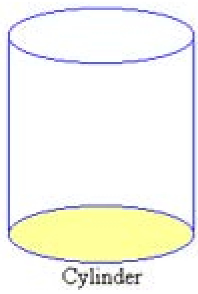
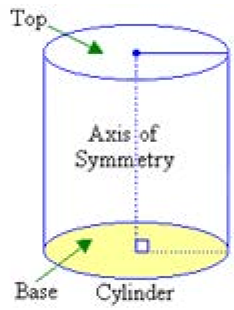
# Chapter 7 Measurement

## 1. Cylinders

Cones and cylinders have curved surfaces as shown below.  So, they are not prisms or polyhedra.



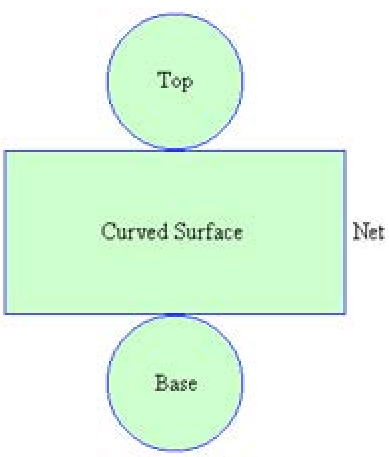

If a line is rotated about a second fixed line while keeping both lines parallel, then a **cylinder** is formed.

Note:

- The fixed line about which the line is rotated is called the **axis of symmetry**.
- The base of the cylinder is a circle and the top of the cylinder is also a circle.
- A cylinder is said to be **right** when the line joining the centre of the base and the centre of the top is perpendicular to the base of the cylinder.
- The **cross-sections** parallel to the base are circles and are all identical.

The **net** of a cylinder consists of three parts:

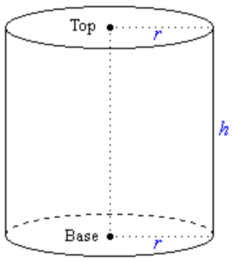
- One circle gives the base and another circle gives the top.
- A rectangle gives the curved surface.



## 2. Curved Surface Area of a Cylinder

Consider a cylindrical can of radius *r* and height *h*.

To determine a formula for the **curved surface area** of a cylindrical can, wrap a sheet of paper snugly around the can and tape it together. Trim the paper at the top and bottom to match the shape of the can. Then slide the paper off the can and cut this paper cylinder parallel to its axis so that it forms the rectangle shown in the following diagram.



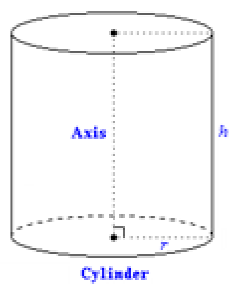


$$
\begin{aligned}
\text{Clearly, the length of the rectangle} &= \text{Circumference of the base} \\
&= 2\pi r \\
\text{The width of the rectangle} &= \text{Height of the cylinder} \\
&= h \\
\therefore \text{ Area of the curved surface} &= \text{Area of the rectangle} \\
&= l \times w \\
&= 2\pi rh
\end{aligned}
$$

Recall:

The **curved surface area** ($CSA$) of a **cylinder** with radius $r$ and height $h$ is given by $2\pi rh$.
That is:

$$CSA = 2\pi rh$$

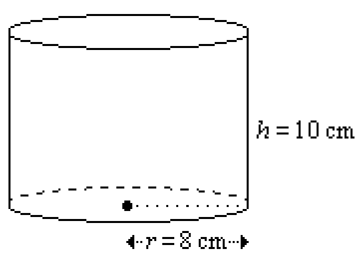



**Example:** Find the area of the curved surface of a cylindrical tin with radius 8 cm and height 10 cm.

Solution:

$$
\begin{aligned}
CSA &= 2\pi rh \\
&= 2 \times \pi \times 8 \times 10 \\
&= 502.65
\end{aligned}
$$

So, the curved surface area is 502.65 $\text{cm}^2$.



