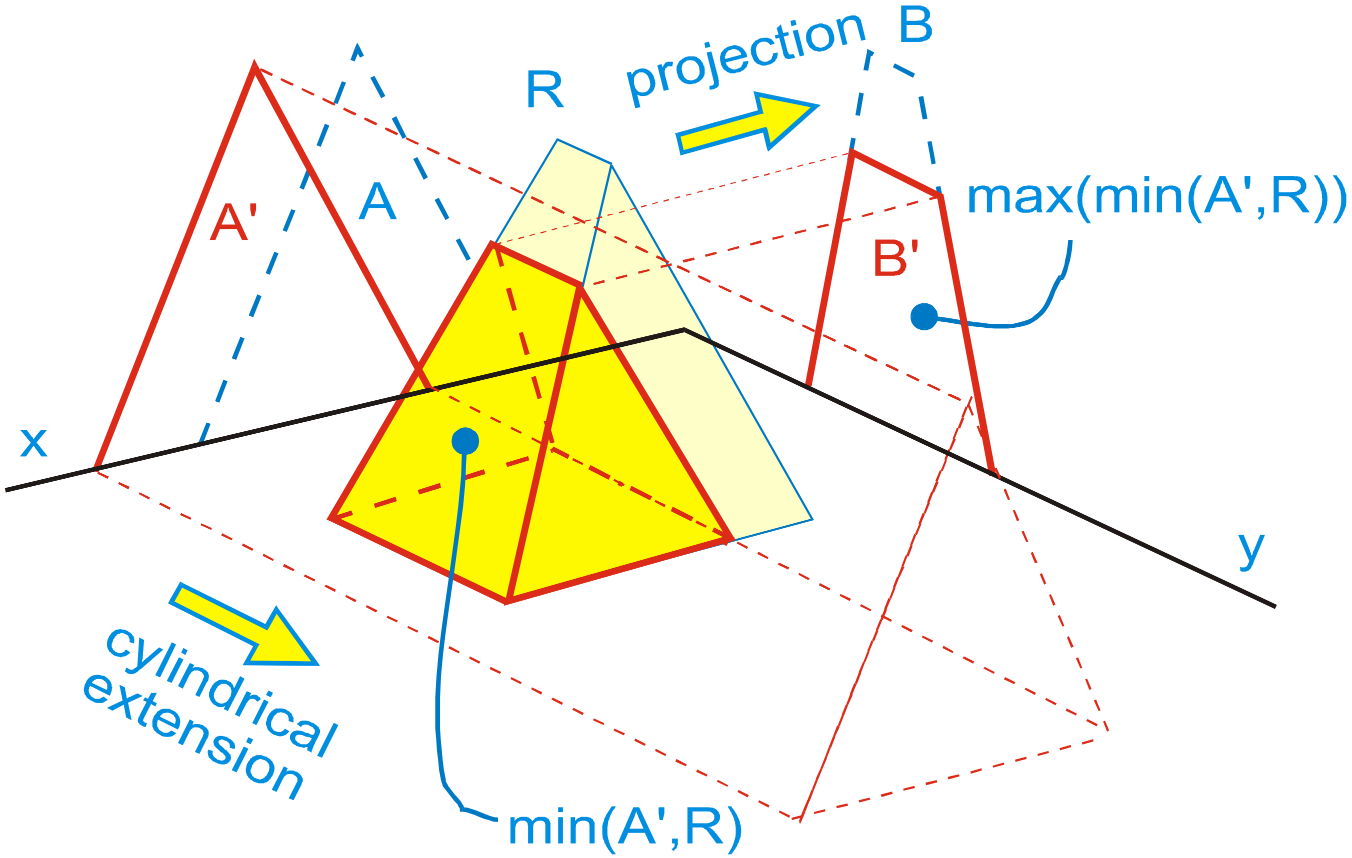
Compositional rule of inference



Example max-min composition

$$\mu_B(y) = \max_x \min\left[\mu_A(x), \mu_{\mathfrak{R}}(x,y)\right]$$

$$\begin{bmatrix} 1.0 & 0.7 & 0.4 & 0.1 & 0.0 \end{bmatrix} \circ \begin{bmatrix} 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 1.0 & 0.1 & 0.0 \\ 0.0 & 1.0 & 0.5 & 0.2 & 0.0 \\ 0.1 & 0.5 & 0.2 & 0.8 & 0.0 \\ 0.3 & 0.0 & 0.0 & 1.0 & 1.0 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.5 & 1.0 & 0.2 & 0.0 \end{bmatrix}$$

Example max-min composition

Young people make long Mobile calls (XxY)

People who make long Mobile calls give a lot of money to clothing (YxZ)

Young people give a lot of money to clothing (XxZ)

$$\begin{bmatrix} 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8 \\ 0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5 \\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.9 \\ 0 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0.8 \\ 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0.2 \end{bmatrix}$$

R ***S*** ***R°S***

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Fuzzy rules and fuzzy relations

- Fuzzy rules like $A \rightarrow B$ are represented as fuzzy relations.
- Often, minimum operator is used to compute the relation.
- The collection of all rules is represented as an aggregated relation using the union operator:

$$R_{\text{tot}} = \bigcup_{i=1}^{N} R_i = \bigvee_{i=1}^{N} R_i$$

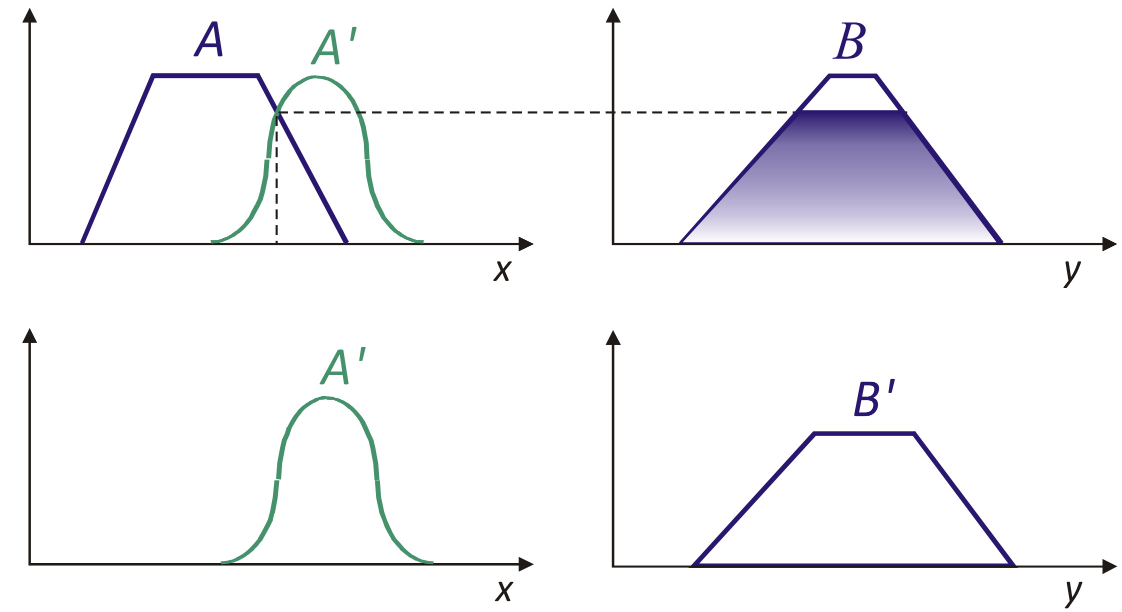
Inference with one rule

If *x* is *A* **then** *y* is *B*

x is *A’*

y is *B’*

$$\mu_{\mathcal{R}}(x,y) = \min(\mu_A(x), \mu_B(y))$$

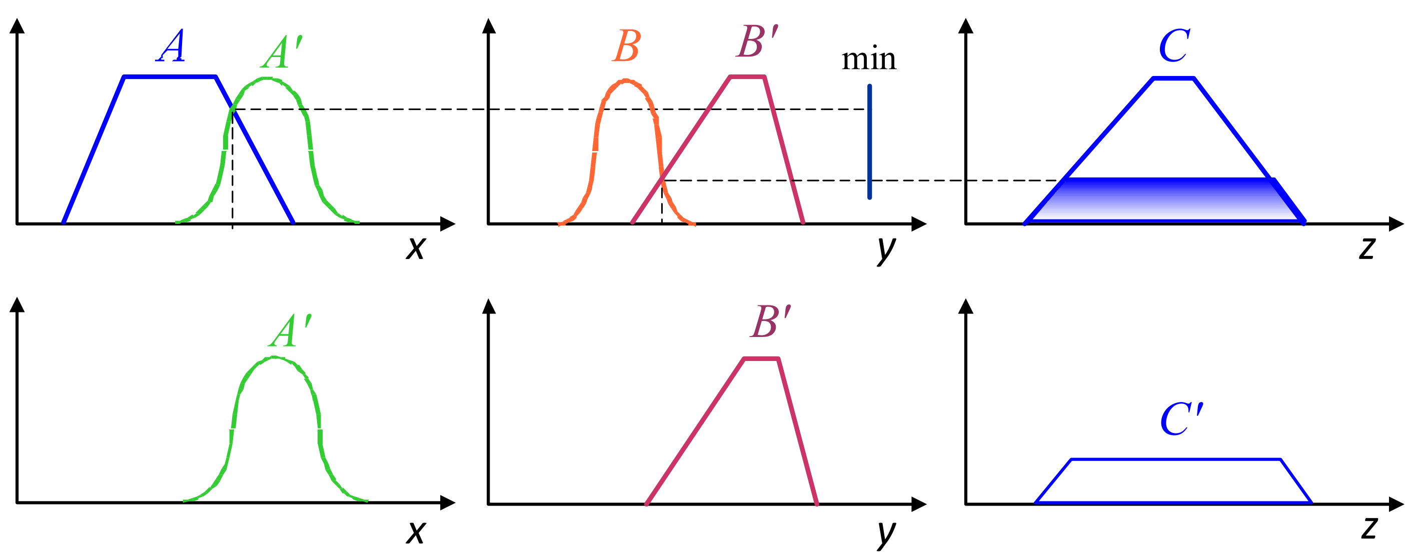


Inference with two antecedents

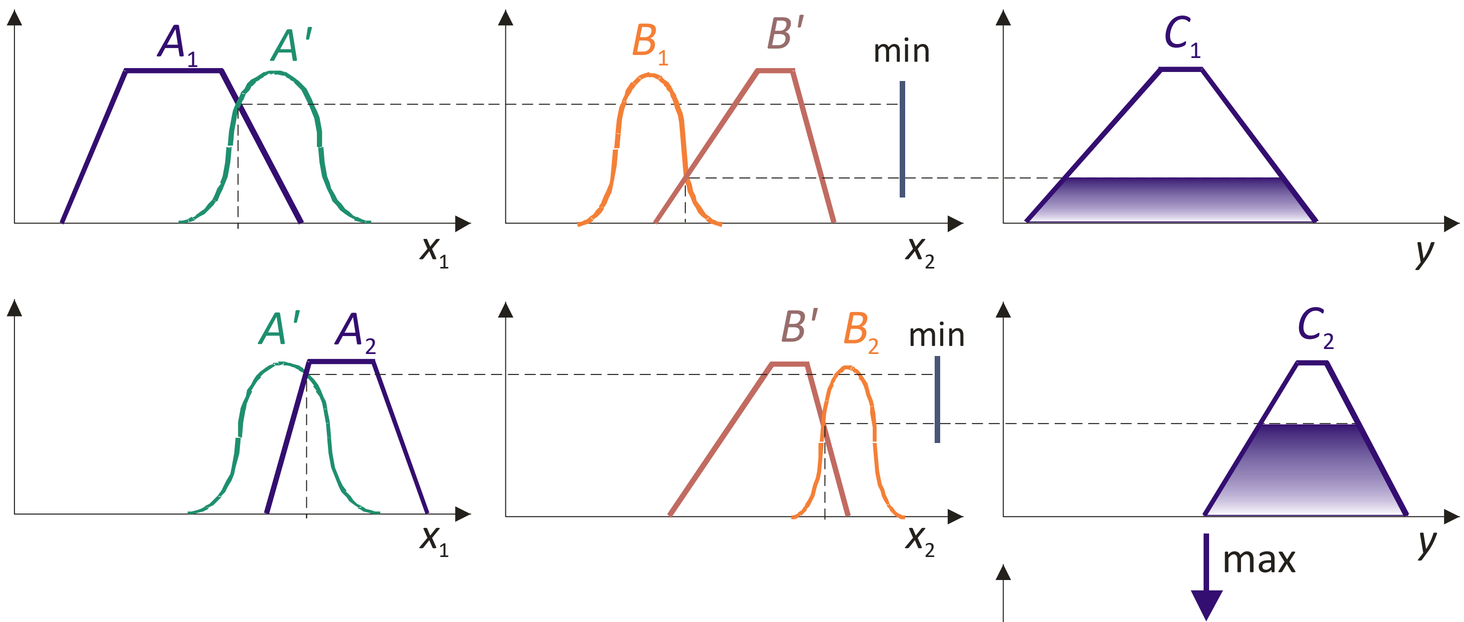
If *x* is *A* and *y* is B **then** *z* is *C*

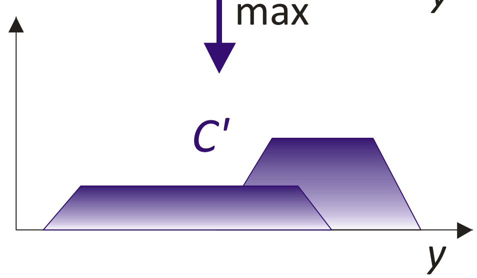
<u>*x* is *A'* and *y* is *B'*</u>.

