Lesson 13: Unit 7 – Identities and Equations (2)

In this lesson we will extend our knowledge of compound angle formulas to include the double angle formulas.
These formulas are special cases of the angle sum formulas studied in the previous module.

Double Angle Formula for Sine

Consider that sin θ) can be expressed as sin(θ+θ).
Using the angle sum formula for sine, sin(A+B)=sin(A)cos(B)+cos(A)sin(B), we have

$$\sin(2\theta) = \sin(\theta + \theta)$$
$$= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)$$
$$= 2\sin(\theta)\cos(\theta)$$

Thus, the double angle formula for sine is **sin(2θ) = 2sin(θ)cos(θ)**.

Double Angle Formula for Cosine

Similarly, we can develop a formula for cos(2ϑ) using the angle sum formula for cosine, cos(A+B)=cos(A)cos(B)−sin(A)sin(B).
We have,

$$\cos(2\theta) = \cos(\theta + \theta)$$
$$= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)$$
$$= \cos^2(\theta) - \sin^2(\theta)$$

Making appropriate substitutions using the Pythagorean identity, $\sin^2(\theta) + \cos^2(\theta) = 1$, the double angle formula for cosine can be expressed in two other ways.
Using $\cos^2(\theta) = 1 - \sin^2(\theta)$,

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
$$= 1 - \sin^2(\theta) - \sin^2(\theta)$$
$$= 1 - 2\sin^2(\theta)$$

Using sin2(θ) = 1− $\cos^2(\theta)$,

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
$$= \cos^2(\theta) - (1 - \cos^2(\theta))$$
$$= 2\cos^2(\theta) - 1$$

Thus, the double angle formula for cosine is given by

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

or $$\boldsymbol{\cos(2\theta) = 1 - 2\sin^2(\theta)}$$

or $$\boldsymbol{\cos(2\theta) = 2\cos^2(\theta) - 1}$$

Double Angle Formula for Tangent

Using the angle sum formula for tangent,

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

express $\tan(2\theta)$ in terms of $\tan(\theta)$.

Using the angle sum formula for tangent, we have

$$\tan(2\theta) = \tan(\theta+\theta) = \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)}$$

$$\boldsymbol{\tan(2\theta)= \frac{2\tan(\theta)}{1-\tan^2(\theta)}}$$

Now, let's see how we might use these formulas.

Example

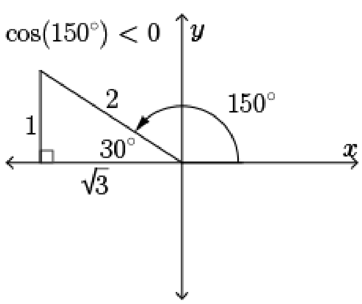
Determine the exact value of each.

a) $\sin^2(75^0) - \cos^2(75^0)$ **b)** $\sin(\pi/8)\cos(\pi/8)$

Solution

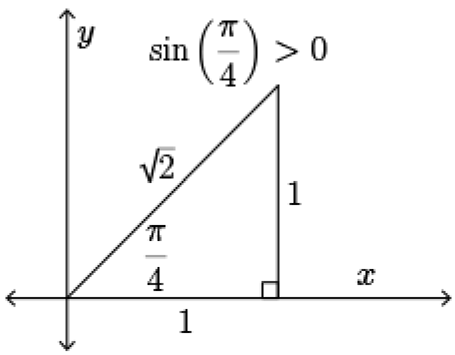
a) $\sin^2(75^0)-\cos^2(75^0)$. The first expression can be simplified using $\cos(2A)=\cos^2(A)-\sin^2(A)$.

$$\begin{aligned}\sin^2(75^0)-\cos^2(75^0) &= -(\cos^2(75^0)-\sin^2(75^0))\\ &= -\cos({}^2(75^0))\\ &= -\cos(150^0)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$



b) $\sin(\pi/8)\cos(\pi/8)$.

