Lesson 9: Unit 5 - Exponential and Logarithmic Functions (2)

Logarithms were first introduced by John Napier in the 17th century for the purpose of simplifying calculations.
This was accomplished with the development of logarithmic tables and, soon after, with logarithmic scales on a slide rule.
With the introduction of the scientific calculator in the mid-1970s, this application of logarithms for computations became somewhat obsolete; however, logarithms are still used today in many areas such as

- scientific formulas and scales (the pH scale in chemistry and the Richter scale for measuring and comparing the intensity of earthquakes),
- astronomy (order of magnitude calculations comparing relative size of massive bodies),
- modelling and solving problems involving exponential growth and decay, and
- many areas of calculus.

We will begin our study of logarithms by introducing and exploring the logarithmic function.
The logarithmic function is simply the inverse of the exponential function.

Exponential function $\xrightarrow[\text{switch } x \text{ and } y]{\text{inverse}}$ Logarithmic function

$$y = c^x,\ c > 0,\ c \neq 1 \qquad\qquad x = c^y$$

$$\downarrow$$

$$y = \log_c(x),\ c > 0,\ c \neq 1$$

If $f(x) = c^x,\ c > 0,\ c \neq 1$, then $f^{-1}(x) = \log_c(x),\ c > 0,\ c \neq 1$.

What Is a Logarithm?

y = $\log_c(x)$

$\Updownarrow$

x = c^y

Equivalent Forms

x = c^y $\Leftrightarrow$ $\log_c(x)$ = y where c>0, c≠1

125 = 5^3 $\Leftrightarrow$ $\log_5(125)$ = 3

$\frac{1}{36}$ = 6^{-2} $\Leftrightarrow$ $\log_6(\frac{1}{36})$ = -2

The value of the logarithm, y, is the exponent to which the base, c, must be raised to yield the value of x.

Thus, $y = c^x$, so x is the value of the power. x is also referred to as the argument of the logarithm.

Example

Evaluate log3(81).

Solution

We ask: “3 raised to what exponent will produce 81?”

$3^?$ = 81

3^4 = 81

Thus, $\log_3$ (81)=4.

Example

Evaluate $\log_{25}$ (5).

Solution

We ask: “25 raised to what exponent will produce 5?”

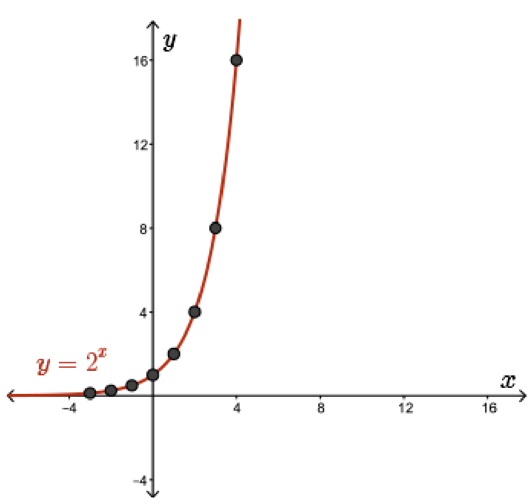
$25^? = 5$

$25^{1/2} = 5$

Thus, $\log_{25}$ (5) = 1/2.

Exponential and Logarithmic Functions

Let's consider the graph of $y=\log_2(x)$.
We will use the fact that this logarithmic function is the inverse of the exponential function $y=2^x$.



Exponential Function

y=2x has domain {x| x ∈R}.

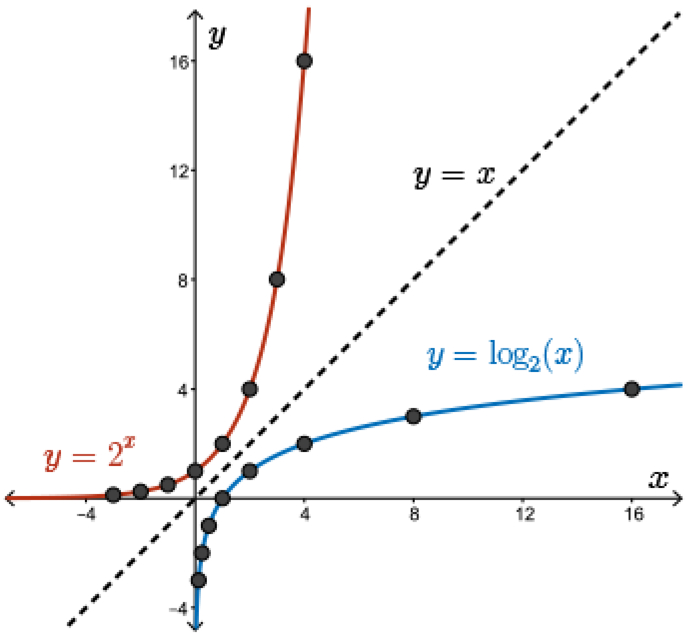
x	y=2^x	(x, y)
−3	18	(−3, 18)
−2	14	(−2, 14)
−1	12	(−1, 12)
0	1	(0, 1)
1	2	(1, 2)
2	4	(2, 4)
3	8	(3, 8)
4	16	(4, 16)

The inverse of $y=2^x$ is given by $x=2^y$, obtained by interchanging the variables, x and y.
Expressing $x=2^y$ in logarithmic form, we have $y=\log_2 (x)$.

