Lesson 1: Unit 1- Functions: Review Transformations and Properties

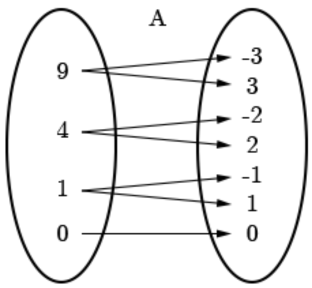
Relations and Function

A **relation** is a set of ordered pairs.

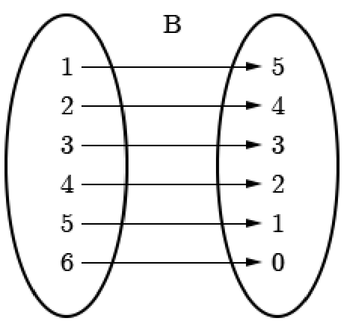
Let $A=\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$.
Let $B=\{(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\}$.
The elements in each set have been listed using **set notation**.

A **function** is a relation, a set of ordered pairs (x, y), **in which for every *x* value, there is only one *y* value**.

By definition, A is not a function, whereas B is a function.
We can represent A and B using mapping diagrams.

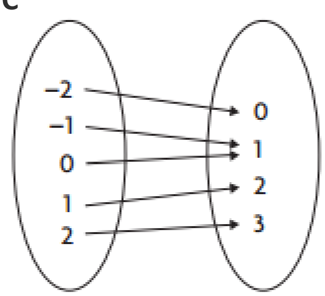


Since at least one x-value maps to more than one y-value, ***A* is not a function**.



Since each x-value maps to exactly one y-value, ***B* is a function**.

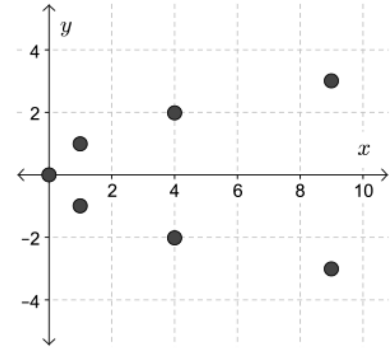
C



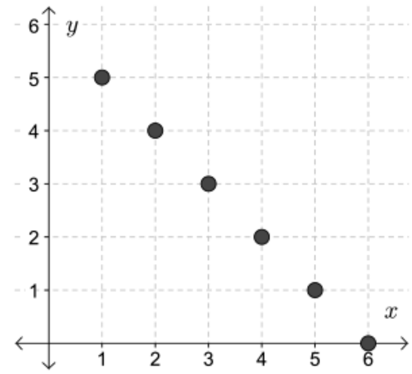
Since each *x*-value maps to exactly one *y*-value, ***C* is a function**.

Representing Relations and Functions Graphically

We have represented *A* and *B* using set notation and mapping diagrams.
We can also represent *A* and *B* on a graph.
A={(9,−3),(4,−2),(1,−1),(0,0),(1,1),(4,2),(9,3)}



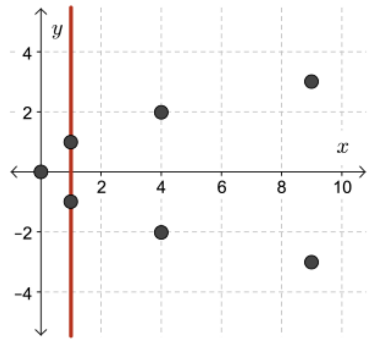
B={(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)}



The Vertical Line Test

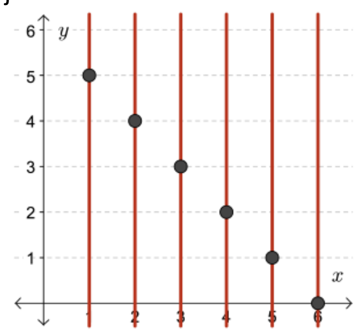
If a vertical line can be drawn anywhere on a graph so that it passes through two or more points on a relation, then the relation is not a function.
If no vertical line can be drawn that passes through more than one point on a relation, then the relation is a function.
This is called **the vertical line test**.

A={(9,−3),(4,−2),(1,−1),(0,0),(1,1),(4,2),(9,3)}



A vertical line can be drawn through the two points, (1,−1) and (1,1). Therefore, *A* is not a function.

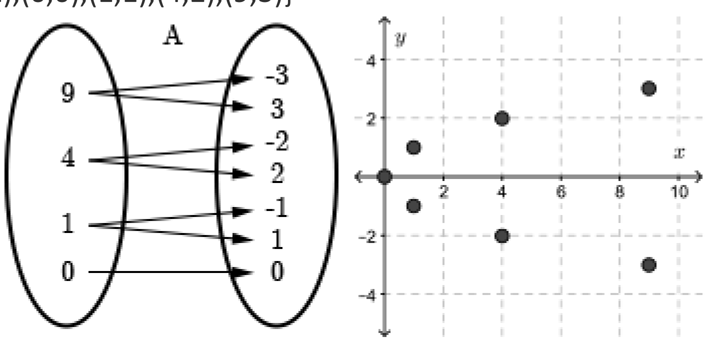
B={(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)}



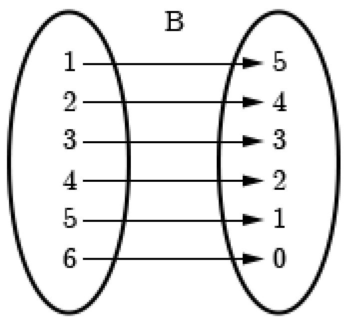
No vertical line can be drawn through any two points on *B*. Therefore, *B* is a function.

