Lesson 1- *Unit 1 – An introduction to calculus (1)*

Calculus is a major branch of mathematics which build on algebra, trigonometry, and analytic geometry. It has wide spread applications in science, engineering, financial mathematics etc.

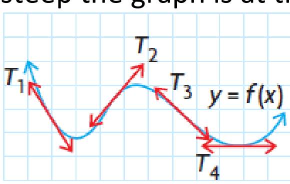
The study of calculus is divided into two fields, differential calculus and integral calculus.

The concept of a limit is essential in differential calculus. We will see that calculating limits is essential in finding the slope of the tangent to a curve at any point on the curve.

We begin our study of calculus by discussing the meaning of a tangent and the related idea of rate of change. This leads us to the study of limits and to the concept of the derivative of a function.

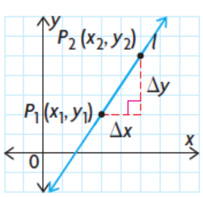
You are familiar with the concept of a **tangent** to a curve. What geometric interpretation can be given to a tangent to the graph of a function at a point *P*?

A tangent is the straight line that most resembles the graph near a point. Its slope tells how steep the graph is at the point of tangency. In the figure below, four tangents have been drawn.



The goal of this section is to develop a method for determining the slope of a tangent at a given point on a curve. We begin with a brief review of lines and slopes.

Lines and Slope



The slope *m* of the line joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined as

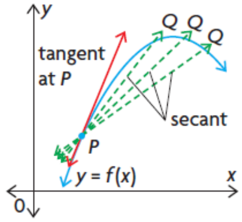
m = $\frac{y_2 - y_1}{x_2 - x_1}$.

The equation of the line *l* in point-slope form is y - y_1 = m(x - x_1).

The equation in slope–*y*-intercept form is y = mx + b where *b* is the *y*-intercept of the line.

To determine the equation of a tangent to a curve at a given point, we first need to know the slope of the tangent. What can we do when we only have one point?

We proceed as follows:



Consider a curve y = f(x) and a point *P* that lies on the curve. Now consider another point *Q* on the curve. The line joining *P* and *Q* is called a **secant**. Think of *Q* as a moving point that slides along the curve toward *P*, so that the slope of the secant *PQ* becomes a progressively better estimate of the slope of the tangent at *P*.

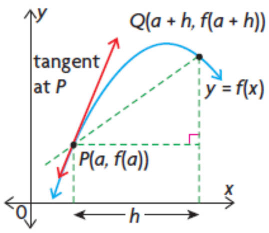
This suggests the following definition of the slope of the tangent:

Slope of a Tangent

The slope of the tangent to a curve at a point *P* is the limiting slope of the secant *PQ* as the point *Q* slides along the curve toward *P*. In other words, the slope of the tangent is said to be the **limit** of the slope of the secant as *Q*
approaches *P* along the curve.

The Slope of a Tangent at an Arbitrary Point

We can now derive a formula for the slope of the tangent to the graph of any function y = f(x).



Let P(a, f(a)) be a fixed point on the graph of and let Q(a+h, f(a+h)) represent any other point on the graph.

The slope of the secant PQ is $\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$

.

Therefore, the slope *m* of the tangent at P(a, f(a)) is

m = $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$**,** if this limit exists.

Ex. Consider the function y = f(x) = x^2- 2x.

a.Find the slope of the tangent line at the generic point P(a, f(a))

b. Find the point where the tangent line is horizontal.

c. Find the point P such that m_P = 2.

d. Find the point P such that the tangent line at P is perpendicular to the line L: x − 3y = 3.

Solution

a.

$\frac{f(a+h)-f(a)}{h} = \frac{2ah+h^2-2h}{h}$ = 2a + h – 2

$m_P = \lim_{h\to 0}(2a+h-2)$ = 2a – 2

b. m = 0 ⇨ a = 1

∴ P(1, -1)

c. m_P = 2 ⇨ a = 2

∴ P(2, 0)

d. $m_P = \frac{1}{3}$

∴ P(-1/2, 5/4)

Ex. Find the equation of the tangent line to the graph of y = f(x) = $\frac{1}{\sqrt{x-1}}$ at the point P(2, 1).