Note: The circular base of the cylinder is drawn as an ellipse.

### 3. Total Surface Area of a Cylinder

Recall:

The **total surface area** ($TSA$) of a **cylinder** with radius $r$ and height $h$ is given by $2\pi r(r+h)$.

That is:

$TSA = 2\pi r(r+h)$

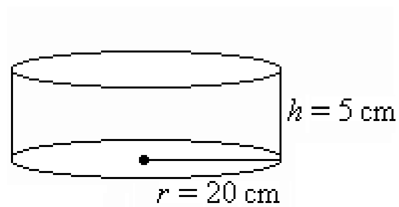
**Example**

Using $\pi = 3.142$, find the total surface area of a cylindrical tin of radius 20 cm and height 5 cm.

Solution:

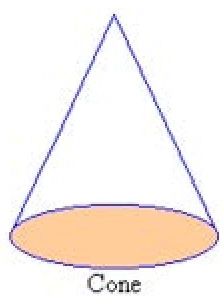
$$\begin{aligned} TSA &= 2\pi r(r+h) \\ &= 2\times 3.142\times 20(20+5) \qquad \{\text{BODMAS}\} \\ &= 2\times 3.142\times 20\times 25 \\ &= 3142.00 \end{aligned}$$

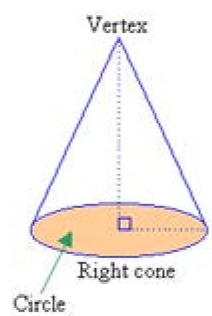
So, the total surface area is 3142 $\text{cm}^2$.




### 4. Cones

If one end of a line is rotated about a second fixed line while keeping the line's other end fixed, then a **cone** is formed.



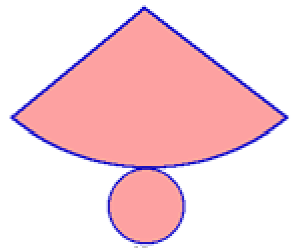

The point about which the line is rotated is called the **vertex** and the base of the cone is a circle.

A cone is said to be **right** when the vertex is directly above the centre of the base.

The **net** of a three-dimensional object is a representation of its faces in two dimensions.

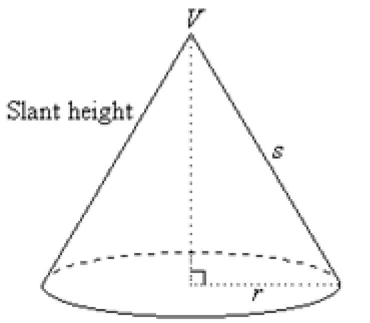
The **net** of a cone consists of the following two parts:

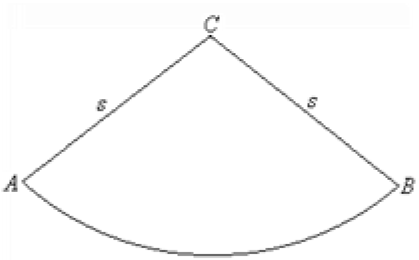
- a circle that gives the base; and
- a sector that gives the curved surface




## 5. Total Surface Area of a Cone

If a **cone** of base radius $r$ and slant height $s$ is cut along the slant height and opened out flat, then the radius of the sector formed is $s$ and the arc length $AB$ is $2\pi r$.




$$\text{Now, } \frac{\text{Area of sector } ABC}{\text{Area of circle with centre at } C} = \frac{\text{Arc length } AB \text{ of sector } ABC}{\text{Circumference of circle with centre at } C}$$

$$\therefore \frac{\text{Area of sector } ABC}{\pi s^2} = \frac{2\pi r}{2\pi s}$$

$$= \frac{r}{s}$$

$$\therefore \text{Area of sector } ABC = \frac{r}{s} \times \pi s^2$$

$$= \pi r s$$

If the curved surface area of a cone is equal to the area of sector $ABC$, then:
The area of the curved surface of a cone = $\pi r s$

$$\therefore \text{Total surface area of the cone} = \text{Area of the base} + \text{Area of curved surface}$$

$$\therefore TSA = \pi r^2 + \pi r s$$

$$TSA = \pi r(r + s)$$

Note:
A cone does not have uniform (or congruent) cross-sections.

