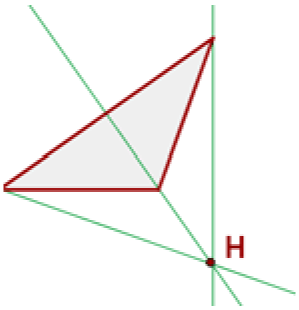
Geometry 3

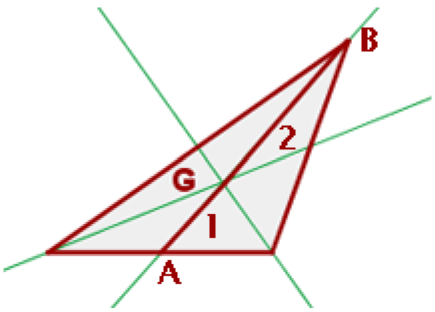
1. **Lines in Triangles**

Orthocenter



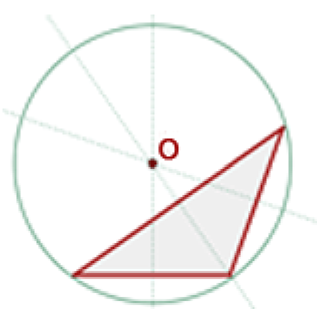
The **orthocenter** is the **point of intersection** of the **three heights** of a triangle. A **height** is each of the **perpendicular lines** drawn from one **vertex to the opposite side** (or its extension).

Centroid



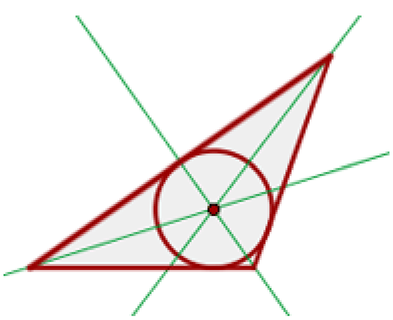
The **centroid** is the point of intersection of the **three medians**. A **median** is each of the **straight lines** that joins the **midpoint** of a side with the **opposite vertex.** The **centroid** divides each **median** into **two segments**, the segment joining the centroid to the vertex multiplied by two is equal to the length of the line segment joining the midpoint to the opposite side. **BG = 2GA**

Circumcenter



The **circumcenter** is the **point of intersection** of the **three perpendicular bisectors**. A **perpendicular bisectors** of a triangle is each line drawn perpendicularly from its midpoint.
The **circumcenter** is the **center** of the **circle inscribed** in the triangle.

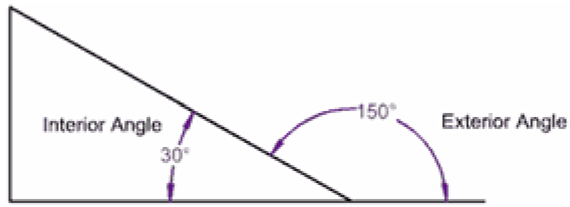
Incenter



The **incenter** is the point of intersection of the three angle bisectors. The angle bisectors of a triangle are each one of the lines that divide an angle into two equal angles.
The **incenter** is the center of the **circle inscribed** in the triangle.

2. Interior and Exterior Angle

An Interior Angle is an angle inside a shape.
The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.



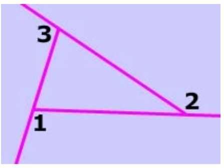
Note: If you add up the Interior Angle and Exterior Angle you get a straight line, 180°.

1) Exterior Angles of Polygons

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.

The sum of the measures of the exterior angles of a polygon, one at each vertex, is 360°.

For examples:
A Polygon is any flat shape with straight sides.



The Exterior Angles of a Polygon add up to 360°.
$\angle 1 + \angle 2 + \angle 3 = 360^0$

2) Interior Angles sum of Polygons

An interior angle of a regular polygon with n sides is $\frac{(n-2)\times 180}{n}$.

