STOR 113 - Decision Models for Business and Economics

Panagiotis Andreou

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"Mathematics is not about numbers, equations, computations, or algorithms; it is about understanding."

William P. Thurston

Contents

Preface

This file serves as a collection of lecture notes for the STOR 113 course taught at the University of North Carolina at Chapel Hill. These notes are based on the content of the book *Finite Mathematics and Applied Calculus*, by Waner and Costenoble, as well as the hand-written notes of Prof. Nilay Argon from the Fall 2023 semester. The objective is to present the relevant theory in a concise manner and suggest a few problems with their solutions, so that students can first practice on their own and then have a reference point for how they did. This is only a first draft of the final document. Any mistakes in this file are solely the responsibility of the author.

Panagiotis Andreou
Chapel Hill, NC
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Chapter 1

Linear Functions

In this chapter, we will study the simplest type of functions, namely linear functions. This serves as a gentle introduction to the more complicated types of functions we will encounter down the road. We begin with some basic definitions and then provide several examples inspired mostly from economics.

1.1 Review

Definition 1.1. A linear function is one that can be written in the form $f(x) = mx+b$, where m is the **slope** and b is the **intercept**.

Given two points on the line, the slope is obtained by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_1 - y_2}{x_1 - x_2}.$$

- If x changes by Δt units, then y changes by $m\Delta t$ units.
- Alternatively, if x changes by Δx units, then y changes by $m\Delta x$ units.

Example 1.1. Suppose that the cost C in dollars of an x-mile taxi ride is a linear function of x:

$$C = 3.5 + 2x.$$

Then,

- The slope (2) represents the variable cost per mile.
- The intercept (3.5) represents the fixed cost or the cost of starting a ride.

Exercise 1.1. Two points $(x_1, y_1) = (32, 0)$ and $(x_2, y_2) = (100, 212)$ are provided. Find the slope m and the intercept b of the line.

Solution.

1. Calculate the slope m:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{212 - 0}{100 - 32} = \frac{212}{68} = 3.1176$$

2. Using the point-slope form of the line equation $y = mx + b$, and substituting for m and any point:

$$y = 3.1176x + b$$

$$0 = 3.1176(32) + b$$

$$b = -99.7632$$

Therefore, the equation of the line is

$$y = 3.1176x - 99.7632.$$

□

Exercise 1.2. Suppose that $(1, 3)$ and $(7, 2)$ are on the line. Find the equation of the line.

Solution. When they ask us to find the equation of a straight line, we have to find the slope m and the intercept b. We start by computing the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{3-2}{1-7} = \frac{1}{-6} = -\frac{1}{6},$$

so the equation becomes

$$y = -\frac{1}{6}x + b.$$

Now, the point (1,3) is on the line, so the equation yields

$$3 = -\frac{1}{6} \cdot 1 + b \Leftrightarrow 3 + \frac{1}{6} = b \Leftrightarrow b = \frac{19}{6}.$$

Therefore, the equation of the line is

$$y = -\frac{1}{6}x + \frac{19}{6}.$$

□

1.2 Linear Functions in Economics

Definition 1.2. A *cost function* specifies the cost C as a function of the number of items x. If $C(x)$ is in the form of $C(x) = mx + b$, then it is called a linear cost function, where m is the marginal cost and b is the fixed cost.

Definition 1.3. *Revenue* is the total payment received, also called gross proceeds. If each item is sold at a price of a, then the revenue function is $R(x) = ax$, which is the revenue function of the revenue.

Definition 1.4. *Profit* is what remains of the revenue after costs are subtracted.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$P(x) = R(x) - C(x)$$

- If $P < 0$, we say that this is a loss.
- If $P = 0$, we say that we break even $(R(x) = C(x))$.
- If $P > 0$, we say that we make profit.

Exercise 1.3 (Break-Even Analysis)**.** Your college newspaper, *The Collegiate Investigator*, has fixed production costs of \$70 per edition and marginal printing and distribution costs of 40 cents per copy. *The Collegiate Investigator* sells for 50 cents per copy.

a. Write down the associated cost, revenue, and profit functions.

b. What profit (or loss) results from the sale of 500 copies of the newspaper?

c. How many copies should be sold to break even?

Solution. a. The three functions are listed below:

- Cost: $C(x) = 70 + 0.4x$, where x is the number of copies sold.
- Revenue: $R(x) = 0.5x$.
- Profit: $P(x) = R(x) - C(x) = 0.5x - (70 + 0.4x) = 0.5x - 70 - 0.4x = 0.1x - 70$.

b. We evaluate the profit function at $x = 500$, so we find a loss of $P(500) = 0.1(500) - 70 = 50 - 70 = -20$ dollars.

c. We solve the equation:

$$P(x) = 0 \Leftrightarrow 0.1x - 20 = 0 \Leftrightarrow x = 700 \text{ copies.}$$

□

Exercise 1.4 (Break-Even Analysis)**.** The Oliver Company plans to market a new product. Based on its market studies, Oliver estimates that it can sell up to 5,500 units in 2005. The selling price will be \$2 per unit. Variable costs are estimated to be 40% of total revenue. Fixed costs are estimated to be \$6,000 for 2005. How many units should the company sell to break even?

Solution. Let x be the number of units sold. The cost function is $C(x) = 6,000 + mx$, where m is the variable cost which is $\frac{40}{100}R(x) = \frac{40}{100} \cdot 2x = 0.8x$. The revenue function is $R(x) = 2x$. Therefore, $C(x) = 6,000 + 0.8x$. The profit function is

$$P(x) = R(x) - C(x) = 2x - (6,000 + 0.8x).$$

We want x so that $P(x) = 0$, which gives us $2x = 6,000 + 0.8x$. Simplifying, $1.2x = 6,000$. Solving for x, we find $x = 5,000$ units. □

1.3 Demand and Supply Functions

- A **demand function** expresses demand q (the amount demanded) as a function of the unit price. If the demand function is in the form

$$q(p) = mp + b,$$

  then it is called a linear demand function. Here, m is the change in demand per unit change in price (generally negative).

- A **supply function** expresses supply q (the number of items a supplier is willing to make available) as a function of the price p. If q is in the form

$$q(p) = mp + b,$$

  then it is called a linear supply function, and m is typically positive.

- Demand and supply are said to be in **equilibrium** when demand equals supply. The corresponding P and q are called **equilibrium price** and equilibrium demand, respectively.

Exercise 1.5 (Equilibrium Price: Cell Phones)**.** Worldwide quarterly sales of Nokia cell phones were approximately $q = -p + 156$ million phones when the wholesale price p was \$$p$.

1. If Nokia was prepared to supply $q = 4p - 394$ million phones per quarter at a wholesale price of \$$p$, what would have been the equilibrium price?
2. The actual wholesale price was \$105 in the fourth quarter of 2004. Estimate the projected shortage or surplus at that price.

Solution.

1. The demand and supply functions are as follows:

$$\text{Demand: } q = -p + 156$$

$$\text{Supply: } q = 4p - 394$$

Setting demand equal to supply, we get

$$-p + 156 = 4p - 394 \Leftrightarrow p = 110\text{dollars}.$$

2. If $p < 110$, we have shortage; otherwise, surplus. Since $p = 105 < 110$, we have shortage. Plug in $p = 105$ into the demand and supply, and then take the difference:

$$\begin{aligned} \text{Demand: } & q = -p + 156 = -105 + 156 = 51 \text{ million phones} \\ \text{Supply: } & q = 4p - 394 = 4(105) - 394 = 420 - 394 = 26 \text{ million phones} \\ \text{Shortage: } & 51 - 26 = 25 \text{ million phones.} \end{aligned}$$

□

1.4 Systems of Two Linear Equations

A linear equation in two unknowns is an equation of the form

$$ax + by = c,$$

where a, b and c are given real numbers, and x and y are unknowns.

Exercise 1.6. Solve for x and y:

$$x - y = 0 \quad (\text{Eqn. } 1) \tag{1.1}$$

$$x + y = 4 \quad (\text{Eqn. } 2) \tag{1.2}$$

Solution. Idea 1: Use one equation to express one of the unknowns in terms of the other. Then use the other equation to solve for a single unknown. This is known as the *the substitution method*.

$$\begin{aligned} &\text{Eqn } 1 \Leftrightarrow x - y = 0 \Leftrightarrow y = x \\ &\text{Eqn } 2 \Leftrightarrow x + x = 4 \Leftrightarrow 2x = 4 \Leftrightarrow x = 2 \end{aligned}$$

Therefore, $y = x = 2$.

Idea 2: Multiply each equation by a constant and add them up to get rid of one unknown.

$$x - y = 0 \Leftrightarrow y = x$$
$$x + y = 4 \Leftrightarrow y = 4 - x$$

Add them together:

$$2x = 4 \Leftrightarrow x = 2$$

Using $x = 2$ in Eqn 1, we get:

$$2 - y = 0 \Leftrightarrow y = 2.$$

□

Exercise 1.7 (Voting)**.** An appropriations bill passed the U.S. House of Representatives with 49 more members voting in favor than against. If all 435 members of the House voted either for or against the bill, how many voted in favor and how many voted against?

Solution. Let x be the number of votes in favor and y be the number of votes against. We can set up the following system of equations:

$$x + y = 435 \quad \text{(The total number of members)}$$
$$x - y = 49 \quad \text{(The difference in votes)}$$

To find the values of x and y, we can use the method of elimination:

$$x + y = 435 \quad \text{(Equation 1)}$$
$$x - y = 49 \quad \text{(Equation 2)}$$

Adding the two equations together to eliminate y, we get:

$$2x = 484 \Leftrightarrow x = \frac{484}{2} \Leftrightarrow x = 242 \quad \text{(Number of votes in favor).}$$

Substituting x back into Equation 1, we find

$$242 + y = 435 \Leftrightarrow y = 435 - 242 \Leftrightarrow y = 193 \quad \text{(Number of votes against).}$$

Therefore, 242 members voted in favor of the bill, and 193 voted against it. □

1.5 Extra Problems

Exercise 1.8. Given two points on a line $(3, 7)$ and $(5, 11)$, determine the equation of the line in the slope-intercept form $y = mx + b$.

Solution. First, calculate the slope m:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 7}{5 - 3} = \frac{4}{2} = 2.$$

Recall the point-slope form

$$y - y_1 = m(x - x_1).$$

Using the slope $m = 2$ and point $(3, 7)$, substitute above to get

$$y - 7 = 2(x - 3).$$

Simplify and solve for y:

$$y = 2x - 6 + 7 = 2x + 1.$$

Therefore, the equation of the line is $y = 2x + 1$. □

Exercise 1.9. The cost to produce x items is represented by the linear function $C(x) = 15x + 300$. Determine the variable cost per item and the fixed cost.

Solution. The cost function $C(x) = 15x + 300$ is in the form $C(x) = mx + b$, where:

- $m = 15$ represents the variable cost per item.
- $b = 300$ represents the fixed cost.

□

Exercise 1.10. A company sells x products at a price of \$25 each. Write the revenue function and calculate the profit if the cost function is $C(x) = 10x + 200$.

Solution. The revenue function $R(x)$ is given by the number of items sold times the price per item:

$$R(x) = 25x.$$

The profit function $P(x)$ is the revenue minus the cost:

$$P(x) = R(x) - C(x) = 25x - (10x + 200) = 15x - 200.$$

Therefore, the profit function is $P(x) = 15x - 200$. □

Exercise 1.11. Determine the number of units x that must be sold to break even if the cost function is $C(x) = 20x + 500$ and the revenue function is $R(x) = 30x$.

Solution. Set the profit function $P(x) = R(x) - C(x)$ equal to zero to find the break-even point:

$$0 = 30x - (20x + 500).$$

Simplify and solve for x:

$$0 = 10x - 500 \Leftrightarrow 10x = 500 \Leftrightarrow x = 50.$$

Thus, 50 units must be sold to break even. □

Exercise 1.12. Two companies have different cost structures for producing items: Company A has a cost function $C_A(x) = 12x + 150$ and Company B has a cost function $C_B(x) = 15x + 100$. Find the number of items x at which both companies have the same costs.