**Example**

A conical tent has a radius of 3 m and a perpendicular height of 4 m. Find:

a. the slant height of the tent

b. the curved surface area of the cone using $\pi = 3.142$.

Solution:

a. By Pythagoras' Theorem from triangle $VCA$,

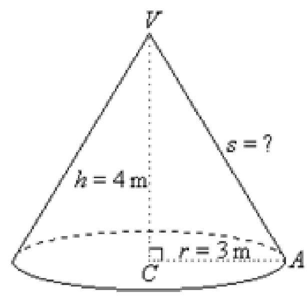
$$s^2 = h^2 + r^2 = 4^2 + 3^2 = 16 + 9 = 25$$

$$\therefore s = \sqrt{25} = 5$$

So, the slant height of the tent is 5 m.

b. $CSA = \pi rs = 3.142 \times 3 \times 5 = 47.13$

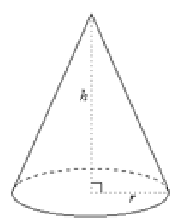
So, the curved surface area is $47.13\ \text{m}^2$.




### 6. Volume of a Cone

The volume of a cone is given by

$$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h$$




**Example**

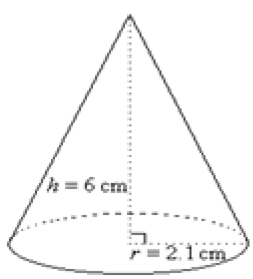
Find the volume of a cone whose base radius is 2.1 cm and height is 6 cm using $\pi = \frac{22}{7}$.

Solution:

$$V = \frac{1}{3}Ah$$

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 2.1^2 \times 6 = 27.72$$

So, the volume is $27.72\ \text{cm}^3$.

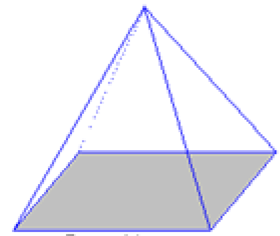



### 7. Pyramid

**Pyramids** are polyhedra because they are made up of plane faces. **Spheres** are not polyhedra because they are curved.

If a line that is fixed at a point, called the **vertex**, is moved around the perimeter of a polygon, then a **pyramid** is formed.

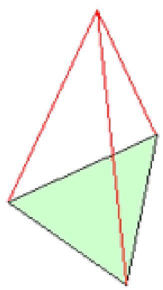
The right solid is pyramid.



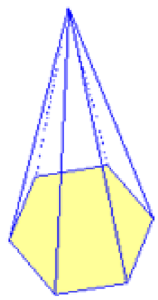
Pyramid

Pyramids are named after their **base**. So, the pyramid shown above is an example of a rectangular pyramid.

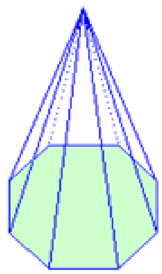
Some other pyramids are shown below.



Triangular Pyramid

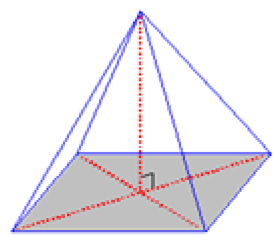


Hexagonal Pyramid



Octagonal Pyramid

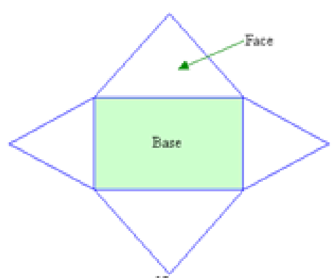
A pyramid is said to be **right** when its vertex is directly above the centre of the base. The **cross-sections** parallel to the base are the same shape but have different sizes.



