Lesson 7: Unit 5 - Exponential and Logarithmic Functions (1)

In This Unit

- We will explore the behaviour of the exponential function and its inverse, the logarithmic function.
- We will use these functions to model situations and solve problems.

First

- We will investigate the graphs of exponential functions.
- We will apply our knowledge of transformations to graph functions of the form $y=af(b(x-h))+k$ where $y=f(x)$ is an exponential function.

Introduction

The general equation of an **exponential function** with a coefficient of 1 is given by

$$y = c^x$$

where $c>0$, $c\neq 1$, $c\in$ R.

An exponential function is different from a power function, such as $y=x^2$ or $y=x^n$, where the variable is the base and the exponent is a constant.

An exponential function, $y=c^x$, has a constant base, c, and a variable exponent, x.

$$y=2^x$$

Examples of exponential functions are $y=2^x$, $y=5^x$, and $y=\left(\frac{3}{7}\right)^x$.

Example 1

Graph $y=2^x$ and describe its behaviour.

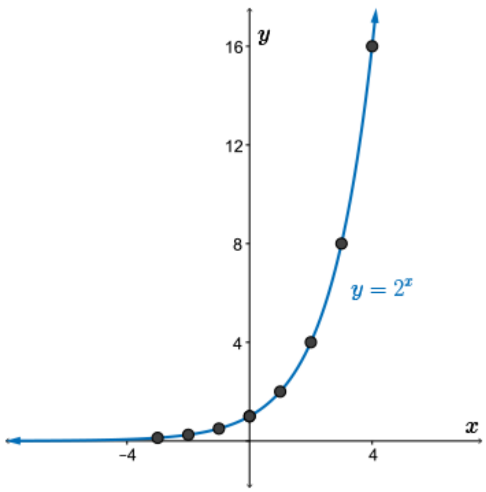
Solution

Domain: $\{x|\ x \in \mathrm{R}\}$

x	y
−3	$2^{-3}=(1/2)^3 = 1/8$
−2	$2^{-2}= (1/2)^2= 1/4$
−1	$2^{-1}= 1/2$
0	$2^0= 1$
1	$2^1= 2$
3/2	$2^{3/2}= (\sqrt{2})^3 \approx 2.828$

2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$

Note that the value of the function grows very quickly as *x* increases in value.
To describe this end behaviour, we say
$y \to \infty$ as $x \to \infty$



As *x* decreases in value, becoming a large negative value, the graph of the function approaches the *x*-axis.
That is,

$$y \to 0 \text{ as } x \to -\infty$$

Since $2^{-k} = (1/2)^k$, the value of the function will remain positive and approach 0 from above as *x* decreases in value ($x \to -\infty$).

The function $y=2^x$ has the following properties:

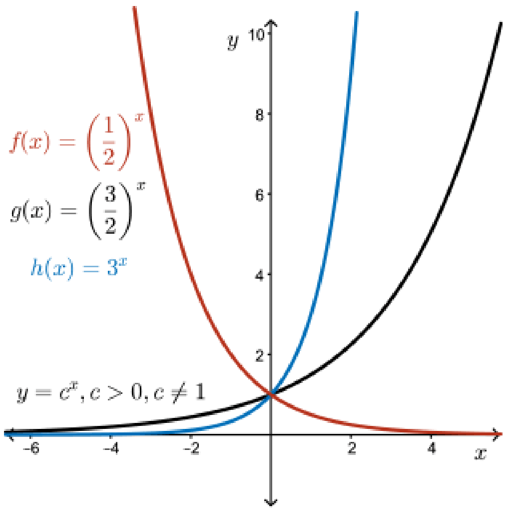
- Domain: $\{x | x \in R\}$
- Range: $\{y | y > 0, y \in R\}$
- Horizontal asymptote: $y=0$
- *y*-intercept: (0,1)
- The function is always increasing

Do all exponential functions behave this way?
What happens to the shape of the curve, $y=c^x$, as the value of the base, c, changes?