Rearranging the double angle formula for sine, we have $\sin(A)\cos(A) = \frac{1}{2}\sin(2A)$. Thus,



$\sin(\pi/8)\cos(\pi/8) = \frac{1}{2}\sin(2(\frac{\pi}{8}))$

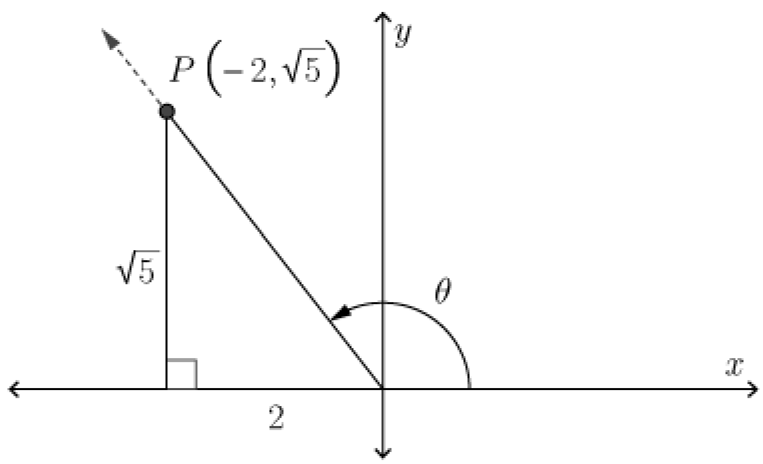
$= \frac{1}{2}\sin(\frac{\pi}{4})$

$= \frac{\sqrt{2}}{4}$

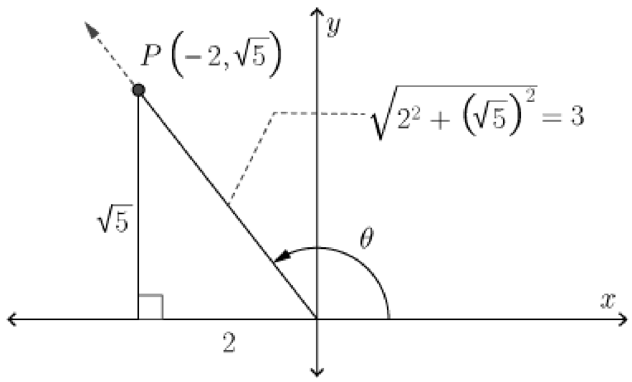
Example

If $\tan(\theta) = -\frac{\sqrt{5}}{2}$ for $\frac{\pi}{2} \le \theta \le \pi$, determine the exact value of $\sin(2\theta)$.

Solution



Since $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, to determine the exact value of $\sin(2\theta)$, we must determine the exact values of $\sin(\theta)$ and $\cos(\theta)$.
Now, $\frac{\pi}{2} \le \theta \le \pi$, so θ is an obtuse angle with a terminal arm in the second quadrant.
Using this information and the fact that $\tan(\theta) = -\frac{\sqrt{5}}{2}$, we know that the terminal arm passes through the point $P(-2, \sqrt{5})$.
The hypotenuse of the right triangle shown has length 3.
Then, $\sin(\theta)=\sqrt{5}/3$ and $\cos(\theta)=-2/3$.



Finally,

$\sin(2\theta)$ = $2\sin(\theta)\cos(\theta)$

= $2(\sqrt{5}/3)(-2/3)$

= $-\dfrac{4\sqrt{5}}{9}$

Therefore, the exact value of $\sin(2\theta)$ is $-4\sqrt{5}/9$.

Example

Determine the exact value of $\cos(\frac{5\pi}{8})$.

Solution

If $\theta = 5\pi/8$, then $2\theta = 2(5\pi/8) = 5\pi/4$. Now, using $\cos(2\theta) = 2\cos^2(\theta)-1$, we have

$\cos(5\pi/4) = 2\cos^2(5\pi/8)-1$.

Rearranging this equation to isolate $\cos(5\pi/8)$, we have

$$\cos(\tfrac{5\pi}{8}) = \pm\sqrt{\frac{\cos(\frac{5\pi}{4})+1}{2}}$$

However, $\cos(\frac{5\pi}{8}) < 0$, so $\cos(\frac{5\pi}{8}) = -\sqrt{\dfrac{\cos(\frac{5\pi}{4})+1}{2}}$

Since $\cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$ we have $\cos(\frac{5\pi}{8}) = -\dfrac{\sqrt{2-\sqrt{2}}}{2}$

Summary

The double angle identities or formulas are special cases of the angle sum formulas, where the two angles are equal.

