Chapter 3 Polynomials 2

1. Factors of Quadratic Trinomials of the Type $ax^2 + bx + c$

1.1 Factoring Trinomials by Decomposition

Investigation:

Expand ($2x + 3)(4x + 5)$ $(2x + 3)(4x + 5)$ $= 2x(4x + 5) + 3(4x + 5)$ $= 8x^2 + 10x + 12x + 15$ $= 8x^2 + 22x + 15$	Now, let's go back → **Factoring** $8x^2 + 22x + 15$ $= 8x^2 + 10x + 12x + 15$ → Decompose 22x into 10x + 12x $= 2x(4x + 5) + 3(4x + 5)$ → Group two by two $= (2x + 3)(4x + 5)$ → Factor

How do we know to decompose the 22x into 10x and 12x? Why not 2x and 20x?

Let's compare the coefficients:

$(2x + 3)(4x + 5)$

$= (2x)(4x) + (2x)(5) + (3x)(4) + (3)(5)$

$= (2)(4)x^2 + [(2)(5) + (3)(4)]x + (3)(5)$

$= ax^2 + bx + c$

$a = 2(4)$ $\qquad$ $b = 2(5) + 3(4)$ $\qquad$ $c = 3(5)$

$ac = (2)(4)(3)(5) = (2)(5)(3)(4)$

What is b? b is the sum of 2(5) and 3(4) which are the factors of ac.

To break down the middle term, we need to find two numbers such that the product must be 8 x 15 = 120 and the sum must be 22.

10 x 12 = 120 and 10 + 12 = 22

That is why we break down 22x into 10x and 12x.

The method is called "**decomposition**" because the coefficient of the middle (*x*) term of the trinomial is broken down (or "decomposed") into two smaller numbers.

Example 1: Factor $9x^2 + 21x – 8$.

Step 1: If necessary, rearrange the expression so the terms are in standard form (i.e. exponents decrease from left to right)**.** The example is already in the correct order.

Step 2: We look for two numbers that have a sum equal to the middle term coefficient, and a product equal to the product of the first and last coefficients.

In this case, we want two numbers with a sum of +21 and a product of $9 \times (-8) = -72$

Step 3: We list pairs of factors of the desired product in an orderly fashion; we don't list any pairs where the smaller factor is greater than the square root of the desired product.

We list the factors of 72 (we ignore the minus sign for now) in pairs; we stop when the smaller factor is 8 (since 9 is bigger than the square root of 72). We get this table:

1	72
2	36
3	24
4	18
6	12
8	9

Step 4: If the product we actually want is *positive*, look for a row in the table where the *sum* of the factors equals the middle term coefficient (ignoring its sign); if the product is *negative*, look for a row where the *difference* equals the middle term coefficient (ignoring its sign).

We have a *negative* product, so we look for a *difference* of 21. In the third row of the table, we find the numbers 3 and 24; these numbers have a difference of 21.

Step 5: Adjust the signs of the two numbers: if the product is *positive*, make the signs of *both* numbers the same as the sign of the middle term coefficient; if the product is *negative*, make the sign of the *bigger* number the same as that of the middle term coefficient, and the sign of the smaller number the *opposite* of that of the middle term coefficient.

The product is negative, so we make the sign of the bigger number (24) positive (since the middle term of the original expression has a positive coefficient) and the sign of the smaller number (3) negative. So our numbers are now –3 and 24.

Step 6: Rewrite the original expression, splitting the middle term into the two parts determined above.

We will split the middle term ($21x$) into the two parts determined by the numbers we obtained in the previous step, i.e. $-3x$ and $24x$: $9x^2 + 21x - 8 = 9x^2 - 3x + 24x - 8$

Step 7: Factor the first two terms and the last two terms separately, by finding a common factor in each pair of terms. If there is no common factor, use 1 (or –1).

The first two terms have a common factor of $3x$, and the last two have a common factor of 8. So we get: $3x(3x – 1) + 8(3x – 1)$

Step 8: The second (bracketed) factor in each of the resulting terms *must* be the same; if not, we've made a mistake. Take out this common factor from each term, and we have the final result.

$$\mathbf{9x^2 + 21x – 8 = (3x – 1)(3x + 8)}$$

Example 2: Factor $2x^2 + x – 6$.