Solution. Set the cost functions equal to find the number of items at which costs are the same:

$$12x + 150 = 15x + 100.$$

Solve for x:

$$150 - 100 = 15x - 12x \Leftrightarrow 50 = 3x \Leftrightarrow x = \frac{50}{3} \approx 16.67.$$

Therefore, at approximately 16.67 items, both companies have the same costs. □

Chapter 2

Non-Linear Models

2.1 Quadratic Functions

A quadratic function is a function that can be written in the form:

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are fixed numbers and $a \neq 0$. The graph of a quadratic function is a parabola. Its main elements are:

- *y-intercept:* The parabola intersects the y-axis, i.e., where $x = 0$. We have $f(0) = c$.
- *Vertex:* The turning point of the parabola. For $a > 0$, the vertex is at the minimum point. For $a < 0$, the vertex is at the maximum point. The vertex can be found using:

$$\text{x-coordinate} = \frac{-b}{2a} \qquad \text{y-coordinate} = f\left(\frac{-b}{2a}\right).$$

- *X-intercepts:* These are points where $f(x) = 0$ which implies $ax^2 + bx + c = 0$.

  Step 1: Find the discriminant $\Delta = b^2 - 4ac$.

  Step 2: If $\Delta = 0$, there is a single x-intercept, which is the vertex:

$$x = \frac{-b}{2a}$$

  For $\Delta < 0$, there are no x-intercepts, which means the parabola does not intersect the x-axis.

  For $\Delta > 0$, there are two x-intercepts:

– The first x-intercept $x = \frac{-b-\sqrt{\Delta}}{2a}$

– The second x-intercept $x = \frac{-b+\sqrt{\Delta}}{2a}$

Exercise 2.1. Consider the quadratic function

$$f(x) = -x^2 - 40x + 500,$$

where $a = -1$, $b = -40$, and $c = 500$. Find the vertex, the y-intercept, and x-intercepts (if any).

Solution.

y-intercept: For $x = 0$ we have $f(0) = 500$.

Vertex: The x-coordinate of the vertex is given by

$$x = \frac{-b}{2a} = \frac{-(-40)}{2(-1)} = \frac{40}{-2} = -20.$$

The y-coordinate of the vertex is found by evaluating $f(-20)$:

$$f(-20) = -(-20)^2 - 40(-20) + 500 = -400 + 800 + 500 = 900.$$

Therefore, the vertex is at $(-20, 900)$.

x-intercepts: Step 1: Compute the discriminant

$$\Delta = b^2 - 4ac = (-40)^2 - 4(-1)(500) = 1600 + 2000 = 3600.$$

Since $\Delta > 0$, there are 2 x-intercepts.

Step 2: Calculate the x-intercepts using

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-40) \pm \sqrt{3600}}{2(-1)}.$$

This simplifies to

$$x = \frac{40 \pm 60}{-2},$$

which gives us two solutions for the x-intercepts:

$$x = \frac{40 + 60}{-2} = -50, \quad x = \frac{40 - 60}{-2} = 10.$$

□

Exercise 2.2 (Website Profit)**.** You operate a gaming website, `www.mudbeast.net`, where users must pay a small fee to log on. When you charged \$2, the demand was

280 log-ons per month. When you lowered the price to \$1.50, the demand increased to 560 log-ons per month.

1. Construct a linear demand function for your website and hence obtain the monthly revenue R as a function of the log-on fee z.
2. Your Internet provider charges you a monthly fee of \$30 to maintain your site. Express your monthly profit P as a function of the log-on fee z, and hence determine the log-on fee you should charge to obtain the largest possible monthly profit.
3. What price to charge to break even?

Solution.

1. Let q be the demand and z the log-on fee. Two points on the line are given: $(2, 280)$ and $(1.50, 560)$. The slope m of the demand function is calculated by

   $$m = \frac{\Delta q}{\Delta z} = \frac{280 - 560}{2 - 1.5} = \frac{-280}{0.5} = -560 \text{ log-ons per dollar.}$$

   The demand function q is

   $$q = mz + b.$$

   To find b, use one of the points:

   $$280 = -560 \cdot 2 + b \Leftrightarrow b = 280 + 2(560) = 1400.$$

   Thus, the demand function is

   $$q = -560z + 1400.$$

   The monthly revenue R is

   $$R = zq = z(-560z + 1400) = -560z^2 + 1400z.$$

2. The profit P is

   $$P = R - C = -560z^2 + 1400z - 30.$$

   For $a < 0$, the parabola is facing downward, so the largest possible monthly profit is achieved at the vertex. The vertex of the parabola occurs at $z = \frac{-b}{2a}$:

   $$z = \frac{-1400}{2(-560)} = \frac{1400}{1120} = \frac{5}{4} \text{ dollars.}$$

The profit at this price point is

$$P\left(\frac{5}{4}\right) = -560\left(\frac{5}{4}\right)^2 + 1400\left(\frac{5}{4}\right) - 30 = \$845 \text{ per month.}$$

3. To find the break-even point, we find the value of z such that $P(z) = 0$. This means we need to find the x-intercepts of $P(z)$.

   Step 1: Compute the discriminant Δ of the quadratic equation.

   $$\Delta = b^2 - 4ac = (1400)^2 - 4(-560)(-30) = 1,892,800.$$

   Since $\Delta > 0$, there are two x-intercepts.

   Step 2: Calculate the x-intercepts using the quadratic formula:

   $$z = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1400 \pm \sqrt{1,892,800}}{2(-560)}.$$

   After simplifying, we find the approximate x-intercepts to be around 0.02 and 2.48. Therefore, if $0.02 < z < 2.48$, the profit is positive, and if $z = 0.02$ or $z = 2.48$, we break even. If $z < 0.02$ or $z > 2.48$, there is a loss.

□

2.2 Exponential Functions

2.2.1 Review of exponents

Say a is any real number and n is any positive integer. Then, a^n means $a \cdot a \cdot \ldots \cdot a$ (n times), where a is the base and n the exponent. In addition, suppose that a is non-zero. Then,

$$a^{-n} = \frac{1}{a^n} = \frac{1}{a} \cdot \frac{1}{a} \cdot \ldots \cdot \frac{1}{a} \quad \text{(n times)}.$$

Identities for exponents:

1. $a^m a^n = a^{m+n}$ e.g., $3^2 \cdot 3^4 = 3^6$
2. $\frac{a^m}{a^n} = a^{m-n}$ $(a \neq 0)$ e.g., $\frac{4^5}{4^2} = 4^3$
3. $(a^n)^m = a^{nm}$ e.g., $(2^2)^3 = 2^6$
4. $(ab)^n = a^n b^n$ e.g., $\left(\frac{4}{3}\right)^2 = 4^2\left(\frac{1}{3}\right)^2 = \frac{16}{9}$

2.3 Exponential Functions and Models

An exponential function has the form

$$f(x) = Ab^x,$$

where A and b are constants with $A \neq 0$, $b > 0$, and $b \neq 1$. Exponential functions are good models for population growth, radioactive decay, growth of financial investments.

Properties:

1. Suppose x increases by Δx:

$$f(x + \Delta x) = Ab^{x+\Delta x} = Ab^x \cdot b^{\Delta x} = f(x) \cdot b^{\Delta x}.$$

   If x increases by 1 unit, then $f(x)$ is multiplied by b units. Exponential functions are "multiplicative", whereas linear functions are additive.

2. $f(0) = Ab^0 = A$ is the y-intercept.
3. If $b > 1$, $y = f(x)$ is said to grow exponentially in x. e.g., $f(x) = 3^x$.
4. If $0 < b < 1$, $y = f(x)$ is said to decay exponentially in x. e.g., $f(x) = \left(\frac{1}{3}\right)^x$.

Exercise 2.3. Given the points $(1, 3)$ and $(3, 6)$ on an exponential curve, find the equation of the form $f(x) = Ab^x$. What are A and b?

Solution. The two points must satisfy the equation:

$$\begin{aligned} f(1) &= 3 \Leftrightarrow 3 = Ab^1 \Leftrightarrow Ab = 3, \\ f(3) &= 6 \Leftrightarrow 6 = Ab^3. \end{aligned}$$

Use these two equations to reduce the number of unknowns (A and b) to one. We can either use substitution or division.

1) Substitution: Substitute $A = \frac{3}{b}$ into the second equation:

$$\frac{3}{b} b^3 = 6 \Leftrightarrow 3b^2 = 6 \Leftrightarrow b^2 = 2 \Leftrightarrow b = \sqrt{2}.$$

Then $A = \frac{3}{\sqrt{2}}$, and the function is

$$f(x) = \frac{3}{\sqrt{2}} (\sqrt{2})^x = 3\sqrt{2} \left(\frac{1}{\sqrt{2}} \right)^x.$$

2) Division: Divide the second equation by the first equation:

$$\frac{Ab^3}{Ab} = \frac{6}{3} \Leftrightarrow b^2 = 2 \Leftrightarrow b = \sqrt{2}.$$

Then, using one of the equations to find A:

$$Ab = 3 \Leftrightarrow A(\sqrt{2}) = 3 \Leftrightarrow A = \frac{3}{\sqrt{2}}.$$

$\square$

Exercise 2.4. A bacteria culture starts with 1,000 bacteria and doubles in size every 3 hours. Find an exponential model for the size of the culture as a function of time t in hours, and use the model to predict how many bacteria there will be after 2 days.

Solution. Let $f(t)$ be the size of the culture after t hours. The model is $f(t) = Ab^t$ where A is the initial amount, and b is the growth rate. Given that $f(0) = A = 1000$ and the culture doubles every 3 hours, we have $f(3) = 2000$. So, $f(3) = 1000b^3 = 2000$ leads to $b^3 = 2$ and hence $b = \sqrt[3]{2}$. Thus, the model is

$$f(t) = 1000\left(\sqrt[3]{2}\right)^t.$$

For 2 days, or 48 hours, we have

$$f(48) = 1000\left(\sqrt[3]{2}\right)^{48}.$$

Calculating $f(48)$, we get $f(48) = 1000\left(\sqrt[3]{2}\right)^{48} = 65,536,000$ bacteria. $\square$

2.3.1 Application of Exponential Functions: Compound Interest

Suppose you invest \$100 in an investment account with an annual yield of 10%. The interest is reinvested at the end of every year unless you withdraw the money. Let t be number of years since the initial investment, and $A(t)$ the value of your investment at time t.

$$A(t) = 100(1.1)^t.$$

This is of the general form

$$A(t) = P(1+r)^t.$$

where P is the present value, r is the annual interest rate. If the interest is reinvested (compounded) n times per year, then the future value after t years will be

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Example 2.1. A bank offers 4.9% interest on a savings account.

- Interest is reinvested monthly $\rightarrow n = 12$.
- \$5,000 is invested.
- How much does this investment worth in 5 years?

Given $r = 0.049$ and $P = 5000$, we compute

$$A(5) = 5000\left(1 + \frac{0.049}{12}\right)^{12\cdot 5} = 5000\,(1.00366)^{60} \approx \$6225.40.$$

Here, each value of $A(t)$ is calculated according to $A(t) = 100 \cdot (1.1)^t$.

t	0	1	2	3	...
$A(t)$	100	110	121	133.1	...

Table 2.1: Investment Growth Table

Example 2.2 (Continuous Compound Interest)**.** A bank offers 4.9% interest on a savings account:

- Interest is reinvested monthly ($n = 12$).
- \$5,000 is invested.
- How much is this investment worth in 5 years?

