A Game Theoretic Fault Detection Filter

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***Abstract*—The fault detection process is approximated with a disturbance attenuation problem. The solution to this problem, for both linear time-varying and time-invariant systems, leads to a game theoretic filter which bounds the transmission of all exogenous signals except the fault to be detected. In the limit, when the disturbance attenuation bound is brought to zero, a complete transmission block is achieved by embedding the nuisance inputs into an unobservable, invariant subspace. Since this is the same invariant subspace structure seen in some types of detection filters, we can claim that the asymptotic game filter is itself a detection filter. One can also make use of this subspace structure to reduce the order of the limiting game theoretic filter by factoring this invariant subspace out of the state space. The resulting lower dimensional filter will then be sensitive only to the failure to be detected. A pair of examples given at the end of the paper demonstrate the effectiveness of the filter for time-invariant and time-varying problems in both full-order and reduced-order forms.**

***Index Terms*— Analytic redundancy, detection filters, differential games, disturbance attenuation, singular optimal control, unknown input observer.**

I. Introduction

A *detection filter* is a Luenberger Observer whose error residuals take on known and predictable characteristics when a failure occurs in the underlying physical system. Because of this, it has long attracted the interest of researchers in the field of fault detection and identification (FDI). The detection filter was first proposed by Beard [1] as part of a "self-reorganizing" control system which adapts to component failures and other unexpected changes in the underlying physical system. Subsequent researchers have refined a geometric interpretation of the filter [2], [3], developed design algorithms [4], [5], and robustified the filter to exogenous inputs [5]. This particular line of inquiry has led to what is now known as the *Beard–Jones Fault Detection Filter*.

The Beard–Jones filter works by imparting a special invariant subspace structure into the observer. This subspace structure is ideal for isolating the effects of different faults because it makes use of the fact that failures drive a system like unexpected inputs. As such, they bias observer residuals and can be associated with reachable subspaces. The detection filter restricts each of these reachable subspaces to lie within an invariant subspace and fixes the set of invariant subspaces containing the faults to be nonoverlapping. The end result is that the effect of a failure is wholly contained within a single invariant subspace. With this structure, simultaneous detection and identification can be achieved by projecting the biased residual onto each of the invariant subspaces. A nonzero projection (or one that exceeds a threshold) indicates a fault; the subspace associated with the projection identifies the fault.

A related approach known as the *unknown input observer* [6], [7] simplifies the detection filter problem by dividing the entire set of faults into two sets: the faults which are to be detected and the faults which are disturbances. Unknown input observers work by making the second set unobservable. One can take this idea one step further and simply factor out the unobservable subspace that contains the disturbing faults. The remaining lower dimensional system can then be monitored with a reduced-order *residual generator* [8]. Unknown input observers are clearly less capable than Beard–Jones filters, since they can identify only one fault out of the complete set of faults, but this is due to a simpler invariant subspace structure that is easily approximated by optimization methods [9], [10].

Approximate methods are desirable because they are potentially more flexible and more robust than current detection filter design techniques, which tend to be based on geometric theory [3], [11], eigenstructure assignment [4], [7], or solving a stringent set of linear equations [12]. These methods lead directly to the desired subspace structure, but they also limit the applicability of detection filters to linear time-invariant systems and can lead to filters with poor robustness to plant parameter variations [10]. By relaxing the design requirements, however, we can determine new structures which admit a wider class of systems.

In this paper, we will introduce an approximate detection filter which follows in line with the unknown input observer—the *game theoretic fault detection filter*. This new filter is realized when we pose and solve a disturbance attenuation problem patterned after the fault detection process. As a result, the game theoretic fault detection filter bounds the transmission of all exogenous signals, save the fault to be detected. Furthermore, we will show that this filter is applicable to time-varying systems and that it becomes an unknown input observer in the limit when the bound in the disturbance attenuation problem is taken to zero. We do not have space in this paper to address the issue of robustness to model uncertainty, but instead direct the reader to [13]–[15].

We begin in Section II by motivating a disturbance attenuation approach to FDI for both linear time-invariant systems and time-varying systems. We then derive the game theoretic fault detection filter in Section III by solving a disturbance attenuation problem patterned after the fault detection process. To show how this filter asymptotically obtains the structure

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of an unknown input observer, we establish the equivalence between the limiting case disturbance attenuation problem and a singular differential game in Section IV. We then derive sufficient conditions for a nonpositive cost in this game in Section V. These sufficiency conditions will turn out to be the key to analyzing the asymptotic game theoretic filter. In Section VI, we solve the singular game, and in Section VII we establish the link between this solution and the unknown input observer. We also derive a reduced-order detection filter which is similar to the residual generator of [8]. In Section VIII, we demonstrate the effectiveness of the game theoretic filter for fault detection in both the full-order and reduced-order versions, and in Section IX we show that the filter can also be applied to a time-varying problem. To the best of our knowledge, the latter is the first example presented in the literature of FDI for a time-varying system.

Remark 1.1: It has recently come to our attention that a similar line of investigation has been carried out by Edelmayer *et al*. [16], [17], though with different emphases. □

Remark 1.2: We will use the term "detection filter" to refer to all of the methods described in this section, reserving the terms "Beard–Jones filter" or "unknown input observer" for when we need to be more specific. Our use of "detection filter" in this way is nonstandard, but no standard appears to exist (see, e.g., [18] and [12]). "Detection filter" is short and to the point and sounds no worse than anything else that has been proposed.

II. The Approximate Detection Filter Design Problem

A. Modeling the Detection Problem

The general class of systems that we will look at are linear, observable, possibly time-varying, and driven by noisy measurements

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + v. \end{aligned} \tag{1}$$

We will also assume that our state matrices have sufficient smoothness to guarantee the existence of derivatives of various order.

Beard [1] showed that failures in the sensors and actuators and unexpected changes in the plant dynamics can be modeled as additive signals

$$\dot{x} = Ax + Bu + F_1\mu_1 + \cdots + F_q\mu_q. \tag{2}$$

Let n be the dimension of the state space. The $n \times p_i$ matrix, $F_i, i = 1\cdots q$, is called a *failure map* and represents the directional characteristics of the ith fault. The $p_i \times 1$ vector μ_i is the *failure signal* and represents the time dependence of the failure. It will always be assumed that each F_i is monic, i.e., $F_i\mu_i \neq 0$ for $\mu_i \neq 0$. We will look at F_i and μ_i in more detail in Section II-B, and we will show the importance of the monicity assumption in Section II-C. Throughout this paper, we will refer to μ_1 as the "target fault" and the other faults, $\mu_j, j = 2\cdots q$, as the "nuisance faults." Without loss of generality, we can represent the entire set of nuisance faults with a single map and vector

$$\dot{x} = Ax + Bu + F_1\mu_1 + F_2\mu_2.$$

Suppose that it is desired to detect the occurrence of the failure μ_1 in spite of the measurement noise v and the possible presence of the nuisance faults μ_2. As described earlier, a detection filter-based solution to this problem

$$\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x}) \tag{3}$$

works by keeping the reachable subspaces of μ_1 and μ_2 in separate and nonintersecting invariant subspaces. Thus, with a properly chosen projector H, we can project the filter residual $(y - C\hat{x})$ onto the orthogonal complement of the invariant subspace containing μ_2 and get a signal

$$z = H(y - C\hat{x}) \tag{4}$$

such that

$$z = 0 \text{ when } \mu_1 = 0 \text{ and } \mu_2 \text{ is arbitrary.} \tag{5}$$

To be useful for FDI, z must also be such that

$$z \neq 0 \text{ when } \mu_1 \neq 0. \tag{6}$$

If we restrict ourselves to time-invariant systems, then (6) will be equivalent to requiring that the transfer function matrix between $\mu_1(s)$ and $z(s)$[1] be left-invertible. Left-invertibility, however, is a severe restriction, and it has no analog for the general time-varying systems that we want to consider here. Previous researchers [5], [8] have, in fact, only required that the mapping from $\mu_1(t)$ to $z(t)$ be *input observable*, i.e., $z \neq 0$ for any μ_1 that is a step input. It is then argued [8] that with input observability z will be nonzero for "almost any" μ_1, since μ_1 is unlikely to remain in the kernel of the mapping to z for all time.

We formulate the approximate detection filter design problem by requiring input observability and relaxing the requirement for strict blocking that is implied by (5). We, instead, only require that the transmission of the nuisance fault be bounded above by a preset level $\gamma > 0$

$$\frac{||z||^2}{||\mu_2||^2} \leq \gamma. \tag{7}$$

Equation (7) is clearly a disturbance attenuation problem. We refer to the solution to the approximate detection filter problem as the *game theoretic fault detection filter*.

Remark 2.1: Detection filters typically make no assumptions like the one we make above, i.e., that the failure signals are L_2. We will show later that in the limiting case we will not need to make such assumptions. □

[1] $\mu_1(s)$ and $z(s)$ are the LaPlace Transforms of the time-domain signals $\mu_1(t)$ and $z(t)$.

B. Modeling Failures

In this section, we will show how to construct failure maps and signals for each type of failure. Existing methods [1], [4], [5] exist for time-invariant systems. For actuator faults and plant changes, these methods can be extended "as is" to time-varying systems. In the actuator fault case, this means that the map is taken to be the corresponding column of the input matrix. In the plant fault case, the map is similarly derived by pulling out the corresponding entries in the state matrix. The failure signals in both cases can be found by choosing an appropriate time function.[2]

Sensor faults require a generalization of the time-invariant result. Because these failures enter the system through the measurements, we can initially model them as an additive input in the measurement equation

$$y = Cx + E_j \mu_j. \tag{8}$$

C is an $m \times n$ matrix, and E_j is an $m \times 1$ unit vector with a one at the jth position, which corresponds to a failure in the jth sensor.

Following [5], we determine the sensor failure map by finding the input to the plant which drives the error state in the same way that μ_j will in (8). This is elegantly accomplished by a Goh transformation on the error space [19]. Defining the estimation error e as $x - \hat{x}$, the filter residual is then

$$r := y - C\hat{x} = Ce$$

when there is no sensor noise (1). When a sensor failure occurs

$$r = Ce + E_j \mu_j. \tag{9}$$

Let f_j be the solution to $E_j = C f_j$. The transformation begins by defining a new error state

$$\bar{e} := e + f_j \mu_j \tag{10}$$

which allows us to rewrite (9) as $r = C\bar{e}$. Assuming a generic form for the observer (3) and a homogeneous dynamic system $\dot{x} = Ax$, we differentiate $\bar{e}$

$$\begin{aligned} \dot{\bar{e}} &= \dot{e} + \dot{f}_j \mu_j + f_j \dot{\mu}_j \\ &= Ae + LC\bar{e} + Af_j\mu_j - Af_j\mu_j + \dot{f}_j\mu_j + f_j\dot{\mu}_j \\ &= (A + LC)\bar{e} + [f_j \quad (Af_j - \dot{f}_j)] \begin{Bmatrix} \dot{\mu}_j \\ -\mu_j \end{Bmatrix} \end{aligned}$$

to get a differential equation for the transformed error trajectory. Clearly, the equivalent input is one which enters the system through

$$F_j = [f_j \quad f_j^*] \tag{11}$$

where $f_j^* = Af_j - \dot{f}_j$. When the system is time-invariant, $\dot{f}_j = 0$ and (11) will match the time-invariant failure map given in [1] and [4]. For our purposes, finding F_j is the key result. The actual time history of the failure signal is not important, and so undue importance should not be attached to the "equivalent" input $[\dot{\mu}_j^T \quad -\mu_j^T]^T$.

[2] For example, hard failures or saturation failures can be modeled as step inputs.

C. Constructing the Failure Signal

We complete our formulation of the disturbance attenuation problem for fault detection by constructing the projector H, which determines the failure signal z (4). For time-invariant systems, this projector is constructed to map the reachable subspace of μ_2 to zero [1], [5]. Hence

$$H = I - C\hat{F}[(C\hat{F})^T C\hat{F}]^{-1}(C\hat{F})^T \tag{12}$$

where

$$\hat{F} = [A^{\beta_1} f_1, \cdots, A^{\beta_{p_2}} f_{p_2}]. \tag{13}$$

The vector $f_i, i = 1 \cdots p_2$ is the ith column of F_2, and the integer β_i is the smallest natural number such that $CA^{\beta_i} f_i \neq 0$. The time-varying extension of this result is

$$H = I - C\hat{F}(t)[(C\hat{F}(t))^T C\hat{F}(t)]^{-1}(C\hat{F}(t))^T. \tag{14}$$

The columns of the matrix

$$\hat{F}(t) = [b_1^{\beta_1}(t), \cdots, b_{p_2}^{\beta_{p_2}}(t)] \tag{15}$$

are constructed with the Goh transformation

$$b_i^1(t) = f_i(t) \tag{16}$$
$$b_i^j(t) = A(t)b_i^{j-1}(t) - \dot{b}_i^{j-1}. \tag{17}$$

In the time-varying case, β_i is the smallest integer for which the iteration above leads to a vector $b_i^{\beta_k}(t)$ such that $C(t)b_i^{\beta_k}(t) \neq 0$ for all $t \in [t_0, t_1]$. It will be assumed that $A(t)$, $C(t)$, and $F_2(t)$ are such that β_i exists. Since the state space has dimension n, β_i is such that $0 \leq \beta_i \leq n - 1$. This restricts the class of admissible systems, but such assumptions seem to be unavoidable when dealing with the time-varying case (see, for example, [20]). The Goh transformation will be introduced explicitly in Section VI, where we will also give an alternate representation of (12) and (14).