z is *C'*



Inference with several rules





If x_1 is A_1 and x_2 is B_1 then y is C_1

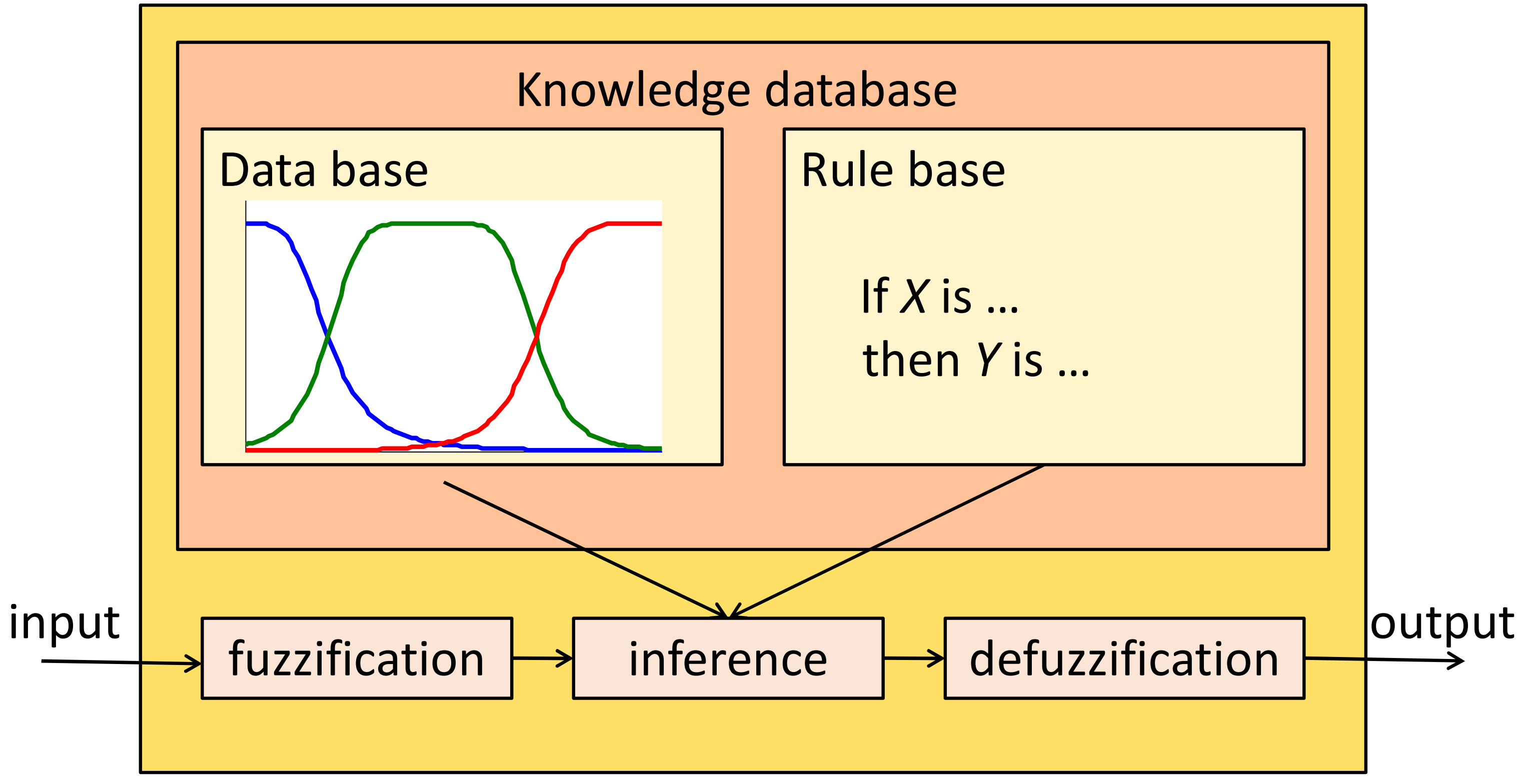
If x_1 is A_2 and x_2 is B_2 then y is C_2

Fuzzy inference system

Multiple names:

- Fuzzy rule-based system
- Fuzzy expert system
- Fuzzy model
- Fuzzy associative memory
- Fuzzy logic controller
- Fuzzy system

Building blocks



Fuzzifier

- Interface between the inputs and the fuzzy system
- Determines the match between a given input and the linguistic terms
- For crisp inputs: computes the membership to linguistic terms
- For fuzzy inputs: computes the maximum membership of the fuzzy input in the linguistic terms

Knowledge base

- Encodes the general relation between the inputs and the outputs
- Rules can be examples, rules of thumb, encoded experience, qualitative relations between variables, etc.
- Rules are often represented as if-then statements

Inference engine

- The reasoning mechanism of the fuzzy system (to infer: to reason/deduce)
- Combines actual inputs with the information encoded in the rule base to compute the fuzzy output of the system
- Usually implements the ***compositional rule of inference*** or some equivalent computation
- Not as context-independent as the inference engine of an expert system

Defuzzifier

- Interface between the fuzzy systems and the output
- Needed when a crisp output is required (e.g. a final decision, a control action, a final advice, etc.)
- Computes a number/symbol that represents the output fuzzy set
- Enhances the interpolation properties of the fuzzy system

