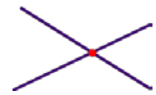
Chapter 1 Linear System Equations

1. Solution of Linear Simultaneous Equations

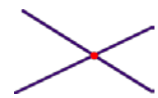
Two linear equations in two variables are said to be **simultaneous** if they are considered at the same time. E.g. $x+y=5$ and $x-y=1$ are simultaneous equations.

The **solution** is where the equations 'meet' or intersect. The red point on the right is the solution of the system.



Case 1: *1 Solution*

This is the most common situation and it involves lines that intersect exactly 1 time.



Case 2: *No Solutions*

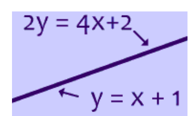
This only happens when the lines are parallel. As you can see, parallel lines are not going to ever meet.

Example of a stem that has no solution:

- **Line 1:** y = 5x +13
- **Line 2:** y = 5x + 12

Case 3: *Infinite Solutions*

This is the rarest case and only occurs when you have the <u>same line</u>.



Consider, for instance, the two lines below (y = 2x+1 and 2y = 4x +2). These two equations are really the same line.
Example of a system that has infinite solutions:

- **Line 1:** y = 2x + 1
- **Line 2:** 2y = 4x + 2

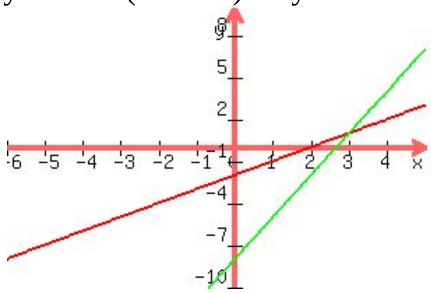
2. Solving Linear Systems Graphically

Simultaneous equations are solved exactly either by the substitution method or the elimination method. An approximate solution can be found by using the graphical method.

The **graphical solution** of the simultaneous equations is given by the point of intersection of the graphs.

Example: solve by graphing
y = x – 2
y = 3x – 8

Graph both lines:
y = x – 2 (red line)   y = 3x – 8 (green line)



Point of intersection (POI) is (3, 1).

Graphing equations in order to identify a specific point of intersection is usually not a precise way to solve systems because it is often difficult to see exactly where two lines intersect (unless you are using a computer-based graphing program that allows you to zoom in on a point). We need to learn how to solve the equations algebraically.

3. Solving Linear Systems by Substitution

A way to solve a linear system algebraically is to use the **substitution method**. To solve simultaneous equations by substitution, we need to find the value of *y* in terms of *x* (or vice versa) for one of the two equations and then substitute this value into the other equation.

Example 1: Solve the following simultaneous equations by using the substitution method:
y – 3x = 0
x + y = 8

Solution: Label the equations as follows:
y – 3x = 0   … (1)
x + y = 8   … (2)

From (1) we have:
$y = 3x$ … (3)

Substituting $y = 3x$ in (2) gives:

$$x + 3x = 8$$
$$4x = 8$$
$$\frac{4x}{4} = \frac{8}{4}$$
$$x = 2$$

When x = 2, y = 3(2) = 6 {From (3)}

So, the solution is (2, 6).

It is always a good idea to check your solution by substitute x = 2 and y = 6 into both equations.

Check:

(1) LS = y – 3x = 6 – 3(2) = 0 RS = 0 LS = RS	(2) LS = x + y = 2 + 6 = 8 RS = 8 LS = RS

Since (2, 6) satisfies both equations, it is the correct solution to the given system of equations.

4. Solving Linear Systems by Elimination

Another way to solve a linear system algebraically is to use the **elimination method**. To solve the simultaneous equations by elimination, make the coefficients of one of the variables the same value in both equations. Then either add the equations or subtract one equation from the other (whichever is appropriate) to form a new equation that contains only one variable. This is referred to as eliminating the variable. Solve the equation thus obtained. Then substitute the value found for the variable in one of the given equations and solve it for the other variable. Write the solution as an *ordered pair*.

