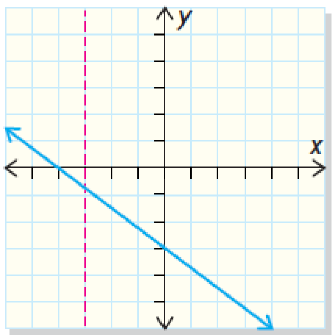
Chapter 3 Transformations of Functions (1)

1. Relation and Function

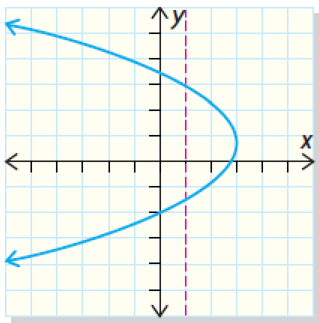
Relation – an identified pattern between two variables that may be represented as ordered pairs, a table of values, a graph, or an equation.

A **function** is a relation in which each value of the independent variable x corresponds to exactly one value of the dependent variable y.

You can use the vertical-line test to check whether a graph represents a function. A graph represents a function if every vertical line intersects the graph in at most one point. This shows that there is only one element in the range for each element of the domain.



A relation that is a function



A relation that is not a function

To write a function using function notation, the form f(x) =… is used to indicate a function, f, which independent variable x. The notation f(3) means the value obtained when x = 3 is substituted, and it is read as "f at 3" or "f of 3".

The symbols g(x) and h(x) are often used to name the outputs of functions, but other letters are also used, such as v(t) for velocity as a function of time.

We call *y* the value of *f* at *x* or the image of *x* under *f*. We also say that *f* maps *x* to *y*.

Example

If h(x) = 2 – 3x and g(x) = x2 + 1, find:

a) h(2) – 3g(1)
b) g(h(2))
c) x when h(x) = g(x)

2. Domain and Range of Functions

Domain – set of values of the independent variable (x-values) of the relation.
Range – set of the dependent variable (y-values) of the relation.

For a function, for each given element of the domain there must be exactly one element in the range.

Functions can be represented in various ways: in words, a table of values, a set of ordered pairs, a mapping diagram, a graph, or an equation.

1) State the domain and range of the following relation. Is the relation a function?

{(2, –3), (4, 6), (3, –1), (6, 6), (2, 3)}

This list of points, being a relationship between certain *x*'s and certain *y*'s, is a relation. The domain is all the *x*-values, and the range is all the *y*-values. Please list the values without duplication:

domain: {2, 3, 4, 6}
range: {–3, –1, 3, 6}

While this is a relation (because *x*'s and *y*'s are being related to each other), you have two points with the same *x*-value: (2, –3) and (2, 3). Since *x* = 2 gives you two possible destinations, then this relation is not a function.

2) State the domain and range of the following relation. Is the relation a function?

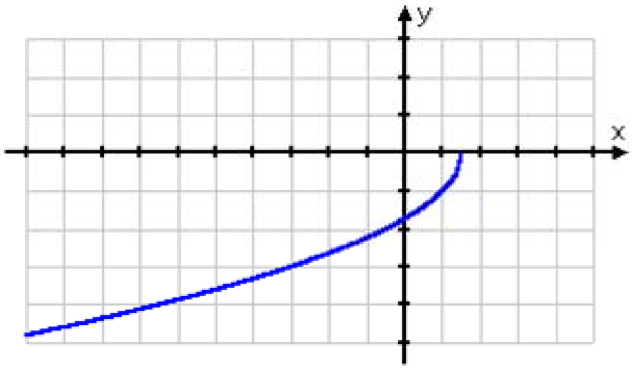
{(–3, 5), (–2, 5), (–1, 5), (0, 5), (1, 5), (2, 5)}

List the *x*-values for the domain and the y-values for the range:
domain: {–3, –2, –1, 0, 1, 2}
range: {5}

It is a function since there is only one y value for each x.

There is one other case for finding the domain and range of functions. They will give you a function and ask you to find the domain (and maybe the range, too).

