Chapter 2 Complex numbers

Review: Radicals

A **radical** is any quantity with a radical symbol, $\sqrt{\ }$. $\sqrt{\ }$ means the "positive square root" of a number.

'4' is the **coefficient.**
Technically, 4 is being multiplied by $\sqrt{10}$.

radical symbol

$$4\sqrt{10}$$

'10' is the **radicand.**
The radicand is the number "in the house".

A **radical expression** is any expression that contains a radical.

To **simplify a radical expression** means:
1) Make the radicand(s) as small as possible
2) Perform every operation (addition, multiplication, etc.) possible
3) Eliminate any fractions in radicand(s)
4) Eliminate any radicals in the denominator of a fraction

Product Property	**Quotient Property**
$\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$, as long as a and b are positive numbers	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, as long as a and b are positive numbers

Example 1: Simplify $\sqrt{12}$.
Using the property above: $\sqrt{12} = \sqrt{4} \bullet \sqrt{3}$

However, we know $\sqrt{4}$ = 2, hence $\sqrt{12} = \boxed{2\sqrt{3}}$

This simplification was made possible because we knew a perfect square (4), divided evenly into the radicand (12). If a person had written $\sqrt{12} = \sqrt{6} \bullet \sqrt{2}$, then no simplifying could be done, because 6 and 2 are not perfect squares.

Thus, the critical part is that one must choose factors that are **perfect squares**.

Example 2: Simplify.

a) $\sqrt{24} = \sqrt{4\times 6} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$

b) $\sqrt{72} = \sqrt{36\times 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$

c) $5\sqrt{27} = 5\sqrt{9\times 3} = 5\sqrt{9}\sqrt{3} = 5(3)\sqrt{3} = 15\sqrt{3}$

Example 3: Simplify $\sqrt{\frac{6}{8}}$.

$\sqrt{\frac{6}{8}} = \sqrt{\frac{3}{4}}$ [Simplify $\frac{6}{8}$ to $\frac{3}{4}$]

$= \frac{\sqrt{3}}{\sqrt{4}}$ [Use the quotient property.]

$= \frac{\sqrt{3}}{2}$ [The denominator can be simplified.]

Example 4: Simplify $\sqrt{\frac{7}{2}}$.

$\sqrt{\frac{7}{2}} = \frac{\sqrt{7}}{\sqrt{2}}$ [Use the quotient property]

$= \frac{\sqrt{7}}{\sqrt{2}}\bullet\frac{\sqrt{2}}{\sqrt{2}}$ [Since the radical still exists in the denominator, both the numerator and denominator will be multiplied by $\sqrt{2}$]

$= \frac{\sqrt{14}}{2}$

Example 5: Simplify $\frac{x\sqrt{y}}{\sqrt{x^3}}$.

$\frac{x\sqrt{y}}{\sqrt{x^3}} = \frac{x\sqrt{y}}{x\sqrt{x}}$ [The denominator can be simplified.]

$= \frac{\sqrt{y}}{\sqrt{x}}$ [The x's on the outside can essentially be cancelled.]

$= \frac{\sqrt{y}}{\sqrt{x}}\bullet\frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{xy}}{x}$

Adding and Subtracting Radicals

The rule is to only add and subtract **like radicals** - radicals that have the same number under the radical symbol. For example: $2\sqrt{3}$, $5\sqrt{3}$, $-10\sqrt{3}$ are like radicals.

Example 1: $3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}$

Example 2: $3\sqrt{2} - 5\sqrt{6} + \sqrt{2} - 2\sqrt{6} + 4$

$= 3\sqrt{2} + \sqrt{2} - 5\sqrt{6} - 2\sqrt{6} + 4$

$= 4\sqrt{2} - 7\sqrt{6} + 4$

Multiply Radicals $c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$

1) Multiply the coefficients and the radicands separately.
2) Simplify the radicand at the end.