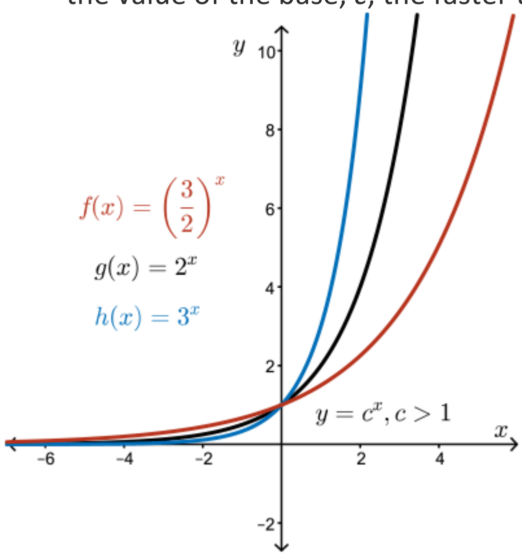
General Observation

An exponential function of the form $y=c^x$, $c>0$, $c\neq1$, has

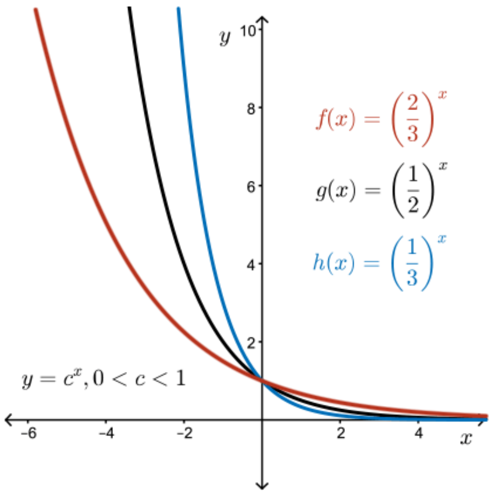
- a domain: $\{x|\ x\in R\}$
- a range: $\{y|\ y>0, y\in R\}$
- no x-intercept and a y-intercept at (0,1), and
- a horizontal asymptote of $y=0$ (the x-axis).



- The graph of an exponential function, $y=c^x$ where $c>1$, is always increasing. The greater the value of the base, c, the faster the curve increases.



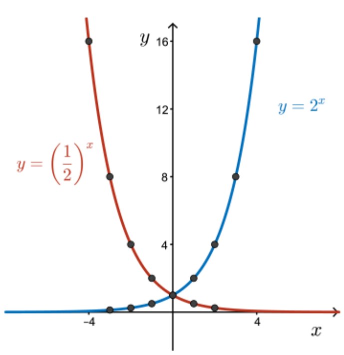
- The graph of an exponential function, $y=c^x$ where $0<c<1$, is always decreasing. The smaller the value of the base, c, the faster the curve decreases.



Consider the table of values and graphs of $y=2^x$ and $y=(1/2)^x$.

x	$y=2^x$	$y=(\frac{1}{2})^x$
−3	$2^{-3}=1/8$	$(1/2)^{-3}=8$
−2	$2^{-2}=1/4$	$(12)^{-2}=4$
−1	$2^{-1}=1/2$	$(12)^{-1}=2$
0	$2^0=1$	$(1/2)^0=1$
1	$2^1=2$	$(1/2)^1=1/2$
2	$2^2=4$	$(1/2)^2=1/4$
3	$2^3=8$	$(1/2)^3=1/8$

The graph of $y= (1/2)^x$ is a reflection of the graph $y=2^x$ in the y-axis.
This observation is supported by the fact that $y= (1/2)^x$ can be expressed as $y=2^{-x}$.



General Observation

$$y=c^x,\ c>0,\ c\neq 1$$

Why must $c>0$ and $c\neq 1$?
When $c=1$, the graph is the horizontal line $y=1$. That is,

$$y=1^x=1, \text{ for all } x \in \mathbb{R}$$

Similarly, when $c=0$, $y=0^x$.
Thus $y=0$ when $x>0$, and y will be undefined when $x<0$ as this implies division by 0.
When $x=0$, $y=0^0$, which some consider to be undefined.
What happens when $c<0$?
For example, consider $y=(-2)^x$:

- When x is even, y is positive; e.g., $(-2)^2=4$, $(-2)^4=16$, …
- When x is odd, y is negative; e.g., $(-2)^3=-8$, $(-2)^5=-32$, …
- When x is 1/2 or 1/4, y is not a real number; e.g., $y=\sqrt{-2}$, $y=\sqrt[4]{-2}$.

This relation lacks continuity and the previously discussed properties of the exponential functions are lost.