Right Rectangular Pyramid

The **net** of the above pyramid consists of:

- a rectangular base; and
- four triangular faces




Net

Note:

- The **cross-sections** parallel to the base have the same shape as the base but different sizes.
- The **net** of a pyramid consists of a base polygon and a number of triangular faces.

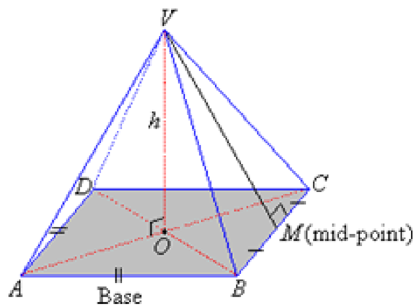
- A **regular pyramid** has a base that is a regular polygon and has faces which are isosceles triangles.

### 8. Total Surface Area of a Pyramid

A **pyramid** is a three-dimensional figure made up of a base and **triangular faces** that meet at the **vertex**, *V*, which is also called the **apex** of the pyramid.

The number of triangular faces depends on the number of sides of the base.

For example, a pyramid with a rectangular base has four triangular faces, whereas a pyramid with a hexagonal face is made up of six triangular faces, and so on.




The lower face *ABCD* is called the **base** and the perpendicular distance from the vertex, *V*, to the base at *O* is called the **height** of the pyramid. The total surface area of a pyramid is the sum of the areas of its faces including its base.

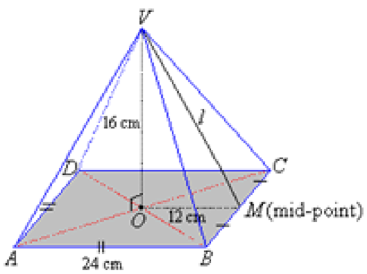
Note:

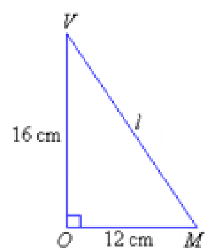
- A square pyramid has four equal triangular faces and a square base.
- A pyramid does not have uniform (or congruent) cross-sections.

**Example**

Find the total surface area of a square pyramid with a perpendicular height of 16 cm and base edge of 24 cm.

Solution:




By Pythagoras' Theorem from right-triangle *VOM*, we have

$l^2 = 16^2 + 12^2 = 256 + 144 = 400 \quad \therefore l = \sqrt{400} = 20$

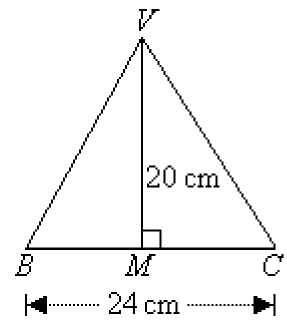
Area of triangle $VBC = \frac{1}{2}bh = \frac{1}{2} \times 24 \times 20 = 240$

Area of square base $= s^2 = 24^2 = 576$

$TSA =$ Area of the base $+ 4 \times$ Area of a triangular face

$= 576 + 4 \times 240 = 576 + 960 = 1536$

So, the total surface area is 1536 cm$^2$.




**9. Volume of a Pyramid**

The volume, *V*, of a pyramid in cubic units is given by

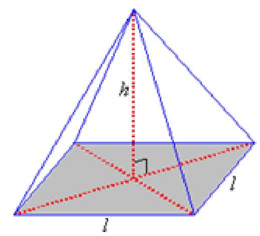
$V = \frac{1}{3}\text{Area of the base} \times \text{Height} = \frac{1}{3}Ah$

where *A* is the area of the base and *h* is the height of the pyramid.