We are now ready to discuss the conditions under which the solution to (7) will also generate an input observable mapping from μ_1 to z. The key requirement is that the system be *output separable*. That is, F_1 and F_2 must be linearly independent and remain so when mapped to the output space by C and A. For time-invariant systems, the test for output separability is

$$\begin{aligned} \operatorname{rank} [CA^{\delta_1}\tilde{f}_1, \cdots, CA^{\delta_{p_1}}\tilde{f}_{p_1}, CA^{\beta_1}f_1, \cdots, CA^{\beta_{p_2}}f_{p_2}] \\ = p_1 + p_2. \end{aligned} \tag{18}$$

As in (13), f_i is the ith column of F_2 and β_i is the smallest integer such that $CA^{\beta_i} f_i \neq 0$. Similarly, $\tilde{f}_j$ is the jth column of F_1, and δ_j is the smallest integer such that $A^{\delta_j}\tilde{f}_j \neq 0$. The integer sum, $p_1 + p_2$, is the total number of columns in F_1 and F_2.

For time-varying systems, the output separability test becomes

$$\begin{aligned} \operatorname{rank} [C(t)\tilde{b}_1^{\delta_1}(t), \cdots, C(t)\tilde{b}_{p_1}^{\delta_{p_1}}(t), C(t)b_1^{\beta_1}(t), \cdots, \\ C(t)b_{p_2}^{\beta_{p_2}}(t)] = p_1 + p_2, \qquad \forall t \in [t_0, t_1] \end{aligned} \tag{19}$$

where the vectors, $b_i^{\beta_i}$ and $\tilde{b}_j^{\delta_j}$ are found from the iteration defined by (16) and (17). The initial vector $\tilde{b}_j^1$ is set equal to the jth column of F_1, and b_i^1 is initialized as the ith column of F_2.

The following proposition connects output separability to input observability and shows the importance of the monicity assumption.

Proposition 2.2: Suppose that we have an approximate detection filter which satisfies (7) and generates the failure signal z given by (4). If F_1 and F_2 are output separable and F_1 is monic, then the mapping $\mu_1(t) \mapsto z(t)$ is input observable. □

Proof: The input observability of the mapping $\mu_1(t) \mapsto z(t)$ is equivalent [5], [8] to the requirement that F_1 be monic and that its image not intersect the unobservable subspace of (HC, A). We have already assumed the former. To show the latter, let us assume the converse, i.e., that there exists a vector

$$\xi_1(t) = \sum_{i=1}^{p_1} \alpha_i \tilde{f}_i(t) \tag{20}$$

such that

$$H(t)C(t)\Phi(t,\tau)\xi_1(\tau) = 0, \qquad \forall t \text{ and } \tau \le t. \tag{21}$$

The vector $\tilde{f}_i$ is the ith column of F_1, and the coefficient α_i is a real number. At least one α_i is nonzero. $\Phi(t,\tau)$ is the state transition matrix of $A(\cdot)$ from τ to t.

Equation (21) implies that $\xi_1(t) \in \ker H(t)C(t)$, since we can set $\tau = t$ and get $\Phi(t,t) = I$. Since the vectors $\tilde{f}_i$ are independent by the monicity assumption, this implies that

$$\tilde{f}_i(t) \in \ker H(t)C(t), \qquad \forall i = 1, \cdots, p_1. \tag{22}$$

If one of these vectors, say $\tilde{f}_{i_0}(t)$, is not also in $\ker C(t)$, then (22) can hold only if $C\tilde{f}_{i_0}(t)$ is linearly dependent upon the vectors $Cb_i^{\beta_i}$, which form the projector $H(t)$. This, in turn, would imply that the output separability test (19) will fail, which implies the proposition via the contrapositive argument. Thus, for argument's sake, let us suppose that all the $\tilde{f}_i$ lie in $\ker C(t)$ so that we can continue with the proof.

Now, because we have assumed that the underlying matrices are smooth enough to allow for derivatives of arbitrary order, a necessary and sufficient condition for (21) is that the derivatives of $HC\Phi\xi_1$ be zero for all t and τ. Thus

$$\begin{aligned}\frac{d}{d\tau}&[H(t)C(t)\Phi(t,\tau)\xi_1(\tau)] \\ &= H(t)C(t)\Phi(t,\tau)[-A(\tau)\xi_1(\tau) + \dot{\xi}_1(\tau)] \\ &= H(t)C(t)\Phi(t,\tau)\xi_2(\tau) = 0, \qquad \forall t, \tau \end{aligned} \tag{23}$$

where $\xi_2 := -A\xi_1 + \dot{\xi}_1$. Again, setting $\tau = t$ in (23) implies that $\xi_2(t) \in \ker H(t)C(t)$. From the definition of ξ_1 and the iteration formulas (16) and (17), we can rewrite ξ_2 as

$$\xi_2 = \alpha_1 \tilde{b}_1^2 + \cdots + \alpha_{p_1} \tilde{b}_{p_1}^2. \tag{24}$$

The same arguments as before will lead us to the conclusion that either $\tilde{b}_j^2(t) \in \ker C(t), \forall j, t$ or that our proposition holds. We will again assume the former for argument's sake.

In the general case, we consider the vector ξ_k which is the kth iteration of formula

$$\xi_j = -A\xi_{j-1} + \dot{\xi}_{j-1} \tag{25}$$

in which the initial vector ξ_1 is given by (20).[3] We can also write ξ_k as

$$\xi_k = \sum_{i=1}^{p_1} \alpha_i \tilde{b}_i^k(t)$$

where $\tilde{b}_i^k$ is the kth step of the iteration (16), (17) with $\tilde{b}_i^1$ taken to be $\tilde{f}_i$, the ith column of F_1. Previously, we saw for the case $k = 1$ that

$$H(t)C(t)\Phi(t,\tau)\xi_k(\tau) = 0, \qquad \forall t \text{ and } \tau \le t \tag{26}$$

implies

$$C(t)\tilde{b}_i^k(t) = 0, \qquad \forall i, t, \tag{27}$$

$$H(t)C(t)\Phi(t,\tau)\xi_{k+1}(\tau) = 0, \qquad \forall t \text{ and } \tau \le t \tag{28}$$

where $\xi_{k+1} := -A\xi_k + \dot{\xi}_k$ is the next step in the iteration (25). The arguments used for $k = 1$ are independent of the particular value of k, which means that (27) and (28) hold for all k. Thus, by induction, we can claim that ξ_1 unobservable through (HC, A) implies that all of the vectors $\tilde{b}_k^i(t), i = 1 \cdots n-1, k = 1 \cdots p_1$ from (16) and (17) lie in the kernel of $C(t)$. This implies that the output separability test matrix

$$[C\tilde{b}_1^{\delta_1}, \cdots, C\tilde{b}_{p_1}^{\delta_{p_1}}, Cb_1^{\beta_1}, \cdots, Cb_{p_2}^{\beta_{p_2}}]$$

will fail to be full rank. Therefore, the contrapositive argument

$$\text{image } F_2 \cap \ker HC \ne 0 \Rightarrow F_1 \text{ and } F_2 \text{ not output separable}$$

implies our proposition. ■

Remark 2.3: Output separability is a necessary condition for the existence of Beard–Jones filters [1], [5]. Edelmayer *et al*. [16], however, show that even without output separability it is still possible to obtain approximate detection filters which can distinguish the target input from the nuisance inputs. There is no guarantee, however, that this will always be the case (see [13]). □

III. A Game Theoretic Solution to the Approximate Detection Filter Design Problem

A. The Disturbance Attenuation Problem

We now turn our attention to the disturbance attenuation problem implied by (7). We begin by defining a disturbance attenuation function

$$D_{af} = \frac{\displaystyle\int_{t_0}^{t_1} ||HC(x - \hat{x})||_Q^2 \, dt}{\displaystyle\int_{t_0}^{t_1} [||\mu_2||_{M^{-1}}^2 + ||v||_{V^{-1}}^2] \, dt + ||x(t_0) - \hat{x}_0||_{P_0}^2} \tag{29}$$

which is simply a ratio of the outputs over the disturbances [21]. Equation (29) is patterned roughly after (7). We have added the sensor noise v and the initial error $x(t_0) - \hat{x}_0$ to the set of disturbance signals to incorporate tradeoffs for noise

[3] Note that this formula is simply the Goh transformation.

rejection and settling time into the problem. M, V, Q, and P_0 are weighting matrices. Note that we do not include the target fault μ_1 at this stage of the design problem, since we are now focusing on nuisance blocking. Our only concern with μ_1 is that it be visible at the output which is what Proposition 2.2 guarantees.

The disturbance attenuation problem is to find the estimate $\hat{x}$ so that for all $\mu_2, v \in L_2[t_0, t_1], x(t_0) \in \mathbb{R}^n$

$$D_{af} \leq \gamma$$

where $\gamma \in \mathbb{R}$ is called the *disturbance attenuation bound*. (C, A) will always be assumed to be an observable pair.

To solve this problem, we convert (29) into a cost function

$$J = \int_{t_0}^{t_1} [||HC(x-\hat{x})||_Q^2 - \gamma(||\mu_2||_{M^{-1}}^2 + ||y - Cx||_{V^{-1}}^2)] \cdot dt - ||x(t_0) - \hat{x}_0||_{\Pi_0}^2 \tag{30}$$

where we have used (1) to rewrite the measurement noise term. Note that we have also rewritten the initial error weighting, defining $\Pi_0 := \gamma^{-1} P_0$. The disturbance attenuation problem is then solved via the differential game

$$\min_{\hat{x}} \max_{y} \max_{\mu_2} \max_{x(t_0)} J \leq 0 \tag{31}$$

subject to

$$\begin{aligned} \dot{x} &= Ax + F_2\mu_2 \\ y &= Cx + v. \end{aligned} \tag{32}$$

B. The Differential Game Solution

We will solve the differential game in two steps beginning with the subproblem

$$\max_{\mu_2} \max_{x(t_0)} J \leq 0$$

subject to (32) with y and $\hat{x}$ fixed. The first step in this solution is to append the problem constraints, which are the system dynamics (32), to the cost (30) through a LaGrange multiplier λ^T

$$J = \int_{t_0}^{t_1} [||HC(x-\hat{x})||_Q^2 - \gamma(||\mu_2||_{M^{-1}}^2 + ||y - Cx||_{V^{-1}}^2) + \lambda^T(Ax + F_2\mu_2 - \dot{x})]\, dt - ||x(t_0) - \hat{x}_0||_{\Pi_0}^2.$$

Integrate $\lambda^T \dot{x}$ by parts and then take the first variation with respect to μ_2 and $x(t_0)$ to get

$$\begin{aligned} \delta J = \int_{t_0}^{t_1} \{ & [(x-\hat{x})^T C^T HQHC + \gamma(y - Cx)^T V^{-1} C + \dot{\lambda}^T \\ & + \lambda^T A]\delta x + [-\gamma \mu_2^T M^{-1} + \lambda^T F_2]\delta\mu_2 \}\, dt \\ & - \lambda(t_1)^T \delta x(t_1) - [(x(t_0) - \hat{x}_0)^T \Pi_0 - \lambda(t_0)^T]\delta x(t_0). \end{aligned} \tag{33}$$

Note that since H is a projector, $H = H^T = H^2$. Equation (33) implies that the first-order necessary conditions to maximize (30) with respect to $x(t_0)$ and μ_2 are

$$\mu_2^* = \frac{1}{\gamma} M F_2^T \lambda \tag{34}$$

$$-\dot{\lambda} = A^T\lambda + C^T HQHC(x - \hat{x}) + \gamma C^T V^{-1}(y - Cx) \tag{35}$$

$$\lambda(t_1) = 0 \tag{36}$$

$$\lambda(t_0) = \Pi_0[x^*(t_0) - \hat{x}_0]. \tag{37}$$

The asterisks in (34) and (37) denote that the extremizing value for the given variable is being used. By substituting the maximizing strategy for μ_2 (34) into the state equation (32), we get a nonhomogeneous two-point boundary value problem (TPBVP)

$$\begin{Bmatrix} \dot{x} \\ \dot{\lambda} \end{Bmatrix} = \begin{bmatrix} A & \frac{1}{\gamma} F_2 M F_2^T \\ -C^T(HQH - \gamma V^{-1})C & -A^T \end{bmatrix} \cdot \begin{Bmatrix} x \\ \lambda \end{Bmatrix} + \begin{Bmatrix} 0 \\ C^T HQHC\hat{x} - \gamma C^T V^{-1} y \end{Bmatrix} \tag{38}$$

by coupling (32) with (35). We will assume solutions x^* and λ^* to (38) such that

$$\lambda^* = \Pi(x^* - x_p). \tag{39}$$

The vector x_p is a measurement-dependent variable which will reduce to the optimal state estimate in the second-half of this game. If we take

$$\Pi(t_0) = \Pi_0 \tag{40}$$

$$x_p(t_0) = \hat{x}_0 \tag{41}$$

then (39) will match the boundary condition for λ at t_0 (37). By substituting (39) back into the TPBVP and working through some minor algebraic manipulations, we find that (39) solves the TPBVP identically if

$$\begin{aligned} -\dot{\Pi} &= A^T\Pi + \Pi A + \frac{1}{\gamma}\Pi F_2 M F_2^T \Pi \\ &\quad + C^T(HQH - \gamma V^{-1})C \end{aligned} \tag{42}$$

$$\begin{aligned} \Pi\dot{x}_p &= \Pi A x_p - [C^T(HQH - \gamma V^{-1})C](\hat{x} - x_p) \\ &\quad + \gamma C^T V^{-1}(y - C\hat{x}). \end{aligned} \tag{43}$$

Equation (42) is clearly a Riccati equation, and its boundary condition is given by (40). Equation (43) looks like an estimator equation, except that it propagates the intermediate variable x_p and not the state estimate $\hat{x}$.