One way to graph $y=\log_2 (x)$ is to interchange the x and y coordinates of each point on y=2x to obtain a corresponding point on its inverse.

$y= 2^x$ (x, y)	$y= \log_2 (x)$ (x, y)
(−3,18)	(18,−3)
(−2,14)	(14,−2)
(−1,12)	(12,−1)
(0,1)	(1,0)
(1,2)	(2,1)
(2,4)	(4,2)
(3,8)	(8,3)
(4,16)	(16,4)

The coordinates of these points support our understanding of logarithms.
Using the point (18, −3), we have $\log_2 (1/8) = -3$, which follows from $2^{-3} = 1/8$.
We now plot these points to obtain the graph of $y=\log_2 x$.



The graph of the inverse of a function is the reflection of the original function in the line *y*=*x*.

Note that not all functions have an inverse function.

In this case, however, $y=\log_2(x)$ is a function since each value of *x* in its domain produces exactly one value for *y* (that is, its graph passes the vertical line test).

Thus, using function notation,

$$f^{-1}(x) = \log_2(x)$$

The graph of $y=2^x$ has a horizontal asymptote of *y*=0.
The graph of $y=\log_2(x)$ has a vertical asymptote of *x*=0.

	Domain	Range
$y=2^x$	$\{x \mid x \in R\}$	$\{y \mid y>0, y \in R\}$
$y=\log_2(x)$	$\{x \mid x>0, x \in R\}$	$\{y \mid y \in R\}$

The graphs of both functions, $y=2^x$ and $y=\log_2(x)$, are increasing.

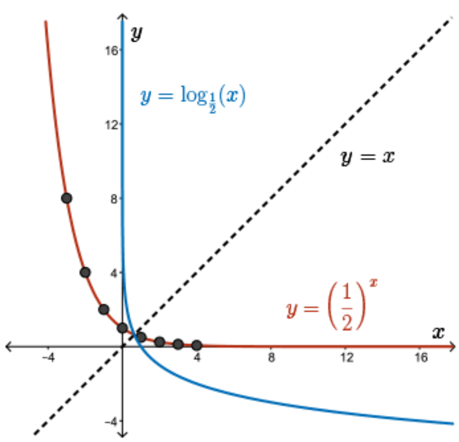
Next let's create the graph of $y=\log_{\frac{1}{2}}(x)$ using the graph of $y=(\frac{1}{2})^x$ and then compare the features of the two graphs.

Determine points on $y=(\frac{1}{2})^x$ and then plot them on the graph.

y= $(\frac{1}{2})^x$	*y*= $\log_{\frac{1}{2}}(x)$
(*x*, *y*)	(*x*, *y*)
(−3,8)	(8,−3)
(−2,4)	(4,−2)
(−1,2)	(2,−1)
(0,1)	(1,0)
(1,1/2)	(1/2,1)
(2,1/4)	(1/4,2)
(3,1/8)	(1/8,3)
(4,1/16)	(1/16,4)

Interchange the coordinates of the points on *y*= $(\frac{1}{2})^x$ to obtain the corresponding points on *y*=$\log_{\frac{1}{2}}(x)$.

Plot the corresponding points to obtain the graph of *y*=$\log_{\frac{1}{2}}(x)$.



The graph of *y*= $\log_{\frac{1}{2}}(x)$ is the reflection of the graph of *y*= $(\frac{1}{2})^x$ in the line *y*=*x*.

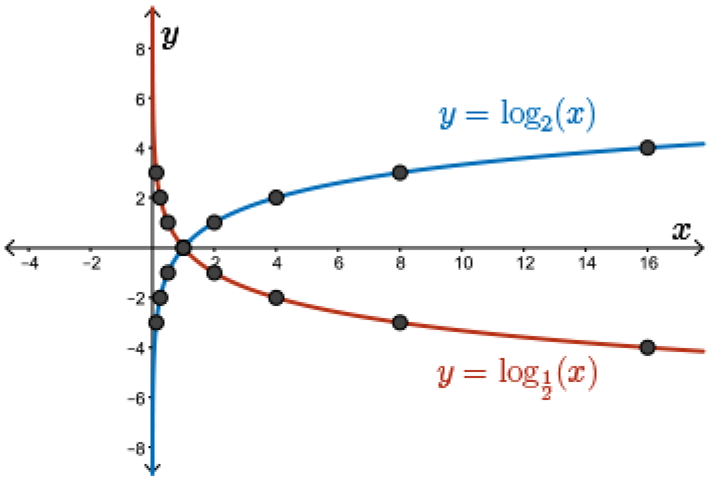
y= $\log_{\frac{1}{2}}(x)$ has a vertical asymptote of *x*=0.

y= $(\frac{1}{2})^x$ has a horizontal asymptote of *y*=0.

In this case, both *y*= $(\frac{1}{2})^x$ and *y*= $\log_{\frac{1}{2}}(x)$ are decreasing functions.

