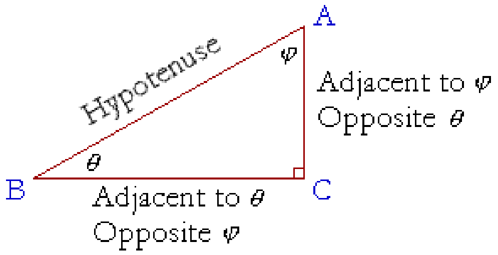
Trigonometry (1)

1. The Trigonometry of an Acute Angle

A right triangle is composed of a right angle, the angle at C, and two acute angles, which are angles less than a right angle.

It is conventional to label the acute angles with Greek letters.



Primary Trig Ratios (sin, cos, tan)

$$\text{sine of } \theta = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\text{cosine of } \theta = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{tangent of } \theta = \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

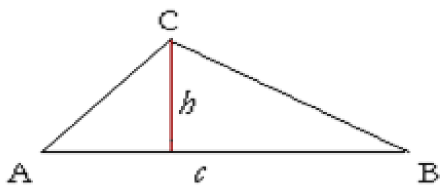
Secondary/Reciprocal Trig Ratios (csc, sec, cot)

Secondary trig ratios are reciprocals of the primary trig ratios.

$$\text{cosecant}\,\theta = \frac{1}{\sin\theta} \qquad \text{secant}\,\theta = \frac{1}{\cos\theta} \qquad \text{cotangent}\,\theta = \frac{1}{\tan\theta}$$

$$\csc\theta = \frac{hypotenuse}{opposite} \qquad \sec\theta = \frac{hypotenuse}{adjacent} \qquad \cot\theta = \frac{adjacent}{opposite}$$

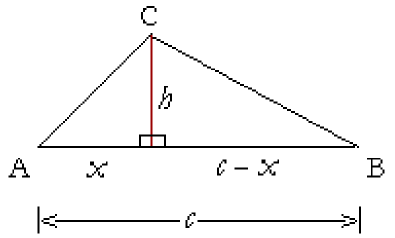
Example 1. The height of a triangle. Every triangle, right-angled or not, will have at least two acute angles.



Let them be the base angles at A and B, so that the base will be the side c. Show that the height h drawn to that base is

$$h = \frac{c}{\cot A + \cot B}$$

Hint: The height h will cut the entire triangle into two right triangles.



Let x be the segment of the base containing the angle A. Then the remaining segment is the difference between the whole c and that segment: $c - x$.

In the right triangle containing the acute angle A,

$$\frac{x}{h} = \cot \text{A}, \quad \text{or} \quad x = h \cot \text{A} \;.\quad .\quad .\quad (1)$$

In the right triangle containing the acute angle B,

$$\frac{c - x}{h} = \cot \text{B}, \quad \text{or} \quad c - x = h \cot \text{B}.$$

On substituting the expression for x from line (1),

$$c - h \cot \text{A} = h \cot \text{B},$$

which implies

$$c = h \cot \text{A} + h \cot \text{B} = h(\cot \text{A} + \cot \text{B}).$$

Therefore, on solving for h,

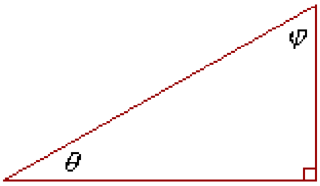
$$h \;=\; \frac{c}{\cot \text{A} + \cot \text{B}}\,.$$

1) Complements

Two angles are called complements of one another if together they equal a right angle. Thus the complement of 60° is 30°. This is the degree system of measurement in which a full circle, made up of four right angles at the center, is called 360°.

Example 2 Name the complement of each angle.

a) 70° **20°** b) 20° **70°** c) 45° **45°** d) θ **90° −** θ



The point about complements is that, in a right triangle, the two acute angles are complementary. For, the three angles of the right triangle are together equal to *two* right angles; therefore, the two acute angles together will equal *one* right angle.

