Notice

The notes are the same as before, but the questions in class and homework are different with before. Please review the notes and do the questions in class and homework.

Counting and Patterns 2

1. Counting

Counting is the action of finding the number of elements of nature numbers.

The traditional way of counting consists of continually increasing a (mental or spoken) counter by a unit for every element of the nature numbers, in some order, while marking (or displacing) those elements to avoid visiting the same element more than once, until no unmarked elements are left; if the counter was set to one after the first object, the value after visiting the final object gives the desired number of elements.

Counting sometimes involves numbers other than one.

For example, when counting money, counting out change, when "counting by twos" (2, 4, 6, 8, 10, 12…) or when "counting by fives" (5, 10, 15, 20, 25…).

2. Counting using tally marks.

Tally marks are a quick way of keeping track of numbers in groups of five.
One vertical line is made for each of the first four numbers; the fifth number is represented by a diagonal line across the previous four.

1	|	6	~~||||~~ |
2	||	7	~~||||~~ ||
3	|||	8	~~||||~~ |||
4	||||	9	~~||||~~ ||||
5	~~||||~~	10	~~||||~~ ~~||||~~

3. Number Patterns

A **sequence** is a pattern of numbers that are formed in accordance with a definite rule.
We can often describe number patterns in more than one way. To illustrate this, consider the following sequence of numbers $\{1, 3, 5, 7, 9, \ldots\}$.

Clearly, the first term of this number pattern is 1; and the terms after the first term are obtained by adding 2 to the previous term. We can also describe this number pattern as a set of odd numbers.

Alternatively, we can represent a number pattern by using special symbols. For the number pattern under consideration, let t_n represent the nth term.

$\therefore$ First term: $t_1 = 1$
Second term: $t_2 = 3$
Third term: $t_3 = 5$
Fourth term: $t_4 = 7$
Fifth term: $t_5 = 9$
... ...

By trial and error, we find that:

$$t_1 = 1 = 2\times1-1$$
$$t_2 = 3 = 2\times2-1$$
$$t_3 = 5 = 2\times3-1$$
$$t_4 = 7 = 2\times4-1$$
$$t_5 = 9 = 2\times5-1$$

By observation, we notice that we can describe this number pattern by the rule

$$t_n = 2n-1$$

where t_n is *n*th term of the sequence.

4. Formula and Tables

A table of values can be generated from the rule

$$t_n = 2n-1$$

as shown below.

When $n = 1$, $t = 2\times1-1 = 2-1 = 1$
When $n = 2$, $t = 2\times2-1 = 4-1 = 3$
When $n = 3$, $t = 2\times3-1 = 6-1 = 5$
When $n = 4$, $t = 2\times4-1 = 8-1 = 7$
When $n = 5$, $t = 2\times5-1 = 10-1 = 9$

n	1	2	3	4	5
t	1	3	5	7	9

5. Finding the Algebraic Rule

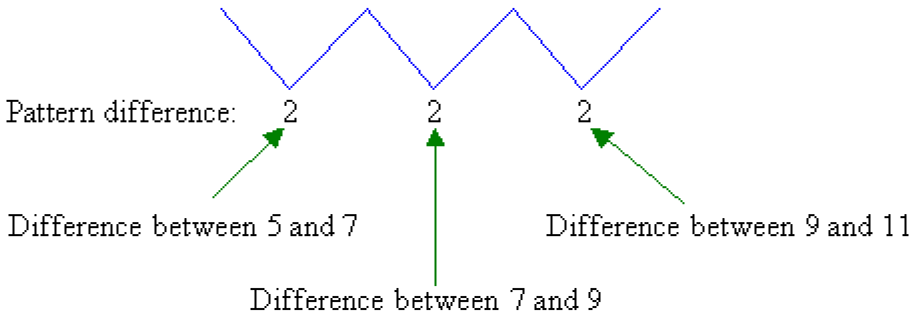
We use algebra to study rules that describe the behavior of everyday things, for example, the behavior of the height of a ball when it is thrown upward or the amount outstanding for a loan after a number of regular repayments.

