Lesson 3: Unit 3- Polynomial Equations and Inequalities (1)

Previously we've study addition, subtraction, and multiplication of polynomials we will extend our knowledge of mathematical operations involving polynomials to include division of polynomials.

Long Division

Example 1

Divide $2x^2+5x-8$ by $x+3$.

Solution
Step 1: Eliminate the first term of the dividend.
The x in the divisor must be multiplied by $2x$ to produce $2x^2$.
Place $2x$ in the quotient above the term with the same degree.

$$2x(x+3)=2x^2+6x$$

Place this product below the dividend, lining up the terms of equal degree.

$$\begin{array}{r|rrrr} & & 2x & & \\ \hline x+3 & 2x^2 & +\ 5x & - & 8 \\ & 2x^2 & +\ 6x & & \\ \cline{2-3} \end{array}$$

Step 2: Subtract $2x^2+6x$ from the dividend to eliminate the first term.
Bring down the next term of the dividend, -8.

$$\begin{array}{r|rrrr} & & 2x & & \\ \hline x+3 & 2x^2 & +\ 5x & - & 8 \\ & 2x^2 & +\ 6x & & \downarrow \\ \cline{2-3} & & -\ 1x & - & 8 \end{array}$$

Step 3: Eliminate the first term of the difference, $-x$. The x in the divisor is multiplied by -1 to produce $-x$.
Place -1 in the quotient above the constant term.
Multiply

$$-1(x+3)=-1x-3$$

and place the product below the last expression, lining up the terms of equal degree.
Subtract to eliminate the x term.
Repeat the steps until the degree of the remainder is less than the degree of the divisor.
The quotient is $2x-1$ and the remainder is -5.

$$
\begin{array}{r|rrrrrrr}
 & & & 2x & - & 1 \\ \hline
x+3 & 2x^2 & + & 5x & - & 8 \\
 & 2x^2 & + & 6x & & \downarrow \\ \cline{2-4}
 & & - & 1x & - & 8 \\
 & & - & 1x & - & 3 \\ \cline{3-6}
 & & & & - & 5
\end{array}
$$

In general,

$$\frac{P(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

or

$$P(x) = d(x)q(x) + r(x)$$

where $P(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

The division question, $(2x^2+5x-8) \div (x+3)$, can now be expressed in terms of the quotient, remainder, and divisor using the statement

$$P(x)/d(x) = q(x) + r(x)/d(x)$$

$$\frac{2x^2+5x-8}{x+3} = 2x-1 + \frac{-5}{x+3}$$

Alternately, using

$$2x^2+5x-8 = (x+3)(2x-1) - 5$$

The latter statement can be used to check if the division was done correctly.

Example 2

Determine the quotient and remainder for $(4x^3-11x-9)\div(2x-3)$, $x \neq \frac{3}{2}$

Solution

$$
\begin{array}{lcr|rrrrrrr}
 & & & 2x^2 & + & 3x & - & 1 \\ \cline{4-8}
 & & 2x-3 & 4x^3 + 0x^2 & - & 11x & - & 9 \\
2x^2(2x-3) & \rightarrow & & 4x^3 - 6x^2 & & & & \\ \cline{4-4}
\text{Subtract} & & & 6x^2 & - & 11x & & \\
3x(2x-3) & \rightarrow & & 6x^2 & - & 9x & & \\ \cline{4-6}
\text{Subtract} & & & & - & 2x & - & 9 \\
-1(2x-3) & \rightarrow & & & - & 2x & + & 3 \\ \cline{5-8}
\text{Subtract} & & & & & & - & 12
\end{array}
$$

The quotient is $2x^2+3x-1$ and the remainder is -12. Therefore,

$$4x^3-11x-9 = (2x-3)(2x^2+3x-1) -12,\ x\neq 3/2$$

Example 3

Determine $3x^4+12x^2-2x^3-5x^2-1, x\neq\pm1$.