Example 1: Solve the following simultaneous equations by using the elimination method:

$$2x+3y=15$$
$$4x-3y=3$$

Solution: Label the equations as follows:

$$2x+3y=15 \qquad \ldots(1)$$
$$4x-3y=3 \qquad \ldots(2)$$

Notice that 3*y* appears in the left-hand side of both equations. Adding the left-hand side of (1) and (2), and then the right-hand sides, gives:

$$2x+3y+4x-3y=15+3$$
$$6x=18$$
$$\frac{6x}{6}=\frac{18}{6}$$
$$x=3$$

Note: We have added equals to equals, and addition eliminates *y*.

Substituting *x* = 3 in (1) gives:

$$\begin{aligned} 2\times 3+3y&=15 \\ 6+3y&=15 \\ 6+3y-6&=15-6 \\ 3y&=9 \\ \frac{3y}{3}&=\frac{9}{3} \\ y&=3 \end{aligned}$$

So, the solution is (3, 3).

Example 2: Solve the following simultaneous equations by using the elimination method:

$$2x+3y=13$$
$$3x+2y=12$$

Solution: Label the equations as follows:

$$2x+3y=13 \qquad \ldots(1)$$
$$3x+2y=12 \qquad \ldots(2)$$

Multiplying (1) by 2 and (2) by 3 gives:

$$4x+6y=26 \qquad \ldots(3)$$
$$9x+6y=36 \qquad \ldots(4)$$

Subtracting (3) from (4) gives:

$$y-3x=0 \text{ and } x+y=8$$

$$\begin{aligned} 5x&=10 \\ \frac{5x}{5}&=\frac{10}{5} \\ x&=2 \end{aligned}$$

Substituting x = 2 in (1) gives:

$$\begin{aligned} 2\times 2+3y&=13 \\ 4+3y&=13 \\ 4+3y-4&=13-4 \\ 3y&=9 \\ \frac{3y}{3}&=\frac{9}{3} \\ y&=3 \end{aligned}$$

So, the solution is (2, 3).

Example 3: Solve the system
x/2 + y/3 = 8
2x/3 + 3y/2 = 17

Start with the first equation $\left(\frac{1}{2}\right)\cdot x+\left(\frac{1}{3}\right)\cdot y=8$

Multiply both sides by the LCD 6 $6\cdot\left(\left(\frac{1}{2}\right)\cdot x+\left(\frac{1}{3}\right)\cdot y\right)=(6)\cdot(8)$

$3\cdot x+2\cdot y=48$

Start with the second equation $\left(\frac{2}{3}\right)\cdot x+\left(\frac{3}{2}\right)\cdot y=17$

Multiply both sides by the LCD 6 $6\cdot\left(\left(\frac{2}{3}\right)\cdot x+\left(\frac{3}{2}\right)\cdot y\right)=(6)\cdot(17)$

$4\cdot x+9\cdot y=102$

Let's start with the new system of linear equations

$3\cdot x+2\cdot y=48$
$4\cdot x+9\cdot y=102$

Now in order to solve this system by using substitution, we need to solve (or isolate) one variable. I'm going to choose y.

Solve for y for the first equation $y=24-\left(\frac{3}{2}\right)\cdot x$

Substitute the expression $24-\left(\frac{3}{2}\right)\cdot x$ into y of the 2nd equation. This will eliminate y so we can solve for x.

$4\cdot x+9\cdot\boxed{\left(24-\left(\frac{3}{2}\right)\cdot x\right)}=102$
$4\cdot x+9\cdot(24)+9\cdot\left(-\frac{3}{2}\right)\cdot x=102$
$4\cdot x+216-\left(\frac{27}{2}\right)\cdot x=102$
$\cancel{\left(\frac{2}{(-19)}\right)\cdot\left(-\frac{19}{2}\right)}\cdot x=\left(-\frac{114}{1}\right)\cdot\left(\frac{2}{(-19)}\right)$ Multiply both sides by $\frac{2}{(-19)}$. This will cancel out $-\frac{19}{2}$ and isolate x.

x = 12

Now that we know that x = 12, lets substitute that in for x to solve for y

Plug in x = 12 into the second equation $48+9\cdot y=102$

$\cancel{\left(\frac{1}{9}\right)\cdot(9)}\cdot y=\left(\frac{54}{1}\right)\cdot\left(\frac{1}{9}\right)$ Multiply both sides by $\frac{1}{9}$. This will cancel out 9 on the left side.

