Chapter 5 Quadratic Equations (1)

1. Quadratic Equations

If the highest power of the pronumeral in an equation is two, then it is said to be a **quadratic equation**.

E.g. $x^2-4=0$, $x^2-9x=0$ and $x^2+7x+10=0$ are quadratic equations.

Solving Quadratic Equations

We will consider the following methods to solve a quadratic equation:

- Factorizing and using the Null Factor Law
- Quadratic formula

The Null Factor Law

Clearly,
$$\begin{aligned} 0\times 9&=0\\ 9\times 0&=0\\ 0\times 0&=0\end{aligned}$$

From this we can infer that:

If the product of any two numbers is zero, then one or both of the numbers is zero.

That is, if $ab = 0$, then $a = 0$ or $b = 0$ (which includes the possibility that $a = b = 0$).

This is called the **Null Factor Law**; and we use it often to solve quadratic equations.

Example 1

Solve the following quadratic equations:

a. $4x(x-8)=0$

b. $(x+5)(x+8)=0$

c. $(6x-42)(7x-63)=0$

d. $(11-5a)(8a+56)=0$

Solution:

a. $4x(x-8)=0$ {Use the Null Factor Law}

$$\begin{aligned} \therefore 4x=0 &\quad\text{or}\quad x-8=0\\ x=0 &\quad\text{or}\quad x=8\end{aligned}$$

b. $(x+5)(x+8)=0$ {Use the Null Factor Law}

$\therefore x+5=0$ or $x+8=0$

$x=-5$ or $x=-8$

c. $(6x-42)(7x-63)=0$ {Use the Null Factor Law}

$\therefore 6x-42=0$ or $7x-63=0$

$6x=42$ or $7x=63$

$\frac{6x}{6}=\frac{42}{6}$ or $\frac{7x}{7}=\frac{63}{7}$

$x=7$ or $x=9$

d. $(11-5a)(8a+56)=0$ {Use the Null Factor Law}

$\therefore 11-5a=0$ or $8a+56=0$

$11=5a$ or $8a=-56$

$\frac{5a}{5}=\frac{11}{5}$ or $\frac{8a}{8}=\frac{-56}{8}$

$a=2\frac{1}{5}$ or $a=-7$

2. Equations Involving Factorization

To solve a quadratic equation, make the right-hand side of the equation zero by transposition. Then factorize the left-hand side, if possible; and then use the Null Factor Law.

Example 2

Solve the following quadratic equations:

a. $9x^2+27x=0$

b. $7x^2-56x=0$

Solution:

a. $9x^2+27x=0$ {Take out the common factor $9x$}

$9x(x+3)=0$ {Use the Null Factor Law}

$9x=0$ or $x+3=0$

$\frac{9x}{9}=\frac{0}{9}$ or $x=-3$

$x=0$ or $x=-3$

Note:

$\frac{0}{a}=0$, where a is any real number except 0.

b. $7x^2 - 56x = 0$ {Take out the common factor $7x$}

$7x(x-8) = 0$ {Use the Null Factor Law}

$7x = 0$ or $x - 8 = 0$

$\frac{7x}{7} = \frac{0}{7}$ or $x = 8$

$x = 0$ or $x = 8$

Example 3

Solve the following quadratic equations:

a. $x^2 - 3x + 2 = 0$

b. $x^2 + 7x + 12 = 0$

Solution:

a. $x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$ {Use the Null Factor Law}

$x - 1 = 0$ or $x - 2 = 0$

$x = 1$ or $x = 2$

b. $x^2 + 7x + 12 = 0$

$(x+3)(x+4) = 0$ {Use the Null Factor Law}

$x + 3 = 0$ or $x + 4 = 0$

$x = -3$ or $x = -4$

Example 4

Solve the following quadratic equations:

a. $a^2 = 6a$

b. $7x^2 = -84x$

c. $x^2 - x = 56$

d. $p(p-1) = 12$

Solution: In order to apply the Null Factor Law, one side must be zero.