Double Angle Formula for Sine
$\sin(2A) = 2\sin(A)\cos(A)$

Double Angle Formula for Tangent
$\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

Double Angle Formulas for Cosine
$\cos(2A) = \cos^2(A) - \sin^2(A)$
$\cos(2A) = 1 - 2\sin^2(A)$
$\cos(2A) = 2\cos^2(A) - 1$
They can be used to simplify trigonometric expressions, prove identities, and determine exact values of trigonometric ratios for angles corresponding to the reference angles $\pi/12$, $\pi/8$, $3\pi/8$, or $5\pi/12$ (15^0, 22.5^0, 67.5^0, or 75^0).

The double angle formulas can also be used to derive other identities or formulas, such as the half-angle formulas.

In the next part of the lesson, "Solving Trigonometric Equations," we will discuss how the double angle formulas and other identities can be used to solve certain trigonometric equations.

- *We will discuss a variety of strategies to solve both linear and nonlinear trigonometric equations algebraically.*
- *We will use fundamental identities and formulas studied in this unit to assist us in solving equations involving more than one trigonometric function.*

Example

Solve $\csc(x)+6=1-3\csc(x)$, where $0 \le x \le 2\pi$, correct to two decimal places.

Solution

First, we identify any non-permissible values of x.
Cosecant of x is undefined when $x=n\pi$, $n \in \mathbb{Z}$.
Thus, $x \ne n\pi$, $n \in \mathbb{Z}$.
To solve this equation, we can use the same techniques used in solving linear equations.
By rearranging the terms and simplifying, we isolate the trigonometric ratio, $\csc(x)$.

$$\csc(x)+6 = 1-3\csc(x)$$

$$4\csc(x) = -5$$

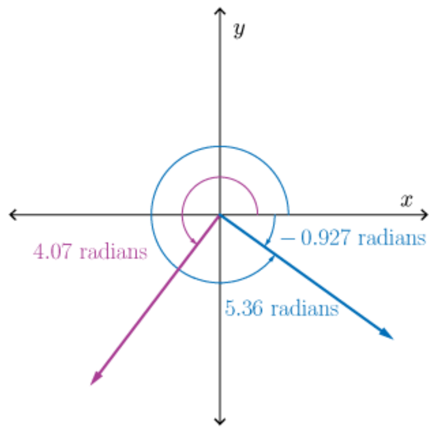
$\csc(x) = -5/4$

$\sin(x) = -4/5$

$x = \sin^{-1}(-4/5)$

$x \approx -0.9273$

This answer provides us with the related acute angle, or reference angle, which is approximately 0.9273 radians.



The sine function is negative for any angle with a terminal arm in quadrant 3 or quadrant 4.

$x \approx \pi + 0.9273$ or $x \approx 2\pi - 0.9273$

To two decimal places of accuracy,

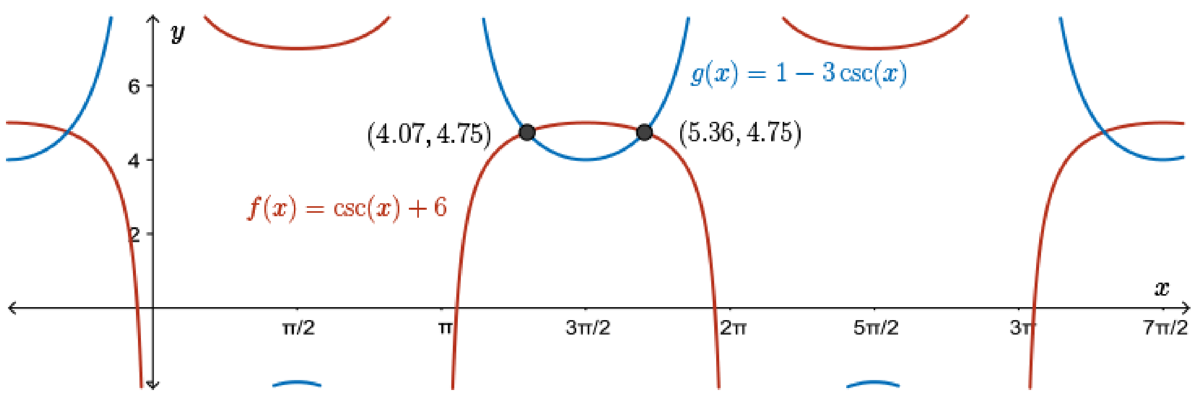
$x \approx 4.07$ radians or $x \approx 5.36$ radians

All possible solutions are defined by

$x \approx 4.0689 + 2\pi n$, $n \in \mathbb{Z}$ or $x \approx 5.3559 + 2\pi n$, $n \in \mathbb{Z}$

However, 4.07 and 5.36 are the only two solutions within the specified domain from $0 \le x \le 2\pi$.

