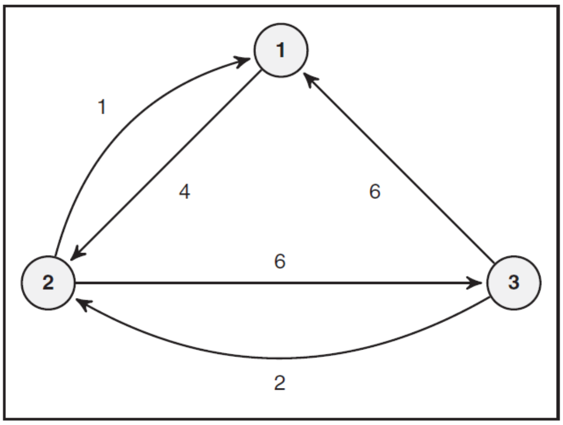
Applied Probability for Computer Science

Exercise list 2: Continuous Time Markov Chains – **Solutions**

Question 1

Consider the CTMC X with transition rates represented in the following graph



(a) Find the generator matrix. Is the chain irreducible?

Solution:

$$\mathbf{G} = \begin{bmatrix} -4 & 4 & 0 \\ 1 & -7 & 6 \\ 6 & 2 & -8 \end{bmatrix}$$

The chain is irreducible, since all states can be reached from all others.

(b) Find the stationary distribution of X.

Solution: The stationary distribution $\boldsymbol{\pi}$ is the solution to the system of equations $\mathbf{A}\boldsymbol{\pi} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 6 \\ 4 & -7 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{\pi} = \texttt{solve}(\texttt{A}, \texttt{b}) = \begin{bmatrix} 0.44 \\ 0.32 \\ 0.24 \end{bmatrix}.$$

(c) Find the transition matrix of the embedded discrete-time chain Z of jumps. *Hint: we discussed the jump chain when we considered simulating paths of a CTMC.*

Solution: The transition matrix for Z is the matrix $\tilde{\mathbf{P}}$ with entries $\tilde{p}_ij = -g_{ij}/g_{ii}$ for $j \neq 0$ and $\tilde{p}_{ii} = 0$ for $i = 1, 2, 3$. Therefore, the required matrix is

$$\tilde{\mathbf{P}} = \begin{bmatrix} 0 & 1 & 0 \\ 1/7 & 0 & 6/7 \\ 3/4 & 1/4 & 0 \end{bmatrix}$$

(d) If $X(0) = 3$, what is the probability that the first jump will be to state 1?

Solution: Notice that $Z_0 = X_0$, so $\mathbb{P}[Z_1 = 1|X_0 = 3] = \mathbb{P}[Z_1 = 1|Z_0 = 3] = 3/4$

(e) If $X(0) = 3$, what is the probability that the second jump will be to state 1?

Solution:

$$\begin{aligned} \mathbb{P}[Z_2 = 1|Z_0 = 3] &= \sum_{j=1}^{3} \mathbb{P}[Z_2 = 1|Z_1 = j]\mathbb{P}[Z_1 = j|Z_0 = 3] \\ &= \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{7} + 0 \cdot \frac{3}{4} = \frac{1}{28} \end{aligned}$$

(f) If $X(0) \sim \boldsymbol{\pi}$, for $\boldsymbol{\pi}$ the stationary distribution, what is the probability that the first jump will be to state 1?

Solution:

$$\begin{aligned} \mathbb{P}[Z_1 = 1] &= \sum_{j=1}^{3} \mathbb{P}[Z_1 = 1|Z_0 = j]\mathbb{P}[Z_0 = j] = \sum_{j=1}^{3} \tilde{p}_{j1}\pi_j \\ &= 0 \cdot 0.44 + \frac{1}{7} \cdot 0.32 + \frac{3}{4} \cdot 0.24 = 0.2257 \end{aligned}$$

(g) If $X(0) \sim \boldsymbol{\pi}$, for $\boldsymbol{\pi}$ the stationary distribution, what is the probability that the process will be in state 1 at time $t = 1$?

