Lesson 6: Unit 4 – *Rational Functions (2)*

Many real life problems can be solved using rational equations.

- Business situations concerning average revenue or average cost.
- Distance-time-velocity problems.
- Concentration conversions in chemistry.
- Work related situations involving productivity rates.

Example

Determine the point(s) of intersection of the functions

$$f(x)=\frac{2}{x^2-1} \text{ and } g(x)=\frac{x}{x+1}$$

and illustrate the situation graphically.

Solution

$\frac{2}{x^2-1}=\frac{x}{x+1} \leftrightarrow \frac{2}{(x+1)(x-1)}=\frac{x}{x+1}$

2 = x(x-1)

x^2-x-2 = 0 $\leftrightarrow$ (x-2)(x+1) = 0

x = -1, 2.

However, $x\neq\pm1$; therefore, x=2 is the only solution.

The solution $x=-1$ is called an **extraneous root** since it is a root of the quadratic formed when simplifying the equation, but it is not a root to the original equation.
There is a point of intersection at x=2.
Substituting this into (1),

$$y=\frac{2}{x^2-1}= 2/3$$

Checking our work by substituting into (2),

$$y=\frac{x}{x+1}= 2/3$$

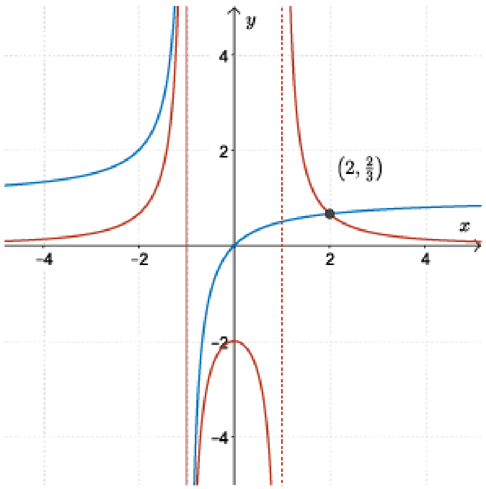
we get the same value obtained from equation (1).

Therefore, the point of intersection of the two functions is (2, 2/3).

From the graph, we can see that the two functions become significantly close near $x=-1$.
Both functions have a vertical asymptote at $x=-1$.
They do not intersect near or at $x=-1$.
There is a point of intersection at x=2.

This example illustrates that the solution to the equation $\frac{2}{x^2-1} = \frac{x}{x+1}$ can be obtained graphically, using technology, by finding the point of intersection of the two functions

$$f(x) = \frac{2}{x^2-1} \text{ and } g(x) = \frac{x}{x+1}$$



Example

Pedro left home at noon and cycled 72 km to his family cottage. His sister, Alexandra, left home on her bike at 1 PM and arrived at the cottage 12 minutes after Pedro. If she cycles, on average, 3 km/h faster than Pedro, how long did it take Pedro to make the trip, and what was his average speed?

Solution

Let *t* represent the time it took, in hours, for Pedro to cycle to the cottage.

	Distance (km)	Cycling Time (h)	Speed (km/h)
Pedro			
Alexandra			

Note: 12 minutes =1260 hours =0.2 hours.
Alexandra can cycle 3 km/h faster than Pedro.

Pedro's speed +3 = Alexandra's speed

$\frac{72}{t} + 3 = \frac{72}{t-0.8}$, $t > 0.8$

$3t^2 - 2.4t - 57.6 = 0$

$t = \frac{2.4 \pm \sqrt{(-2.4)^2 - 4(3)(57.6)}}{2(3)}$

So, $t=4.8$ or $t=-4$, but $t>0.8$, so $t=4.8$ hours and $t=-4$ is an extraneous root.
Thus, Pedro's speed is 724.8=15 km/h.
Therefore, it took Pedro 4 hours and 48 minutes, at an average speed of 15 km/h, to bike to his family cottage.

Example

Solve $\frac{3}{x+1} > -\frac{2}{x-2}$

Algebraic Solution

- Collect all terms to the left side of the inequality and 0 to the right side.
- Simplify the expression on the left side by finding a common denominator and adding the terms.
- Once simplified, factor the numerator, if possible, always leaving the denominator in factored form.
- Create an interval table and identify the sign of each factor in the rational expression within each interval.
- The zero value of each factor in the numerator and denominator indicates when a sign change may occur for that factor.

$$\frac{3}{x+1} > -\frac{2}{x-2}$$

$$\frac{3}{x+1} + \frac{2}{x-2} > 0$$

$$\frac{3(x-2)}{x+1} + \frac{2(x+1)}{x-2} > 0$$

$$\frac{5x-4}{(x+1)(x-2)} > 0$$

$x=-1$ $\quad x=\frac{4}{5}$ $\quad x=2$

	$x<-1$	$-1<x<\frac{4}{5}$	$\frac{4}{5}<x<2$	$x>2$
$5x-4$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$\frac{5x-4}{(x+1)(x+2)}$	$-$	$+$	$-$	$+$

Thus, $\frac{5x-4}{(x+1)(x-2)} > 0$ when $-1<x<4/5$ or $x>2$.

The solution to the inequality is then $\{x \mid -1<x<4/5 \text{ or } x>2,\ x \in R\}$.

Using interval notation the solution is given by $x \in (-1, 4/5) \cup (2, \infty), x \in R$.

Example: Application

An open top box is to be made with the following conditions:

- The length of the base is 10 cm longer then the width of the base.
- The height of the box must be greater than the width.
- The volume must be 375 cm3.

Determine specific restrictions, if any, on the width, length, and height of the box.

Solution

Let x represent the width of the box in centimetres.
The length of the box will be $x+10$ cm. Let h represent the height of the box in centimetres.
Note: $x>0$, $h>0$.
If the volume is 375 cm3, then

$$x(x+10)(h) = 375$$

$$h = 375/x(x+10)$$

Now, the height of the box must be greater than the width, so $h>x$.

Substituting the expression for h, we determine the conditions on the width x such that

$$375/x(x+10) > x$$

Since $x>0$, then $x(x+10)>0$. We can then multiply both sides of the inequality by $x(x+10)$ without changing the inequality condition.
$x^3 + 10x^2 - 375 < 0$
Use the factor theorem to factor this cubic. We may use the rational roots theorem to determine test values. These are any integer factors of 375.

Observe that if we let $P(x)=x^3+10x^2-375$, then $P(5)=0$ and so $x-5$ is a factor.
$(x-5)(x^2+15x+75)<0$
The quadratic factor has no real roots ($b^2-4ac=-75$).
Hence, $(x^2+15x+75)>0$ for all $x>0$.
Therefore, $x-5<0$ since $(x-5)(x^2+15x+75)<0$.
Thus, $0<x<5$ and $10<x+10<15$.

Therefore, the width of the box must be between 0 cm and 5 cm, and the length must be between 10 cm and 15 cm.

Since $h=375/x(x+10)$ and $0<x<5$, then $h>3755/(5+10) = 5$.
This means the height is greater than 5, but is ultimately dependent on width.

We can visualize this situation by graphing the width function $w(x) = x$, $x>0$ and the height function $h(x)=375/x(x+10)$, $x>0$.

The two functions intersect at (5, 5) and $h(x)>w(x)$ when $x<5$.