Given $r = 0.049$ and $P = 5000$, we find

$$A(5) = 5000\left(1 + \frac{0.049}{12}\right)^{12\cdot 5} = 5000(1.00366)^{60} \approx \$6225.40.$$

As n gets larger, $A(t)$ also gets larger. So we prefer n as a larger number. Let's make n go to infinity. Suppose we invest \$1 for 1 year at 100% interest rate compounded n times during the year. Then, the future value is

$$A(1) = 1\left(1 + \frac{1}{n}\right)^n$$

As $n \to \infty$, $\left(1 + \frac{1}{n}\right)^n$ approaches e (Euler's number), approximately 2.71828. What if we invest P dollars at an annual interest rate of r as number of compounding goes to infinity? This is continuous compounding, given by the formula

$$A(t) = Pe^{rt}.$$

2.4 Logarithmic Functions and Models

The base b logarithm of x, $\log_b x$, is the power to which we need to raise b in order to get x. The logarithm $\log_b x = y$ means $b^y = x$ for $x > 0$, $b > 0$, and $b \neq 1$.

Example 2.3.

$$\begin{aligned}
\log_3 9 = 2 \quad &\Leftrightarrow \quad 3^2 = 9, \\
\log_5 125 = 3 \quad &\Leftrightarrow \quad 5^3 = 125, \\
\log_{10} 0.01 = -2 \quad &\Leftrightarrow \quad 10^{-2} = \frac{1}{100} = 0.01, \\
\log_{25} 5 = \frac{1}{2} \quad &\Leftrightarrow \quad 25^{\frac{1}{2}} = 5, \\
\log_5 1 = 0 \quad &\Leftrightarrow \quad 5^0 = 1.
\end{aligned}$$

The property $\log_b 1 = 0$ holds for any base $b > 1$.

Common Logarithm: Base $= 10$

$$\log_{10} x = \log x$$

Natural Logarithm: Base $= e$

$$\log_e x = \ln x$$

2.4.1 Logarithmic Identities

Let $a \neq 1$, $a > 0$, $b > 0$, $x > 0$, $y > 0$.

1. $\log_b(xy) = \log_b x + \log_b y$; e.g., $\log_2 64 = \log_2(32 \cdot 2) = \log_2 32 + \log_2 2 = 5 + 1 = 6$.
2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$; e.g., $\log_3 \left(\frac{1}{4}\right) = \log_3 1 - \log_3 4$.
3. $\log_b(x^r) = r \log_b x$; e.g., $\log_5 5^3 = 3 \log_5 5$.
4. $\log_b b = 1$.
5. $\log_b \left(\frac{1}{x}\right) = \log_b(x^{-1}) = -\log_b x$.
6. Change of base formula:

$$\log_b x = \frac{\log_a x}{\log_a b} = \frac{\ln x}{\ln b} = \frac{\log_2 x}{\log_2 b}$$

   e.g., $\log_7 1 = \frac{\ln 1}{\ln 7} = \frac{0}{\ln 7} = 0$.

7. $\log_b(b^x) = x \log_b b = x$; e.g., $\log_3 3^7 = 7$.

8. $b^{\log_b x} = x$; e.g., $4^{\log_4 5} = 5$ or $e^{\ln x} = x$.

Exercise 2.5. How long will it take a \$500 investment to be worth \$700 if it is continuously compounded at 10% per year?

Solution. The formula for continuous compounding is $A(t) = Pe^{rt}$. Given are:

$$r = 0.1, P = 500, A(t) = 700.$$

We want to solve for t:

$$700 = 500e^{0.1t}$$

Divide both sides by 500:

$$\frac{700}{500} = e^{0.1t}$$

Take the natural logarithm of both sides:

$$\ln\left(\frac{700}{500}\right) = \ln(e^{0.1t})$$

Simplify using the property $\ln(e^x) = x$:

$$\ln\left(\frac{7}{5}\right) = 0.1t$$

Solve for t:

$$t = \frac{\ln(7/5)}{0.1}$$

This yields:

$$t \approx 3.36 \text{ years.}$$

□

2.5 Extra Problems

Exercise 2.6. Given the quadratic function

$$f(x) = 2x^2 - 8x + 6,$$

find the vertex, axis of symmetry, y-intercept, and x-intercepts.

Solution. The quadratic function is in the standard form $f(x) = ax^2 + bx + c$ where $a = 2$, $b = -8$, and $c = 6$.

Vertex and Axis of Symmetry: The x-coordinate of the vertex is given by:

$$x = \frac{-b}{2a} = \frac{-(-8)}{2 \cdot 2} = 2.$$

Substituting $x = 2$ into $f(x)$ gives the y-coordinate of the vertex:

$$f(2) = 2(2)^2 - 8(2) + 6 = 8 - 16 + 6 = -2.$$

Thus, the vertex is $(2, -2)$ and the axis of symmetry is $x = 2$.

Y-intercept:

$$f(0) = 2(0)^2 - 8(0) + 6 = 6.$$

X-intercepts: Solve $f(x) = 0$:

$$2x^2 - 8x + 6 = 0 \Leftrightarrow x^2 - 4x + 3 = 0 \Leftrightarrow (x-3)(x-1) = 0 \Leftrightarrow x = 3 \text{ or } x = 1.$$

The x-intercepts are $(1, 0)$ and $(3, 0)$. □

Exercise 2.7. A company sells q items at a price of $(200 - q)$ dollars each. Find the price that maximizes revenue and determine the maximum revenue.

Solution. The revenue function is given by

$$R(q) = q(200 - q) = 200q - q^2.$$

This is a quadratic function in standard form where $a = -1$, $b = 200$, and $c = 0$.

Maximum Revenue: The revenue is maximized at the vertex of the parabola. The q-coordinate of the vertex (quantity for maximum revenue) is:

$$q = \frac{-b}{2a} = \frac{-200}{2 \cdot (-1)} = 100.$$

Substitute $q = 100$ into the revenue function to find the maximum revenue:

$$R(100) = 100(200 - 100) = 10000.$$

The maximum revenue is \$10,000 when the price per item is $200 - 100 =$ \$100. □

Exercise 2.8. A ball is thrown upward with an initial velocity of 30 m/s from a height of 2 meters. The height $h(t)$ of the ball at time t seconds is given by

$$h(t) = -4.9t^2 + 30t + 2.$$

Find the maximum height reached by the ball and the time it takes to reach the ground.

Solution. The maximum height is given by the vertex of the parabola:

$$t = \frac{-b}{2a} = \frac{-30}{2 \cdot (-4.9)} \approx 3.06 \text{ seconds.}$$

Substituting $t \approx 3.06$ into $h(t)$ gives:

$$h(3.06) = -4.9(3.06)^2 + 30(3.06) + 2 \approx 47.18 \text{ meters.}$$

For the time to reach the ground, solve $h(t) = 0$:

$$-4.9t^2 + 30t + 2 = 0.$$

Using the quadratic formula, $t \approx 6.14$ seconds. □

Exercise 2.9. A certain radioactive substance decays at a rate of 5% per hour. Initially, there are 100 grams of the substance. Model the decay and determine the amount remaining after 10 hours.

Solution. Let $f(t)$ be the amount of substance remaining at time t. The decay model is

$$f(t) = 100\,(1 - 0.05)^t = 100(0.95)^t.$$

After 10 hours, the amount remaining is

$$f(10) = 100(0.95)^{10} \approx 59.87 \text{ grams.}$$

□

Exercise 2.10. A population of bacteria doubles every 3 hours. If the initial population is 200, find the number of bacteria after 24 hours.

Solution. The exponential growth model is:

$$f(t) = 200 \cdot 2^{\frac{t}{3}}.$$

After 24 hours:

$$f(24) = 200 \cdot 2^{\frac{24}{3}} = 200 \cdot 2^8 = 51200.$$

There will be 51,200 bacteria after 24 hours. □

Chapter 3

Derivatives

3.1 Average Rate of Change

The average rate of change of $f(x)$ over the interval $[a, b]$ is

$$\frac{\text{change in } f}{\text{change in } x} = \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a},$$

which is the *slope* of the secant line that passes through $(a, f(a))$ and $(b, f(b))$.

Example 3.1. Let $f(x) = x^2 - 3x + 5$. What is the average rate of change over $[1, 3]$?

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 3(3) + 5 - (1 - 3 + 5)}{2} = \frac{2}{2} = 1$$

As h becomes closer to zero, $f(3 + h) - f(3)$ over h seems to be getting closer to 8. Therefore, $f(a + h) - f(a)$ over h converges to the number that is called the instantaneous rate of change of f at a. This process of letting h get closer and closer to zero is called *taking the limit* as $h \to 0$, denoted by $\lim_{h\to 0}$. If the average rate of change approaches a fixed number for both positive and negative values of h, then this number is the instantaneous rate of change of f at a.

Leibniz's Notation

The average rate of change is given by

$$\frac{\Delta f}{\Delta x}.$$

The instantaneous rate of change is given by

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

For a function $f(x)$, its derivative $f'(x)$ is defined as

$$f'(x) = \frac{df}{dx}.$$

If f is a function of x, its derivative function f' is the function whose value $f'(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.1}$$

3.2 Algebraic Viewpoint of the Derivative

We will find the exact value of the derivative using the definition (3.1).

Example 3.2. For the parabolic function $f(x) = x^2$, we compute:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x. \end{aligned}$$

Thus, the derivative of $f(x) = x^2$ is $f'(x) = 2x$, which is a linear function.

Example 3.3. For a general quadratic function $f(x) = ax^2 + bx + c$, we compute:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\ &= \lim_{h \to 0} \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 - bx - c}{h} \\ &= \lim_{h \to 0} \frac{2axh + ah^2 + bh}{h} = \lim_{h \to 0} (2ax + ah + b) \\ &= \lim_{h \to 0} (2ax + b) = 2ax + b. \end{aligned}$$

Thus, the derivative of the general quadratic function is the linear function $f'(x) = 2ax + b$. We can use this to compute the vertex in an easier way than we did above. The key realization is that the vertex occurs at the point x satisfying $f'(x) = 0$. Solving this equation, we get

$$2ax + b = 0 \Leftrightarrow x = -\frac{b}{2a},$$

which coincides with the our calculations in a previous section.

Exercise 3.1. Find the derivative function $f'(x)$ of $f(x) = \frac{1}{x}$ and evaluate it at $a = 1$.

Solution. Use the definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \lim_{h\to 0} \frac{-1}{x^2+hx} = -\frac{1}{x^2}.$$

Now, evaluate $f'(x)$ at $x=1$:

$$f'(1) = \frac{-1}{1^2} = -1.$$

□

3.3 Derivatives of Powers, Sums, and Constant Multipliers

Finding derivatives using the limit definition is tedious. There are some simple rules that we can use to find derivatives easily.