Function	**Domain**	**Range**
y= $\left(\frac{1}{2}\right)^x$	{*x*| *x* ∈R}	{*y*| *y*>0, *y* ∈R}
y= $\log_{\frac{1}{2}}(x)$	{*x*| *x*>0, *x* ∈R}	{*y*| *y* ∈R}

How does the graph of y= $log_{\frac{1}{2}}(x)$ *compare to the graph of y=*$log_2(x)$*?*

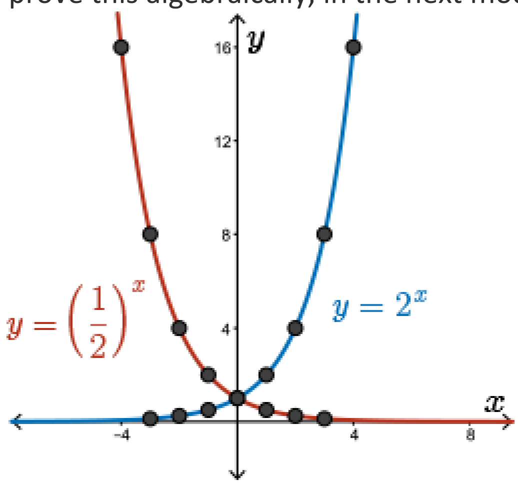


Placing the graph of each function on the same set of axes, we see *y*= $\log_{\frac{1}{2}}(x)$ is a reflection of *y*= $\log_2(x)$ in the *x*-axis.

Using our understanding of transformations, we can say

$$\log_{\frac{1}{2}}(x) = -\log_2(x)$$

We will prove this algebraically, in the next module, after studying the laws of logarithms.



Recall that the graphs of corresponding exponential function, studied earlier in this unit, are reflections of each other in the *y*-axis.

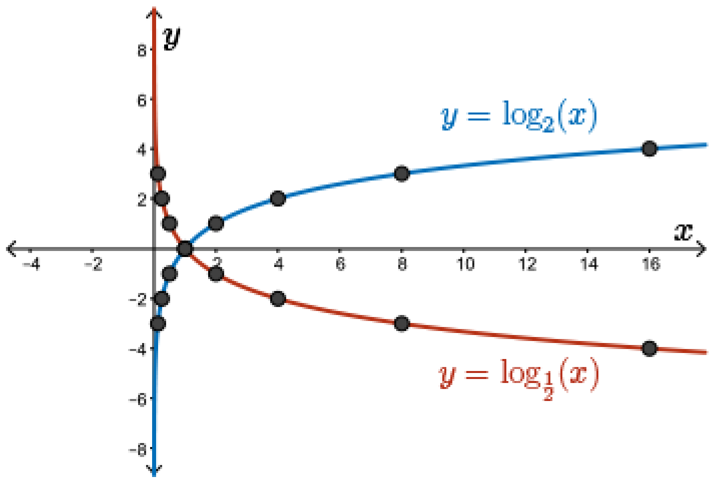
That is,

$$2^x = (1/2)^{-x}$$

This is easily proven using the exponent laws:

$$(1/2)^{-x} = [(1/2)^{-1}]^x = 2^x$$

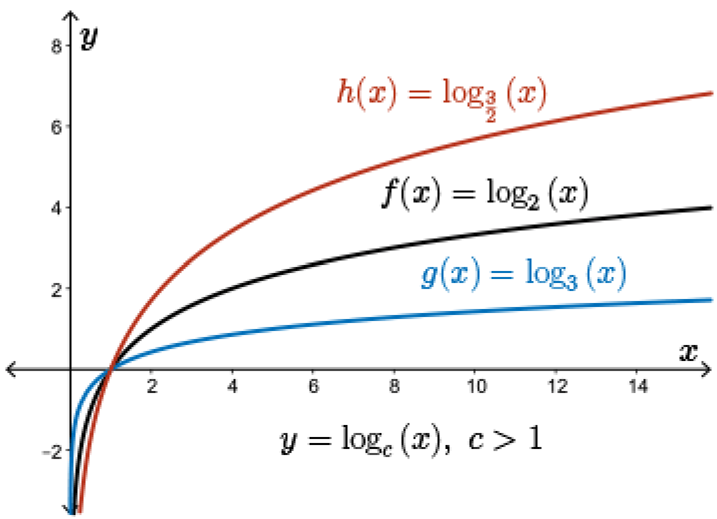
Now, let's summarize our Observations



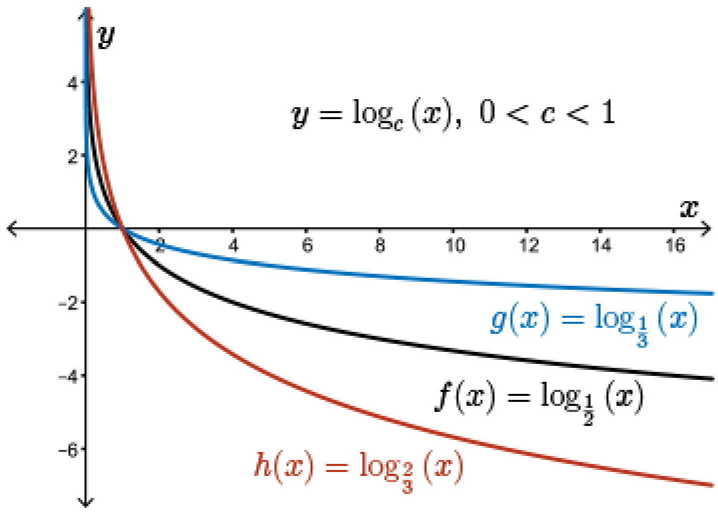
The graphs of logarithmic functions of the form $y=\log_c(x)$, $c>0$, $c\neq1$, have

- domain $\{x|\ x>0,\ x\in\mathbb{R}\}$ and range $\{y|\ y\in\mathbb{R}\}$,
- an x-intercept at (1,0) and no y-intercept, and
- a vertical asymptote of $x=0$ (the y-axis).