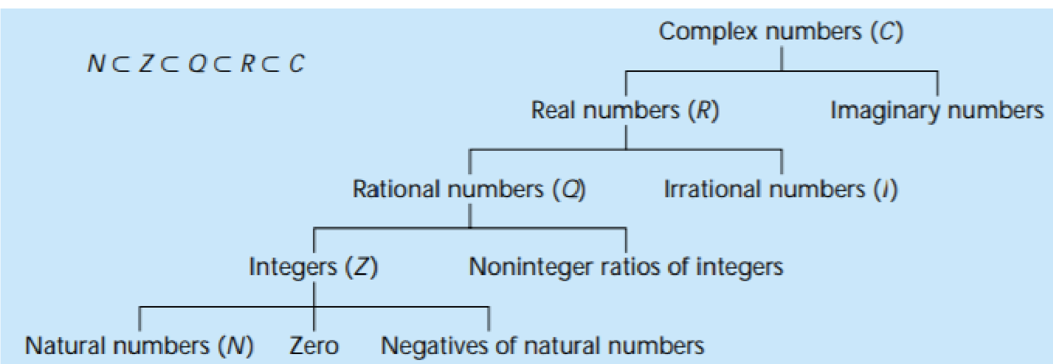
Example 1: Multiply $3\sqrt{2} \bullet 4\sqrt{5}$.
Multiply "3 and 4", and multiply "2 and 5" to get $\boxed{12\sqrt{10}}$.
The radicand, 10, cannot be simplified anymore, so $12\sqrt{10}$ is the final answer.

Example 2: $-4\sqrt{6} \times 2\sqrt{6} = -8\sqrt{36} = -4 \times 6 =$ -48

Example 3: $(2+\sqrt{6})(3-\sqrt{12})$

$= 2\times 3 - 2\times\sqrt{12} + \sqrt{6}\times 3 - \sqrt{6}\times\sqrt{12}$
$= 6 - 2\sqrt{12} + 3\sqrt{6} - \sqrt{6\times 12}$
$= 6 - 2\sqrt{4\times 3} + 3\sqrt{6} - \sqrt{6\times 6\times 2}$
$= 6 - 4\sqrt{3} + 3\sqrt{6} - 6\sqrt{2}$

1. The Complex Numbers



Are the real numbers not sufficient?

If we desire that every integer has an inverse element, we have to invent rational numbers and many things become much simpler.

If we desire every polynomial equation to have a root, we have to extend the real number field R to a larger field C of 'complex numbers', and many statements become more homogeneous.

A complex number is a number that has two components: a real part and an imaginary part, written as: $a + bi$. We call 'a' the **real part** and 'bi' the **imaginary part** of the complex number.

The imaginary is defined to be: $i = \sqrt{-1}$ Then: $i^2 = \left(\sqrt{-1}\right)^2 = -1$

Now, you may think you can do this: $i^2 = \left(\sqrt{-1}\right)^2 = \sqrt{(-1)^2} = \sqrt{1} = 1$

However, it does not work with complex numbers.

In general, we can write any complex number in the form of a + b*i*. We designate any complex number by the letter *z*. As such z = a + b*i*.

Key Words

Names for Particular Kinds of Complex Numbers

Imaginary Unit:	i	
Complex Number:	$a + bi$	a and b real numbers
Imaginary Number:	$a + bi$	$b \neq 0$
Pure Imaginary Number:	$0 + bi = bi$	$b \neq 0$
Real Number:	$a + 0i = a$	
Zero:	$0 + 0i = 0$	
Conjugate of $a + bi$:	$a - bi$	

Example: Simplify.

a) $\sqrt{-9} = \sqrt{9 \bullet (-1)} = \sqrt{9}\sqrt{-1} = \sqrt{9} \bullet \boldsymbol{i} = 3i$

b) $\sqrt{-25} = \sqrt{25 \bullet (-1)} = \sqrt{25}\sqrt{-1} = 5i$

c) $\sqrt{-18} = \sqrt{9 \bullet 2 \bullet (-1)} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3\sqrt{2}i$

d) $-\sqrt{-6} = -\sqrt{6 \bullet (-1)} = -\sqrt{6}\sqrt{-1} = -\sqrt{6}i$

e) $(i)(2i)(-3i)$

$= (2 \cdot -3)(i \cdot i \cdot i)$

$= (-6)(i^2 \cdot i)$

$=(-6)(-1 \cdot i)$

$= (-6)(-i) = \boldsymbol{6i}$

Note this last problem e). Within it, you can see that $i^3 = -i$, because $i^2 = -1$. Continuing, we get:

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

This pattern of powers, signs, 1's, and *i*'s is a cycle: →

$i^1 = i$	$i^5 = i^1 = i$
$i^2 = -1$	$i^6 = i^2 = -1$
$i^3 = -i$	$i^7 = i^3 = -i$
$i^4 = 1$	$i^8 = i^4 = 1$

In other words, to calculate any high power of i, you can convert it to a lower power by taking the closest multiple of 4 that's no bigger than the exponent and subtracting this multiple from the exponent. For example, a common trick question on tests is something along the lines of "Simplify i^{99}", the idea being that you'll try to multiply i ninety-nine times and you'll run out of time, and the teachers will get a good giggle at your expense in the faculty lounge. Here's how the shortcut works:

$$i^{99} = i^{96+3} = i^{(4\times 24)+3} = i^3 = -i$$

That is, $i^{99} = i^3$, because you can just lop off the i^{96}. (Ninety-six is a multiple of four, so i^{96} is just 1, which you can ignore.) In other words, you can divide the exponent by 4 (using long division), discard the answer, and use only the remainder. This will give you the part of the exponent that you care about.

