Intelligent Systems

Exercises

Fuzzy Sets

2. Consider the fuzzy set young described by:
   A = 1.0/5 + 1.0/10 + 0.8/20 + 0.5/30 + 0.2/40 + 0.1/50
   and the fuzzy set B defined by the membership function:

$$\mu(x) = \frac{1}{1+x^2}$$

   Determine, justifying:

   a) $\mathit{support}(A), \mathit{support}(B)$

   b) $\mathit{core}(A), \mathit{core}(B)$

   c) $A_{\alpha|\alpha=0.7}$, $B_{\alpha|\alpha=0.8}$

   d) $A^{+}_{\alpha^{+}|\alpha=0.7}$, $B^{+}_{\alpha^{+}|\alpha=0.8}$

   Answer:

   a) $\mathit{supp}(A) = \{x \in X | \mu_A(x) > 0\} = \{5, 10, 20, 30, 40, 50\}$
   $\mathit{supp}(B) = \mathcal{R}$

   b) $\mathit{core}(A) = \{x \in X | \mu_A(x) = 1\} = \{5, 10\}$
   $\mathit{core}(B) = \{0\}$

   c) $A_{\alpha|\alpha=0.7} = \{x \in X | \mu_A(x) \geq \alpha\} = \{5, 10, 20\}$
   $B_{\alpha|\alpha=0.8} = \quad [-0.5, 0.5]$

   d) $A^{+}_{\alpha^{+}|\alpha=0.7} = \{x \in X | \mu_A(x) > \alpha\} = \{5, 10, 20\}$
   $B^{+}_{\alpha^{+}|\alpha=0.8} = \quad]-0.5, 0.5[$

3. Consider the set of pencils given by: X = P1, P2, P3, P4, P5, P6, and the fuzzy sets *long pencils* and *short pencils* described by fuzzy sets C and D, respectively:

   $C = \{0.1/P1, 0.2/P2, 0.4/P3, 0.6/P4, 0.8/P5, 1.0/P6\}$

   $D = \{1.0/P1, 0.6/P2, 0.4/P3, 0.3/P4, 0.1/P5\}$

   a) Determine the union of the two sets using the max and the probabilistic union;

b) Determine the intersection of the two sets using the min and the product;

c) Comment the results obtained in a) and b).

Answer:

a) $E_{max} = C \cup D = \{1.0/P1, 0.6/P2, 0.4/P3, 0.6/P5, 0.8/P5, 1.0/P6\}$

$E_{prob} = C \cup D = \{1.0/P1, 0.68/P2, 0.64/P3, 0.72/P5, 0.82/P5, 1.0/P6\}$

b) $E_{min} = C \cap D = \{0.1/P1, 0.2/P2, 0.4/P3, 0.3/P5, 0.1/P5\}$

$E_{prob} = C \cap D = \{0.1/P1, 0.12/P2, 0.16/P3, 0.18/P5, 0.08/P5\}$

4. Consider two fuzzy sets A and B such that $core(A) \cap core(B) = \emptyset$. The fuzzy set $C = A \cap B$ can be normal? What is the necessary condition between the supports of A and B such that $\#(C) > 0$? Justify your answers.

   Answer: A normal set has a height of 1. If $core(A) \cap core(B) = \emptyset$ then one of the sets (or both) is not normal. Thus, their intersection can never be normal.

   The cardinality of a fuzzy set, #, is the sum of membership values of all elements of a fuzzy set. To assure that $\#(C) > 0$ then $supp(A) \cap supp(B) \neq \emptyset$

5. Consider the two fuzzy sets in the Universe of Discourse $X = \{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$:

$$\mu_A(x) = \frac{1}{1 + |x|} \quad \text{and} \quad \mu_B(x) = 1 - \frac{|x|}{20}$$

   a) Are the membership functions valid in the given Universe?

   b) Compute the α–cuts of A and B for $\alpha = 0.3$;

   c) Define the previous set of α–cuts of A:

      i) by enumerating its elements;

      ii) by a property of its elements;

      iii) by using a membership function;

   d) Using Zadeh's operators, compute $C = A \cap B$ and $D = A \cap B$. Are C and D convex sets? Justify.