Transformations of Exponential Functions

We will now apply our prior knowledge of transformations to sketch the graphs of exponential functions of the form

$$y=a(c^{b(x-h)})+k$$

Given

$$y=af(b(x-h))+k$$

can you recall the role of the parameters a,b,h, and k in transforming the function $y=f(x)$?

- If $a<0$, then $y=c^x$ is reflected in the x-axis.
- $y=c^x$ is stretched vertically about the x-axis by a factor of $|a|$.
- If $b<0$, then $y=c^x$ is reflected in the y-axis.
- $y=c^x$ is stretched horizontally about the y-axis by a factor of $1/|b|$.
- $y=c^x$ is translated horizontally h units.
  If $h>0$, then $y=c^x$ is translated right.
  If $h<0$, then $y=c^x$ is translated left.
- $y=c^x$ is translated vertically k units.
  If $k>0$, then $y=c^x$ is translated up.
  If $k<0$, then $y=c^x$ is translated down.

The transformation of each point is defined by the mapping

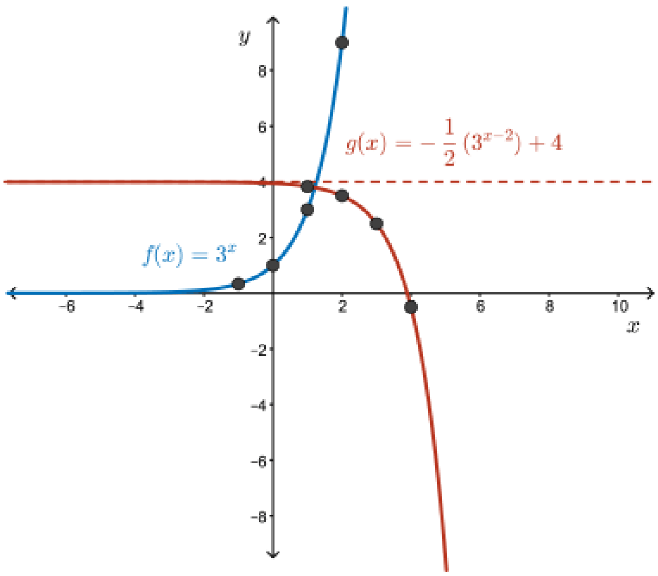
$$(x, y) \rightarrow \left(\frac{1}{b}x+h,\ ay+k\right)$$

When applying the transformations to the graph of the function, the stretches and/or reflections must be performed first (in any order) prior to the translations.

Example

Sketch the graph of $f(x)=3^x$. Describe the transformations applied to $f(x)$ to obtain the graph of the function $g(x) = -\frac{1}{2}(3^{x-2}) + 4$ and graph the transformed function.

Solution



Using the points $(-1,1/3)$,$(0,1)$,$(1,3)$, and $(2,9)$, we sketch the graph of $f(x)=3^x$.
Identify the transformations applied to $f(x)$ to obtain $g(x)$.
The graph of $g(x) = -\frac{1}{2}(3^{x-2}) + 4$ is obtained by applying the transformations, in order:

- a reflection in the x-axis ($a=-1/2<0$)
- a vertical stretch about the x-axis by a factor of $1/2$($|a|=1/2$)
- a horizontal translation right 2 units ($h=2$)
- a vertical translation up 4 units ($k=4$)

The points on the new function can be obtained from the mapping: $(x, y) \rightarrow (x + 2, -\frac{1}{2}y + 4)$.
Specifically,
$(-1, 1/3) \rightarrow (1, 23/6)$
$(0, 1) \rightarrow (2, 3.5)$
$(1, 3) \rightarrow (3, 2.5)$
$(2, 9) \rightarrow (4, -0.5)$
Thus, $g(x)$ is a decreasing function with a horizontal asymptote of $y=4$.
The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y<4, y \in \mathbb{R}\}$.

In order to solve problems involving exponential functions, we will need the skills necessary to solve exponential equations.
Exponential equations are equations in which the variable occurs in the exponent.
Next we will discuss methods of solving exponential equations using the laws of exponents to obtain common bases.

Example

Solve $2^{x-3} = 8\sqrt{2}$.