2) Cofunctions

There are three pairs of cofunctions:

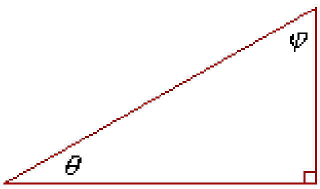
The sine and the cosine
The secant and the cosecant
The tangent and the cotangent

And here is the significance of a cofunction:

***A function of any angle is equal to the cofunction of its complement**

This means that sin 80°, for example, will equal cos 10°.
The cosine is the cofunction of the sine. And 10° is the complement of 80°.

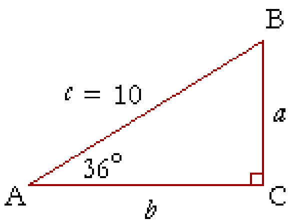
Example 3. Answer in terms of cofunctions.

	a) cos 5° = **<u>sin 85°</u>** b) tan 60° = **<u>cot 30°</u>** c) csc 12° = **<u>sec 78°</u>** d) sin (90° − θ) = **<u>cos θ</u>** e) cot θ = **<u>tan (90° − θ)</u>**

2. Trigonometry of right triangles

To SOLVE A TRIANGLE means to find all three sides and all three angles.

Example 1. Given an acute angle and one side.

Solve the right triangle ABC if angle A is 36°, and side c is 10 cm.	

Solution. Since angle A is 36°, then angle B is 90° − 36° = 54°. To find an unknown side, say a, proceed as follows:

$$\frac{\text{Unknown}}{\text{Known}} = \frac{a}{10} = \sin 36° = .588$$

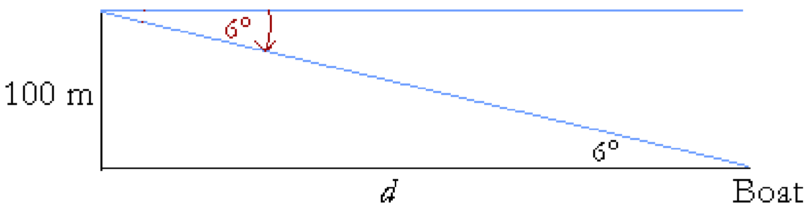
$a = 10 \times .588$ cm = 5.88 cm

Then we need to find b. The easiest way to do that is to use Pythagorean Theorem, but let us try to use the reciprocal trigonometric ratios.

sec 36° = 10 / b

b = 8.09 cm

Example 2. Find the distance of a boat from a lighthouse if the lighthouse is 100 meters tall, and the angle of depression is 6°.



Solution. The angle of depression is the angle below straight ahead -- horizontal -- that an observer must look in order to see something below the observer. Thus in order to see the boat, the lighthouse keeper must look down 6°.

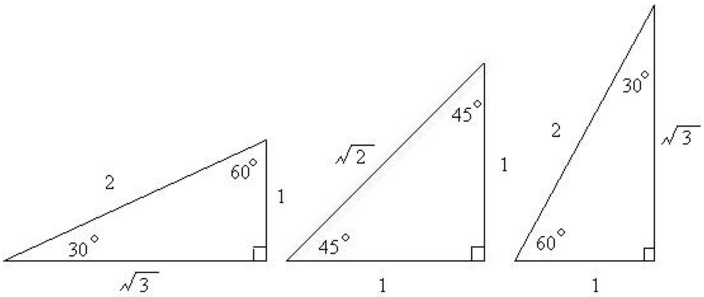
Now, the triangle formed by the lighthouse and the distance d of the boat from the lighthouse, is right-angled. And since the angle of depression is 6°, then the alternate angle is also 6°.

If d is the distance of a boat from the lighthouse, then

$$\frac{d}{100} = \cot 6° = 9.514,$$

Therefore, d = 951.4 meters.