By finding a pattern in observed values (i.e. measurements), we are often able to discover a rule that allows us to make accurate predictions.

6. Using a Difference Pattern

When we try to discover an algebraic rule for <u>ordered pairs</u>, we can find the difference between two successive values of y. This allows us to find a rule as illustrated below.
Consider the following table.

x	1	2	3	4
y	5	7	9	11



We notice that the values of x increase by just one at a time and the difference between the successive values for y is 2.

So, the rule starts off with $y = 2x$. Will this give a correct answer from the table? Let us check.
When $x = 1, y = 2\times x = 2\times 1 = 2$

The answer is no. From the table, when $x = 1$ the value of y should be 5. How do we change our answer from 2 to 5? We should add 3.
$\therefore$ The rule becomes $y = 2x + 3$

Check the rule to see if it is correct:

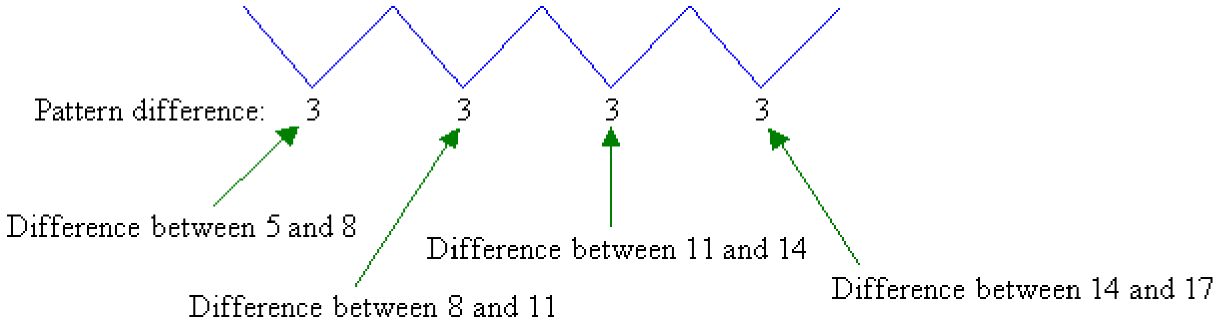
When $x = 1,\ y = 2\times 1 + 3 = 2 + 3 = 5$
When $x = 2,\ y = 2\times 2 + 3 = 4 + 3 = 7$
When $x = 3,\ y = 2\times 3 + 3 = 6 + 3 = 9$
When $x = 4,\ y = 2\times 4 + 3 = 8 + 3 = 11$
So, our rule $y = 2x + 3$ is correct.

Example: Discover the rule for the following table of values:

x	1	2	3	4	5
y	5	8	11	14	17

Solution:
In the given table, the x-values increase by 1 for each ordered pair.

x	1	2	3	4	5
y	5	8	11	14	17



Find the difference between the successive values of *y*. That is:
$8-5=11-8=14-11=17-14=3$

The difference between successive values of *y* is always 3. So, the rule is of the form
$y=3\times x+\square$
To determine the value of $\square$, we check the rule for $x=1$.
When $x=1, y=5$ and $y=3\times 1+\square=3+\square$
$\therefore 5=3+\square$, $5-3=3+\square-3$, $\therefore \square=2$.
Hence the rule is $y=3x+2$