Substituting the maximizing values for μ_2 and $x(t_0)$, (34) and (37) into the original cost function (30), gives us

$$\begin{aligned} \overline{J} = \int_{t_0}^{t_1} [& ||x - \hat{x}||_{C^T HQHC}^2 - ||\lambda||_{(1/\gamma)F_2 M F_2^T}^2 \\ & - \gamma ||y - Cx||_{V^{-1}}^2]\, dt - ||\lambda(t_0)||_{\Pi_0^{-1}}^2. \end{aligned} \tag{44}$$

The second half of the game is then

$$\min_{\hat{x}} \max_{y} \overline{J} \leq 0$$

subject to (43). By adding the identically zero term

$$||\lambda(t_0)||_{\Pi(t_0)^{-1}}^2 - ||\lambda(t_1)||_{\Pi(t_1)^{-1}}^2 + \int_{t_0}^{t_1} \frac{d}{dt}||\lambda(t)||_{\Pi^{-1}}^2\, dt = 0$$

and manipulating the terms in the cost, we find that $\overline{J}$ can be simplified into a pair of quadratic terms

$$\overline{J} = \int_{t_0}^{t_1} [||\hat{x} - x_p||^2_{C^T HQHC} - \gamma ||y - Cx_p||^2_{V^{-1}}]\, dt. \quad (45)$$

From (45), the minimizing strategy for $\hat{x}$ and the maximizing strategy for y are clearly

$$\begin{aligned} \hat{x}^* &= x_p \\ y^* &= Cx_p. \end{aligned} \quad (46)$$

This implies that our game optimal estimate $\hat{x}$ is found from

$$\dot{\hat{x}} = A\hat{x} + \gamma \Pi^{-1} C^T V^{-1}(y - C\hat{x}), \qquad \hat{x}(t_0) = \hat{x}_0 \quad (47)$$

which is simply the extremizing value of $\hat{x}$ (46) applied to (43) and (41). Π is found by propagating (42) with the initial condition (40).

Remark 3.1: The derivation presented in this section follows Banavar and Speyer [22]. □

C. Steady-State Results

In many cases, it is desired to extend finite-time solutions to the steady-state condition. Whenever it is possible to find such a solution, the optimal estimator will be given by (47) with Π being the solution of the algebraic Riccati equation [23]

$$0 = A^T \Pi + \Pi A + \frac{1}{\gamma} \Pi F_2 M F_2^T \Pi + C^T (HQH - \gamma V^{-1}) C. \quad (48)$$

However, unlike linear quadratic optimal control problems, there are no conditions which guarantee the existence of a unique nonnegative definite stabilizing solution to the steady-state game Riccati equation, except in the special case where A is asymptotically stable [23].

IV. Finding the Limiting Solution: Singular Differential Game Theory

We motivated the disturbance attenuation problem of Section III by showing how it approximates the detection filter problem. It is clear, however, that when the disturbance attenuation bound is zero, the two problems are equivalent. It is logical to then ask whether the solution to the disturbance attenuation problem likewise becomes a detection filter at this limit. The answer to this question is by no means obvious, since it is not clear that a limiting case solution even exists.

It is a well-known phenomena of game Riccati equations such as (42) that positive semidefinite symmetric solutions exist only for values of γ larger than a critical value γ_{crit}. This would seem to immediately imply the nonexistence of limiting solutions. However, we can prevent the onset of the γ_{crit} phenomenon by taking the weighting V to zero along with γ so that their product γV^{-1} does not disappear in the limit. This, in and of itself, does not resolve the existence question, but it does turn the limiting case problem into a singular optimization problem since the game cost loses the input term $\gamma ||\mu_2||^2_{M^{-1}}$, i.e.,

$$\begin{aligned} J^* = \lim_{\gamma \to 0} J = \int_{t_0}^{t_1} &[||x - \hat{x}||^2_{C^T HQHC} - ||y - Cx||^2_{\overline{V}^{-1}}]\, dt \\ &- ||x(t_0) - \hat{x}||^2_{\overline{\Pi}_0}. \end{aligned} \quad (49)$$

We define $\overline{V}^{-1} := \lim_{\gamma \to 0} \gamma V^{-1}$ and $\overline{\Pi}_0 := \lim_{\gamma \to 0} \Pi_0$. This is a problem that we can solve.

Remark 4.1: Singular optimization theory has a rich legacy dating back to the beginning of the modern control period. Much of the work from this period is summarized nicely in the book by Bell and Jacobson [24], which is the source for many of the singular optimal control techniques that we will use in the subsequent sections. □

V. Conditions for the Nonpositivity of the Game Cost

In this section, we will determine the properties of the limiting case filter by converting the nonpositivity condition on the game cost (31) into an equivalent linear matrix inequality condition. The latter falls out when we manipulate the cost function to look like a simple quadratic

$$J(\hat{x}, x(t_0), \mu_2, v) = \int_{t_0}^{t_1} \xi^T W \xi \, dt.$$

The vector ξ will consist of linear combinations of the game elements. The nonpositivity of the cost then hinges on the sign definiteness of W. In singular optimal control theory, W is called the "dissipation matrix" because its nonnegative definiteness guarantees that the system will be dissipative [20], [24], [25]. For our purposes, W needs to be *nonpositive definite*, or opposite in sign to the dissipation matrix, in order to guarantee a nonpositive game cost. A nonpositive game cost, in turn, implies that the disturbance attenuation objective is satisfied, giving us a sufficiency condition for an attenuating solution.

This sufficiency condition, however, is strongly tied to the game solution. Results from the game solution are used[4] in several places to construct W, and the sufficiency condition is really nothing more than the first half of the saddle point inequality that is implicit in every differential game

$$\begin{aligned} J(\hat{x}^*, x(t_0), \mu_2, \mu_3, v) &\le J(\hat{x}^*, x^*(t_0), \mu_2^*, \mu_3^*, v^*) \\ &= 0 \le J(\hat{x}, x^*(t_0), \mu_2^*, \mu_3^*, v^*). \end{aligned}$$

As before, the asterisk indicates that the game optimal strategy is being used for that element.

We begin by appending the dynamics of (32) to the cost (30) through the LaGrange Multiplier[5] $(x - \hat{x})^T \Pi$

$$\begin{aligned} J = \int_{t_0}^{t_1} &[||x - \hat{x}||^2_{C^T HQHC} - \gamma ||\mu_2||^2_{M^{-1}} - \gamma ||y - Cx||^2_{V^{-1}} \\ &+ (x - \hat{x})^T \Pi (Ax + F_2 \mu_2 - \dot{x})] dt - ||x(t_0) - \hat{x}_0||^2_{\Pi_0}. \end{aligned} \quad (50)$$

[4] One can think of this as $\hat{x}$ playing its optimal strategy before the adversaries get to play theirs.

[5] Note that this form of the LaGrange Multiplier comes from the TPBVP solution in Section III, (39).

Add and subtract $(x-\hat{x})^T\Pi A\hat{x}$ and $(x-\hat{x})^T\Pi\dot{\hat{x}}$ to (50) and collect terms to get

$$\begin{aligned}J=\int_{t_0}^{t_1}&\{\|x-\hat{x}\|^2_{\Pi A+C^THQHC}-\gamma\|\mu_2\|^2_{M^{-1}}\\&-\gamma\|y-Cx\|^2_{V^{-1}}+(x-\hat{x})^T\Pi F_2\mu_2-(x-\hat{x})^T\\&\cdot\Pi(\dot{x}-\dot{\hat{x}})+(x-\hat{x})^T[\Pi A\hat{x}-\Pi\dot{\hat{x}}]\}\,dt\\&-\|x(t_0)-\hat{x}_0\|^2_{\Pi_0}.\end{aligned}$$

Integrate $(x-\hat{x})^T\Pi(\dot{x}-\dot{\hat{x}})$ in the above by parts and substitute the state equation (32) into the appropriate places. Add and subtract $\hat{x}^TA^T\Pi(x-\hat{x})$ to the result and collect terms to get

$$\begin{aligned}J=\int_{t_0}^{t_1}&\{\|x-\hat{x}\|^2_{\dot{\Pi}+\Pi A+A^T\Pi+C^THQHC}-\gamma\|\mu_2\|^2_{M^{-1}}\\&+(x-\hat{x})^T\Pi F_2\mu_2+\mu_2^TF_2^T\Pi(x-\hat{x})-\gamma\|y-Cx\|^2_{V^{-1}}\\&+(x-\hat{x})^T[-\Pi\dot{\hat{x}}+\Pi A\hat{x}]+[-\Pi\dot{\hat{x}}+\Pi A\hat{x}]^T(x-\hat{x})\}\\&\cdot dt-\|x(t_0)-\hat{x}_0\|^2_{\Pi_0-\Pi(t_0)}-\|x(t_1)-\hat{x}(t_1)\|^2_{\Pi(t_1)}.\end{aligned}\tag{51}$$

Finally, by expanding $\|y-Cx\|^2_{V^{-1}}$ into $\|(y-C\hat{x})-C(x-\hat{x})\|^2_{V^{-1}}$ and collecting terms, (51) can be rewritten as

$$\begin{aligned}J=\int_{t_0}^{t_1}&\{\|x-\hat{x}\|^2_{\dot{\Pi}+\Pi A+A^T\Pi+C^THQHC}-\gamma\|\mu_2\|^2_{M^{-1}}\\&+(x-\hat{x})^T\Pi F_2\mu_2+\mu_2^TF_2^T\Pi(x-\hat{x})-\gamma\|y-C\hat{x}\|^2_{V^{-1}}\\&+(x-\hat{x})^T[-\Pi\dot{\hat{x}}+\Pi A\hat{x}+\gamma C^TV^{-1}(y-C\hat{x})]\\&+[-\Pi\dot{\hat{x}}+\Pi A\hat{x}+\gamma C^TV^{-1}(y-C\hat{x})]^T(x-\hat{x})\}\,dt\\&-\|x(t_0)-\hat{x}_0\|^2_{\Pi_0-\Pi(t_0)}-\|x(t_1)-\hat{x}(t_1)\|^2_{\Pi(t_1)}.\end{aligned}\tag{52}$$

Using the estimator equation (47) we can eliminate the terms in the third and fourth lines of (52). The remainder can then be compactly written as

$$\begin{aligned}J=\int_{t_0}^{t_1}&[\xi^TW(\Pi)\xi-\gamma\|y-C\hat{x}\|^2_{V^{-1}}]\,dt\\&-\|x(t_0)-\hat{x}_0\|^2_{\Pi_0-\Pi(t_0)}-\|x(t_1)-\hat{x}(t_1)\|^2_{\Pi(t_1)}\end{aligned}$$

where

$$\xi:=\begin{Bmatrix}(x-\hat{x})\\ \mu_2\end{Bmatrix}$$

and

$$W(\Pi):=\begin{bmatrix}\dot{\Pi}+A^T\Pi+\Pi A+C^T(HQH-\gamma V^{-1})C & \Pi F_2\\ F_2^T\Pi & -\gamma M^{-1}\end{bmatrix}.\tag{53}$$

Thus, for matrices $\Pi\geq 0$ such that

$$W(\Pi)\leq 0$$
$$\Pi_0-\Pi(t_0)\geq 0\tag{54}$$
$$\Pi(t_1)\geq 0\tag{55}$$

we will have $J\leq 0$. For $\gamma>0$, it is easy to see that the Riccati equation (42) of the previous section is embedded in $W(\Pi)$[6]; but, unlike (42), $W(\Pi)$ can be evaluated in the limit $\gamma\to 0$. In fact, from (53), it is clear that the sufficient conditions for the limiting case cost J^* (49) to be nonpositive are

$$\Pi F_2=0\tag{56}$$
$$\dot{\Pi}+A^T\Pi+\Pi A+C^T(HQH-\overline{V}^{-1})C\leq 0.\tag{57}$$

The boundary conditions are (55) and a modification of (54) to account for a possible jump in the value of Π from t_0 to t_0^+ (we will show this in the next section).

Equation (56) clearly shows that, in the limit, the Riccati matrix Π obtains a nontrivial null space which contains the image of the nuisance map F_2. Moreover, those familiar with singular optimal control theory will recognize (56) and (57) as conditions seen previously for the singular linear quadratic regulator [24].

[6] In fact, the solution of (42) is the solution of $W(\Pi)$ which minimizes its rank [25]

VI. THE SOLUTION TO THE SINGULAR DIFFERENTIAL GAME

In this section, we solve the singular differential game obtained in Section IV. The key result will be a Riccati equation for the limiting case problem. We will find, in subsequent sections, that the solution to this equation is central to understanding the structure of the limiting case game theoretic fault detection filter.

To get our Riccati equation, we must recast the limiting case problem to look like the differential game solved in Section III. This is done with the Goh transformation, which is the same technique that was used in Section II to construct various elements of the disturbance attenuation problem. The transformation begins when we define a new nuisance fault vector ϕ_1 and a new state vector α_1

$$\phi_1(t):=\int_{t_0}^{t}\mu_2(\tau)\,d\tau\tag{58}$$
$$\alpha_1:=x-F_2\phi_1.\tag{59}$$

Differentiating (59) gives us

$$\dot{\alpha}_1=A\alpha_1+B_1\phi_1\tag{60}$$
$$B_1:=AF_2-\dot{F}_2.\tag{61}$$

Equation (60) is the new state equation. The need for the subscripts, e.g., B_1, will become obvious later. Substituting (58) and (59) into the limiting case game cost (49) gives us

$$\begin{aligned}J^*=\int_{t_0}^{t_1}&[\|\alpha_1-\hat{x}\|^2_{C^THQHC}-\|y-C\alpha_1\|^2_{\overline{V}^{-1}}\\&+(y-C\alpha_1)^T\overline{V}^{-1}CF_2\phi_1+\phi_1^TF_2^TC^T\overline{V}^{-1}\\&\cdot(y-C\alpha_1)-\|\phi_1\|^2_{F_2^TC^T\overline{V}^{-1}CF_2}]\,dt\\&-\|\alpha_1(t_0)+F_2\phi_1(t_0^+)-\hat{x}_0\|^2_{\Pi_0}.\end{aligned}\tag{62}$$

At first glance, the term $\phi_1(t_0^+)$ may seem odd, since $\phi_1(t)$ is defined to be the integral of μ_2 from t_0 to t. However, μ_2 need not be an L_2 function, or even a function at all, since it no longer appears in the game cost (49). Thus, we must be able to account for impulsive jumps in ϕ_1 at t_0. Classical results from singular optimization theory, in fact, contain many examples of control inputs [24] or state estimates [26], [27] which are impulsive at the initial time.