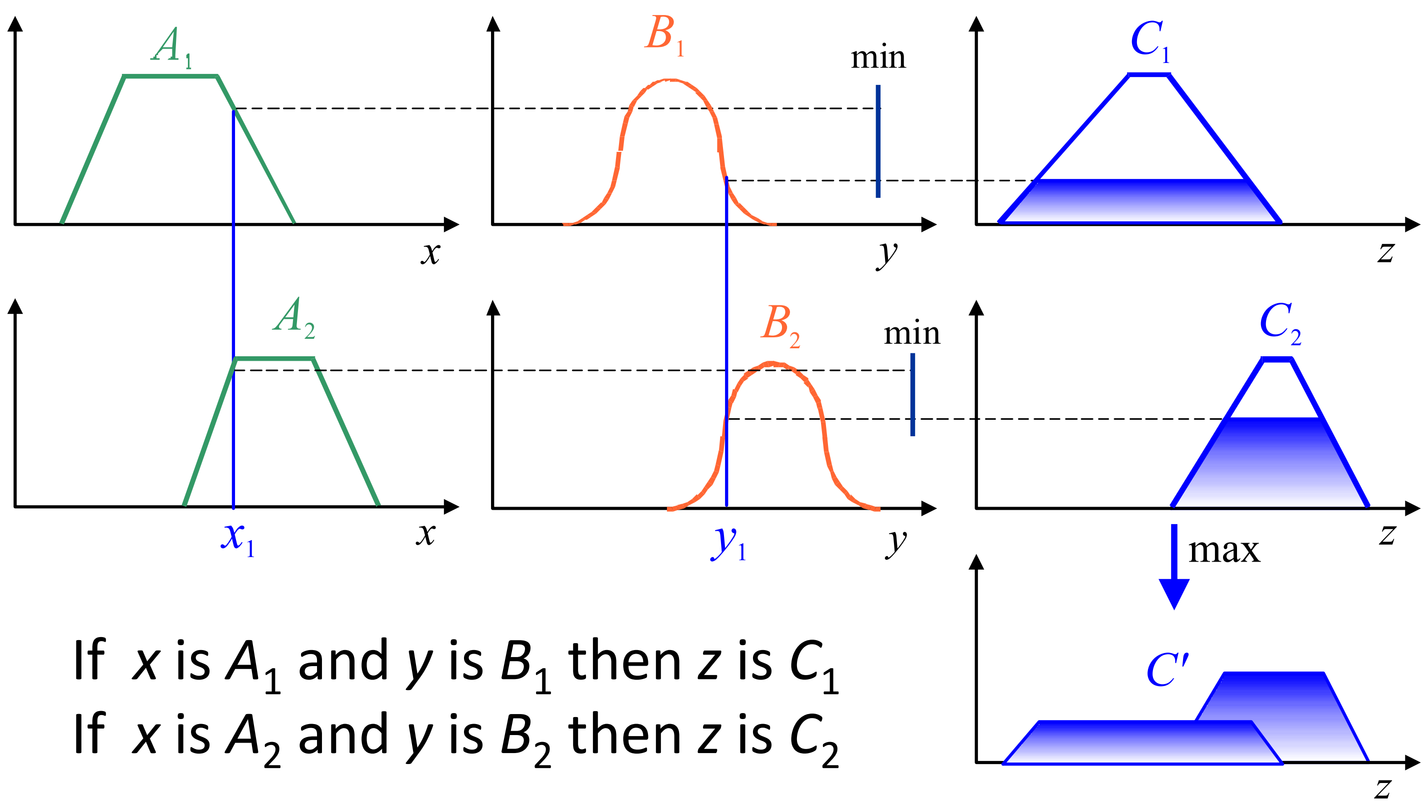
Mamdani fuzzy models

Five major steps:

- Fuzzification
- Degree of fulfillment
- Inference
- Aggregation
- Defuzzification

➢ Mamdani reasoning is mathematically equivalent to the compositional rule of inference.

Mamdani fuzzy inference



If x is A_1 and y is B_1 then z is C_1
If x is A_2 and y is B_2 then z is C_2

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Defuzzification methods

Main types:

- Centroid of area (COA)

$$z_{\mathrm{COA}} = \frac{\int_z \mu_{C'}(z) z \, dz}{\int_z \mu_{C'}(z) \, dz}$$

- Mean of maxima (MOM)

$$z_{\mathrm{MOM}} = \frac{\int_{z'} z \, dz}{\int_{z'} dz}, \quad Z' = \{ z \mid \mu_{C'}(z) \text{ is maximum} \}$$

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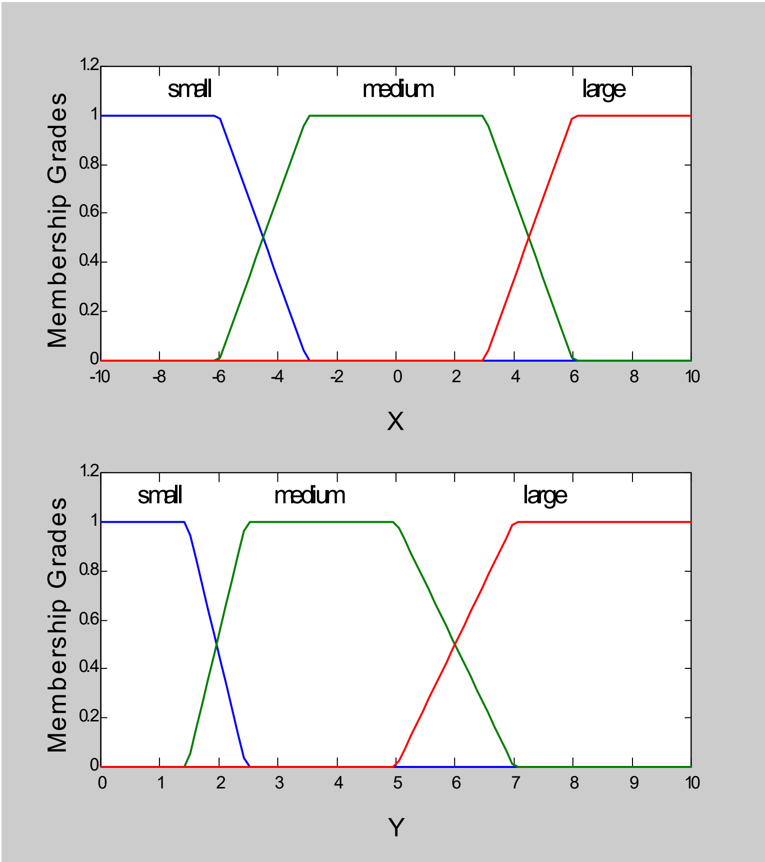
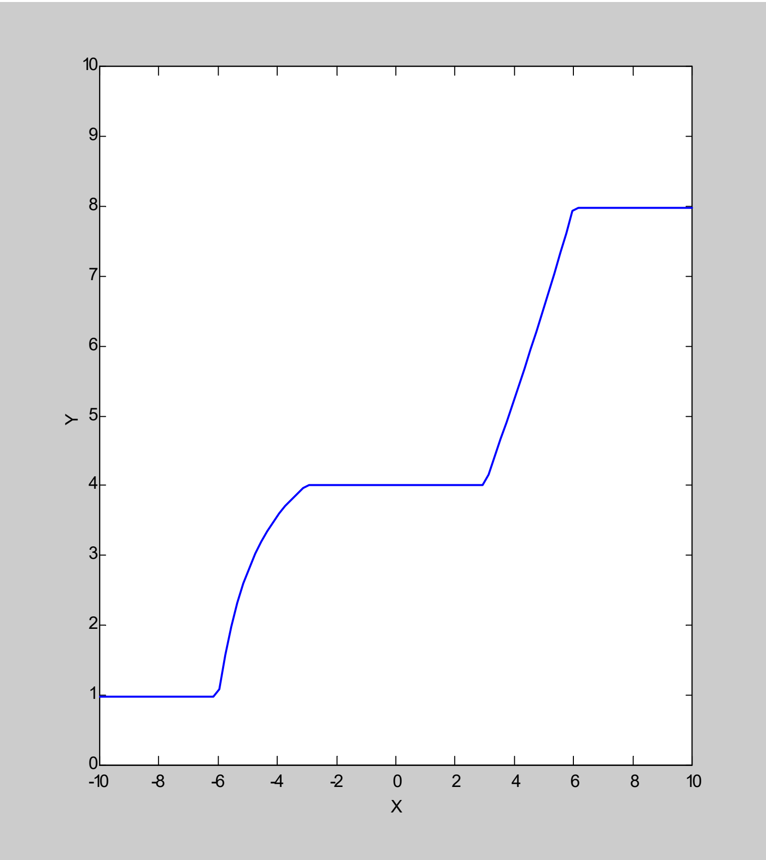
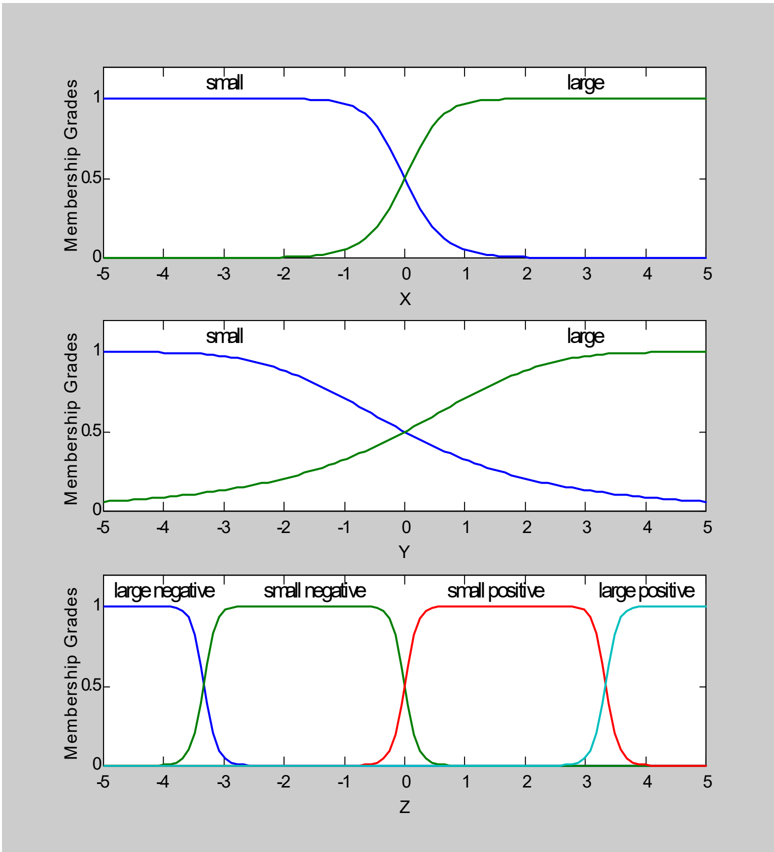
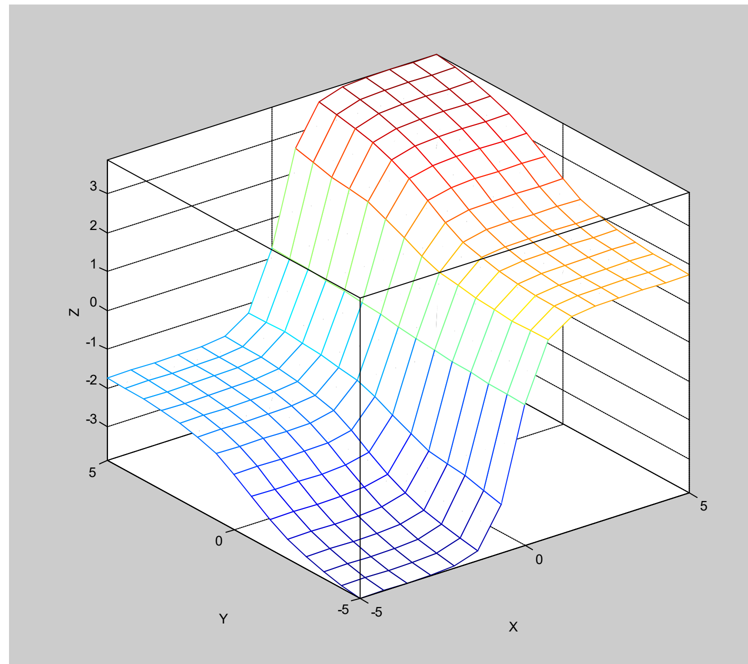
Mamdani – single input