   Answer:

   a) For the given universe X, $\mu_A \in [\frac{1}{9}, 1]$ and $\mu_B \in [\frac{3}{5}, 1]$, so both are valid.

   b) $A_{\alpha|\alpha=0.3} = \{-2, 0, 2\}$

   $B_{\alpha|\alpha=0.3} = \{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$

   c) Define the previous set of α–cuts of A:

i) $A_{0.3} = \{-2, 0, 2\}$

ii) $A_{0.3} = \{x : |x| < 4\}$

iii)

$$\mu_{A_{0.3}}(x) = \begin{cases} 1, \text{ if } x \in [-2, 2] \\ 0, \text{ otherwise}\} \end{cases}$$

d)

$C = A \cap \bar{B} = \{0.11/-8, 0.14/-6, 0.2/-4, 0.1/-2, 0/0, 0.1/2, 0.2/4, 0.14/6, 0.11/8\}$

$D = A \cap B = \{0.11/-8, 0.14/-6, 0.2/-4, 0.33/-2, 1/0, 0.33/2, 0.2/4, 0.14/6, 0.11/8\}$

C is not convex and D is convex.

6. Prove that Morgan's law $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ is valid for Zadeh's operators (union, intersection and complement).

Answer:

$$\begin{aligned} \overline{(A \cup B)} &= 1 - A \cup B = 1 - max(\mu_a(x), \mu_B(x)) = 1 - (-min(-\mu_a(x), -\mu_B(x))) \\ &= 1 + min(-\mu_a(x), -\mu_B(x)) = min(1 - \mu_a(x), 1 - \mu_B(x)) = \overline{A} \cap \overline{B} \end{aligned}$$

Fuzzy Relations

8. Compute the cylindrical extension of the fuzzy set $A = 0.2/x_1 + 0.8/x_2$ to $X \times Y$, with $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$.

Answer: $cext_Y(A) =$

	y_1	y_2	y_3
x_1	0.2	0.2	0.2
x_2	0.8	0.8	0.8

9. Consider the fuzzy relation $\mathcal{R}$ given by:

	y_1	y_2	y_3	y_4
x_1	0.8	0.9	0.6	0.1
x_2	0.2	0.4	0.7	0.8
x_3	0.1	0.2	0.5	0.2

Obtain the projections of $\mathcal{R}$ onto X and Y respectively.

Answer:

$proj_X(\mathcal{R}) = 0.9/x_1 + 0.8/x_2 + 0.5/x_3$

$proj_Y(\mathcal{R}) = 0.8/y_1 + 0.9/y_2 + 0.7/y_3 + 0.8/y_4$

10. Consider that the fuzzy relation in 9 corresponds to "x is considerably bigger than y". Consider also the fuzzy set expressing "x is small" given by $A = 0.3/x1, 1/x2, 0.8/x3$. Obtain the value of the expression "x is considerably bigger than y and x is small".

Answer: "x is considerably bigger than y and x is small" $= \mathcal{R} \cap proj_Y(A)$

$cext_Y(A) =$

	y_1	y_2	y_3	y_4
x_1	0.3	0.3	0.3	0.3
x_2	1	1	1	1
x_3	0.8	0.8	0.8	0.8

$\mathcal{R} \cap cext_Y(A) =$

	y_1	y_2	y_3	y_4
x_1	0.3	0.3	0.3	0.1
x_2	0.2	0.4	0.7	0.8
x_3	0.1	0.2	0.5	0.2

12. Consider the following fuzzy inference system:

    1. If x is Small then y is Big
    2. If x is Medium then y is Small
    3. If x is Big then y is Medium

    a) Describe the basic steps of Mamdani inference.

    b) Use Fig. 1 below to describe these steps for an input $x = 6$.

    c) Compute the defuzzified output using the center of gravity method, when the domain (Universe of Discourse) of the output is $X = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

    Answer:

    b) Fig. 2

    c)

$$z_{COG} = \frac{\sum_z \mu_{C'}(z)z}{\sum_z \mu_{C'}(z)}$$

$z_{COG} = \frac{0(0.25)+1(0.25)+2(0.25)+3(0.5)+4(0.75)+5(0.5)+6(0)+7(0)+8(0)}{0.25+0.25+0.25+0.5+0.75+0.5+0+0+0} = 3.1$

13. (a) Define a type-zero and a type-one Takagi-Sugeno models. What is an affine Takagi-Sugeno model?

    (b) Describe briefly the necessary steps to derive a Takagi-Sugeno model of a system from numeric data.