**10. Volume of a Square-based Pyramid**

The volume of a **square-based pyramid** is given by

$V = \frac{1}{3}Ah = \frac{1}{3}l^2h$




**Example**

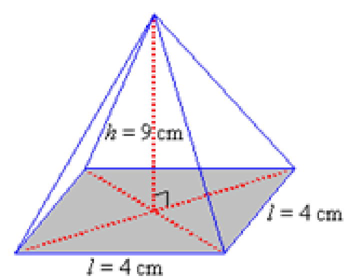
A pyramid has a square base of side 4 cm and a height of 9 cm. Find its volume.

Solution:

$V = \frac{1}{3}Ah$

$= \frac{1}{3}l^2h = \frac{1}{3} \times 4^2 \times 9 = 48$

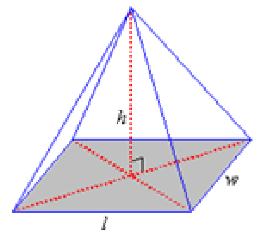
So, the volume is 48 cm$^3$.



### 11. Volume of a Rectangular-based Pyramid

The volume of a **rectangular-based pyramid** is given by

$$V = \frac{1}{3}Ah = \frac{1}{3}lwh$$



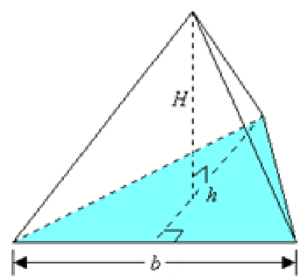

**Example**
Find the volume of a rectangular-based pyramid whose base is 8 cm by 6 cm and height is 5 cm.

Solution: V = $\frac{1}{3}$ x 8 x 6 x 5 = 80 cm$^2$
So the volume is 80 cm$^2$.

### 12. Volume of a Triangular Pyramid

The volume of a **triangular pyramid** is given by

$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}\left(\frac{1}{2}bh\right)H = \frac{1}{6}bhH$$

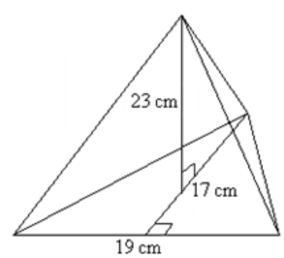



**Example**
Find the volume of the following triangular pyramid, rounding your answer to two decimal places.

Solution:

$$V = \frac{1}{3}AH$$
$$= \frac{1}{3}(\frac{1}{2}bh)H = \frac{1}{6}bhH = \frac{1}{6}\times 19\times 17\times 23$$
$$= 1238.17 \quad \{\text{Correct to 2 decimal places}\}$$

So, the volume is 1238.17 cm$^3$.



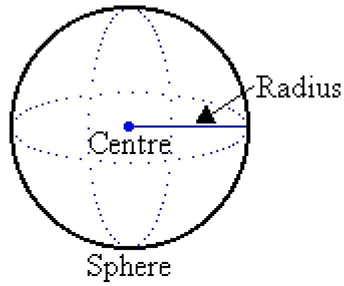

### 13. Spheres

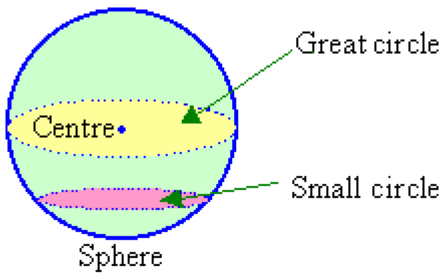
If a semicircle is revolved about its diameter, then a **sphere** is formed.

The following solid is a sphere.

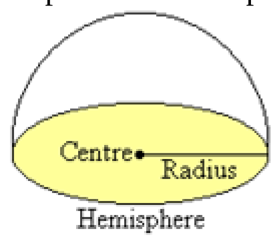
The cross-sections of a sphere are circles.

The cross-sections that pass through the centre of the sphere are called **great circles** whilst other cross-sections are called **small circles**.