Mamdani – two inputs



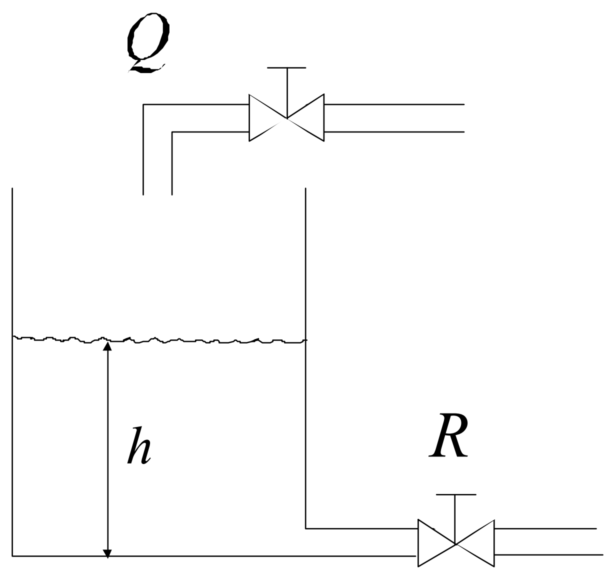


Example: control of liquid level

If level is *low* **then** *increase* valve opening

If level is *OK* **then** *maintain* valve opening

If level is *high* **then** *decrease* valve opening



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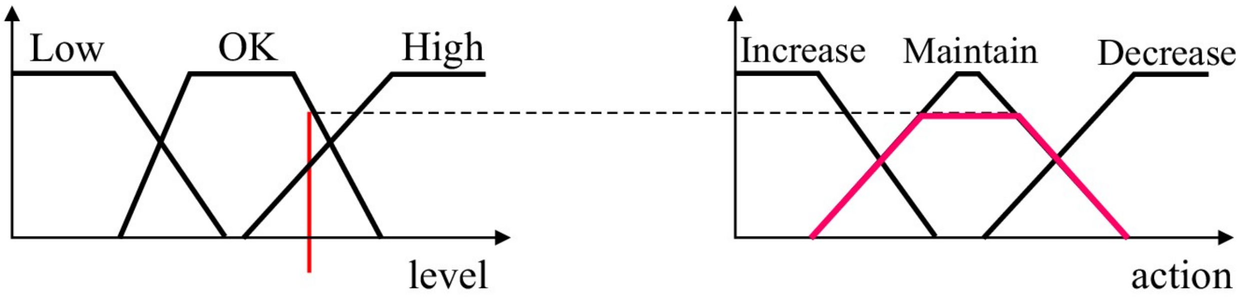
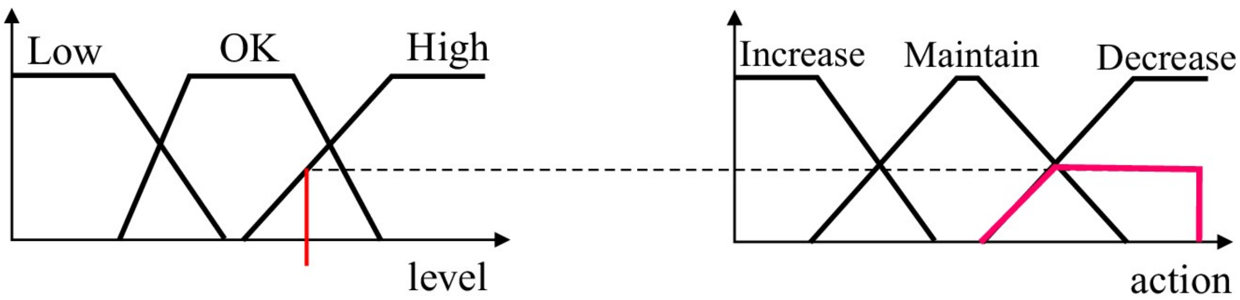
Fired fuzzy rules