Solution

$$\begin{array}{r|rrrrr}
 & & & 3x^2 & -\ 2x & +\ 15 \\ \cline{2-6}
x^2+0x-1 & 3x^4 & -\ 2x^3 & +\ 12x^2 & +\ 0x & -\ 5 \\
 & 3x^4 & +\ 0x^3 & -\ 3x^2 & & \\ \cline{2-4}
 & & -\ 2x^3 & +\ 15x^2 & +\ 0x & \\
 & & -\ 2x^3 & +\ 0x^2 & +\ 2x & \\ \cline{3-5}
 & & & 15x^2 & -\ 2x & -\ 5 \\
 & & & 15x^2 & +\ 0x & -\ 15 \\ \cline{4-6}
 & & & & -\ 2x & +\ 10
\end{array}$$

Synthetic Division

Synthetic division is a quick and easy method for division of polynomials when the divisor is a binomial of the form $(x-n)$.

Example 1

Divide $2x^2+5x-8$ by $x+3$.

Solution

To perform division of $\frac{2x^2+5x-8}{x+3}$, $x\neq-3$, create a frame as below.
Place the value of n from the divisor outside the frame and the coefficients of the dividend inside the frame, along the top. Leave a blank line below the numbers.

$$\begin{array}{c|l}
x+3 & 2x^2+5x-8 \\
\uparrow & \\
n=-3 &
\end{array}
\quad\rightarrow\quad
\begin{array}{r|rrr}
-3 & 2 & 5 & -8 \\
 & & & \\ \hline
\end{array}$$

Bring the first coefficient down below the frame to begin forming the quotient.

$$\begin{array}{r|rrr}
-3 & 2 & 5 & -8 \\
 & \downarrow & & \\ \hline
 & 2 & &
\end{array}$$

Multiply the constant, −3, from the divisor by the 2 of the quotient, then place this product, −6, under the next coefficient in the frame.

$$\begin{array}{r|rrr} -3 & 2 & 5 & -8 \\ & \downarrow & -6 & \\ \hline & 2 & & \end{array}$$

Add −6 to the coefficient, 5, and place the sum, −1, below the −6.

$$\begin{array}{r|rrr} -3 & 2 & 5 & -8 \\ & \downarrow & -6 & \\ \hline & 2 & -1 & \end{array}$$

Then repeat by multiplying the −3 by the −1 and placing the product, 3, below the next coefficient in the frame.

Complete the process by adding the −8 and 3 and placing the sum, −5, below the 3.

$$\begin{array}{r|rrr} -3 & 2 & 5 & -8 \\ & \downarrow & -6 & 3 \\ \hline & 2 & -1 & -5 \end{array}$$

This last value −5 is the remainder. The two preceding values are the coefficients of the quotient, arranged in order of decreasing powers of *x*.

Working backwards, the −1 is the constant term and 2 is the coefficient of the *x* term.
The next value moving left, if there was one, would be the coefficient of the *x*2 term, and so on.
The quotient is 2*x*−1 and the remainder is the last value, −5.
Therefore,
$\frac{2x^2+5x-8}{x+3}$= 2x-1 + $\frac{-5}{x+3}$, *x*≠ 3.

Example 2

Divide $2x^4-3x^3-12x^2+8$ by $x-4$ using synthetic division.

Solution

$$\begin{array}{r|rrrrr} 4 & 2 & -3 & -12 & 0 & 8 \\ & \downarrow & 8 & 20 & 32 & 128 \\ \hline & 2 & 5 & 8 & 32 & 136 \end{array}$$

Reading the bottom line, the remainder is 136 and the quotient is $2x^3+5x^2+8x+32$.

Therefore,

$$2x^4-3x^3-12x^2+8x-4 = (2x^3+5x^2+8x+32)(x-4) + 136,\ x\neq4$$

Example 3

Divide $4x^3-11x-9$ by $2x-3$ using synthetic division.