The graph of a logarithmic function where ***c*>1** is always increasing. The greater the value of the base, *c*, the slower the curve increases as *x* increases.



The graph of a logarithmic function where **0<*c*<1** is always decreasing. The smaller the value of the base, *c*, the slower the curve decreases as *x* increases.



Transformations of Logarithmic Functions

The parameters *a*, *b*, *h*, and *k* in the equation *y*= $a\log_c b(x-h)$+ *k* correspond to the following transformations:

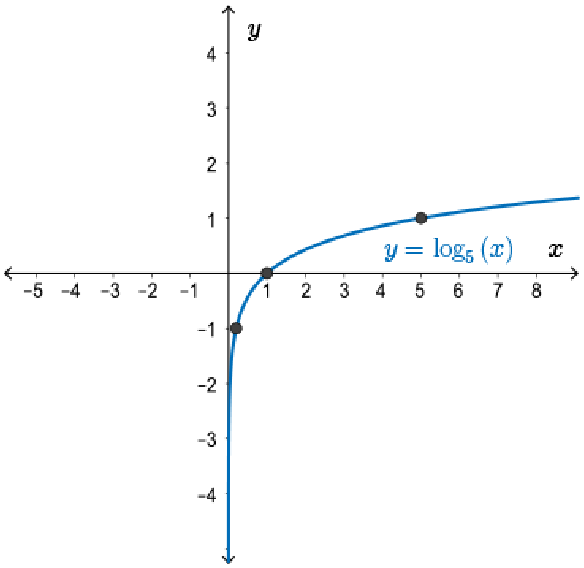
- If *a*<0, *y*= $\log_c(x)$ is reflected in the *x*-axis.
- *y*= $\log_c(x)$ is stretched vertically about the *x*-axis by a factor of |*a*|.
- If *b*<0, *y*=$\log_c(x)$ is reflected in the *y*-axis.
- *y*= $\log_c(x)$ is stretched horizontally about the *y*-axis by a factor of 1|*b*|.
- *y*= $\log_c(x)$ is translated horizontally *h* units.
  - If *h*>0, then *y*= $\log_c(x)$ is translated right.
  - If *h*<0, then *y*= $\log_c(x)$ is translated left.
- *y*= $\log_c(x)$ is translated vertically *k* units.
  - If *k*>0, then *y*= $\log_c(x)$ is translated up.
  - If *k*<0, then *y*= $\log_c(x)$ is translated down.

The transformation of each point is defined by the mapping $(x, y) \rightarrow (\frac{1}{b}x + h, ay + k)$.

Example

a. Graph the function $f(x)=2\log_5(3-x)+1$. Identify the domain, range, and any asymptote of the function. Is the function increasing or decreasing?
b. Determine the equation $y=f^{-1}(x)$ and sketch its graph.

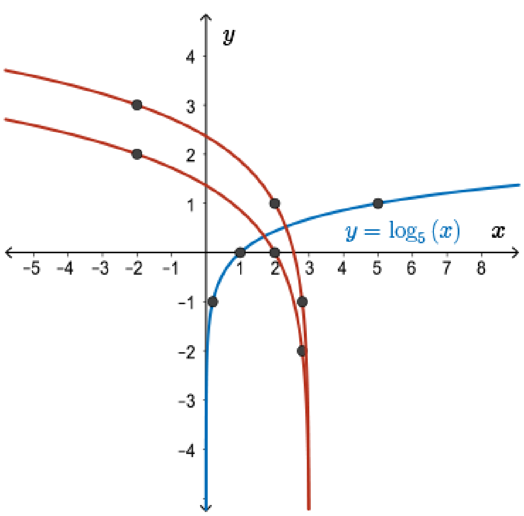
Solution



a. Start with the graph of $y=\log_5(x)$.
The domain of the function is $\{x| \ x>0, x \in R\}$.

x	$y = \log_5(x)$
1/5	$\log_5(1/5) = -1$
1	$\log_5(1) = 0$
5	$\log_5(5) = 1$
25	$\log_5(25) = 2$

