Chapter 5 Quadratic Equations (2)

1. The Quadratic Formula

Recall that:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

A quadratic equation can be obtained by using the square completion method, as illustrated below.

$$ax^2+bx+c=0 \qquad \{\text{Subtract } c \text{ from both sides}\}$$

$$ax^2+bx+c-c=0-c$$

$$ax^2+bx=-c \qquad \{\text{Divide both sides by } a\}$$

$$x^2+\frac{b}{a}x=-\frac{c}{a} \qquad \{\text{Add half the coefficient of } x\text{, squared}\}$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a} \qquad \{\text{Use } a^2+2ab+b^2=(a+b)^2\}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a} \qquad \{\text{Simplify RHS}\}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \qquad \{\text{Take square root of both sides}\}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a} \qquad \left\{\text{Subtract } \frac{b}{2a} \text{ from both sides}\right\}$$

$$x+\frac{b}{2a}-\frac{b}{2a}=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is called the **quadratic formula**.

Example 1

Solve $x^2 - 3x - 4 = 0$ for x.

Solution:

Comparing $x^2 - 3x - 4 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -3$, $c = -4$.

Now, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\therefore\ x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2 \times 1}$$

$$= \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2}$$

$$= \frac{3+5}{2} \quad \text{or} \quad x = \frac{3-5}{2}$$

$$= \frac{8}{2} \quad \text{or} \quad = \frac{-2}{2}$$

$$= 4 \quad \text{or} \quad = -1$$

Alternative way:

$$\begin{aligned}
x^2 - 3x - 4 &= 0 && \{\text{Factorise the LHS}\} \\
\therefore\ (x-4)(x+1) &= 0 && \{\text{Use the Null Factor Law}\} \\
x - 4 = 0 \quad &\text{or} \quad x + 1 = 0 \\
x = 4 \quad &\text{or} \quad x = -1
\end{aligned}$$

Note: If the LHS of an equation cannot be factorized, then we use the quadratic formula. The quadratic formula can be used to solve any quadratic equation.

Example 2

Solve $2x^2 - 7x - 6 = 0$ for x.

Solution:

Comparing $2x^2 - 7x - 6 = 0$ with $ax^2 + bx + c = 0$ gives $a = 2$, $b = -7$, $c = -6$.

$$\text{Now,}\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore\ x=\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(-6)}}{2\times 2}$$

$$=\frac{7\pm\sqrt{49+48}}{4}=\frac{7\pm\sqrt{97}}{4}=\frac{7\pm 9.8489}{4}$$

$$=\frac{7+9.8489}{4}\quad\text{or}\quad x=\frac{7-9.8489}{4}$$

$$=\frac{16.8489}{4}\quad\text{or}\quad x=\frac{-2.8489}{4}$$

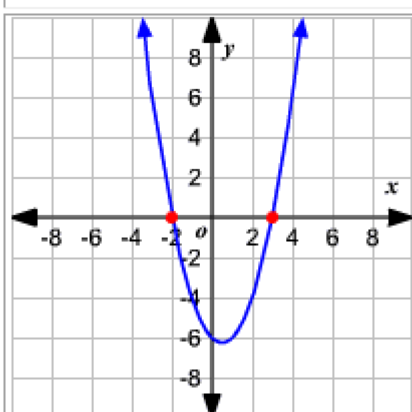
$$=4.2122\quad\text{or}\quad x=-0.7122$$

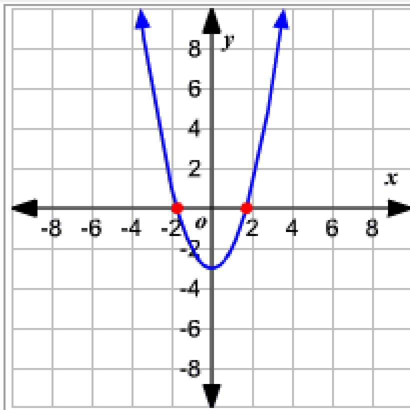
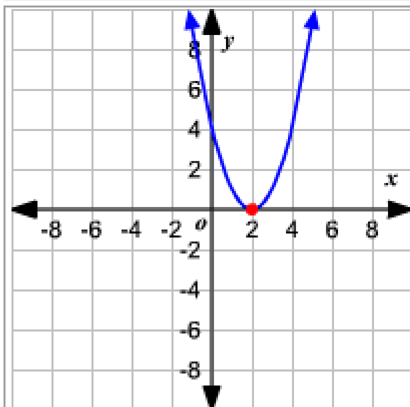
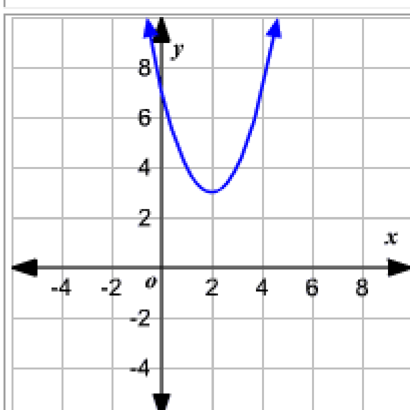
Note: In the above example, the LHS of the equation cannot be factorized. So, we use the quadratic formula to solve the equation.