The power rule: If n is any constant and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Examples:

$$\begin{aligned}
f(x) &= x^2 && \Rightarrow\ f'(x) = 2x^{2-1} = 2x,\\
f(x) &= x^5 && \Rightarrow\ f'(x) = 5x^{5-1} = 5x^4,\\
f(x) &= x && \Rightarrow\ f'(x) = 1x^{1-1} = 1,\\
f(x) &= 1 && \Rightarrow\ f'(x) = 0 \text{ (since } x^0 = 1 \text{ and the derivative of a constant is 0)},\\
f(x) &= \frac{1}{x} = x^{-1} && \Rightarrow\ f'(x) = (-1)x^{-1-1} = -\frac{1}{x^2},\\
f(x) &= \frac{1}{x^4} = x^{-4} && \Rightarrow\ f'(x) = (-4)x^{-4-1} = -4x^{-5},\\
f(x) &= \sqrt{x} = x^{\frac{1}{2}} && \Rightarrow\ f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Recall Leibniz's notation:

$$\frac{d}{dx}f(x) = f'(x)$$

Examples using Leibniz's notation:

$$\frac{d}{dx}x^3 = 3x^2,$$
$$\frac{d}{dt}\left(\frac{1}{t}\right) = \frac{d}{dt}t^{-1} = -t^{-2} = -\frac{1}{t^2}.$$

3.4 Derivatives for Sums and Constant Multipliers

If $f(x)$ and $g(x)$ are any two functions with derivatives $f'(x)$ and $g'(x)$, and if c is some constant, then:

1. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$
2. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) = cf'(x)$

Examples:

1. $\frac{d}{dx}[cx^2] = c\frac{d}{dx}x^2 = c(2x) = 2cx$
2. $\frac{d}{dx}[cx] = c\frac{d}{dx}x = c$
3. $\frac{d}{dx}[c] = 0$
4. $\frac{d}{dx}[x+x^2] = \frac{d}{dx}x + \frac{d}{dx}x^2 = 1 + 2x$
5. $\frac{d}{dx}[ax^2+bx+c] = \frac{d}{dx}ax^2 + \frac{d}{dx}bx + \frac{d}{dx}c = a\frac{d}{dx}x^2 + b\frac{d}{dx}x + 0 = 2ax + b$

3.5 Marginal Analysis

Recall that the cost function $C(x)$ gives the total cost C as a function of the number of items or products x. The marginal cost function is the derivative of the cost function $C(x)$, i.e., $C'(x)$. It measures the rate of change of cost with respect to x. The units for marginal cost are units of cost per item, \$/item. The cost of one more item when you produce/sell x items is given by $C(x+1) - C(x)$. Thus, the marginal cost is

$$C'(x) = \lim_{h\to 0} \frac{C(x+h) - C(x)}{h}.$$

The function $C'(x)$ is commonly used as an approximation for $C(x+1) - C(x)$.

Example 3.4. The marginal cost of producing a single item is \$500, with a fixed cost of \$250. The cost function is

$$C(x) = 250 + 500x$$

Here, 250 is the fixed cost and $500x$ is the variable cost. The marginal cost function is:

$$C'(x) = 0 + 500 = 500\,\$/\text{item}.$$

Using the marginal cost to approximate the cost of one more item:

$$C(x+1) - C(x) = 250 + 500(x+1) - (250 + 500x) = 500\,\$/\text{item}.$$

In this case, we have $C'(x) = C(x+1) - C(x)$ (exactly equal).

Exercise 3.2. Given the cost function $C(x) = 10,000 + 5x - 0.001x^2$, find the marginal cost at the production level of $x = 1000$.

Solution. The marginal cost function is given by

$$C'(x) = \frac{d}{dx}(10,000 + 5x - 0.001x^2) = 0 + 5 - 0.002x,$$

so we find

$$C'(1000) = 5 - 0.002(1000) = \$4.8 \text{ per item.}$$

This is the marginal cost at $x = 1000$. For the cost of one additional item, we compute

$$\begin{aligned}
&C(1001) - C(1000) \\
&\quad = [10,000 + 5(1001) - 0.001(1001)^2] - [10,000 + 5(1000) - 0.001(1000)^2] \\
&\quad = \$4.799.
\end{aligned}$$

In this example, $C'(1000)$ is a good approximation for the cost of producing the 1001st item. □

3.5.1 Marginal Revenue and Profit

Recall that:

- The revenue function $R(x)$ gives the revenue realized by the sales of x units of a certain commodity.
- The marginal revenue is the derivative $R'(x)$ of the revenue function $R(x)$. It approximates revenue to be obtained from the sale of an additional item.
- Similarly, the marginal profit function $P'(x)$ measures the approximate change in profit if an additional unit is to be sold.
- Average cost: $\bar{C}(x) = \frac{C(x)}{x}$ represents the average cost per item when x items are produced or sold.

Exercise 3.3. The cost, in thousands of dollars, of airing x television commercials during a Super Bowl game is given by $C(x) = 20 + 4,000x + 0.05x^2$.

1. Find the marginal cost function and use it to estimate how fast the cost is increasing when $x = 4$. Compare this with the exact cost of airing the fifth commercial.
2. Find the average cost function $\bar{C}$ and evaluate $\bar{C}(4)$. What does the answer tell you?

Solution.

1. The marginal cost function, $C'(x)$, can be found by taking the derivative of the cost function:

$$C'(x) = \frac{d}{dx}(20 + 4,000x + 0.05x^2) = 4,000 + 0.1x.$$

This represents the cost of airing an additional commercial. To estimate how fast the cost is increasing when $x = 4$, we evaluate $C'(4)$:

$$C'(4) = 4,000 + 0.1(4) = 4,000.4 \,\text{thousands of dollars.}$$

This is the approximate additional cost for the fifth commercial.

2. The average cost function $\bar{C}(x)$ is defined as the total cost divided by the number of items (or commercials, in this case):

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{20 + 4,000x + 0.05x^2}{x}.$$

We can evaluate this function for $x = 4$ to find the average cost per commercial:

$$\bar{C}(4) = \frac{20 + 4,000(4) + 0.05(4)^2}{4} = 4,005.2 \,\text{thousands of dollars per commercial.}$$

This indicates that the average cost of airing the first four commercials is \$4,005,200 per commercial.

3.6 Product and Quotient Rules of Differentiation

If $f(x)$ and $g(x)$ are differentiable functions of x, then so is their product $f(x)g(x)$, and

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In words: "The derivative of a product is the derivative of the first times the second, plus the first times the derivative of the second."

Example 3.5. Consider the function $y = 3x(4x^2 - 1)$.

Solution. First way: Simplify before differentiating:

$$y = 12x^3 - 3x$$

$$\frac{dy}{dx} = 12(3x^2) - 3 = 36x^2 - 3.$$

Second way: Use the product rule with $f(x) = 3x$ and $g(x) = 4x^2 - 1$:

$$f'(x) = 3 \quad \text{and} \quad g'(x) = 8x$$

By the product rule,

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x) = 3(4x^2 - 1) + 3x(8x) = 12x^2 - 3 + 24x^2 = 36x^2 - 3.$$

Example 3.6. Consider the function $y = (2x^{0.5} + 4x - 5)(x^{-1})$.

Solution. By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= (2(0.5)x^{-0.5} + 4)(x^{-1}) + (2x^{0.5} + 4x - 5)(-x^{-2}) \\ &= (x^{-0.5} + 4)(x^{-1}) + (2x^{0.5} + 4x - 5)(1 - x^{-2}).\end{aligned}$$

Example 3.7. Consider the function $y = (\sqrt{x} + 1)(\sqrt{x} + \frac{1}{x})$.

Solution. Using the product rule,

$$\begin{aligned}\frac{dy}{dx} &= (\frac{1}{2}x^{-\frac{1}{2}})(\sqrt{x} + \frac{1}{x}) + (\sqrt{x} + 1)(\frac{1}{2}x^{-\frac{1}{2}} - x^{-2}) \\ &= \frac{1}{2} + x^{-\frac{5}{2}} + \frac{1}{2}x - 2x^{-3} \\ &= 1 - \frac{3}{2}x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-3}.\end{aligned}$$

3.7 Quotient Rule

If $f(x)$ and $g(x)$ are differentiable functions of x, then so is their quotient $\frac{f(x)}{g(x)}$ (provided that $g(x) \neq 0$), and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

In words: "The derivative of a quotient is the derivative of the top times the bottom, minus the top times the derivative of the bottom, all over the bottom squared."

Example 3.8. Consider the function $y = \frac{2x+4}{3x-1}$.

$$f(x) = 2x + 4, \quad f'(x) = 2$$

$$g(x) = 3x - 1, \quad g'(x) = 3$$

By the quotient rule,

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} = \frac{2(3x - 1) - (2x + 4)(3)}{(3x - 1)^2} = \frac{-14}{(3x - 1)^2}.$$

Example 3.9. Consider the function $y = \frac{x^2-4x}{x^2+x+1}$.

$$f(x) = x^2 - 4x, \quad f'(x) = 2x - 4$$

$$g(x) = x^2 + x + 1, \quad g'(x) = 2x + 1$$

By the quotient rule,

$$\frac{dy}{dx} = \frac{(2x-4)(x^2+x+1) - (x^2-4x)(2x+1)}{(x^2+x+1)^2} = \frac{5(x^2-1)}{(x^2+x+1)^2}.$$

Exercise 3.4. The monthly sales of Sunny Electronics' new sound system are given by $q(t) = 2,000 - 100t^2$ units per month, t months after its introduction. The price Sunny charges is $p(t) = 1,000 - 2t$ dollars per sound system, t months after introduction.

Solution. To find the rate of change of monthly sales, the rate of change of the price, and the rate of change of monthly revenue $r(t)$ 5 months after the introduction of the sound system, we can calculate the following:

- $q(t)$ represents the monthly sales.
- $p(t)$ represents the price per unit.
- $r'(t)$ represents the rate of change of monthly revenue.

The revenue function $r(t)$ is the product of $p(t)$ and $q(t)$, i.e., $r(t) = p(t)q(t)$. The derivatives are as follows:

$$q'(t) = \frac{d}{dt}(2000 - 100t^2) = -200t$$

$$p'(t) = \frac{d}{dt}(1000 - 2t) = -2$$

Thus, the rate of change of monthly sales 5 months after the introduction is

$$q'(5) = -200 \cdot 5 = -1000 \,\text{units/month},$$

which indicates that the monthly sales were decreasing at a rate of 1000 units per month 5 months after introduction. Similarly, the rate of change of the price 5 months after the introduction is

$$p'(5) = -2 \cdot 5 = -10 \,\text{dollars/month},$$

which means that the unit price of an item was decreasing at a rate of 10 dollars per month 5 months after introduction. Finally, to find the rate of change of monthly revenue $r'(t)$, we use the product rule:

$$r'(t) = p'(t)q(t) + p(t)q'(t).$$

Substituting the values for $t = 5$ gives us

$$\begin{aligned} r'(5) &= p'(5)q(5) + p(5)q'(5) \\ &= (-10)(2000 - 100 \cdot 5^2) + (1000 - 2 \cdot 5)(-1000) \\ &= -10(2000 - 2500) + (990)(-1000) \\ &= 10(500) - 990000 \\ &= 5000 - 990000 \\ &= -985000 \,\text{dollars/month.} \end{aligned}$$

□

3.8 The Chain Rule

The chain rule is one of the most useful techniques in calculus for differentiating compositions of functions. It states that if a variable y depends on a variable u which itself depends on a variable x, then the rate of change of y with respect to x can be found by multiplying the rate of change of y with respect to u by the rate of change of u with respect to x. This rule is particularly useful because it allows the differentiation of complex functions by breaking them down into simpler parts. In particular, if f is a differentiable function of u and u is a differentiable function of x, then f is also a differentiable function of x, where

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}.$$

Example 3.10. We will compute the derivative

$$\frac{d}{dx}(2x + 3)^5.$$

Let $f(u) = u^5$ and $u = 2x + 3$, then

$$\frac{d}{dx}f(u) = \frac{df}{du} \cdot \frac{du}{dx} = 5u^4 \cdot 2 = 10(2x + 3)^4.$$

Example 3.11. For the function $f(x) = (x^2 + 2x)^4$, we apply the chain rule to find its derivative. Let $u = x^2 + 2x$, so $f(x) = u^4$. The derivatives are given by

$$\frac{df}{du} = 4u^3,$$

$$\frac{du}{dx} = 2x + 2.$$

Thus, by the chain rule,

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 4u^3(2x+2) = 4(x^2+2x)^3(2x+2) = 8(x^2+x)(x+1)^3.$$

Example 3.12. Given $h(x) = \left(\frac{1}{x^2+1}\right)^3$, find $\frac{dh}{dx}$. Let $u = x^2 + 1$, hence $h(u) = \frac{1}{u^3}$. Then,

$$\frac{dh}{du} = -3u^{-4},$$
$$\frac{du}{dx} = 2x.$$

Using the chain rule,

$$\frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx} = -3u^{-4} \cdot 2x = -6x \cdot (x^2+1)^{-4}.$$

Example 3.13. For the revenue function $r(x) = (0.1x^2 - 4.2x + 9.5)^{1.5}$, the derivative $r'(x)$ is obtained by differentiating the outside function with respect to the inside function and then multiplying by the derivative of the inside. We have

$$r'(x) = 1.5(0.1x^2 - 4.2x + 9.5)^{0.5} \cdot (0.2x - 4.2),$$

where 1.5 is the derivative of the outside function raised to the 1.5 power, and $0.2x - 4.2$ is the derivative of the inside function $0.1x^2 - 4.2x + 9.5$.