Solution

$2^{x-3} = 8\sqrt{2}$

$2^{x-3} = 2^3(2^{1/2})$

$2^{x-3} = 2^{7/2}$

If two powers with the same base are equivalent, then their exponents must be equivalent.
Equating the exponents,

$$x-3 = 7/2$$
$$x = 13/2$$

Example

Solve $25^{x-2} = 125^{2x-4}$.

Solution

$25^{x-2} = 125^{2x-4}$
$(5^2)^{x-2} = (5^3)^{2x-4}$
$5^{2x-4} = 5^{6x-12}$
Equating the exponents,
$2x-4 = 6x-12$
$x = 2$

Verifying,

$$\text{L.S.} = 25^{x-2} = 25^{2-2} = 25^0 = 1$$
$$\text{R.S.} = 125^{2x-4} = 125^{2(2)-4} = 125^0 = 1$$

Since L.S. = R.S., we have that $x=2$.

Example

Solve $3^{2x} - 10(3^x) + 9 = 0$.

Solution

This equation is quadratic in form.

The first term is $3^{2x} = (3^x)^2$. Let $a = 3^x$.

$(3^x)^2 - 10(3^x) + 9 = 0$

$a^2 - 10a + 9 = 0$

$(a-1)(a-9) = 0$

$a=1$ or $a=9$

Then, let $a=3^x$. Thus, $3^x = 1$ or $3^x = 9$.
This means that $x=0$ or $x=2$.

Summary

- Some exponential equations can be solved algebraically by expressing the power on each side of the equation with a common base and using the property that $x=y$ when $c^x = c^y$. Exponent laws may need to be applied to simplify the exponential expressions first.
- Exponential equations with powers that cannot be expressed using a common base can be solved using graphing technology, or an approximate solution can be obtained using a systematic trial and error method.

Exponential functions are closely related to geometric sequences. That is, sequences of the form

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$$

in which the ratio of any term to its preceding term is a non-zero constant.

$$t_{n+1} / t_n = r,\ r \neq 0,\ n \geq 1,\ n \in Z$$

An exponential function can be used to model a wide variety of real-life situations; from financial growth of investments to depreciation in the value of a vehicle; from population growth to radioactive decay.

Now we will explore various applications of the exponential function and solve related problems.

Example

An unstable element Fermium-253 (^{253}Fm) has a half-life of 3 days.
The mass M, in milligrams, of a given amount of 253Fm remaining after t days is modelled by the given table.

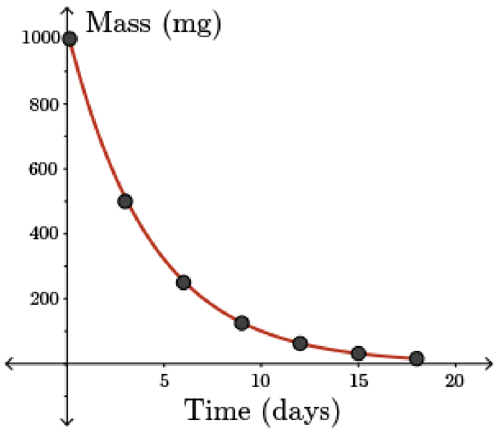
t (days)	M (mg)
0	1000

3	500
6	250
9	125
12	62.5
15	31.25
18	15.625

Determine the equation of a function that models this relationship.

Solution

Half-life is the time required for a quantity to decrease to half its original amount.
Half-life is used to measure decay when working with radioactive substances in nuclear physics and chemistry.
If we consider the graph associated with the given table, the function to model this data appears to be a decreasing exponential function (with the value of the base between 0and 1).



t (days)	M (mg)	1^{st} Difference ΔM	2^{nd} Difference $\Delta^2 M$
0	1000	$500-1000$ $=-500$	$-250-(-500)$ $=250$
3	500	-250	125
6	250	-125	62.5
9	125	-62.5	31.25
12	62.5	-31.25	15.625
15	31.25	-15.625	
18	15.625		

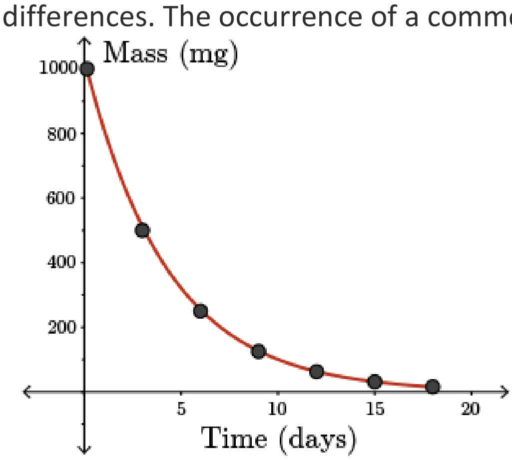
When working with polynomial functions, we first used finite differences to help us identify a relationship, given a table of values.