$y=\frac{54}{9}$ → y = 6

So our solution is (12, 6)

5. Special Solution Set

Example 1: Solve the following linear system

$7x + 2y = 16$
$-21x - 6y = 24$

Neither of these equations is particularly easier than the other for solving. I'll get fractions, no matter which equation and which variable I choose. I guess I'll take the first equation, and I'll solve it for *y*, because at least the 2 (from the "2*y*") will divide evenly into the 16.

$$7x + 2y = 16$$
$$2y = -7x + 16$$
$$y = -\left(\tfrac{7}{2}\right)x + 8$$

Now I'll plug this into the other equation:

$$-21x - 6\left(-\left(\tfrac{7}{2}\right)x + 8\right) = 24$$
$$-21x + 21x - 48 = 24$$
$$0x - 48 = 24$$
$$0x = 72$$

What times 0 gives us 72?

All my math was right, but I can't find a number such that when it multiplies 0 gives us 72. Keep in mind that, when solving, you're trying to find where the lines intersect. What if they don't intersect? We knew, from the previous lesson, that this system represents two parallel lines graphically. A system of equations is called an **inconsistent system of equations** if there is no solution because the lines are parallel.

Solution: no solution

Example 2: Solve the following system.

$y = 36 - 9x$
$3x + {}^{y}/_{3} = 12$

The first equation is already solved for *y*, so I'll use the substitution method to substitute that into the second equation:

$3x + (36 - 9x)/3 = 12$
$3x + 12 - 3x = 12$
$0x + 12 = 12$
$0x = 0$ or $12 = 12$ (Identity; it is always true no matter what x is)

What multiplies 0 give us 0? x can be anything!

I did substitute the *first* equation into the *second* equation, so this unhelpful result is not because of some screw-up on my part. It's just that this is what a **dependent system** looks like when you try to find a solution. Remember that, when you're trying to solve a system, you're trying to use the second equation to narrow down the choices of points on the first equation. You're trying to find the one single point that works in both equations. But in a dependent system, the "second" equation is really just another copy of the first equation, and *all* the points on the one line will work in the other line. In other words, I got an unhelpful result because the second line equation didn't tell me anything new. This tells me that the system is actually dependent, and that the solution is the whole line.

A **system of equations** is **consistent** if there is at least one set of values for the unknowns that satisfies every equation in the system. If a consistent system has an infinite number of solutions, it is **dependent**. When you graph the **equations**, both **equations** represent the same line.

Solution: Infinite many of solutions.

Class practice:

a) $4x - 4y = -4$	b) $2y = 16 + 4x$	c) $y = x + 4$
$3x + 2y = 12$	$3x + 5y = 27$	$3x - 3y = -12$
Ans: $(x, y) = (2, 3)$	$(x, y) = (-1, 6)$	Infinite many of solutions

6. Problem Solving

There are many situations in which scientists, mathematicians and engineers have to deal with simultaneous equations. To solve problems involving simultaneous equations, introduce two variables and form two equations. Then solve the equations and state your answer in words.

Example 1: The denominator of a certain fraction is 1 more than the numerator, and the sum of the numerator and the denominator is 7. Find the fraction.

Solution:

Let the numerator be x and the denominator be y; i.e. the fraction is $\frac{x}{y}$. Now, 1 more than the numerator is $x+1$, and the sum of the numerator and denominator is $x+y$. So, we are given that

$$y = x+1 \qquad \ldots(1)$$
$$x+y = 7 \qquad \ldots(2)$$

Substituting $y = x + 1$ in (2) gives:

$$\begin{aligned} x+x+1 &= 7 \\ 2x+1 &= 7 \\ 2x+1-1 &= 7-1 \\ 2x &= 7-1 \\ 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

Substituting $x = 3$ in (1) gives: y = 3 + 1 = 4

Example 2: The length of a rectangular orchard is 40 m more than the width, and the perimeter is 200 m. Find the length and width. Hence determine the area of the orchard.

