Counting and Patterns

1. Counting

Counting is the action of finding the number of elements of nature numbers.

The traditional way of counting consists of continually increasing a (mental or spoken) counter by a unit for every element of the nature numbers, in some order, while marking (or displacing) those elements to avoid visiting the same element more than once, until no unmarked elements are left; if the counter was set to one after the first object, the value after visiting the final object gives the desired number of elements.

Counting sometimes involves numbers other than one.

For example, when counting money, counting out change, when "counting by twos" (2, 4, 6, 8, 10, 12…) or when "counting by fives" (5, 10, 15, 20, 25…).

2. Counting using tally marks.

Tally marks are a quick way of keeping track of numbers in groups of five.

One vertical line is made for each of the first four numbers; the fifth number is represented by a diagonal line across the revious four.

1	𝍷	6	𝍸𝍷
2	𝍷𝍷	7	𝍸𝍷𝍷
3	𝍷𝍷𝍷	8	𝍸𝍷𝍷𝍷
4	𝍷𝍷𝍷𝍷	9	𝍸𝍷𝍷𝍷𝍷
5	𝍸	10	𝍸𝍸

3. Number Patterns

A **sequence** is a pattern of numbers that are formed in accordance with a definite rule.
We can often describe number patterns in more than one way. To illustrate this, consider the following sequence of numbers {1, 3, 5, 7, 9, …}.

Clearly, the first term of this number pattern is 1; and the terms after the first term are obtained by adding 2 to the previous term. We can also describe this number pattern as a set of odd numbers.

Alternatively, we can represent a number pattern by using special symbols. For the number pattern under consideration, let t_n represent the *n*th term.

∴ First term: $t_1 = 1$
Second term: $t_2 = 3$
Third term: $t_3 = 5$
Fourth term: $t_4 = 7$
Fifth term: $t_5 = 9$
… …

By trial and error, we find that:

$t_1 = 1 = 2\times1-1$

$t_2 = 3 = 2\times2-1$

$t_3 = 5 = 2\times3-1$

$t_4 = 7 = 2\times4-1$

$t_5 = 9 = 2\times5-1$

By observation, we notice that we can describe this number pattern by the rule

$t_n = 2n-1$

where t_n is *n*th term of the sequence.

4. Formula and Tables

A table of values can be generated from the rule

$t_n = 2n-1$

as shown below.

When $n = 1$, $t = 2\times1-1 = 2-1 = 1$

When $n = 2$, $t = 2\times2-1 = 4-1 = 3$

When $n = 3$, $t = 2\times3-1 = 6-1 = 5$

When $n = 4$, $t = 2\times4-1 = 8-1 = 7$

When $n = 5$, $t = 2\times5-1 = 10-1 = 9$

n	1	2	3	4	5
t	1	3	5	7	9

5. Finding the Algebraic Rule

We use algebra to study rules that describe the behavior of everyday things, for example, the behavior of the height of a ball when it is thrown upward or the amount outstanding for a loan after a number of regular repayments.

By finding a pattern in observed values (i.e. measurements), we are often able to discover a rule that allows us to make accurate predictions.

6. Using a Difference Pattern

When we try to discover an algebraic rule for ordered pairs, we can find the difference between two successive values of *y*. This allows us to find a rule as illustrated below.
Consider the following table.

x	1	2	3	4
y	5	7	9	11

Pattern difference: 2 2 2

Difference between 5 and 7

Difference between 9 and 11

Difference between 7 and 9

We notice that the values of x increase by just one at a time and the difference between the successive values for y is 2.

So, the rule starts off with $y = 2x$. Will this give a correct answer from the table? Let us check.

When $x = 1, y = 2 \times x = 2 \times 1 = 2$

The answer is no. From the table, when $x = 1$ the value of y should be 5. How do we change our answer from 2 to 5? We should add 3.