Solution

Carrying out the steps of synthetic division, we have

$$\begin{array}{c|cccc} \frac{3}{2} & 4 & 0 & -11 & -9 \\ & \downarrow & 6 & 9 & -3 \\ \hline & 4 & 6 & -2 & -12 \end{array}$$

The quotient is $4x^2+6x-2$ and the remainder is -12, when dividing $4x^3-11x-9$ by $x-\frac{3}{2}$, so

$$\frac{4x^3-11x-9}{x-\frac{3}{2}} = 4x^2+6x-2 - \frac{12}{x-\frac{3}{2}}$$

Working the original division question

$$\frac{4x^3-11x-9}{2x-3} = \frac{1}{2}\left(\frac{4x^3-11x-9}{x-\frac{3}{2}}\right)$$

$$= \frac{1}{2}\left(4x^2+6x-2 - \frac{12}{x-\frac{3}{2}}\right)$$

$$= 2x^2+3x-1-\frac{12}{2x-3},\ x\neq 3/2$$

This example demonstrates the challenge faced when working with a linear divisor of the form $mx-n$, where $m\neq 1$.
There may be less opportunity for an error to be made or for a step to be missed if long division is used in these situations.
A more complicated version of synthetic division can be performed when the divisor is non-linear. Again, it is likely much easier to just use long division.

Challenge Example

When $12x^3-14x^2+px-1$ is divided by $qx-2$, the quotient is $4x^2+rx+3$ and the remainder is 5.
Find p, q, and r.

Solution

Using $P(x)=d(x)q(x)+r(x)$
where $P(x)=12x^3-14x^2+px-1$, $q(x)=4x^2+rx+3$, $d(x)=qx-2$, and $r(x)=5$.
Substituting,

$12x^3-14x^2+px-1 = (qx-2)(4x^2+rx+3)+5$

$= 4qx^3-8x^2+qrx^2-2rx+3qx-6+5$

$= (4q)x^3+(qr-8)x^2+(3q-2r)x-1$

$$12x^3-14x^2+px-1=(4q)x^3+(qr-8)x^2+(3q-2r)x-1$$

For the left side to equal the right side for all values of x, the coefficients of the terms must be equal.

$4q = 12$ (1)

$qr - 8 = -14$ (2)

$3q - 2r = p$ (3)

From (1),

$q = 3$.

Substituting into (2),

$r = -2$

Substituting q and r into (3),

$p = 13$

Therefore, $q=3$, $r=-2$, and $p=13$.

Summary

- Long division or synthetic division can be used to determine the quotient and remainder when a polynomial (dividend) is divided by another polynomial (divisor) of the same or a lesser degree.
- To carry out the division, ensure the terms of the polynomial are in descending order of degree. Missing powers of x in the dividend or divisor are included using a coefficient of 0 to keep work accurate and aligned correctly.
- The division is completed when the degree of the remaining terms after subtraction is less than the degree of the divisor.
- The result of the division $P(x) \div d(x)$, where $P(x)$ is the polynomial dividend and $d(x)$ is the divisor, is given by

$$\frac{P(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

  where $q(x)$ is the quotient and $r(x)$ is the remainder.
- The polynomial dividend can be expressed in terms of the quotient, divisor and remainder using

$$P(x)=d(x)q(x)+r(x)$$

To check the result of the division, multiply the quotient by the divisor and add the remainder. The result should be the dividend.

Next we will take a standard form polynomial function and express it in factored form and we will locate the zeros of the function and sketch its graph.
To factor a polynomial of degree greater than 2 requires an understanding of an important mathematical theorem called the factor theorem.

Since the factor theorem follows directly from the remainder theorem, we will need to begin with the remainder theorem.

The Remainder Theorem

Consider the following polynomial divisions, $P(x)\div(mx-n)$.