Note that these values are permissible values of the cosecant function.
Therefore, the roots in the domain $0 \le x \le 2\pi$ are approximately 4.07 and 5.36.
We will now verify the solutions to this equation using graphing technology.
One way to do this is to determine the point of intersection of the two functions, $f(x) = \csc(x) + 6$ and $g(x) = 1 - 3\csc(x)$.



Example

Determine the exact roots of $\sqrt{2}\sin(x)\cos(x) = \cos(x)$, where $-\pi \le x \le 2\pi$.

Solution

Collect the terms to one side of the equation and factor,

$$\sqrt{2}\sin(x)\cos(x) = \cos(x)$$

$$\sqrt{2}\sin(x)\cos(x) - \cos(x) = 0$$

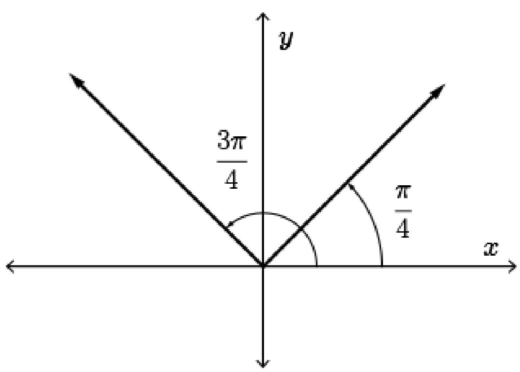
$$\cos(x)(\sqrt{2}\sin(x) - 1) = 0$$

$\cos(x)=0$ or $\sqrt{2}\sin(x)-1=0$

We know $\cos(x)=0$ when $x=\frac{\pi}{2}+n\pi$, $n \in Z$.

Thus, for $-\pi \le x \le 2\pi$, $x= -\pi/2$, $\pi/2$, and $3\pi/2$.

We know $\sqrt{2}\sin(x)-1= 0$ when $\sin(x)=\frac{\sqrt{2}}{2}$.



The reference angle is $\pi/4$ and sine is positive for any angle with a terminal arm in quadrant 1 or quadrant 2.

For $0 \le x \le 2\pi$, $x = \pi/4$ and $x = \pi-\pi/4 = 3\pi/4$.

For $-\pi \le x \le 0$, $\sin(x) \le 0$ so no additional solutions lie in this part of the required domain.

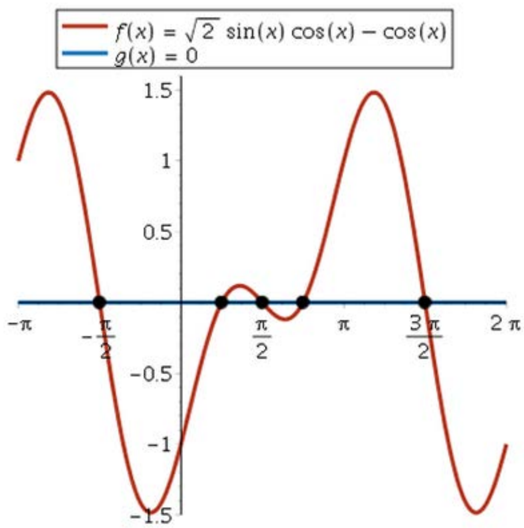
Thus, $x = \pi/4$ and $x = 3\pi/4$.

Therefore, the exact roots of $\sqrt{2}\sin(x)\cos(x) = \cos(x)$, where $-\pi \le x \le 2\pi$ are $-\pi/2$, $\pi/4$, $\pi/2$, $3\pi/4$, and $3\pi/2$.

One way to verify this solution graphically using technology, is to graph

$$f(x) = \sqrt{2}\sin(x)\cos(x) - \cos(x)$$

and determine the zeros or x-intercepts of the function.



The graph presented here allows for a quick visual check of the solution.
The zeros occur occur when $f(x) = 0$; that is, when $\sqrt{2}\sin(x)\cos(x)-\cos(x) = 0$.
Zeros shown in the domain $-\pi \le x \le 2\pi$ are $-\pi/2$, $\pi/4$, $\pi/2$, $3\pi/4$, and $3\pi/2$.