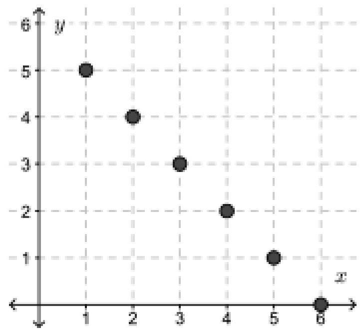
Domain and Range

A={(9,−3),(4,−2),(1,−1),(0,0),(1,1),(4,2),(9,3)}



B={(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)}





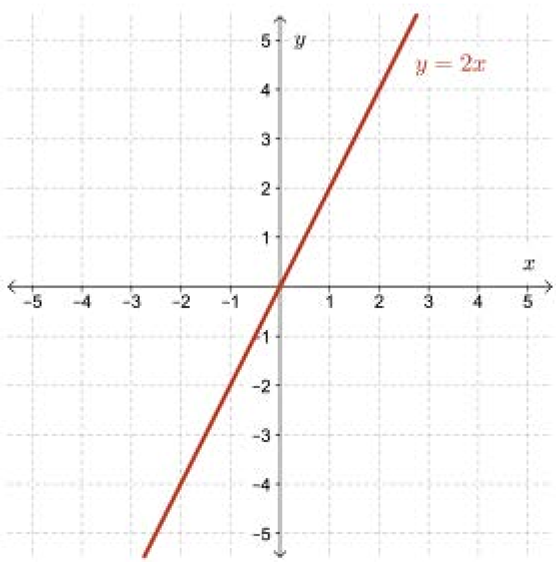
The set of all possible values of the independent variable, *x*, is called the **domain**.

The set of all possible values of the dependent variable, *y*, is called the **range**.

For *A*={(9,−3),(4,−2),(1,−1),(0,0),(1,1),(4,2),(9,3)} the domain is {0,1,4,9}, and the range is {−3,−2,−1,0,1,2,3}.
For *B*={(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)} the domain is {1,2,3,4,5,6}, and the range is {0,1,2,3,4,5}.

Using the terminology of domain and range, a function is a relation in which each element in the domain corresponds to exactly one element in the range.

Example 1: State the domain and range of the following. State, with justification, whether or not each relation is a function.

a. 

Solution

This relation is a function since it passes the vertical line test.
The domain is the set of real numbers. We write this as follows:

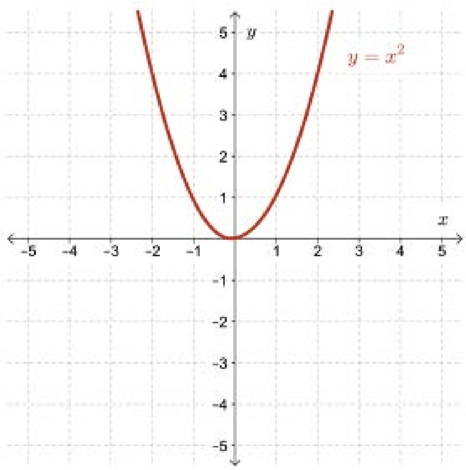
Domain is {*x*| *x* ∈ R}

We read this, “The domain is the set of *x* values such that *x* is an element of the set of real numbers.”

In set notation, the symbol, “|”, is a mathematical short form for “such that” and the symbol, “∈”, stands for “element of.”

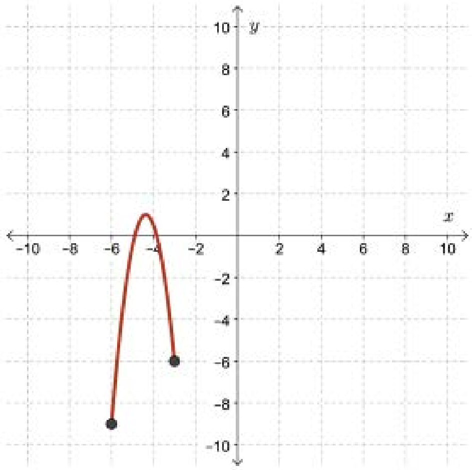
The range is the set of real numbers. We write this as follows:

Range is $\{y \mid y \in R\}$.

b. 

Solution

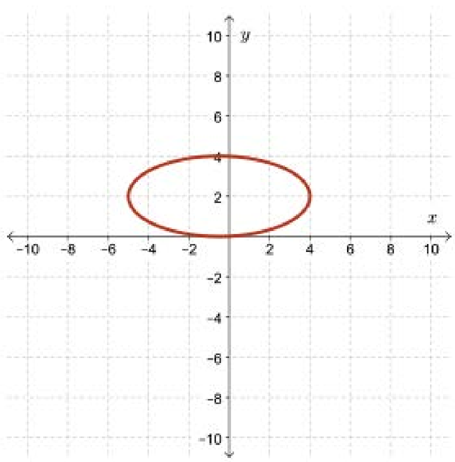
This relation is a function since it passes the vertical line test.
The domain is the set of real numbers. There is no real number which cannot be squared. We write this as follows:
Domain is $\{x|x \in R\}$.
The range is the set of real numbers which are greater than or equal to zero since squaring any real number results in a real number that is positive or zero. We write this as follows:
Range is $\{y|y \geq 0, y \in R\}$.



c.

