Linear Function

1. **Linear Equation**

A **linear equation** is an equation whose graph is a straight line.

A **linear equation in one variable** is an equation that simply involves x.

There are no terms involving x^2, x^3, $x^{1/2}$ etc. Each term has a degree of at most 1.

All operations, such as addition or multiplication, involve only x and numerical constants.

For examples:
$3x + 4 = 5$ is an example of a linear equation.
$2(x + 1) = 6(x - 4)$ is also a linear equation.

These equations can be solved very easily by performing algebraic operations to the equation to isolate x.

A **linear equation in two variables** is, as the name suggests, an equation that involves 2 variables.

The standard form of this type of equation is $Ax + By = C$, where, if at all possible, A, B, and C are integers, and A is non-negative, and, A, B, and C have no common factors other than 1.

For example, $3x + y = 7$ is a linear equation in two variables. $y = 2x + 1/3$ is also an example, since it can be rewritten as $2x - y = -1/3$ (or equivalently $6x - 3y = -1$).

Example: Change the linear relation $y = (5/6)x + 7/4$ to standard form.

Our first step is to multiply both sides by the least common multiple of 6 and 4, namely 12.

$$12\mathbf{y} = 12(\ (5/6)\mathbf{x} + 7/4\ )$$
$$12\mathbf{y} = (12)(5/6)\mathbf{x} + (12)(7/4)$$
$$12\mathbf{y} = 10\mathbf{x} + 21$$

Now we have removed all fractions, we bring the **x** term to the left side.

$$^{-}10\mathbf{x} + \mathbf{y} = 21$$

And, because the coefficient of **x** is negative, we multiply both sides by $^{-}1$.

$$(^{-}1)(^{-}10\mathbf{x} + 12\mathbf{y}) = (^{-}1)(21)$$

This produces the desired result, the standard form

$$10\mathbf{x} - 12\mathbf{y} = -21$$

2. Slope-Intercept Form

Linear equations in two variables can also be expressed in the slope-intercept form $y = mx + b$.

The **slope** of a line, represented by the variable m, is defined as the ratio of change in values of y to change in value of x.

Example: determine an equation of the line with m = 3 and y-intercept 2.

Solution: b = 2 and m = 3, the y = 3x + 2

3. Point-slope Form

The slope is also known as rise over run. For any two points (x_1, y_1), (x_2, y_2) on a line L, the formula for calculating the slope of L is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **y-intercept**, denoted b, is the point (0, b), where the line intercepts the y-axis.

There are two variations of the slope-intercept form of a line. The first is called the **point-slope form** of the equation of a line.

This formula is used when the slope and a single point on the line are given in the problem.

The formula for the point-slope form is:

$$(y - y_1) = m(x - x_1)$$

Example: determine an equation of a line through point (3, 2) with slope m = 2.

Solution: (x_1, y_1) = (3, 2) and m = 2, so y – 2 = 2 (x − 3), this equation can also be expressed in standard form: 2x − y − 4 = 0.

4. Two-point Form

Given two points (x_1, y_1) and (x_2, y_2), then the equation of the line can be expressed as

$$(y - y_1) = \frac{(y_1 - y_2)}{(x_1 - x_2)}(x - x_1) \quad \text{or } y - y_1 = m(x - x_1), \text{ here } m = \frac{(y_1 - y_2)}{(x_1 - x_2)}$$

Example: given two points P_1 (2, 3) and P_2 (−1, 2), determine the equation of the line.

Solution: $m = \frac{(y_1 - y_2)}{(x_1 - x_2)} = \frac{(3-2)}{(2-(-1))} = \frac{1}{3}$,

so $y - 3 = \frac{1}{3}(x - 2)$,

this equation can be expressed in standard form: x − 3y + 7 = 0.

The other variation is the **two-point form** of the equation of a line. This is simply the point-slope formula, with the slope formula substituted for m.

This formula should be used when two points on a line are given in the problem. The formula for the two-point form is:

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

The vertical line x = a represents a special case of the slope-intercept form of a line.

Since $(x_2 - x_1) = 0$, the slope of the line is $m = (y_2 - y_1)/0$.