1. $\dfrac{x^3-5x^2-x-10}{x-2}$

$$\begin{array}{rl}
 & \; x^2 - 3x - 7 \\ \cline{2-2}
x-2 & \big)\; x^3 - 5x^2 - x - 10 \\
 & \; x^3 - 2x^2 \\ \cline{2-2}
 & \; -3x^2 - x \\
 & \; -3x^2 + 6x \\ \cline{2-2}
 & \; -7x - 10 \\
 & \; -7x + 14 \\ \cline{2-2}
 & \; -24
\end{array}$$

2. $\dfrac{x^3-28x-41}{x+4}$

$$\begin{array}{rl}
 & \; x^2 - 4x - 12 \\ \cline{2-2}
x+4 & \big)\; x^3 + 0x^2 - 28x - 41 \\
 & \; x^3 + 4x^2 \\ \cline{2-2}
 & \; -4x^2 - 28x \\
 & \; -4x^2 - 16x \\ \cline{2-2}
 & \; -12x - 41 \\
 & \; -12x - 48 \\ \cline{2-2}
 & \; 7
\end{array}$$

3. $\dfrac{2x^3-3x^2-8x-3}{2x+1}$

$$\begin{array}{rl}
 & \; x^2 - 2x - 3 \\ \cline{2-2}
2x+1 & \big)\; 2x^3 - 3x^2 - 8x - 3 \\
 & \; 2x^3 + x^2 \\ \cline{2-2}
 & \; -4x^2 - 8x \\
 & \; -4x^2 - 2x \\ \cline{2-2}
 & \; -6x - 3 \\
 & \; -6x - 3 \\ \cline{2-2}
 & \; 0
\end{array}$$

Now consider the value of the dividend, $P(x)$, when $x=n/m$ (Note that $m=1$ in cases 1 and 2).

In example 1 the divisor is x-2 so,
$P(x) = x^3-5x^2-x-10$
$P(2) = -24$ is the remainder.

Similarly, in example 2
$P(x) = x^3-28x-41$
$P(-4) = 7$ also the remainder.

In the 3rd example the remainder is $P(-\frac{1}{2})$.

The Remainder Theorem

When a polynomial, $P(x)$, is divided by a linear polynomial, $(x-n)$, the value of the remainder is given by the value of $P(n)$. This extends to include linear factors of the form $(mx-n)$, where m and n are integers and $m\neq 0$. When a polynomial function $P(x)$ is divided by $(mx-n)$, the remainder is $P(\frac{n}{m})$.

Example 1

Find the remainder when $P(x)=8x^3+20x^2-16x-25$ is divided by the following linear polynomials.
a. $x+3$

b. $2x-3$

Solution

a.

$$P(-3) = 8(-3)^3+20(-3)^2-16(-3)-25=-13$$

Therefore, the remainder is −13.

b.

$$P(3/2) = 8(3/2)^3+20(3/2)^2-16(3/2)-25 = 23$$

Therefore, the remainder is 23.

Example 2

The polynomial 2*x*4−5*x*3+*kx*2+10*x*−3 has a remainder of 21 when divided by ($x+4$). Find the value of k.

Solution

Let $P(x)=2x^4-5x^3+kx^2+10x-3$.
If the remainder is 21 when the polynomial is divided by ($x+4$), then $P(-4)=21$.
Substituting,

$$2(-4)^4-5(-4)^3+k(-4)^2+10(-4)-3 = 512+320+16k-40-3 = 21$$
$$k = -48$$

Example 3

For what value of k will the polynomial $2x^3+kx^2-5x+2$ have the same remainder when divided by $x-2$ as it will when divided by $x+1$?

Solution

Let $P(x)=2x^3+kx^2-5x+2$.
$P(2)$ is the remainder when dividing by $x-2$, while $P(-1)$ is the remainder when dividing by $x+1$.
Since these remainders are the same, then $P(2)=P(-1)$.
Therefore, k = -1.

Next theorem is a direct result of the remainder theoem.