Special Angles



	0°	30°	45°	60°	90°
sin	$\sin 0° = 0$	$\sin 30° = \frac{1}{2}$	$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\sin 60° = \frac{\sqrt{3}}{2}$	$\sin 90° = 1$
cos	$\cos 0° = 1$	$\cos 30° = \frac{\sqrt{3}}{2}$	$\cos 45° = \frac{\sqrt{2}}{2}$	$\cos 60° = \frac{1}{2}$	$\cos 90° = 0$
tan	$\tan 0° = 0$	$\tan 30° = \frac{\sqrt{3}}{3}$	$\tan 45° = 1$	$\tan 60° = \sqrt{3}$	$\tan 90° = \mathit{undefined}$

Example: Determine the exact value of (sin 45°) (cos 45°) + (sin 30°) (sin 60°).

$$(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

I substituted the exact values of each trigonometric ratio.

$$= \frac{2}{4} + \frac{\sqrt{3}}{4}$$

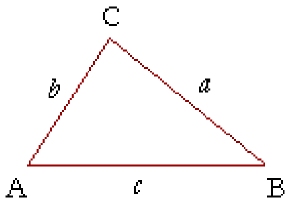
I evaluated the expression by multiplying, then adding the numerators.

$$= \frac{2 + \sqrt{3}}{4}$$

The exact value is $\frac{2 + \sqrt{3}}{4}$.

3. The Sine Law

The Sine Law states the following:
The sides of a triangle are to one another in the same ratio as the <u>sines</u> of their opposite angles.



This means that in the oblique triangle ABC, side a, for example, is to side b as the sine of angle A is to the sine of angle B.

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Similarly,

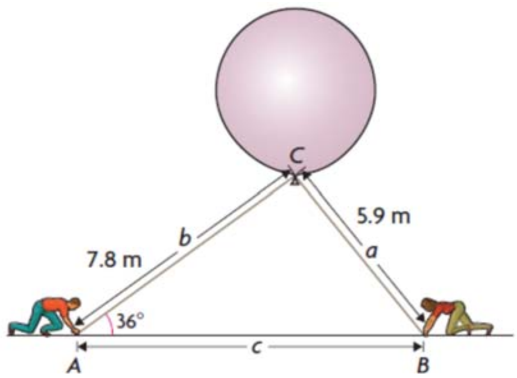
$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

And so on, for any pair of sides and their opposite angles.

Strictly, A, B, C are points. They are the vertices of the triangle. "Angle A" is a brief way of saying "The angle at the point A." By "sin A," then, we mean "The sine of the angle at the point A."

Example: Albert and Belle are part of a scientific team studying thunderclouds. The team is about to launch a weather balloon into an active part of a cloud. Albert's rope is 7.8 m long and makes an angle of 36° with the ground. Belle's rope is 5.9 m long. Determine the distance between Albert and Belle.

Solution: Assuming that Albert and Belle are on Opposite Sides of the Balloon, then we have the following diagram:



a / sin A = b / sin B

5.9 / sin 36 = 7.8 / sin B

5.9 sin B = 7.8 sin 36

$\angle$B = 51 degrees

$\angle$C = 180 – 36 – 51 = 93 degrees

a / sin A = c / sin C

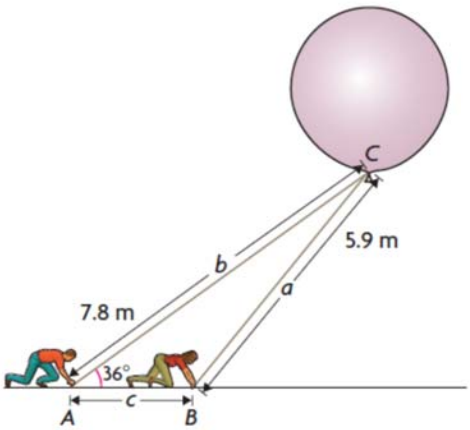
5.9 / sin 36 = c / sin 93

c sin 36 = 5.9 sin 93

c = 10 m

Therefore, if Albert and Belle are on opposite sides of the balloon, they are about 10 m apart.

What if they are on the same side of the balloon?



a / sin A = b / sin B

5.9 / sin 36 = 7.8 / sin B

∠B = 51 degrees

However, ∠B in this diagram is obtuse.

∠B = 180 – 51 = 129 degrees

∠C = 180 – 36 – 129 = 15 degrees

a / sin A = c / Sin C

5.9 / sin 36 = c / sin 15

c sin 36 = 5.9 sin 15

c = 2.6 m

If Albert and Belle are on the same side of the balloon, they are about 2.6 m apart.