Solution: Let the length and width of the rectangular orchard be l m and w m respectively. Then:

$$l = w+40 \qquad \ldots(1)$$

$$\begin{aligned} \text{Perimeter} &= 200 \\ \therefore\ 2(l+w) &= 200 \qquad \{\text{Divide both sides by } 2\} \\ l+w &= 100 \qquad \ldots(2) \end{aligned}$$

Substituting $l = w + 40$ in (2) gives:

$$\begin{aligned} w+40+w &= 100 \\ 2w+40 &= 100 \\ 2w+40-40 &= 100-40 \\ 2w &= 60 \\ \frac{2w}{2} &= \frac{60}{2} \\ w &= 30 \end{aligned}$$

Substituting $w = 30$ in (1) gives:

$$l = 30+40 \ = 70$$

So, the length and width are 70 m and 30 m respectively.

Now, $A = lw = 70 \times 30 = 2100$

So, the area of the orchard is 2100 m^2

7. More Application Problems

Interest and Money Problems

In this section, the interest and money problems should seem familiar. The difference is that we will be making use of two variables when setting up the algebraic equations.

Example 1: A roll of 32 bills contains only \$5 bills and \$10 bills. If the value of the roll is \$220, then how many of each bill are in the roll?

Solution: Begin by identifying the variables.

Let x represent the number of \$5 bills.
Let y represent the number of \$10 bills.

When using two variables, we need to set up two equations. The first equation is created from the fact that there are 32 bills.

$$x + y = 32$$

The second equation sums the value of each bill: the total value is \$220.

$$\$5 \cdot x + \$10 \cdot y = \$220$$

Present both equations as a system; this is our algebraic setup.

$$\begin{cases} x + y = 32 \\ 5x + 10y = 220 \end{cases}$$

Here we choose to solve by elimination, although substitution would work just as well. Eliminate *x* by multiplying the first equation by −5.

$$\begin{cases} x + y = 32 \\ 5x + 10y = 220 \end{cases} \overset{\times(-5)}{\Rightarrow} \begin{cases} -5x - 5y = -160 \\ 5x + 10y = 220 \end{cases}$$

Now add the equations together:

$$\begin{array}{r} -5x-5y=-160 \\ +\quad 5x+10y=220 \\ \hline 5y=60 \\ \dfrac{5y}{5}=\dfrac{60}{5} \\ y=12 \end{array}$$

Once we have *y*, the number of \$10 bills, back substitute to find *x*.

$$\begin{aligned} x+y&=32 \\ x+12&=32 \\ x+12-12&=32-12 \\ x&=20 \end{aligned}$$

Therefore, there are twenty \$5 bills and twelve \$10 bills. The check is left to the reader.

Example 2: A total of \$6,300 was invested in two accounts. Part was invested in a CD at a $4\frac{1}{2}$% annual interest rate and part was invested in a money market fund at a $3\frac{3}{4}$% annual interest rate. If the total simple interest for one year was \$267.75, then how much was invested in each account?

Solution:

Let x represent the amount invested at $4\frac{1}{2}\% = 4.5\% = 0.045$

Let y represent the amount invested at $3\frac{3}{4}\% = 3.75\% = 0.0375$

The total amount in both accounts can be expressed as

$$x + y = 6{,}300$$

To set up a second equation, use the fact that the total interest was \$267.75. Recall that the interest for one year is the interest rate times the principal (I=prt=pr·1=pr). Use this to add the interest in both accounts. Be sure to use the decimal equivalents for the interest rates given as percentages.

$$\begin{array}{ccccc} \textit{interest from the CD} & + & \textit{interest from the fund} & = & \textit{total interest} \\ 0.045x & + & 0.0375y & = & 267.75 \end{array}$$

These two equations together form the following linear system:

$$\begin{cases} x + y = 6{,}300 \\ 0.045x + 0.0375y = 267.75 \end{cases}$$

Eliminate *y* by multiplying the first equation by −0.0375.

$$\begin{cases} x + y = 6{,}300 \\ 0.045x + 0.0375y = 267.75 \end{cases} \overset{\times(-0.0375)}{\Rightarrow} \begin{cases} -0.0375x - 0.0375y = -236.25 \\ 0.045x + 0.0375y = 267.75 \end{cases}$$

