Analytic Geometry

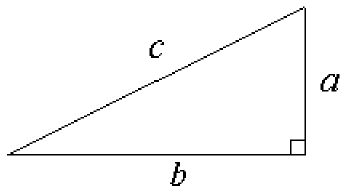
1. Basic Definitions

1) Distance Formula

Recall Pythagoras' Theorem:

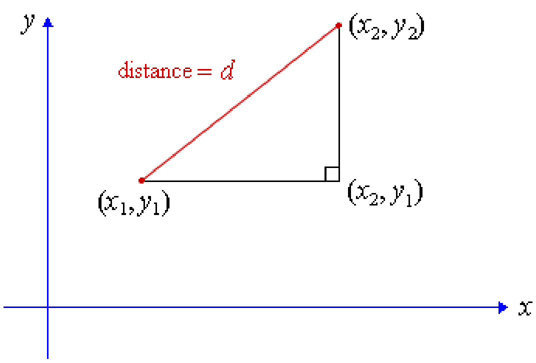
For a right-angled triangle with hypotenuse length *c*,

$c = \sqrt{a^2 + b^2}$



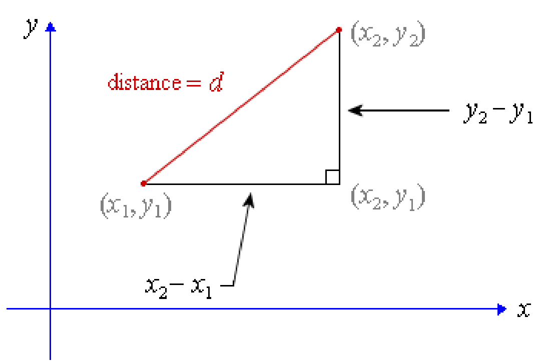
We use this to find the distance between any two points (x_1, y_1) and (x_2, y_2) on the **cartesian plane**:

The cartesian plane was named after Rene Descartes. It is also called the ***x-y* plane**.
See more about Descartes in Functions and Graphs.



The point (x_2, y_1) is at the right angle. We can see that:

- The distance between the points (x_1, y_1) and (x_2, y_1) is simply $x_2 - x_1$ and
- The distance between the points (x_2, y_2) and (x_2, y_1) is simply $y_2 - y_1$.



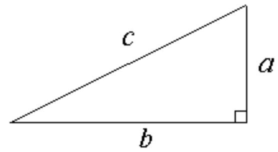
Using Pythagoras' Theorem we have the distance between (x_1, y_1) and (x_2, y_2) given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2) Gradient (or slope)

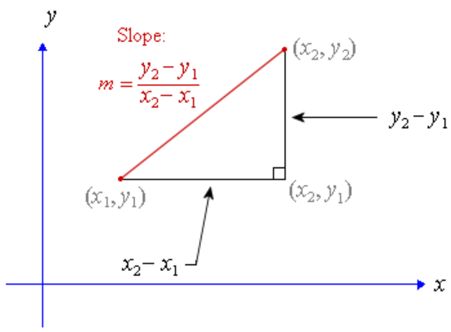
The **slope** of a line is defined as

$$\frac{\text{vertical rise}}{\text{horizontal run}} = \frac{a}{b}$$



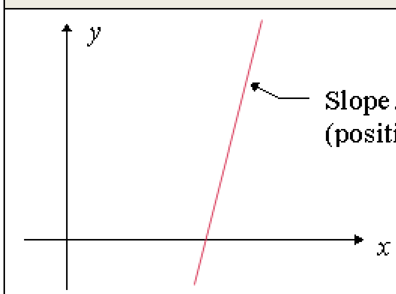
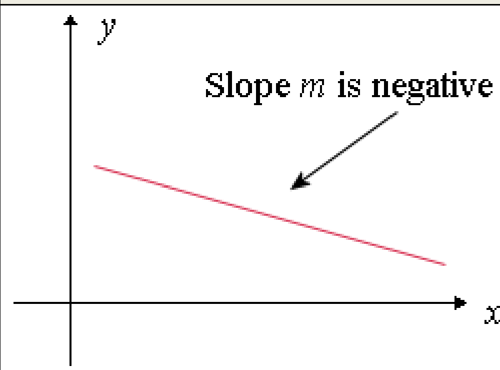
In this triangle, the gradient of the line is given by: $\frac{a}{b}$

In general, for the line joining the points (x_1, y_1) and (x_2, y_2):

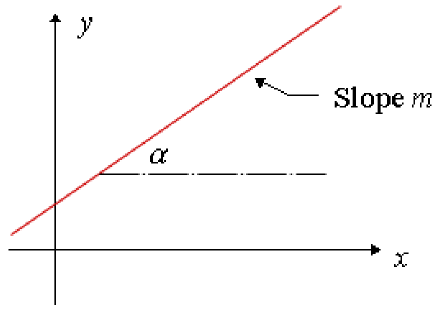


We see from the diagram above, that the **slope** (usually written *m*) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In general, a **positive slope** indicates the value of the dependent variable **increases** as we go left to right:	A **negative slope** means that the value of the dependent variable is **decreasing** as we go left to right:
 [The **dependent variable** in the above graph is the *y*-value.]	