Solution

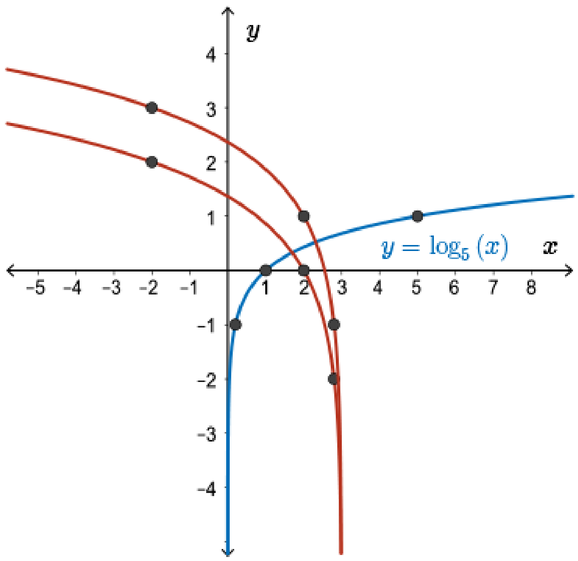


Express $f(x)= 2\log_5(3-x) + 1$ as

$$f(x)=2\log_5[-(x-3)] + 1$$

to correctly identify the transformations that must be applied to the graph of $y=\log 5(x)$.
We may obtain the graph of $f(x)$ from the graph of $y=\log 5(x)$ by applying the following transformations, in order:

- First, apply a vertical stretch about the x-axis by a factor of 2.
- Next, apply a reflection in the y-axis.
- Then, apply a horizontal translation right 3 units.
- Finally, apply a vertical translation up 1 unit.



The domain is $\{x|\ x<3,\ x \in R\}$.
This is seen graphically but can be obtained algebraically since

$$-(x-3) > 0$$
$$x-3 < 0$$
$$x < 3$$

The range is $\{y|\ y \in R\}$.
The function is decreasing and has a vertical asymptote of $x=3$.

b. To determine the equation of the inverse of $f(x) = 2\log_5[-(x-3)] + 1$, we must interchange the x and variables in the equation of $f(x)$ and solve for y.

Inverse:

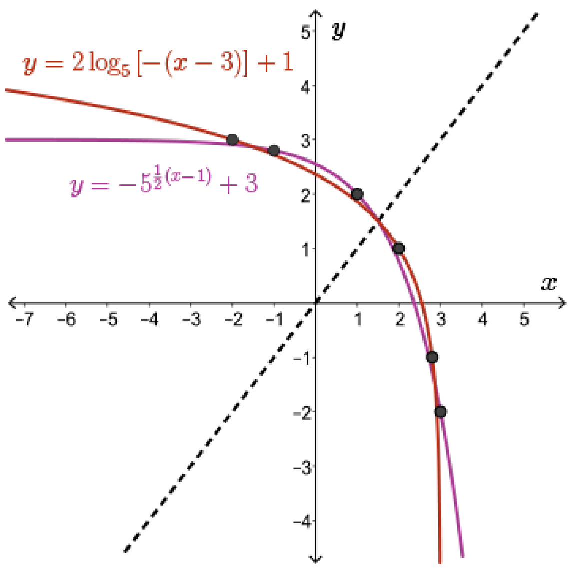
$$x = 2\log_5[-(y-3)] + 1$$
$$x-1 = 2\log_5[-(y-3)]$$
$$\frac{x-1}{2} = \log_5[-(y-3)]$$
$$5^{\frac{x-1}{2}} = -(y-3)$$

$$y = -5^{\frac{x-1}{2}} + 3$$

$$\therefore f^{-1}(x) = -5^{\frac{x-1}{2}} + 3$$

The graph of $f^{-1}(x) = -5^{\frac{x-1}{2}} + 3$ can be obtained by

- interchanging the x and y coordinates of the points on $f(x)$ to obtain points on $f{-}1(x)$,
- reflecting the graph of $f(x)=2\log_5[-(x-3)]+1$in the line $y=x$, or
- applying the appropriate transformations to the graph of $y=5^x$
  - a reflection in the x-axis
  - a horizontal stretch about the y-axis by a factor of 2
  - a horizontal translation right 1 unit
  - a vertical translation up 3 units



Summary

- The value of a logarithm is an exponent. The statement $\log_c(m)=n$ is equivalent to $c^n=m$.
- The inverse of $y=c^x$, $c>0$, $c\neq1$ is $y=\log_c(x)$, $c>0$, $c\neq1$. Their graphs are reflections of each other in the line $y=x$.
- The function $y=\log_c(x)$, $c>0$, $c\neq1$

  - has a domain of $\{x \mid x>0, x \in R\}$ and a range of $\{y \mid y \in R\}$,
  - has an x-intercept at (1,0) and a vertical asymptote of $x=0$ (y-axis), and
  - is increasing when $c>1$ and decreasing when $0<c<1$.

The parameters a, b, h, and k of $y = a\log_c[b(x-h)] + k$ correspond to transformations that map points, (x, y), on $y = c^x$ to $(1/bx + h, ay + k)$; exactly as they did for functions of the form $y = af[b(x-h)] + k$.

Introduction

When working with exponents, we employ a variety of properties and laws to help simplify and evaluate exponential expressions.

Properties	**Laws/Rules**
$c^0 = 1, c \neq 0$	Product of Powers: $(c^m)(c^n) = c^{m+n}$
$c^{-n} = \frac{1}{c^n}$	Quotient of Powers: $\frac{c^m}{c^n} = c^{m-n}$
$c^{\frac{1}{n}} = \sqrt[n]{c}$	A Power of a Power: $(c^m)^n = c^{mn}$

Common Logarithms

A logarithm base 10 is called a **common logarithm**. For simplicity, log10(x) is often written log(x), with base 10understood.