Example

Determine the roots of $\cos(x)-\sin(x) = \cos(2x)$ where $-\pi \le x \le \pi$.

Solution

Replacing $\cos(2x)$ with $\cos^2(x)-\sin^2(x)$, we have

$$\begin{aligned}
\cos(x)-\sin(x) &= \cos(2x) \\
\cos(x)-\sin(x) &= \cos^2(x)-\sin^2(x) \\
\cos(x)-\sin(x) &= (\cos(x)+\sin(x))(\cos(x)-\sin(x)) \\
0 &= (\cos(x)+\sin(x))(\cos(x)-\sin(x))-(\cos(x)-\sin(x)) \\
0 &= (\cos(x)-\sin(x))(\cos(x)+\sin(x)-1)
\end{aligned}$$

So, $\cos(x)-\sin(x)=0$ or $\cos(x)+\sin(x)-1=0$.

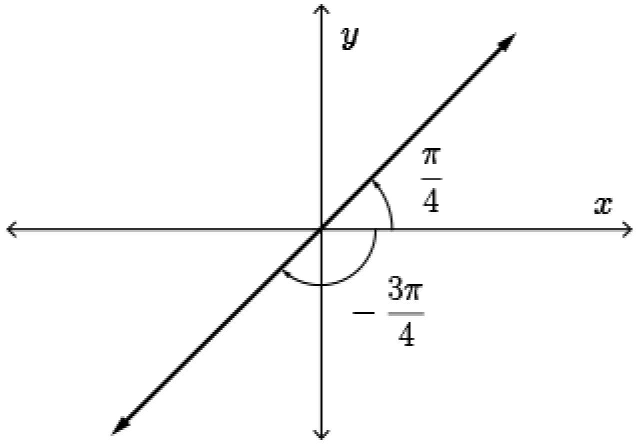
Solving the first equation:

$$\cos(x)-\sin(x) = 0$$
$$\sin(x) = \cos(x)$$

Dividing each side by $\cos(x)$, we obtain

$$\tan(x) = 1$$

The reference angle of the solution is $\pi/4$.



Tangent is positive for any angle with a terminal arm in quadrant 1 or quadrant 3.
For $-\pi \le x \le \pi$, $\tan(x)=1$ when $x=\pi/4$ or $x=-3\pi/4$.

Solving the second equation, $\cos(x) + \sin(x) - 1 = 0$:

$$\cos(x) + \sin(x) = 1$$

$$(\cos(x)+\sin(x))^2 = 1^2$$

$$\mathrm{Cos}^2(x) + 2\sin(x)\cos(x) + \sin^2(x) = 1$$

$$2\sin(x)\cos(x)+1 = 1$$

$$2\sin(x)\cos(x) = 0$$

Replacing $2\sin(x)\cos(x)$ with $\sin(2x)$, we have

$$\sin(2x)=0$$

To determine the values of x in the domain, $-\pi \le x \le \pi$, find values for $2x$ where $-2\pi \le 2x \le 2\pi$.
We know $\sin(\theta) = 0$ when $\theta = n\pi$, $n \in \mathrm{Z}$ so

$$2x= -2\pi, -\pi, 0, \pi, 2\pi$$

$$x = -\pi, -\pi/2, 0, \pi/2, \pi$$

Since squaring was used to solve the equation $\cos(x) + \sin(x) - 1 = 0$, we must check for extraneous roots (roots that are not solutions to the original equation).
Which of these values for x, $-\pi$, $-\pi/2$, 0, $\pi/2$, and π, are actual solutions to $\cos(x)+\sin(x)-1=0$?

We can show that

$$\cos(-\pi)+\sin(-\pi)-1 \ne 0,\ \cos(-\pi/2)+\sin(-\pi/2)-1 \ne 0, \text{ and } \cos(\pi)+\sin(\pi)-1 \ne 0.$$

However, $x=0$ and $x=\pi/2$ satisfy $\cos(x)+\sin(x)-1=0$, so 0 and $\pi/2$ are the roots of this equation in the specified domain.
Therefore, the roots of the equation, $\cos(x)-\sin(x) = \cos(2x)$, in the domain
$-\pi \le x \le \pi$ are $-3\pi/4$, 0, $\pi/4$, and $\pi/2$.