Solution

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{\sqrt{a+h-1}}-\frac{1}{\sqrt{a-1}}}{h}$$

$$= \frac{\frac{1}{\sqrt{a+h-1}}-\frac{1}{\sqrt{a-1}}}{h}\,\frac{\frac{1}{\sqrt{a+h-1}}+\frac{1}{\sqrt{a-1}}}{\frac{1}{\sqrt{a+h-1}}+\frac{1}{\sqrt{a-1}}}$$

$$= \frac{\frac{1}{a+h-1}-\frac{1}{a-1}}{h(\frac{1}{\sqrt{a+h-1}}+\frac{1}{\sqrt{a-1}})} = \frac{\frac{a-1-a-h+1}{(a+h-1)(a-1)}}{h(\frac{1}{\sqrt{a+h-1}}+\frac{1}{\sqrt{a-1}})}$$

$$= \frac{\frac{-h}{(a+h-1)(a-1)}}{h(\frac{1}{\sqrt{a+h-1}}+\frac{1}{\sqrt{a-1}})} = \frac{\frac{-1}{(a+h-1)(a-1)}}{(\frac{1}{\sqrt{a+h-1}}+\frac{1}{\sqrt{a-1}})}$$

a = 2

$$\lim_{h\to 0}\frac{\frac{-1}{(a+h-1)(a-1)}}{(\frac{1}{\sqrt{a+h-1}}+\frac{1}{\sqrt{a-1}})} = -\frac{1}{2}$$

∴ y = $-\frac{1}{2}$x + 2

Rate of Change

Consider a function and two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on its graph.

The average rate of change of the function over the interval $[x_1,\ x_2]$ is defined by the quotient:

ARC = $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Geometrically, the average rate of change is the slope of the secant line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Instantaneous Rate of Change

As h → 0 the Average Rate of Change approaches to the Instantaneous Rate of Change (IRC):

IRC = $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Note: *The Instantaneous Rate of Change (IRC) is the same as the slope of the tangent line at the point P(a, f (a)).*

The limit of a Function

Consider the following table of values for f(x) = x^2 where x is less than 2 but increasing and getting closer and closer to 2:

x	1	1.9	1.99	1.999	1.9999
$f(x)$	1	3.61	3.9601	3.996 00	3.999 60

We say that as x approaches 2 from the left, f(x) approaches 4.from below.

We construct a similar table of values where x is greater than 2 but decreasing and getting closer and closer to 2:

x	3	2.1	2.01	2.001	2.0001
$f(x)$	9	4.41	4.0401	4.004 00	4.000 40

In this case we say as x approaches 2 from the right, f(x) approaches 4 from above.

In summary, we can now say that as x approaches 2 from either direction, f(x) approaches a limit of 4, and write

$\lim_{x \to 2} x^2 = 4$

Informal **definition of a limit**

The following definition of a limit is informal but adequate to the purpose of this course:

If f(x) can be made as close as we like to some real number A by making x sufficiently close (but not equal to) a, then we say that f(x) has a **limit** of A as x approaches a, and we write

$\lim_{x \to a} f(x) = \text{A}.$

In this case, f(x) is said to converge to A as x approaches a.

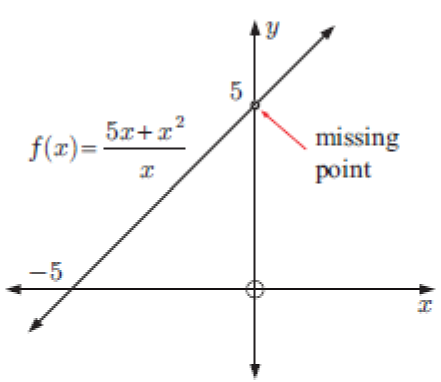
It is important to note that in defining the limit of *f* as *x* approaches *a, x* does NOT reach a.

The limit is defined for x close *but not equal* to a. Whether the function f is defined or not at x = a is not important to the definition of the limit of f as x approaches a. What is important is the behaviour of f as x gets very close to a.

