Chapter 5 Analytic Geometry (3)

1. Equation of Parallel and Perpendicular Lines

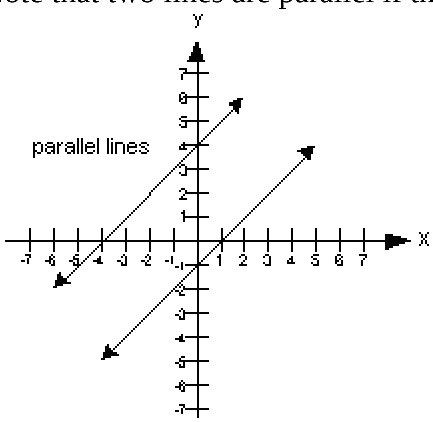
Parallel Lines and Their Slopes

Straight lines will be parallel if they have the same slope.

The following are equations of parallel lines: $y = 3x + 1$ and $y = 3x - 8$.
They have the same slope 3.

So if two lines are **parallel**, then they have the **same slope**. That is $m_1 = m_2$
In other words, the slopes of parallel lines are equal.

Note that two lines are parallel if there slopes are equal and they have **different y - intercepts.**



If the y-intercepts are the same, then two lines are **coinciding – exactly the same line**.

Perpendicular Lines and Their Slopes

Straight lines will be perpendicular if

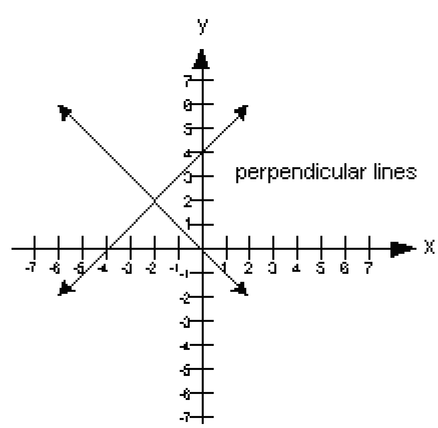
• their slopes have opposite signs -- one positive and one negative, and
• they are reciprocals of one another.

That is: If *m* is the slope of one line, then a **perpendicular** line has slope −1/m or

$$m_1 \times m_2 = -1$$

In other words, perpendicular slopes are **negative reciprocals of each other.**

For example, if a line has slope 4, then every line that is perpendicular to it has slope −¼.



Example 1: Given the line $2x - 3y = 9$ and the point $(4, -1)$, find lines through the point that are

(a) parallel to the given line.

(b) perpendicular to the given line.

In other words, they've given me a reference line — $2x - 3y = 9$ — that I'll be comparing to, and some point somewhere else on the plane — namely, $(4, -1)$. Then they want me to find the line through $(4, -1)$ that is parallel to (that has the same slope as) $2x - 3y = 9$. On top of that, they then want me to find the line through $(4, -1)$ that is perpendicular to (that has a slope that is the negative reciprocal of the slope of) $2x - 3y = 9$.

Clearly, the first thing I need to do is solve "$2x - 3y = 9$" for "$y =$", so that I can find my reference slope:

$$2x - 3y = 9$$
$$-3y = -2x + 9$$
$$y = (2/3)\, x - 3$$

So the reference slope from the reference line is $m = 2/3$.

a) Since a parallel line has an identical slope, then the parallel line through $(4, -1)$ will have slope $m = 2/3$. Hey, now I have a point and a slope! So I'll use the point-slope form to find the line:

$$y - (-1) = (2/3)\,(x - 4)$$
$$y + 1 = (2/3)\, x - 8/3$$
$$y = (2/3)\, x - 8/3 - 3/3$$

$y = (2/3)\, x - 11/3$This is the parallel line that they asked for.

b) For the perpendicular line, I have to find the perpendicular slope. The reference slope is $m = 2/3$, and, for the perpendicular slope, I'll flip this slope and change the sign. Then the perpendicular slope is $m = -3/2$. So now I can do the point-slope form. Note that the only change from the calculations I just did is that the slope is different now.

$$y - (-1) = (-3/2)\,(x - 4)$$
$$y + 1 = (-3/2)\, x + 6$$

$$y = (-3/2)\, x + 5$$

Then the full solution to this exercise is:

Parallel line: $\boldsymbol{y = (2/3)\, x - 11/3}$

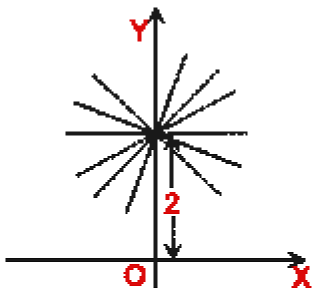
perpendicular line: $\boldsymbol{y = (-3/2)\, x + 5}$

2. Families of Lines

We always need at least two pieces of information to find the unique equation. If we are only given one, then we will be able to find a family of lines – all possible lines that satisfies the condition. We will have an unknown constant (parameter) in our equation.

Example 1: Determine the equation for the family of lines having y-intercept 2.

The equation $y = mx + 2$, for different real values of m, represents a family of lines with y-intercept 2 units. A few members of this family are shown in figure below.