3) Determine the domain and range of the given function: $-\sqrt{-2x+3}$



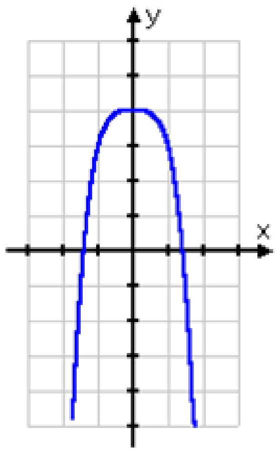
The domain is all values that *x* can take on. The only problem you have with this function is that you cannot have a negative inside the square root. So set the insides greater-than-or-equal-to zero, and solve. This will be the domain:

$-2x + 3 \geq 0$
$-2x \geq -3$
$2x \leq 3$
$x \leq 3/2 = 1.5$

Then the domain is "$\{x |\ x \leq 3/2, x \in \mathbb{R}\}$ ".
The range is all negative square root, which is always negative since the square root is always positive. The range is $\{y |\ y \leq 0, y \in \mathbb{R}\}$.

The domain and the range can also be found from the graph. While the graph goes down very slowly, you know that, eventually, you can go as low as you like (by picking an *x* that is sufficiently big). Also, from your experience with graphing, you know that the graph will never start coming back up.

4) Determine the domain and range of the given function:
$y = -x^4 + 4$



This is just a garden-variety polynomial. There are no denominators (so no division-by-zero problems) and no radicals (so no square-root-of-a-negative problems). There are no problems with a polynomial. There are no values that you can't plug in for x. When you have a polynomial, the answer is always that the domain is "$\{x |\ x \in \mathbb{R}\}$".

The range will vary from polynomial to polynomial. The graph goes only as high as $y = 4$, but it will go as low as you like. Then: The range is "$\{y |\ y \leq 4, y \in \mathbb{R}\}$".

Interval notation

The interval of numbers between *a* and *b*, including *a* and *b*, is often denoted [*a*, *b*]. The two numbers are called the *endpoints* of the interval.
To indicate that one of the endpoints is to be excluded from the set, the corresponding square bracket can be replaced with a parenthesis.

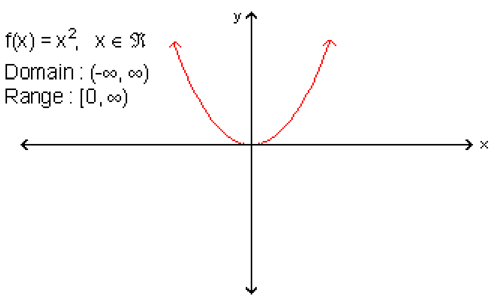
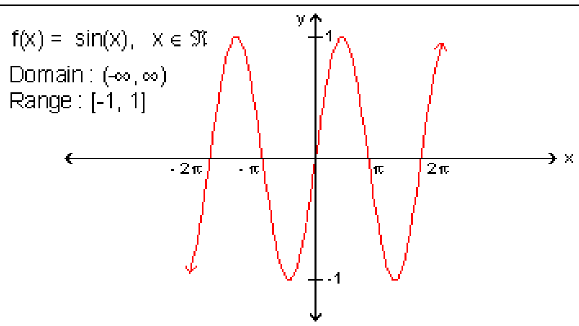
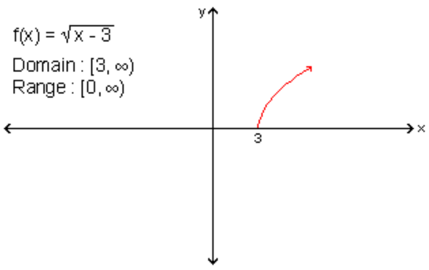
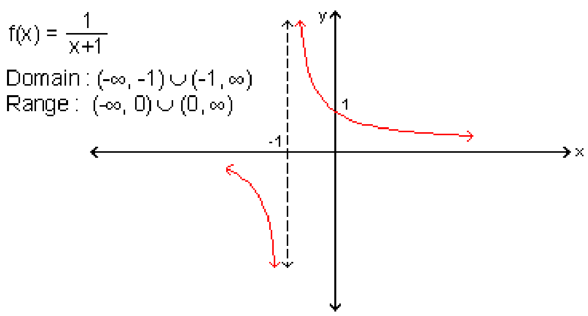
$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\},$$
$$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\},$$
$$(a,b] = \{x \in \mathbb{R} \mid a < x \le b\},$$
$$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}.$$

In both styles of notation, one may use an infinite endpoint to indicate that there is no bound in that direction. Specifically, one may use $a = -\infty$ or $b = +\infty$ (or both). For example, $(0, +\infty)$ is the set of all positive real numbers, and $(-\infty, +\infty)$ is the set of real numbers.

The ***union*** of sets is the set of elements that is in the first set ***"or"*** the second set. The symbol for Union is $\cup$.