Common logarithms were the first logarithms introduced to carry out complicated calculations in the decimal number system (base 10).

The log function $\boxed{\text{log}}$ on your calculator works in base 10.

Evaluate.
a. log(0.001)
b. log(70)
c. log(−100)

Solution

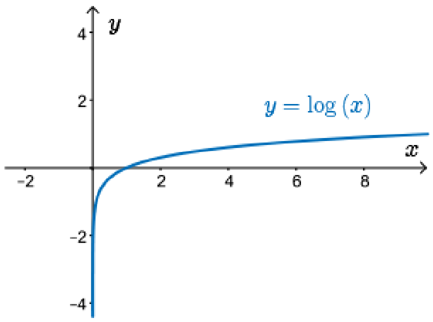
a. Evaluating this logarithm, we have

$$\begin{aligned}\log_{10}(0.001) &= \log_{10}(1/1000)\\ &= \log_{10}(10^{-3})\\ &= -3\end{aligned}$$

You can also use the log function on your calculator $\boxed{\text{log}}$. This works in base 10 (common base).

b. log(70)=1.84509804…
Since this answer is the value of an exponent, it is important to keep 3 or 4 decimal places in your answer when rounding.
Therefore, log(70)≈1.845. This means that $10^{1.845} \approx 70$.

c. The calculator returns an error when asked to determine log(−100).
Why is this?
If we let log(−100)=*n*, then $10^n = -100$.
There is no value for *n* such that 10*n* produces a negative value since the base is positive.
Also, remember that the domain of the logarithmic function $y=\log_c(x)$, $c>0$, $c\neq1$ is $\{x|\ x>0, x\in R\}$.



Example

Evaluate.

a. $4^{\log_4 64}$

b. $5^{\log_5 \frac{1}{25}}$

c. $3^{\log_3 20}$

Solution

a. We have

$$4^{\log_4 64} = 4^3$$
$$= 64$$

b. We have

$$5^{\log_5 \frac{1}{25}} = 5^{-2}$$
$$= 1/25$$

Notice $4^{\log_4 64}$=64 and $5^{\log_5 \frac{1}{25}}$=1/25. It would seem that $3^{\log_3 20}$ should equal 20.
c. If we let $3^{\log_3 20}=n$ and convert to logarithmic form, we have

$$\log_3(n) = \log_3(20) \quad \therefore n=20$$

Thus, $3^{\log_3 20}$=20.

In general, $c^{\log_c(n)} = n$ for all $c>0$, $c\neq1$, $n>0$.
Similarly, $\log_c(c^n) = n$ for all $c>0$, $c\neq1$, $n \in R$.

Example

Solve for x.

a. $\log_{16}(\sqrt{8}) = x$

b. $\log_x(15) = 2$

c. $\log_4(x) = 0$

Solve each of these equations using the equivalent exponential form:

$$\log_c(m) = n \iff m = c^n \text{ when } c>0,\ c\neq 1,\ m>0$$

Solution

a. $\log_{16}(\sqrt{8}) = x \implies 16^x = 8^{1/2}$

$(2^4)^x = (2^3)^{1/2}$

$4x = 3/2$

$\therefore x = 3/8$

b. $\log_x(15) = 2 \implies x^2 = 15$

$x = \pm\sqrt{15}$ but $x>0$

$\therefore x = \sqrt{15}$

c. $\log_4(x) = 0 \implies 4^0 = x$

$\therefore x = 1$

Note: $\mathbf{\log_c(1) = 0}$, for all $c>0$, $c\neq 1$. This follows from the exponential property $c^0 = 1$, $c\neq 0$.

Logarithm of a Product

Product Law

$\mathbf{\log_c(xy) = \log_c(x) + \log_c(y)}$, where $c>0$, $c\neq 1$, $x>0$, $y>0$.

This is the logarithmic form of the exponent law $(c^m)(c^n) = c^{m+n}$.

Proof:

Let $\log_c(x) = m$ and $\log_c(y) = n$.

Then, $c^m = x$ and $c^n = y$.

Now,

$$\begin{aligned}\log_c(xy) &= \log_c(c^m c^n)\\ &= \log_c(c^{m+n})\\ &= m+n\\ &= \log_c(x) + \log_c(y)\end{aligned}$$

Logarithm of a Quotient

Quotient Law

$\mathbf{log}_c(\frac{x}{y})$ = $\mathbf{log}_c(x) - \mathbf{log}_c(y)$, where $c>0$, $c\neq1$, $x>0$, $y>0$.

This is the logarithmic form of the exponent law $\frac{c^m}{c^n} = c^{m-n}$.

Proof:
Let $\log_c(x) = m$ and $\log_c(y) = n$.
Rewriting each in exponential form gives $c^m = x$ and $c^n = y$.
Now,
$\log_c(\frac{x}{y}) = \log_c(\frac{c^m}{c^n})$
$= \log_c(c^{m-n})$
$= m-n$
$= \log_c(x) - \log_c(y)$

Logarithm of a Power

Power Law

$\mathbf{log}_c(x^n) = n\mathbf{log}_c(x)$, where $c>0$, $c\neq1$, $x>0$

This is the logarithmic form of the exponent law $(c^m)^n = c^{mn}$.