a. $a^2 = 6a$ {Make the RHS zero by subtracting $6a$ from both sides}

$a^2 - 6a = 0$ {Take out the common factor a}

$a(a-6) = 0$ {Use the Null Factor Law}

$a = 0$ or $a - 6 = 0$

$a = 0$ or $a = 6$

b.
$$7x^2 = -84x \quad \{\text{Make the RHS zero by adding } 84x \text{ to both sides}\}$$
$$7x^2 + 84x = 0 \quad \{\text{Take out the common factor } 7x\}$$
$$7x(x+12) = 0 \quad \{\text{Use the Null Factor Law}\}$$
$$7x = 0 \quad \text{or} \quad x+12 = 0$$
$$x = 0 \quad \text{or} \quad x = -12$$

c.
$$x^2 - x = 56 \quad \{\text{Make the RHS zero by subtracting 56 from both sides}\}$$
$$x^2 - x - 56 = 0 \quad \{\text{Factorise the LHS}\}$$
$$(x-8)(x+7) = 0 \quad \{\text{Use the Null Factor Law}\}$$
$$x - 8 = 0 \quad \text{or} \quad x+7 = 0$$
$$x = 8 \quad \text{or} \quad x = -7$$

d.
$$p(p-1) = 12 \quad \{\text{Remove brackets}\}$$
$$p^2 - p = 12 \quad \{\text{Subtract 12 from both sides}\}$$
$$p^2 - p - 12 = 0 \quad \{\text{Factorise the LHS}\}$$
$$(p-4)(p+3) = 0 \quad \{\text{Use the Null Factor Law}\}$$
$$p - 4 = 0 \quad \text{or} \quad p+3 = 0$$
$$p = 4 \quad \text{or} \quad p = -3$$

3. Use of the Difference of Two Squares

Recall that:
$a^2 - b^2 = (a+b)(a-b)$
A solution of a quadratic equation may be obtained by using DOTS, as illustrated below.

Example: Solve $x^2 - 81 = 0$ for x.

Solution:

$$x^2 - 81 = 0$$
$$x^2 - 9^2 = 0 \qquad (\textit{Use } a^2 - b^2 = (a+b)(a-b)$$
$$(x+9)(x-9) = 0 \quad (\textit{Use the Null Factor Law})$$
$$\therefore\ x+9 = 0 \quad \textit{or} \quad x-9 = 0$$
$$x = -9 \quad \textit{or} \quad x = 9$$

4. Use of Substitution

Sometimes the solution of an equation can be simplified by the use of **substitution**, as illustrated in the following example.

Example 5

Solve $(x+3)^2+8(x+3)+15=0$ for x.

Solution:

$(x+3)^2+8(x+3)+15=0$

Replace $x+3$ by a

$\therefore a^2+8a+15=0$ {Factorise the LHS}

$(a+3)(a+5)=0$ {Use the Null Factor Law}

$a+3=0$ or $a+5=0$

$a=-3$ or $a=-5$

Now replace a by $x+3$

$x+3=-3$ or $x+3=-5$

$x=-6$ or $x=-8$

5. Quadratic Equations Containing Fractions

The solution of an equation containing fractions is obtained by multiplying all terms by the lowest common denominator, making the RHS equal to zero, factorizing the LHS and using the Null Factor Law.

Example 6

Solve $x+3=\dfrac{10}{x}$ for x.

Solution:

$x+3=\dfrac{10}{x}$ {LCD = x. So, multiply both sides by x}

$x(x+3)=x\times\dfrac{10}{x}$

$x^2+3x=10$

$x^2+3x-10=0$ {Factorise the LHS}

$(x+5)(x-2)=0$ {Use the Null Factor Law}

$x+5=0$ or $x-2=0$

$x=-5$ or $x=2$

6. Use of a Square Root

Square root $\left(\sqrt{\ }\right)$ is the inverse operation of square $(\ )^2$. So, to solve an equation we can take the square root of both sides of the equation, as illustrated below.