Solution

This relation is a function since it passes the vertical line test.
This example is different from the previous example in that there are definite starting and ending points on the graph.
It would appear from the graph that the domain is the set of real numbers from −6 to −3, inclusive. We write this as follows:
Domain is $\{x|-6\le x\le -3, x\in R\}$.
It would appear from the graph that the range is also restricted to real numbers between −9 and 1, inclusive. We write this as follows:
Range is $\{y|-9\le y\le 1, y\in R\}$.



d.

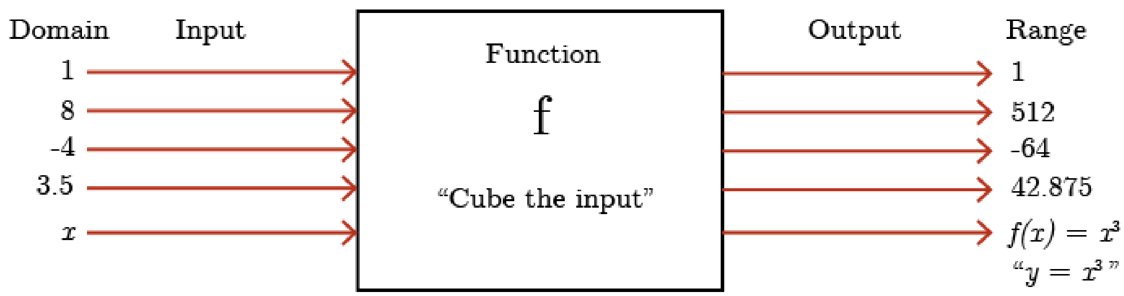
Solution

This relation is not a function since it fails the vertical line test. For example, a vertical line drawn along the *y*-axis intersects the relation twice.

It would appear from the graph that the domain is the set of real numbers from −5 to 4, inclusive. We write this as follows:
Domain is $\{x|-5\leq x\leq 4, x\in R\}$.
It would appear from the graph that the range is also restricted to real numbers between 0 and 4, inclusive. We write this as follows:
Range is $\{y|0\leq y\leq 4, y\in R\}$.

Function Notation

Think of a function as a machine. The machine inputs acceptable values from the domain, and then performs some operation to produce an output value in the range.
In the function machine illustrated below, an input is provided and the function cubes the input to produce an output.
We use a special notation called function notation to represent this. In this case, $f(x)=x^3$.
We read $f(x)$ as "f at x" or "f of x".



So, $f(8)$ tells the function to cube 8 producing an output of 512. Therefore, $f(8)=512$.
$f(x)=x^3$ can be written as the equation $y=x^3$; $f(x)$ and y are interchangeable.
Function notation is helpful in illustrating the value that is being substituted in the function for x.
If $f(x)=x^3$, then we know that $f(3.5)$ is asking us to evaluate the function when $x=3.5$. It follows that $f(3.5)=3.5^3=42.875$.

Example

Let $f(x)=2x+1$ and $g(x)=x^2+4x-2$.
a. Evaluate
  i. $f(-6)$
  ii. $g(-3)$

Solution

i. Since $f(x)=2x+1$,

$f(-6)=2(-6)+1=-11$

ii. Since $g(x)=x^2+4x-2$,

$g(-3)=(-3)2+4(-3)-2=9-12-2=-5$

b. For what value(s) of x does $f(x)=-7$?

Solution

We are looking for the value of x which, when substituted into the function, produces an output value of −8.
Since $f(x)=-8$,

$$2x+1=-7$$
$$2x=-8$$
$$x=-4$$

Therefore, when $x=-4$, $f(-4)=-7$.

c. For what value(s) of x does $f(x)=g(x)$?

Solution

$$f(x)=g(x)$$
$$2x+1=x^2+4x-2$$
$$0=x^2+2x-3$$
$$0=(x+3)(x-1)$$

Therefore, $x=-3$ or $x=1$.
It follows that $f(-3)=g(-3)=-5$ and $f(1)=g(1)=3$.
d. Evaluate $f(g(5))$.

Solution

We always evaluate the work inside brackets first.
So in this case, we evaluate $g(5)$ by substituting 5 for x into $g(x)$, and then substitute that result for x into $f(x)$.
Since $g(x)=x^2+4x$,

$$g(5)=(5)^2+4(5)-2=25+20-2=43$$

Now,

$$f(g(5))=f(43)=2(43)+1=87$$

Therefore, $f(g(5))=87$.

e. Simplify $f(g(x))$.

Solution

Since $f(x)=2x+1$ and $g(x)=x^2+4x-2$,

$$
\begin{aligned}
f(g(x)) &= f(x^2+4x-2)\\
&= 2(x^2+4x-2)+1\\
&= 2x^2+8x-3
\end{aligned}
$$

f. Simplify $g(f(x))$.

Solution

Since $g(x) = x^2+4x-2$ and $f(x)=2x+1$,

$$
\begin{aligned}
g(f(x)) &= g(2x+1)\\
&= (2x+1)^2+4(2x+1)-2\\
&= 4x^2+12x+3
\end{aligned}
$$

Generally speaking, $f(g(x))\neq g(f(x))$.
However, there are functions $f(x)$ and $g(x)$ for which $f(g(x))=g(f(x))$.

Composite Functions

$$f(x)=2x+3$$
$$g(x)=x^2+4x$$
$$f(g(x))=2x^2+8x+3 \text{ and } g(f(x))=4x^2+20x+21$$

$f(g(x))$ and $g(f(x))$ are examples of composite functions.
Composition of functions is the process of combining two or more functions where one function is performed first and the result is substituted in place of x into the next function, and so on. When we substitute one function into another function, we create a composite function.

$f(g(x))$ is read "*f* of *g* of *x*". It is also written $(f\circ g)(x)$.