Since division by 0 is undefined, the slope of the line is undefined. The line is also expressed in terms of x, rather than y.

Due to these properties, vertical lines cannot be expressed using the slope-intercept form.

Examples

I) Find the equation of the line that has a slope of -4 and passes through the point (0, 2). Write the equation in standard and slope-intercept form.

Given that m = -4 and $(x_1, y_1) = (0, 2)$, we substitute these values into the slope-intercept formula.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m(x_2 - x_1) &= (y_2 - y_1) \\ m(x - x_1) &= (y - y_1) \\ (-4)(x - 0) &= (y - 2) \\ -4x &= y - 2 \\ y &= -4x + 2 \end{aligned}$$

∴ In standard form, the equation of the line is $4x + y - 2 = 0$, while in slope-intercept form, the equation of the line is $y = -4x + 2$.

II) Find the equation of the line that passes through the points (1, 7) and (-2, 1). Write the equation in standard and slope-intercept form.

We are given $(x_1, y_1) = (1, 7)$, and $(x_2, y_2) = (-2, 1)$. We need to determine the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 7}{-2 - 1} = \frac{-6}{-3} = 2$$

Now we can substitute the values for slope and either of the points into the slope-intercept equation.

$$\begin{aligned} m(x - x_1) &= (y - y_1) \\ (2)(x - 1) &= (y - 7) \\ 2x - 2 &= y - 7 \\ y &= 2x + 5 \end{aligned}$$

∴ In standard form, the line is $2x - y + 5 = 0$, while in slope intercept form, the equation of the line is $y = 2x + 5$.

<u>Note:</u> For another example of finding the equation of a line, see question #1 in the Additional Examples section at the bottom of the page.

5. Line Theorems

Two lines are parallel if they have equal slopes. Parallel lines never cross each other. The distance between two parallel lines is always the same for every point along the lines.

Two lines are perpendicular, meaning their angle of intersection is 90°, if their slopes are negative reciprocals of each other. For lines L_1 and L_2 with slopes m_1 and m_2, respectively,

$$L_1 \perp L_2 \quad \longleftrightarrow \quad m_1 = -\frac{1}{m_2} \quad \text{or} \quad m_2 = -\frac{1}{m_1} \quad \text{or} \quad m_1 m_2 = -1$$

The midpoint of a line segment joining any two points (x_1, y_1) and (x_2, y_2) is the point given by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The distance, D, between any two points (x_1, y_1) and (x_2, y_2) is given by:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

6. Length of segment

The length of a line segment can be found by Pythagorean Theorem given two points P_1 (x_1, y_1) and $P_2(x_2, y_2)$, then the segment joining P_1 and P_2 may be expressed by following formula:

$$L = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example: find the length of the line segment joining points (3, 2) and (−1, 4)

Solution: $L = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(3 - (-1))^2 + (2 - 4)^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 4\sqrt{5}$

7. Midpoint of a line segment

We can calculate the coordinates of the midpoint of a line segment if the coordinates of the endpoints are given.

The coordinates of the midpoint M of the segment with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

8. Graphing Linear Equations

Linear equations in one variable have graphs of either horizontal or vertical lines, depending on which co-ordinate plane variable is used.

For equations of the form x = a, the graph is a vertical line, intersecting the x-axis at (a,0).
For equations of the form y = b, the graph is a horizontal line intersecting the y-axis at (0,b).

Linear equations are simply straight lines. If you can determine two points that lie on a line, then the graph is the line that connects those two points. This is sometimes referred to as the **two point method** of graphing linear equations.

The most common points to determine from the equation are the x and y intercepts, (x, 0) and (0, y), respectively. These points can be determined by substituting x = 0 and y = 0 into the equation and finding the corresponding y and x values.

The **slope-intercept method** can also be used to graph linear equations.

The equation should be written in slope-intercept form so the slope and y-intercept can be easily identified.

After plotting the y-intercept on the graph, a second point can be obtained using the ratio of the slope. The graph is simply the line that connects these points.

Examples

I) Graph the line $4x + y - 2 = 0$ by finding two points on the line.