For example, f(x) = $\frac{5x+x^2}{x}$ and we wish to find the limit as x approaches 0 it is tempting for us to simply substitute x = 0 into f(x). Not only that we get the meaningless value of $\frac{0}{0}$, but also we destroy the basic limit method. Observe that if f(x) = $\frac{5x+x^2}{x} = \frac{x(5+x)}{x}$ then

f(x) = $\begin{cases} 5+x & if\ x \neq 0 \\ is\ undefined & if\ x = 0 \end{cases}$

The graph of it is shown below. It is the straight line y = x + 5 with the point (0, 5) missing, called a point of discontinuity of the function.



However, even though this point is missing, the limit of f as x → 0 does exist and it is 5.

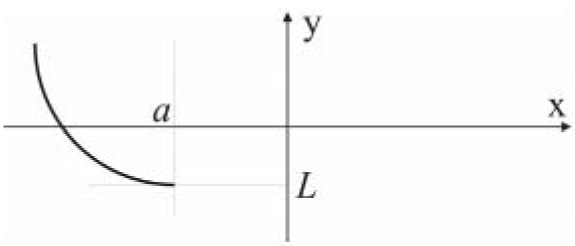
We write $\lim_{x \to 0} f(x) = 5$.

We can now define two other concepts.

Left-Hand Limit If the values of y = f (x) can be made arbitrarily close to L by taking x sufficiently close to a with x < a , then:

$\lim_{x \to a^-} f(x) = L$

Read: The limit of the function f (x) as x approaches a from the left is L.



Notes:

1. The function may be or not defined at a .

2. DNE stands for Does Not Exist.

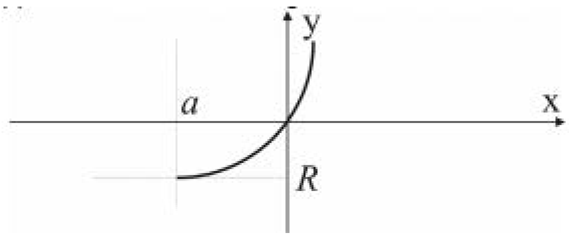
3. L must be a number.

4. ∞ is not a number.

Right-Hand Limit If the values of y = f (x) can be made arbitrarily close to L by taking x sufficiently close to a with x > a , then:

$\lim_{x \to a^+} f(x) = R$

Read: The limit of the function f (x) as x approaches a from the right is R.

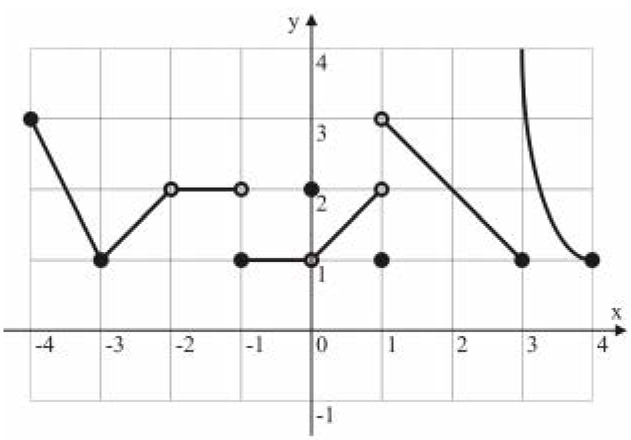


Notes:

1. R must be a number. ∞ is not a number.

2. The function may be or not defined at a.

Ex. Use the function y = f (x) defined by the following graph to find each limit.



1. $\lim_{x\to-4^-} f(x)$ DNE

2. $\lim_{x\to-4^+} f(x)$ = 3

3. $\lim_{x\to-3^+} f(x)$= 1

4. $\lim_{x\to-3^-} f(x)$= 1

5. $\lim_{x\to-2^-} f(x)$= 2

6. $\lim_{x\to-2^+} f(x)$= 2

7. $\lim_{x\to0^+} f(x)$= 1

8. $\lim_{x \to 0^-} f(x) = 1$

9. $\lim_{x \to 1^+} f(x) = 3$

10. $\lim_{x \to 1^-} f(x) = 2$

11. $\lim_{x \to 3^-} f(x) = 1$

12. $\lim_{x \to 3^+} f(x) = +\infty$

13. $\lim_{x \to 3} f(x)$ DNE

14. $\lim_{x \to -2} f(x) = 2$

Now to recap,

1. If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$ does exist and L = R = l.

2. If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$ Does Not Exist (DNE).

3. l must be a number. ∞ is not a number.