If $F_2^T C^T \overline{V}^{-1} C F_2 > 0$, we have a new differential game

$$\min_{\hat{x}} \max_{y} \max_{\phi_1} \max_{\alpha_1(t_0)} J^* \le 0$$

subject to (60). This is the essentially the same game that we examined in Section III, the only difference being new cross-terms between the nuisance input and measurement noise. We should note that we are only able to recover the game from Section III because the projector H (12), (14) has been constructed so that $HCF_2 = 0$. Without this property, there would be cross-terms involving $\hat{x}$ and ϕ_1 in (62), which changes the game.

We begin by appending the state dynamics (60) to the cost (62) through a LaGrange multiplier λ^T

$$\begin{aligned} J^* = \int_{t_0}^{t_1} & [||\alpha_1 - \hat{x}||^2_{C^T HQHC} - ||y - C\alpha_1||^2_{\overline{V}^{-1}} \\ & - ||\phi_1||^2_{F_2^T C^T \overline{V}^{-1} C F_2} + (y - C\alpha_1)^T \overline{V}^{-1} C F_2 \phi_1 \\ & + \phi_1^T F_2^T C^T \overline{V}^{-1} (y - C\alpha_1) + \lambda^T (A\alpha_1 + B_1 \phi_1 - \dot{\alpha}_1)] \\ & \cdot dt - ||\alpha_1(t_0) + F_2 \phi_1(t_0^+) - \hat{x}_0||^2_{\overline{\Pi}_0}. \end{aligned} \tag{63}$$

By maximizing (63) with respect to $\alpha_1(t_0)$ and ϕ_1, we find that the first-order necessary conditions are

$$\begin{aligned} \dot{\lambda} &= -A^T \lambda + C^T \overline{V}^{-1} C F_1 \phi_1 - C^T \overline{V}^{-1} (y - C\alpha_1) \\ &\quad - C^T HQHC(\alpha_1 - \hat{x}) & (64) \\ \phi_1 &= (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} B_1^T \lambda + (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} \\ &\quad \cdot F_2^T C^T \overline{V}^{-1} (y - C\alpha_1) & (65) \\ \lambda(t_0^+) &= \overline{\Pi}_0 [\alpha_1(t_0) + F_2 \phi_1(t_0) - \hat{x}_0] & (66) \\ \lambda(t_1) &= 0 \\ \phi_1(t_0^+) &= (F_2^T \overline{\Pi}_0 F_2)^{-1} F_2^T \overline{\Pi}_0 [\alpha_1(t_0) - \hat{x}_0]. & (67) \end{aligned}$$

Note that (67) results when we maximize the initial error term $||\alpha_1(t_0) + F_2 \phi_1(t_0^+) - \hat{x}_0||^2_{\overline{\Pi}_0}$ with respect to $\phi_1(t_0^+)$. Substituting this term into (66) gives us

$$\lambda(t_0^+) = [\overline{\Pi}_0 - \overline{\Pi}_0 F_2 (F_2^T \overline{\Pi}_0 F_2)^{-1} F_2^T \overline{\Pi}_0](\alpha_1(t_0) - \hat{x}_0). \tag{68}$$

Using (65), we can rewrite our state equation (60) and our co-state equation (64) as

$$\begin{aligned} \dot{\alpha}_1 &= [A - B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} C] \alpha_1 \\ &\quad + B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} B_1^T \lambda + B_1 (F_2^T C^T \overline{V}^{-1} \\ &\quad \cdot C F_2)^{-1} F_2^T C^T \overline{V}^{-1} y & (69) \\ \dot{\lambda} &= -[A - B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} C]^T \lambda \\ &\quad - C^T HQHC(\alpha - \hat{x}) - C^T [\overline{V}^{-1} - \overline{V} C F_2 (F_2^T C^T \\ &\quad \cdot \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1}](y - C\alpha). & (70) \end{aligned}$$

It will turn out that our notation can be greatly simplified if we rearrange the term in front of $(y - C\alpha)$ in (70)

$$\begin{aligned} & \overline{V}^{-1} - \overline{V}^{-1} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} \\ &\quad = \overline{V}^{-(1/2)} [I - \overline{V}^{-(1/2)} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} \\ &\qquad \cdot F_2^T C^T \overline{V}^{-(1/2)}] \overline{V}^{-(1/2)}. \end{aligned} \tag{71}$$

Since the term inside the square brackets on the right-hand side of (71) is a projector, we can rewrite (71) as

$$\begin{aligned} & \overline{V}^{-1} - \overline{V}^{-1} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} \\ &\quad = \overline{V}^{-(1/2)} [I - \overline{V}^{-(1/2)} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} \\ &\qquad F_2^T C^T \overline{V}^{-(1/2)}] \\ &\qquad [I - \overline{V}^{-(1/2)} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} \\ &\qquad F_2^T C^T \overline{V}^{-(1/2)}] \overline{V}^{-(1/2)}. \end{aligned}$$

Pulling the $\overline{V}^{-(1/2)}$ terms through the projectors and in toward the center gives us

$$\begin{aligned} & \overline{V}^{-1} - \overline{V}^{-1} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} \\ &\quad = [I - \overline{V}^{-1} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T] \\ &\qquad \cdot \overline{V}^{-1} [I - C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1}] \\ &\quad = \overline{H}^T \overline{V}^{-1} \overline{H} \end{aligned}$$

where we have defined

$$\overline{H} := I - C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1}. \tag{72}$$

Aside from greatly condensing our equations, $\overline{H}$ has the same effect as a projector since it is idempotent, i.e., $\overline{H}^2 = \overline{H}$. Thus, its eigenvalues will be either one or zero. $\overline{H}$, in fact, maps vectors onto the same subspace as H, since $\overline{H} C F_2 = 0$. If V is of the form νI, then $\overline{H}$ is identically H. This becomes evident if one looks at (14) or (12) and substitutes F_2 in for $\hat{F}$.

Using (72), we can write (69) and (70) compactly as a TPBVP, as shown in the equation at the bottom of the page.

If we assume a solution of the form

$$\lambda = S(\alpha_1 - \alpha_p) \tag{73}$$

where α_p is analogous to x_p from Section III, then differentiating (73) gives us

$$\begin{aligned} -\dot{S} &= S[A - B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} C] \\ &\quad + [A - B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} C]^T S \\ &\quad + C^T (HQH - \overline{H}^T \overline{V}^{-1} \overline{H}) C \\ &\quad + S B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} B_1^T S \end{aligned} \tag{74}$$

$$\begin{aligned} \begin{Bmatrix} \dot{\alpha}_1 \\ \dot{\lambda} \end{Bmatrix} &= \begin{bmatrix} A - B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} C & B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} B_1^T \\ C^T (\overline{H}^T \overline{V}^{-1} \overline{H} - HQH) C & -A^T + C^T \overline{V}^{-1} C F_2 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} B_1^T \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \lambda \end{Bmatrix} \\ &\quad + \begin{Bmatrix} B_1 (F_2^T C^T \overline{V}^{-1} C F_2)^{-1} F_2^T C^T \overline{V}^{-1} y \\ -C^T (\overline{H})^T \overline{V}^{-1} \overline{H} y + C^T HQHC\hat{x} \end{Bmatrix} \end{aligned}$$

and

$$S\dot{\alpha}_p = SA\alpha_p + SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} F_2^T C^T \cdot \overline{V}^{-1}(y - C\alpha_p) + C^T HQHC(\alpha_p - \hat{x}) + C^T(\overline{H})^T \cdot \overline{V}^{-1}\overline{H}(y - C\alpha_p) \tag{75}$$

with the boundary conditions

$$\begin{aligned} S(t_0^+) &= \overline{\Pi}_0 - \overline{\Pi}_0 F_2 (F_2^T \overline{\Pi}_0 F_2)^{-1} F_2^T \overline{\Pi}_0 \\ \alpha_p(t_0^+) &= \hat{x}_0. \end{aligned} \tag{76}$$

The boundary conditions fall out when we match (73) with the boundary condition for λ at t_0^+ (68). Note that

$$\begin{aligned} S(t_0^+)F_2 &= \overline{\Pi}_0 F_2 - \overline{\Pi}_0 F_2 (F_2^T \overline{\Pi}_0 F_2)^{-1} F_2^T \overline{\Pi}_0 F_2 \\ &= \overline{\Pi}_0 F_2 - \overline{\Pi}_0 F_2 = 0 \end{aligned} \tag{77}$$

which shows that $S(t)$ takes an impulsive jump at t_0. Equation (74) is sometimes referred to in singular optimal control literature as the *Goh Riccati equation*. The existence of the game optimal estimator (75) will hinge upon the existence of a nonnegative definite symmetric solution to this equation. To get the final form of (75), we need to solve the second half of the game

$$\min_{\hat{x}} \max_{y} \overline{J}^* \le 0$$

$$\begin{aligned} \overline{J}^* = \int_{t_0}^{t_1} \Big[&\|\alpha_1 - \hat{x}\|_{C^T HQHC} - \|y - C\alpha_1\|_{\overline{H}^T \overline{V}^{-1} \overline{H}} \\ &- \|\alpha_1 - \alpha_p\|_{SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} B_1^T S} \Big] dt \\ &- \|\alpha_1(t_0^+) - \hat{x}_0\|_{\overline{\Pi}_0 - \overline{\Pi}_0 F_2 (F_2^T \overline{\Pi}_0 F_2)^{-1} F_2^T \overline{\Pi}_0}. \end{aligned}$$

$\overline{J}^*$ is the singular game cost (62) with the extremizing value of ϕ_1—including $\phi_t(t_0^+)$—substituted into the appropriate places. For simplicity, we define

$$\begin{aligned} \hat{e} &= \alpha_1 - \hat{x} \\ \hat{v} &= y - C\alpha_1 \\ \hat{r} &= \alpha_1 - \alpha_p \end{aligned}$$

so that the game is now

$$\min_{\hat{e}} \max_{\hat{v}} \overline{J}^* \le 0$$

$$\begin{aligned} \overline{J}^* = \int_{t_0}^{t_1} \Big[&\|\hat{e}\|_{C^T HQHC} - \|\hat{v}\|_{\overline{H}^T \overline{V}^{-1} \overline{H}} \\ &- \|\hat{r}\|_{SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} B_1^T S} \Big] dt \\ &- \|\hat{r}(t_0^+)\|_{\overline{\Pi}_0 - \overline{\Pi}_0 F_2 (F_2^T \overline{\Pi}_0 F_2)^{-1} F_2^T \overline{\Pi}_0} \end{aligned} \tag{78}$$

subject to

$$\begin{aligned} S\dot{\hat{r}} &= S(\dot{\alpha}_1 - \dot{\alpha}_p) \\ &= \{S[A - B_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} F_2^T C^T \overline{V}^{-1} C] \\ &\quad + SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} B_1^T S \\ &\quad + C^T(HQH - \overline{H}^T \overline{V}^{-1} \overline{H})C\}\hat{r} - C^T(\overline{H})^T \\ &\quad \cdot \overline{V}^{-1} \overline{H}\hat{v} - C^T HQHC\hat{e}. \end{aligned} \tag{79}$$

After appending (79) to (78) through the LaGrange multiplier ψ^T and taking the first variation, we find that the first-order necessary conditions to extremize the cost with respect to $\hat{e}$ and $\hat{r}$ are

$$0 = C^T HQHC(\hat{e} - \hat{r}) \tag{80}$$

$$0 = \overline{H}^T \overline{V}^{-1} \overline{H}(\hat{v} + C\hat{r}) \tag{81}$$

$$\begin{aligned} S\dot{\psi} &= S[A - B_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} F_2^T C^T \overline{V}^{-1} C]\psi \\ &\quad + SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} B_1^T S\hat{r} \end{aligned} \tag{82}$$

$$0 = (\overline{\Pi}_0 - \overline{\Pi}_0 F_2 (F_2^T \overline{\Pi}_0 F_2)^{-1} F_2^T \overline{\Pi}_0)[\psi(t_0^+) - \hat{r}(t_0^+)]. \tag{83}$$

Because of (76), (83) can be rewritten as

$$0 = S(t_0^+)[\psi(t_0^+) - \hat{r}(t_0^+)]. \tag{84}$$

Using (79)–(81), we get

$$\begin{aligned} S\dot{\hat{r}} &= \{S[A - B_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} F_2^T C^T \overline{V}^{-1} C] \\ &\quad + SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} B_1^T S\}\hat{r} + [C^T(\overline{H})^T \overline{V}^{-1} \\ &\quad \cdot \overline{H}C - C^T HQHC](\hat{r} - \psi). \end{aligned} \tag{85}$$

Equation (84) tells us that

$$\psi = \hat{r} \bmod \ker S \text{ at } t = t_0^+$$

which implies that

$$\psi = \hat{r} \bmod \ker HC \ (= \ker \overline{H}C) \text{ at } t = t_0^+$$

since $\ker S = \ker HC$ at $t = t_0^+$ (77). Equation (85), therefore, simplifies to

$$\begin{aligned} S\dot{\hat{r}} &= S[A - B_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} F_2^T C^T \overline{V}^{-1} C \\ &\quad + SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} B_1^T S]\hat{r} \end{aligned} \tag{86}$$

at $t = t_0^+$, which is the same equation as (82). Hence

$$\psi = \hat{r} \bmod \ker S$$

over the entire time interval since they are equal ($\bmod \ker S$) at t_0^+ and are propagated by the same equations for $t > t_0^+$. This means that

$$\hat{e} = \hat{r} \bmod \ker S, \qquad \forall t$$

which implies that

$$\hat{x} = \alpha_p \bmod \ker S, \qquad \forall t.$$

The optimal estimate is thus given by

$$\begin{aligned} S\dot{\hat{x}} &= SA\hat{x} + SB_1(F_2^T C^T \overline{V}^{-1} CF_2)^{-1} F_2^T C^T \overline{V}^{-1}(y - C\hat{x}) \\ &\quad + C^T(\overline{H})^T \overline{V} \overline{H}(y - C\hat{x}) \end{aligned} \tag{87}$$

where S is the solution of (74). While it is possible to generate an estimate from (87) in full-order form, it may not be practical since S is not invertible. We will see in the next section, however, that an easily implementable reduced-order estimator can be derived from these results.

