INF5620 Final Project

Andreas Nygård Osnes

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This project consider the non-linear diffusion model:

$$\rho u_t = \nabla \cdot (\alpha(u) \nabla u) + \mathbf{f}(\mathbf{x},\ t) \tag{1}$$

with initial condition

$$u(\mathbf{x}, 0) = I(x)$$

and boundary condition

$$\frac{\partial u}{\partial n} = 0$$

ρ is a constant and $\alpha(u)$ is some known known function of u.

a)

We now introduce a Backward Euler discretization in time. This gives us:

$$\frac{u^n - u^{n-1}}{\Delta t} = \frac{1}{\rho}\nabla \cdot (\alpha(u^n)\nabla u^n) + \frac{1}{\rho}\mathbf{f}(\mathbf{x}, t) \tag{2}$$

$$u^n - \frac{\Delta t}{\rho}\nabla \cdot (\alpha(u^n)\nabla u^n) = u^{n-1} + \Delta t \mathbf{f}(\mathbf{x}, t) \tag{3}$$

The variational formulation of the initial condition is found by recognizing that the residual is $R = u(\mathbf{x}, 0) - I(x)$. The error should be orthogonal to each function v, which gives:

$$(u - I, v) = 0$$

$$\underline{\underline{(u, v) = (I, v)}}$$

The variariational formulation for the spatial problem is found by considering equation 3. We define the residual $R = LHS - RHS$ and use the Galerkin method which gives:

$$(R, v) = 0$$

Inserting the expression for R we get:

$$(u^n, v) - \frac{\Delta t}{\rho}(\nabla \cdot [\alpha(u^n)\nabla u^n], v) = \left(u^{n-1}, v\right) + \Delta t (\mathbf{f}(\mathbf{x}, t), v) \tag{4}$$

on the second term on the left-hand side we apply partial integration:

$$(\nabla \cdot \alpha(u^n)\nabla u^n, v) = \int_\Omega [\nabla \cdot \alpha(u^n)\nabla u^n] v d\Omega = [\alpha(u^n)\nabla u^n v] - \int_\Omega \alpha(u^n)\nabla u^n \nabla v d\Omega = -\int_\Omega \alpha(u^n)\nabla u^n \nabla v d\Omega$$

where we have used that the term $\alpha(u^n)\nabla u^n v$ evaluated at the boundaries is zero, by the boundary condition. Equation 4 then becomes:

$$\underline{\underline{(u^n, v) + (\alpha(u^n)\nabla u^n, \nabla v) = \left(u^{n-1}, v\right) + \Delta t (\mathbf{f}(\mathbf{x}, t), v)}} \tag{5}$$

Equation 5 is the variational formulation of the spatial problem.

b)

To formulate a Picard iteration at the PDE level we consider again equation 2. We now replace the argument to α with $u\_$,which symbolizes the previous iteration value of u, to linearize this term and rename $u^n \rightarrow u$. We then get:

$$u = u^{n-1} + \frac{\Delta t}{\rho} \nabla \cdot \left(\alpha \left(u\_ \right) \nabla u \right) + \frac{\Delta t}{\rho} \mathbf{f} \left(\mathbf{x}, t \right)$$

where the iteration is started by setting $u\_ = u^{n-1}$. The analogue to equation 5 in the Picard iteration then becomes:

$$\left(u, v \right) + \frac{\Delta t}{\rho} \left(\alpha \left(u\_ \right) \nabla u, \nabla v \right) = \left(u^{n-1}, v \right) + \Delta t \left(\mathbf{f} \left(\mathbf{x}, t \right), v \right) \tag{6}$$

c)

If we restrict equation 6 to only a single iteration, it simply becomes:

$$\left(u^n, v \right) + \frac{\Delta t}{\rho} \left(\alpha \left(u^{n-1} \right) \nabla u^n, \nabla v \right) = \left(u^{n-1}, v \right) + \Delta t \left(\mathbf{f} \left(\mathbf{x}, t \right), v \right)$$

This is easily implemented in a python program by using the FEniCS software.

g)

The group finite element method consists of approximating a non-linear term $f\left(u\right)$ by:

$$f\left(u\right) \approx \sum_{j=0}^{N_n} f\left(u_j\right) \phi_j\left(x\right)$$

where u_j is the value of u at node x_j. This is unfinished...