Find the x- and y-intercepts of the line by substituting $(x,0)$ and $(0,y)$ into the equation.

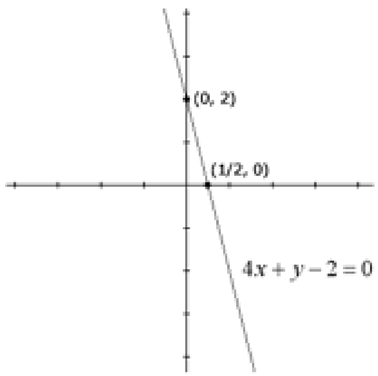
$$4(x) + (0) - 2 = 0 \qquad 4(0) + (y) - 2 = 0$$
$$4x = 2 \qquad y - 2 = 0$$
$$x = \frac{1}{2} \qquad y = 2$$

II) Graph the line $2x + 4y - 14 = 0$ using the slope-intercept method.

Write the equation in slope-intercept form.

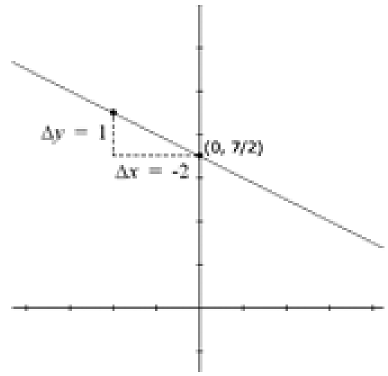
$$2x + 4y - 14 = 0 \longrightarrow y = -\frac{1}{2}x + \frac{7}{2}$$

The x-intercept is $(1/2, 0)$ and y-intercept is $(0, 2)$. The graph can be found by plotting these two points and connecting them with a line, as shown below.



$2x + 4y - 14 = 0 \longrightarrow y = -\frac{1}{2}x + \frac{7}{2}$

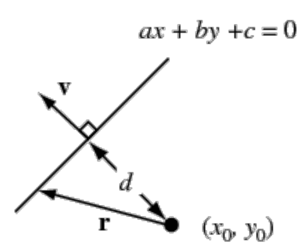
From the equation, we are given that the slope $m = -1/2$ and the y-intercept is $(0, 7/2)$. The slope tells us that another point can be found 1 unit upward and 2 units to the left of the y-intercept at $(-2, 9/2)$. The graph can be determined by plotting these two points and connecting them with a line, as shown below.



9. Distance From a Point to a Line

In the case of a line in the plane given by the equation ax + by + c =0, where a, b and c are real constants with a and b not both zero, the shortest distance from the line to a point (x_0, y_0) is:

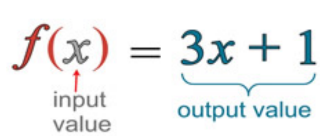
$$\operatorname{distance}(ax + by + c = 0, (x_0, y_0)) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$



10. Function Notation

Function notation is the way a function is written. It is meant to be a precise way of giving information about the function without a rather lengthy written explanation.

The most popular function notation is $f(x)$ which is read "f of x".
This is NOT the multiplication of f times x..



Traditionally, functions are referred to by single letter names, such as *f*, *g*, *h* and so on.

$f(x) = x^2 + 1 \qquad g(x) = x - 7 \qquad h(b) = 3b^2 - 2b + 1 \qquad S(t) = \frac{1}{2}t^2 - 3t + 1$

You used to say "$y = 2x + 3$; solve for y when $x = -1$". Now you say "$f(x) = 2x + 3$; find $f(-1)$" (pronounced as "*f*-of-*x* is 2*x*plus three; find *f*-of-negative-one"). You do exactly the same thing in either case: you plug in –1 for *x*, multiply by 2, and then add the 3, simplifying to get a final value of +1.

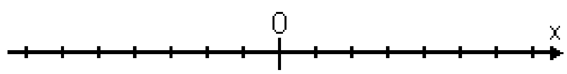
Example: Given $f(x) = x^2 + 2x - 1$, find $f(-3)$.

$$\begin{aligned} f(-3) &= (-3)^2 + 2(-3) - 1 \\ &= 9 - 6 - 1 \\ &= 2 \end{aligned}$$

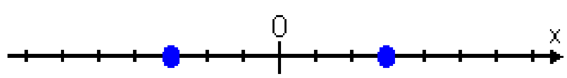
11. Absolute Value

The **absolute value** of a number is its distance from 0 on the number line.