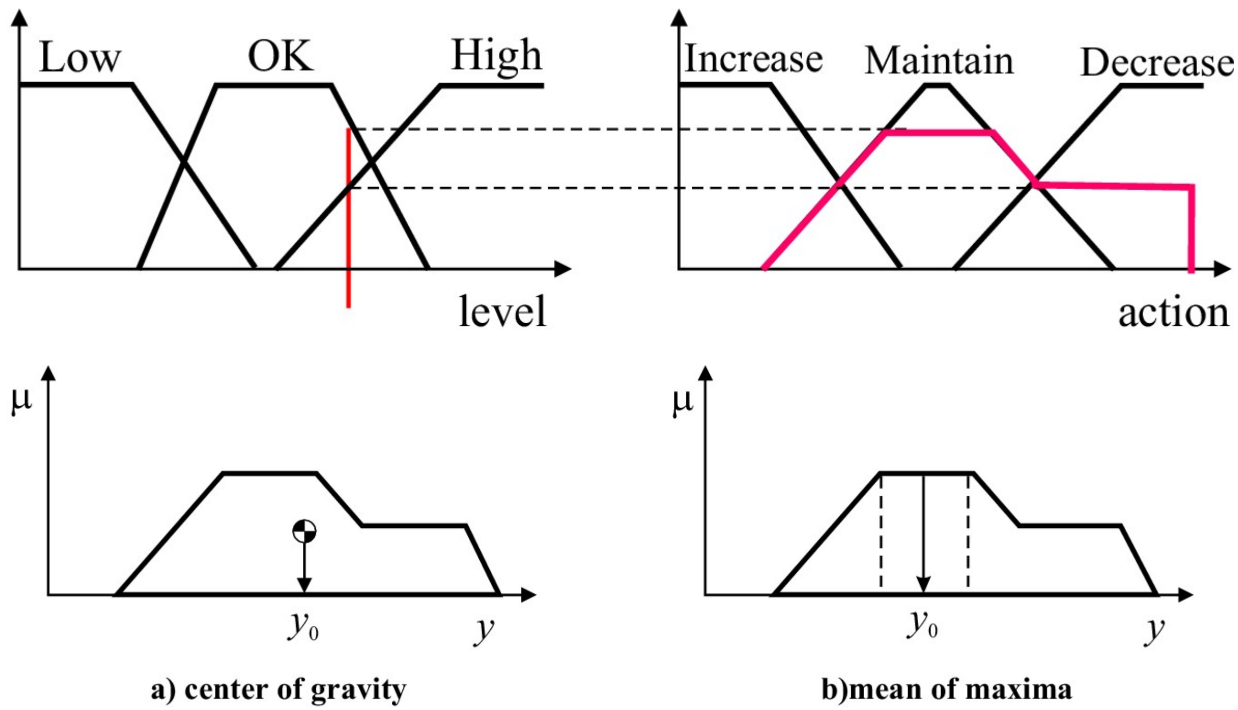


If level is *OK* **then** *Maintain* valve opening



If level is *High* **then** *Decrease* valve opening

Aggregation and defuzzifucation



Takagi-Sugeno fuzzy models

- Fuzzy antecedents, crisp consequents

  If $\boldsymbol{x}$ is A then $y = f(\boldsymbol{x})$

- **Zero-order Sugeno**: constant consequent

  If $\boldsymbol{x}$ is A then $y = b$

- **First-order Sugeno**: linear consequent

  If $\boldsymbol{x}$ is A then $y = \boldsymbol{ax} + b$

- Overall output is a weighted average of individual rule outputs:

$$\hat{y} = \frac{\sum_{k=1}^{K} w_k y_k}{\sum_{k=1}^{K} w_k}$$

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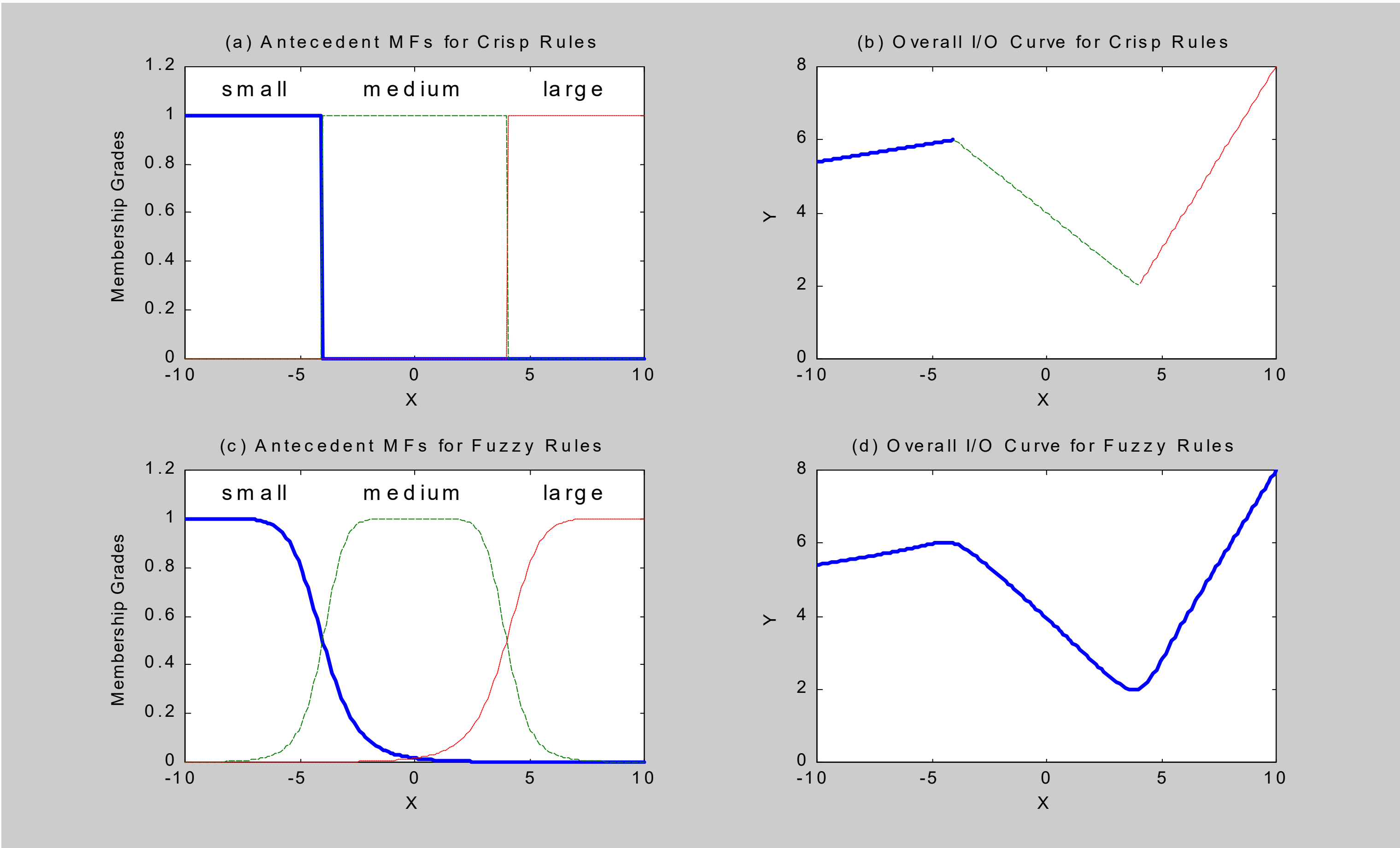
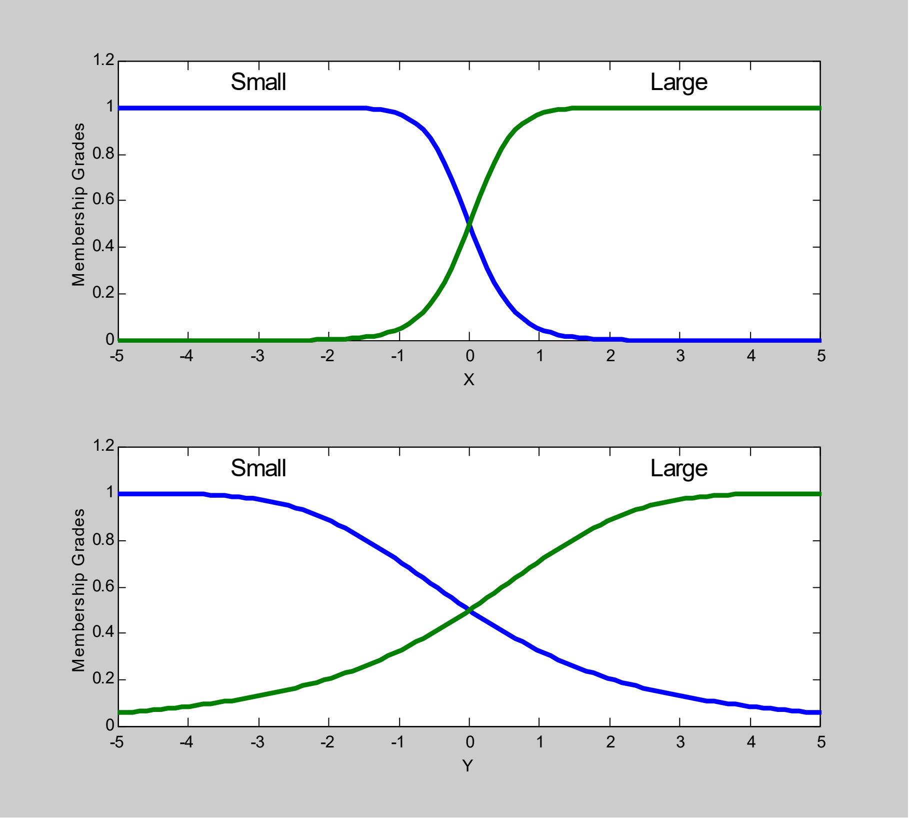
Fuzzy vs. crisp rule set

