Simple Harmonic Motion
Unit 2: Momentum, Impulse and Energy
Grade 12 Physics

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Quick Review of Hooke's Law

Hooke's law for an ideal spring shows that the force a compressed or stretched spring exerts on objects connected to it is proportional to its displacement:

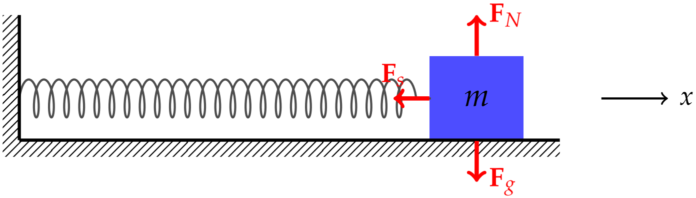
$$\boxed{\mathbf{F}_s = -k\mathbf{x}}$$

The **spring constant** k (or **Hooke's constant**, **force constant**) is the stiffness of the spring. The elastic potential energy stored in the spring is:

$$\boxed{U_e = \frac{1}{2}kx^2}$$

Mass on a Spring

Consider the forces acting on a mass connected horizontally to a spring



The net force is due only to spring force $\mathbf{F}_s = -k\mathbf{x}$. This is true both when the spring is in compression or extension. (In the diagram above, the spring is in extension.) We can apply Newton's 2nd law in the direction of motion, i.e.:

$$-kx = ma$$

This is a standard problem in basic calculus.

A Simple Solution

The solution to this *second-order ordinary dfferential equation* is quite simple:

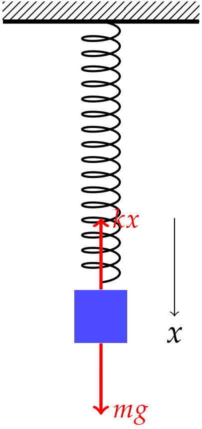
$$x(t) = A\sin(2\pi f t + \phi)$$
$$v(t) = 2A\pi f\cos(2\pi f t + \phi)$$
$$a(t) = -4A\pi^2 f^2\sin(2\pi f t + \phi)$$

f is the frequency of vibration, A is the amplitude, and ϕ is a phase shift that depends on the initial condition. The frequency f and period T are given by:

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}} \qquad \boxed{T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}}$$

Note that f and T do not depend on amplitude A.

Vertical Spring-Mass System



For a vertical spring-mass system, the analysis is *slightly* more complicated, but the result is very similar:

$$x(t) = A\sin(2\pi f t + \phi) + B$$
$$a(t) = -4A\pi^2 f^2 \sin(2\pi f t + \phi)$$

B is just the stretching of the spring due to its weight, given by:

$$B = \frac{mg}{k}$$

and frequency f is the same as the horizontal case

Conservation of Energy in a Spring-Mass System

Since acceleration is non constant for spring-mass problems (kinematics require calculus) we instead use **conservation of energy** to solve these problems.

- If there is no friction, the forces by the spring on the mass (and by the mass on the spring) are *internal* forces, and energy in the system is conserved:

$$K_1 + U_{e,1} + U_{g,1} = K_2 + U_{e,2} + U_{g,2}$$

- For the horizontal spring-mass system, the total energy is:

$$E_{\text{total}} = \frac{1}{2}kA^2$$

Simple Example

Example 1: A mass suspended from a spring is oscillating up and down. Consider the following two statements:

1. At some point during the oscillation, the mass has zero velocity but it is accelerating
2. At some point during the oscillation, the mass has zero velocity and zero acceleration.

(a) Both occur at some time during the oscillation
(b) Neither occurs during the oscillation
(c) Only (1) occurs
(d) Only (2) occurs

Another Example

Example 2: An object of mass $5\,\mathrm{kg}$ hangs from a spring and oscillates with a period of $0.5\,\mathrm{s}$. By how much will the equilibrium length of the spring be shortened when the object is removed.

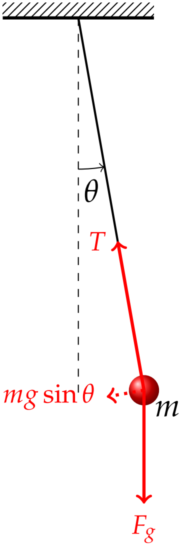
(a) $0.75\,\mathrm{cm}$

(b) $1.50\,\mathrm{cm}$

(c) $3.13\,\mathrm{cm}$

(d) $6.20\,\mathrm{cm}$

What About a Pendulum?



- Pendulums also exhibit simple harmonic motion
- For a pendulum, there are two forces acting on the mass: weight $F_g = mg$ and tension T
- When the mass is deflected by an angle θ, there is a component of gravity pointing towards the rest position, with magnitude $mg\sin\theta$
- For small angles (less than 15°), $\sin\theta \approx \theta$ so this force is proportional θ (like in Hooke's law where $F_s \propto x$)

Equation for the Pendulum

For Small Angles ($\theta < 15^\circ$)

Using some calculus, we find that the solution for $\theta(t)$ to be very similar to the spring-mass system:

$$\boxed{\theta(t) = \theta_{\max} \sin(2\pi f t + \phi)}$$

where frequency f is given by

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}}$$

and ϕ is a phase shift based on the initial condition of the pendulum.

Pendulum Example Problem

Example 3: A simple pendulum consists of a mass m attached to a light string of length l. If the system is oscillating through small angles, which of the following is true

(a) The frequency is independent of the acceleration due to gravity, g.

(b) The period depends on the amplitude of the oscillation.

(c) The period is independent of the mass m.

(d) The period is independent of the length l.

Another Pendulum Example

Example 4: A bucket full of water is attached to a rope and allowed to swing back and forth as a pendulum from a fixed support. The bucket has a hole in its bottom that allows water to leak out. How does the period of motion change with the loss of water?

(a) The period does not change.

(b) The period continuously decreases.

(c) The period continuously increases.

(d) The period increases to some maximum and then decreases again.

Think About g

Example 5: A little girl is playing with a toy pendulum while riding in an elevator. Being an astute and educated young lass, she notes that the period of the pendulum is $T = 0.5\,\mathrm{s}$. Suddenly the cables supporting the elevator break and all of the brakes and safety features fail simultaneously. The elevator plunges into free fall. The young girl is astonished to discover that the pendulum has:

(a) continued oscillating with a period of $0.5\,\mathrm{s}$.

(b) stopped oscillating entirely.

(c) decreased its rate of oscillation to have a longer period.

(d) increased its rate of oscillation to have a lesser period.