Lesson 12: Unit 6 - Applications of vectors(2)

What is a force?

A **force** can be described in terms of a *direction* and a *magnitude* (the strength of the force). This leads to a natural mathematical interpretation of force as a vector.

The magnitude of a force is measured in **newtons (N)**.
A newton is equal to the force that would give a 1 kg mass an acceleration of 1 metre per second every second.

$$1 \text{ N} = 1 \text{ kg}(\text{ms2})$$

As an example, gravity causes objects (at the Earth's surface) to accelerate at 9.8ms2 as they fall.

The magnitude of the gravitational force is the product of an object's mass and its acceleration. This is defined by Newton's second law, $F=ma$.
Thus, the gravitational force on a 1 kg object is approximately 9.8 N; on a 2 kg object, it is approximately 2 kg ×9.8ms2=19.6 N.

It is important to be able to determine the net effect of several forces acting on an object at once.
The object's state of motion is determined by this net force.

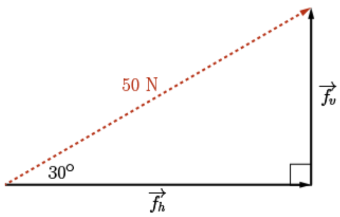
The net force can be represented by one single force, which has the same effect as all the forces acting together.
This single force is called the **resultant**, and can be determined by using vector addition.

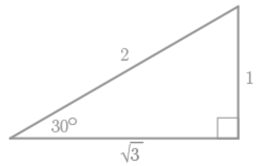
Let's take a look to some examples.

Ex. Erin pulls a wagon with a force of 50 N. The handle on the wagon makes an angle of 30∘ to the horizontal.

- **a.** How much of Erin's force is being exerted to pull the wagon forward?
- **b.** How much force is tending to lift the wagon upwards?

Solution
We resolve the 50 N force into its *horizontal* and *vertical* components.



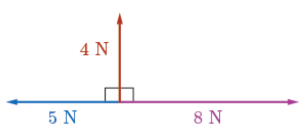


$\cos(30^0)$ = $\frac{|\vec{f_h}|}{50}$ ⇨ $|\vec{f_h}|$ = 50 $\cos(30^0)$ = $25\sqrt{3}$ N
This is the magnitude of the horizontal component of the force.

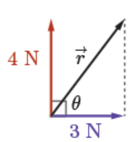
$\sin(30^0)$ = $\frac{|\vec{f_v}|}{50}$ ⇨ $|\vec{f_v}|$ = 50$\sin(30^0)$ = 25 N
This is the vertical component of the force.

Ex. Find the magnitude and direction of the resultant of the following forces acting on an object:
$\vec{u}$ has magnitude 8 N east; $\vec{v}$ has magnitude 4 N north; and $\vec{w}$ has magnitude 5 N west.

Solution
First draw a diagram and arrange the vectors tail to tail.



The resultant, $\vec{r}$, of all three forces is equivalent to the net force of the resultant of two vectors with the third.
Observe that $\vec{u}$ is in the east direction and $\vec{w}$ is in the west direction.
Thus, we determine the resultant of the 8 N east force and the 5 N west force to be a 3 N east force.



$\tan(\vartheta)$=43 and so $\vartheta \approx 53^0$
$|\vec{r}|^2 = 3^2 + 5^2 = 25$
$|\vec{r}|$= 5 N
The resultant of the three given forces is 5 N and its direction is approximately east 53^0 north.
Alternatively, we could say $\vec{r}$'s direction is north 37^0east.

Ex. Could forces of 5 N, 18 N, and 11 N act on an object to produce equilibrium?

Solution

Vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are in equilibrium if $\vec{u}+\vec{v}+\vec{w}=\vec{0}$.

First let's assume that these three forces are not parallel.

Then to create equilibrium, the three vectors $\vec{u}, \vec{v}$, and $\vec{w}$ with magnitudes 5 N, 18 N, and 11 N, must form a triangle.

The triangle inequality gives that $|\vec{u}+\vec{v}| \leq |\vec{u}|+|\vec{v}|$.

This must be true for every possible assignment of the three vectors to the three magnitudes, 5 N, 18 N, and 11 N.

For example, if $|\vec{u}+\vec{v}|=5$ and $|\vec{u}|=18$ and $|\vec{v}|=11$, then the triangle inequality is satisfied.

In particular, if $|\vec{u}|=5$ N and $|\vec{v}|=11$ N, then the resultant $|\vec{u}+\vec{v}|=18$ N.

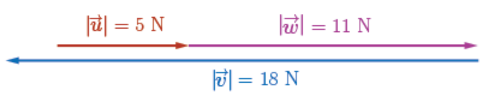
Then

$$|\vec{u}+\vec{v}|=18 \nleq 16=5+11=|\vec{u}|+|\vec{v}|$$

We conclude, it is impossible to form a triangle with these three non-parallel forces. Thus, equilibrium cannot be achieved.

What if the three forces are parallel?