Next, add the equations together to eliminate the variable *y*.

$$\begin{array}{rl} & -0.0375x - 0.0375y = -236.25 \\ + & 0.045x + 0.0375y = 267.75 \\ \hline & 0.0075x \qquad\qquad = 31.5 \end{array}$$

$$\frac{0.0075x}{0.0075} = \frac{31.5}{0.0075}$$

$$x = 4{,}200$$

Back substitute.

$$\begin{aligned} x + y &= 6{,}300 \\ 4{,}200 + y &= 6{,}300 \\ 4{,}200 + y - 4{,}200 &= 6{,}300 - 4{,}200 \\ y &= 2{,}100 \end{aligned}$$

Therefore, $4,200 was invested at 412% and $2,100 was invested at 334%.

Mixture Problems

Mixture problems often include a percentage and some total amount. It is important to make a distinction between these two types of quantities. For example, if a problem states that a 20-ounce container is filled with a 2% saline (salt) solution, then this means that the container is filled with a mixture of salt and water as follows:

	Percentage	**Amount**
Salt	2% = 0.02	0.02(20 ounces) = 0.4 ounces
Water	98% = 0.98	0.98(20 ounces) = 19.6 ounces

In other words, we multiply the percentage times the total to get the amount of each part of the mixture.

Example 1: A 2% saline solution is to be combined and mixed with a 5% saline solution to produce 72 ounces of a 2.5% saline solution. How much of each is needed?

Solution:

Let x represent the amount of 2% saline solution needed.
Let y represent the amount of 5% saline solution needed.

The total amount of saline solution needed is 72 ounces. This leads to one equation,

$$x + y = 72$$

The second equation adds up the amount of salt in the correct percentages. The amount of salt is obtained by multiplying the percentage times the amount, where the variables *x* and *y* represent the amounts of the solutions.

$$\begin{array}{ccccc} \textit{salt in } 2\% \textit{ solution} & + & \textit{salt in } 5\% \textit{ solution} & = & \textit{salt in the end solution} \\ 0.02x & + & 0.05y & = & 0.025(72) \end{array}$$

The algebraic setup consists of both equations presented as a system:

$$\begin{cases} x + y = 72 \\ 0.02x + 0.05y = 0.025(72) \end{cases}$$

Solve.

$$\begin{cases} x + y = 72 \\ 0.02x + 0.05y = 0.025(72) \end{cases} \xRightarrow{\times(-0.02)} \begin{cases} -0.02x - 0.02y = -1.44 \\ 0.02x + 0.05y = 1.8 \end{cases}$$

$$\begin{array}{rr} & -0.02x - 0.02y = -1.44 \\ + & 0.02x + 0.05y = 1.8 \\ \hline & 0.03y = 0.36 \\ & \dfrac{0.03y}{0.03} = \dfrac{0.36}{0.03} \\ & y = 12 \end{array}$$

Back substitute.

$$
\begin{aligned}
x + y &= 72 \\
x + 12 &= 72 \\
x + 12 - 12 &= 72 - 12 \\
x &= 60
\end{aligned}
$$

Therefore, we need 60 ounces of the 2% saline solution and 12 ounces of the 5% saline solution.

Class practice: A 50% alcohol solution is to be mixed with a 10% alcohol solution to create an 8-ounce mixture of a 32% alcohol solution. How much of each is needed?

Answer: Therefore, to obtain 8 ounces of a 32% alcohol mixture we need to mix 4.4 ounces of the 50% alcohol solution and 3.6 ounces of the 10% solution.

Uniform Motion Problems (Distance Problems)

Recall that the distance traveled is equal to the average rate, which is the speed, times the time traveled at that speed, d = s·t. These uniform motion problems usually have a lot of data, so it helps to first organize that data in a chart and then set up a linear system. In this section, you are required to use two variables.

Example 1: An executive traveled a total of 8 hours and 1,930 miles by car and by plane. Driving to the airport by car, she averaged 60 miles per hour. In the air, the plane averaged 350 miles per hour. How long did it take her to drive to the airport?

Solution: We are asked to find the time it takes her to drive to the airport; this indicates that time is the unknown quantity.