The idea of composite functions will be developed in the future in this course.
We will answer questions like, ***"When is it possible to compose two functions?"***, and, "***Do two functions exist such that f(g(x))=g(f(x))?"***

Sometimes when solving mathematical problems, the solution is a set of numbers lying in an interval.
There are several ways to describe these intervals.

- We can write the solution algebraically using set notation.

$$\{x|x<8, x\in \mathbb{R}\}$$

- We can display the solution graphically on a number line.

- We can use a special notation called interval notation.

$$x \in (-\infty, -1] \cup [1, +\infty),\ x \in R$$

Describing Solutions Algebraically

A solution can be described algebraically using various symbols.

Symbol	Meaning	Word Example	Algebraic Solution
<	less than	x less than 9	$\{x \mid x<9,\ x\in R\}$
>	greater than	x greater than −22	$\{x \mid x>-22,\ x\in R\}$
≥	greater than or equal to	x greater than or equal to 17	$\{x \mid x\geq 17,\ x\in R\}$
≤	less than or equal to	x less than or equal to 97	$\{x \mid x\leq 97,\ x\in R\}$
≠	not equal to	x not equal to -3.5	$\{x \mid x\neq -3.5,\ x\in R\}$

Example

Describe the set of real numbers from −4 to 5 that includes −4, but does not include 5.

Solution

$$\{x \mid -4\leq x<5,\ x\in R\}$$

We read this, "the set of all *x* such that −5 is less than or equal to *x*, which is less than 3, and *x* is an element of the set of real numbers."

Example

Describe the set of real numbers which does not include -2.

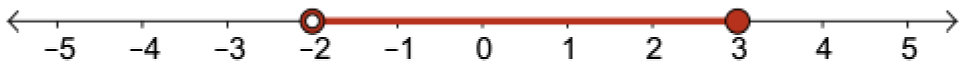
Solution

$$\{x \mid x\neq -2,\ x\in R\}$$

We read this, "the set of all *x* such that *x* is not equal to -2 and *x* is an element of the set of real numbers."

Describing Solutions Graphically

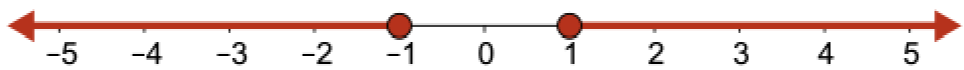
An interval can be described graphically using a number line.



A solid dot means that the endpoint is included in the interval.
An open dot means that the endpoint is excluded from the interval.
The example shown illustrates the solution which could be written in set notation as

$$\{x|-2<x\leq 3,\ x\in \mathrm{R}\}$$

Another interval is illustrated on the following number line:



Using set notation, this solution would be written

$$\{x|x\leq -1 \text{ or } x\geq 1,\ x\in \mathrm{R}\}$$

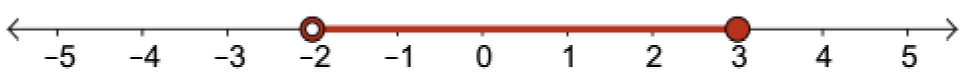
Note
The use of the word or tells us that there is a union of two solutions. In this case, x can take on values less than or equal to −1, or x can take on values greater than or equal to 1. Alternatively, x cannot take on values between −1 and 1.

Describing Solutions Using Interval Notation

We have described the solution, "x greater than −2 and x less than or equal to 3" using set notation

$$\{x|-2<x\leq 3,\ x\in \mathrm{R}\}$$

and graphically with a number line

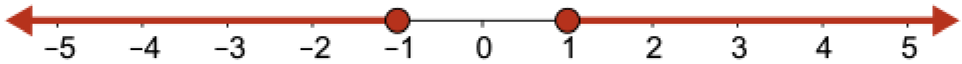


The same solution can be described using interval notation as follows:

$$x\in(-2,3],\ x\in \mathrm{R}$$

This interval goes from the smallest number, −2, to the largest number, 3.
The round bracket to the left of −2 indicates that −2 is excluded from the interval.
The square bracket to the right of 3 indicates that 3 is included in the interval.
If an interval is written $x\in[a,b)$, $x\in\mathrm{R}$, then a would be included in the interval and b would be excluded from it.
If the interval were written $x\in(a,b)$, $x\in\mathrm{R}$, then neither endpoint would be included in the interval.
If the interval were written $x\in[a,b]$, $x\in\mathrm{R}$, then both endpoints would be included in the interval.

We also illustrated the solution, "x less than or equal to −1 or x greater than or equal to 1" on a number line.



Algebraically, using set notation, this solution would be written $\{x|x \le -1 \text{ or } x \ge 1, x \in R\}$.
Using interval notation, this solution would be written

$$x \in (-\infty, -1] \cup [1, +\infty), x \in R$$

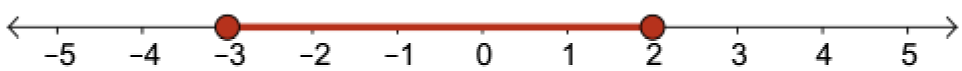
- The solution includes all numbers less than or equal to −1; however, we can never reach $-\infty$. Therefore, $-\infty$is not included in the interval.
- The solution includes all numbers greater than or equal to 1; however, we can never reach $+\infty$. Therefore, $+\infty$ is not included in the interval.
- The symbol $\cup$ represents the union of the two sets.

Example

Represent "all real numbers from −3 to 2, inclusive" in the following ways: algebraically, on a number line, and using interval notation.