Practice: Simplify i^{17}.

$i^{17} = i^{16+1} = i^{4 \cdot 4 + 1} = i^1 = \boldsymbol{i}$

Equal Complex Numbers

Two complex numbers (a, b) and (c, d) are equal if and only if (a = c and b = d).

So a + bi = c + di   <=>   a = c and b = d

2. Addition and Subtraction of Complex Numbers

We define the sum of complex numbers in a trivial way.

$(a, b) + (a', b') = (a + a', b + b')$ or $(a + bi) + (a' + b'i) = (a + a') + (b + b')i$

$(a + bi) - (c + di) = (a - c) + (b - d)i$ and $a + (-b)i = a - bi$

The same algebra rules that we learned for polynomials apply for complex numbers - the concepts of like terms and the distributive rule apply. For example, the terms $7i$ and i are like terms; the terms 6 and $6i$ are unlike terms.

Example: Simplify

a) $(2 + 3i) + (1 - 6i) = (2 + 1) + (3i - 6i) = 3 + (-3i) = 3 - 3i$

b) $(5 - 2i) - (-4 - i) = 5 - 2i + 4 + i = (5 + 4) + (-2i + i) = (9) + (-1i) = \mathbf{9 - \boldsymbol{i}}$

3. Multiplication and Division of Complex Numbers

We define the product of complex numbers to be: (a, b) × (c, d) = (ac - bd, ad + bc) using FOIL which stands for "**F**irsts, **O**uters, **I**nners, **L**asts"

Example 1: Simplify

a) $(2 - i)(3 + 4i) = (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i)$

$= 6 + 8i - 3i - 4i^2 = 6 + 5i - 4(-1)$

$= 6 + 5i + 4 = 10 + 5i$

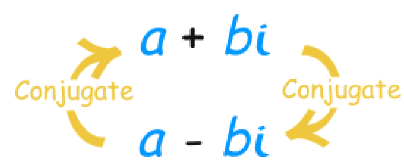
b) $(2 + 3i)(1 + 2i) = -4 + 7i$

Complex numbers have conjugates which means that the two complex numbers have the same real component and the imaginary components are "negative opposites". If we designate z = a + bi, then we designate the complex conjugate as a – bi.

A prerequisite skill is the idea of "rationalizing the denominator" - in other words, we have roots in the denominator that we can algebraically "remove" or change.

Specifically, we do not want a term containing i in the denominator, so we must "remove it" using algebraic concepts (recall $i^2 = -1$ and recall the product of conjugates)

When dividing by (a + bi), we multiply the numerator and the denominator with the complex conjugate of the denominator (a – bi).



Example 2: Simplify

a) $\frac{5}{2i} = \frac{5}{2i} \bullet \frac{i}{i} = \frac{5i}{2(-1)} = -\frac{5i}{2}$

b) $\frac{3}{2+i}$

If you multiply top and bottom by i, you get:

$$\frac{3}{2+i} = \frac{3}{2+i} \cdot \frac{i}{i} = \frac{3i}{2i+i^2} = \frac{3i}{2i-1} = \frac{3i}{-1+2i}$$

You will still have i at the bottom. Therefore, we need to multiply the conjugate.

$$\begin{aligned}(a+bi)(a-bi) &= a^2 - abi + abi - (bi)^2 \\ &= a^2 - b^2(i^2) \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2 \end{aligned}$$

→ A REAL NUMBER!

$$\frac{3}{2+i} \bullet \frac{2-i}{2-i} = \frac{3(2-i)}{4-(-1)} = \frac{6-3i}{5} = \frac{6}{5} - \frac{3}{5}i$$

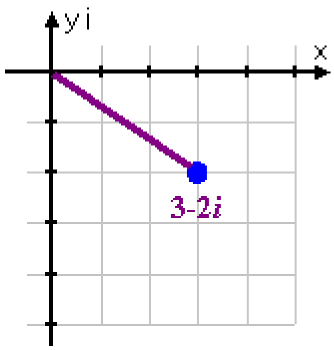
Note that when we multiply the conjugate, difference of square formula is applied to the denominator.