The Factor Theorem

The Factor Theorem

For a given polynomial, $P(x)$, ($x-n$) is a factor of $P(x)$ if and only if $P(n)=0$. This means that if $P(n)=0$, then ($x-n$) is a factor of $P(x)$. Conversely, if ($x-n$) is a factor of $P(x)$, then $P(n)=0$. This extends to include linear factors of the form ($mx-n$), where m and n are integers and $m\neq 0$. ($mx-n$) is a factor of $P(x)$ if and only if $P(\frac{n}{m})=0$.

Example 4

Using the factor theorem, which of the following are factors of $2x^3-3x^2-2x+3$?
a. $x-1$
b. $x+3$
c. $2x-3$

Solution

Let $P(x)=2x^3-3x^2-2x+3$.
a. $P(1) = 0$
Therefore, $x-1$ is a factor.
b. $P(-3)= -72$
Therefore, $x+3$ is NOT a factor.
c. $P(3/2) = 0$
Therefore, $2x-3$ is a factor.

Example 5

Find k such that $x+4$ is a factor of $x^4+2x^3-7x^2+kx-8$.

Solution

Let $P(x)=x^4+2x^3-7x^2+kx-8$.
If $x+4$ is a factor, then $P(-4)=0$. Therefore,
$k = 2$.

Example 6

An unknown polynomial, $f(x)$, of degree 12 yields a remainder of 2 when divided by $x-1$ and a remainder of 5 when divided by $x+2$. A factor of the unknown polynomial is $x+1$. Find the remainder when $f(x)$ is divided by $(x-1)(x+1)(x+2)$.

Solution

Using the division statement

$$f(x)=d(x)q(x)+r(x)$$
$$f(x)=(x-1)(x+1)(x+2)q(x)+r(x)$$

When dividing $f(x)$ by $(x-1)(x+1)(x+2)$, the remainder is at most degree 2 (quadratic), since the divisor is cubic.
Let $r(x)=ax^2+bx+c$, where a, b, and c could be 0.
Hence, $f(x)=(x-1)(x+1)(x+2)q(x)+(ax2+bx+c)$.
When $f(x)$ is divided by $x-1$, the remainder is 2.

$f(1)=(1-1)(1+1)(1+2)q(1)+a(1)2+b(1)+c \Rightarrow \quad a+b+c=2 \quad (1)$
When $f(x)$ is divided by $x+2$, the remainder is 5.
$f(-2)=(-2-1)(-2+1)(-2+2)q(-2)+a(-2)2+b(-2)+c \Rightarrow \quad 4a-2b+c=5 \quad (2)$
When $f(x)$ is divided by $x+1$, the remainder is 0 since $x+1$ is a factor.
$f(-1)=(-1-1)(-1+1)(-1+2)q(-1)+a(-1)2+b(-1)+c \Rightarrow \quad a-b+c=0 \quad (3)$
This is a system with three equations in three unknowns.
Solving this system we get,
a = 2, b = 1, and c = -1.

Factoring Polynomials

The ability to factor a polynomial, for example $2x^2+7x-15=(2x-3)(x+5)$, is essential to graphing polynomial functions and solving polynomial equations.

When a polynomial is in factored form, the zeros of the function, or the roots of the equation, are easily identifiable

You have previously developed skills to factor quadratic polynomials, identifying linear factors that produce rational roots or zeros, as shown below.

$$f(x)= 2x^2+7x-15= (2x-3)(x+5)$$

Example

Factor $x^3-4x^2-3x+18$.

We will employ the factor theorem as we begin this process. Recall:

Factor Theorem

For a given polynomial, $P(x)$, $(x-n)$ is a factor of $P(x)$ if and only if $P(n)=0$. This means that if $P(n)=0$, then $(x-n)$ is a factor of $P(x)$. Conversely, if $(x-n)$ is a factor of $P(x)$, then $P(n)=0$. This can extend to include any linear factor. In other words, $(mx-n)$ is a factor of $P(x)$ if and only if $P(n/m)=0$.

