# Homogeneity fusion for Sparse Grouped Network VAR models

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# Motivation

- Multivariate time series show up in many modern applications:
  - Rumor spreading in a social network
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- Sample size: $dT$
- If $d < T$, use standard OLS estimator.
- If $d > T$, OLS is impossible. Instead, assume structure:
  - Sparsity (Lasso) [Basu and Michailidis, 2015]
  - Low Rank (factor models) [Stock and Watson, 2011]
  - Network models with few parameters [Zhu et al., 2017]

# Network Time Series

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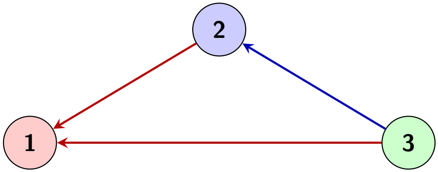


$$Y_{1,t} = \nu_1 Y_{1,t-1} + \varepsilon_{1,t}$$
$$Y_{2,t} = \beta_{21} Y_{1,t-1} + \nu_2 Y_{2,t-1} + \varepsilon_{2,t}$$
$$Y_{3,t} = \beta_{31} Y_{1,t-1} + \beta_{32} Y_{2,t-1} + \nu_3 Y_{3,t-1} + \varepsilon_{3,t}$$

# Related Work

- **Network VAR models** introduced by [Zhu et al., 2017], modeling temporal dependence through networks.

- Types of **group/community** structures incorporated in [Zhu and Pan, 2020, Chen et al., 2023, Zhu et al., 2023].

- **Unknown networks:** community detection in [Gudhmundsson and Brownlees, 2021] (Blockbuster algorithm) and network estimation in [Martin et al., 2024] (NIRVAR model).

# The GNAR model

- **Unknown** directed network with adjacency matrix $A = (a_{ij})_{1 \leq i,j \leq d}$.
- Latent partition $\mathbb{G} = \{G_1, \ldots, G_K\}$ of $\{1, \ldots, d\}$ into $K$ groups. We denote the group membership of node $i$ by $g_i$.

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- GNAR model [Zhu et al., 2023]:

$$Y_{i,t} = \sum_{j=1}^{d} \phi_{g_i g_j} \frac{a_{ij}}{n_i} Y_{j,t-1} + \nu_{g_i} Y_{i,t-1} + \varepsilon_{i,t}, \quad i \in [d],\ t \in [T],$$

where $n_i = \sum_{j=1}^{d} a_{ij}$ the out-degree of $i$, and $\varepsilon_{i,t} \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2)$.

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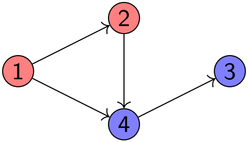
- *Remark:* When $K = 1$, the model reduces to the original model of [Zhu et al., 2017] and there are two parameters: network effect $\phi$ and autoregressive effect $\nu$.

# Example




Updating equations:

$$\begin{cases} Y_{1,t} = \nu_{1} Y_{1,t-1} + \dfrac{\phi_{11}}{2} Y_{2,t-1} + \dfrac{\phi_{12}}{2} Y_{4,t-1} + \varepsilon_{1,t} \\ Y_{2,t} = \nu_{1} Y_{2,t-1} + \phi_{12} Y_{4,t-1} + \varepsilon_{2,t} \\ Y_{3,t} = \nu_{2} Y_{3,t-1} + \varepsilon_{3,t} \\ Y_{4,t} = \phi_{22} Y_{3,t-1} + \nu_{2} Y_{4,t-1} + \varepsilon_{4,t} \end{cases}$$

# Fusion estimator

We want to solve the following optimization problem:

## Fusion estimator

$$\underset{\substack{A\in\{0,1\}^{d\times d},\mathbb{G}\in[K]^d \\ \Phi\in\mathbb{R}^{K\times K},\boldsymbol{\nu}\in\mathbb{R}^K}}{\operatorname{argmin}} \sum_{t=2}^{T}\sum_{i=1}^{d}\left(Y_{i,t}-\sum_{j=1}^{d}\phi_{g_i g_j}\frac{a_{ij}}{n_i}Y_{j,t-1}-\nu_{g_i}Y_{i,t-1}\right)^2$$

$$\text{s.t.}\quad \sum_{i=1}^{d}\sum_{j=1}^{d}a_{ij}\le s$$

# Theoretical goals

1. The misspecification probability for grouping/network goes to 0, i.e.,

$$\mathbb{P}(\mathbb{G}(\hat{B}) \neq \mathbb{G}(B^*)) \to 0, \quad \text{as } d, T \to \infty,$$

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   with an explicit convergence rate.

2. Estimation and prediction errors go to 0, i.e.,

$$\|\hat{\boldsymbol{\beta}}^{\mathsf{ol}} - \boldsymbol{\beta}^*\|_2 \to 0 \quad \text{and} \quad \|Z(\hat{\boldsymbol{\beta}}^{\mathsf{ol}} - \boldsymbol{\beta}^*)\|_2 \to 0,$$

   with specified convergence rates.

# Roadmap

1. Bound the misspecification probability:

$$\mathbb{P}\left(\mathbb{G}(\hat{B}) \neq \mathbb{G}(B^*)\right)$$

$$\leq \sum_{\omega \in \{\mathbb{G}(B) \mid B \in \Theta(K^*, s^*), \mathbb{G}(B) \neq \mathbb{G}(B^*)\}} \mathbb{P}\left( \min_{\substack{\widetilde{B} \in \Theta(K^*, s^*) \\ \mathbb{G}(\widetilde{B}) = \omega}} \|\mathbf{Y} - Z\widetilde{\boldsymbol{\beta}}\|_2^2 < \|\mathbf{Y} - Z\hat{\boldsymbol{\beta}}^{\mathrm{ol}}\|_2^2 \right)$$

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2. Handle each term

$$\mathbb{P}\left( \min_{\substack{\widetilde{B} \in \Theta(K^*, s^*) \\ \mathbb{G}(\widetilde{B}) = \omega}} \|\mathbf{Y} - Z\widetilde{\boldsymbol{\beta}}\|_2^2 < \|\mathbf{Y} - Z\hat{\boldsymbol{\beta}}^{\mathsf{ol}}\|_2^2 \right) = \mathbb{P}\left( \|\mathbf{Y} - Z\hat{\boldsymbol{\beta}}\|_2^2 < \|\mathbf{Y} - Z\hat{\boldsymbol{\beta}}^{\mathsf{ol}}\|_2^2 \right)$$

$$= \mathbb{P}\left(2\boldsymbol{\varepsilon}'(I - P_{\mathbb{G}(B)})Z\boldsymbol{\beta}^* + \|(I - P_{\mathbb{G}(B)})Z\boldsymbol{\beta}^*\|_2^2 - \boldsymbol{\varepsilon}'(P_{\mathbb{G}(B)} - P_{\mathbb{G}(B^*)})\boldsymbol{\varepsilon} < 0\right),$$

where $P_{\mathbb{G}(B)}$ is a $dT \times dT$ projection matrix including all lagged values.

# Remarks

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   - Conditional estimation does not guarantee solution to the optimization problem of interest.
   - Our fusion estimator handles all components jointly, allowing for better theoretical guarantees.

# Mixed Integer Programming

- *Idea:* formulate the original optimization problem as a MIP and use Gurobi to solve it, as in [Bertsimas et al., 2016].