Clearly, $(+3)^2 = 9$

$(-3)^2 = 9$

$\therefore$ If $x^2 = 9$,

then $x^2 = (\pm 3)^2$

Taking the square root of both sides gives

$x = \pm 3$

$\therefore\ x = 3$ or $x = -3$

We write the solution as follows

$x^2 = 9$ {Take square root of both sides}

$\therefore\ x = \pm\sqrt{9}$

$\therefore\ x = \pm 3$

7. Solve by Completing the Squares

Example

Solve:

a. $x^2 = 100$ b. $6x^2 = 150$ c. $2x^2 + 1 = 73$

d. $\dfrac{2x^2}{3} = 24$ e. $\dfrac{x^2}{8} - 2 = 0$ f. $(x-4)^2 = 25$

Solution:

a. $x^2 = 100$ $\therefore\ x = \pm\sqrt{100}$ $\therefore\ x = \pm 10$

b. $6x^2 = 150$ $\dfrac{6x^2}{6} = \dfrac{150}{6}$ $x^2 = 25$

$\therefore\ x = \pm\sqrt{25}$ $\therefore\ x = \pm 5$

c. $2x^2 + 1 = 73$ $2x^2 + 1 - 1 = 73 - 1$ $2x^2 = 72$

$\dfrac{2x^2}{2} = \dfrac{72}{2}$ $x^2 = 36$ $\therefore\ x = \pm\sqrt{36}$ $\therefore\ x = \pm 6$

d. $\frac{2x^2}{3} = 24$ $\quad$ $3 \times \frac{2x^2}{3} = 3 \times 24$ $\quad$ $2x^2 = 72$

$\frac{2x^2}{2} = \frac{72}{2}$ $\quad$ $x^2 = 36$ $\quad$ $\therefore x = \pm\sqrt{36}$ $\quad$ $\therefore x = \pm 6$

e. $\frac{x^2}{8} - 2 = 0$ $\quad$ $\frac{x^2}{8} - 2 + 2 = 0 + 2$ $\quad$ $\frac{x^2}{8} = 2$

$8 \times \frac{x^2}{8} = 8 \times 2$ $\quad$ $x^2 = 16$ $\quad$ $\therefore x = \pm\sqrt{16}$ $\quad$ $\therefore x = \pm 4$

f. $(x-4)^2 = 25$ $\quad$ $\therefore x - 4 = \pm\sqrt{25}$ $\quad$ $\therefore x - 4 = \pm 5$

$x - 4 + 4 = \pm 5 + 4$ $\quad$ $\therefore x = 9$ or $x = -1$

Example: Solve (a) $x^2 + 6x - 2 = 0$ (b) $2x^2 - 4x + 3 = 0$

(a) $x^2 + 6x - 2 = 0$
$x^2 + 6x = 2$
$x^2 + 6x + 9 = 2 + 9$
$(x + 3)^2 = 11$
$x + 3 = \pm\sqrt{11}$
$x = -3 \pm\sqrt{11}$

The roots are $-3 + \sqrt{11}$ and $-3 - \sqrt{11}$.

(b) $2x^2 - 4x + 3 = 0$
$2(x^2 – 2x) = -3$
$2(x^2 – 2x + 1 – 1) = -3$
$x^2 – 2x + 1 – 1 = -3/2$
$(x – 1)^2 = -3/2 + 1$
$(x – 1)^2 = -1/2$
$x – 1 = \pm\sqrt{-\frac{1}{2}}$
$x = 1 \pm\sqrt{-\frac{1}{2}}$
There are no real solutions.

8. Problem Solving

Quadratic equations are often used to solve problems in science or in the design of objects to be constructed. Many word problems involve:

- distance, speed and time
- area of a paddock or path
- projectiles
- construction of bridges
- logos

The following approach may be useful to solve and model word problems.
Draw a diagram if necessary and let the unknown quantity be *x* or some other suitable pronumeral.

Using the given information, write an equation in terms of *x* and solve it.
Check the reasonableness of the solution and state the answer in words.