I have a = 2, b = 1, and c = –6, so ac = (2)(–6) = –12. So I need to find factors of –12 that add up to +1. The pairs of factors for 12 are 1 and 12, 2 and 6, and 3and 4. Since –12 is negative, I need one factor to be positive and the other to be negative (because positive times negative is negative). This means that I'll want to use the pair "3 and 4", and I'll want the 3 to be negative, because –3 + 4 = +1.

$\mathbf{2x^2 + x – 6}$	As we can see, it can be grouped into two ways where both will produce the same answer.	$\mathbf{2x^2 + x – 6}$
$\mathbf{= 2x^2 – 3x + 4x – 6}$		$\mathbf{= 2x^2 + 4x – 3x – 6}$
$\mathbf{= x(2x – 3) + 2(2x – 3)}$		$\mathbf{= 2x(x + 2) – 3(x + 2)}$
$\mathbf{= (2x – 3)(x + 2)}$		$\mathbf{= (x + 2)(2x – 3)}$

Practice: Factor $2x^2 - 5x + 3$

$2x^2 - 5x + 3$	**Prod = 2 x 3 = 6**	**-2 x (-3) = 6**
$= 2x^2 - 2x – 3x + 3$	**Sum = -5**	**-2 + (-3) = -5**
$= 2x(x – 1) – 3(x – 1)$		
$\mathbf{= (x - 1)\,(2x - 3)}$		

1.2 Factoring Trinomials by Criss-Cross

Example 1: Factor $7x^2 + 9x + 2$
Factors of the first term $7x^2$ are $7x$ and x
Factors of the last term 2 are 1 and 2

Place the factors one above the other as shown below:

$$\begin{array}{ll} 7x & +1 \\ x & +2 \\ \hline & 15x \end{array}$$

Cross multiply: $7x \times 2 + x \times 1 = 14x + x = 15x$. We reject this pair as the middle term is $9x$.

Always keep the left side and flip the right side.

$$\begin{array}{ll} 7x & +2 \\ x & +1 \\ \hline & 9x \end{array}$$

Cross multiply: $7x \times 1 + x \times 2 = 7x + 2x = 9x$. We accept this pair as the middle term is 9x.

The solution is read across: $7x^2 + 9x + 2 = (7x + 2)(x + 1)$

2. Use of a Common Factor

If a quadratic trinomial has a common factor, take it out and place it in front of the brackets. Then use the cross-multiplication method to factorize the quadratic trinomial.

Example 1: Factor $12x^2 - 14x - 6$

Solution:

$12x^2 - 14x - 6$ {Take out the common factor 2}

$= 2(6x^2 - 7x - 3)$

Factors of $6x^2$ are $6x \times x$, $2x \times 3x$.

Factors of -3 are 1×-3, -1×3.

Place the linear factors one above the other.

$$\begin{array}{lcccc} 6x & +1 & -3 & -1 & +3 \\ x & -3 & +1 & +3 & -1 \\ \hline & -17x & 3x & 17x & -3x \end{array}$$

$$\begin{array}{lcc} 2x & +1 & -3 \\ 3x & \underline{-3} & \underline{+1} \\ & -3x & -7x \end{array}$$

Middle term

$$\therefore\ 6x^2 - 7x - 3 = (2x-3)(3x+1)$$

$$\text{So, } 12x^2 - 14x - 6 = 2(2x-3)(3x+1)$$

Check:

$$\begin{aligned} 2(2x-3)(3x+1) &= 2[2x(3x+1) - 3(3x+1)] \\ &= 2[6x^2 + 2x - 9x - 3] \\ &= 2(6x^2 - 7x - 3) \\ &= 12x^2 - 14x - 6 \end{aligned}$$

Note: The common factor 2 was taken out first to make the factorization simpler.

Example 2: Factor $18a^2 + 3a - 6$

Solution:

$$18a^2 + 3a - 6 \qquad \{\text{Take out the common factor } 3\}$$

$$= 3(6a^2 + a - 2)$$

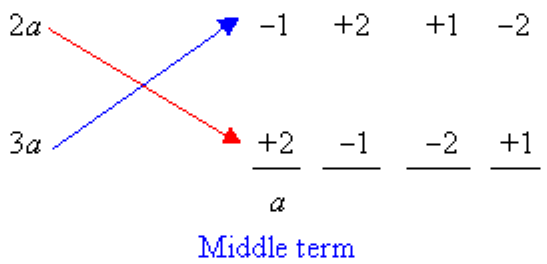
Factors of $6a^2 = 6a \times a,\ 2a \times 3a$

Factors of $-2 = -1 \times 2,\ 1 \times -2$

Place the linear factors one above the other

$$\begin{array}{lcccc} 6a & -1 & +2 & +1 & -2 \\ a & \underline{+2} & \underline{-1} & \underline{-2} & \underline{+1} \\ & 11a & -4a & -11a & 4a \end{array}$$

Now try the factors 2*a* and 3*a*.



