Chapter 1 Exponential Rules

1. The Exponent Rules and Powers with Integral Bases

- **Multiplying Like Bases With Exponents (The Product Rule for Exponents)**

Let's first start by using the definition of exponents to help you to understand how we get to the law for multiplying like bases with exponents:

$x^2 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x = x^5$

Note that 2 + 3 = 5, which is the exponent we ended up with. We had 2 x's written in a product plus another 3 x's written in the product for a total of 5 x's in the product. To indicate that we put the 5 in the exponent.

In general, $x^m \cdot x^n = x^{m+n}$

In other words, when you multiply like bases you add your exponents.

The reason is, exponents count how many of your base you have in a product, so if you are continuing that product, you are adding on to the exponents.

Example 1: Use the product rule to simplify the expression $y^4 \cdot y^7$.

$y^4 \cdot y^7 = y^{4+7} = y^{11}$

Example 2: Use the product rule to simplify the expression $(-2a^3b^2)(4a^5b^3)$.

$(-2a^3b^2)(4a^5b^3) = -8a^{3+5}b^{2+3} = -8a^8b^5$

- **Zero as an exponent $x^0 = 1$**

Except for 0, any base raised to the 0 power simplifies to be the number 1.

Example 3: Evaluate $(2xyz)^0$.

$(2xyz)^0 = 1$

Example 4: Evaluate $3x^0$.

$3x^0 = 3(1) = 3$

- **Dividing Like Bases With Exponents (Quotient Rule for Exponents)**

Let's first start by using the definition of exponents to help you to understand how we get to the law for dividing like bases with exponents:

$$\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = x \cdot x \cdot x = x^3$$

Note how 5 - 2 = 3, the final answer's exponent. When you multiply you are adding on to your exponent, so it should stand to reason that when you divide like bases you are taking away from your exponent.

In general, $\dfrac{x^m}{x^n} = x^{m+n}$

In other words, when you divide like bases you subtract their exponents.

Keep in mind that you always take the numerator's exponent minus your denominator's exponent, NOT the other way around.

Example 5: Use the quotient rule to simplify the expression $\dfrac{y^{10}}{y^3}$.

$$\frac{y^{10}}{y^3} = y^{10-3} = y^7$$

Example 6: Use the quotient rule to simplify the expression $\dfrac{-25a^7b^4}{5a^2b}$.

$$\frac{-25a^7b^4}{5a^2b} = -5a^{7-2}b^{4-1} = -5a^5b^3$$

- **Base Raised to Two Exponents (Power Rule for Exponents)**

Let's first start by using the definition of exponents as well as the law for multiplying like bases to help you to understand how we get to the law for raising a base to two exponents:
$(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x^6$

Note how 2 times 3 is 6, which is the exponent of the final answer. We can think of this as 3 groups of 2, which of course would come out to be 6.

In general, $(x^m)^n = x^{mn}$

In other words, when you raise a base to two exponents, you multiply those exponents together. Again, you can think of it as *n* groups of *m* if it helps you to remember.

Example 7: Use the power rule for exponents to simplify the expression $(y^4)^2$.

$$(y^4)^2 = y^{(4)(2)} = y^8$$

- **A Product Raised to an Exponent (Power of a Product Rule)**

Let's first start by using the definition of exponents to help you to understand how we get to the law for raising a product to an exponent:

$(xy)^3 = (xy)(xy)(xy) = x^3y^3$

Note how both bases of your product ended up being raised by the exponent of 3.

In general, $(xy)^n = x^n y^n$

In other words, when you have a PRODUCT (not a sum or difference) raised to an exponent, you can simplify by raising each base in the product to that exponent.

Example 8: Use the power of a product rule to simplify the expression $(ab)^5$.

$(ab)^5 = a^5b^5$

- **A Quotient Raised to an Exponent (Power of a Quotient Rule)**

Let's first start by using the definition of exponents to help you to understand how we get to the law for raising a quotient to an exponent: $\left(\frac{x}{y}\right)^3 = \frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y} = \frac{x^3}{y^3}$

Since, division is really multiplication of the reciprocal, it has the same basic idea as when we raised a product to an exponent.

In general, $(\frac{x}{y})^n = \frac{x^n}{y^n}$

In other words, when you have a QUOTIENT (not a sum or difference) raised to an exponent, you can simplify by raising each base in the numerator and denominator of the quotient to that exponent.

Example 9: Use the power of a quotient rule to simplify the expression $\left(\frac{2}{x}\right)^5$.