Example 3.14. Consider the function $f(x) = \{1 + [1 + (1+2x)^3]^3\}^2$. We apply the chain rule repeatedly to find its derivative.

$$\begin{aligned}
f'(x) &= 2\{1 + [1 + (1+2x)^3]^3\} \cdot \frac{d}{dx}\{1 + [1 + (1+2x)^3]^3\} \\
&= 2\{1 + [1 + (1+2x)^3]^3\} \cdot 3[1 + (1+2x)^3]^2 \cdot \frac{d}{dx}[1 + (1+2x)^3] \\
&= 6\{1 + [1 + (1+2x)^3]^3\}^2 \cdot 3(1+2x)^2 \cdot \frac{d}{dx}(1+2x) \\
&= 18\{1 + [1 + (1+2x)^3]^3\}^2 \cdot (1+2x)^2 \cdot 2 \\
&= 36\{1 + [1 + (1+2x)^3]^3\}^2 \cdot (1+2x)^2.
\end{aligned}$$

Note that each step involves applying the chain rule to differentiate the outer function and then multiply by the derivative of the inner function, continuing inward.

Exercise 3.5 (Marginal Profit Analysis)**.** Paramount Electronics has an annual profit given by

$$P = -100,000 + 5,000q - 0.25q^2 \text{ dollars},$$

where q is the number of laptop computers it sells each year. The number of laptop computers it can make and sell each year depends on the number n of electrical engineers Paramount employs, according to the equation

$$q = 30n + 0.01n^2.$$

Use the chain rule to find $\frac{dP}{dn}$ when $n = 10$, and interpret the result.

Solution. We apply the chain rule:

$$\frac{dP}{dn} = \frac{dP}{dq} \cdot \frac{dq}{dn}, \quad \text{where } \frac{dP}{dq} = 5000 - 0.5q \text{ and } \frac{dq}{dn} = 30 + 0.02n.$$

Now, evaluating these derivatives at $n = 10$ gives us:

$$\begin{aligned} q &= 30 \cdot 10 + 0.01 \cdot 10^2 = 300 + 1 = 301 \text{ number of laptops}, \\ \left.\frac{dP}{dq}\right|_{q=301} &= 5000 - 0.5 \cdot 301 = 5000 - 150.5, \\ \left.\frac{dq}{dn}\right|_{n=10} &= 30 + 0.02 \cdot 10 = 30 + 0.2 = 30.2. \end{aligned}$$

Thus, the marginal profit with respect to the number of engineers is:

$$\begin{aligned} \left.\frac{dP}{dn}\right|_{n=10} &= (5000 - 150.5)(30.2) \\ &= (4849.5)(30.2) \\ &= 146,454.9 \text{ dollars per engineer.} \end{aligned}$$

This result suggests that if we increase the number of engineers from 10 to 11, the profit will increase approximately by \$146,454.9. □

3.9 Derivatives of Exponential Functions

Given that the derivative of e^x with respect to x is e^x itself, and for any positive number b, the derivative of b^x is $b^x \ln(b)$, we can solve the following examples.

Example 3.15. We will find the derivative of e^{x^3-2x}. Apply the chain rule for derivatives. Here, $f = e^u$, where $u = x^3 - 2x$.

$$\frac{d}{dx} e^{x^3-2x} = e^{x^3-2x} \cdot \frac{d}{dx}(x^3 - 2x) = e^{x^3-2x}(3x^2 - 2).$$

Example 3.16. We will find the derivative of 4^x. Use the general rule for the derivative of b^x. Then, apply the chain rule for the derivative of the inner function:

$$\frac{d}{dx}4^x = 4^x \ln(4) \cdot \frac{d}{dx}(x) = 4^x \ln(4).$$

Generalize: For any function of the form $b^{f(x)}$, where b is a constant and $f(x)$ is a function of x, the derivative is:

$$\frac{d}{dx}b^{f(x)} = b^{f(x)} \ln(b) \cdot \frac{df}{dx}.$$

Example 3.17. Given $r(x) = (2x-1)^2$, find $r'(x)$. Apply the chain rule for the outer function and the power rule for the inner function. Then simplify the expression.

$$r'(x) = 2 \cdot 2(e^{2x-1}) \cdot \frac{d}{dx}(e^{2x-1}) = 4e^{2x-1} \cdot e^{2x-1} \cdot \frac{d}{dx}(2x-1) = 4e^{2(2x-1)}.$$

Alternative approach: first rewrite the function to make the chain rule application more clear. Then take the derivative.

$$r(x) = e^{4x-2}, \quad r'(x) = e^{4x-2} \cdot \frac{d}{dx}(4x-2) = 4e^{4x-2}.$$

Example 3.18. We will find the derivative of the function $g(x) = e^x + e^{-x}$ using the quotient rule.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \\ &= \frac{\left(e^x + e^{-x}\right)\frac{d}{dx}\left(e^x - e^{-x}\right) - \left(e^x - e^{-x}\right)\frac{d}{dx}\left(e^x + e^{-x}\right)}{\left(e^x + e^{-x}\right)^2} \\ &= \frac{\left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2} \\ &= \cdots = -\frac{4}{\left(e^x - e^{-x}\right)^2}. \end{aligned}$$

Exercise 3.6. If \$10,000 is invested in a savings account offering 4% per year, compounded continuously, we want to know how fast the balance is growing after 3 years.

Solution. Given that the account balance $A(t)$ at time t in years is modeled by the equation

$$A(t) = Pe^{rt},$$

where P is the principal amount, r is the rate of growth, and e is the base of the natural logarithm. The amount $A(t)$ with $P = 10,000$ dollars and $r = 0.04$ per year is

$$A(t) = 10,000e^{0.04t}.$$

We want to find the rate of growth of the balance after 3 years, which is the derivative of $A(t)$ with respect to time t, denoted as $A'(t)$.

$$A'(t) = \frac{d}{dt}\left(10,000e^{0.04t}\right) = 10,000e^{0.04t} \cdot \frac{d}{dt}(0.04t)$$
$$= 10,000e^{0.04t} \cdot 0.04 = 400e^{0.04t}.$$

Evaluating at $t = 3$ years, we get

$$A'(3) = 400e^{0.04 \cdot 3} = 400e^{0.12} \approx 400e^{0.12} \approx \$451/\text{year}.$$

□

Exercise 3.7 (Population Growth)**.** The population of Lower Anchovia was 4,000,000 at the start of 2010 and was doubling every 10 years. We want to find out how fast the population was growing per year at the start of 2010.

Solution. Given the exponential growth model for the population,

$$P(t) = A \cdot b^t,$$

where $P(t)$ is the population t years after 2010, A is the initial population and b is the growth factor. From the information that the population doubles every 10 years, we have:

$$P(0) = 4,000,000 = A \cdot b^0 \quad \Leftrightarrow \quad A = 4,000,000$$
$$P(10) = 8,000,000 = A \cdot b^{10} \quad \Leftrightarrow \quad 8,000,000 = 4,000,000 \cdot b^{10} \quad \Leftrightarrow \quad 2 = b^{10}$$

Thus, we can solve for b and get

$$b = \sqrt[10]{2}.$$

The population model becomes

$$P(t) = 4,000,000 \cdot 2^{t/10}.$$

We calculate the rate of change of the population at the start of 2010 ($t = 0$) by differentiating $P(t)$ with respect to t:

$$P'(t) = 4,000,000 \cdot \ln(2) \cdot \frac{2^{t/10}}{10}.$$

Thus, the rate of change of the population at the start of 2010 is

$$P'(0) = 4,000,000 \cdot \ln(2) \cdot \frac{1}{10} = 400,000 \ln(2).$$

This evaluates to approximately 277,259 people per year. $\square$

3.10 Extra Problems

Exercise 3.8. Calculate the average rate of change of the function $f(x) = x^3 - 6x^2 + 9x + 1$ over the interval $[0, 3]$.

Solution. The average rate of change of $f(x)$ from $x = 0$ to $x = 3$ is given by

$$\frac{f(3) - f(0)}{3 - 0} = \frac{(3^3 - 6 \cdot 3^2 + 9 \cdot 3 + 1) - (0^3 - 6 \cdot 0^2 + 9 \cdot 0 + 1)}{3}$$

$$= \frac{(27 - 54 + 27 + 1) - 1}{3} = \frac{0}{3} = 0.$$

The average rate of change of the function over the interval $[0, 3]$ is 0. $\square$

Exercise 3.9. Find the derivative of the function $f(x) = \sqrt{x}$ using the definition of the derivative.

Solution. The derivative $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}.$$

Using the conjugate to simplify, we have

$$f'(x) = \lim_{h \to 0} \frac{(\sqrt{x + h} - \sqrt{x})(\sqrt{x + h} + \sqrt{x})}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Thus, the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$. $\square$

Exercise 3.10. Use the product rule to find the derivative of $f(x) = (x^2 + 2x)(3x - 4)$.

Solution. Let $u(x) = x^2 + 2x$ and $v(x) = 3x - 4$. Then,

$$u'(x) = 2x + 2 \text{ and } v'(x) = 3.$$

Using the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = (2x + 2)(3x - 4) + (x^2 + 2x)(3)$$

$$= 6x^2 - 8x + 6x - 8 + 3x^2 + 6x = 9x^2 + 4x - 8.$$

Therefore, the derivative is $f'(x) = 9x^2 + 4x - 8$. $\square$

Exercise 3.11. Find the derivative of the function $f(x) = \frac{x^2 - 1}{x + 2}$ using the quotient rule.

Solution. Let $u(x) = x^2 - 1$ and $v(x) = x + 2$. Then,

$$u'(x) = 2x \text{ and } v'(x) = 1.$$

Using the quotient rule, we compute

$$\begin{aligned} f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{2x(x+2) - (x^2-1)(1)}{(x+2)^2} \\ &= \frac{2x^2 + 4x - x^2 + 1}{(x+2)^2} = \frac{x^2 + 4x + 1}{(x+2)^2}. \end{aligned}$$

Thus, the derivative of $f(x) = \frac{x^2-1}{x+2}$ is $f'(x) = \frac{x^2+4x+1}{(x+2)^2}$. □

Exercise 3.12. Find the second derivative of the function $f(x) = 3x^4 - 5x^3 + 2x - 7$.

Solution. First, find the first derivative:

$$f'(x) = 12x^3 - 15x^2 + 2.$$

Now, find the second derivative:

$$f''(x) = 36x^2 - 30x.$$

□

Exercise 3.13. Given the function $f(x) = \sin(x^2 + 2x)$, find the derivative.

Solution. To find the derivative of $f(x) = \sin(x^2 + 2x)$, we use the chain rule. Let $u(x) = x^2 + 2x$. Then, the derivative of $u(x)$ is:

$$u'(x) = 2x + 2.$$

The chain rule states that

$$f'(x) = \cos(u(x)) \cdot u'(x) = \cos(x^2 + 2x) \cdot (2x + 2).$$

Therefore, the derivative of $f(x)$ is

$$f'(x) = (2x + 2)\cos(x^2 + 2x).$$

□

Exercise 3.14. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Solution. To find $\frac{dy}{dx}$ for the equation $x^3 + y^3 = 6xy$, we use implicit differentiation.