Solution: The required probability is

$$\mathbb{P}[X_1 = 1] = \sum_{j=1}^{3} \mathbb{P}[X_1 = 1|X_0 = j]\mathbb{P}[X_0 = j] = \sum_{j=1}^{3} p_{j1}(1)\pi_j,$$

where the $p_{j1}(1)$ form the first column of the transition matrix

$$\mathbf{P}_1 = e^{\mathbf{G}} = \texttt{expm(G)} = \begin{bmatrix} 0.44 & 0.32 & 0.24 \\ 0.44 & 0.32 & 0.24 \\ 0.44 & 0.32 & 0.24 \end{bmatrix}.$$

Therefore, $\mathbb{P}[X_1 = 1] = 0.44 \sum_{j=1}^{3} \pi_j = 0.44$.

Question 2

Let $\lambda, \mu > 0$ and let X be a CTMC on $S = \{1, 2\}$ with generator

$$\mathbf{G} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

(a) Find $\mathbb{P}[X(t) = 2 | X(0) = 1, X(3t) = 1]$

Solution: Using the Markov property and the homogeneity of the process, the required probability is

$$\frac{\mathbb{P}[X(t) = 2, X(3t) = 1 | X(0) = 1]}{\mathbb{P}[X(3t) = 1 | X(0) = 1]} = \frac{p_{12}(t)p_{21}(2t)}{p_{11}(3t)}$$

(b) Find $\mathbb{P}[X(t) = 2 | X(0) = 1, X(3t) = 1, X(4t) = 1]$

Solution: Following the same reasoning, the required probability is the same as for the previous point:

$$\frac{\mathbb{P}[X(t) = 2, X(3t) = 1, X(4t) = 1 | X(0) = 1]}{\mathbb{P}[X(3t) = 1, X(4t) = 1 | X(0) = 1]} = \frac{p_{12}(t)p_{21}(2t)p_{11}(t)}{p_{11}(3t)p_{11}(t)} = \frac{p_{12}(t)p_{21}(2t)}{p_{11}(3t)}$$

(c) Find the probabilities in points (a) and (b) when $\mu = \lambda = 1$.

Solution: The transition semigroup for the process is

$$\mathbf{P}_t = \frac{1}{\lambda + \mu} \begin{bmatrix} \lambda + \mu e^{-t(\lambda+\mu)} & \mu - \mu e^{-t(\lambda+\mu)} \\ \lambda - \lambda e^{-t(\lambda+\mu)} & \mu + \lambda e^{-t(\lambda+\mu)} \end{bmatrix} = \begin{bmatrix} \frac{1+e^{-2t}}{2} & \frac{1-e^{-2t}}{2} \\ \frac{1-e^{-2t}}{2} & \frac{1+e^{-2t}}{2} \end{bmatrix},$$

Therefore,

$$\begin{aligned} \mathbb{P}[X(t) = 2 | X(0) = 1, X(3t) = 1] &= \mathbb{P}[X(t) = 2 | X(0) = 1, X(3t) = 1, X(4t) = 1] \\ &= \frac{(1 + e^{-2t}/2)^2}{(1 - e^{-2t})/2} = \frac{(1 + e^{-2t})^2}{2(1 - e^{-2t})} \end{aligned}$$

Question 3

Describe the jump chain for a birth-death process with birth rates λ_n and death rates μ_n. *Hint: we discussed the jump chain when we considered simulating paths of a CTMC.*

Solution: The jump chain is a discrete time Markov chain on $S = \{0, 1, 2, \dots\}$ satisfying, for $i \geq 1$,

$$\mathbb{P}[Z_{n+1} = j | Z_n = i] = \begin{cases} \frac{\lambda_i}{\lambda_i + \mu_i} & \text{if } j = i + 1, \\ \frac{\mu_i}{\lambda_i + \mu_i} & \text{if } j = i - 1 \end{cases},$$

and $\mathbb{P}[Z_{n+1} = 1 | Z_n = 0] = 1$ for all $n \geq 0$.

Question 4

An asymmetric simple random walk in continuous time on the non-negative integers,

$S = \{0, 1, 2, \ldots\}$, with retention at 0 is a CTMC X with transition semigroup $\mathbf{P}_t$ given by

$$\mathbb{P}[X(t+h) = j | X(t) = i] = \begin{cases} \lambda h + o(h) & \text{if } j = i+1, i \geq 0 \\ \mu h + o(h) & \text{if } j = i-1, i \geq 1 \\ 1 & \text{if } i = 0, j = 1 \\ o(h) & \text{otherwise} \end{cases},$$

for $\lambda, \mu > 0$.

(a) Find the infinitesimal generator of the process

Solution:

$$\mathbf{G} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \ldots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \ldots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(b) Find the stationary distribution of the process and state any required conditions for its existence.