$(\frac{2}{x})^5 = \frac{2^5}{x^5} = \frac{32}{x^5}$

- **Negative Exponent**

A negative exponent just means that the base is on the wrong side of the fraction line, so you need to flip the base to the other side. For instance, "x^{-2}" just means "x^2, but underneath, as in $\frac{1}{x^2}$".

Example: Write the following using only positive exponents.

a) $x^{-4} = \frac{1x^{-4}}{1} = \frac{1}{x^4}$

b) $\frac{x^2}{x^{-3}} = \frac{1x^2}{1x^{-3}} = \frac{1x^2x^3}{1} = x^5$

c) $2x^{-1} = \frac{2x^{-1}}{1} = \frac{2}{x^1} = \frac{2}{x}$

- **Simplifying an Exponential Expression**

When simplifying an exponential expression, write it so that each base is written one time with one exponent. In other words, write it in the most condense form you can.

A lot of times you are having to use more than one rule to get the job done. As long as you are using the rule appropriately, you should be fine.

Example 1: Simplify $\frac{(5a^3)(2a^4)}{(a^2)}$.

$$\frac{(5a^3)(2a^4)}{(a^2)} = \frac{10a^{3+4}}{a^2} = \frac{10a^7}{a^2} = 10a^{7-2} = 10a^5$$

Example 2: Simplify $\frac{(3x^4y^5)^2}{x^2y^5}$.

$$\frac{(3x^4y^5)^2}{x^2y^5} = \frac{3^2x^{4\cdot2}y^{5\cdot2}}{x^2y^5} = \frac{9x^8y^{10}}{x^2y^5} = 9x^{8-2}y^{10-5} = 9x^6y^5$$

Example 3: Simplify $\frac{(-4x^3y^{-1})^2(5x^3y^{-2})^0}{(2x^4y)^3}$

Step 1: Apply the Zero-Exponent Rule. In this case, after applying the zero-exponent rule and multiplying by 1, that term is essentially gone.	$\frac{(-4x^3y^{-1})^2(1)}{(2x^4y)^3}$
Step 2: Apply the Power Rule. In this case, I kept the –2 in parentheses because I did not want to lose the negative sign.	$\frac{(-4)^2x^6y^{-2}}{2^3x^{12}y^3}$

Step 3: Apply the Negative Exponent Rule. Move every negative exponent in the numerator to the denominator and vice versa.	$\frac{(-4)^2x^6}{y^2 2^3 x^{12} y^3}$
Step 4: Apply the Product Rule.	$\frac{(-4)^2x^6}{2^3 x^{12} y^5}$
Step 5: Apply the Quotient Rule. In this case, the x's ended up in the denominator.	$\frac{(-4)^2}{2^3 x^6 y^5}$
Step 6: Raise each coefficient (or number) to the appropriate power and then simplify or reduce any remaining fractions. In this case, the fraction does reduce.	$\frac{16}{8x^6y^5} = \frac{2}{x^6y^5}$

Example 4: Simplify $\frac{(6x^3y^{-2})^{-2}(3x^4y^{-5})^2}{(2x^4y^2)^{-3}}$

Step 1: Apply the Zero-Exponent Rule. In this case, there are no zero powers.	$\frac{(6x^3y^{-2})^{-2}(3x^4y^{-5})^2}{(2x^4y^2)^{-3}}$
Step 2: Apply the Power Rule.	$\frac{6^{-2}x^{-6}y^4 3^2 x^8 y^{-10}}{2^{-3}x^{-12}y^{-6}}$
Step 3: Apply the Negative Exponent Rule. Move every negative exponent in the numerator to the denominator and vice versa.	$\frac{y^4 3^2 x^8 2^3 x^{12} y^6}{6^2 x^6 y^{10}}$
Step 4: Apply the Product Rule. In this case, we can apply the rule to the x's and y's in the numerator.	$\frac{3^2 2^3 x^{20} y^{10}}{6^2 x^6 y^{10}}$
Step 5: Apply the Quotient Rule. In this case, the x's ended up in the numerator and the y's ended up canceling out.	$\frac{3^2 2^3 x^{14}}{6^2}$
Step 6: Raise each coefficient (or number) to the appropriate power and then simplify or reduce any remaining fractions. In this case, the numbers in the numerator get multiplied together and then the fraction gets reduce.	$\frac{9\cdot 8x^{14}}{36} = \frac{72x^{14}}{36} = 2x^{14}$

Practice: Simplify.

a) $\frac{-32x^5y^4}{(4xy^2)^2}$ b) $\left(\frac{12x^4y}{-3xy}\right)^2$ c) $\frac{6x^{-2}y^3\cdot 2^{-1}x^{-3}y^{-1}}{(3x^0y^3)^{-3}}$