$(-\infty,-1) \cup (4,\infty)$ = $x > 4$ ***"or"*** $x < -1$.

More examples of functions with use of interval notation

f(x) = x²	**f(x) = sin x**
	
f(x) = $\sqrt{x-3}$	**f(x) =** $\dfrac{1}{x+1}$
	

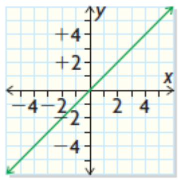
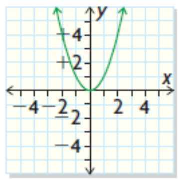
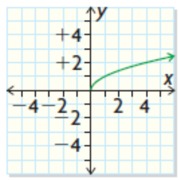
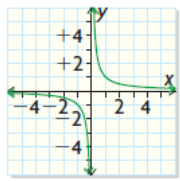
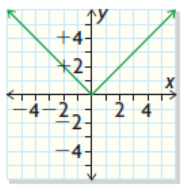
3. Parent Function

In mathematics, every function can be classified as a member of a **family**. Each member of a family of functions is related to the simplest, or most basic, function sharing the same characteristics. This function is called the **parent function.**

Parent functions include, but are not limited to,

$f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \frac{1}{x}$, and $f(x) = |x|$.

Each has unique characteristics that define the shape of its graph.

Equation of Function	Name of Function	Sketch of Graph
$f(x) = x$	linear function	
$f(x) = x^2$	quadratic function	
$f(x) = \sqrt{x}$	square root function	
$f(x) = \frac{1}{x}$	reciprocal function	
$f(x) = |x|$	absolute value function	

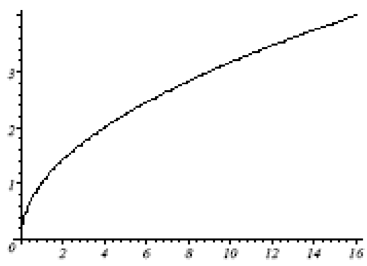
4. Horizontal and Vertical translation of Functions

1) Horizontal translation: y = f (x − a)

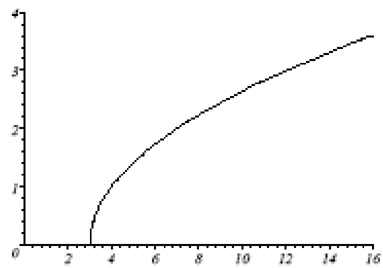
If we begin with the formula y = f (x) and we replace x with x−a where a is a constant, then we get the formula y = f (x − a). The graph of y = f (x − a) is obtained by translating the graph of y = f (x) to the right by a units if a is a positive number or to the left by |a| units if a is a negative number. If the point (x, y) is on the graph of y = f (x), then the point (x + a, y) is on the graph of y = f (x − a).

Example 1:

Starting with the formula y = $\sqrt{x}$, if we replace x with x − 3, then we get the formula y = $\sqrt{x-3}$. The graph of y = $\sqrt{x-3}$ is obtained by translating the graph of y = $\sqrt{x}$ to the right by 3 units.



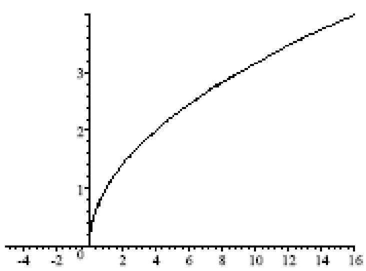
Graph of $y = \sqrt{x}$



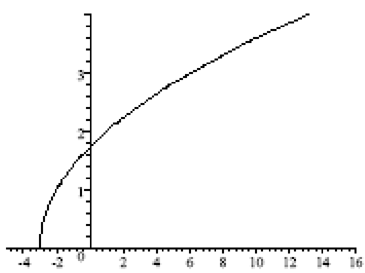
Graph of $y = \sqrt{x-3}$

Example 2:

Starting with the formula y = $\sqrt{x}$, if we replace x with x − (−3) = x+3, then we get the formula y = $\sqrt{x+3}$. The graph of y = $\sqrt{x+3}$ is obtained by translating the graph of y = $\sqrt{x}$ to the left by 3 units.