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- General MIP formulation:

$$\min \left( \mathbf{x}' Q \mathbf{x} + \mathbf{x}' \mathbf{y} \right) \quad \text{s.t.} \quad \mathbf{x} \in \mathcal{X},$$

  where $\mathcal{X}$ is the feasible set and the vector **x** includes:

  - *Binary variables:* $x_i \in \{0, 1\}$ for $i \in \mathcal{I}$
  - *Continuous variables:* $x_i \geq 0$ for $i \notin \mathcal{I}$

# MIP formulation of GNAR optimization problem

$$\underset{\substack{B\in\mathbb{R}^{d\times d},\, A\in\{0,1\}^{d\times d}\\ \Phi\in\mathbb{R}^{K\times K},\, \boldsymbol{\nu}\in\mathbb{R}^{K},\\ \Omega\in\{0,1\}^{d\times d\times K\times K}}}{\operatorname{argmin}} \sum_{t=2}^{T}\sum_{i=1}^{d}\left(Y_{i,t}-\sum_{j=1}^{d} b_{ij}Y_{j,t-1}\right)^2$$

$$\begin{aligned}
\text{s.t.}\quad & a_{ij}\in\{0,1\},\, a_{ii}=0,\, n_i=\sum_{j=1}^{d} a_{ij},\quad i,j\in[d]\\
& \omega_{ijkl}\in\{0,1\},\quad i,j\in[d],\, k,l\in[K]\\
& \sum_{k=1}^{K}\sum_{j=1}^{d}\sum_{l=1}^{K}\omega_{ijkl}=1,\quad i\in[d]\\
& \sum_{j=1}^{d}\sum_{l=1}^{K}\omega_{ijkl}(b_{ii}-\nu_k)=0,\quad i\in[d],k\in[K]\\
& \omega_{ijkl}\left(n_i b_{ij}-\phi_{kl}a_{ij}\right)=0,\quad i,j\in[d],\, i\neq j,\, k,l\in[K]\\
& \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\leq s.
\end{aligned}$$

# Implementation

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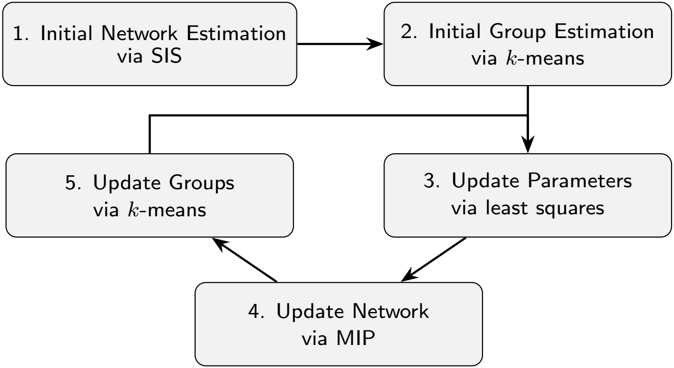
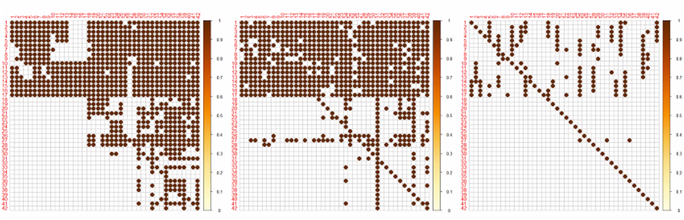
- Instead, follow the iterative scheme:



# Real Data: FRED-MD Example

- *Data:* 134 monthly U.S. macroeconomic variables.
- We focus on a subset of 42 variables in two groups:
  - Group 6: Interest & Exchange Rates (22 nodes)
  - Group 7: Prices (20 nodes)
- We treat variables as nodes in a network and apply our method to estimate:
  - The edge structure (adjacency matrix)
  - Group memberships



*Estimated adjacency matrices. Left: Fusion, Middle: Best Subset, Right: Adaptive Lasso*

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- We formulated our problem as a mixed integer program and developed a practical **iterative algorithm** to solve this problem at scale.
- Applied to real macroeconomic data, our method reveals interesting networks and groupings, complementing existing empirical approaches.

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