    Answer:

    (a) Type-zero has a constant consequent. If $\mathbf{x}$ is A then $y = b$

    Type-one has a linear consequent. If $\mathbf{x}$ is A then $y = \mathbf{ax} + b$

    An affine function is a function of type $y = \mathbf{ax} + b$, so an affine Takagi-Sugeno model is a type-one Takagi-Sugeno model.

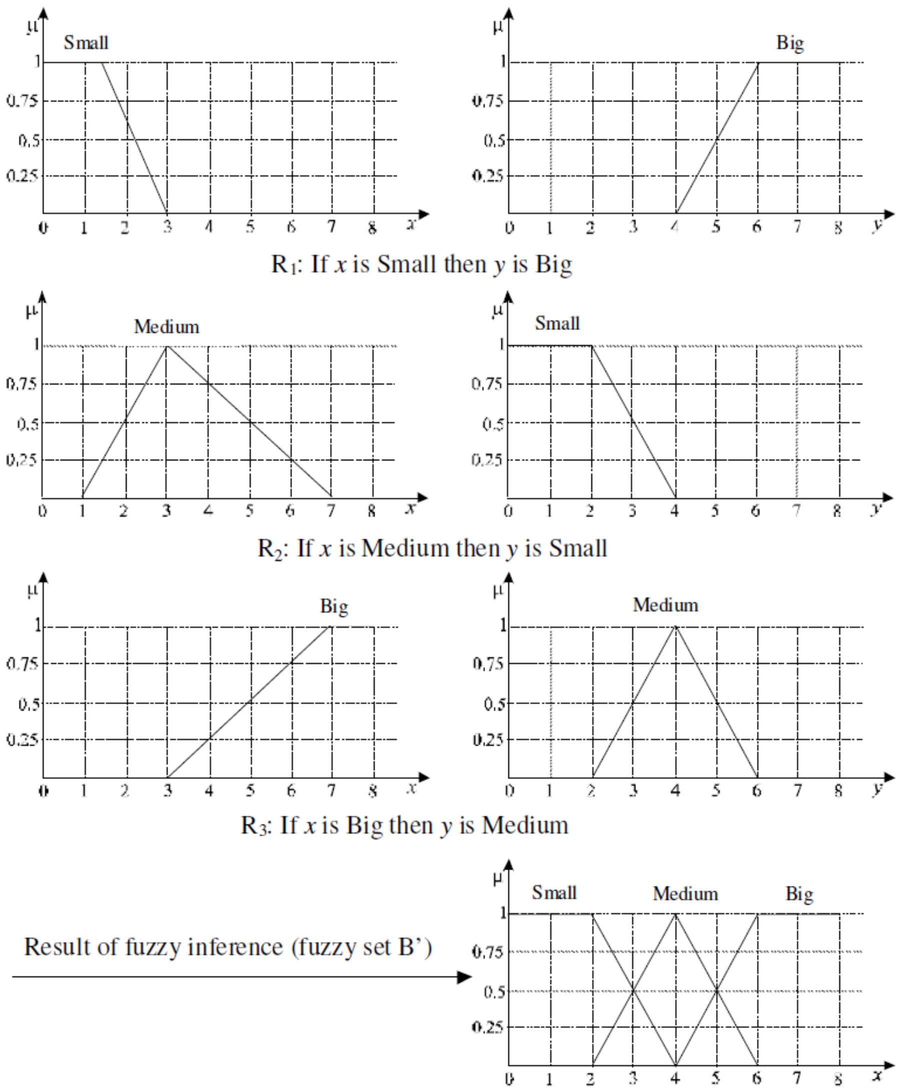

Figure 1: Mamdani inference.