Solution: We recognize X as a birth-death process with birth rates $\lambda_i = \lambda$ for all $i \geq 0$ and $\mu_i = \mu$ for all $i \geq 1$. Since $\lambda_0 = \lambda > 0$, the process is irreducible and the stationary distribution, as seen in class, is given by:

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \pi_0 = \left(\frac{\lambda}{\mu}\right)^n \pi_0, \quad n \geq 1,$$

where

$$\pi_0 = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right)^{-1} = \left(1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\frac{1}{1 - \lambda/\mu}\right)^{-1} = 1 - \frac{\lambda}{\mu}.$$

A necessary condition for the convergence of the series and, therefore, for the existence of the stationary distribution is $\lambda/\mu < 1 \Leftrightarrow \lambda < \mu$.

(c) Write a short code in R to simulate and plot a path from X with $\lambda = 1$ and $\mu = 3$, starting at $X(0) = 0$ in the time-frame $[0, 100)$. Estimate the average proportion of time that the process spends at state 0. Compare it to the mean proportion of time that the process spends at state 0 in the steady state. *Don't forget to set the seed of the random number generator for reproducibility!*

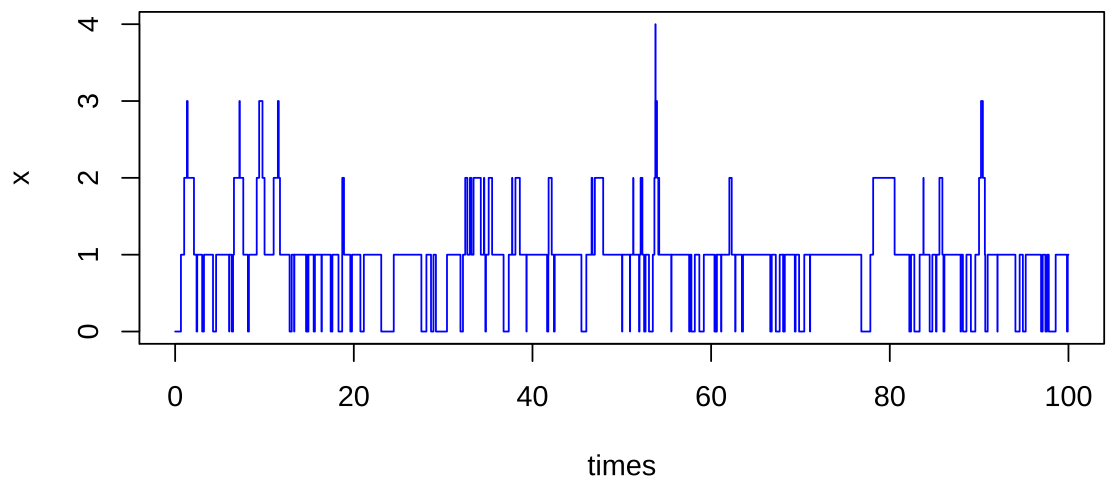
Solution:

```
lambda <- 1
mu <- 3
T <- 100
# Initialize
set.seed(3653)
n <- 0
t <- 0
times <- 0
x_t <- 0
```

```
x <- x_t
# Time spent at state 0
t_0 <- 0
# Simulate the path:
u <- rexp(1, lambda)
t <- u
while (t <= T){
    if (x_t==0){
        x_t = 1
        t_0 <- t_0 + u
    } else{
        x_t <- x_t + sample(c(-1,1), 1, prob =
         c(mu/(lambda+mu),lambda/(lambda+mu)))
    }
    if(x_t==0){
        u <- rexp(1, lambda)
    }else{
        u <- rexp(1, lambda+mu)
    }
    x <- c(x, x_t)
    n <- n+1
    t <- t + u
    times <- c(times,t)
}
x[n+1] <- x[n]
if (x[n+1]==0){
    t_0 <- t_0 - (times[n+1]-T)
}
times[n+1] <- T
n <- n-1

# Plot
plot(times, x, type="s",col="blue", yaxt="n")
axis(side=2, at=0:max(x))

# Estimated proportion of time at 0
t_0/T
```



Using this random seed the estimated proportion of time at state 0 is 0.6240655. The theoretical proportion of time at the steady state is $\pi_0 = 1 - 1/3 = 0.6666667$.