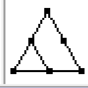
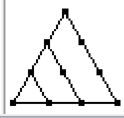
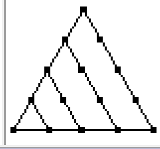
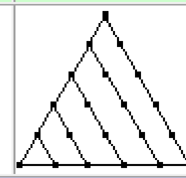
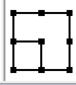
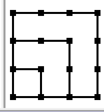
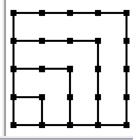
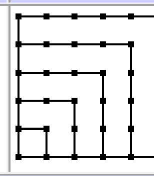
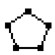
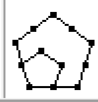
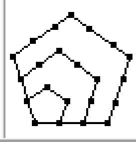
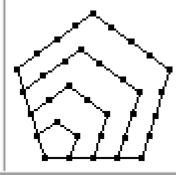
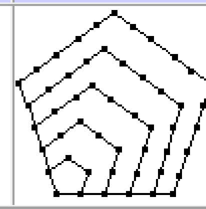
Check the rule to see if it is correct:
When $x=2$, $y=3\times 2+2=6+2=8$
When $x=3$, $y=3\times 3+2=9+2=11$
When $x=4$, $y=3\times 4+2=12+2=14$
When $x=5$, $y=3\times 5+2=15+2=17$
So, our rule is correct.

7. Polygonal Numbers

Polygonal Numbers are really just the number of vertexes in a figure formed by a certain polygon. The first number in any group of Polygonal Numbers is always 1, or a point. The second number is equal to the number of vertexes of the polygon.

A formula that will generate the n^{th} x-gonal number (for example: the 2^{nd} 3-gonal, or triangular number) is:

$$\frac{n^2-n}{2}\times(x-2)+n$$

Type	1st	2nd	3rd	4th	5th	6th
Triangular						
Value	1	3	6	10	15	21
Square						
Value	1	4	9	16	25	36
Pentagonal						

8. Permutation

Fundamental Counting Rule - If there are a ways to pick something and b ways to pick something else, then in total, there are a × b total ways you can pick both!

Example: Alison wants to make a bracelet out of coloured beads. She has 6 beads, each a different colour. How many different bracelets can she make, if she uses all the beads?

There are 6 ways to pick the first bead. There are 5 ways to pick the second bead. There are 4 ways to pick the third bead, and so on… There is only 1 way to pick the last bead.
Therefore, in total, there are 6 x 5 x 4 x 3 x 2 x 1 = 720 ways to make different bracelets.

A product of the form $n \times (n - 1) \times (n - 2) \cdot \cdot \cdot \times 2 \times 1$ can be written shorthand as **n!** - this is read as **"n factorial".**

In the previous example, the order is important - different order, different item.

Permutation - A permutation is an arrangement of a certain number of distinct types of items, where the **order of the items matter.**

More Examples
- Numbers - the arrangement of the digits matter (repetition allowed)
- Locker "combinations" (should be permutations)
- Winners in a marathon - the order of the runners matters!

Example: How many bracelets can Alison make if she has 30 different coloured beads and wants to use 25 of them?

Alison had 30 beads but only wants to use 25 of them (so she is NOT using (30 - 25) = 5 of them). Using the fundamental counting rule, we had the following expression: $30 \times 29 \times 28 \times \cdot \cdot \cdot \times 6$

We can write this in a different way: $30 \times 29 \times 28 \times \cdot \cdot \cdot \times 6 \times [5 \times 4 \times 3 \times 2 \times 1] \div [5 \times 4 \times 3 \times 2 \times 1]$. So what we get is actually $30! \div (30 - 25)! = 30! \div 5! = 30!/5!$.
This is a shorter way of writing out that long expression!

What did we do? We just divided (# of objects we have)! by (# of objects we did not use)!
How many objects did we NOT use? Simple - just subtract the number of objects you DID use (k) from the number of objects you didn't use (n). This is just (n − k).

If we are counting permutations of n DISTINCT objects, where we use k of them without reusing an object, the amount is "n pick k"
Total Permutations
$= n!/(n - k)!$
= (# of distinct objects you HAVE)! / (# of distinct objects you AREN'T using)!
$= {}_nP_k$

(If we use ALL the objects, where n = k, then we just use n! - we define 0! = 1)
For permutations of n DISTINCT objects, where we use k of them and are allowed to reuse an object, the number is simply "$n \times n \times \cdot \cdot \cdot \times n$]= n^k".