If $F_2^T C^T \overline{V}^{-1} CF_2$ fails to be positive definite, we will still have a singular problem, but our results from above will be invalid, since (74) requires that the inverse $(F_2^T C^T \overline{V}^{-1} CF_2)^{-1}$

exists. We must, therefore, continue to transform our problem until it is nonsingular. The way we proceed, however, depends on the type of singularity with which we are dealing.

1) If $F_2^T C^T \overline{V}^{-1} C F_2$ is *totally* singular, i.e., equal to zero, we can repeat the Goh transformation

$$\begin{aligned}\phi_2(t) &= \int_{t_0}^{t} \phi_1(\tau)\, d\tau \\ \alpha_2 &= \alpha_1 - B_1\phi_1 \\ \dot{\alpha}_2 &= A\alpha_2 + B_2\phi_2 \\ B_2 &= AB_1 - \dot{B}_1 \end{aligned} \tag{88}$$

since we have the same problem with which we started.

2) If $F_2^T C^T \overline{V}^{-1} C F_2$ is only *partially* singular, i.e., singular but nonzero, we can always assume that this quantity takes the form

$$F_2 C^T \overline{V}^{-1} C F_2 = \begin{bmatrix} Q^a & 0 \\ 0 & 0 \end{bmatrix}$$

where Q_a is positive definite, since we can reshuffle the state space if needed [28]. In fact, we can also group the nuisance input ϕ_1 accordingly

$$\dot{\alpha}_1 = A\alpha_1 + B_1^a \phi_1^a + B_1^b \phi_1^b$$

so that the game cost will be singular with respect to ϕ_1^b but not ϕ_1^a. The transformation proceeds on ϕ_1^b only

$$\begin{aligned}\phi_2^b(t) &= \int_{t_0}^{t} \phi_1^b(\tau)\, d\tau \\ \alpha_2 &= \alpha_1 - B_1^b \phi_2^b. \end{aligned}$$

The new input matrix is then

$$B_2 = [B_1^a \quad AB_1^b - \dot{B}_1^b] \tag{89}$$

and the new game is formed as before.

In both cases, we can stop the transformation process if $B_1^T C^T \overline{V}^{-1} C B_1$ is positive definite. Otherwise, we must continue the transformation in the appropriate manner. When we finally get a B_k such that $B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1}$ is positive definite, the differential game will be

$$\min_{\hat{x}} \max_{y} \max_{\phi_k} \max_{\alpha_k(t_0)} J^* \le 0$$

$$\begin{aligned} J^* = \int_{t_0}^{t_1} &[||\alpha_k - \hat{x}||^2_{C^T HQHC} - ||y - C\alpha_k||^2_{\overline{V}^{-1}} \\ &+ (y - C\alpha_k)^T \overline{V}^{-1} C B_{k-1}\phi_k + \phi_k^T B_{k-1}^T C^T \\ &\cdot \overline{V}^{-1}(y - C\alpha_k) - ||\phi_k||^2_{B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1}}]\, dt \\ &- ||\alpha_k(t_0) + \overline{B}\hat{\phi}(t_0^+) - \hat{x}_0||^2_{\overline{\Pi}_0} \end{aligned}$$

subject to

$$\dot{\alpha}_k = A\alpha_k + B_k\phi_k$$

where

$$\alpha_k = x + \overline{B}\hat{\phi}(t_0^+).$$

$\overline{B}$ is a matrix formed by eliminating the redundant columns of the composite matrix $[F_2 \cdots B_{k-1}]$, and $\hat{\phi}(t_0^+)$ is the corresponding vector $[\phi_1^T \cdots \phi_k^T]^T$ with the redundant elements eliminated. These columns and elements must be removed to avoid accounting for the same signal more than once. The potential for repeating elements exists because in the partially singular case the columns of B_{j-1}, which correspond to the nonsingular inputs (denoted B_j^a), are simply carried over unchanged to the next input matrix B_j, e.g., (89).

The corresponding Goh Riccati equation for the general problem is

$$\begin{aligned} -\dot{S} = &S[A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1}^T C^T \overline{V}^{-1} C] \\ &+ [A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1}^T C^T \overline{V}^{-1} C]^T \\ &\cdot S + C^T(HQH - \overline{H}^T \overline{V}^{-1} \overline{H})C \\ &+ SB_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_k^T S \end{aligned} \tag{90}$$

with the boundary condition

$$S(t_0^+) = \overline{\Pi}_0 - \overline{\Pi}_0 \overline{B}(\overline{B}^T \overline{\Pi}_0 \overline{B})^{-1} \overline{B}^T \overline{\Pi}_0. \tag{91}$$

The general form of the estimator is

$$\begin{aligned} S\dot{\hat{x}} = &SA\hat{x} + SB_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1}^T \\ &\cdot C^T \overline{V}^{-1}(y - C\hat{x}) + C^T(\overline{H})^T \overline{V}^{-1} \overline{H}(y - C\hat{x}). \end{aligned} \tag{92}$$

Implicit in the general form of the Goh Riccati equation is the assumption that the projector H will be such that $HCB_i = 0, \forall i < k-1$, where k is the iteration step which leads to a nonsingular problem. If one examines the construction of H, however, he will see that this is always the case. B_{k-1} is constructed by operating on the columns of the nuisance map F_2 with the Goh transformation until they lie outside the kernel of C (61), (88), (89). This is the same way that the matrices, $\hat{F}$ or $\hat{F}(t)$, from the definition of H, (14), (12), are derived. Thus, one could equivalently define H as

$$H = I - CB_{k-1}[(CB_{k-1})^T CB_{k-1}]^{-1}(CB_{k-1})^T$$

and $\overline{H}$ as

$$\overline{H} = I - CB_{k-1}[B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1}]^{-1} B_{k-1}^T C^T \overline{V}^{-1}. \tag{93}$$

We can relate the Goh Riccati equation to our game Riccati equation and to the limiting case analysis through the following theorem taken directly from [24].

Theorem 6.1: The solution to (90), S, is such that

$$SB_{i-1} = 0, \forall t \in [t_0, t_1] \tag{94}$$

$$\dot{S} + SA + A^T S + C^T(HQH - \overline{V}^{-1})C \le 0. \tag{95}$$

□

Proof: Rewrite (90) as

$$\begin{aligned}\dot{S} + SA + A^T S + C^T(HQH - \overline{H}^T\overline{V}^{-1}\overline{H})C \\ = -SB_k(B_{k-1}^T C^T \overline{V}^{-1} CB_{k-1})^{-1} B_k^T S \\ + SB_k(B_{k-1}^T C^T \overline{V}^{-1} CB_{k-1})^{-1} B_{k-1}^T C^T \overline{V}^{-1} C \\ + C^T \overline{V}^{-1} CB_{k-1}(B_{k-1}^T C^T \overline{V}^{-1} CB_{k-1})^{-1} B_k^T S.\end{aligned} \tag{96}$$

Using (93), we can expand $\overline{H}$ in (96) and collect terms to get

$$\begin{aligned}\dot{S} + SA + A^T S + C^T(HQH - \overline{V}^{-1})C \\ = -[SB_k - C^T\overline{V}^{-1}CB_{k-1}] \times (B_{k-1}^T C^T \overline{V}^{-1} CB_{k-1})^{-1} \\ \cdot [B_k^T S - B_{k-1}^T C^T \overline{V}^{-1} C].\end{aligned} \tag{97}$$

Equation (97) clearly implies our first claim, (95). To prove our second claim, (94), we first rearrange (97) as

$$\begin{aligned}\dot{S} = -SA - A^T S - C^T(HQH - \overline{V}^{-1})C \\ - [SB_k - C^T\overline{V}^{-1}CB_{k-1}] \\ \times (B_{k-1}^T C^T \overline{V}^{-1} CB_{k-1})^{-1}[B_k^T S - B_{k-1}^T C^T \overline{V}^{-1} C].\end{aligned} \tag{98}$$

Pre-multiply both sides of (98) by B_{k-1}^T and then subtract $\dot{B}_{k-1}^T S$ from both sides of the result to get

$$\begin{aligned}-B_{k-1}^T\dot{S} - \dot{B}_{k-1}^T S \\ = -B_{k-1}^T SA - \dot{B}_{k-1}^T S + B_{k-1}^T A^T S - B_{k-1}^T C^T \overline{V}^{-1} C \\ - [B_{k-1}^T SB_k - B_{k-1}^T C^T \overline{V}^{-1} CB_{k-1}^T] \\ (B_{k-1}^T C^T \overline{V}^{-1} \cdot CB_{k-1}^{-1})^{-1}[B_k^T S - B_{k-1}^T C^T \overline{V}^{-1} C].\end{aligned}$$

If we collect terms, making use of the fact that $B_k = AB_{k-1} - \dot{B}_{k-1}$, we get

$$\begin{aligned}\frac{d}{dt}[B_{k-1}^T S] = -B_{k-1}^T S[A + B_k(B_{k-1}^T C^T \overline{V}^{-1} CB_{k-1})^{-1} \\ \cdot (B_k^T S - B_{k-1}^T C^T \overline{V}^{-1} C)].\end{aligned} \tag{99}$$

Equation (99) is a homogeneous differential equation in the variable $B_{k-1}^T S$. The boundary condition to (99) is derived from (91), where we make use of fact that $S(t_0^+)$ is a projector that maps B_{k-1} to zero since $\overline{B}$ contains the columns of B_{k-1}

$$\begin{aligned}B_{k-1}^T S(t_0^+) = B_{k-1}^T \overline{\Pi}_0 - B_{k-1}^T \overline{\Pi}_0 \overline{B}(\overline{B}^T \overline{\Pi}_0 \overline{B})^{-1} \overline{B}^T \overline{\Pi}_0 \\ = 0.\end{aligned} \tag{100}$$

Equations (99) and (100) together imply (94). ■

The same arguments used to prove Theorem 6.1 are used in [28] to show that (94) implies $SF_2 = 0, \forall t \in [t_0, t_1]$. Hence, the Goh Riccati solution satisfies all of the sufficient conditions for nonpositivity.

Remark 6.2: Equations (67) and (100) show that the nuisance input acts impulsively at t_0 to move the error trajectory onto the singular arc defined by (94). The projector H restricts z to this singular arc so that the transformed game becomes the same game that was solved in Section III, except restricted to the singular arc. In this sense, S is the limit of Π on the singular arc. □

Remark 6.3: The requirement that $B_i^T C^T \overline{V}^{-1} CB_i$ be invertible is the Generalized Legendre–Clebsch condition [24]. Note that taking V to zero along with γ so that γV^{-1} has a nonzero limit is crucial to satisfying this condition. □

VII. The Relationship Between the Limiting Game Filter and Detection Filters

A. A Reduced-Order Detection Filter from the Limiting Game Solution

The results that we will present in this subsection are most easily derived for time-invariant systems. We will, therefore, limit ourselves to this case to simplify the presentation.