This is call the **ambiguous case.**

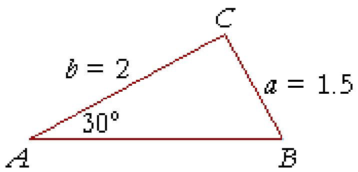
The Ambiguous Case

The so-called ambiguous case arises from the fact that an acute angle and an obtuse angle have the same sine. If we had to solve

$$\sin x = \frac{1}{2}\sqrt{2},$$

for example, both $x = 45°$ or $x = 135°$ would satisfy the equation.

In the following example, we will see how this ambiguity could arise.



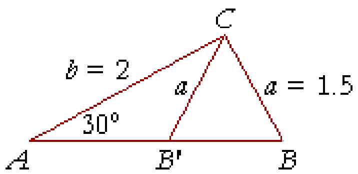
In triangle *ABC*, angle $A = 30°$, side $a = 1.5$ cm, and side $b = 2$ cm. Let us use the law of sines to find angle *B*.

$$\frac{\sin B}{\sin 30^\circ} = \frac{2}{1.5}$$

Since: $\sin 30° = 1/2, \sin B = \frac{1}{2} \times \frac{20}{15} \approx 0.666$, $B \approx 42°$.

But the sine of an angle is equal to the sine of its supplement. That is, 0.666 is also the sine of $180° - 42° = 138°$.

This problem has *two* solutions. Not only is angle *CBA* a solution,



but so is angle *CB'A*, which is the supplement of angle *CBA*. (We can see that it is the supplement in the isosceles triangle *CBB'*; angle *CB'A* is the supplement of angle *CB'B*, which is equal to angle *CBA*.)

Given two sides of a triangle *a, b*, then, and the acute angle opposite one of them, say angle *A*, under what conditions will the triangle have two solutions, only one solution, or no solution?

Let us first consider the case $a < b$. Upon applying the law of sines, we arrive at this equation:

$$\sin B = \sin A \cdot \frac{b}{a}$$

Now, since $\frac{h}{b} = \sin A$, where *h* is the height of the triangle (Fig. 1),

Then, $b \sin A = h$.

On replacing this in the right-hand side of equation 1), it becomes

$$\sin B = \frac{h}{a}$$

There are now three possibilities:

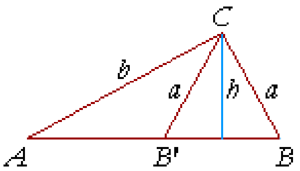


Fig. 1

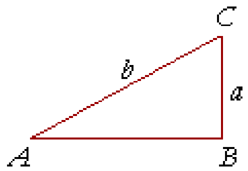


Fig. 2

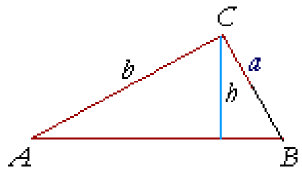


Fig. 3

$\frac{h}{a} < 1$, which implies h < *a* (Fig. 1),

$\frac{h}{a} = 1$, which implies h = *a* (Fig. 2),

$\frac{h}{a} > 1$, which implies h > *a* (Fig. 3).

In the first of these -- ***h*** or ***b* sin *A*** < *a* -- there will be two triangles.
In the second -- ***h*** or ***b* sin *A* = *a*** -- there will be one right-angled triangle.
And in the third -- ***h*** or ***b* sin *A* > *a*** -- there will be no solution.

Example: Let *a* = 2 cm, *b* = 6 cm, and angle *A* = 60°. How many solutions are there for angle *B*?

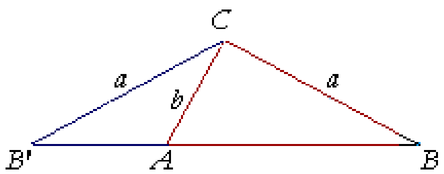
Answer: We must calculate the height, *b* sin *A*. If it is less than *a*, there will be two solutions. If it is equal to *a*, there will be one solution. And if it is greater than *a*, there will be no solution.

Now, $\sin 60^0 = \frac{\sqrt{3}}{2}$, therefore: $b \sin A = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

Since *a* = 2, then b sin *A* > *a*. ($\sqrt{3} = 1.732$).

There is no solution.

Finally, we will consider the case in which angle *A* is acute, and *a* > *b*.