Solution

Algebraically, in set notation, we write $\{x|-3 \le x \le 2, x \in R\}$.
On a number line, the solution is shown as



Using interval notation, this solution would be written $x \in [-3, 2], x \in R$.

Example

Represent "all real numbers except −1 and 4." in the following ways: algebraically, on a number line, and using interval notation.

Solution

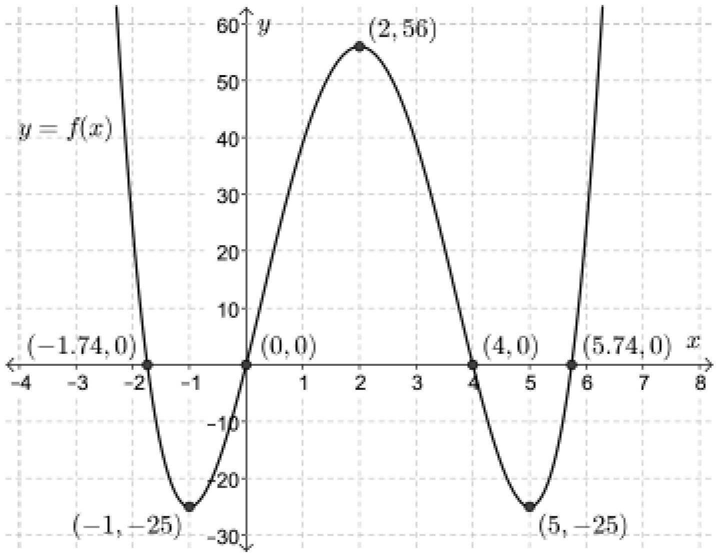
Algebraically, in set notation, we write $\{x|x \ne -1, x \ne 4, x \in R\}$.
On a number line, the solution is shown as

Using interval notation, this solution would be written $x \in (-\infty, -1) \cup (-1, 4) \cup (4, +\infty), x \in R$.

***Note:** $+\infty$ will sometimes be written without the + symbol, simply as ∞.*

An Illustrative Example

The graph shows some function $y=f(x)$.



Note the following about the graph.

- There are four x-intercepts: $x=-1.74$, $x=0$, $x=4$, and $x=5.74$.
- Between $x=-1.74$ and $x=0$, the graph reaches a local minimum point at (−1,−25).
- Between $x=0$ and $x=4$, the graph reaches a local maximum point at (2,56).
- Between $x=4$ and $x=5.74$, the graph reaches a local minimum point at (5,−25).
- As values of x get smaller and smaller, the value of the function gets larger and larger.
- As values of x get larger and larger, the value of the function gets larger and larger.

Note

At this time, do not be concerned with where the points on the graph came from. We will use the graph to illustrate some definitions which will be used throughout the course. We will also use the graph to illustrate a use for interval notation.

A function y=f(x) is **positive on an interval** if the value of f(x) is greater than 0 (i.e., y>0) for all values of x in the interval.

Geometrically, the graph of the function resides **above the x-axis** in the given interval.

In our example, $f(x)$ is above the x-axis for values of $x<-1.74$, for $0<x<4$, and for $x>5.74$.

In interval notation, we can write the positive interval

$$x\in(-\infty,-1.74)\cup(0,4)\cup(5.74,\infty)$$

A function $y=f(x)$ is **negative on an interval** if the value of $f(x)$ is less than 0 (i.e., $y<0$) for all values ofx in the interval.
Geometrically, the graph of the function resides **below the x-axis** in the given interval.

In our example, $f(x)$ is below the x-axis for $-1.74<x<0$ and $4<x<5.74$.
In interval notation, we can write the negative interval

$$x\in(-1.74,0)\cup(4,5.74).$$

Increasing and Decreasing Intervals

A function $f(x)$ is **increasing on an interval** if the value of $y=f(x)$ increases as the value of x increases.
More precisely, a function f is increasing on an interval, (a,b), if for every $\mathbf{x_1,x_2\in(a,b)}$ **with** $\mathbf{x_1<x_2,\ f(x_1)\le f(x_2)}$**.**
Geometrically, as one moves from left to right along the curve, the value of the y-coordinate never decreases.
The function is said to be **strictly increasing** if $\mathbf{f(x_1)<f(x_2)}$ **for** $\mathbf{x_1<x_2}$ in the interval.

In our example, $f(x)$ is increasing for $-1<x<2$ and $x>5$.
In interval notation, we can write the increasing interval

$$x\in(-1,2)\cup(5,\infty)$$

A function $f(x)$ is **decreasing on an interval** if the value of $y=f(x)$ decreases as the value of xincreases.
More precisely, a function f is decreasing on an interval, (a,b), if for every $\mathbf{x_1,x_2\in(a,b)}$ **with** $\mathbf{x_1<x_2,\ f(x_1)\ge f(x_2)}$**.**
Geometrically, as one moves from left to right along the curve, the value of the y-coordinate never increases.
The function is said to be **strictly decreasing** if $\mathbf{f(x_1)>f(x_2)}$ **for** $\mathbf{x_1<x_2}$ in the interval.

In our example, $f(x)$ is decreasing for $x<-1$ and $2<x<5$.
In interval notation, we can write the decreasing interval

$$x\in(-\infty,-1)\cup(2,5)$$

Maxima and Minima

A **local maximum** is a point on a graph whose y value is *greater than or equal to the y values* of all other points near it. The function changes from increasing to decreasing, as x increases, at a local maximum.

A **local minimum** is a point on a graph whose y value is *less than or equal to the y values* of all other points near it. The function changes from decreasing to increasing, as x increases, at a local minimum.