$\therefore$ The rule becomes $y = 2x + 3$

Check the rule to see if it is correct:

When $x = 1,\ y = 2 \times 1 + 3 = 2 + 3 = 5$

When $x = 2,\ y = 2 \times 2 + 3 = 4 + 3 = 7$

When $x = 3,\ y = 2 \times 3 + 3 = 6 + 3 = 9$

When $x = 4,\ y = 2 \times 4 + 3 = 8 + 3 = 11$

So, our rule $y = 2x + 3$ is correct.

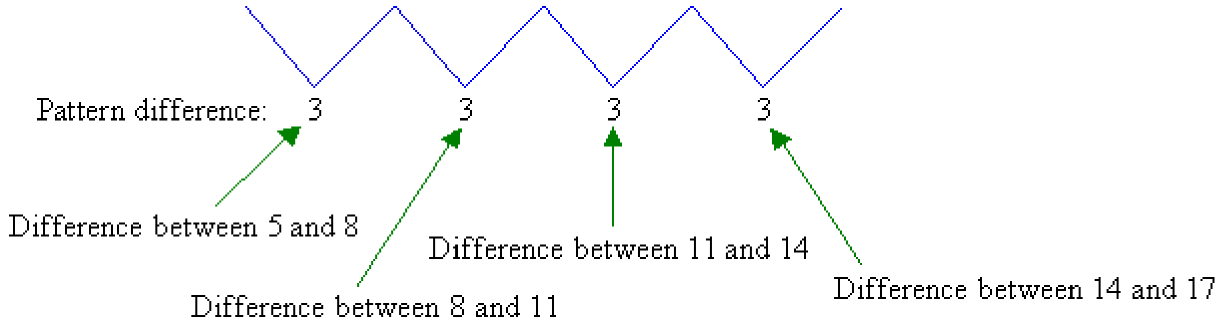
Example: Discover the rule for the following table of values:

x	1	2	3	4	5
y	5	8	11	14	17

Solution:

In the given table, the x-values increase by 1 for each ordered pair.

x	1	2	3	4	5
y	5	8	11	14	17



Find the difference between the successive values of *y*. That is:
$8 - 5 = 11 - 8 = 14 - 11 = 17 - 14 = 3$

The difference between successive values of *y* is always 3. So, the rule is of the form
$y = 3 \times x + \square$

To determine the value of $\square$, we check the rule for $x = 1$.
When $x = 1, y = 5$ and $y = 3 \times 1 + \square = 3 + \square$
$\therefore 5 = 3 + \square$, $5 - 3 = 3 + \square - 3$, $\therefore \square = 2$.
Hence the rule is $y = 3x + 2$

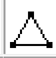
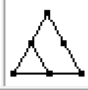
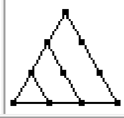
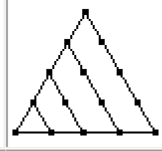
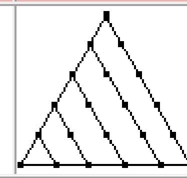
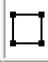
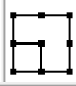
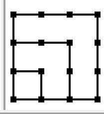
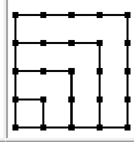
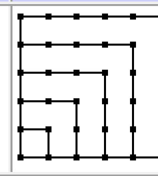
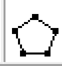
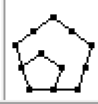
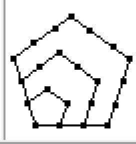
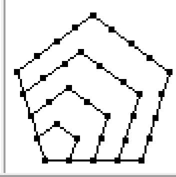
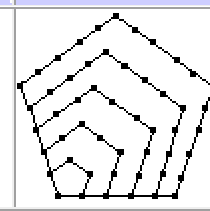
Check the rule to see if it is correct:
When $x = 2$, $y = 3 \times 2 + 2 = 6 + 2 = 8$
When $x = 3$, $y = 3 \times 3 + 2 = 9 + 2 = 11$
When $x = 4$, $y = 3 \times 4 + 2 = 12 + 2 = 14$
When $x = 5$, $y = 3 \times 5 + 2 = 15 + 2 = 17$
So, our rule is correct.

