

Notice

The notes are the same as before, but the questions in class and homework are different with before. Please review the notes and do the questions in class and homework.

Counting and Patterns 1

1. Counting

Counting is the action of finding the number of elements of nature numbers.

The traditional way of counting consists of continually increasing a (mental or spoken) counter by a unit for every element of the nature numbers, in some order, while marking (or displacing) those elements to avoid visiting the same element more than once, until no unmarked elements are left; if the counter was set to one after the first object, the value after visiting the final object gives the desired number of elements.

Counting sometimes involves numbers other than one.

For example, when counting money, counting out change, when "counting by twos" (2, 4, 6, 8, 10, 12...) or when "counting by fives" (5, 10, 15, 20, 25...).

2. Counting using tally marks.

Tally marks are a quick way of keeping track of numbers in groups of five.

One vertical line is made for each of the first four numbers; the fifth number is represented by a diagonal line across the previous four.

1		6	
2		7	
3		8	
4		9	
5		10	

3. Number Patterns

A **sequence** is a pattern of numbers that are formed in accordance with a definite rule.

We can often describe number patterns in more than one way. To illustrate this, consider the following sequence of numbers {1, 3, 5, 7, 9, ...}.

Clearly, the first term of this number pattern is 1; and the terms after the first term are obtained by adding 2 to the previous term. We can also describe this number pattern as a set of odd numbers.

Alternatively, we can represent a number pattern by using special symbols. For the number pattern under consideration, let t_n represent the n th term.

∴ First term: $t_1 = 1$
 Second term: $t_2 = 3$
 Third term: $t_3 = 5$
 Fourth term: $t_4 = 7$
 Fifth term: $t_5 = 9$
 ...

By trial and error, we find that:

$$\begin{aligned} t_1 &= 1 = 2 \times 1 - 1 \\ t_2 &= 3 = 2 \times 2 - 1 \\ t_3 &= 5 = 2 \times 3 - 1 \\ t_4 &= 7 = 2 \times 4 - 1 \\ t_5 &= 9 = 2 \times 5 - 1 \end{aligned}$$

By observation, we notice that we can describe this number pattern by the rule

$$t_n = 2n - 1$$

where t_n is n th term of the sequence.

4. Formula and Tables

A table of values can be generated from the rule

$$t_n = 2n - 1$$

as shown below.

$$\text{When } n = 1, t = 2 \times 1 - 1 = 2 - 1 = 1$$

$$\text{When } n = 2, t = 2 \times 2 - 1 = 4 - 1 = 3$$

$$\text{When } n = 3, t = 2 \times 3 - 1 = 6 - 1 = 5$$

$$\text{When } n = 4, t = 2 \times 4 - 1 = 8 - 1 = 7$$

$$\text{When } n = 5, t = 2 \times 5 - 1 = 10 - 1 = 9$$

n	1	2	3	4	5
t	1	3	5	7	9

5. Finding the Algebraic Rule

We use algebra to study rules that describe the behavior of everyday things, for example, the behavior of the height of a ball when it is thrown upward or the amount outstanding for a loan after a number of regular repayments.

By finding a pattern in observed values (i.e. measurements), we are often able to discover a rule that allows us to make accurate predictions.

6. Using a Difference Pattern

When we try to discover an algebraic rule for ordered pairs, we can find the difference between two successive values of y . This allows us to find a rule as illustrated below.

Consider the following table.

x	1	2	3	4
y	5	7	9	11

Diagram illustrating the difference pattern:

Pattern difference: 2 2 2

Difference between 5 and 7 Difference between 7 and 9 Difference between 9 and 11

We notice that the values of x increase by just one at a time and the difference between the successive values for y is 2.

So, the rule starts off with $y = 2x$. Will this give a correct answer from the table? Let us check.

$$\text{When } x = 1, y = 2 \times x = 2 \times 1 = 2$$

The answer is no. From the table, when $x = 1$ the value of y should be 5. How do we change our answer from 2 to 5? We should add 3.

$$\therefore \text{The rule becomes } y = 2x + 3$$

Check the rule to see if it is correct:

$$\text{When } x = 1, y = 2 \times 1 + 3 = 2 + 3 = 5$$

$$\text{When } x = 2, y = 2 \times 2 + 3 = 4 + 3 = 7$$

$$\text{When } x = 3, y = 2 \times 3 + 3 = 6 + 3 = 9$$

$$\text{When } x = 4, y = 2 \times 4 + 3 = 8 + 3 = 11$$

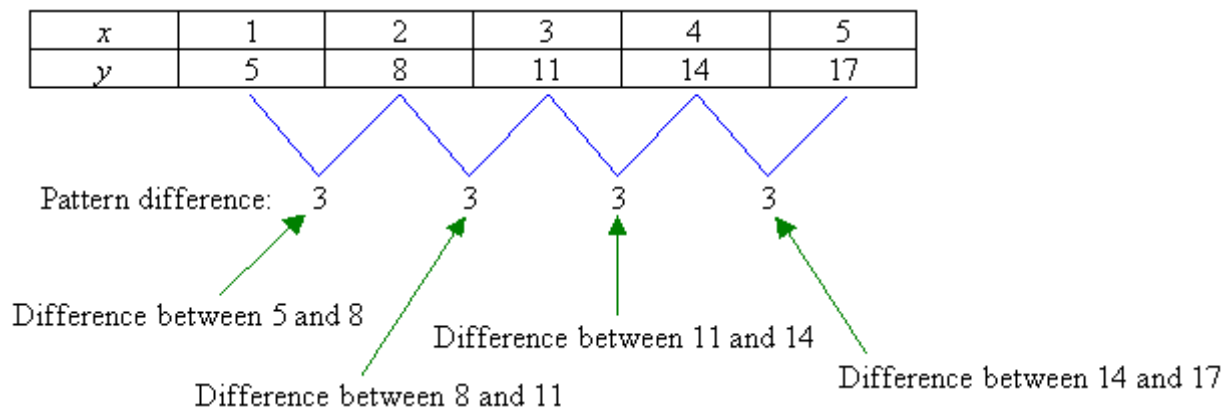
So, our rule $y = 2x + 3$ is correct.

Example: Discover the rule for the following table of values:

x	1	2	3	4	5
y	5	8	11	14	17

Solution:

In the given table, the x -values increase by 1 for each ordered pair.



Find the difference between the successive values of y . That is:

$$8 - 5 = 11 - 8 = 14 - 11 = 17 - 14 = 3$$

The difference between successive values of y is always 3. So, the rule is of the form

$$y = 3 \times x + \square$$

To determine the value of \square , we check the rule for $x = 1$.

$$\text{When } x = 1, y = 5 \text{ and } y = 3 \times 1 + \square = 3 + \square$$

$$\therefore 5 = 3 + \square, 5 - 3 = 3 + \square - 3, \therefore \square = 2.$$

Hence the rule is $y = 3x + 2$

Check the rule to see if it is correct:

$$\text{When } x = 2, y = 3 \times 2 + 2 = 6 + 2 = 8$$

$$\text{When } x = 3, y = 3 \times 3 + 2 = 9 + 2 = 11$$

$$\text{When } x = 4, y = 3 \times 4 + 2 = 12 + 2 = 14$$

$$\text{When } x = 5, y = 3 \times 5 + 2 = 15 + 2 = 17$$











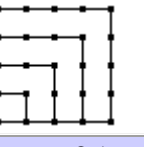
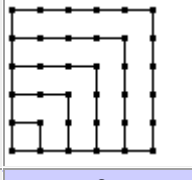






So, our rule is correct.

7. Polygonal Numbers

Polygonal Numbers are really just the number of vertexes in a figure formed by a certain polygon. The first number in any group of Polygonal Numbers is always 1, or a point. The second number is equal to the number of vertexes of the polygon.

A formula that will generate the n^{th} x -gonal number (for example: the 2nd 3-gonal, or triangular number) is:

$$\frac{n^2 - n}{2} \times (x - 2) + n$$

Type	1 st	2 nd	3 rd	4 th	5 th	6 th
Triangular						
Value	1	3	6	10	15	21
Square						
Value	1	4	9	16	25	36
Pentagonal						

8. Permutation

Fundamental Counting Rule - If there are a ways to pick something and b ways to pick something else, then in total, there are $a \times b$ total ways you can pick both!

Example: Alison wants to make a bracelet out of coloured beads. She has 6 beads, each a different colour. How many different bracelets can she make, if she uses all the beads?

There are 6 ways to pick the first bead. There are 5 ways to pick the second bead. There are 4 ways to pick the third bead, and so on... There is only 1 way to pick the last bead.

Therefore, in total, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways to make different bracelets.

A product of the form $n \times (n - 1) \times (n - 2) \cdots \times 2 \times 1$ can be written shorthand as **$n!$** - this is read as “ **n factorial**”.

In the previous example, the order is important - different order, different item.

Permutation - A permutation is an arrangement of a certain number of distinct types of items, where the **order of the items matter**.

More Examples

- Numbers - the arrangement of the digits matter (repetition allowed)
- Locker “combinations” (should be permutations)
- Winners in a marathon - the order of the runners matters!

Example: How many bracelets can Alison make if she has 30 different coloured beads and wants to use 25 of them?