A hemisphere is a half sphere

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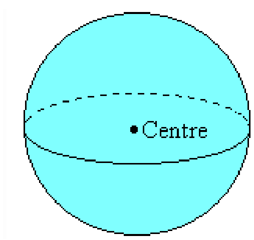
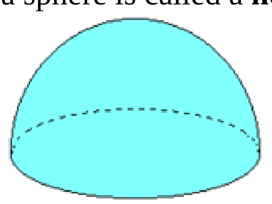
## 14. Surface Area of a Sphere

A **sphere** is a body bounded by a surface whose every point is **equidistant** (i.e. the same distance) from a fixed point, called the centre. For example, a shot (a heavy iron ball) is a solid sphere and a tennis ball is a hollow sphere.

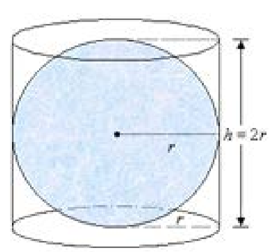



One-half of a sphere is called a **hemisphere**.



Archimedes discovered that a cylinder that circumscribes a sphere, as shown in the following diagram, has a curved surface area equal to the surface area, *S*, of the sphere.

$$\text{Surface area of sphere} = CSA \text{ of a cylinder}$$
$$\therefore S = 2\pi rh$$
$$= 2\pi r \times 2r$$
$$S = 4\pi r^2$$



**Example 1**

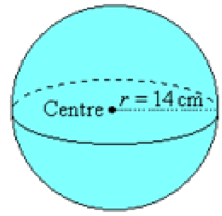
Find the surface area of a sphere of diameter 28 cm using $\pi = \dfrac{22}{7}$.

Solution:

$d = 28$ cm $\qquad r = \dfrac{1}{2}d = \dfrac{1}{2} \times 28 = 14$ cm

Now, $S = 4\pi r^2 = 4 \times \dfrac{22}{7} \times 14^2 = 2464$

So, the surface area is $2464\ \text{cm}^2$.




**Example 2**

A spherical ball has a surface area of $2464\ \text{cm}^2$. Find the radius of the ball, correct to 2 decimal places, using $\pi = 3.142$.

Solution:

Let the radius be $r$ cm.

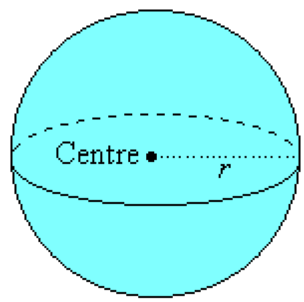
$$S = 4\pi r^2$$

$$2464 = 4\pi r^2 = 4 \times 3.142 \times r^2 = 12.568 r^2$$

$$\frac{2464}{12.568} = \frac{12.568 r^2}{12.568}$$

$$\therefore\ r^2 = 196.054 \qquad r = \sqrt{196.054} = 14.00$$

So, the radius of the ball is 14 cm.



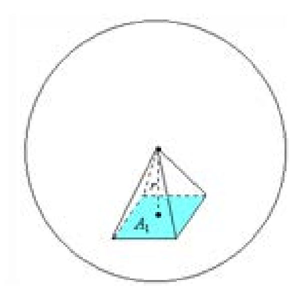

Note:

We often use a circle to represent a sphere. The three-dimensional figure of a sphere is drawn by a circle and an ellipse as illustrated above.

### 14. Volume of a Sphere

If four points on the surface of a sphere are joined to the centre of the sphere, then a pyramid of perpendicular height *r* is formed, as shown in the diagram.

Consider the solid sphere to be built with a large number of such solid pyramids that have a very small base which represents a small portion of the surface area of a sphere.