To begin, we consider the initial condition, $S(t_0)$, (91), to the Goh Riccati equation (90). This matrix is symmetric nonnegative definite with a nontrivial nullspace which means that there exists a nonsingular orthonormal transformation Γ such that

$$\Gamma^T S(t_0)\Gamma = \begin{bmatrix}\overline{S}(t_0) & 0 \\ 0 & 0\end{bmatrix}. \tag{101}$$

$\overline{S}(t_0)$ is positive definite and symmetric. Applying the same transformation to our system matrices gives us

$$\begin{aligned}C\Gamma = [C_1 \quad C_2], \qquad \Gamma^T A \Gamma = \begin{bmatrix}A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \\ \Gamma^T B_{k-1} = \begin{bmatrix}D_1 \\ D_2\end{bmatrix}, \qquad \Gamma^T B_k = \begin{bmatrix}G_1 \\ G_2\end{bmatrix}.\end{aligned}$$

Note that since $SB_{k-1} = 0$, applying the transformation Γ within the product SB_{k-1} gives us $\Gamma^T \underline{S}\Gamma\Gamma^T B_{k-1} = \overline{S}D_1 = 0$, which implies that $D_1 = 0$ since $\overline{S}$ is positive definite. Hence

$$\Gamma^T B_{k-1} = \begin{bmatrix}0 \\ D_2\end{bmatrix}.$$

We can apply this transformation to the estimator state

$$\hat{\eta} = \begin{Bmatrix}\hat{\eta}_1 \\ \hat{\eta}_2\end{Bmatrix} = \Gamma^T \hat{x}$$

to get a reduced-order estimator by transforming (92)

$$\begin{aligned}&\begin{bmatrix}\overline{S} & 0 \\ 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\hat{\eta}}_1 \\ \dot{\hat{\eta}}_2\end{Bmatrix} \\ &= \begin{bmatrix}\overline{S} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\begin{Bmatrix}\hat{\eta}_1 \\ \hat{\eta}_2\end{Bmatrix} + \begin{Bmatrix}C_1^T \\ C_2^T\end{Bmatrix}\overline{H}^T \\ &\cdot \overline{V}^{-1}\overline{H}\left(y - [C_1 \quad C_2]\begin{Bmatrix}\hat{\eta}_1 \\ \hat{\eta}_2\end{Bmatrix}\right) - \begin{bmatrix}\overline{S} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}G_1 \\ G_2\end{bmatrix} \\ &\cdot \left([0 \quad D_2^T]\begin{bmatrix}C_1^T \\ C_2^T\end{bmatrix}\overline{V}^{-1}[C_1 \quad C_2]\begin{bmatrix}0 \\ D_2\end{bmatrix}\right)^{-1}[0 \quad D_2^T] \\ &\cdot \begin{bmatrix}C_1^T \\ C_2^T\end{bmatrix}\overline{V}^{-1}(y - C_1\hat{\eta}_1 - C_2\hat{\eta}_2).\end{aligned} \tag{102}$$

From (102), we get equations

$$\begin{aligned}\overline{S}\dot{\hat{\eta}}_1 = \overline{S}A_{11}\hat{\eta}_1 + \overline{S}A_{12}\hat{\eta}_2 \\ + C_1^T(\overline{H})^T\overline{V}^{-1}\overline{H}(y - C_1\hat{\eta}_1 - C_2\hat{\eta}_2)\end{aligned} \tag{103}$$

$$\begin{aligned}- \overline{S}G_1(D_2^T C_2^T \overline{V}^{-1} C_2 D_2)^{-1} D_2^T C_2^T \\ \cdot \overline{V}^{-1}(y - C_1\hat{\eta}_1 - C_2\hat{\eta}_2)\end{aligned} \tag{104}$$

$$0 = C_2^T(\overline{H})^T\overline{V}^{-1}\overline{H}(y - C_1\hat{\eta}_1 - C_2\hat{\eta}_2). \tag{105}$$

Since the residual signal $y - C_1\hat{\eta}_1 - C_2\hat{\eta}_2$ is arbitrary, (105) implies that

$$\overline{H}C_2 = 0. \tag{106}$$

Since $\ker S \subset \ker HC$ at t_0^+, by (77) we can take a vector $\xi \in \ker S(t_0^+)$ and then pre- and post-multiply (90) by ξ^T and ξ, respectively, to get $\xi^T \dot{S}(t_0^+)\xi = 0$. This implies that

$$\Gamma^T \dot{S}(t_0^+)\Gamma = \begin{bmatrix} \dot{\overline{S}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, if we apply the Goh transformation to the Goh Riccati equation, (90), we get three equations:

$$0 = C_2^T(HQH - \overline{H}^T\overline{V}^{-1}\overline{H})C_2 \tag{107}$$

$$\begin{aligned} 0 = {} & \overline{S}[A_{12} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1}C_2] \\ & + C_1^T(HQH - \overline{H}^T\overline{V}^{-1}\overline{H})C_2 \end{aligned} \tag{108}$$

$$\begin{aligned} \dot{\overline{S}} = {} & \overline{S}[A_{11} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1}C_1] \\ & + [A_{11} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1}C_1]^T\overline{S} \\ & + \overline{S}G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}G_1^T\overline{S} \\ & + C_1^T(HQH - \overline{H}^T\overline{V}^{-1}\overline{H})C_1. \end{aligned} \tag{109}$$

Equation (109) is a reduced-order Goh Riccati equation with a boundary condition taken from (101). Equations (106) and (107) imply that

$$HC_2 = 0. \tag{110}$$

By applying (106) and (110) to (108), we find that

$$\overline{S}[A_{12} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1}C_2] = 0$$

which means that

$$A_{12} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1}C_2 = 0 \tag{111}$$

since $\overline{S}$ is positive definite. Together (106), (110), and (111) reduce (104) to

$$\begin{aligned} \dot{\hat{\eta}}_1 = {} & A_{11}\hat{\eta}_1 + G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^T \\ & \cdot C_2^T\overline{V}^{-1}(y - C_1\hat{\eta}_1) + \overline{S}^{-1}C_1^T(\overline{H})^T\overline{V}^{-1} \\ & \cdot \overline{H}(y - C_1\hat{\eta}_1). \end{aligned} \tag{112}$$

Equations (106), (110), and (111) also imply that for all $t \in [t_0, t_1]$, the kernel of the Goh Riccati matrix does not increase or decrease in size. Thus, (90) reduces to (109).

To get the error equation, we define

$$\eta = \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} := \Gamma^T\alpha_k, \qquad e_1 := \eta_1 - \hat{\eta}_1, \qquad e_2 := \eta_2 - \hat{\eta}_2.$$

If we transform the state dynamics (32) and the measurements (1) we get

$$\dot{\eta}_1 = A_{11}\eta_1 + A_{12}\eta_2 + G_1\phi_k \tag{113}$$

$$\begin{aligned} \dot{\eta}_2 &= A_{21}\eta_1 + A_{22}\eta_2 + G_2\phi_k \\ y &= C\alpha_k + C\overline{B}\hat{\phi} + v = C_1\eta_1 + C_2\eta_2 + C_2D_2\phi_k + v. \end{aligned} \tag{114}$$

The expression on the farthest right-hand side of (114) results from the fact that the only columns in $\overline{B}$ which lie outside the kernel of C are those which correspond to B_{k-1}. Using (112)–(114), the error equation for our reduced-order estimator is

$$\begin{aligned} \dot{e}_1 = {} & [A_{11} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_1^T\overline{V}^{-1}C_1 \\ & - \overline{S}^{-1}C_1^T(\overline{H})^T\overline{V}^{-1}\overline{H}C_1]e_1 \\ & + [A_{12} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1}C_2]\eta_2 \\ & + G_1\phi_k - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1} \\ & \cdot C_2(D_2\phi_k + v) - \overline{S}^{-1}C_1^T(\overline{H})^T\overline{V}^{-1} \\ & \cdot \overline{H}(C_2D_2\phi_k + v). \end{aligned} \tag{115}$$

By using (106) and (111), (115) can be further simplified to

$$\begin{aligned} \dot{e}_1 = {} & [A_{11} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^TC_1^T\overline{V}^{-1}C_1 \\ & - \overline{S}^{-1}C_1^T(\overline{H})^T\overline{V}^{-1}\overline{H}C_1]e_1 - [G_1(D_2^TC_2^T\overline{V}^{-1} \\ & \cdot C_2D_2)^{-1}D_2^TC_2^T\overline{V}^{-1}C_2 + \overline{S}^{-1}C_1^T(\overline{H})^T\overline{V}^{-1}\overline{H}]v. \end{aligned} \tag{116}$$

Equation (116) clearly shows that the error e_1 is not influenced by the nuisance fault. In practice, however, we will not be able to monitor e_1 directly, since our information about the system comes from the filter residual $(y - C\hat{\eta}_1)$, which is not free from the influence of η_2. The use of the projector H on this residual, however, will eliminate this influence. The failure signal for the reduced-order filter is then

$$z = H(y - C_1\hat{\eta}_1).$$

It is fairly easy to show.

Theorem 7.1: Reduced-order estimator (112) is asymptotically stable. □

Proof: From (116), it is clear that the stability of the reduced-order estimator depends on the eigenvalues of the closed-loop matrix

$$\begin{aligned} \overline{A}_{cl} = {} & A_{11} - G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}D_2^T \\ & \cdot C_1^T\overline{V}^{-1}C_1 - \overline{S}^{-1}C_1^T(\overline{H})^T\overline{V}^{-1}\overline{H}C_1. \end{aligned} \tag{117}$$

We can show that $\overline{A}_{cl}$ is Hurwitz, i.e., its eigenvalues are in the open left-half plane, by rewriting (109) as

$$\begin{aligned} \dot{\overline{S}} + \overline{SA}_{cl} + \overline{A}_{cl}^T\overline{S} = {} & -C_1^T(HQH + \overline{H}^T\overline{V}^{-1}\overline{H})C_1 \\ & - \overline{S}G_1(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}G_1^T\overline{S}. \end{aligned} \tag{118}$$

The first term on the right-hand side of (118) is negative semidefinite. The second term is negative definite, since $\overline{S}$ and $(D_2^TC_2^T\overline{V}^{-1}C_2D_2)^{-1}$ are positive definite and G_1 has independent columns by construction. Therefore

$$\dot{\overline{S}} + \overline{SA}_{cl} + \overline{A}_{cl}^T\overline{S} < 0.$$

Lyapunov's Stability theorem [29] then implies that $\overline{A}_{cl}$ is Hurwitz. ■

Remark 7.2: The difficulty in deriving a reduced-order filter for the time-varying case lies in finding a way to transform S so that the reduced-order estimator (112) and the reduced-order Riccati equation (109) remain valid over the entire time interval. Note that Theorem 6.1, (94), gives insight on how to form the transformation matrix and that [30] gives similar insights on the propagation structure. □

Remark 7.3: Taking the limit of γV^{-1} allows us to give a rigorous accounting of the measurement uncertainty v in the lower dimensional subspace determined by the singular arc. The authors of [8] claim that they can choose a reduced-order filter structure to deal with such issues, though their analyses do not explicitly consider sensor noise, making it unclear how such noise projects onto their reduced state space. □

B. The Invariant Subspace Structure of the Limiting Case Filter

Unknown input observers work by placing the image of the nuisance failure map into the unobservable subspace. We will show that the asymptotic game filter works in the same way. Getting to this result, however, will require that we first introduce some concepts from geometric control theory [31], [32].

Geometric control theory gets its name from its use of abstract subspaces and operators, without specific bases, to define and solve problems in systems theory. Of particular importance in this theory are subspaces which are invariant with respect to operators. If we consider a system (1), (32), defined by the triple, (C, A, F_2), and let $\mathcal{X}$ denote the state space, then a subspace $\mathcal{W} \subset \mathcal{X}$ is said to be *A-invariant* if for every $x \in \mathcal{W}, Ax \in \mathcal{W}$. This can be equivalently symbolized as $A\mathcal{W} \subset \mathcal{W}$. $\mathcal{W}$ is said to be (C, A)*-invariant* if there exists an output injection feedback matrix L such that $(A + LC)\mathcal{W} \subset \mathcal{W}$.

Another important element in geometric control theory, and in control theory in general, is the concept of *invariant zeros*. Invariant zeros are the complex numbers λ which cause the matrix

$$P(\lambda) = \begin{bmatrix} A - \lambda I & F_2 \\ C & 0 \end{bmatrix}$$

to lose column rank [33]. Associated with each zero is an *invariant zero direction* z such that

$$\begin{bmatrix} A - \lambda I & F_2 \\ C & 0 \end{bmatrix} \begin{Bmatrix} z \\ w \end{Bmatrix} = 0. \tag{119}$$

When (119) holds, the vector w is such that $w = Kz$ for some matrix K [34]. Invariant zeros behave like multivariable analogs to the transfer function zeros of classical control theory. They play a fundamental role in determining the limits of performance for optimal control systems [35], [36], and they are also essential to defining special (C, A)-invariant subspaces, called (C, A)*-unobservability subspaces*, which are used in Beard–Jones Fault Detection Filters [3], [5] and in reduced-order residual generators [8] for design.

The following theorem tells us that $\ker S$ is a (C, A)-invariant subspace, and it also tells us something of $\ker S$'s invariant zero structure. The second claim will require the following lemma.

Lemma 7.4: Let λ and z be an invariant zero and zero direction for the triple (C, A, F_2). Let $\mathcal{W} \subset \mathcal{X}$ be a (C, A)-invariant subspace with the image of F_2 contained in $\mathcal{W}$, and let L be any map such that $(A + LC)\mathcal{W} \subset \mathcal{W}$. Then $z \notin \mathcal{W}$ implies that λ is an eigenvalue of $(A + LC)$ restricted to the factor space $\mathcal{X}/\mathcal{W}$.

Proof: See [5, Proposition 2.9]. ■

Theorem 7.5:

1) $\ker S$ is a (C, A)-invariant subspace.
2) The invariant zero directions corresponding to the right-half plane and $j\omega$-axis zeros of (C, A, F_2) lie in $\ker S$. □

Proof: Consider the "state matrix"

$$A_{cl} = SA - SB_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1}^T C^T \overline{V}^{-1} C + C^T (\overline{H})^T \overline{V} \overline{H} C$$

of the full-order estimator found from the singular game solution (92). If we pre- and post-multiply A_{cl} by Γ^T and Γ respectively, we get

$$\Gamma^T A_{cl} \Gamma = \begin{bmatrix} \overline{S} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{A}_{cl} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \tag{120}$$

where $\overline{A}_{cl}$ is as defined in (117) and

$$\begin{aligned} \overline{A}_{12} &:= A_{12} - G_1(D_2^T C_2^T \overline{V}^{-1} C_2 D_2)^{-1} D_2^T C_2^T \overline{V}^{-1} C_2 \\ \overline{A}_{21} &:= A_{21} - G_2(D_2^T C_2^T \overline{V}^{-1} C_2 D_2)^{-1} D_2^T C_2^T \overline{V}^{-1} C_1 \\ \overline{A}_{22} &:= A_{22} - G_2(D_2^T C_2^T \overline{V}^{-1} C_2 D_2)^{-1} D_2^T C_2^T \overline{V}^{-1} C_2. \end{aligned}$$

From (111), $\overline{A}_{12} = 0$, which means that (120) simplifies to

$$\Gamma^T A_{cl} \Gamma = \begin{bmatrix} \overline{S} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{A}_{cl} & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}. \tag{121}$$

Equation (121) clearly implies that $\ker S$ is (C, A)-invariant, since the error e, if it initially lies in $\ker S$, must be of the form

$$e(t_0) = \begin{Bmatrix} 0 \\ \underline{e}(t_0) \end{Bmatrix}$$

in the basis corresponding to the transformation Γ. In the absence of exogenous inputs, (121) implies that $e(t)$ will then be propagated by way of

$$\dot{e}(t) = \begin{bmatrix} \overline{S} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \overline{A}_{22} \underline{e}(t) \end{Bmatrix}.$$

This clearly shows that the error trajectory will never leave $\ker S$.