The key to remember here is - all of these formulas come from the Fundamental Counting Rule!

Example: Collector Christian has a collection of 7 toy train cars and creates a train using 4 of them. How many different ways could he have made the train?

$\frac{7!}{(7-4)!} = \frac{7\times6\times5\times4\times3!}{3!}$ = 7 x 6 x 5 x 4 = 840

9. Combination

What if order doesn't matter?
How many possible 3 digit lottery tickets are there if numbers from 1-4 are used (no repeats)?

List out all permutations:
(1, 2, 3   1, 3, 2   2, 3, 1   2, 1, 3   3, 1, 2   3, 2, 1) → {1, 2, 3}
(1, 2, 4   1, 4, 2   2, 1, 4   2, 4, 1   4, 1, 2   4, 2, 1) → {1, 2, 4}
(1, 3, 4   1, 4, 3   3, 1, 4   3, 4, 1   4, 1, 3   4, 3, 1) → {1, 3, 4}
(2, 3, 4   2, 4, 3   3, 2, 4   3, 4, 2   4, 2, 3   4, 3, 2) → {2, 3, 4}

Using permutation method:

Problem... {1, 3, 4} is the same ticket as {1, 4, 3}. We double counted.
Group all permutations which use the same digits. How many are in each group?

The number of different lottery tickets is the number of groups of permutations that use the same digits. This is the same as making order irrelevant! What did we do? We divided the total number by the number of elements in each group - how many are each in group? The number of permutations of k elements - this is exactly k! So we divide by k!

Combination - A selection of items where **order does not matter** is known as a combination. There are ${}_nP_k$ total permutations; but each grouping has k! elements, so we have to divide by k!. Then there are
$[{}_nP_k] \div k! = {}_nP_k / k! = n! / (n-k)! \div k! = n! / k!(n-k)!$ combinations in total.

We write this as ${}_nC_k$, and say "n choose k". So the actual number of lottery tickets is ${}_4C_3 = 4$ - from 4 possible numbers, you are CHOOSING three of them, where order DOES NOT matter.

Example: There is a bag of 12 gumballs, each of different colour. How many handfuls of 3 gumballs can you make?
Since the order doesn't matter, ${}_{12}C_3 = \frac{12!}{3!(12-3)!} = \frac{12\times11\times10\times9!}{3!9!} = \frac{12\times11\times10}{3\times2\times1} = 220$

►Questions in class

Counting

1. A 4 × 4 × 4 cube consisting of smaller cubes is painted and then broken apart. How many of the smaller cubes will have exactly 2 painted sides?

2. How many three digit numbers can be constructed using the digits 1, 2, 3, 4 and 5 if the same digit cannot appear twice in a row in any of the numbers?

3. How many digits are needed to write all of the integers from 1 to 1000 inclusive? For example, to write the numbers from 1 to 10 inclusive, one would need 11 digits.

4. How many different numbers can be constructed using the digits 0, 1, 2, 2? All of the digits must be used each time and no number can begin with 0.

5. The positive integers, starting with 1, are written in order 123456789101112.... What is the digit appearing in the 100th place?

6. How many numbers from the set {−5, −4, –3, –2, –1, 0, 1, 2, 3} satisfy the inequality $-3x^2 < -14$?

Pattern

7. In the sequence 1, 3, 3, 3, 5, 5, 5, 5, 5, 7, 7, …? What is the 100th number?

8. What is the next number in the sequence 2, 1, 3, 4, 7, 11, …?

9. Starting at 9 and counting by 8's, Samuel counts 9, 17, 25, …. What kind of numbers that will be counted?

10. A sequence is formed in the following way: The first two numbers of the sequence are 1 and 3. Each subsequent number is the sum of the previous two members of the sequence. What is the third even number of the sequence?