Differentiating both sides with respect to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}.$$

Rearranging to solve for $\frac{dy}{dx}$, we isolate $\frac{dy}{dx}$ terms on one side:

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2.$$

Factoring out $\frac{dy}{dx}$ gives

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2.$$

Thus,

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}.$$

This expression gives the derivative of y with respect to x implicitly defined by the equation $x^3 + y^3 = 6xy$. □

Exercise 3.15. A ladder 10 meters long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 meter per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 meters from the wall?

Solution. Let x be the distance from the wall to the bottom of the ladder, and y be the height of the top of the ladder above the ground. The relationship between x and y is given by the Pythagorean theorem:

$$x^2 + y^2 = 10^2 = 100.$$

Differentiating both sides with respect to t (time), we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

We know $\frac{dx}{dt} = 1$ meter per second and $x = 6$ meters when we want to find $\frac{dy}{dt}$. Solving for y using the Pythagorean theorem gives

$$y^2 = 100 - 36 = 64 \Leftrightarrow y = 8 \text{ meters}.$$

Substituting $x = 6$, $y = 8$, and $\frac{dx}{dt} = 1$ into the differentiated equation yields

$$2(6)(1) + 2(8)\frac{dy}{dt} = 0 \Leftrightarrow 12 + 16\frac{dy}{dt} = 0.$$

Solving for $\frac{dy}{dt}$, we get

$$16\frac{dy}{dt} = -12 \Leftrightarrow \frac{dy}{dt} = -\frac{12}{16} = -\frac{3}{4} \text{ meters per second.}$$

Therefore, the top of the ladder is sliding down the wall at a rate of $-\frac{3}{4}$ meters per second when the bottom is 6 meters from the wall. $\square$

Chapter 4

Optimization of Functions

4.1 Maxima and Minima

We will use derivatives to optimize functions, seeking their relative maxima and minima. These critical points correspond to dips or peaks in the graph of a function defined on a domain $[a, b]$.

Definition 4.1.

- A function f has a *relative (local) maximum* at $x = c$ if there exists an open interval (r, s) containing c such that $f(x) \leq f(c)$ for all x between r and s.
- A function f has a *relative (local) minimum* at $x = c$ if there exists an open interval (r, s) containing c such that $f(x) \geq f(c)$ for all x between r and s.
- A function f has an *absolute (global) maximum* at $x = c$ if $f(x) \leq f(c)$ for all x in the domain of f.
- A function f has an *absolute (global) minimum* at $x = c$ if $f(x) \geq f(c)$ for all x in the domain of f.
- A *critical point* of a function f is a point in the domain of f where either:
  1. $f'(x) = 0$ (called stationary points)
  2. $f'(x)$ does not exist (called singular points).

Locating Candidates for Relative Extrema

To find relative extrema, we first identify all critical points, which include stationary and singular points. These are candidates for relative extrema.

Note: These critical points are only candidates for relative extrema. It is possible that they may not be actual extrema.

Locating Candidates for Absolute Extrema

- All local (relative) extrema are candidates for global (absolute) extrema.
- Endpoints are also candidates for global (absolute) extrema. (Closed intervals have endpoints; open intervals do not.)

Example 4.1. Find all the relative and global extrema of $f(x) = x^3 - 6x^2 + 1$ defined on the interval $[-1, 5]$.

Solution.
Step 1: Find critical points.

1. Stationary points: $f'(x) = 0$

$$f'(x) = 3x^2 - 12x = 3x(x-4)$$

$$\Rightarrow 3x(x-4) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

2. Singular points: None since $f'(x)$ is defined for all x in the domain.

Step 2: Identify whether a critical point is a relative extremum or not. We'll use the first derivative test to determine the sign of the derivative off to the left and right of each critical point.

Left of c	Right of c	Conclusion
$+$	$-$	Relative maximum
$-$	$+$	Relative minimum
$-$	$-$	Not a relative extremum
$+$	$+$	Not a relative extremum

Let's apply the test to the above example:

x	Sign of f' left of x	$f'(x)$	Sign of f' right of x	Conclusion
0	$+$	0	$-$	Relative max
4	$-$	0	$+$	Relative min

Conclusion: For $f(x) = x^3 - 6x^2 + 1$,

- $x = 0$ is a relative max.
- $x = 4$ is a relative min.

Step 3: Candidates for global extrema.

1. All relative extrema: $x = 0$ and $x = 4$.
2. End points of the interval: $x = -1$ and $x = 5$.

Now we evaluate the function $f(x) = x^3 - 6x^2 + 1$ at each of these points:

x	-1	0	4	5
$f(x)$	$f(-1)$	$f(0)$	$f(4)$	$f(5)$

Calculating the values:

$$\begin{aligned}
f(-1) &= (-1)^3 - 6(-1)^2 + 1 = -1 - 6 + 1 = -6 \\
f(0) &= 0^3 - 6 \cdot 0^2 + 1 = 1 \\
f(4) &= 4^3 - 6 \cdot 4^2 + 1 = 64 - 96 + 1 = -31 \\
f(5) &= 5^3 - 6 \cdot 5^2 + 1 = 125 - 150 + 1 = -24.
\end{aligned}$$

□

Example 4.2. Consider the function $f(x) = x^3$.

Step 1: Critical points. First, we find the derivative of the function:

$$f'(x) = 3x^2.$$

To find the stationary points, we set the derivative equal to zero:

$$f'(x) = 0 \Leftrightarrow 3x^2 = 0 \Leftrightarrow x = 0.$$

Step 2: First-derivative test. Next, we apply the first-derivative test around the critical point at $x = 0$:

Left of 0	At 0	Right of 0
$f'(x) > 0$	$f'(0) = 0$	$f'(x) > 0$

Since the sign of the derivative does not change from positive to negative or vice versa, $x = 0$ is not a relative extremum.

Conclusion: The function $f(x) = x^3$ does not have any relative extrema. Furthermore, since the function is cubic and the derivative is always positive except at $x = 0$, there are no global extrema on the real number line. □

Example 4.3. Given the function $h(t) = 2t^3 + 3t^2$ with domain $[-2, \infty)$, we want to find the exact location of all the relative and absolute extrema.

Step 1: Identify Critical Points. First, we find the derivative of the function:

$$h'(t) = 6t^2 + 6t.$$

To find the stationary points, we set the derivative equal to zero:

$$h'(t) = 0 \Leftrightarrow 6t(t+1) = 0 \Leftrightarrow t = 0 \quad \text{or} \quad t = -1.$$

Step 2: First-Derivative Test. We then analyze the sign of the first derivative around the critical points to determine their nature:

Interval	Sign of $h'(t)$	Behavior of $h(t)$	Conclusion
$t < -1$	$+$	Increasing	
$t = -1$	0	Stationary Point	Local Max
$-1 < t < 0$	-	Decreasing	
$t = 0$	0	Stationary Point	Local Min
$t > 0$	$+$	Increasing	

Step 3: Global Extrema. We evaluate the function at the critical points and the endpoints to find the global extrema:

t	$h(t)$
-2	$h(-2)$
-1	$h(-1)$
0	$h(0)$

After evaluating $h(t)$, we find that $t = -1$ gives the local maximum and $t = 0$ gives the local minimum. Considering the endpoint $t = -2$, we determine the global minimum. $\square$

Example 4.4. We want to find the exact location of all the relative and absolute extrema of the function $f(x) = \sqrt[3]{x^3 - 3x}$.

First, we compute the derivative of the function:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x^3 - 3x\right)^{\frac{1}{3}} = \frac{1}{3}\left(x^3 - 3x\right)^{-\frac{2}{3}} \cdot \frac{d}{dx}(x^3 - 3x) \\
&= \frac{1}{3}\left(x^3 - 3x\right)^{-\frac{2}{3}} \cdot (3x^2 - 3) = (3x^2 - 3) \cdot \frac{1}{3}\left(x^3 - 3x\right)^{-\frac{2}{3}} = \frac{x^2 - 1}{(x^3 - 3x)^{\frac{2}{3}}}.
\end{aligned}
$$

Step 1: Critical Points. To identify critical points we look for where $f'(x) = 0$ or does not exist:

$$f'(x) = 0 \Leftrightarrow 3x^2 - 3 = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = \pm 1$$

Step 2: First-Derivative Test. We use the first-derivative test to classify the critical points:

Sign of f'	Interval
$+$	$(-\infty, -1)$
$-$	$(-1, 0)$
$+$	$(0, 1)$

$$- \quad (1, \infty)$$

From this test, we can see that $x = -1$ is a local maximum and $x = 1$ is a local minimum.

Step 3: Global Extrema. To find the global extrema, we evaluate the function at the critical points and endpoints of the interval.

$$f(-1) = 2.598$$
$$f(1) = -1.598$$

Finally, we conclude that $x = -1$ gives a local maximum and $x = 1$ gives a local minimum. There are no endpoints to consider for this function, since the domain is $(-\infty, \infty)$. □

4.1.1 Application of Maxima and Minima

Exercise 4.1. Minimize $F = x^2 + y^2$ under the constraint $x + 2y = 10$.

Solution. Use substitution to reduce the problem into one of optimization of a function of a single variable. Given the constraint $x+2y = 10$, we can express x as $x = 10-2y$. Substituting into F gives

$$F = (10 - 2y)^2 + y^3.$$

We want to find the global minimum of F. To do this, we first differentiate F with respect to y:

$$\frac{dF}{dy} = 2(10 - 2y)(-2) + 2y = -40 + 10y.$$

Set the derivative equal to zero to find stationary points:

$$10y - 40 = 0 \Leftrightarrow y = 4.$$

This gives us the stationary point at $y = 4$. We observe that F is decreasing in $(-\infty, 4)$ and increasing in $(4, \infty)$, so $y = 4$ is a global minimum for $F(y)$. Evaluating F at $y = 4$, we get

$$F = (10 - 2 \cdot 4)^2 + 4^2 = 20.$$

Thus, setting $y = 4$ to minimize F gives us $x = 10 - 2y = 10 - 8 = 2$.
Conclusion: F is minimized when $y = 4$ and $x = 2$. □

Exercise 4.2. Assume that the demand for tuna in a small coastal town is given by the equation

$$p = \frac{500,000}{q^{1.5}},$$

where p is the price in dollars and q is the number of pounds of tuna that can be sold in a month. The town's fishery wishes to sell at least 5,000 pounds of tuna per month.

(a) How much should the town's fishery charge for tuna to maximize monthly revenue? Assume that the revenue function R is given by $R = pq$ and that we are considering $q \in [5,000, \infty)$.

(b) How much tuna will it sell per month at that price?

(c) What will be its resulting revenue?

Solution.

(a) The monthly revenue R is given by $R = pq$. To maximize revenue, we set up the equation:

$$R = p \cdot q = \frac{500,000}{q^{1.5}} \cdot q = 500,000 \cdot q^{-0.5}.$$

To find the maximum, we take the derivative of R with respect to q and set it to zero:

$$\frac{dR}{dq} = -250,000 \cdot q^{-1.5}.$$

Solving for q when $\frac{dR}{dq} = 0$, we find there are no stationary points because the derivative does not change sign for $q \in [5,000, \infty)$. Thus, we conclude that $q = 5,000$ is the quantity that maximizes revenue.

(b) The corresponding price p when $q = 5,000$ pounds of tuna is sold is given by

$$p = \frac{500,000}{5000^{1.5}} = \$1.41 \text{ per pound.}$$

(c) The resulting revenue R when selling 5,000 pounds at the price of \$1.41 per pound is

$$R = p \cdot q = (1.41) \cdot (5,000) = \$7,070 \text{ per month.}$$

□

Exercise 4.3. The *FeatureRich Software Company* sells its graphing program, Dogwood, with a volume discount. If a customer buys x copies, then he or she pays $\frac{8500}{\sqrt{x}}$ per copy. It cost the company \$10,000 to develop the program and \$2 to manufacture each copy. If a single customer were to buy all the copies of Dogwood, how many copies would the customer have to buy for FeatureRich Softwares average profit per copy to be maximized? How are average profit and marginal profit related at this number of copies?