Question 5

Consider a tree-state CTMC X used as a simplified model for weather. The state space is $\{rain, snow, clear\}$. Assume that rainfall lasts, on average, 3 hours at a time. When it snows, the duration, on average, is 6 hours. And the weather stays clear, on average, for 12 hours. Furthermore, changes in weather states are described by the stochastic transition matrix

$$\tilde{\mathbf{P}} = \begin{array}{c} \\ \text{rain} \\ \text{snow} \\ \text{clear} \end{array} \begin{array}{c} \begin{array}{ccc} \text{rain} & \text{snow} & \text{clear} \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1/4 & 3/4 & 0 \end{bmatrix} \end{array}$$

(a) Find the infinitesimal generator for the process.

Solution: For simplicity, we label the states, rain = 1, snow = 2, clear = 3 and let U_i the holding time at state i. Then, $\mathbb{E}[U_1] = 3$ hours, $\mathbb{E}[U_2] = 6$ hours and $\mathbb{E}[U_3] = 12$ hours. Given that $U_i \sim \text{Exp}(-g_{ii})$, the main diagonal of the infinitesimal generator is given by $g_{11} = -1/3$, $g_{22} = -1/6$, $g_{33} = -1/12$.

For $i \neq j$, the (conditional) transition probability from state i to state j, given a jump at time t is $\tilde{p}_i j = -g_{ij}/g_{ii}$. Therefore, $g_{ij} = -g_{ii}\tilde{p}_{ij}$ for all $i = 1, 2, 3$, and $j \neq i$. So the generator for the process is:

$$\mathbf{G} = \begin{bmatrix} -1/3 & 1/6 & 1/6 \\ 1/8 & -1/6 & 1/24 \\ 1/48 & 1/16 & -1/12 \end{bmatrix}$$

(b) Find the one day transition probabilities for the process.

Solution: The one day transition probabilities for the process are given by

$$\mathbf{P}_{24} = e^{24\,\mathbf{G}} = \texttt{expm}(24 * \texttt{G}) = \begin{bmatrix} 0.1615 & 0.3473 & 0.4912 \\ 0.1645 & 0.3529 & 0.4826 \\ 0.1573 & 0.3401 & 0.5025 \end{bmatrix}$$

(c) Find the stationary distribution of the process.

Solution: The stationary distribution $\boldsymbol{\pi}$ is the solution to the system of equations $\mathbf{A}\boldsymbol{\pi} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -1/3 & 1/8 & 1/48 \\ 1/6 & -1/6 & 1/16 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{\pi} = \texttt{solve}(\texttt{A}, \texttt{b}) = \begin{bmatrix} 0.1605 \\ 0.3457 \\ 0.4938 \end{bmatrix}.$$

(d) What is the probability of rain at any given time?

Solution: We can assume that the process is at the steady state (i.e. that weather patterns haven't changed for a long period of time). Therefore, the probability of rain is $\pi_1 = 0.1605$.

(e) Assuming it is currently raining, what is the probability it will be raning at the same time tomorrow?

Solution: $\mathbf{P}[X_{t+24} = 1 | X_t = 1] = p_{11}(24) = 0.1615$.

Question 6

Jobs arrive at a computer according to a Poisson process with intensity λ. The central processor handles them one by one in order of arrival, and each has an exponentially distributed runtime with parameter μ, the runtimes of different jobs being independent of each other and of the arrival process. Let $X(t)$ be the number of jobs in the system (either running or wiating) at time t, where $X(0) = 0$.

(a) Explain why X is a Markov chain, and write down its generator.

Solution: The interarrival times and runtimes are independent and exponentially distributed. The lack-of-memory property of the exponential distribution guarantees that X has the Markov property. The state space is $S = \{0, 1, 2, \ldots\}$ and the generator is

$$\mathbf{G} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(b) Show that a stationary distribution, π exists if and only if $\lambda < \mu$, and find it in this case.

Solution: We recognize **G** as the generator of a birth-death process with birth rates $\lambda_i = \lambda$ for all $i \geq 0$ and death rates $\mu_i = \mu$ for all $i \geq 1$. Therefore, the stationary distribution π exists if and only if

$$1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_0 i - 1}{\mu_1 \mu_2 \cdots \mu_i} = 1 + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^i = \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu} \right)^i < \infty.$$

This is a geometric series, and it is known to converge (see Exercize 4) if and only if $\lambda/\mu < 1 \quad \Leftrightarrow \quad \lambda < \mu$. The stationary distribution is given in question 4b.