7. Polygonal Numbers

Polygonal Numbers are really just the number of vertexes in a figure formed by a certain polygon. The first number in any group of Polygonal Numbers is always 1, or a point. The second number is equal to the number of vertexes of the polygon.

A formula that will generate the n^{th} x-gonal number (for example: the 2^{nd} 3-gonal, or triangular number) is:

$$\frac{n^2 - n}{2} \times (x - 2) + n$$

Type	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**
Triangular						
Value	1	3	6	10	15	21
Square						
Value	1	4	9	16	25	36
Pentagonal						

8. Factorials
One the most important uses of factorials are problems in which total outcomes must be determined. There is an easier way that can summarize this set of multiplication statements: 100 × 99 × 98 × 97 × 96 × ... × 5 × 4 × 3 × 2 × 1. We just have to write 100!. A common mathematical notation in counting is the **factorial notation (!).**

The factorial of some positive integer, written as n! is represented as: (n) × (n − 1) × (n − 2) × ... × (2) × (1)

Example: 4! = 4 × 3 × 2 × 1 = 24

Essentially all we need to do is multiply every number from 1 to n to find n!.

But factorials have some restrictions and special cases that we must look at:
• 0! is a special case that we must remember. We can say that **0! = 1.**
• We can only find the factorial of integers (whole numbers). For example 3.14! or π! is not possible and therefore has no answer.
• We cannot find the factorial of negative integers. Therefore we cannot get an answer for (−5)!

Example: Simplify $\frac{6!}{3!}$
Solution: Without using our calculator and expanding the factorials we get:

$$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

We can see that in both the numerator and denominator, there exists a "3×2×1". So let's cancel them out. This leaves us with: $6 \times 5 \times 4 = 120$

Example: Simplify $\frac{6!\times 3!}{8!}$
Solution: Without using our calculator and expanding the factorials we get:

$$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

We can see that in both the numerator and denominator, there exists a "$6 \times 5 \times 4 \times 3 \times 2 \times 1$". So let's cancel them out. This leaves us with:

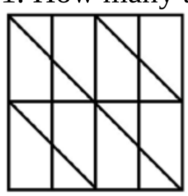
$$\frac{3 \times 2 \times 1}{8 \times 7}$$

Let's cancel out a 2 from the numerator and denominator:

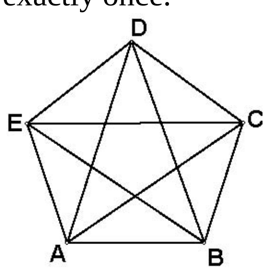
$$\frac{3 \times 1}{4 \times 7} = \frac{3}{28}$$

Questions in class

1. How many triangles can be found in the figure shown below?



2. How many different ways can one travel from A to E in the figure shown? A path must travel in a straight line turning only at the points A, B, C, D, E and must go through every lettered point exactly once.



3. How many different 4 digit numbers can be made by ordering the digits 1, 2, 3, 3?

4. Find the number of two-digit positive integers whose digits total 7.

5. How many whole numbers between 99 and 999 contain exactly one 0?

6. In Canada we use coins of the following denominations: \$0.01, \$0.05, \$0.10, \$0.25, \$1 and \$2. In how many ways can we obtain a sum of \$3.15 using exactly 10 coins?

7. Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?

8. How many four-digit whole numbers are there such that the leftmost digit is odd, the second digit is even, and all four digits are different?

9. How many two-digit numbers have digits whose sum is a perfect square?

10. In our volleyball league, the numbers on the jerseys may be formed using one or two digits chosen from 1, 2, 3, 4, 5. Numbers with repeated digits are allowed but the single digit numbers are not allowed. What is the total number of possible jersey numbers for each team?