The second part of our theorem follows from Lemma 7.4. Applied to our case, this lemma tells us that any zero directions which do not lie in $\ker S$ will show up as eigenvalues of the submatrix $\overline{A}_{cl}$ in (121). In Theorem 7.1, however, we showed that $\overline{A}_{cl}$ is asymptotically stable. Therefore these zero directions cannot lie outside of $\ker S$. ■

Since the image of F_2 lies in $\ker S$, the following theorem implies that the limiting case game filter has the structure of an unknown input observer.

Theorem 7.6: $\ker S$ is contained in the unobservable subspace of (HC, A), where H is the projector defined by (12) or (14). □

Proof: Let ξ be a constant vector which lies in $\ker S$ at $t = t_0^+$. Define

$$\Gamma^T \xi = \begin{Bmatrix} 0 \\ \underline{\xi} \end{Bmatrix}$$

and pre- and post-multiply the Goh Riccati equation (90) by ξ^T and ξ, respectively, to get

$$0 = \xi^T \dot{S} \xi + \xi^T C^T (HQH - \overline{H}^T \overline{VH}) C\xi. \tag{122}$$

However

$$\xi^T \Gamma\Gamma^T \dot{S} \Gamma\Gamma^T \xi = [0 \quad \underline{\xi}^T] \begin{bmatrix} \dot{\underline{S}} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \underline{\xi} \end{Bmatrix} = 0 \tag{123}$$

which implies that

$$\xi^T C^T (HQH - \overline{H}^T \overline{VH}) C\xi = 0. \tag{124}$$

Since HC and $\overline{H}C$ project onto the same space,[7] we can define $\overline{\xi} := HC\xi = \overline{H}C\xi$ and then rewrite (124) as

$$||\overline{\xi}||^2_{Q - \overline{V}^{-1}} = 0.$$

So long as $Q \neq \overline{V}^{-1}$, this implies that $||\overline{\xi}|| = 0$, which, in turn, implies that

$$\xi \in \ker HC = \ker \overline{H}C. \tag{125}$$

We will now consider the general case. We define the vector ξ_k to be

$$\xi_k := [A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1}^T C^T \overline{V}^{-1} C]^k \xi$$

and assume that $\xi_k \in \ker S$. Note that this implies that

$$\Gamma \xi_k = \begin{Bmatrix} 0 \\ \underline{\xi}_k \end{Bmatrix}$$

which, in turn, means that $\dot{S}\xi_k = 0$, since we can transform $\dot{S}\xi_k$ into a form similar to (123). ξ_k, lying in the kernels of both S and $\dot{S}$, then implies that ξ_k also lies in $\ker HC$ since we can pre- and post-multiply (90) by ξ_k^T and ξ_k, respectively, to get

$$\xi_k^T C^T (HQH - \overline{H}^T \overline{VH}) C\xi_k = 0.$$

Thus, if we post-multiply (90) by ξ_k, the remainder

$$\begin{aligned} 0 =& S[A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1} C^T \overline{V}^{-1} C]\xi_k \\ =& S[A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1} C^T \overline{V}^{-1} C]^{k+1}\xi \end{aligned}$$

tells us that

$$\begin{aligned} \xi_{k+1} := [A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1} C^T \\ \cdot \overline{V}^{-1} C]^{k+1}\xi \end{aligned}$$

lies in the kernel of S. The same arguments as before then lead to $\xi_{k+1} \in \ker \dot{S}$ and $\xi_{k+1} \in \ker HC$. Thus ξ_k, lying the kernel of S, implies it also lies in the kernel of HC and that ξ_{k+1} lies in the kernel of S. By induction, we can claim that this holds for all k, in particular, $k = 0, \cdots, n-1$, so that

$$\begin{aligned} &HC\xi = 0 \\ &HC[A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1} C^T \overline{V}^{-1} C] \\ &\quad \cdot \xi = 0 \\ &\quad \vdots = \vdots \\ &HC[A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1} C^T \overline{V}^{-1} C]^{n-1} \\ &\quad \cdot \xi = 0. \end{aligned}$$

This leads us to the conclusion that our original vector ξ lies in the unobservable subspace of the pair $(HC, A - B_k(B_{k-1}^T C^T \overline{V}^{-1} C B_{k-1})^{-1} B_{k-1} C^T \overline{V}^{-1} C)$. Since output injection feedback cannot change the observability of a system, we can claim that ξ lies in the unobservable subspace of (HC, A). Since ξ is an arbitrary vector in $\ker S$, this implies the proposition. ■

Remark 7.7: The embedding subspace of the residual generator in [8] is a (C, A)-unobservability subspace. These subspaces are distinguished by the fact that they contain all of the invariant zero directions associated with the triple, (C, A, F_2). This characteristic allows for complete eigenvalue assignment on both the (C, A)-unobservability subspace and its factor space. Our invariant subspace $\ker S$, on the other hand, can only be said to be (C, A)-invariant since we cannot account for all of the invariant zeros with Theorem 7.5. The pole assignment freedom in [8], however, is needed to meet ancillary design objectives in the residual generator which can be met in the game theoretic filter through different choices of the weighting matrices.

Remark 7.8: The proof to Theorem 7.5 is patterned after the proof to [37, Lemma 3.1(b)]. It is also possible to prove the second part of Theorem 7.5 using the proof given in [35, Appendix B, Part 3]. Other reported results concerning the null space of Riccati Solutions are cited in [38]. □

VIII. Example 1: Accelerometer Fault Detection for the F16XL

A. Problem Statement

To demonstrate the effectiveness of the game theoretic filter, the F16XL example of [11] is re-examined. The aircraft dynamics are linearized about trimmed level flight at 10 000 ft altitude and Mach 0.9. For simplicity, a reduced-order five-state model of the longitudinal dynamics (including a first-order wind gust model) is considered

$$\begin{aligned} \dot{x} &= Ax + B_{wg} w_{wg} \\ y &= Cx + v. \end{aligned}$$

The five components of the state vector are

$$x = \begin{Bmatrix} u \\ w \\ q \\ \theta \\ w_g \end{Bmatrix} \begin{matrix} \text{long. velocity (ft/s)} \\ \text{normal velocity (ft/s)} \\ \text{pitch rate (deg/s)} \\ \text{pitch (deg)} \\ \text{wind gust (ft/s)} \end{matrix} \tag{126}$$

[7] See the discussion which follows (72).

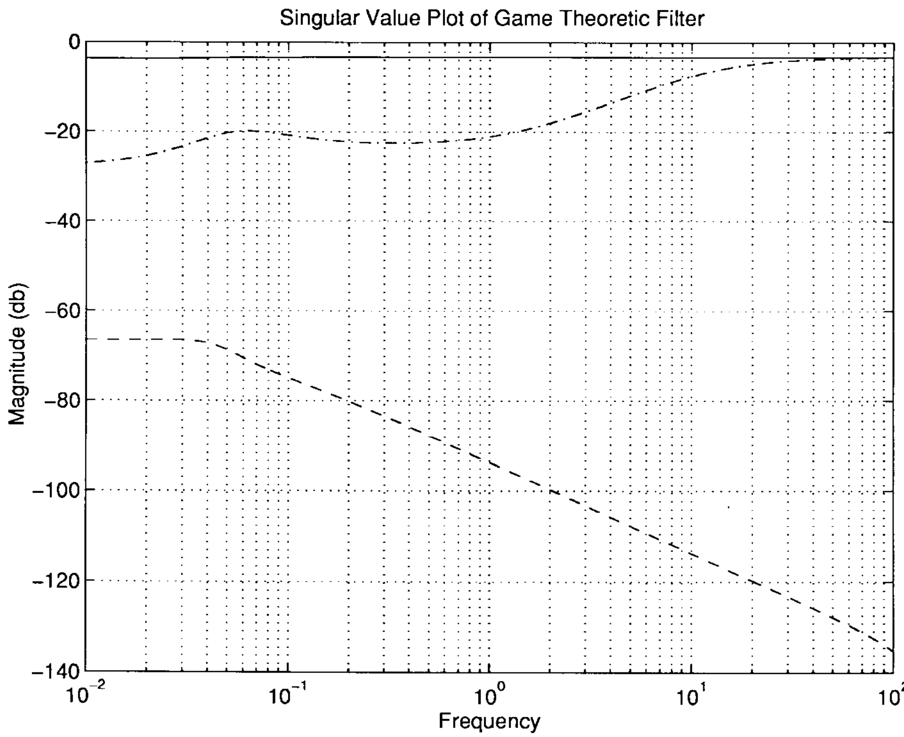

Fig. 1. Singular value plot of accerometer fault versus singular value plot of wind gust (solid line—output due to μ_{A_z}; dashed line—output due to μ_{wg}; dashed-dot line—output due to μ_{A_z} for filter with unity sensor noise weighting).

with the measurements

$$y = \begin{Bmatrix} q \\ \theta \\ A_z \\ A_x \end{Bmatrix} \begin{matrix} \text{pitch rate (deg/s)} \\ \text{pitch (deg)} \\ \text{long. acceleration (ft/s}^2\text{)} \\ \text{normal acceleration (ft/s}^2\text{).} \end{matrix} \tag{127}$$

The input w_{wg} is wind gust and v is the sensor noise. The system matrices are

$$A = \begin{bmatrix} -0.0674 & 0.0430 & -0.8886 & -0.5587 & 0.0430 \\ 0.0205 & -1.4666 & 16.5800 & -0.0299 & -1.4666 \\ 0.1377 & -1.6788 & -0.6819 & 0 & -1.6788 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.1948 \end{bmatrix} \tag{128}$$

$$B_{wg}^T = [0 \quad 0 \quad 0 \quad 0 \quad 2.0156] \tag{129}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.0139 & 1.0517 & 0.1485 & -0.0299 & 0 \\ -0.0677 & 0.0431 & 0.0171 & 0 & 0 \end{bmatrix}. \tag{130}$$

It is desired to detect a normal accelerometer fault A_z in the presence of the wind gust disturbance and the sensor noise. Following the modeling techniques described in Section II, we convert the accelerometer fault into an input to the system

$$\dot{x} = Ax + F_{A_z}\mu_{A_z} + F_{wg}\mu_{wg} \tag{131}$$

$$y = Cx + v \tag{132}$$

where

$$F_{A_z} = \begin{bmatrix} 0.6003 & 0 \\ 0.9429 & -1.3706 \\ 0 & -1.5003 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In order to distinguish the accelerometer fault from the wind gust disturbance, we model the wind gust as the nuisance fault. Hence, F_{wg} is simply B_{wg} and μ_{wg} is the wind gust input w_{wg}. A quick check shows that F_{A_z} and F_{wg} are output separable. Finally, we generate the residual projector

$$\begin{aligned} H &= I - (CAF_{wg})[(CAF_{wg})^T(CAF_{wg})]^{-1}(CAF_{wg})^T \\ &= \begin{bmatrix} 0.5330 & 0 & -0.4982 & -0.0264 \\ 0 & 1 & 0 & 0 \\ -0.4982 & 0 & 0.4685 & -0.0281 \\ -0.0264 & 0 & -0.0281 & 0.9985 \end{bmatrix}. \end{aligned} \tag{133}$$

B. Full-Order Filter Design

The full-order filter design problem boils down to finding a solution to the game Riccati equation (42) which results in a filter with acceptable performance. Acceptable, in this example, means that the filter transmits the target fault and attenuates the nuisance fault so that there is good separation between the respective transmission levels. Since the inverse of Π is used in the filter gain, we directly solve for this inverse using a variation of (48)

$$\begin{aligned} 0 = {} & \Pi^{-1}A^T + A\Pi^{-1} + \frac{1}{\gamma}F_2 M F_2^T \\ & + \Pi^{-1}(C^T H^T Q H - \gamma V^{-1})C\Pi^{-1} \end{aligned} \tag{134}$$