3) Inclination



We have a line with slope *m* and the angle that the line makes with the *x*-axis is α. From trigonometry, we recall that the tan of angle α is given by:

$$\tan\alpha = \frac{opposite}{adjacent}$$

Now, since slope is also defined as opposite/adjacent, we have:

$$\tan\alpha = \frac{opposite}{adjacent} = m$$

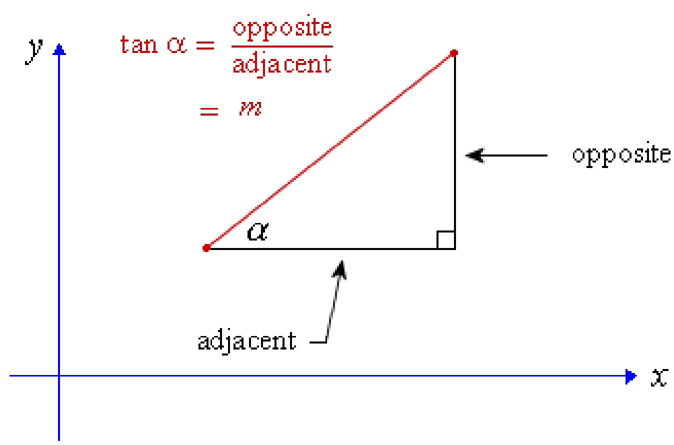
This gives us the result:

$\tan\alpha = m$

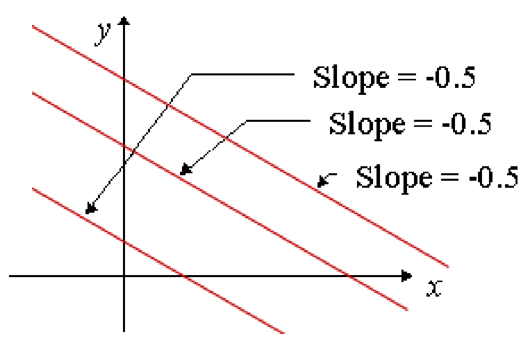
Then we can find angle α using

$\alpha = \arctan\ m$ (That is, $\alpha = \tan^{-1} m$)

This angle α is called the **inclination** of the line.



4) Parallel Lines

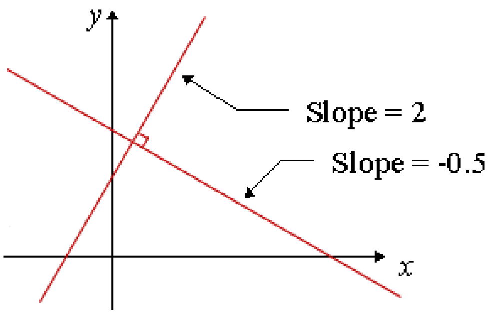


Lines which have the same slope are *parallel*.

If a line has slope m_1 and another line has slope m_2 then the lines are **parallel** if

$m_1 = m_2$

5) Perpendicular Lines



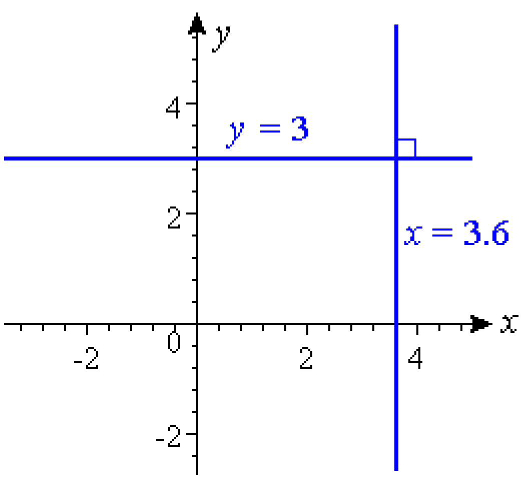
If a line has slope m_1 and another line has slope m_2 then the lines are **perpendicular** if
$m_1 \times m_2 = -1$

In the example at right, the slopes of the lines are 2 and - 0.5 and we have:
2 × - 0.5 = - 1
So the lines are perpendicular.

What if one of the lines is parallel to the *y*-axis?

For example, the line $y = 3$ is parallel to the *x*-axis and has slope 0. The line $x = 3.6$ is parallel to the *y*-axis and has an *undefined* slope.

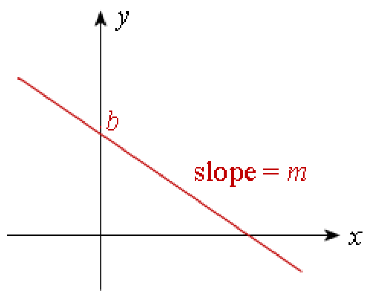
The lines are clearly perpendicular, but we cannot find the product of their slopes. In such a case, we cannot draw a conclusion from the product of the slopes, but we can see immediately from the graph that the lines are perpendicular.