Example 2: Find the equation of the family of lines with x-intercept -4.

Since the x-intercept of the family is given to be -4, therefore, each member of the family passes through the point (-4, 0).

By using point-slope form, the equation of such a family of lines is $y – 0 = m (x - (-4))$
i.e. $y = m (x + 4)$, where m is a parameter.

Note. The above equation of the family does not give the vertical line through the point (-4, 0). However, the equation of this line is $x = -4$ i.e. $x + 4 = 0$.

Therefore, $y = m(x + 4)$ or $x = -4$.

3. Application

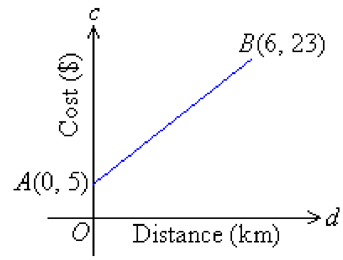
A linear model is a linear equation that represents a real-world scenario. You can write the equation for a linear model in the same way you would write the slope-intercept equation of a line. The y-intercept of a linear model is the quantity that does not depend on x. The slope is the quantity that changes at a constant rate as x changes. The change must be at a constant rate in order for the equation to be a linear model.

Example 1: The cost of transporting documents by courier is given by the line segment drawn in the diagram. Find the slope and the y-intercept of the line segment and describe its meaning.

Solution:

Let $(d_1, c_1) = (0, 5)$ and $(d_2, c_2) = (6, 23)$.

$$m = \frac{c_2 - c_1}{d_2 - d_1}$$
$$= \frac{23 - 5}{6 - 0}$$
$$= \frac{18}{6}$$
$$= 3$$



So, the slope of the line is 3. This means that the cost of transporting documents is \$3 per km travelled to deliver the documents. The y-intercept is 5. This means that there is a fixed charge of \$5, i.e. it costs \$5 for the courier to arrive.

Example 2: A machine salesperson earns a base salary of \$40,000 plus a commission of \$300 for every machine he sells.

a) Write an equation that shows the total amount of income the salesperson earns, if he sells x machines in a year.

The y-intercept is \$40,000; the salesperson earns a \$40,000 salary in a year and that amount does not depend on x. The slope is \$300 because the salesperson's income increases by \$300 for each machine he sells. The linear model representing the salesperson's total income is
y = \$300x + \$40,000

(a) What would be the salesperson's income if he sold 150 machines?

If the salesperson were to sell 150 machines, let x = 150 in the linear model;

300(150) + 40,000 = 85,000.

His income would be **\$85,000.**

(b) How many machines would the salesperson need to sell to earn a \$100,000 income?

To find the number of machines he needs to sell to earn a \$100,000 income, let y = 100,000 and solve for x:

y = 300x + 40,000
100,000 = 300x + 40,000
60,000 = 300x
x = 200

To earn a \$100,000 income the salesperson would need to sell **200** machines.

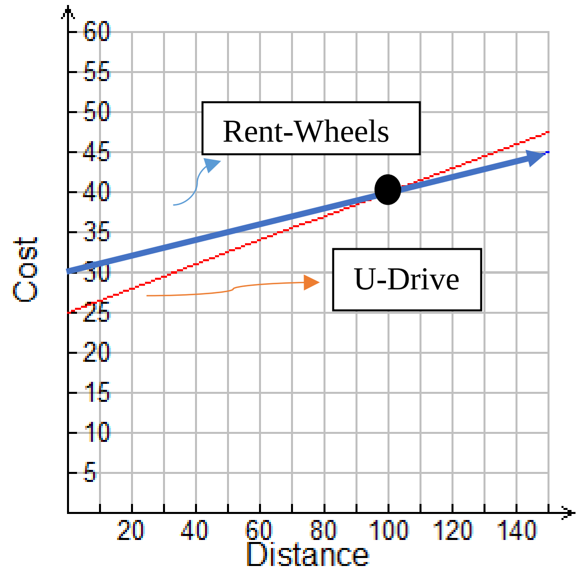
Example 3: To rent a car for one day, two companies offer different rates.

- Rent-Wheels charges $30 each day plus an additional charge of 10¢/km driven.
- U-Drive charges $25 each day plus an additional charge of 15¢/km driven.

a) Write an equation to model each company's charge.

Let C be the total cost and x be number of km driven.
For Rent-Wheels, $C = 30 + 0.1x$
For U-Drive, $C = 25 + 0.15x$

b) Graph both equations on the same set of axes.



c) What is the break-even point for the two companies?

Graphically, we can see that the break-even point is the point of intersection of the two lines, which is at (100, 40).

Algebraically, we can set the total cost equal and solve for x.
$30 + 0.1x = 25 + 0.15x$
$5 = 0.05x$
$x = 100$

If 100 km is driven, the total cost would be the same which is $40 for both companies.

d) When is it less expensive to rent a car from Rent-Wheels?

If more than 100 km is drive, it would be less expansive to rent a car from Rent-Wheels.

e) When is it more expensive to rent a car from Rent-Wheels?

If less than 100 km is drive, it would be more expansive to rent a car from Rent-Wheels.