Proof:

Let $\log_c(x) = m$ then $x = c^m$.

Now,

$\log_c(x^n) = \log_c((c^m)^n)$

$= \log_c(c^{mn})$

$= mn$

$= n\log_c(x)$

Example

Simplify each expression to a single logarithm.
a. $\log_3(x) + \log_3(x-1) - \log_3(2x)$
b. $\log(2p) - 2\log(q) + 3\log(pq)$

Solution

a. $\log_3(x) + \log_3(x-1) - \log_3(2x)$

$=\log_3 (x(x-1)) - \log_3 (2x)$

$=\log_3 (x(x-1)/2x)$

$=\log_3 (\frac{x-1}{2})$

Note that $x>1$.

b. $\log(2p) - 2\log(q) + 3\log(pq)$

$=\log(2p) - \log(q^2) + \log((pq)^3)$

$=\log(\frac{2p}{q^2}) + \log(p^3q^3)$

$=\log(2p^4q)$

Note that $p>0$ and $q>0$.

In the next example we will demonstrate how logarithms can be used to solve exponential equations.

Example

Use logarithms to solve $5^x = 30$.

Solution

Method 1:

$$5^x = 30$$

$$\log(5^x) = \log(30)$$

$$x\log(5) = \log(30)$$

$$x = \log(30)/\log(5)$$

$$x \approx 2.113$$

Check: 52.113=29.98635...

Method 2:

$5^x = 30$ can be re-expressed in logarithmic form

$$\log_5 (30) = x$$

Thus, $x = \log_5 (30)$. We can also conclude that $\log_5 (30) = \log(30)/\log(5)$, the solution obtained in our first approach.

Change of Base Formula

$\mathbf{log_c(x)} = \frac{\mathbf{log_a(x)}}{\mathbf{log_a(c)}}$, where *a* and *c*>0, *a* and *c*≠1,*x*>0

Proof:
Let $\log_c(x) = n$, so $c^n = x$.
Then,

$\log_a(c^n) = \log_a(x)$

$n\log_a(c) = \log_a(x)$

$n = \log_a(x)/\log_a(c)$ but $n = \log_n(x)$

$\log_c(x) = \frac{\log_a(x)}{\log_a(c)}$

This formula allows us to convert any logarithm to a different base. For example, $\log_3(20)$ can be calculated using the common log function $\boxed{\text{log}}$ on the calculator by converting the base 3 logarithm to base 10.

$$\text{Log}_3(20) = \log(20)/\log(3)$$

Therefore, $\log_3(20) \approx 2.7268$.

Summary

1. The logarithmic function, $y=\log_c(x)$, is the inverse of the exponential function, $y = c^x$, where $c>0$, $c\neq1$.
2. The statement $\log_c(m) = n$ is equivalent to the statement $m = c^n$.
3. The logarithm, $\log c(x)$, is defined only when $c>0$, $c\neq1$ and $x>0$.
4. A common logarithm is a logarithm base 10. When the base is not provided, such as $\log(x)$, it is assumed to be a common logarithm.
5. The following statements hold true for logarithms with $c>0$ and $c\neq1$:

- $c^{\log_c n} = n$, $n>0$
- $\log_c(c^n) = n$
- $\log_c(1) = 0$
- $\log_c(m) = \log_c(n)$ if and only if $m=n$, m and $n>0$

6. For $c>0$, $c\neq1$, $x>0$, $y>0$,

- $\log_c(xy) = \log_c(x) + \log_c(y)$ (Product Law)
- $\log_c(xy) = \log_c(x) - \log_c(y)$ (Quotient Law)
- $\log_c(x^n) = n\log_c(x)$ (Power Law)

7. The formula for converting a logarithm from one base to another is $\log_c(x) = \log_a(x) / \log_a(c)$. This formula is often used to convert to base 10.

All of the above properties and laws of logarithms can be used to simplify logarithmic expression and solve exponential and logarithmic equations.

Next, we will solve exponential and logarithmic equations using the properties and laws of exponents and logarithms and we will apply these skills to solve problems involving exponential growth and decay.

Example

Solve $5^{x-4} = 3(4^{2x+1})$.

Solution

5^{x-4}	$=3(4^{2x+1})$	
$\log(5^{x-4})$	$= \log[3(4^{2x+1})]$	Take the base 10 logarithm of each side of the equation.
$\log(5^{x-4})$	$= \log(3) + \log(4^{2x+1})$	Apply the product law of logarithms.
$(x-4)\log(5)$	$= \log(3) + (2x+1)\log(4)$	Apply the power law of logarithms.
$x\log(5)-4\log(5)$	$=\log(3)+2x\log(4)+\log(4)$	Expand using the distributive property.
$x\log(5)-2x\log(4)$	$=\log(3)+\log(4)+4\log(5)$	Collect like terms.
$x(\log(5)-2\log(4))$	$=\log(3)+\log(4)+4\log(5)$	Factor x.
x	$= (\log(3)+\log(4)+4\log(5)) / (\log(5)-2\log(4))$	Isolate x.