2. Working with Scientific Notations using Exponential Rules

a) Multiply Two Numbers Written in Scientific Notation:

$(9 \times 10^{-1}) \times (3 \times 10^{10}) = 2.7 \times 10^{10}$

Multiplications and divisions can be done in any order - take advantage of this! First, multiply the two coefficients and then multiply the two powers of ten by adding their exponents: since -1 + 10 = 9, then $10^{-1} \times 10^{10} = 10^9$. Finally, combine your two answers and convert to scientific notation: $27 \times 10^9 = 2.7 \times 10^{10}$. In symbols:

$(9 \times 10^{-1}) \times (3 \times 10^{10})$
$= (9 \times 3) \times (10^{-1} \times 10^{10})$
$= (27) \times (10^9)$
$= 2.7 \times 10^{10}$

b) Divide Two Numbers Written in Scientific Notation:

$(3.5 \times 10^{-6}) / (5 \times 10^{-2}) = 7 \times 10^{-5}$

Distribute the division across both the coefficients and the powers of ten. Next, divide the two coefficients: 3.5/5 = (35/10)/5 = (35/5)/10 = 7/10 = 0.7. Then, divide the two powers of ten by subtracting their exponents: since -6 - (-2) = -6 + 2 = -4, then $10^{-6} / 10^{-2} = 10^{-4}$. Finally, combine your two answers and convert to scientific notation. In symbols:

$(3.5 \times 10^{-6}) / (5 \times 10^{-2})$
$= (3.5 / 5) \times (10^{-6} / 10^{-2})$
$= (0.7) \times (10^{-4})$
$= 7 \times 10^{-5}$

c) Add Two Numbers Written in Scientific Notation:

$4.9 \times 10^2 + 7.9 \times 10^3 = 8.39 \times 10^3$

First, factor out one of the powers of ten; either will work, but the smaller one may be easiest. This involves dividing both numbers by the power of ten and multiplying the whole quantity by the same power of ten. To divide one power of ten by another, simply subtract the two exponents (see Multiplication/Division). Next, convert the two numbers from scientific notation to real numbers. Now add the two numbers normally. Finally convert to scientific notation if the coefficient is less than 1 or greater than 10.

$4.9 \times 10^2 + 7.9 \times 10^3$
$= (4.9 \times 10^2/10^2 + 7.9 \times 10^3/10^2) \times 10^2$
$= (4.9 \times 10^0 + 7.9 \times 10^1) \times 10^2$
$= (4.9 + 79) \times 10^2$
$= 83.9 \times 10^2$
$= 8.39 \times 10^3$

Another way to perform any operation on two scientific notation numbers is to convert both to normal numbers, then perform the operation and finally convert the result back to scientific notation. This method is cumbersome, however, if either exponent is very large or very small. Here it works beautifully.

$4.9 \times 10^2 + 7.9 \times 10^3$
$= 490 + 7900$
$= 8390$
$= 8.39 \times 10^3$

d) Subtract Two Numbers Written in Scientific Notation:
$4.9 \times 10^{-6} - 7.9 \times 10^{-5} = -7.41 \times 10^{-5}$

As with addition, start by factoring out one of the powers of ten. Next, convert both scientific notation numbers to real numbers. Subtract the two numbers normally and convert to scientific notation if the coefficient is not between 1 and 10 (or 1 and -10).

$4.9 \times 10^{-6} - 7.9 \times 10^{-5}$
$= (4.9 \times 10^{-6} / 10^{-6} - 7.9 \times 10^{-5} / 10^{-6}) \times 10^{-6}$
$= (4.9 \times 10^{0} - 7.9 \times 10^{1}) \times 10^{-6}$
$= (4.9 - 79) \times 10^{-6}$
$= - (79 - 4.9) \times 10^{-6}$
$= -74.1 \times 10^{-6}$
$= -7.41 \times 10^{-5}$

Practice in class

1. Simplify

a) $(3a^3b^2)^{-2}(2a^4b)^3$
b) $32x^5z \div (2x^2z^{-2})^{-3}$
c) $(\frac{3x^{-2}y}{2y^{-1}})^{-2}$
d) $\frac{p^{-2}q}{c^3} \div \left(\frac{p^5q^{-2}}{c^3}\right)^{-1}$

2. Evaluate using scientific notation.

a) $\frac{0.000039 \times 2970000000}{26600000}$

b) $\frac{4920000000 \times 0.000081}{0.0000367}$

c) $\frac{0.0003^2}{1500 \times 0.00005}$