Solution. The average profit function $P(x)$ is given by

$$P(x) = \frac{500\sqrt{x} - 10000 - 2x}{x}.$$

To maximize $P(x)$, we first differentiate $P(x)$ with respect to x to find the stationary points:

$$\frac{dP}{dx} = \frac{500}{2} \cdot x^{-\frac{1}{2}} - \frac{10000}{x^2} - 2.$$

Setting $\frac{dP}{dx} = 0$, we solve for x to find the point that maximizes $P(x)$. The derivative simplifies to

$$-250 \cdot x^{-\frac{3}{2}} + 10000 \cdot x^{-2} = 0 \Leftrightarrow 250 \cdot x^{-\frac{3}{2}} = 10000 \cdot x^{-2}$$

$$\Leftrightarrow x^{\frac{1}{2}} = 40 \Leftrightarrow x = 1600.$$

After finding the critical point at $x = 1600$, we test for the nature of the extremum using the first-derivative test.

x	Sign of $P'(x)$	
0	+	
1600	0	Local max
∞	$-$	

This shows that $x = 1600$ is a candidate for a global maximum, which is confirmed by examining the sign of $P'(x)$ around 1600. Now, evaluating the average profit function $P(x)$ at $x = 1600$:

$$P(1600) = \$4.25 \text{ per copy.}$$

If 1600 copies are sold, the average profit will be maximized. Next, we find the marginal profit at $x = 1600$:

$$P'(x) = 500x^{-\frac{1}{2}} - 10000 - 2$$

$$P'(1600) = 250\left(\frac{1}{\sqrt{1600}}\right) - 2 = \$4.25 \text{ per copy.}$$

At this value of x that maximizes $P(x)$, the average profit equals the marginal profit.

□

4.2 The Second Derivative

The second derivative of a function $f(x)$, denoted by $f''(x)$, is defined as the derivative of the derivative of $f(x)$. If y is a function of x, then the second derivative of y with respect to x is denoted by $\frac{d^2y}{dx^2}$.

Examples:

1. Let $y = 3x^2 - 6$. We compute:
$$\frac{dy}{dx} = 6x$$
$$\frac{d^2y}{dx^2} = 6$$
This is the first and second derivative, respectively.

2. Let $f(x) = e^{-(x-1)} - x$. We compute:

$$f'(x) = -e^{-(x-1)}\frac{d}{dx}(-(x-1)) = -e^{-(x-1)}(-1)$$

$$f''(x) = -\left(-e^{-(x-1)}\right)\frac{d}{dx}(-(x-1)) = -(-1)e^{-(x-1)} = e^{-(x-1)}.$$

4.2.1 Concavity and Convexity

Let f be a differentiable function on $[a, b]$.

1. If $f'' > 0$ on (a, b), f is *convex* on (a, b).
2. If $f'' < 0$ on (a, b), f is *concave* on (a, b).

Definition 4.2. A point on the graph of a function at which concavity changes is called an **inflection point**.

Candidates for Inflection Points: Any point in the domain of f such that $f''(x) = 0$ or $f''(x)$ is not defined. If f'' changes sign from left to right of the candidate point, then it is an inflection point.

Example 4.5. Consider the function $S(t) = 100 + 60t^2 - t^3$. Find all inflection points.

$$S'(t) = 120t - 3t^2,$$
$$S''(t) = 120 - 6t.$$

Setting the second derivative equal to zero to find critical points:

$$S''(t) = 0 \Leftrightarrow 120 - 6t = 0 \Leftrightarrow t = 20.$$

This is the only candidate for an inflection point. Sign of S'': Analyzing the sign of the second derivative around the candidate:

t	$\cdots$	20	$\cdots$
$S''(t)$	$+$	0	$-$

Therefore, $t = 20$ is an inflection point as S'' changes sign from positive to negative.

Note on the Second Derivative Test

We can use the second derivative to determine whether a stationary point is a relative minimum or maximum, assuming that the second derivative exists. If c is a stationary point, i.e., $f'(c) = 0$, then we distinguish the following cases:

- If $f''(c) < 0$, c is a local maximum.
- If $f''(c) > 0$, c is a local minimum.
- If $f''(c) = 0$ and $f'(c) = 0$, then c is not a relative minimum or maximum.

The second derivative test is an alternative to the first-derivative test for stationary points and is still needed for singular points.

Example 4.6. Consider the function $g(x) = x^3 - 6x^2 + 6x$. Then,

$$g'(x) = 3x^2 - 12x + 6.$$

Stationary points are found where $g'(x) = 0$. The second derivative is

$$g''(x) = 6x.$$

Evaluating at the stationary points:

$$g''(-2) = -12 < 0, \quad \text{which implies that} \quad x = -2 \text{ is a local maximum.}$$

$$g''(2) = 12 > 0, \quad \text{which implies that} \quad x = 2 \text{ is a local minimum.}$$

□

4.3 Elasticity

Price elasticity is a measure of how demand changes as price changes. Suppose that $q = f(p)$ gives us the demand equation for a product where q is the quantity and p is the price. Suppose that we increase the price p by a very small amount Δp:

$$\text{percentage increase in price} = \frac{\Delta p}{p} \cdot 100\%.$$

An increase in price of Δp will result in a decrease in q say by Δq:

$$\text{percentage decrease in price} = -\frac{\Delta q}{q} \cdot 100\%.$$

We define **elasticity** E as

$$E = \frac{\text{percentage decrease in quantity (demand)}}{\text{percentage increase in price}} = -\frac{\Delta q}{q} \cdot 100\% \cdot \frac{p}{\Delta p \cdot 100\%} = -\frac{\Delta q}{\Delta p} \cdot \frac{p}{q}.$$

If Δp is very close to zero, i.e., $\Delta p \to 0$, we have

$$E = -\frac{dq}{dp} \cdot \frac{p}{q}.$$

The price elasticity of demand $E = -\frac{dq}{dp} \cdot \frac{p}{q}$ measures how sensitive the demand is to changes in price. Note that $E > 0$, since $dq/dp < 0$.

Definition 4.3. We consider three cases:

1. If $E < 1$, then the demand is **inelastic**.
2. If $E = 1$, then the demand is **elastic** (very sensitive to changes in price).
3. If $E > 1$, then the demand has **unit elasticity**.

Example 4.7. Suppose that the demand equation for a product is given by

$$q = 10,000 - 5p,$$

where p is the price in dollars. The derivative of q with respect to p is $\frac{dq}{dp} = -5$. This gives us the elasticity E as follows:

$$E = -\frac{dq}{dp} \cdot \frac{p}{q} = -(-5) \cdot \frac{p}{10,000 - 5p} = \frac{5p}{10,000 - 5p}.$$

We can express E as a function of p:

$$E = \frac{p}{2,000 - p}.$$

We consider a few values of p to determine elasticity:

- If $p = 500$, then
$$E = \frac{500}{2,000 - 500} = \frac{500}{1500} = \frac{1}{3} < 1.$$
This implies that the demand is inelastic (not sensitive to changes in price).
- If $p = 1500$, then
$$E = \frac{1500}{2,000 - 1500} = \frac{1500}{500} = 3 > 1.$$
This implies that the demand is elastic (sensitive to changes in price).
- If $p = 1000$, then
$$E = \frac{1000}{2,000 - 1000} = 1.$$

This implies that the demand has unit elasticity.

4.3.1 Relation of Elasticity to Revenue Maximization

Revenue is the product of price and quantity, given by

$$R = p \cdot q.$$

To find how R changes with p, we calculate the derivative of R with respect to p:

$$\frac{dR}{dp} = \frac{d(p \cdot q)}{dp} = q + p \cdot \frac{dq}{dp}.$$

By the product rule, this can be written as

$$\frac{dR}{dp} = q + p \cdot \frac{dq}{dp} = q\left(1 + \frac{p}{q} \cdot \frac{dq}{dp}\right).$$

We define elasticity E as

$$E = -\frac{dq}{dp} \cdot \frac{p}{q}.$$

Substituting the value of E in the derivative of R we get

$$\frac{dR}{dp} = q(1 - E).$$

We have three cases:

Case 1: If $E > 1$, that is, the demand is elastic, $\frac{dR}{dp} < 0$. An increase in price will decrease revenue $\Rightarrow$ decrease price.

Case 2: If $E < 1$, that is, the demand is inelastic, $\frac{dR}{dp} > 0$. An increase in price will increase revenue $\Rightarrow$ increase price.

Case 3: If $E = 1$, i.e., the demand has unit elasticity, $\frac{dR}{dp} = 0 \Rightarrow$ this price p at which $E = 1$ is a stationary point. For $E < 1$, $\frac{dR}{dp} > 0$. For $E > 1$, $\frac{dR}{dp} < 0$. Hence, this p maximizes R.

Exercise 4.4. The consumer demand equation for tissues is given by $q = (100 - p)^2$, where p is the price per case of tissues and q is the demand in weekly sales.

a) Determine the price elasticity of demand E when the price is set at \$30, and interpret your answer.

b) At what price should tissues be sold to maximize the revenue?

c) Approximately how many cases of tissues would be demanded at that price?

Solution.

a) Given the demand equation $q = (100-p)^2$, the price elasticity of demand is defined as

$$E = -\frac{dq}{dp} \cdot \frac{p}{q}.$$

For $p = 30$,

$$E = -\frac{d[(100-p)^2]}{dp} \cdot \frac{p}{q} = -2(100-p)(-1) \cdot \frac{p}{(100-p)^2} = 2 \cdot \frac{(100-p)p}{(100-p)^2} = \frac{2p}{100-p}$$

$$E = \frac{2(30)}{100-30} = \frac{60}{70} = \frac{6}{7} \approx 0.86 < 1.$$

This indicates that the demand is inelastic, meaning that a 1% increase in price will result in less than a 1% decrease in the quantity demanded, suggesting that an increase in price might not significantly decrease demand.

b) Revenue R is maximized when $E = 1$, that is when

$$E = \frac{2p}{100-p} = 1.$$

Solving for p gives us

$$2p = 100 - p \Leftrightarrow 3p = 100 \Leftrightarrow p = \frac{100}{3} \approx \$33.33.$$

c) For $p = \$33.33$, we get

$$q = (100-p)^2 = (100-33.33)^2 \approx 4444 \text{ cases of tissues.}$$

4.3.2 Income Elasticity of Demand

Let $q(x)$ denote the probability that a person will purchase a product, where x is the income of a potential customer. The income elasticity of demand is defined to be

$$E = \frac{dq}{dx} \cdot \frac{x}{q}.$$

Note that there is no negative sign in front of this elasticity formula, because $\frac{dq}{dx} > 0$.

Example 4.8. The likelihood that a child will attend a live theatrical performance can be modeled by the following demand equation:

$$q = 0.01(-0.0078x^2 + 1.5x + 4.1),$$

where q is the fraction of children with annual household income x thousand dollars who will attend a live dramatic performance at a theater during the year. Compute the income elasticity of demand at an income level of \$20,000 and interpret the result. (Round your answer to two significant digits.)

Solution. The income elasticity of demand is given by

$$E = \frac{dq}{dx} \cdot \frac{x}{q}.$$

Calculating the derivative of q with respect to x:

$$\frac{dq}{dx} = 0.01\left(-0.0156x + 1.5\right).$$

Substituting the values into the elasticity formula:

$$E = \frac{0.01(-0.0156x + 1.5)}{0.01(-0.0078x^2 + 1.5x + 4.1)} \cdot x = \frac{-0.0156x + 1.5}{-0.0078x^2 + 1.5x + 4.1} \cdot x.$$

At $x = 20$ (representing \$20,000), we calculate $E \approx 0.77$. This means that if the household income increases by 1%, the fraction of kids attending will increase by 0.77% approximately.

4.4 Functions of Several Variables

Definition 4.4. A real-valued function f of $x, y, z, \ldots$ is a rule for obtaining a new number $f(x, y, z, \ldots)$ from the values of a sequence of independent variables $(x, y, z, \ldots)$. If there are two or more variables that can change a function, then that function is called a *multivariate function* or a function of multiple (several) variables.