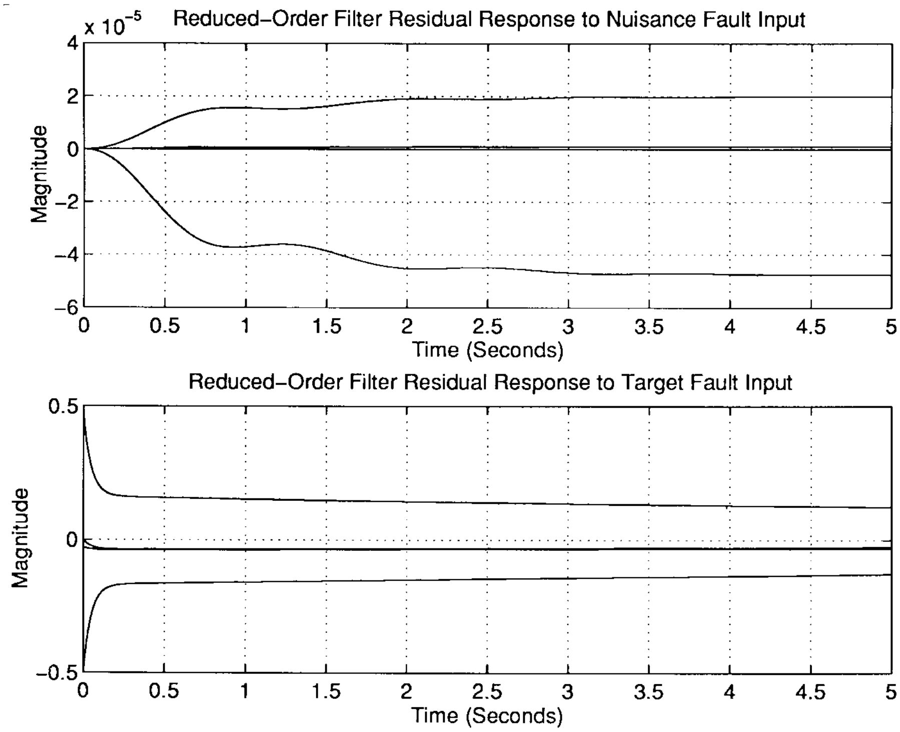

Fig. 2. Reduced-order detection filter performance for F-16XL example.

with the following values of γ and the weighting matrices:

$$\gamma = 5 \times 10^{-7}, \qquad Q = M = I$$
$$\gamma^{-1}V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10\,000 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{135}$$

The transmission levels of the resulting filter are given in Fig. 1. The peculiar form of $\gamma^{-1}V$ is necessitated by the fact that in the true system, the target fault is a sensor fault which appears in the measurements

$$y = Cx + E_{A_z}\mu_{A_z}.$$

Thus, it has a direct feedthrough to the failure signal

$$z = H(y - C\hat{x}) = HC(x - \hat{x}) + HE_{A_z}\mu_{A_z}.$$

The effect of this feedthrough can be seen in Fig. 1, which is a plot of the singular values versus frequency of the transfer matrix between the inputs μ_{A_z} and μ_{wg} to the failure signal z. The solid line is the transmission of the target fault with $\gamma^{-1}V$ as given in (135). The dashed-dot line, which runs directly below the solid line, is the target fault transmission when $\gamma^{-1}V = I$. As Fig. 1 shows, the direct feedthrough of the target fault prevents the transmission from rolling off at higher frequencies and can also detrimentally effect the DC gain of the target fault transmission, as evidenced by the dashed-dot line. By choosing $\gamma^{-1}V$ as it is in (135), however, the contribution of this sensor channel to the state estimate will be kept small so that the direct feedthrough will drive the failure signal in the event of an accelerometer failure. In general, we may not always be able to choose such an extreme form for $\gamma^{-1}V$ and still get a solution to the game Riccati equations. In those cases, one simply has to do the best that he can and rely on post-processing of the residual to help with the fault declaration.

The dashed line in Fig. 1 is the nuisance fault transmission. It can been seen that at least 60 dB of separation exists between the output due to the wind gust and the output due to the accelerometer fault. This should be satisfactory.

C. Reduced-Order Filter Design

A reduced-order filter was found using a steady-state version of the filter presented in Section III-A. The filter was designed with the weightings

$$V = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 200 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad Q = I. \tag{136}$$

As Fig. 2 shows, this reduced-order filter effectively blocks the nuisance fault transmission while preserving the target fault transmission. Moreover, if one compares the form of V used above to that used in (135), he will see that the reduced-order filter does not require an extreme form of that weighting matrix in order to work well.

IX. Example 2: Position Sensor Fault Detection in the Face of Mass-Rate Uncertainty for a Simple Rocket (Time-Varying System)

A. Introduction and Problem Statement

In this section, we present, possibly, the first example of detection filtering applied to a time-varying system. Our example is taken from [39] and is a rocket moving in the vertical direction with height $h(t)$ and velocity $v(t)$. The rocket is propelled against gravity g by thrust generated from expelled fuel mass

$$F_{\text{thrust}} = -V_e u(t) \tag{137}$$

$$u(t) = \dot{m}(t). \tag{138}$$

$\dot{m}$ is the rate of change of the mass due to spent fuel, and V_e is the exit velocity of the fuel through the nozzle. Kinematics gives us $\dot{h}(t) = v(t)$, and Newton's Second Law of Motion gives us

$$\dot{v}(t) = -g + \frac{V_e u(t)}{m(t)}. \tag{139}$$

Defining $x_1(t) = h(t), x_2(t) = v(t)$, and $x_3(t) = m(t)$, we get

$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{Bmatrix} = f(t) = \begin{bmatrix} x_2(t) \\ -g + V_e u(t)/x_3(t) \\ u(t) \end{bmatrix} \tag{140}$$

as our state equation. If we assume that the mass rate $u(t)$ is nominally a constant, $\overline{u}(t) = u_0$, then integrating each of the state equations in turn gives us

$$\begin{Bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \\ \overline{x}_3(t) \end{Bmatrix} = \begin{bmatrix} -\frac{g}{2}t^2 + \frac{m_0 V_e}{u_0}\left[\left(1 + \frac{u_0}{m_0}t\right)\ln\left(1 + \frac{u_0}{m_0}t\right) - \frac{u_0}{m_0}t\right] \\ -gt + V_e \ln\left(1 + \frac{u_0}{m_0}t\right) \\ m_0 + u_0 t \end{bmatrix} \tag{141}$$

as the nominal solution to (140). m_0 is the initial mass of the rocket. If the true mass rate of the rocket, however, is $u(t) = u_0 + \delta u(t)$, where δu is some "small," time-varying perturbation, then the system will be perturbed away from the nominal state $x(t) = \overline{x}(t) + \delta x(t)$. Using a Taylor expansion of (140) about (141) and neglecting terms higher than first-order, we find that behavior of the system about the nominal trajectory is

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{V_e u_0}{(m_0 + u_0 t)^2} \\ 0 & 0 & 0 \end{bmatrix} \delta x(t) + \begin{Bmatrix} 0 \\ \frac{V_e}{m_0 + u_0 t} \\ 1 \end{Bmatrix} \delta u(t). \tag{142}$$

Finally, we will assume that we have sensors that measure the height and velocity of the rocket so that our state-space system is completed with the measurement equation

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x.$$

With these measurements, our system is observable.

B. Position Sensor Fault Detection in the Face of Mass-Rate Uncertainty

Suppose that we want to detect a position sensor fault despite uncertainty about the mass rate input $u(t)$. We can apply the game theoretic detection filter to this problem by treating the perturbation $\delta u(t)$ as the nuisance fault

$$F_2(t) = \begin{Bmatrix} 0 \\ \frac{V_e}{m_0 + u_0 t} \\ 1 \end{Bmatrix} \qquad \mu_2(t) = \delta u(t). \tag{143}$$

To check for output separability, we first need to find the failure map for the position sensor fault. As described in Section II-B, we begin with

$$y = Cx + E\mu_2$$

where $E^T = [1 \quad 0]$. The first column of the sensor failure map is found as the solution to the equation

$$E = Cf$$

which can easily be verified to be $f^T = [1 \quad 0 \quad 0]$. The second column is found from $Af - \dot{f}$ which in this case turns out to be zero. Thus, we only need a single column failure map for the position sensor. The output separability test is then

$$M(t) = [CF_2 \quad Cf] = \begin{bmatrix} 0 & 1 \\ \frac{V_e}{m_0 + u_0 t} & 0 \end{bmatrix}$$

which is full rank so long as $u_0 t \neq -m_0$, which we will take to be the case in our example (note that u_0 is a negative quantity since it represents the rate of mass loss). Finally, the failure signal projector H is

$$\begin{aligned} H &= I - CF_2[(CF_2)^T(CF_2)]^{-1}F_2^T C^T \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{Bmatrix} 0 \\ \frac{V_e}{m_0 + u_0 t} \end{Bmatrix} \frac{(m_0 + u_0 t)^2}{V_e^2} \begin{bmatrix} 0 & \frac{V_e}{m_0 + u_0 t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{144}$$

From here, one obtains the detection filter by propagating the equations

$$\delta\dot{\hat{x}} = A\delta\hat{x} + PC^T V^{-1}(y - C\delta\hat{x}) \tag{145}$$

$$\dot{P} = PA^T + AP - C^T(V^{-1} - HQH)C + \frac{1}{\gamma}F_2 M F_2^T. \tag{146}$$

A failure is then declared whenever the failure signal

$$z = H(y - C\delta\hat{x})$$

exceeds some *a priori* chosen threshold. After some trial and error, the weightings

$$V = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.045 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P(t_0) = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \qquad M = 10\,000 \tag{147}$$

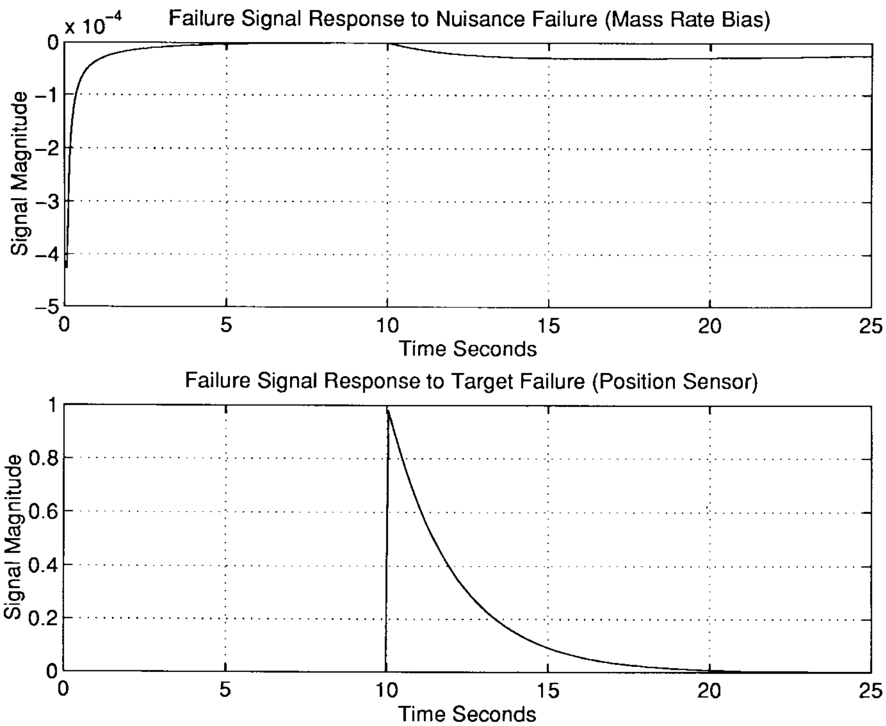

Fig. 3. Game theoretic fault detection filter performance for position sensor fault detection (failures occur at $t = 10$ s).

were chosen along with $\gamma = 0.25$. The initial conditions were arbitrarily picked to be

$$\delta x(t_0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad e(t_0) = \begin{Bmatrix} 0 \\ -0.3 \\ 0.2 \end{Bmatrix}. \tag{148}$$

A nonzero initial condition on the error state was chosen to demonstrate filter convergence properties. The physical parameters of the rocket were taken from [40, pp. 263, 264] and are the characteristics of the first-stage Minuteman Missle Motor

$$m_0 = 50{,}550 \text{ lb-mass}, \qquad u_0 = -855\frac{\text{lb-mass}}{\text{s}}$$
$$V_e = -5180.\frac{\text{ft}}{\text{sec}}. \tag{149}$$

As we discussed in the previous example, Q and V are chosen to maximize the low-frequency transmission of the target fault. In this example, we were limited by the sensitivity of the Riccati Solution to different choices of Q and V.

The rocket dynamics along with the filter were simulated from $t_0 = 0$ s to $t_1 = 25$ s. In Fig. 3, the response of the failure signal generated by the game theoretic filter is displayed for a hard failure of the position sensor at $t = 10$ s and for a step bias in the mass rate, also occurring at $t = 10$ s. The detrimental effect of the direct feedthrough upon the transmission of the target fault can clearly be seen in this figure. The response to the target fault has a transient quality which dies away noticeably after 5 s. The magnitude of the target fault response, however, is still quite a bit greater than the nuisance fault transmission and remains so for a substantial period of time. Thus, a reasonably designed post-processing scheme should be able to detect and declare a sensor fault. We should note that a different sensor suite might improve our ability to detect a position sensor fault since, with the current set, the position bias is unobservable to the velocity sensor. The initial response at the beginning of Fig. 3 is the transient response of the filter to the nonzero initial condition. It must be noted that in this example the Riccati Matrix loses definiteness past $t = 50$ s. For this application, this may not be a liability since the rocket motor is on for only a brief period of time.

X. Conclusions

In this paper, we posed and solved a disturbance attenuation problem which closely approximates the actions of a fault detection filter. The end product is a game theoretic filter which acts as an approximate unknown input observer. We also showed that this approximation can be made more and more exact until, in the limit, the game theoretic filter becomes an unknown input observer exactly. A related result is that a reduced-order observer can also be obtained from the limiting case.

The disturbance attenuation-based approach that we have introduced here leads to filters which are more flexible, more robust, and more applicable than existing fault detection structures. Time-varying systems, in fact, can be monitored for the first time. Finally, in the course of our limiting case analysis, we showed that singular optimization theory can be used to analyze the asymptotic properties of game theoretic estimators. It is possible that the application of singular optimization theory to other disturbance attenuation problems can lead to similar insights.

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