Example 1: A cyclist takes (x + 3) hours to travel a distance of 108 kilometres when cycling at x kilometres per hour. Find:
a. the value of x
b. The time take to complete the journey

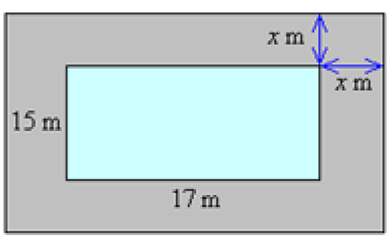
Solution:

a. $$\text{Distance} = \text{Speed} \times \text{Time}$$
$$108 = x(x+3)$$
$$108 = x^2+3x$$

$$\therefore\ x^2+3x-108=0$$
$$(x-9)(x+12)=0$$
$$x-9=0 \quad \text{or} \quad x+12=0$$
$$x=9 \quad \text{or} \quad x=-12$$
$$x=9 \quad \text{as} \quad x\neq -12$$

b. $$\text{So, the time taken} = (x+3) \text{ hours}$$
$$= (9+3) \text{ hours} \qquad (\because x=9)$$
$$= 12 \text{ hours}$$

Example 2: A swimming pool 17 m by 15 m is surrounded by a concrete path, as shown in the following diagram. If the area of the path is 68 m2, find the width of the path.



Solution:

$$\text{Area of the path} = \text{Area of the outer rectangle} - \text{Area of the inner rectangle}$$
$$68 = (17+2x)(15+2x) - 17\times 15$$
$$68 = 255+34x+30x+4x^2-255$$

$$68 = 4x^2 + 64x$$

$$4x^2 + 64x - 68 = 0 \qquad \{\text{Take out the common factor of } 4\}$$

$$4(x^2 + 16x - 17) = 0 \qquad \{\text{Divide both sides by } 4\}$$

$$x^2 + 16x - 17 = 0$$

$$(x-1)(x+17) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x + 17 = 0$$

$$x = 1 \quad \text{or} \quad x = -17$$

$$x = 1 \quad \text{as} \quad x \neq -17$$

Example 3: A right-angled triangular tray is to be constructed so that its length is 5 cm more than its base. If the area of the tray has to be 12 cm^2, find the width of the base.

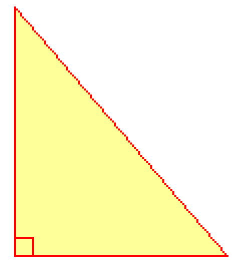
Solution:

Let the width of the base be x cm.

$\therefore$ Length of the tray $= (x+5)$ cm

Now, area $= \frac{1}{2}\text{base} \times \text{length}$

$$\therefore\ 12 = \frac{1}{2}x(x+5)$$



Multiply both sides by 2

$$2 \times 12 = 2 \times \frac{1}{2}x(x+5)$$

$$24 = x(x+5)$$

$$24 = x^2 + 5x$$

$$x^2 + 5x - 24 = 0$$

$$(x-3)(x+8) = 0$$

$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0$$

$$x = 3 \quad \text{or} \quad x = -8$$

$$x = 3, \text{ as } x = -8 \text{ is not physically feasible}$$

So, the width of the base is 3 cm.

Check:

$b = 3$ cm, $l = 8$ cm

$$A = \frac{1}{2}bl = \frac{1}{2} \times 3 \times 8 = 12 \text{ cm}^2$$

Example 4: The number of square metres in a square room's floor area exceeds the number of metres in the room's perimeter by 60. Find the side-length of the room.

Solution:

Let x m be the side-length of the room.

$\therefore$ Area $= x^2$

Perimeter $= 4x$

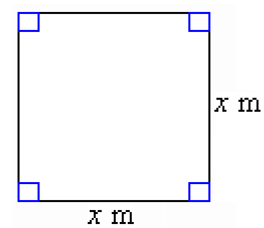
Now, $x^2 - 4x = 60$

$$x^2 - 4x - 60 = 0$$

$$(x-10)(x+6) = 0$$

$$x - 10 = 0 \quad \text{or} \quad x + 6 = 0$$

$$x = 10, \text{ as } x = -6 \text{ is not a reasonable answer.}$$



So, the side-length of the room is 10 m.

Check:

Area $-$ Perimeter $= 10^2 - 4 \times 10 = 100 - 40 = 60$