Example:

To find the measure of an interior angle of a regular octagon, which has 8 sides, apply the formula above as follows:
((8-2) × 180) /8 = 135°

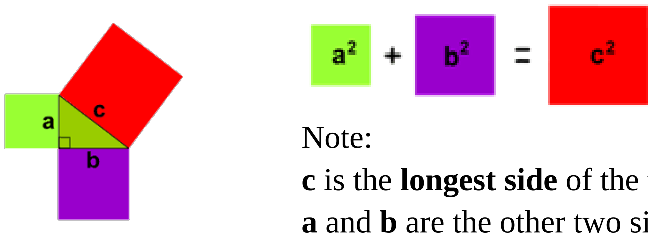
The sum of the measures of the interior angles of a polygon with *n* sides is (n-2)180.

3. Pythagorean Triples

A "Pythagorean Triple" is a set of positive integers, **a**, **b** and **c** that fits the rule:
$a^2 + b^2 = c^2$

Triangles

And when you make a triangle with sides **a**, **b** and **c** it will be a right angled triangle:



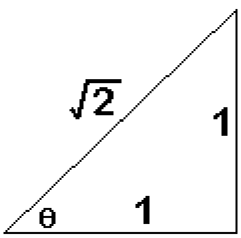
Note:
c is the **longest side** of the triangle, called the "hypotenuse"
a and **b** are the other two sides

4. Special Triangles

There are two special triangles in studies of triangles. One is the 30°-60°-90° triangle. The other is the isosceles right triangle. They are special because, with simple geometry, we can know the ratios between their sides.

1) 45°-45°-90° Triangle

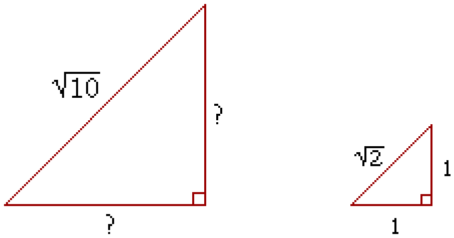
Theorem. *In a 45°-45°-90° triangle the sides are in the ratio 1 : 1 : $\sqrt{2}$.*



We choose a length of one unit for the matching sides because this is simplest, and then we get the $\sqrt{2}$ value by using the Pythagorean Theorem.

Example: In an isosceles right triangle, the hypotenuse is $\sqrt{10}$ inches. How long are the legs?

Sketch the triangles and place the ratio numbers.

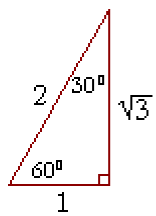


How has the side corresponding to $\sqrt{2}$ been multiplied?

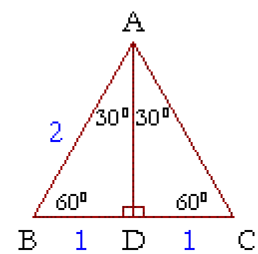
According to the rule for multiplying radicals, it has been multiplied by $\sqrt{5}$. Therefore, all the sides will be multiplied by $\sqrt{5}$. The legs are $1 \cdot \sqrt{5} = \sqrt{5}$.

2) 30°-60°-90° Triangle

Theorem. *In a 30°-60°-90° triangle the sides are in the ratio 1 : 2 : $\sqrt{3}$.*



Here is the proof that in a 30°-60°-90° triangle the sides are in the ratio 1 : 2 : $\sqrt{3}$. It is based on the fact that a 30°-60°-90° triangle is *half* of an equilateral triangle.



Proof: Draw the equilateral triangle ABC. Then each of its equal angles is 60°.

Draw the straight line AD bisecting the angle at A into two 30° angles.
Then AD is the perpendicular bisector of BC. Triangle ABD therefore is a 30°-60°-90° triangle.

Now, since BD is equal to DC, then BD is half of BC.

This implies that BD is also half of AB, because AB is equal to BC. That is,

BD : AB = 1 : 2

From the Pythagorean Theorem, we can find the third side AD:

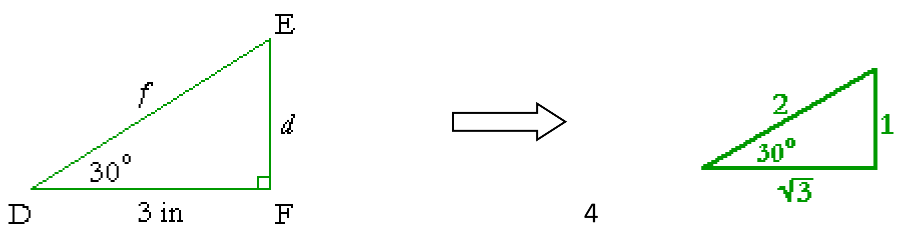
$AD^2 + 1^2 = 2^2$

$AD^2 \quad = 4 - 1 = 3$

$AD \quad = \sqrt{3}$.