Note:
It is important that there is only one term on each side of the equation when taking the logarithm of each side.
These terms can consist of factors in a product or a quotient.
The product and quotient law of logarithms allow us to manipulate these, as was shown in this example.

However, moving all terms involving variable x to one side (a common step when solving equations) and then taking the logarithm of each side would not allow us to continue. For example,

$5^{x-4} = 3(4^{2x+1})$

$5^{x-4} - 3(4^{2x+1}) = 0$

$\log(5^{x-4} - 3(4^{2x+1})) = \log(0)$

There is no algebraic way to simplify the logarithm of a difference or the logarithm of a sum (on the left side), and the logarithm of 0 is undefined (on the right side).

Example

A sample of radioactive material had a mass of 56.8 grams. After 5 years, the material decayed to a mass of 20.5grams. Determine the half-life of the substance.

Solution

We will begin with the general form of the equation used to model exponential growth and decay,

$$Q(t) = Q_0(r)^{\frac{t}{T}}$$

where r is the growth/decay factor, Q is the quantity remaining after time t, and Q_0 is the initial quantity. T is the growth/decay period and t is measured in the same units as T.
In this situation, the decay factor is r= 1/2, Q_0 = 56.8 g, and Q= 20.5 g after t= 5 years.
We must determine T, the half-life period.

Solution

20.5	$=56.8(1/2)^{5/T}$	
$(1/2)^{5/T}$	$=20.556.8$	Isolate the exponential expression.
$\log(0.5)^{5/T}$	$=\log(20.556.8)$	Take the logarithm of both sides of the equation.
$(5/T)\log(0.5)$	$=\log(20.556.8)$	Apply the power law.
T	$=5\log(0.5)/\log(20.556.8)\approx 3.4$	Rearrange to solve for T.

Therefore, the half-life of the substance is approximately 3.4 years (since t is in years).

Example

Solve $\log_3(x-1) = \log_3(x^2) - \log_3(x+3)$. Identify any restrictions on x.

Solution

$\log_3(x-1) = \log_3(x^2) - \log_3(x+3)$.
We have

$x-1>0$ and $x^2>0$ and $x+3>0$

Thus, $x>1$.
Using the quotient law for logarithms, express the right side of the equation as a single logarithm.

$$\log_3 (x-1) = \log_3 (x^2/(x+3))$$

For these two base 3 logarithms to be equal, the value of the powers (the arguments of each) must be equal.

$$x-1 = x^2/(x+3)$$

Then
$x^2 + 2x - 3 = x^2$
$x = 3/2$
which satisfies $x>1$.
Therefore, $x=3/2$.

Example

Solve $2\log_4 (x) - \log_4 (4x+3) = -1$.

Solution

Since $x>0$ and $4x+3>0$, then $x>0$ and $x>-3/4$. Therefore, $x>0$.

$\log_4(x^2) - \log_4(4x+3) = -1$	Simplify the left side using power and quotient laws.
$\text{Log}_4(x^2/ (4x+3)) = -1$	
$x^2/(4x+3) = 4^{-1}$	Express in exponential form.
$x^2/ (4x+3) = 1/4$	Solve the resulting equation.
$4x^2 = 4x+3$	
$4x^2-4x-3 = 0$	
$(2x-3)(2x+1) =0$	
$x=3/2$ or $x =-1/2$	

However, $x>0$. Therefore, $x=3/2$.
The root $-1/2$ is an extraneous root. It is a root of the rational equation formed when converting to exponential form, but it is not a root to the original logarithmic equation.
We must be careful to watch for extraneous roots when solving logarithmic equations since there are always restrictions placed on the variable.

Example

Algebraically, determine all points of intersection of the two functions
$f(x) = \log_2(2x-2)$ (1)
$g(x) = 5-\log_2(x-1)$ (2)
Illustrate graphically.

Solution

To find the points of intersection, set $f(x) = g(x)$.

$\log_2(2x-2) = 5-\log_2(x-1),\ x>1$	Identify any restrictions on *x*.
$\log_2(2x-2)+\log_2(x-1) = 5$	Collect logarithmic terms on one side.
$\log_2[(2x-2)(x-1)] = 5$	Apply the product law for logarithms.
$(2x-2)(x-1) = 2^5$	Convert to exponential form.

Then solve the resulting equation.

So $x=5,-3$, but $x>1$. Therefore, $x=5$ ($x=-3$ is an extraneous root).

There is one point of intersection at $x=5$.
Find the *y*-coordinate of the point by determining $f(5)$ or $g(5)$.

$$f(5) = \log_2(2(5)-2) = \log_2(8) = 3$$
$$g(5) = 5-\log_2(5-1) = 5-2 = 3$$

Therefore, the two functions intersect at (5, 3).
Transformations can also be applied to $y=\log_2(x)$ to graph $f(x)=\log_2(2x-2)$ and $g(x)=5-\log_2(x-1)$.
To obtain the graph of $f(x) = \log_2(2(x-1))$:

- Apply a horizontal stretch about the *y*-axis by factor of 1/2.
- Apply a horizontal translation 1 unit right.

To obtain the graph of $g(x) = -\log_2(x-1) + 5$:

- Apply a reflection in the *x*-axis.
- Apply a horizontal translation 1 unit right.
- Apply a vertical translation 5 units up.