Example 4.9. Consider the function $f(x, y) = 3x^2y - 2 + y^3$.

$$\begin{aligned} f(0, 3) &= 3(0)(3)^2 - 2 + 2(3) = 0 - 2 + 27 = 25 \\ f(3, 0) &= 3(3)^2(0) - 2 + 0^3 = -2. \end{aligned}$$

Definition 4.5. A linear multivariate function is of the form

$$f(x_1, x_2, \ldots, x_n) = a_0 + a_1x_1 + a_2x_2 + \ldots + a_nx_n,$$

where $a_0, a_1, \ldots, a_n$ are constants and $x_1, x_2, \ldots, x_n$ are independent variables.

Example 4.10. Your sales of online video and audio clips are booming. Your Internet provider wants to get in on the action and has offered you unlimited technical assistance and consulting if you agree to pay them 3 cents for every video clip and 4 cents for every audio clip you sell on the site. Further, the provider agrees to charge you only \$10 per month to host your site. Set up a monthly cost function for the scenario, and describe each variable.

Solution. Let x denote the number of video clips, y the number of audio clips, and $c(x, y)$ the monthly cost function in dollars. Then,

$$c(x, y) = 10 + 0.03x + 0.04y.$$

□

4.5 Partial Derivatives

If f is a function of x, then $\frac{df}{dx}$ gives us how fast f changes with respect to x. If f is a function of multiple variables, then "partial derivatives" tell us how fast f changes as one of the variables while all other variables are kept fixed.

Definition 4.6. The partial derivative of f with respect to x is the derivative of f with respect to x when all other variables are treated as constants.

Partial derivatives are denoted by $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots$

Example 4.11. We compute the partial derivatives of several functions.

1. Consider the function $f(x, y) = x^3 + 2y^2$. Then,

$$\frac{\partial f}{\partial x} = 3x^2 + 0 = 3x^2,$$
$$\frac{\partial f}{\partial y} = 0 + 4y = 4y.$$

2. Consider the function $z = x^2y^2 + x^3y + x$. Then,

$$\frac{\partial z}{\partial x} = y^2 \cdot (2x) + y \cdot (3x^2) + 1 = 2xy^2 + 3x^2y + 1,$$
$$\frac{\partial z}{\partial y} = x^2 \cdot (2y) + x^3 \cdot 1 = 2x^2y + x^3.$$

3. Consider the function $g(x, y) = e^{x^4y+y}$. Then,

$$\frac{\partial g}{\partial x} = e^{x^4y+y} \cdot \frac{\partial(x^4y + y)}{\partial x} = e^{x^4y+y} \cdot (4x^3y) = 4x^3ye^{x^4y+y},$$

$$\frac{\partial g}{\partial y} = e^{x^4y+y} \cdot \frac{\partial(x^4y + y)}{\partial y} = e^{x^4y+y} \cdot (x^4 + 1) = (x^4 + 1)e^{x^4y+y}.$$

4. Consider the function $f(x, y, z) = x^4y^3 + 2xy$. Then,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^4y^3) + \frac{\partial}{\partial x}(2xy) = 4x^3y^3 + 2y,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^4y^3) + \frac{\partial}{\partial y}(2xy) = 3x^4y^2 + 2x,$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^4y^3) + \frac{\partial}{\partial z}(2xy) = 0 + 0 = 0.$$

Exercise 4.5. Your weekly cost (in dollars) to manufacture x cars and y trucks is given by the cost function:

$$C(x, y) = 200,000 + 6,000x + 4,000y - 100,000e^{-0.01(x+y)}.$$

What is the marginal cost of a car? Of a truck? How do these marginal costs behave as total production $x + y$ increases?

Solution. To find the marginal cost of manufacturing cars and trucks, we compute the partial derivatives of the cost function with respect to x and y, respectively. The marginal cost of a car is given by the partial derivative of C with respect to x:

$$\begin{aligned}
\frac{\partial C}{\partial x} &= \frac{\partial}{\partial x}(200,000 + 6,000x + 4,000y - 100,000e^{-0.01(x+y)}) \\
&= 0 + 6,000 + 0 - 100,000\frac{\partial}{\partial x}e^{-0.01(x+y)} \\
&= 6,000 - 100,000 \cdot (-0.01) \cdot e^{-0.01(x+y)} \cdot \frac{\partial}{\partial x}(x + y) \\
&= 6,000 - 100,000 \cdot (-0.01) \cdot e^{-0.01(x+y)} \cdot (1) \\
&= 6,000 + 1,000e^{-0.01(x+y)}.
\end{aligned}$$

Similarly, the marginal cost of a truck is given by the partial derivative of C with respect to y:

$$\begin{aligned}
\frac{\partial C}{\partial y} &= \frac{\partial}{\partial y}(200,000 + 6,000x + 4,000y - 100,000e^{-0.01(x+y)}) \\
&= 0 + 0 + 4,000 - 100,000\frac{\partial}{\partial y}e^{-0.01(x+y)} \\
&= 4,000 - 100,000 \cdot (-0.01) \cdot e^{-0.01(x+y)} \cdot \frac{\partial}{\partial y}(x + y) \\
&= 4,000 - 100,000 \cdot (-0.01) \cdot e^{-0.01(x+y)} \cdot (1) \\
&= 4,000 + 1,000e^{-0.01(x+y)}.
\end{aligned}$$

As the total production $x + y$ increases, the term $e^{-0.01(x+y)}$ approaches zero. This implies that the marginal cost for both a car and a truck will decrease as production increases, due to the diminishing exponential term. Graphs for both marginal costs with respect to total production would show a decreasing trend. □

4.6 Extra Problems

Exercise 4.6. Given a piece of wire 100 cm long, what dimensions should a rectangle have to maximize the enclosed area?

Solution. Let the length of the rectangle be x cm, and the width be y cm. The perimeter constraint is $2x + 2y = 100$, which simplifies to $x + y = 50$. Solving for y, we have $y = 50 - x$. The area A of the rectangle is given by

$$A = xy = x(50 - x) = 50x - x^2.$$

To find the maximum area, take the derivative of A with respect to x and set it to zero:

$$\frac{dA}{dx} = 50 - 2x = 0 \Leftrightarrow x = 25.$$

Since $y = 50 - x$, then $y = 25$ as well. Therefore, the rectangle should be a square with sides of 25 cm to maximize the area. □

Exercise 4.7. A company wants to manufacture cylindrical cans that will hold 500 ml of liquid. What dimensions should the can have to minimize the cost of the metal to make the cans? Assume no waste material.

Solution. Let r be the radius and h be the height of the cylinder. The volume V of the cylinder is given by $V = \pi r^2 h$. Given $V = 500$, we have

$$\pi r^2 h = 500 \Leftrightarrow h = \frac{500}{\pi r^2}.$$

The surface area S of the cylinder (which approximates the cost) is:

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{500}{\pi r^2} = 2\pi r^2 + \frac{1000}{r}.$$

Minimize S by taking its derivative with respect to r and setting it to zero:

$$\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2} = 0 \Leftrightarrow 4\pi r^3 = 1000 \Leftrightarrow r^3 = \frac{250}{\pi} \Leftrightarrow r \approx 4.2\,\text{cm}.$$

Then, $h = \frac{500}{\pi(4.2)^2} \approx 9.0\,\text{cm}$. □

Exercise 4.8. A companys profit P in thousands of dollars is given by $P(x) = -3x^2 + 36x - 96$, where x is the number of units sold in thousands. Find the number of units the company should sell to maximize its profit.

Solution. To find the maximum profit, take the derivative of P with respect to x and set it to zero:

$$\frac{dP}{dx} = -6x + 36 = 0 \Leftrightarrow x = 6.$$

To confirm it is a maximum, check the second derivative:

$$\frac{d^2P}{dx^2} = -6 < 0,$$

which indicates a maximum. The company should sell 6,000 units to maximize its profit. □

Exercise 4.9. Find the point on the curve $y = x^2 + 4$ that is closest to the point $(4, 1)$.

Solution. The distance D from a point $(x, x^2 + 4)$ on the curve to the point $(4, 1)$ is given by

$$D^2 = (x - 4)^2 + (x^2 + 4 - 1)^2.$$

Minimize D^2 (to avoid the square root) by taking its derivative and setting it to zero:

$$\frac{d}{dx}\left[(x-4)^2 + (x^2+3)^2\right] = 2(x-4) + 2(x^2+3)(2x) = 0.$$

Solving this equation (using numerical methods or a calculator if necessary) yields $x \approx 3.1$, and therefore $y \approx 13.6$. □

Exercise 4.10. A theater finds that they sell 120 tickets when ticket prices are \$50 each. Past data suggest that for each \$1 decrease in price, five more tickets are sold. What ticket price maximizes revenue?

Solution. Let x be the decrease in ticket price. Then, the price per ticket is $50 - x$ and the number of tickets sold is $120 + 5x$. The revenue R is

$$R = (50 - x)(120 + 5x) = 6000 + 250x - 5x^2.$$

Maximize R by setting its derivative to zero:

$$\frac{dR}{dx} = 250 - 10x = 0 \Leftrightarrow x = 25.$$

The ticket price that maximizes revenue is $50 - 25 = \$25$. □

Exercise 4.11. Find the dimensions of a box with a fixed volume of 1000 cm^3 that minimizes the surface area.

Solution. Let the dimensions be x, y, and z. Given $xyz = 1000$, assume $x = y$ for simplicity (a cube or nearly cube will minimize surface area):

$$x^2z = 1000 \Leftrightarrow z = \frac{1000}{x^2}.$$

The surface area S is

$$S = 2(x^2) + 4(xz) = 2x^2 + 4x\frac{1000}{x^2} = 2x^2 + \frac{4000}{x}.$$

Minimize S by taking its derivative:

$$\frac{dS}{dx} = 4x - \frac{4000}{x^2} = 0 \Leftrightarrow 4x^3 = 4000 \Leftrightarrow x = \sqrt[3]{1000} = 10\,\text{cm}.$$

Then, $z = \frac{1000}{100} = 10\,\text{cm}$. All dimensions are 10 cm. □

Exercise 4.12. Maximize the volume of a cylinder that can be inscribed in a sphere of radius 10 cm.

Solution. Let the radius of the cylinder be r and its height be h. The sphere's equation is $r^2 + \left(\frac{h}{2}\right)^2 = 10^2$, so

$$h = 2\sqrt{100 - r^2}.$$

The volume V of the cylinder is

$$V = \pi r^2 h = \pi r^2(2\sqrt{100 - r^2}) = 2\pi r^2\sqrt{100 - r^2}.$$

To maximize V, take its derivative:

$$\frac{dV}{dr} = 4\pi r\sqrt{100 - r^2} - \frac{2\pi r^3}{\sqrt{100 - r^2}}.$$

Setting $\frac{dV}{dr} = 0$ and solving yields $r = \frac{10\sqrt{2}}{2} \approx 7.07\,\text{cm}$ and $h \approx 14.14\,\text{cm}$. □

Exercise 4.13. Find the area of the largest rectangle that can be inscribed under the curve $y = 16 - x^2$ in the first quadrant.

Solution. Let the vertices of the rectangle be at $(x, 0)$, $(x, 16 - x^2)$, $(0, 16 - x^2)$, and $(0, 0)$. The area A of the rectangle is

$$A = x(16 - x^2).$$

Maximize A by taking its derivative:

$$\frac{dA}{dx} = 16 - 3x^2 = 0 \Leftrightarrow x^2 = \frac{16}{3} \Leftrightarrow x = \sqrt{\frac{16}{3}} \approx 2.31.$$

The maximum area is when $x \approx 2.31$, so

$$A = 2.31(16 - (2.31)^2) \approx 27.23\,\text{cm}^2.$$

□