Graph of $y = \sqrt{x}$

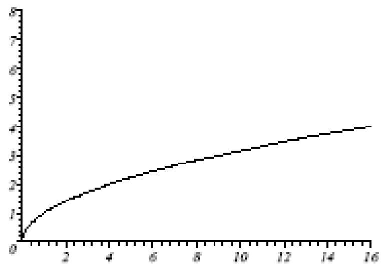


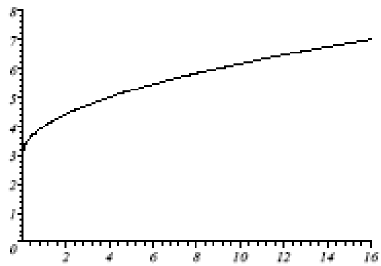
Graph of $y = \sqrt{x+3}$

2) Vertical translation: y = f (x) + a.
If we begin with the formula y = f (x) and we replace y with y −a where a is a constant, then we get the formula y − a = f (x) which can also be written as y = f (x) + a. The graph of y = f (x) + a is obtained by translating the graph of y = f (x) up by a units if a is a positive number or down by |a| units if a is a negative number. If the point (x, y) is on the graph of y = f (x), then the point (x, y + a) is on the graph of y = f (x) + a.

Example 3:

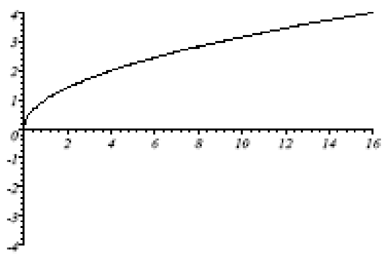
Starting with the formula y = $\sqrt{x}$, if we replace y with y − 3, then we get the formula y −3 = $\sqrt{x}$ which can also be written as y = $\sqrt{x}$ +3. The graph of y = √x + 3 is obtained by translating the graph of y = $\sqrt{x}$ up by 3 units.

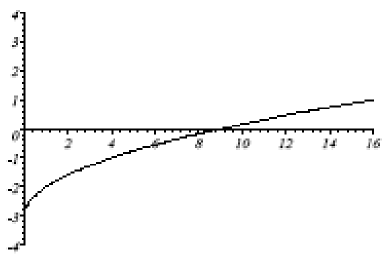

Graph of $y = \sqrt{x}$


Graph of $y = \sqrt{x} + 3$

Example 4:

Starting with the formula y = $\sqrt{x}$, if we replace y with y − (−3) = y +3, then we get the formula y + 3 = $\sqrt{x}$ which can also be written as y = $\sqrt{x}$ − 3. The graph of y = − 3 is obtained by translating the graph of y = $\sqrt{x}$ down by 3 units.


Graph of $y = \sqrt{x}$


Graph of $y = \sqrt{x} - 3$

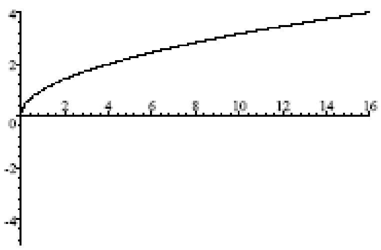
3) Horizontal and Vertical Transformations:

y = $\sqrt{4x-6}$ **− 5.**

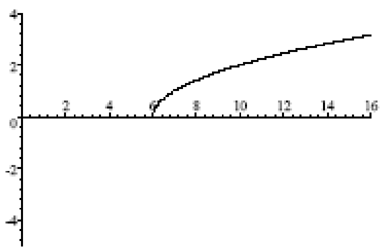
Starting with a "basic" function such as y = $\sqrt{x}$, we can perform a sequence of transformations to obtain the graph of a similar but "less basic" function. As an example, let us perform a sequence of transformations that lead to the graph of y = $\sqrt{4x-6} - 5$.

Step 1.

We begin with y = √x and replace x with x − 6 to obtain y = $\sqrt{4x-6}$. The graph of y = $\sqrt{x-6}$ is a translation of the graph of y = $\sqrt{x}$ to the right by 6 units.



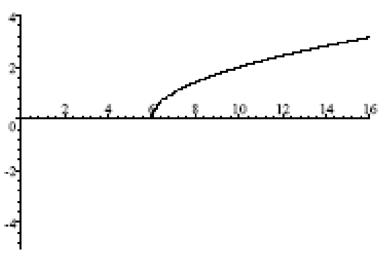
Graph of $y = \sqrt{x}$



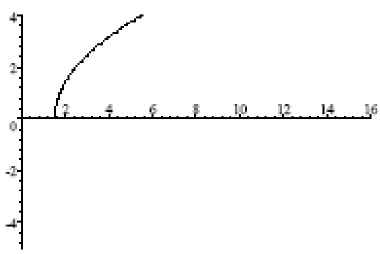
Graph of $y = \sqrt{x-6}$

Step 2.