It is apparent from the finite differences table that this data cannot be modelled by a polynomial function. A constant n^{th} difference cannot be obtained.
The 1st differences are not constant, but are related by a factor of 1/2; for instance, −250 is 1/2 of −500.
This same relationship occurs with the 2^{nd} differences and the values of the mass, M.

In this relationship, the ratio of consecutive values of Min the table is constant.
That is, $M(3)/M(0)=M(6)/M(3)=M(9)/M(6)$ and so on.
This will happen for exponential behaviour when the change in the independent variable is constant. In this case, $\Delta t=3$.

t (days)	M (mg)	Ratio
0	1000	$\frac{M(3)}{M(0)} = \frac{500}{1000} = \frac{1}{2}$
3	500	$\frac{M(6)}{M(3)} = \frac{250}{500} = \frac{1}{2}$
6	250	$\frac{1}{2}$
9	125	$\frac{1}{2}$
12	62.5	$\frac{1}{2}$
15	31.25	$\frac{1}{2}$
18	15.625	

This common ratio of 1/2 is also present within the set of first differences and second differences. The occurrence of a common ratio is a way of identifying exponential behaviour.



To determine the equation we begin with the general form of the exponential function having base 1/2.

$$y= (1/2)^x \rightarrow y = a(1/2)^{b(x-h)} + k$$

The horizontal asymptote in this situation is $M= 0$ since $M \rightarrow 0$ as $t \rightarrow \infty$.

There is no vertical translation of the function $y= (1/2)^x$ and so $k= 0$.
A horizontal translation will not affect the shape of the exponential graph and can be converted to a vertical stretch, as demonstrated in an earlier module.
Therefore, we let $h=0$ and consider only the vertical and horizontal stretch factors to define this function.
The equation to model this data can thus be determined using $M(t) = a(1/2)^{bt}$ where $t≥0$ is the time in days.
The initial mass of the substance, denoted $M0$, is 1000 mg, so $M(0)=1000$.

$$1000=a(1/2)^{b(0)} \therefore a=1000$$

Using $M(3)=500$, we can solve for b.

$$500 = 1000(1/2)^{b(3)} \therefore b=1/3$$

Thus, $M(t)=1000(12)^{t/3}$, $t≥0$.

In General

1. For **half-life** in general, the quantity of a substance, M, remaining after time t, is given by

$$M=M_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$$

where M_0 is the initial amount and H is the half-life of the substance.
Note: *Time t must be expressed in the same units as H.*

Similarly, biologists work with a doubling period when describing the growth of cells in a petri dish.

2. A **doubling period** is understood as the time required for the population to double.

In this situation, the population P after time t is given by

$$P=P_0(2)^{t/D}$$

where P_0 is the initial population and D is the doubling period (in the same units as t).

3. In general, if a quantity Q is multiplied by a factor r, over every constant period of time T, then

$$Q(t)=Q_0(r)^{t/T}$$

where Q_0 is the initial quantity (i.e. $Q0=Q(0)$) and t is the time measured in the same units as T.
Q is increasing when $r>1$ (r is the growth factor) and decreasing when $0<r<1$ (r is the decay factor).

Example

A speedboat is purchased for $45000 and sold eight years later for $20000.
Determine the average rate of depreciation per year on the value of the boat.

Solution

The value (in $) of the boat, V, is given by $V = V_0(r)^{t/T}$ where V_0 is the original value and r is the decay factor.
We are given V = $20000, V_0 = $45000, and t = 8 years.
T=1 since we are asked to find the rate of depreciation per year.
$20\,000 = 45\,000(r)^8$
$r = \sqrt[8]{\frac{4}{9}}$, $r>0$
$r \approx 0.9036$

Since the boat retains approximately 0.9036 or 90.36% of its value each year, then the rate of depreciation is given by

$$1 - 0.9036 = 0.0964$$

The average annual rate of depreciation was approximately 9.64%.