If there is more than one local minimum, we refer to them as local minima.
If there is more than one local maximum, we refer to them as local maxima.

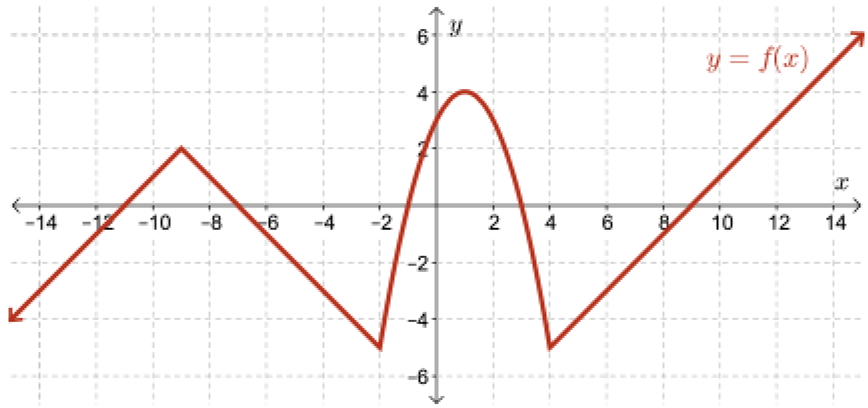
On the graph, the increasing intervals are shown in **blue** and the decreasing intervals are shown in **red**.
At (−1,−25), the function changes from decreasing to increasing, so (−1,−25) is a local minimum.
At (2,56), the function changes from increasing to decreasing, so (2,56) is a local maximum.
At (5,−25), the function changes from decreasing to increasing, so (5,−25) is a local minimum.

End Behaviour

The **end behaviour** of a function refers to what happens to the function for extreme positive and negative values of *x*.
Mathematicians write "as $x \to \infty$" whenever *x* approaches very large values.
They write "as $x \to -\infty$" whenever *x* approaches very small values (which are large negative values).
We use "$\to$" for the word "approaches".
In our example, as $x \to -\infty$, the value of $y=f(x)$ tends to infinity.
Expressed mathematically, we write "as $x \to -\infty$, $f(x) \to \infty$" and read
"as *x* approaches $-\infty$, $f(x)$ approaches ∞."
Similarly, as $x \to \infty$, the value of $y=f(x)$ tends to infinity.
Expressed mathematically, we write "as $x \to \infty$, $f(x) \to \infty$."

Example

For the function $y=f(x)$, identify the *x*-intercepts, *y*-intercepts, positive and negative intervals, increasing and decreasing intervals, local maxima and local minima, and end behaviour. You may assume that the key points on the graph have integral coordinates.



Solution

***x*-intercepts:**
$x=-11, x=-7, x=-1, x=3, x=9$

Positive Intervals:
$x \in (-11,-7) \cup (-1,3) \cup (9,\infty)$

Increasing Intervals:
$x \in (-\infty,-9) \cup (-2,1) \cup (4,\infty)$

Local Maxima:
$(-9,2), (1,4)$
y-intercept:
$y=3$

Negative Intervals:
$x \in (-\infty,-11) \cup (-7,-1) \cup (3,9)$

Decreasing Intervals:
$x \in (-9,-2) \cup (1,4)$

Local Minima:
$(-2,-5), (4,-5)$

End Behaviour:
as $x \to -\infty, y \to -\infty$; as $x \to \infty, y \to \infty$

Summary

- We introduced the notion of solutions that lie in an interval.
- We have shown three ways to write such intervals: set notation, a number line, and interval notation.
- We made several definitions which will be used throughout this course.

Now we are going to look at linear inequalities.

In past experiences, we have solved a variety of different equations. In particular, we have solved linear equations.
We have discovered that we can perform the same operation on both sides of the equation without affecting the equality.

Is the same true for inequalities?

When solving inequalities, you can add or subtract the same number from each side of the inequality and the inequality will remain true.

You can multiply or divide both sides of the inequality by the same positive number and the inequality will remain true.

However, when you multiply or divide both sides of an inequality by the same negative number, the resulting inequality becomes false.
To make the inequality true again, reverse the direction of the inequality symbol.

Example 1

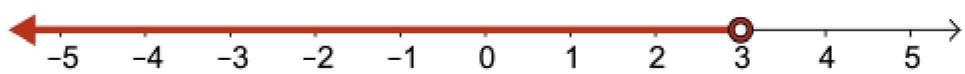
Solve $-3x+1>-8$, where $x \in R$. Write the solution using interval notation and illustrate the solution on a number line.

Solution

$$-3x+1 > -8$$

Subtract 1 from both sides: $-3x > -9$

Divide both sides by -3: $x < 3$



Using interval notation, $x \in (-\infty,3)$, $x \in R$.
If we choose any value of x less than 3, the inequality will be true.
If we choose any value of x greater than or equal to 3, the inequality will be false.

When $x=1, -3(1)+1-2>-8>-8$ TRUE

When $x=5, -3(5)+1-14 \ngtr -8 \ngtr -8$ FALSE

Example 2

Solve $-4x-1 \le 2x+5$, where $x \in R$. Write the solution using interval notation and illustrate the solution on a number line.

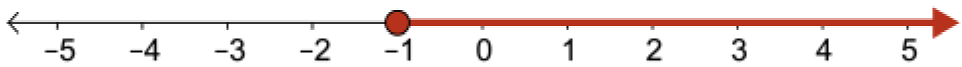
Solution

$$-4x-1 \le 2x+5$$

Subtract $2x$ from both sides: $-6x-1 \le 5$

Add 1 to both sides: $-6x \leq 6$

Divide both sides by -6: $x \geq -1$



Using interval notation, $x \in [-1, +\infty)$, $x \in R$

Example 3

Solve $-1 < \frac{4-x}{2} < 3$, $x \in R$. Write the solution using interval notation and illustrate the solution on a number line.