We begin with y = $\sqrt{x-6}$ and replace x with 4x to obtain y = $\sqrt{4x-6}$. The graph of y = $\sqrt{4x-6}$ is a horizontal compression of the graph of y = $\sqrt{x-6}$ by a factor of 1/4.



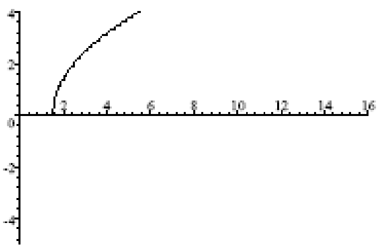
Graph of $y = \sqrt{x-6}$



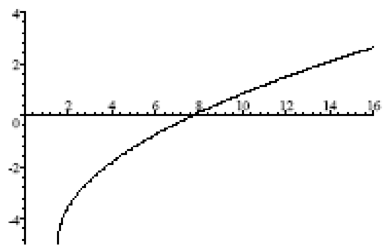
Graph of $y = \sqrt{4x-6}$

Step 3.

We begin with y = $\sqrt{4x-6}$ and replace y with y + 5 to obtain y + 5 = $\sqrt{4x-6}$ which is the same as y = $\sqrt{4x-6}$ − 5. The graph of y = $\sqrt{4x-6}$ −5 is a translation of the graph of y = $\sqrt{4x-6}$ downward by 5 units.



Graph of $y = \sqrt{4x-6}$



Graph of $y = \sqrt{4x-6} - 5$

5. Reflections

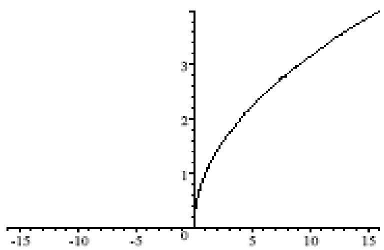
1) Reflections through the y Axis

If we begin with the formula y = f (x) and we replace x with −x, thenwe get the formula y = f (−x). The graph of y = f (−x) is obtained by reflecting the graph of y = f (x) through the y axis. If the point (x, y) is on the graph of y = f (x), then the point (−x, y) is on the graph of y = f (−x).

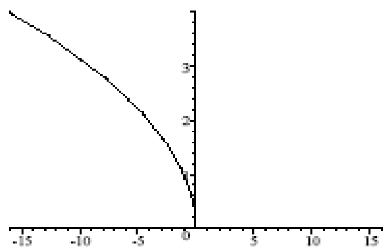
Example 5

Starting with the formula y = $\sqrt{x}$, if we replace x with −x, then we get the formula

y = $\sqrt{x}$. The graph of y = $\sqrt{x}$ is obtained by reflecting the graph of y = $\sqrt{x}$ through the y axis.



Graph of $y = \sqrt{x}$



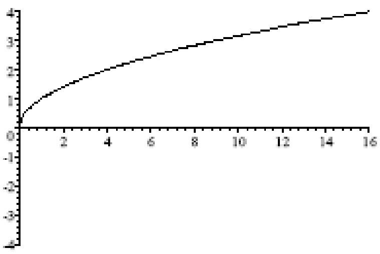
Graph of $y = \sqrt{-x}$

3) Reflections through the x Axis

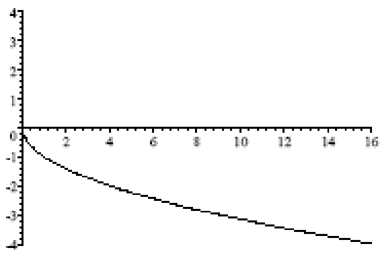
If we begin with the formula y = f (x) and we replace y with −y, then we get the formula −y = f (x) which can also be written as y = −f (x). The graph of y = −f (x) is obtained by reflecting the graph of y = f (x) through the x axis. If the point (x, y) is on the graph of y = f (x), then the point (x, −y) is on the graph of y = −f (x).

Example 6

Starting with the formula y = $\sqrt{x}$, if we replace y with −y, then we get the formula y = $-\sqrt{x}$. The graph of y = $-\sqrt{x}$ is obtained by reflecting the graph of y = $\sqrt{x}$ through the x axis.



Graph of $y = \sqrt{x}$



Graph of $y = -\sqrt{x}$