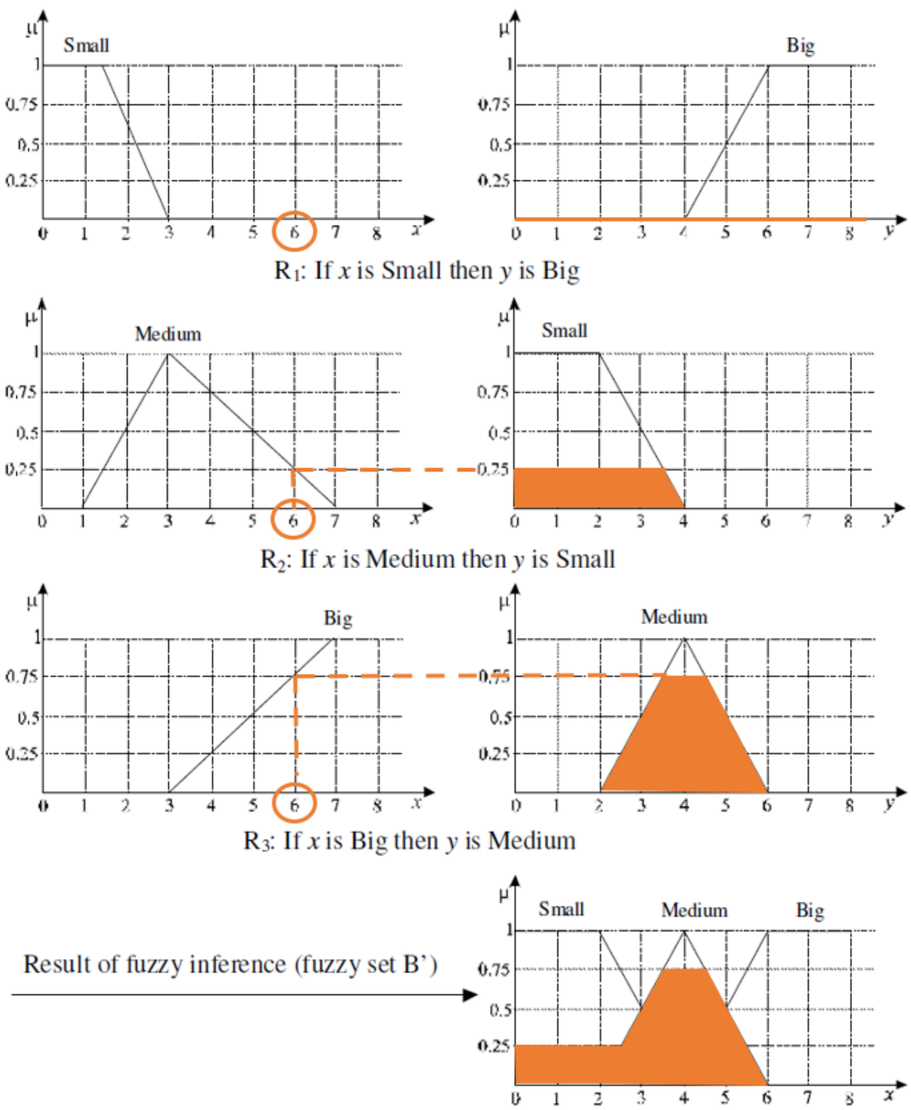

Figure 2: Mamdani inference.

14. Consider the following fuzzy rules:

    1) If x is A_1 and y is B_1 then $z = c_1$.

    2) If x is A_2 and y is B_2 then $z = c_2$.

    3) If x is A_3 and y is B_3 then $z = c_3$.

    4) If x is A_4 and y is B_4 then $z = c_4$.

    How to compute the global output z?

    Answer:

    Same steps as in the previous exercise. For each rule, membership value is the minimum of either A or B ("and"). Other defuzzification methods could be used, such as centroid of area (COA).

18. Consider a zero-order Takagi Sugeno (singleton) model with the following rules:

    1) If x is Small then $y = b_1$,

    2) If x is Big then $y = b_2$,

    and the membership functions given in Fig. 3. Consider also that:

    $x_1 = 1$, $y_1 = 3$

    $x_2 = 5$, $y_2 = 4.5$

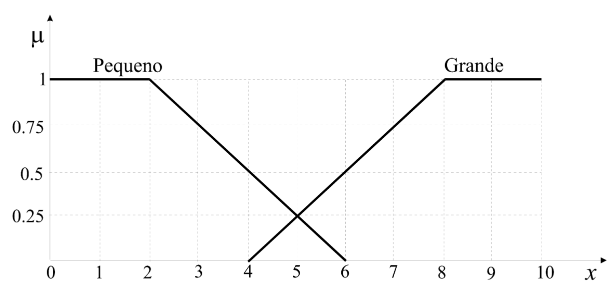

Figure 3: Membership functions.

Compute the consequent parameters b_1 and b_2 such that the model has the minimum squared error using the given data. What is the value of this error?

Answer:

$$\hat{y} = \frac{\sum_{k=1}^{K} w_k y_k}{\sum_{k=1}^{K} w_k}$$

$b_1 = 3$, $b_2 = 6$

The error is zero.