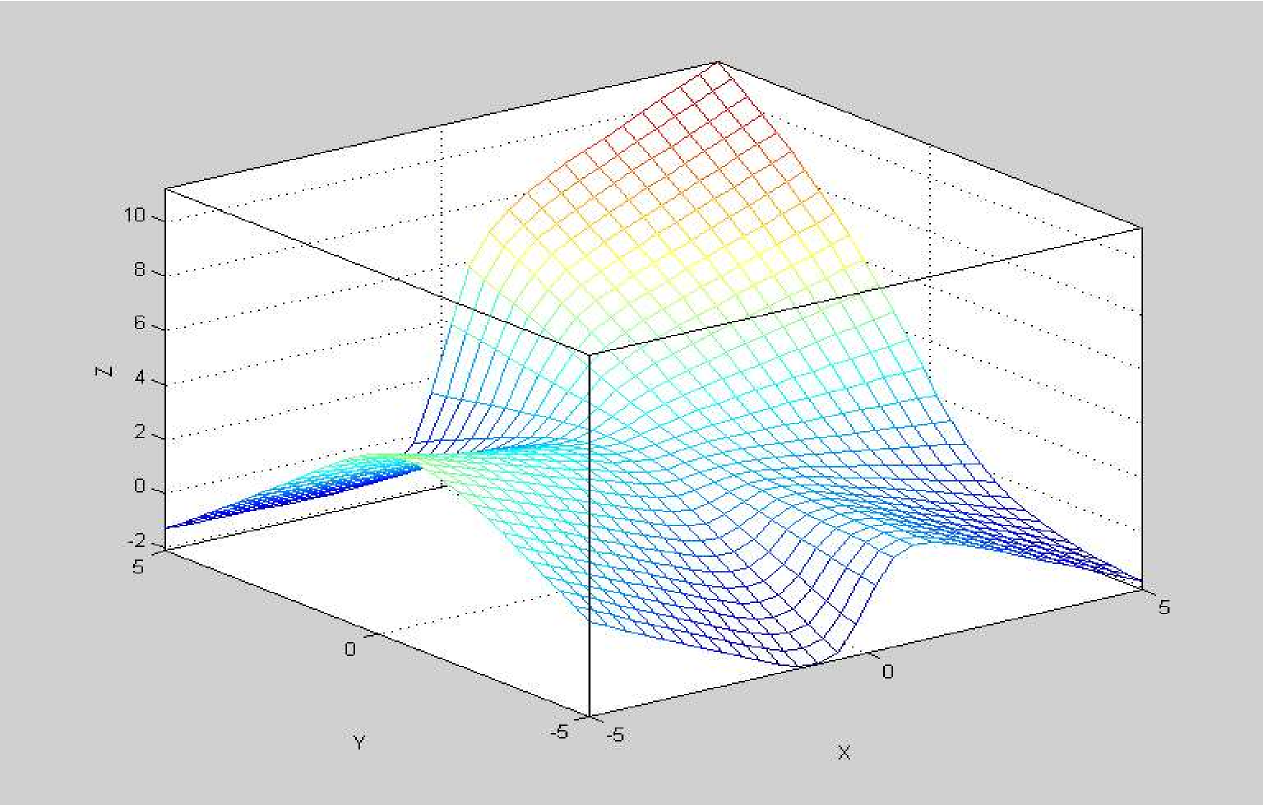


Sugeno – two inputs

- **If** *x* is Small **and** *y* is Small **then**
  $z = -x + y + 1$
- **If** *x* is Small **and** *y* is Large **then**
  $z = -y + 3$
- **If** *x* is Large **and** *y* is Small **then**
  $z = -x + 3$
- **If** *x* is Large **and** *y* is Large **then**
  $z = x + y + 2$