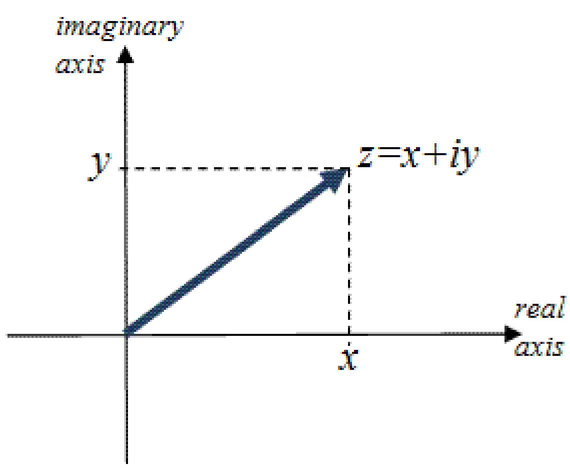
c) $\dfrac{-5+10i}{3-\sqrt{-16}} = \dfrac{-5+10i}{3-4i} = \dfrac{(-5+10i)(3+4i)}{(3-4i)(3+4\mathrm{i})} = \dfrac{-15-20i+30i-40}{9+16} = \dfrac{-55+10i}{25} = \dfrac{-11+2i}{5}$

d) $\dfrac{4-3i}{5+\sqrt{8}i} = \dfrac{(4-3i)(5-\sqrt{8}\,\mathrm{i})}{(5+\sqrt{8}i)(5-\sqrt{8}\,\mathrm{i})} = \dfrac{20-4\sqrt{8}i-15i-3\sqrt{8}}{25+8} = \dfrac{20-6\sqrt{2}}{33} - \dfrac{8\sqrt{2}+15}{33}i$

4. Graph of a Complex Number

You can graph complexes, but not in the *x*, *y*-plane. You need the "complex" plane. For the complex plane, the *x*-axis is where you plot the real part, and the *y*-axis is where you graph the imaginary part. A complex number can now be shown as a point. For instance, for the complex number 3 – 2*i*, you would graph it like this: You know how the number line goes **left-right**? Well let's have the imaginary numbers go **up-down**:



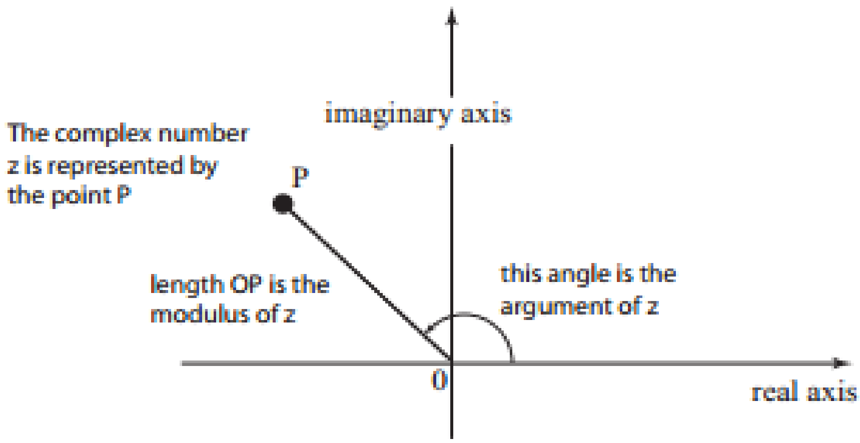


This leads to an interesting fact: When you learned about regular ("real") numbers, you also learned about their order (this is what you show on the number line). But *x*,*y*-points don't come in any particular order. You can't say that one point "comes after" another point in the same way that you can say that one number comes after another number. For instance, you can't say that (4, 5) "comes after" (4, 3)" in the way that you can say that 5 comes after 3. Pretty much all you can do is compare "size", and, for complex numbers, "size" means "how far from the origin". To do this, you use the Distance Formula, and compare which complexes are closer to or further from the origin. This "size" concept is called "**the modulus**". For instance, looking at our complex number plotted above, its modulus is computed by using the Distance Formula:

$$|3-2i| = \sqrt{(3)^2+(2)^2} = \sqrt{9+4} = \sqrt{13} \approx 3.61$$

Note that all points at this distance from the origin have the same modulus. All the points on the circle with radius sqrt(13) are viewed as being complex numbers having the same "size" as 3 – 2*i*.