Let's look at the number line:



The absolute value of *x*, denoted "$| x |$" (and which is read as "the absolute value of *x*"), is the distance of *x* from zero. This is why absolute value is never negative; absolute value only asks "how far?" not "in which direction?" This means not only that $| 3 | = 3$, because 3 is three units to the right of zero, but also that $| -3 | = 3$, because –3 is three units to the left of zero.



Warning: The absolute-value notation is *bars*, not parentheses or brackets. Use the proper notation; the other notations do *not* mean the same thing.

It is important to note that the absolute value bars do NOT work in the same way as do parentheses. Whereas $-(-3) = +3$, this is NOT how it works for absolute value:

Example: Simplify $-| -3 |$.

Given $-| -3 |$, I first handle the absolute value part, taking the positive and converting the absolute value bars to parentheses:

$$-| -3 | = -(+3)$$

Now I can take the negative through the parentheses:

$$-|-3| = -(3) = -3$$

As this illustrates, if you take the negative of an absolute value, you will get a negative number for your answer.

Example: Simplify | 2 + 3(–4) |.

$$|2 + 3(-4)| = |2 - 12| = |-10| = 10$$

An absolute value function is a function that contains an algebraic expression within absolute value symbols.

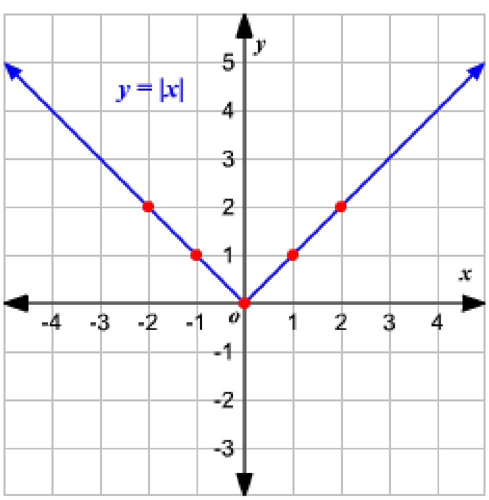
The absolute value parent function, written as $f(x) = |x|$, is defined as

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

To graph an absolute value function, choose several values of *x* and find some ordered pairs.

x	$y = \lvert x \rvert$
-2	2
-1	1
0	0
1	1
2	2

Plot the points on a coordinate plane and connect them. Observe that the graph is V-shaped.



Questions in class

1. Find the equation of the perpendicular bisector of the line segment joining the points (1, 2) and (-2, 1).

2. Find the real number *a* for which the lines $-4x+3y=7$ and $ax-2y=4$ will be perpendicular.

3. Find a point on the y-axis that is equidistant from (1, 1) and (5, 5).

4. Let f(x) be a linear function with f(1) = 2 and f(3) = 8. Find f(5).

5. Find the area of the region bounded by the graphs of $y = |x| + 1$ and $y = 5$.

6. Find an equation for the perpendicular bisector of the line segment connecting the points (-2, 5) and (3, -7).

7. The *x*-axis, the *y*-axis, and the line through the point (1, 3) having slope -3 form a triangle, find the area.

8. Find the distance from the point (5, 6) to the line $x + y = 3$.

9. Suppose that for any integer *n*, $f(n) = \begin{cases} n+3 & if.n.is.odd \\ \frac{n}{2} & if.n.is.even \end{cases}$

If *k* is odd and $f(f(f(k))) = 27$, find the sum of the digits in *k*.

10. Given points *A* = (-2, 5) and *B* = (6, -1), find the point *C* from the list below so that *A*, *B*, and *C* are collinear.

(A) *C* = (10, -4) (B) *C* = (2, -2) (C) *C* = (4, 3) (D) *C* = (-6, 9) (E) *C* = (0, 0)