If $A_1, A_2, A_3, A_4, \ldots, A_n$ represent the base areas (of pyramids) on the surface of a sphere, then:

$$\begin{aligned} V &= \text{Volume of the sphere} \\ &= \text{Sum of the volumes of all pyramids} \\ &= \frac{1}{3}A_1 r + \frac{1}{3}A_2 r + \frac{1}{3}A_3 r + \frac{1}{3}A_4 r + \ldots + \frac{1}{3}A_n r \\ &= \frac{1}{3}(A_1 + A_2 + A_3 + A_4 + \ldots + A_n) r \\ &= \frac{1}{3}(\text{Surface area of the sphere}) r \\ &= \frac{1}{3} \times 4\pi r^2 \times r \ = \frac{4}{3}\pi r^3 \end{aligned}$$

$\therefore$ The volume, $V$, of a sphere in cubic units is given by

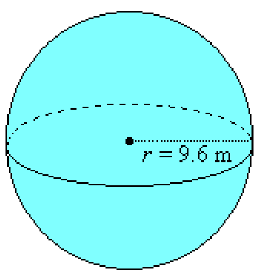
$$V = \frac{4}{3}\pi r^3$$

where $r$ is the radius of the sphere.

**Example:** Find the volume of a sphere of radius 9.6 m, rounding your answer to two decimal places.

Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 9.6^3 \\ &= \frac{4}{3} \times \pi \times 884.736 \\ &= 3705.97 \end{aligned}$$




So, the volume is 3705.97 $m^3$.

## 15. Optimizing Measurement

**Optimization –** the process of finding values that make a given quantity the greatest (or least) possible given certain conditions.

### Perimeter and Area Relationship of a Rectangle

Brandon needs to find the dimensions that will maximize the rectangular area of an enclosure with a perimeter of 32m.

| Rectangle | Width (m) | Length (m) | Perimeter (m) | Area ($m^2$) |
|---|---|---|---|---|
| 1 | 1 | 15 | 32 | 15 |
| 2 | 2 | 14 | 32 | 28 |
| 3 | 3 | 13 | 32 | 39 |
| 4 | 4 | 12 | 32 | 48 |
| 5 | 5 | 11 | 32 | 55 |
| 6 | 6 | 10 | 32 | 60 |
| 7 | 7 | 9 | 32 | 63 |
| 8 | 8 | 8 | 32 | 64 |

➔ **Maximum area occurs when the length = width.**

**When given the perimeter, the maximum area occurs when the rectangle is a square.**

*How about 3 sides only?*
Now Brandon decides to use one of the existing walls as a boundary for the enclosure. He still has 32m of fencing. What is the greatest possible area of the rectangle? What is its dimension?

| Rectangle | Width (m) | Length (m) | Sum of the 3 sides (m) | Area ($m^2$) |
|---|---|---|---|---|
| 1 | 1 | 32-2x1 = 30 | 32 | 1x30 = 30 |
| 2 | 2 | 32-2x2 = 28 | 32 | 56 |
| 3 | 3 | 26 | 32 | 78 |
| 4 | 4 | 24 | 32 | 96 |
| 5 | 5 | 22 | 32 | 110 |
| 6 | 6 | 20 | 32 | 120 |
| 7 | 7 | 18 | 32 | 126 |
| 8 | 8 | 16 | 32 | 128 |

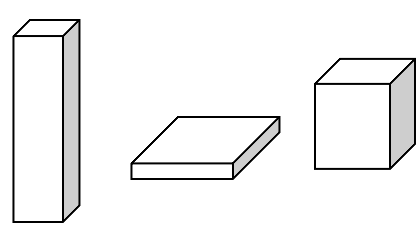
➔ **Maximum area occurs when the length = 2 x width.**

**When given the perimeter, the maximum area occurs for 3 sided rectangle is when the length is twice the width.**

In 3-dimensional figures, in order to save materials, we need to **minimize surface area** instead of perimeter. We also want to **maximize volume** instead of area.