Therefore in a 30°-60°-90° triangle the sides are in the ratio 1 : 2 : $\sqrt{3}$.

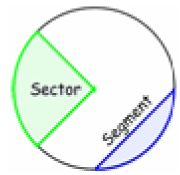
Example: In the right triangle DFE, angle D is 30°, and side DF is 3 inches. How long are sides *d* and *f* ?



Draw a similar triangle in the same orientation. Then see that the side corresponding to $\sqrt{3}$ was multiplied by $\sqrt{3}$. Therefore, each side will be multiplied by $\sqrt{3}$. Side *d* will be $1 \cdot \sqrt{3} = \sqrt{3}$. Side *f* will be $2\sqrt{3}$.

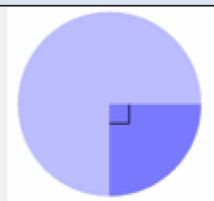
5. Circle Sector and Segment

Slices
There are two main "slices" of a circle:
The "pizza" slice is called a **Sector**.
And the slice made by a chord is called a **Segment**.



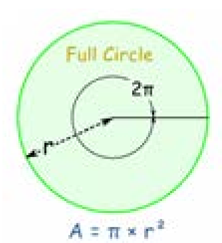
1) Common Sectors

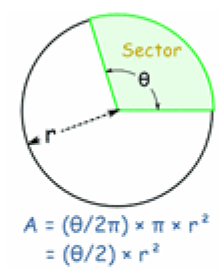
The Quadrant and Semicircle are two special types of Sector:

Quarter of a circle is called a **Quadrant**	Half a circle is called a **Semicircle.**
	

2) Area of a Sector

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.
Note: I am using radians for the angles.





This is the reasoning:

- A circle has an angle of 2π and an Area of: πr^2
- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

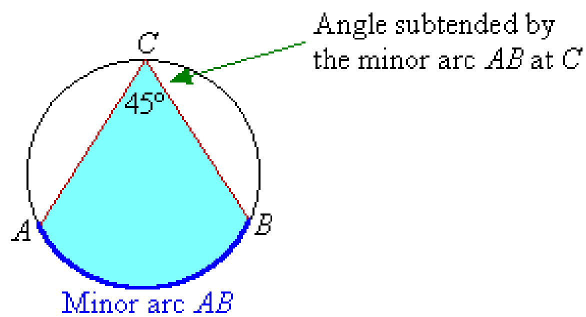
Area of Sector = $\frac{1}{2} \times \theta \times r^2$ *(when θ is in radians)*
Area of Sector = $\frac{1}{2} \times (\theta \times \pi/180) \times r^2$ *(when θ is in degrees)*

6. Angle at the Circumference

If the end points of an arc are joined to a third point on the circumference of a circle, then an angle is formed. It is called the **inscribed angle**.

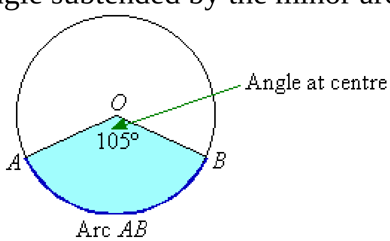
For example, the minor arc *AB* subtends an angle of 45º at *C*. The angle *ACB* is said to be the angle subtended by the minor arc *AB* (or simply arc *AB*) at *C*.

The angle *ACB* is an angle at the circumference standing on the arc *AB*.



7. Angle at the Centre

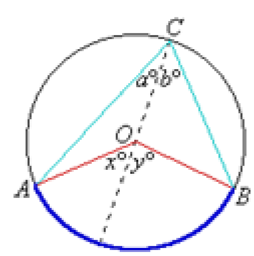
If the end points of an arc are joined to the centre of a circle, then an angle is formed.
For example, the minor arc *AB* subtends an angle of 105º at *O*. The angle *AOB* is said to be the angle subtended by the minor arc *AB* (or simply arc *AB*) at the centre *O*.



The angle *AOB* is an angle at the centre *O* standing on the arc *AB*, it is also called the **central angle**.

Angle at Centre Theorem

Use the information given in the diagram to prove that the angle at the centre of a circle is twice the angle at the circumference if both angles stand on the same arc.