$\therefore\ 6a^2 + a - 2 = (2a-1)(3a+2)$

So, $18a^2 + 3a - 6 = 3(2a-1)(3a+2)$

Check:

$$\begin{aligned}3(2a-1)(3a+2) &= 3[2a(3a+2)-(3a+2)]\\ &= 3[6a^2+4a-3a-2]\\ &= 3(6a^2+a-2)\\ &= 18a^2+3a-6\end{aligned}$$

Practice in class: Place the correct *signs* to give the middle term.

a) $2x^2 + 7x - 15 = (2x - 3)(x + 5)$
b) $2x^2 - 7x - 15 = (2x + 3)(x - 5)$

Note: When the constant term is negative, as in parts a), b), c), then the <u>signs</u> in each factor will be different. But when the constant term is positive, as in part d), the signs will be the same. Usually, however, that happens by itself.

Practice in class: Factor these trinomials.

a) $3x^2 + 8x + 5 = (3x + 5)(x + 1)$
b) $3x^2 + 16x + 5 = (3x + 1)(x + 5)$
c) $2x^2 - 7x + 5 = (2x - 5)(x - 1)$
d) $2x^2 - 11x + 5 = (2x - 1)(x - 5)$
e) $3x^2 + x - 10 = (3x - 5)(x + 2)$

3. Sum and Differences of Cubes

In this section, formulas and procedures will allow you to factor expressions in the form $x^3 - y^3$ and also $x^3 + y^3$. Recall from previous sections that $x^2 - y^2 = (x - y)(x + y)$ and that $x^2 + y^2$ cannot be factored. Begin with the multiplication problems:

a) $(x - y)(x^2 + xy + y^2) = x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 = x^3 - y^3$
b) $(x + y)(x^2 - xy + y^2) = x^3 - x^2y + xy^2 + x^2y - xy^2 + y^3 = x^3 + y^3$

This derives the formulas, known as the "sum and difference of cubes formulas."

SUM AND DIFFERENCE OF CUBES FORMULAS

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) \qquad x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Translated into words, this means the sum or difference of two cubes can be factored into the product of a binomial times a trinomial. Begin by taking the cube root of the perfect cubes.

In the difference formula, the binomial is "the first minus the second." Then use this binomial to build the trinomial that follows: take the "square of the first" plus the "product of the first and second" plus the "square of the second."

The sum formula is similar, in that the formula begins with the binomial that is "the first plus the second". Then use this binomial to build the trinomial that follows, which except for one sign is the same as the difference formula: take the "square of the first" minus the "product of the first and second" plus the "square of the second."

These formulas are really easy to remember. Notice that the $x^3 - y^3$ formula begins with $(x - y)$. The $x^3 + y^3$ formula begins with $(x + y)$. Next, notice that the trinomial factor in both formulas is the same except for one sign. This trinomial factor in each formula involves the "square of the first," the "product of the two," and the "square of the second." The first sign in the trinomial is the opposite of the sign of the binomial, and the last sign is always positive. Finally, you will *never* be able to factor the resulting trinomial (by ordinary trinomial methods), so you need not even try (at this level).

Before getting into the exercises, be sure to be familiar with the perfect cubes: $1^3 = 1$; $2^3 = 8$; $3^3 = 27$; $4^3 = 64$; and $5^3 = 125$. Be able to recite these from memory: 1, 8, 27, 64, 125.

Example 1: Factor completely, using the sum and difference of cubes formulas.

1. $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$
2. $x^3 + 125 = (x + 5)(x^2 - 5x + 25)$

Practice: Factor the following expressions.

1. $8x^3 - 27y^3$
2. $125y^3 - 8x^3$
3. $8x^3 + 1$
4. $125y^3 - 1$
5. $5x^4 + 40x$
6. $10x^5y + 80x^2y^4$
7. $3x^5y^5 - 81x^2y^2$