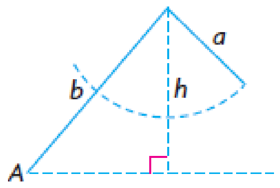


In this case, there is only one solution, namely, the angle *B* in triangle *CBA*. For, in triangle *CAB'*, the angle *CAB'* is obtuse.

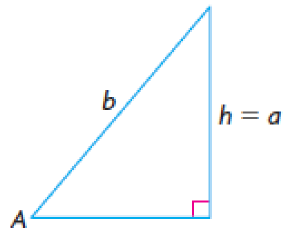
Summary

In the ambiguous case, if $\angle A$, a, and b are given and $\angle A$ is acute, there are four cases to consider. In each case, the height of the triangle is $h = b \sin A$.

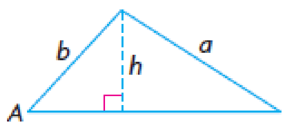
If $\angle A$ is acute and $a < h$, no triangle exists.



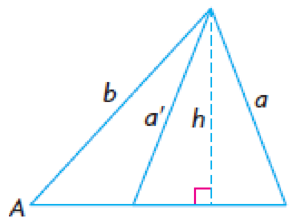
If $\angle A$ is acute and $a = h$, one right triangle exists.



If $\angle A$ is acute and $a > b$, one triangle exists.

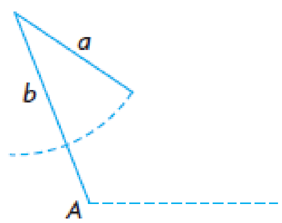


If $\angle A$ is acute and $h < a < b$, two triangles exist.

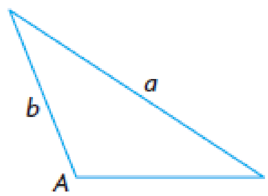


If $\angle A$, a, and b are given and $\angle A$ is obtuse, there are two cases to consider.

If $\angle A$ is obtuse and $a < b$ or $a = b$, no triangle exists.



If $\angle A$ is obtuse and $a > b$, one triangle exists.



4. The Law of Cosines

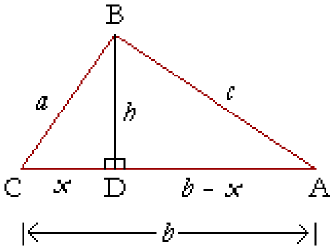
In particular, when we know two sides of a triangle and their included angle, then the Law of Cosines enables us to find the third side.

Thus if we know sides a and b and their included angle θ, then the Law of Cosines states:

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

(The Law of Cosines is a extension of the Pythagorean theorem; because if θ were a right angle, we would have $c^2 = a^2 + b^2$.)

Proof of the Law of Cosines



Let ABC be a triangle with sides *a, b, c*. We will show

$$\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C.}$$

Draw BD perpendicular to CA, separating riangle ABC into the two right triangles BDC, BDA. BD is the height h of triangle ABC.

Call CD x. Then DA is the whole b minus the segment x: $b - x$. Also, since

$$\frac{x}{a} = \cos C \quad \text{then} \quad \boldsymbol{x = a \cos C} \; . \; . \; . \; . \; . \; . \; . \; . \; \textbf{(1)}$$

Now, in the right triangle BDC, according to the Pythagorean Theorem,

$$h^2 + x^2 = a^2, \quad \text{so that} \quad \boldsymbol{h^2 = a^2 - x^2.} \; . \; . \; . \; . \; . \; . \; \textbf{(2)}$$

In the right triangle BDA,

$$c^2 = h^2 + (b - x)^2 = h^2 + b^2 - 2bx + x^2.$$

For h^2, let us substitute line (2):

$$c^2 = a^2 - x^2 + b^2 - 2bx + x^2 = a^2 + b^2 - 2bx.$$

Finally, for x, let us substitute line (1):

$$\boldsymbol{c^2 = a^2 + b^2 - 2b \cdot a \cos C.}$$

In the same way, we could prove that

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A} \quad \text{and}$$

$$\boldsymbol{b^2 = a^2 + c^2 - 2ac \cos B.}$$