Alison had 30 beads but only wants to use 25 of them (so she is NOT using $(30 - 25) = 5$ of them). Using the fundamental counting rule, we had the following expression: $30 \times 29 \times 28 \times \dots \times 6$

We can write this in a different way: $30 \times 29 \times 28 \times \dots \times 6 \times [5 \times 4 \times 3 \times 2 \times 1] \div [5 \times 4 \times 3 \times 2 \times 1]$. So what we get is actually $30! \div (30 - 25)! = 30! \div 5! = 30!/5!$.

This is a shorter way of writing out that long expression!

What did we do? We just divided (# of objects we have)! by (# of objects we did not use)!
How many objects did we NOT use? Simple - just subtract the number of objects you DID use (k) from the number of objects you didn't use (n). This is just $(n - k)$.

If we are counting permutations of n DISTINCT objects, where we use k of them without reusing an object, the amount is "n pick k"

Total Permutations

$$= n!/(n - k)!$$

$$= (\text{\# of distinct objects you HAVE})! / (\text{\# of distinct objects you AREN'T using})!$$

$$= {}_n P_k$$

(If we use ALL the objects, where $n = k$, then we just use $n!$ - we define $0! = 1$)

For permutations of n DISTINCT objects, where we use k of them and are allowed to reuse an object, the number is simply " $n \times n \times \dots \times n = n^k$ ".

The key to remember here is - all of these formulas come from the Fundamental Counting Rule!

Example: Collector Christian has a collection of 7 toy train cars and creates a train using 4 of them. How many different ways could he have made the train?

$$\frac{7!}{(7-4)!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 7 \times 6 \times 5 \times 4 = 840$$

9. Combination

What if order doesn't matter?

How many possible 3 digit lottery tickets are there if numbers from 1-4 are used (no repeats)?

List out all permutations:

(1, 2, 3	1, 3, 2	2, 3, 1	2, 1, 3	3, 1, 2	3, 2, 1) $\rightarrow \{1, 2, 3\}$
(1, 2, 4	1, 4, 2	2, 1, 4	2, 4, 1	4, 1, 2	4, 2, 1) $\rightarrow \{1, 2, 4\}$
(1, 3, 4	1, 4, 3	3, 1, 4	3, 4, 1	4, 1, 3	4, 3, 1) $\rightarrow \{1, 3, 4\}$
(2, 3, 4	2, 4, 3	3, 2, 4	3, 4, 2	4, 2, 3	4, 3, 2) $\rightarrow \{2, 3, 4\}$

Using permutation method:

Problem... {1, 3, 4} is the same ticket as {1, 4, 3}. We double counted.

Group all permutations which use the same digits. How many are in each group?

The number of different lottery tickets is the number of groups of permutations that use the same digits. This is the same as making order irrelevant! What did we do? We divided the total number by the number of elements in each group - how many are each in group? The number of permutations of k elements - this is exactly k! So we divide by k!

Combination - A selection of items where **order does not matter** is known as a combination. There are ${}_nP_k$ total permutations; but each grouping has k! elements, so we have to divide by k!. Then there are

$[{}_nP_k] \div k! = {}_nP_k / k! = n! / (n - k)! \div k! = n! / k!(n - k)!$ combinations in total.

We write this as ${}_nC_k$, and say “n choose k”. So the actual number of lottery tickets is ${}_4C_3 = 4$ - from 4 possible numbers, you are CHOOSING three of them, where order DOES NOT matter.

Example: There is a bag of 12 gumballs, each of different colour. How many handfuls of 3 gumballs can you make?

Since the order doesn't matter, ${}_{12}C_3 = \frac{12!}{3!(12-3)!} = \frac{12 \times 11 \times 10 \times 9!}{3!9!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$

► **Question in class**

○ **Counting**

1. How many animals do I have if all but two are dogs, all but two are cats and all but two are hamsters?
2. How many times is the digit 3 used in writing all of the integers from 1 to 100?
3. What is the digit in the ones' place in the expansion of 3^{47} ?
4. In the game of 6**9, a person chooses 6 different integers from 1 to 9. If the order in which the numbers are chosen is not important, how many choices are possible?
5. In how many ways can 7 different colours be arranged in a row if the first colour is always red, the last colour is always blue and the middle colour is always green?
6. How many integers between 1 and 1000 (inclusive) do not contain the digits 8 or 9?
7. One needs 4221 digits to number the pages of a book. How many pages does this book contain?
8. I have 6 different books, 3 with red covers and 3 with blue covers. In how many different ways can I arrange these books on a shelf so that no two books of the same colour are next to each other?

○ **Patterns**

1. What is the next number in the list $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, \dots$?
2. What is the next term in the sequence $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}, \dots$?
3. What is the next term in the sequence 1, 3, 5, 11, 21, 43 ... ?