4. The function may be or not defined at a.

*If the function is defined by a formula (algebraic expression) then the limit of the function at a point a may be determined by **substitution**:*

$\lim_{x \to a} f(x)$ = f (a)

Notes:

1. In order to use substitution, the function must be defined on both sides of the number a.

2. Substitution does not work if you get one of the following 7 indeterminate cases:

$\infty - \infty \quad 0 \times \infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^{\infty} \quad 0^0 \quad \infty^0$

Ex. Compute each limit.

a. $\lim_{x\to 1^-} \frac{x^2}{x+1} = \frac{1^2}{1+1} = \frac{1}{2}$

b. $\lim_{x\to 1^+} \frac{x^2}{x+1} = \frac{1^2}{1+1} = \frac{1}{2}$

c. $\lim_{x\to 1} \frac{x^2}{x+1} = \frac{1^2}{1+1} = \frac{1}{2}$

d. $\lim_{x\to 2^-} \sqrt{x-2}$ = DNE

e. $\lim_{x\to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0$

f. $\lim_{x\to 2} \sqrt{x-2}$ = DNE

If the function changes formula at a then:

1. Use the appropriate formula to find first the left-side and the right-side limits.

2. Compare the left-side and the right-side limits to conclude about the limit of the function at a.

Ex. Consider f(x) = $\begin{cases} 2x-3, & x<2 \\ 0, & x=2 \\ x^2-1, & x>2 \end{cases}$

a. $\lim_{x\to 2} f(x)$

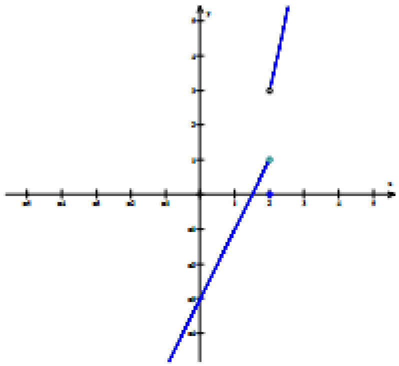
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x-3) = 1$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-1) = 0$

$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^-} f(x) \therefore \lim_{x\to 2} f(x)$ = DNE

b. $\lim_{x\to 0} f(x) = \lim_{x\to 0} (2x-3)$ = -3

c. Draw a diagram to illustrate the situation.



Limits Properties

We assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then:

1. $\lim_{x \to a} k$ = k

2. $\lim_{x \to a} x$ = a

3. $\lim_{x \to a}[f(x) \pm g(x)]$ = $\lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

4. $\lim_{x \to a} cf(x)$ = c$\lim_{x \to a} f(x)$

5. $\lim_{x \to a} \frac{f(x)}{g(x)}$ = $\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, $\quad \lim_{x \to a} g(x) \neq 0$

6. $\lim_{x \to a}[f(x)g(x)]$ = $\left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$

7. $\lim_{x \to a}[f(x)]^n$ = $\left[\lim_{x \to a} f(x)\right]^n$

8. If P(x) is a polynomial function, then

$\lim_{x \to a} P(x)$ = P(a)

9. **Squeeze Theorem** *(This result goes by a variety of names including The Sandwiching Theorem, The Pinching Theorem)*

Let assume that g(x) ≤ f (x) ≤ h(x) on an open interval containing the number x = a and $\lim_{x \to a} g(x)$ = $\lim_{x \to a} h(x)$ = L Then: $\lim_{x \to a} f(x)$ = L.

Ex. Given that $\lim_{x \to 2} f(x)$ = -1 and $\lim_{x \to 2} g(x)$ = 16 use the limits properties to find