Given:
$\angle AOB$ and $\angle ACB$ stand on the same arc; and O is the centre of the circle.
To prove:
$\angle AOB = 2\angle ACB$

Proof:

From ΔOAC, $x = a + a$ {Exterior angle of a triangle}

$\therefore x = 2a$ ... (1)

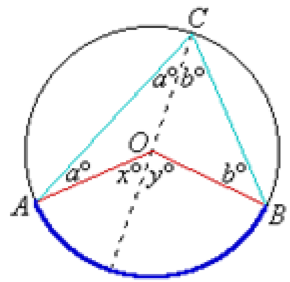
From ΔOBC, $y = b + b$ {Exterior angle of a triangle}

$\therefore y = 2b$ ... (2)

Adding (1) and (2) gives:

$x + y = 2a + 2b = 2(a + b)$ $\therefore \angle AOB = 2\angle ACB$

As required.



In general: The angle at the centre of a circle is twice the angle at the circumference if both angles stand on the same arc. This is called the **Angle at Centre Theorem**.

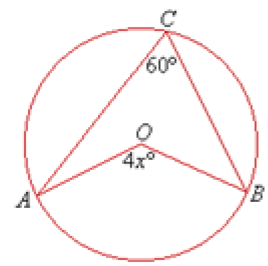
Example 1: Find the value of unknown in the following circle centred at *O*.

Solution:

$4x = 2 \times 60$ {Angle at Centre Theorem}

$4x = 120$

$\frac{4x}{4} = \frac{120}{4}$ $x = 30$



8. Angle in a Semi-Circle

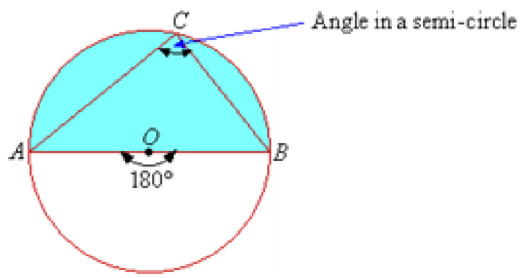
Let $\angle AOB = 180°$.

Then $\angle ACB$ is in a semi-circle.

By the Angle at Centre Theorem, we have:

$\angle AOB = 2\angle C$ $2\angle C = 180°$

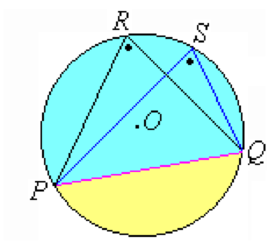
$\therefore \angle C = 90°$



In general: The angle in a semi-circle is a right angle.

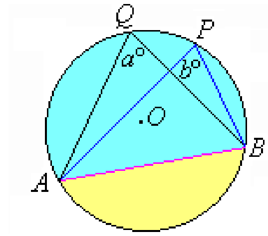
9. Angles in the Same Segment

In the left diagram**,** $\angle PRQ$ and $\angle PSQ$ are in the major segment. So, we say that angle *PRQ* and angle *PSQ* are in same segments.



Theorem

Use the information given in the diagram to prove that the angles in the same segment of a circle are equal. That is, *a* = *b*.

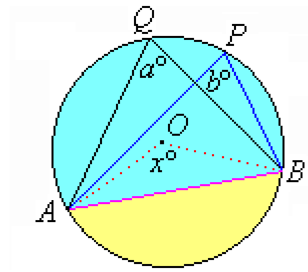


Given: $\angle APB$ and $\angle AQB$ are in the same segment; and *O* is the center of the circle.
To prove: $\angle APB = \angle AQB$.

Construction: Join *O* to *A* and *B*.
Proof:

Let $\angle AOB = x°$.

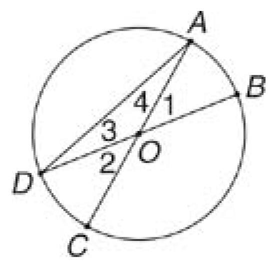
Clearly, $x = 2a$	{Angle at Centre Theorem}
$x = 2b$	{Angle at Centre Theorem}
$\therefore\ 2a = 2b$	{Transitive}
$\therefore\ a = b$	As required.



In general: Angles in the same segment of a circle are equal.