**Minimizing Surface Area (from a fixed Volume) of Rectangular Based Prisms**

| V | Length x of Square base | Volume = l w h Find height h | SA = $2x^2 + 4xh$ |
|---|---|---|---|
| 512 | x = 5 | h = $512 / 5^2 = 20.48$ | 459.6 |
| | x = 6 | h = 14.2 | 412.8 |

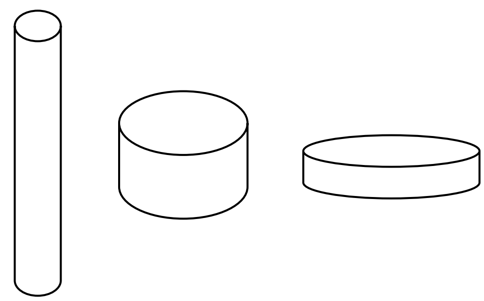


| | | | |
|---|---|---|---|
| | x = 7 | h = 10.4 | 389.2 |
| | x = 8 | h = 8 | 384 |
| | x = 9 | h = 6.3 | 388.8 |
| | x = 10 | h = 5.12 | 404.8 |

**Conclusion: The minimum surface area of a rectangular based prism for a given surface area, always occurs when it is a cube. ie. SA = $6x^2$, V = $x^3$**

**Minimizing Surface Area (from a fixed Volume) of Cylinders**

| V | Radius r of the base | Volume = $\pi r^2 h$ Find height h | SA = $2\pi r^2 + 2\pi rh$ |
|---|---|---|---|
| 400 | r = 2 | h = 400 / $\pi(2)^2$ = 31.8 | 424.7 |
| | r = 3 | h = 14.15 | 323.3 |
| | r = 4 | h = 7.96 | 300.6 |
| | r = 5 | h = 5.1 | 317.3 |
| | r = 6 | h = 3.54 | 359.6 |



➔ **Minimum SA occurs when h = 2r.**

SA = $2\pi r^2 + 2\pi rh$
= $2\pi r^2 + 2\pi r(2r)$
= $2\pi r^2 + 4\pi r^2$
= $6\pi r^2$

V = $\pi r^2 h$
= $\pi r^2(2r)$
= $2\pi r^3$

**Conclusion: The maximum volume of a cylinder for a given surface area, always occurs when h = 2r = d ie. SA = $6\pi r^2$, V = $2\pi r^3$**

**Maximizing Volume (from a fixed Surface Area) of Rectangular Based Prisms**

| SA | Length x of Square base | SA = $2x^2 + 4xh$ Find height h | Volume = l w h |
|---|---|---|---|
| 96 | x = 1 | 96 = 2 x $1^2$ + 4 x 1 x h<br>h = 23.5 | 23.5 |
| | x = 2 | h = 11 | 44 |
| | x = 3 | h = 6.5 | 58.5 |
| | x = 4 | h = 4 | 64 |
| | x = 5 | h = 23 | 57.5 |
| | x = 6 | h = 1 | 36 |

**Conclusion: The maximum volume of a rectangular based prism for a given surface area, always occurs when it is a cube. ie. SA = $6x^2$, V = $x^3$.**

**Maximizing Volume (from a fixed Surface Area) of Cylinders**

| SA | Radius r of the base | SA = $2\pi r^2 + 2\pi rh$ Find height h | Volume = $\pi r^2 h$ |
|---|---|---|---|
| 75 | r = 1 | 75 = $2\pi(1)2 + 2\pi(1)h$<br>h = 10.94 | 34.3 |
| | r = 2 | h = 3.97 | 50.2 |
| | r = 3 | h = 1 | 27.7 |

**Conclusion: The maximum volume of a cylinder for a given surface area, always occurs when h = 2r = d ie. SA = $6\pi r^2$, V = $2\pi r^3$**

**In Summary:**

**Rectangular Prism:**
Given a fixed SA, the closed rectangular prism that maximizes the volume is a cube.
Given a fixed volume, the closed rectangular prism that minimizes the SA is a cube.
SA = $6x^2$ V = $x^3$

**Cylinder:**
Given a fixed SA, the closed cylinder that maximizes the volume is with h = 2r = d.
Given a fixed volume, the closed cylinder that minimizes the SA is with h = 2r = d.
SA = $6\pi r^2$ V = $2\pi r^3$