To produce equilibrium, we still need the resultant of the sum of these three vectors to equal $\vec{0}$.



We see that this is not possible either.

Thus, forces 5 N, 18 N, and 11 N acting on an object could never produce equilibrium.

In general, it is important to note that we did not need to consider the two cases (non-parallel vectors and parallel vectors) separately.

For three non-parallel vectors $\vec{u}, \vec{v}$, and $\vec{w}$ to produce equilibrium, then $|\vec{u}+\vec{v}|<|\vec{u}|+|\vec{v}|$.

For three parallel vectors $\vec{u}, \vec{v}$, and $\vec{w}$ to produce equilibrium, then $|\vec{u}+\vec{v}|=|\vec{u}|+|\vec{v}|$.

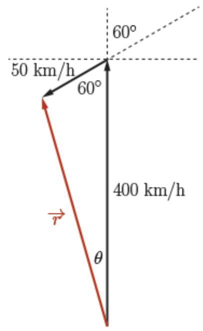
That is, using the triangle inequality $|\vec{u}+\vec{v}| \leq |\vec{u}|+|\vec{v}|$ accounts for both of these cases.

Velocity is another example of vectors.

Ex. An airplane heading due north at 400 km/h encounters a 50 km/h wind from north 60∘ east. Determine the resultant velocity (magnitude and heading) of the plane.

Solution

Start by drawing a diagram.



$|\vec{r}|^2 = 50^2 + 400^2 - 2(50)(400)\cos(60^0)$
$|\vec{r}|^2 \approx 377.5$ km/h

Solving for the direction using the sine law:

$$\sin(\vartheta)/50 = \sin(60\circ)/377.5 \approx 7\circ$$

Therefore, the resultant velocity of the plane is approximately 377.5 km/h on a course of north 7∘ west.

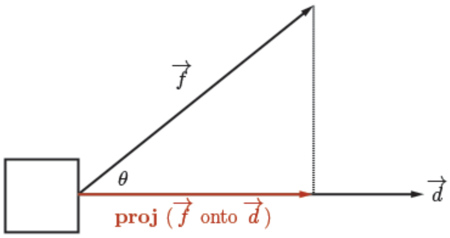
The heading is the direction the nose of the plane is pointed. The course or track is the resultant.

Work

Work is said to have been done when a force acts on an object to displace it in the direction of the force.

Mathematically, work is defined to be the scalar quantity equal to the component of the force that acts in the direction of the displacement multiplied by the magnitude of that displacement.

- $\vec{f}$ is the force acting on the object.
- $\vec{d}$ is the displacement of the object.
- ϑ is the angle between $\vec{f}$ and $\vec{d}$.



Work $= |$**proj** $(\vec{f}$ onto $\vec{d})||\vec{d}|$

$= |\vec{f}|\cos(\vartheta)|\vec{d}|$, for $0\circ \leq \vartheta \leq 90\circ$

$=|\vec{f}\,||\vec{d}\,|\cos(\vartheta)$

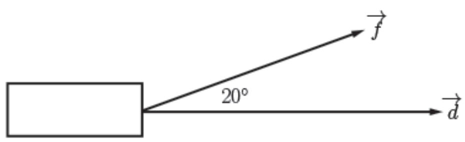
$=\vec{f}\cdot\vec{d}$

Note that the final definition of work is the dot product, $\vec{f}\cdot\vec{d}$, of the force and displacement vectors, and **not** the magnitude.

This allows for the possibility that work may be a negative quantity. Note that this happens when $\vec{d}$ and the **proj** ($\vec{f}$ onto $\vec{d}$) are in opposite directions.
The units of measurement for work are defined to be **newton-metres (Nm)** or **joules (J)**, where 1 Nm =1 J, so long as displacement is measured in metres and force in newtons.

Ex. A wagon is pulled a distance of 200 m by a 160 N force applied at an angle of 20∘ to the road. Calculate the work done by the force.

Solution



$W=|\vec{f}\,||\vec{d}\,|\cos(\vartheta)$

$=(160)(200)\cos(20^\circ)$

$\approx 30\,070$ J

Ex. Find the amount of work done by a 5 N force in moving an object from A (−2,1) to B (7,8), where the force is applied at a 30∘ angle to AB at A. Assume the distance moved is in metres.

Solution

First, we determine the direction vector

$$\overrightarrow{AB}=(9,7)$$

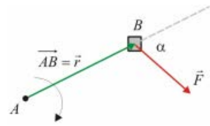
Therefore, W = $|\vec{f}||\vec{d}|\cos(\theta)=\frac{5}{2}\sqrt{390}$J.

Torque

The torque (rotational or turning effect) about the point A , created by a force $\vec{F}$ acting on an object located at the point B is given by:

$\vec{\tau} = \overrightarrow{AB} \times \vec{F} = \vec{r} \times \vec{F}$

$|\vec{\tau}|$= rFsin(α)



Where α = <($\vec{F}$,$\vec{r}$)

$\vec{r} = \overrightarrow{AB}$