Also, it is true that for all equal inscribed angles in the same circle, subtending arcs must be equal.

Given angle 3 = angle 4, then we can conclude that arc AB = arc CD

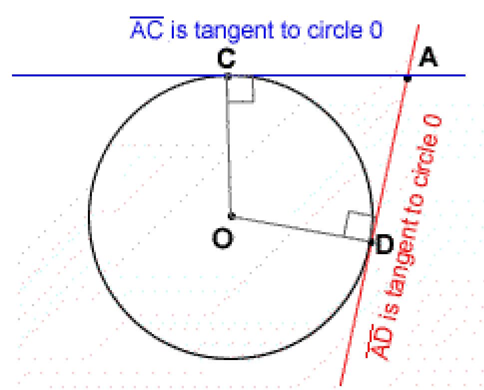


10. Tangent to a Circle

A tangent to a circle is perpendicular to the radius at the point of tangency.

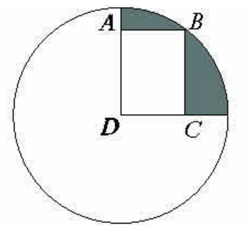
- This is a very useful property when the radius that connects to the point of tangency is part of a right angle, because the trigonometry and the Pythagorean Theorem apply to right triangles.

- A **tangent** intersects a circle at one point.
  - C and **D** are the **points of tangency** to circle O
  - AC and AD are tangent to circle O.
- **Perpendicular** means at right angles (meet at 90º).
  - OC and OD are radii of the circle O.
  - OC is perpendicular to AC.
  - OD is perpendicular to AD.

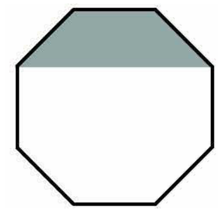


Questions in class

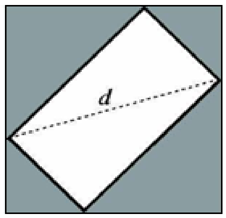
1. ABCD is a rectangle, *D* is the centre of the circle, and *B* is on the circle. If *AD* = 4 and *CD* = 3, then what is the area of the shaded region?



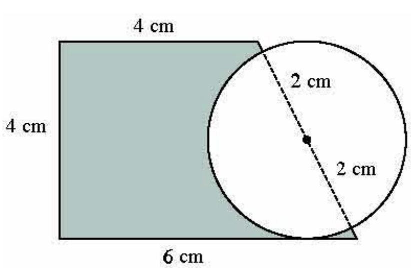
2. The figure to the right is a regular octagon. What fraction of its area is shaded?



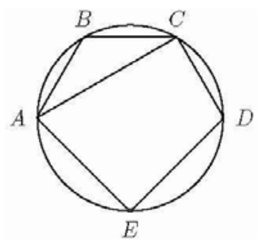
3. An isosceles right triangle is removed from each corner of a square piece of paper, so that a rectangle remains. The removed triangles are shown as gray in the picture below. Find the length of the diagonal d, if the sum of the areas of the triangles cut off is 200 square units.



4. The figure is a right trapezoid with side lengths 4 cm, 4 cm and 6 cm as labeled. The circle has radius 2 cm. What is the area, in cm^2, of the shaded region?

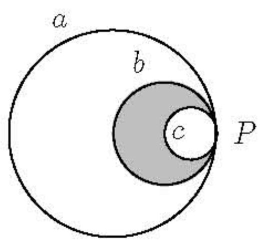


5. A (non-regular) pentagon ABCDE is inscribed in a circle, as shown. Segment AD is a diameter of the circle, sides AB, BC, and CD are equal, and sides AE and DE are equal. What is the angle CAE in degrees?



6. Points *A*, *B* and *C* are collinear. Point *B* is the midpoint of the line segment *AC.* Point *D* is a point not collinear with the other points for which *DA* = *DB* and *DB* = *BC* = 10. Then what is the DC?

7. Three circles, a, b, and c, are tangent to each other at point P, as shown. The center of b is on c and the center of a is on b. what is the ratio of the area of the shaded region to the total area of the unshaded regions enclosed by the circles?



8. The corners of a square of side x are cut off so that a regular octagon remains. What is the length of each side of the resulting octagon?

9. In the triangle shown, ∠BAD = α, AB = AC and AD = AE. Find ∠CDE in terms of α.