Solution

In this solution, we will work directly with the three parts of the inequality. When we perform an operation, we will do it to each part of the inequality.

$-1 < \frac{4-x}{2} < 3$

$-2 < 4-x < 6$ multiply each part by 2

$-6 < -x < 2$ subtract 4 from each part

$6 > x > -2$ multiply each part by -1

Using interval notation, $x \in (-2, 6)$, $x \in R$.

Example 4

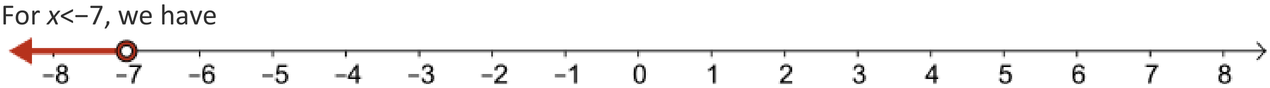
Solve $9+4x<3x+2\leq-1$, $x \in R$. Write the solution using interval notation and illustrate the solution on a number line.

Solution

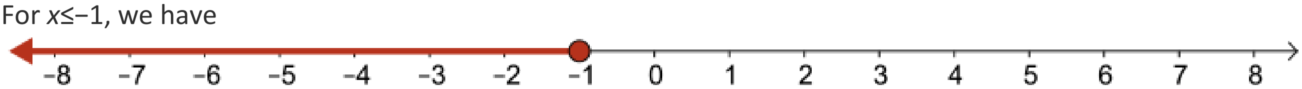
We want $9+4x<3x+2$ **and** $3x+2\leq-1$. Whatever solution we find must satisfy both inequalities.

$x < -7$ and $x \leq -1$

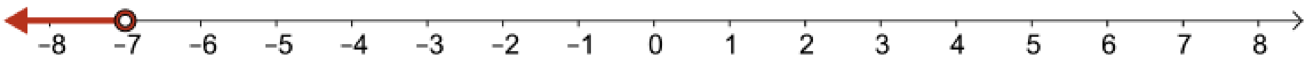
For $x<-7$, we have



For $x\leq-1$, we have



We want the intersection of these two solutions. That is, we want all numbers that satisfy both inequalities at the same time.



Using interval notation, $x \in (-\infty, -7)$, $x \in R$.

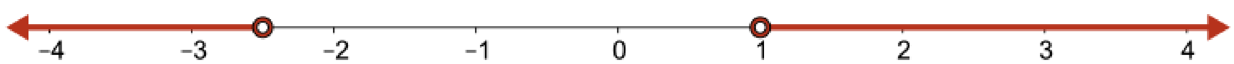
Example 5

Solve $(2x+5)(x-1)>0$, $x \in R$. Write the solution using interval notation and illustrate the solution on a number line.

Solution

To obtain a product greater than zero, we multiply two positive quantities **or** two negative quantities.
We want ($2x+5>0$ **and** $x-1>0$) **or** ($2x+5<0$ **and** $x-1<0$).

Note
Notice that if $2x+5>0$ and $x-1<0$, then $(2x+5)(x-1)<0$.
Similarly, if $2x+5<0$ and $x-1>0$, then $(2x+5)(x-1)<0$.
$2x+5>0$ **and** $x-1>0$ **or** $2x+5<0$ **and** $x-1<0$.
$x>-5/2$ **and** $x>1$ **or** $x<-5/2$ **and** $x<1$.
$x>1$ or $x<-5/2$.
$x \in (-\infty, -52) \cup (1, \infty)$, $x \in R$.



Example 6

Solve $\frac{3}{x} < 5$, $x \in R$. Write the solution using interval notation and illustrate the solution on a number line.

Solution

Case 1: $x>0$ and $3/x<5$

$3<5x$ multiply both sides by a positive number, namely x

$3/5<x$ divide both sides by 5

Since $x>0$ and $x>3/5$, then $x>3/5$.

Case 2: $x<0$ and $3/x<5$

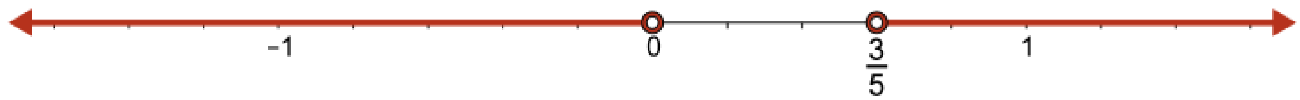
$3>5x$ multiply both sides by a negative number, namely x

$3/5>x$ divide both sides by 5

Since $x<0$ and $x<3/5$, then $x<0$.
From Case 1, we have $x>3/5$.
From Case 2, we have $x<0$.

It is **either** the solution from Case 1 or the solution from Case 2, so we want the union of the two solutions.

On a number line the solution is:



Using interval notation, $x \in (-\infty, 0) \cup (3/5, \infty)$, $x \in R$.
To verify the solution, substitute values of x less than 0 or greater than 3/5 into the inequality. The resulting inequality will be true.
If you substitute values for x greater than 0 and less than or equal to 3/5, then the resulting inequality will be false.