$\lim_{x\to 2} \frac{-5f(x)+g(x)}{\sqrt[4]{g(x)}-f(x)}$

Solution

$$\lim_{x\to 2} \frac{-5f(x)+g(x)}{\sqrt[4]{g(x)}-f(x)} = \frac{\lim_{x\to 2}(-5f(x)+g(x))}{\lim_{x\to 2}(\sqrt[4]{g(x)}-f(x))}$$

$$= \frac{\lim_{x\to 2}(-5f(x))+\lim_{x\to 2} g(x)}{\lim_{x\to 2}\left(\sqrt[4]{g(x)}\right)-\lim_{x\to 2} f(x)}$$

$$= \frac{-5\lim_{x\to 2} f(x)+\lim_{x\to 2} g(x)}{\sqrt[4]{\lim_{x\to 2} g(x)} - \lim_{x\to 2} f(x)}$$

$$= \frac{-5(-1)+16}{\sqrt[4]{16}-(-1)}$$

$= 7$

Ex. Compute a. $\lim_{x\to 0} \frac{x^2-2x+1}{x-1} = \frac{0^2-2(0)+1}{0-1}$ = -1

b. $\lim_{x\to 1} \frac{x^2-2x+1}{x-1} = \frac{0}{0}$ (indeterminate case)

In this case substitution doesn't work.

The indeterminate form $\frac{0}{0}$ may be eliminated by **factoring** and canceling out the common factor that generates zeros:

$$\lim_{x\to a} \frac{F(x)}{G(x)} = \lim_{x\to a} \frac{(x-a)f(x)}{(x-a)g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Note: Canceling out the common factor (x − a) is a correct operation because $\lim_{x\to a} f(x)$ means that x approaches a but is not equal to a.

So, to come back to our question $\lim_{x\to 1} \frac{x^2-2x+1}{x-1} = \lim_{x\to 1} \frac{(x-1)^2}{x-1} = \lim_{x\to 1}(x-1) = 0$

When dealing with the indeterminate form $\frac{0}{0}$ you may use **the conjugate radicals** to cancel out the common factor that generates zeros.

Ex. Compute $\lim_{x\to 1}\frac{\sqrt{x+25}-5}{x}$.

Solution

Substitution does not work (case 0/0).

$$\lim_{x\to 1}\frac{\sqrt{x+25}-5}{x} = \lim_{x\to 1}\left(\frac{\sqrt{x+25}-5}{x}\times\frac{\sqrt{x+25}+5}{\sqrt{x+25}+5}\right)$$

$$= \lim_{x\to 0}\frac{x+25-25}{x(\sqrt{x+25}+5)}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{x+25}+5}$$

$$= \frac{1}{10}$$

Another way of dealing with the case 0/0 is by **changing the variable**, the process of canceling the common factor may be simplified.

Ex. Change the variable to compute $\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

Solution

$\sqrt{\boldsymbol{x}} = \boldsymbol{x}^{\frac{1}{2}}$ $\quad$ $\sqrt[3]{x} = \boldsymbol{x}^{\frac{1}{3}}$

LCM = **6**

u = $\boldsymbol{x}^{\frac{1}{6}}$ $\quad$ x→1 ⇨ u→ $\mathbf{1}^{\frac{1}{6}}$**=1**

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \lim_{u\to 1}\frac{u^3-1}{u^2-1} = \lim_{u\to 1}\frac{(u-1)(u^2+u+1)}{(u-1)(u+1)} = \frac{3}{2}$$

Ex. Compute each limit.

a. $\lim_{x\to 0} x|x|$ $\quad$ b. $\lim_{x\to 0}\frac{|x|}{x}$ $\quad$ c. $\lim_{x\to 0}[x]$ $\quad$ ($[x]$ is the integer part of x)

Solution

$a.$ $|x| = \begin{cases} -x & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$ ⇨ x$|x| = \begin{cases} -x^2 & if\ x < 0 \\ x^2 & if\ x \geq 0 \end{cases}$

$\lim_{x\to 0^-} x|x|$ = $\lim_{x\to 0^-} (-x^2)$ = 0

$\lim_{x\to 0^+} x|x|$ = $\lim_{x\to 0^+} (x^2)$ = 0

$\lim_{x\to 0} x|x|$ = 0

$b.$ $\frac{|x|}{x} = \begin{cases} -1 & if\ x < 0 \\ 1 & if\ x > 0 \end{cases}$

$\lim_{x\to 0^-} \frac{|x|}{x}$ = -1

$\lim_{x\to 0^+} \frac{|x|}{x}$ = 1

∴ $\lim_{x\to 0} \frac{|x|}{x}$ = DNE

$c.$ $[x]$ = -1 if -1≤ $x < 0$ and $[x]$ = 0 if 0≤ $x < 1$ ∴ $\lim_{x\to 0}[x]$= DNE