The same situation occurs with the ***x*- and *y*-axes**. They are perpendicular, but we cannot calculate the product of the 2 slopes, since the slope of the *y*-axis is undefined.

2. The Straight Line

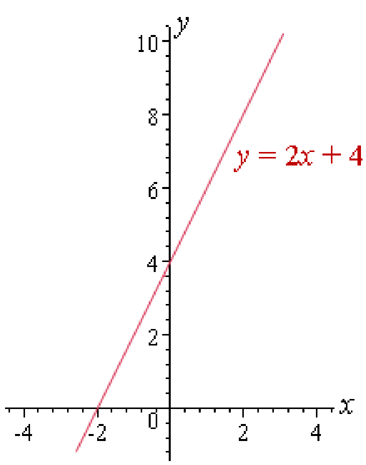
1) Slope-Intercept Form of a Straight Line



The slope-intercept form (otherwise known as "gradient, y-intercept" form) of a line is given by:
$y = mx + b$

This tells us the slope of the line is m and the y-intercept of the line is b.

For example: The line $y = 2x + 4$ has slope $m = 2$ and y-intercept $b = 4$.

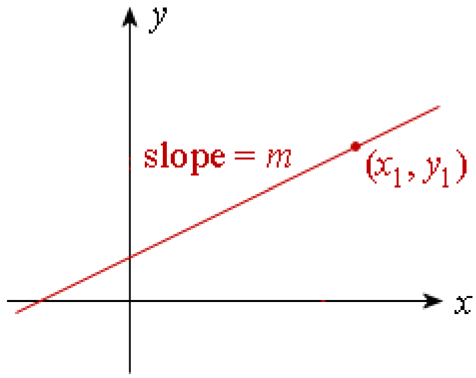


We do not need to set up a table of values to sketch this line. Starting at the y-intercept ($y = 4$), we sketch our line by going up 2 units for each unit we go to the right (since the slope is 2 in this example). To find the x-intercept, we let $y = 0$.

$2x + 4 = 0, \quad x = -2$

We notice that this is a *function*. That is, each value of x that we have gives one corresponding value of y.

2) Point-Slope Form of a Straight Line



We need other forms of the straight line as well. A useful form is the **point-slope form** (or point - gradient form).

We use this form when we need to find the equation of a line passing through a point (x_1, y_1) with slope *m*:

$(y_2 - y_1) = m(x_2 - x_1)$

3) General Form

Another form of the straight line which we come across is **general form:**

***Ax* + *By* + *C* = 0**

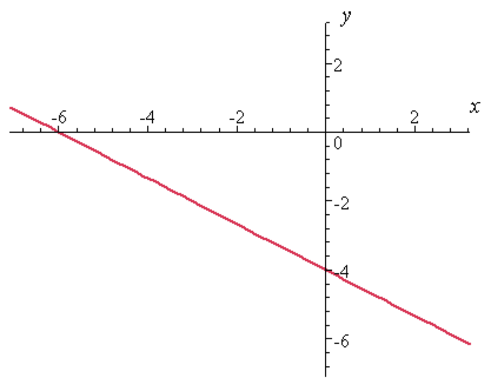
It can be useful for drawing lines by finding the *y*-intercept (put *x* = 0) and the *x*-intercept (put *y* = 0).

For example: Draw the line $2x + 3y + 12 = 0$.

If $x = 0$, we have: $3y + 12 = 0$, so $y = -4$.

If $y = 0$, we have: $2x + 12 = 0$, so $x = -6$.

So the line is:



Note that the *y*-intercept is -4 and the *x*-intercept is -6.

Questions in class

1. Find the real number a for which the lines $-4x + 3y = 7$ and $ax - 2y = 4$ will be perpendicular.

2. Find a point on the y-axis that is equidistant from (1, 1) and (5, 5).

3. Let f(x) be a linear function with $f(1) = 2$ and $f(3) = 8$. Find f(5).

4. Find the area of the region bounded by the graphs of $y = |x| + 1$ and $y = 5$.

5. The *x*-axis, the *y*-axis, and the line through the point (1, 3) having slope -3 form a triangle, what is the area with the triangle.

6. Find the distance from the point (5, 6) to the line $x + y = 3$.

7. If the point (x, y) is one-third of the way from (-2, 6) to (6,-8), then what is $x + y$?

8. Find the distance from the point (2, 3) to the line $x - y = 5$.

9. A line has *y*-intercept (0, 3) and is perpendicular to the line $2x + y = 3$. Find the *x*-intercept of the line.

10. The line $3x + my = 3m$ cuts a triangular portion from the first quadrant whose area is 6. What is the value of m?

11. Find the x-coordinate of the point on the x-axis equidistant to $(1, -1)$ and $(-5, -5)$.