2. Discriminant

The expression b^2-4ac is called the **discriminant**.

The discriminant can be used to confirm the number of x -intercepts and the type of solutions of the quadratic equation.

Value of Discriminant	**Type and Number of x-intercepts (roots)**	**Example of Graph of Related Function**
$b^2-4ac>0$; b^2-4ac is a perfect square	2 real and rational	 $f(x)=x^2-x-6$

$b^2-4ac>0$; b^2-4ac is not a perfect square	2 real and irrational	 $f(x)=x^2-3$
$b^2-4ac=0$	1 real	 $f(x)=(x-2)^2$
$b^2-4ac<0$	No x-intercepts	 $f(x)=x^2-4x+7$

Example 1: Find the nature of roots for $3x^2 – 5x + 7 = 0$
a = 3, b = -5, c = 7
b2 – 4ac = (-5)2 – 4(3)(7) = -59 < 0

Therefore, the roots are not real. They are complex numbers.

Example 2: For what values of k does $x^2 + 12x – k = 0$ have real and unequal roots?
For the equation, $a = 1$, $b = 12$, and $c = -k$
$b^2 – 4ac = 122 – 4(1)(-k) = 144 + 4k$

For real and unequal roots, $b^2 – 4ac > 0$
$144 + 4k > 0$
$4k > -144$
$\mathbf{k > -36}$

3. Problem Solving

We now consider several applications that make use of quadratic equations. First, the strategy for solving word problems is below.

Strategy for Solving Word Problems

1. Read the problem carefully—several times if necessary—that is, until you understand the problem, know what is to be found, and know what is given.

2. Let one of the unknown quantities be represented by a variable, say *x*, and try to represent all other unknown quantities in terms of *x*. This is an important step and must be done carefully.

3. If appropriate, draw figures or diagrams and label known and unknown parts.

4. Look for formulas connecting the known quantities to the unknown quantities.

5. Form an equation relating the unknown quantities to the known quantities.

6. Solve the equation and write answers to *all* questions asked in the problem.

7. Check and interpret all solutions in terms of the original problem—not just the equation found in step 5—since a mistake may have been made in setting up the equation in step 5.

Example 1: The path of a basketball after it is thrown from a height of 1.5 m above the ground is given by the equation $h = -0.25d^2 + 2d + 1.5$, where h is the height, in metres, and d is the horizontal distance, in metres.

a) How far has the ball travelled horizontally, to the nearest tenth of a metre, when it lands on the ground?

When it lands on the ground, $h = 0$

$0 = -0.25d^2 + 2d + 1.5$

a = -0.25; b = 2; c = 1.5

$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2 \pm \sqrt{2^2 - 4(-0.25)(1.5)}}{2(-0.25)}$

= -0.7 or 8.7 m

Since *d* represents distance, it must be positive. The basketball has travelled a horizontal distance of about 8.7 m when it lands on the ground.

b) Find the horizontal distance when the basketball is at a height of 4.5 m above the ground.

$4.5 = -0.25d^2 + 2d + 1.5$

$0 = -0.25d^2 + 2d - 3$

a = -0.25; b = 2; c = -3

$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2 \pm \sqrt{2^2 - 4(-0.25)(-3)}}{2(-0.25)}$

= 2 or 6 m

The basketball will be at a height of 4.5 m twice along its parabolic path: on the way up at a horizontal distance of 2 m and on the way down at a horizontal distance of 6 m.

Example 2: Find a possible quadratic equation in standard form with each pair of roots.

a) $x_1 = 3$, $x_2 = -5$

$a(x - 3)(x + 5) = 0$

a can be any number, for convenience, we choose a = 1.

$x^2 + 2x - 15 = 0$

b) $x = \frac{14 \pm \sqrt{140}}{4}$

$a(x - \frac{14+\sqrt{140}}{4})(x - \frac{14-\sqrt{140}}{4}) = 0$

a can be any number, for convenience, we choose a = 1.

$x^2 - 2(7)x + (\frac{196-140}{16}) = 0$

$x^2 - 14x + 7/2 = 0$

$2x^2 - 28x + 7 = 0$