Opinion dynamics on non-sparse networks with community structure

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- Update opinions according to the recursion

$$\mathbf{R}_i^{(k)} = c\sum_{j=1}^{n} C_{ij}\mathbf{R}_j^{(k-1)} + \mathbf{W}_i^{(k)} + (1-c-d)\mathbf{R}_i^{(k-1)},$$

where $\{\mathbf{W}_i^{(k)} : k \geq 0\}$ are i.i.d. and $0 < c+d \leq 1$.

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$$\boldsymbol{\mathcal{R}}_i^{(k)} = \sum_{t=0}^{k-1}(1-c-d)^t \mathbf{W}_i^{(k-t)} + 1(k \geq 2) \sum_{t=1}^{k-1} \sum_{s=1}^{t} a_{s,t} (M^s \bar{W})_{J_i \bullet}$$
$$+ \sum_{s=1}^{k} a_{s,k} (M^s \bar{R})_{J_i \bullet} + (1-c-d)^k \mathbf{R}_i^{(0)},$$

for $k \geq 1$ and $i \in V_n$, where $a_{s,t} = \binom{t}{s}(1-c-d)^{t-s}c^s$.

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- The media signals are the only ones that persist in the limit, meaning they determine polarization in the system!

Theorem (A., Olvera-Cravioto '24)

Suppose $\theta_n \geq (6H\Lambda_n)^2 \Delta_n \log n$. Then, there exists a constant $\Gamma < \infty$ such that

$$\sup_{k\geq 0} \mathbb{E}_n \left[\left\| R^{(k)} - \mathcal{R}^{(k)} \right\|_\infty \right] \leq \Gamma \left(\sqrt{\frac{\log n}{\theta_n}} + \max_{1\leq r,s\leq K} \left| \frac{\pi_s^{(n)} \pi_r - \pi_s \pi_r^{(n)}}{\pi_r^{(n)} \pi_s} \right| \right).$$

Moreover, for any sequence $\theta_n \to \infty$, we have

$$\sup_{k\geq 0} \max_{i\in V_n} \mathbb{E}_n \left[\left\| \mathbf{R}_i^{(k)} - \boldsymbol{\mathcal{R}}_i^{(k)} \right\|_1 \right] \xrightarrow{P} 0, \quad n \to \infty.$$

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- *Intuition*: the average number of neighbors that any vertex has grows with θ_n. The larger the number of neighbors, the more their aggregate contributions behave as the average opinion.
- Since the rows in the limiting process $\{\mathcal{R}^{(k)} : k \geq 1\}$ are independent of each other, Theorem 1 yields that the trajectories of the process $\{R^{(k)} : k \geq 0\}$ are asymptotically independent, i.e., the system exhibits *propagation of chaos*.