This means that our test values must satisfy this condition.
Remember: a rational number is any number that can be written in the form a/b, where a and b are integers and $b\neq0$. Therefore, the rational root theorem encompasses integer roots.

Solution

Let $P(x)=x3-4x2-3x+18$.

- We will always look for a common factor first: this will help to simplify the process. There is no common factor in this example.

- **Using the rational root theorem**, we create a list of all possible test values for later use with the factor theorem.

We list all combinations of the factors of the constant term, 18, divided by the factors of the leading coefficient, 1.
For the rational root q/r, q must divide 18, and thus must come from the set $\{\pm1,\pm2,\pm3,\pm6,\pm9,\pm18\}$.
Similarly, r must divide into 1, giving two possibilities, $\{\pm1\}$.

$$q/r\in\{\pm1/1,\ \pm2\ 1,\ \pm3\ 1,\ \pm6\ 1,\ \pm9\ 1,\ \pm18\ 1\}$$

This list, of course, can be simplified to just the factors of 18.

$$q/r\in\{\pm1,\pm2,\pm3,\pm6,\pm9,\pm18\}$$

- **Using the factor theorem**, we look for a value, $x=n$, from the test values such that $P(n)=0$.

You may want to consider the coefficients of the terms of the polynomial and see if you can cut the list down.
Sometimes, if the value of the coefficients or constant term is large, you may want to start with a larger test value.
In this example, we have a mixture of positive and negative coefficients, so there is not necessarily a restriction on the sign of the test value.
With a quick look, we can see that ±1 will not work since the constant value is significantly larger than the coefficients of the first three terms.

$$P(2)=(2)3-4(2)2-3(2)+18=4$$

Therefore, $(x-2)$ **is not** a factor.

$$P(-2)=(-2)3-4(-2)2-3(-2)+18=0$$

Therefore, $(x+2)$ **is** a factor.

- We use $(x+2)$ as the first factor of the polynomial and we can determine the corresponding factor by long division (or synthetic division).

Using long division(or the synthetic one), we now have $x^3-4x^2-3x+18=(x+2)(x^2-6x+9)$

We can continue by factoring the quadratic. The fully factored form of the cubic would be

$x^3-4x^2-3x+18 = (x+2)(x-3)(x-3) = (x+2)(x-3)^2$

Example

Factor $x4-3x^3-13x^2+3x+12$.

Solution

$$(x-1)(x^3-2x^2-15x-12)$$

To factor $x^3-2x^2-15x-12$, we use the rational root theorem and the factor theorem.
Let $g(x)=x^3-2x^2-15x-12$.
Note that $g(-1)=(-1)3-2(-1)2-15(-1)-12=0$, so $(x+1)$ is a factor.

Identifying the first and last term gives

$$X^3-2x^2-15x-12=(x+1)(x^2+\blacksquare x-12)$$

So we have $x^3+x^2-12x-12$ and thus we need $-3x^2-3x=-3x(x+1)$.
Hence, $x^4-3x^3-13x^2+3x+12=(x-1)(x+1)(x^2-3x-12)$.
The quadratic cannot be factored further.
Therefore, $x^4-3x^3-13x^2+3x+12$ in factored form is $(x-1)(x+1)(x^2-3x-12)$.

Summary

- To factor a polynomial, $P(x)$, of degree 3 or greater, begin with the factor theorem. Look for a value n, such that $P(n)=0$.
- To employ the rational root theorem (also called the rational zero theorem), the test values for n must be of the form qr, where q is a factor of the constant term of $P(x)$ and r is a factor of the leading coefficient. If $P(qr)=0$, then $rx-q$ is a factor of $P(x)$.
- To determine the other factor(s) of $P(x)$, divide $rx-q$ into the polynomial using long division (or synthetic division) or use the “have and need” method.
- This process may need to be repeated several times depending on the degree of the polynomial.
- It is possible that no linear factors will be found depending on the nature of the polynomial.