Games

►Introduction

People have been playing games for the length of recorded history. The uses of these games have ranged from innocently passing the time of day to the settling of disputes in place of war. There are not very many people that haven't spent time playing tic-tac-toe, checkers or chess. However, it was only during the early 20th century that mathematicians started analyzing games as a separate area of mathematics, which is now known as game theory.

A *game* usually consists of 2 players who are trying to reach some specified goal. In a 2-person game, the two players take turns playing as described by the rules of the game.

For instance, in Tic-tac-toe, the objective of each player is to create either 3 X's or 3 O's in a row, column or diagonal. The *rules* in this game are that the two players alternately draw an X or an O in an unoccupied space of a 3 by 3 grid until either all spaces are used, or one player achieves their goal. If a player has a means of winning the game regardless of their opponent's moves, then we say that there is a *winning strategy* for that particular player. It shouldn't take too long to convince yourself that neither player has a winning strategy for tic-tac-toe. Thus, you can only win this game if your opponent errs.

In *mathematical* games, there is no element of chance and all of the game's information is available to both players. For instance, the use of dice or cards has some random element, so mathematical games never make use of dice or cards. For the game's information to be available to both players, both players can examine all things that could affect the game. For example, most games with cards allow a player to view his own cards, but not those of his opponent.

Therefore, these games are not 'mathematical' by our definition. Tic-tac-toe is an example of a mathematical game. (*An interesting result about mathematical games is that if a tie cannot occur, then one of the two players must have a winning strategy*.) If player A cannot *force* a win from a given position, then B must have some strategy to prevent A from winning. Since there is no way to tie, this means B's strategy forces a win for himself!

A typical problem will give both the rules and the goal of the game and ask which player has a winning strategy. This can be stated in many ways such as "What is X's best move from this position?" or "Who has a winning strategy?" or "If it is X's turn, who will win the game?"

For each question, it is not enough to simply state who will win or what move should be made.

A solution must give reasons as to why a certain player can guarantee a win, or why a given move is part of a winning strategy.

►Questions in class

Game 1.
Consider a matchstick game, where the players were asked to choose 1 to 4 matches from 1 of 2 piles of 13 and 17 matchsticks, the winner being the player who takes the last matchstick.

Game 2.
Consider a rook starting on the lower left hand corner of a standard 8 by 8 chess board. On any turn the rook may be moved to the right or upwards, *but not both*, as many squares as the player wishes. Al and Bert take turns moving the rook with Al going first, the winner being the player who advances the rook to the upper right square (diagonally opposite the starting square). What is Bert's winning strategy? Who should win if the board were 8 by 10?

Game 3.
In a game, Xavier and Yolanda take turns calling out whole numbers. The first number called must be a whole number between and including 1 and 9. Each number called after the first must be a whole number which is 1 to 10 greater than the previous number called.

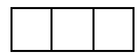
(a) The first time the game is played, the person who calls the number 15 is the winner. Explain why Xavier has a winning strategy if he goes first and calls 4.

(b) The second time the game is played, the person who calls the number 50 is the winner. If Xavier goes first, how does he guarantee that he will win?

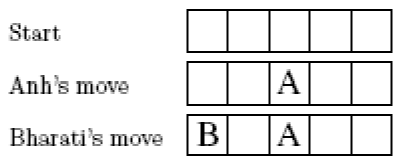
Game 4.

A game begins with a row of empty boxes. On a turn, a player can put his or her initial in 1 box or in 2 adjacent boxes. (Boxes are called adjacent if they are next to each other.) Anh and Bharati alternate turns. Whoever initials the last empty box wins the game.

(a) The game begins with a row of 3 boxes. Anh initials the middle box. Explain why this move guarantees him a win no matter what Bharati does.

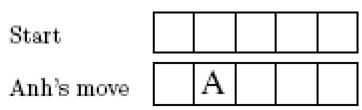


(b) Now the game begins with a row of 5 boxes. Suppose that the following moves have occurred:



Show a move that Anh can make next in order to guarantee that he will win. Explain how this move prevents Bharati from winning.

(c) Again the game begins with a row of 5 boxes. Suppose that the following move has occurred.



Show the two possible moves that Bharati can make next to guarantee she wins. Explain how each of these moves prevents Anh from winning.