Let x represent the time it took to drive to the airport.
Let y represent the time spent in the air.

	Distance =	***Rate*** ×	***Time***
Travel by car		60 mph	x
Travel by air		350 mph	y
Total	1,930 mi		8 hours

Use the formula D=r·t to fill in the unknown distances.

Distance traveled in the car : $D = r \cdot t = 60 \cdot x$
Distance traveled in the air : $D = r \cdot t = 350 \cdot y$

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

The distance column and the time column of the chart help us to set up the following linear system.

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

$$60x + 350y = 1{,}930 \qquad x + y = 8$$

$$\begin{cases} x + y = 8 & \leftarrow \textit{total time traveled} \\ 60x + 350y = 1{,}930 & \leftarrow \textit{total distance traveled} \end{cases}$$

Solve.

$$\begin{cases} x + y = 8 \\ 60x + 350y = 1{,}930 \end{cases} \overset{\times(-60)}{\Rightarrow} \begin{cases} -60x - 60y = -480 \\ 60x + 350y = 1{,}930 \end{cases}$$

$$\begin{array}{rl} & -60x - 60y = -480 \\ + & \underline{\;\;60x + 350y = 1{,}930} \\ & 290y = 1450 \\ & \dfrac{290y}{290} = \dfrac{1450}{290} \\ & y = 5 \end{array}$$

Now back substitute to find the time it took to drive to the airport *x*:

$$\begin{aligned} x + y &= 8 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$

Therefore, it took her 3 hours to drive to the airport.

It is not always the case that time is the unknown quantity. Read the problem carefully and identify what you are asked to find; this defines your variables.

Example 2: Flying with the wind, an airplane traveled 1,365 miles in 3 hours. The plane then turned against the wind and traveled another 870 miles in 2 hours. Find the speed of the airplane and the speed of the wind.

Solution: There is no obvious relationship between the speed of the plane and the speed of the wind. For this reason, use two variables as follows:

Let x represent the speed of the airplane.
Let w represent the speed of the wind.

Use the following chart to organize the data:

	Distance =	***Rate*** ×	***Time***
Flight with wind	1,365 mi		3 hrs
Flight against wind	870 mi		2 hrs
Total			

With the wind, the airplane's total speed is x+w. Flying against the wind, the total speed is x−w.

	Distance =	***Rate*** ×	***Time***
Flight with wind	1,365 mi	$x + w$	3 hrs
Flight against wind	870 mi	$x - w$	2 hrs
Total	2,235 mi		5 hrs

Use the rows of the chart along with the formula D=r·t to construct a linear system that models this problem. Take care to group the quantities that represent the rate in parentheses.

	Distance =	***Rate*** ×	***Time***	
Flight with wind	1,365 mi	$x + w$	3 hrs	$1{,}365 = (x + w) \cdot 3$
Flight against wind	870 mi	$x - w$	2 hrs	$870 = (x - w) \cdot 2$
Total	2,235 mi		5 hrs	

$$\begin{cases} 1{,}365 = (x + w) \cdot 3 & \leftarrow \textit{distance traveled with the wind} \\ 870 = (x - w) \cdot 2 & \leftarrow \textit{distance traveled against the wind} \end{cases}$$

If we divide both sides of the first equation by 3 and both sides of the second equation by 2, then we obtain the following equivalent system:

$$\begin{cases} 1{,}365 = (x+w)\cdot 3 \\ 870 = (x-w)\cdot 2 \end{cases} \begin{matrix} \overset{\div 3}{\Rightarrow} \\ \underset{\div 2}{\Rightarrow} \end{matrix} \begin{cases} 455 = x+w \\ 435 = x-w \end{cases}$$

$$\begin{aligned} x + w &= 455 \\ + \quad x - w &= 435 \\ \hline 2x \quad &= 890 \\ \frac{2x}{2} \quad &= \frac{890}{2} \\ x &= 445 \end{aligned}$$

Back substitute.

$$\begin{aligned} x + w &= 455 \\ 445 + w &= 455 \\ w &= 10 \end{aligned}$$

Therefore, the speed of the airplane is 445 miles per hour and the speed of the wind is 10 miles per hour.