Any complex number, z, can be represented by a point in the complex plane as shown in the following diagram.



We can join point P to the origin with a line segment, as shown. We associate with this line segment two important quantities. The length of the line segment, that is OP, is called the **modulus** of the complex number. The angle from the positive axis to the line segment is called the **argument** of the complex number, z.

The modulus and argument are fairly simple to calculate using trigonometry.

We define modulus or absolute value of a + bi as $\sqrt{a^2+b^2}$ by Pythagorean Theorem.

We write this modulus of a + bi as |a + bi|, and argument of a + bi as arg(a + bi).

Example: Find the modulus and argument of z = 4 + 3i.

Solution: The complex number z = 4 + 3i is shown in Figure 2. It has been represented by the point Q which has coordinates (4, 3). The modulus of z is the length of the line OQ which we can find using Pythagorean Theorem.

$OQ^2 = 4^2 + 3^2 = 16 + 9 = 25$ and hence OQ = 5.

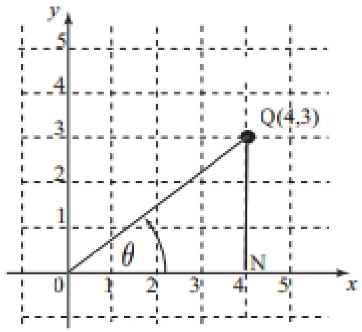


Figure 2. The complex number $z = 4 + 3i$.

Hence the modulus of z = 4 + 3i is 5.
To find the argument we must calculate the angle between the x axis and the line segment OQ. We have labelled this θ in Figure 2.

By referring to the right-angled triangle OQN in Figure 2 we see that

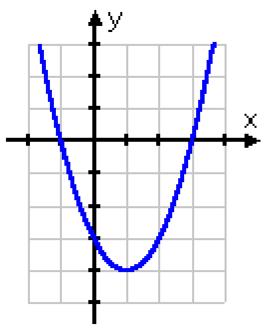
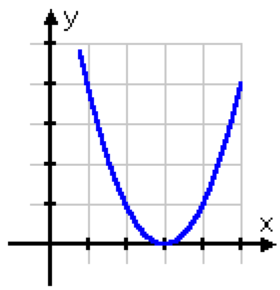
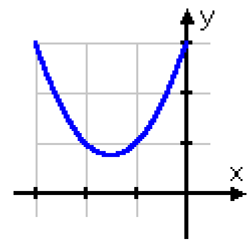
$\tan\theta = 3/4$
$\theta = \tan^{-1}(3/4) = 36.97^\circ$

To summarize, the modulus of z = 4 + 3i is 5 and its argument is θ = 36.97°. There is a special symbol for the modulus of z; this is |z|. So, in this example, |z| = 5.

We also have an abbreviation for argument: we write arg(z) = 36.97°.

5. Complex Numbers and the Quadratic Formula

Remember that the Quadratic Formula solves "$ax^2 + bx + c = 0$" for the values of *x*. Also remember that this means that you are trying to find the x-intercepts of the graph. When the Formula gives you a negative inside the square root, you can now simplify the zero by using complex numbers. The answer you come up with is a valid "zero" or "root" or "solution" for "$ax^2 + bx + c = 0$", because, if you plug it back into the quadratic, you'll get zero after you simplify. But you cannot graph a complex number on the *x*, *y*-plane. So this "solution to the equation" is not an *x*-intercept. In other words, you can make this connection between the Quadratic Formula, complex numbers, and graphing:

$x^2 - 2x - 3$	$x^2 - 6x + 9$	$x^2 + 3x + 3$
$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)}}{2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = \frac{-2}{2}, \frac{6}{2} = -1, 3$	$x = \frac{6 \pm \sqrt{(-6)^2 - 4(9)}}{2} = \frac{6 \pm \sqrt{36-36}}{2} = \frac{6 \pm \sqrt{0}}{2} = \frac{6 \pm 0}{2} = 3$	$x = \frac{-3 \pm \sqrt{(3)^2 - 4(3)}}{2} = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm \sqrt{-3}}{2} = -\frac{3}{2} \pm \frac{\sqrt{3}i}{2}$
a positive number inside the square root	zero inside the square root	a negative number inside the square root
two real solutions	one (repeated) real solution	two complex solutions
		
two distinct x-intercepts	one (repeated) x-intercept	no x-intercepts (In fact, there are 2 imaginary roots)