Sugeno – two inputs



Approximation capability

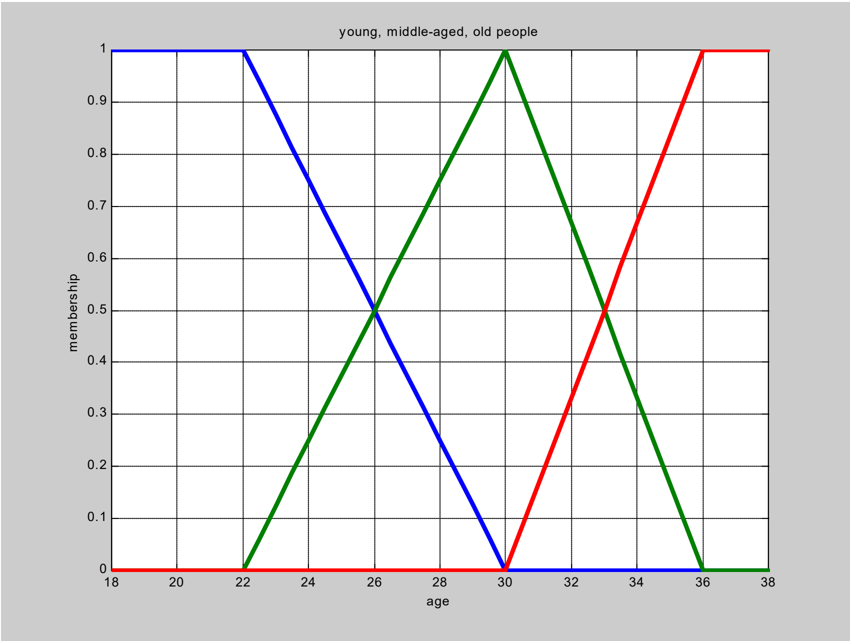
- Fuzzy systems are general function approximators (c.f. neural networks)
- Accuracy of a mapping augments by increasing the number of rules (examples) in the rule base
- Best results are obtained when the number of linguistic terms in the input and the output are increased (a finer partition)
- Using too many linguistic terms diminishes the transparency of fuzzy systems

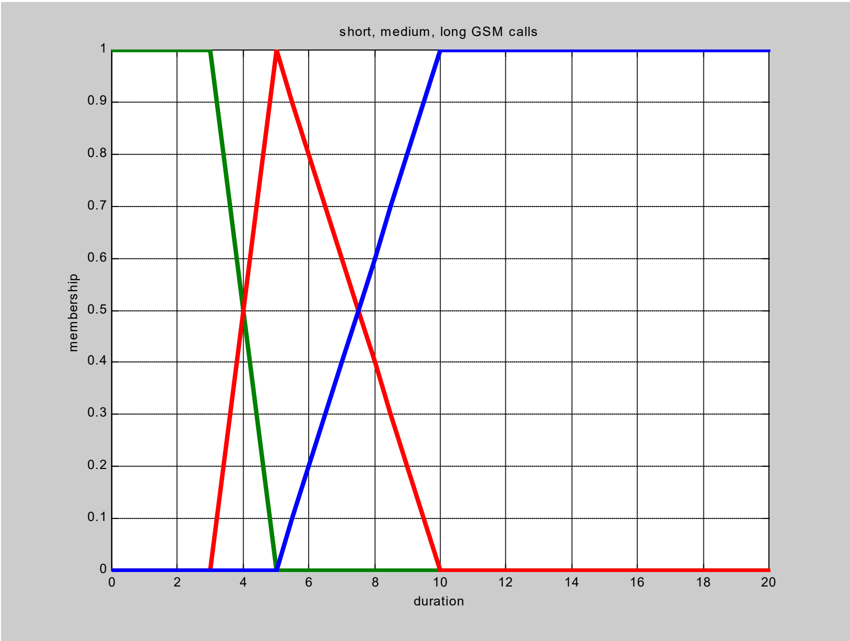
Interpolation properties

- Multiple rules in a fuzzy system may fire (become active) because fuzzy sets overlap
- Fuzzy rules represent typical cases or examples of the relation between two quantities
- The reasoning mechanism interpolates between the rules to determine the system output

Example

1. ***Young people make long Mobile calls***
2. ***Middle aged people make short Mobile calls***
3. ***Old people make medium-long Mobile calls***





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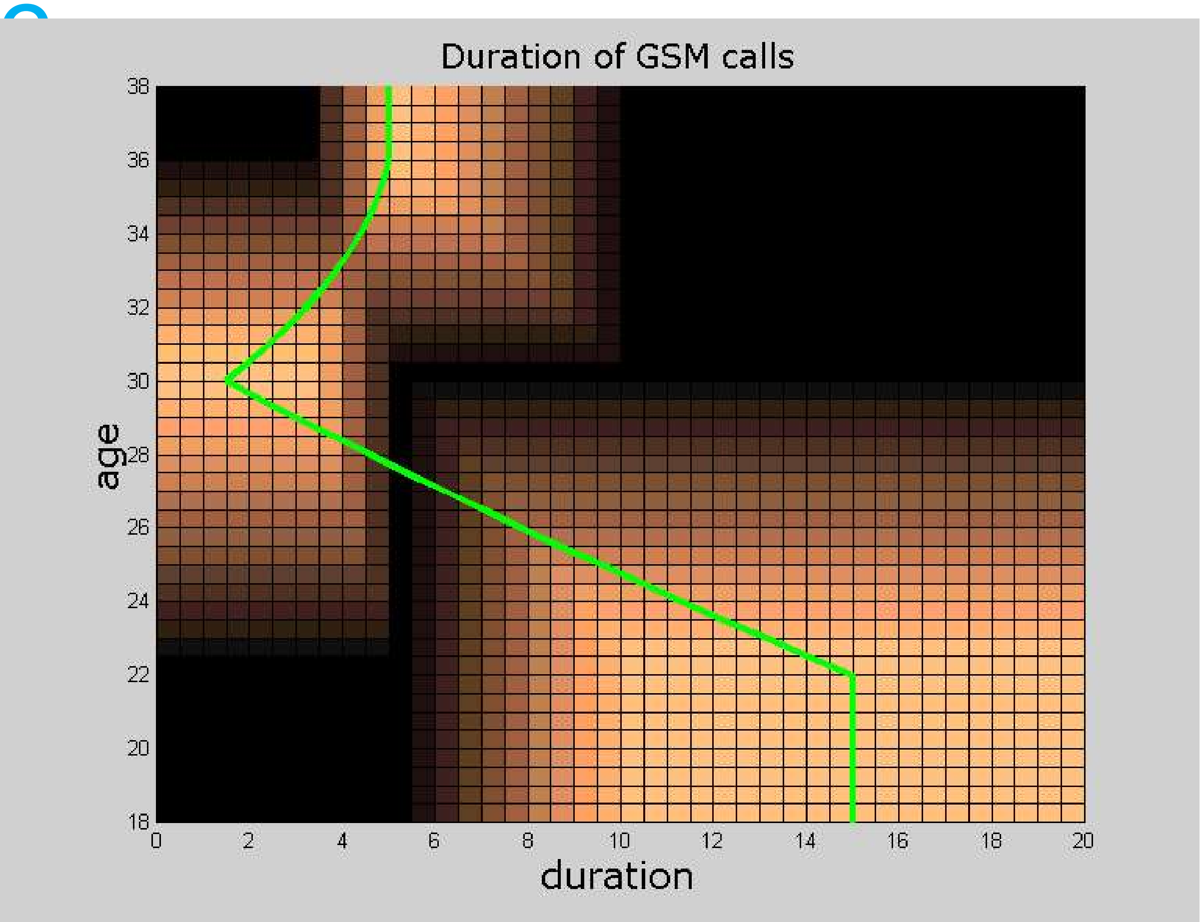
Example



Duration of GSM calls
age
duration
38
36
34
32
30
28
26
24
22
20
18
0
2
4
6
8
10
12
14
16
18
20

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