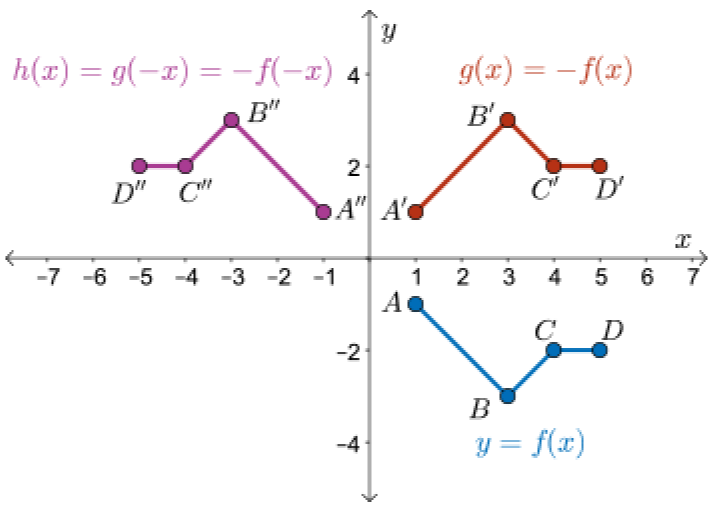
Transformations Review

In previous modules, we have examined specific transformations separately.

Translations

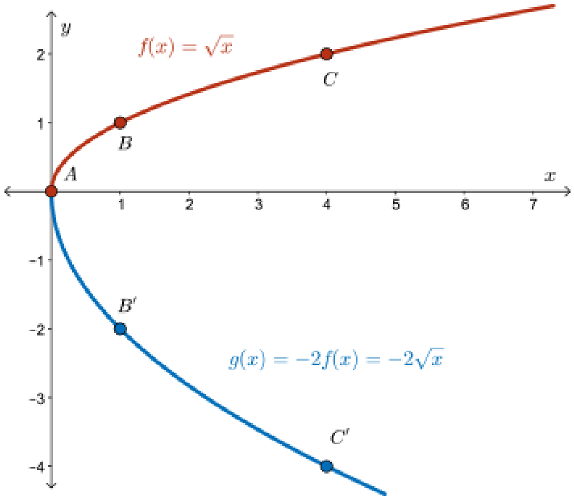
We saw that for some function, y=f(x), g(x)=f(x−h)+k is a translation of y=f(x) horizontally h units (to the right if $h>0$ or to the left if $h<0$) and vertically k units (up if $k>0$ and down if $k<0$).

Reflections



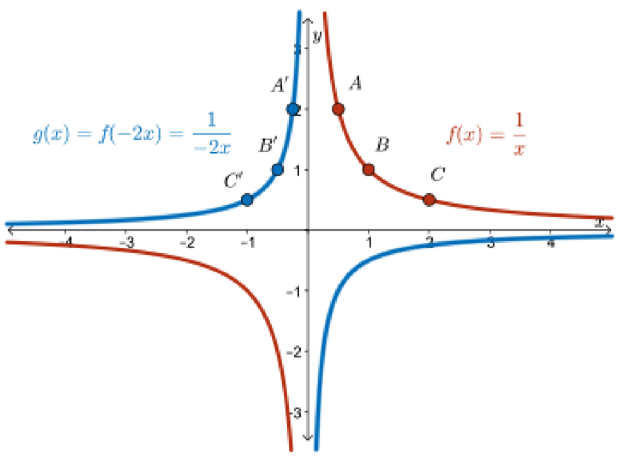
We then saw that for some function, $y=f(x)$, $g(x)=-f(x)$ is a reflection of $y=f(x)$ about the x-axis.
And we saw that for some function, $y=f(x)$, $g(x)=f(-x)$is a reflection of $y=f(x)$ about the y-axis.
This generalizes so that for some function, $y=f(x)$, $g(x)=af(bx)$ is at least a reflection of $y=f(x)$ about the x-axis if $a<0$ and about the y-axis if $b<0$. (Horizontal and/or vertical stretches may also be involved.)

Vertical Stretches



Next, we saw that for some function, *y*=*f*(*x*), *g*(*x*) = *af*(*x*), where *a*>0, is a vertical stretch of *y*=*f*(*x*) about the *x*-axis by a factor of *a*.
We know that there is a reflection if *a*<0, so we generalize.
For some function, *y*=*f*(*x*), *g*(*x*)=*af*(*x*) is a vertical stretch of *y*=*f*(*x*) about the *x*-axis by a factor of |*a*|.
If |*a*|>1, *g*(*x*) stretches away from the *x*-axis.
If |*a*|<1, *g*(*x*) compresses towards the *x*-axis.
On the graph, *g*(*x*)=−2*f*(*x*) is a reflection of *y*=*f*(*x*)about the *x*-axis and a vertical stretch of *y*=−*f*(*x*) about the *x*-axis by a factor of 2.
Recall that *A* is called an invariant point since it is on the line of reflection.

Horizontal Stretches



Finally, we saw that for some function, *y*=*f*(*x*), *g*(*x*) = *f*(*bx*), where *b*>0, is a horizontal stretch of *y*=*f*(*x*) about the *y*-axis by a factor of 1/*b*.
We know that there is a reflection if *b*<0, so we generalize.
For some function *y*=*f*(*x*), *g*(*x*) = *f*(*bx*) is a horizontal stretch of *y*=*f*(*x*) about the *y*-axis by a factor of 1/|*b*|.

If $|b|>1$, $g(x)$ compresses towards the y-axis.
If $|b|<1$, $g(x)$ stretches away from the y-axis.
On the graph, $g(x) = f(-2x)$ is a reflection of $y=f(x)$about the y-axis and a horizontal stretch of $y=f(-x)$ about the y-axis by a factor of 1/2.