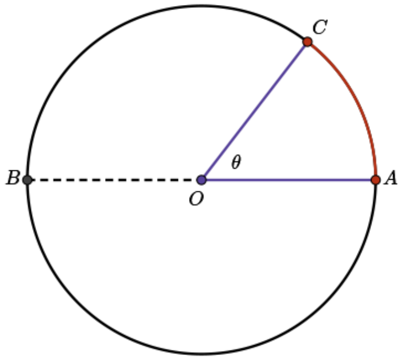
Lesson 10: Unit 6 – Trigonometric Functions (1)

Angles Measured in Degrees



The circle has centre *O*.

- *OA*, *OB*, and *OC* are radii and *AB* is a diameter.
- *AC* is a part of the circumference and it's called an arc.
- *OAC* is a sector of the circle.
- ∠*AOC* is a sector angle or central angle.

θ is said to be the angle subtended at the centre of the circle by arc *AC*.

There is a relationship between the size of a central angle and the arc length that it is subtended by.

The larger the central angle, the longer the arc length; the longer the arc length, the larger the central angle subtended by it.
To this point, most students may have only experienced angles that have been measured in degrees.

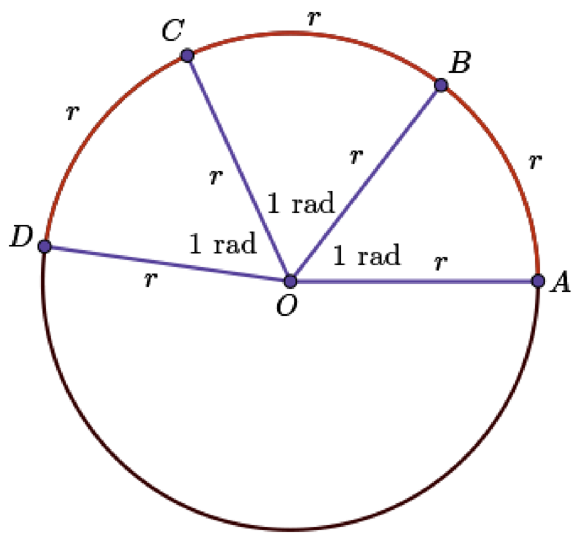
1 **degree** is the measure of an angle subtended at the centre of a circle by an arc whose length is $\frac{1}{360}$ of the circumference of the entire circle.

If the arc length is $\frac{17}{360}$ of the circumference of the circle, then the angle subtended at the centre of the circle by that arc measures 17^0.

Angles can be measured in units other than degrees.

Radians Defined

1 **radian** is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.



In other words, in a circle with radius r, the arc length, which subtends an angle of 1 radian, is r.
The unit abbreviation for radians is **rad**.
In the diagram, a circle with radius r and center O is shown.
Let arc length $AB=OA=OB=r$, the radius of the circle.

Now, arc length $AC=2r$ and it follows that $\angle AOC=2$ rad.
Next, extend the arc from C to D so that the arc length
$CD=$ arc length $BC=r$.
Now, arc length $AD=3r$ and it follows that $\angle AOD=3$ rad.
To determine the number of radii that make up a specific arc length, divide the arc length by the length of the radius.
This would also be the size of the subtended angle in radians.
That is, if θ is a central angle subtended by an arc of a circle, then θ = arc length/radius and θ is measured in radians.

Relating Radians to Degrees

What is the relationship between radians and degrees?

We know that for, θ in radians, $\theta = \frac{\text{arc length}}{\text{radius}}$.
For a circle with radius r, we also know that the circumference is $2\pi r$.
So how many radians are there in a complete revolution?

Using the formula, with arc length $=2\pi r$ and radius $r>0$, $\theta = 2\pi r/r = 2\pi$ rad.
But we also know that a complete revolution is 360^0. So,

$$2\pi \text{ rad} = 360^0 \tag{1}$$

Dividing both sides of (1) by 2, we obtain the relation

$$\pi \text{ rad} = 180^0 \tag{2}$$

Dividing both sides of equation (2) by π, we can estimate the size of 1 rad in terms of degrees

$$1 \text{ rad} = \frac{180}{\pi} \approx 57.3^0$$

Dividing both sides of equation (2) by 180, we can estimate the size of 1^0 in terms of radians

$$1^0 = \frac{\pi}{180}\text{rad} \approx 0.017 \text{ rad}$$

Converting Between Radians and Degrees

There are many ways to convert from radians to degrees or degrees to radians.
No matter which method you choose, the result that π rad $=180^0$ is typically used.
Before looking at methods used for conversion, one observation must be made at this time.
It is very common to omit the units for radian measure. *Angles given without units are assumed to be in radians*.
So our result is generally written $\pi=180^0$.
This is the convention that will be followed in this lesson.

First Method

The first method involves the use of equivalent proportions.
If asked to convert from radians to degrees, the proportion could be written

$$\frac{\text{unknown angle measured in degrees}}{\text{known angle measured in radians}} = \frac{180^0}{\pi}$$

If asked to convert from degrees to radians, the proportion could be written

$$\frac{\text{unknown angle measured in radians}}{\text{known angle measured in degrees}} = \frac{\pi}{180^0}$$

Example

a. Convert $3\pi/4$ to degrees.

b. Convert 25^0 to radians.

Solution

Let x represent the measure of the unknown angle.

a. $\frac{x}{\frac{3\pi}{4}} = \frac{180^0}{\pi}$

$x = (3\pi/4)(180^0/\pi)$

$x = 135^0$

b. $\frac{x}{25^0} = \frac{\pi}{180^0}$

$x = \frac{5\pi}{36}\text{rad} \sim 0.436 \text{ rad}$

Second Method

The second method basically multiplies the given angle measure by one.

If you multiply a radian measure by $\frac{180^0}{\pi}$, the radians divide out and you will be in degrees.

Since $180^0 = \pi$, $\frac{180^0}{\pi} = 1$.

For example, convert $2\pi/3$ to degrees.

$$2\pi/3 = (2\pi/3)(180^0/\pi) = 120^0$$

In a very similar way, we can multiply a degree measure by π180∘ to convert from degrees to radians.

Convert 300^0 to radians.

$$300^0 = 300^0(\pi/180^0) = 5\pi/3 \text{ rad}$$

Note that $5\pi/3$ is an exact answer.

To convert to an approximate answer, multiply 5 by π and divide by 3. So $5\pi/3$ rad ≈5.24 rad.

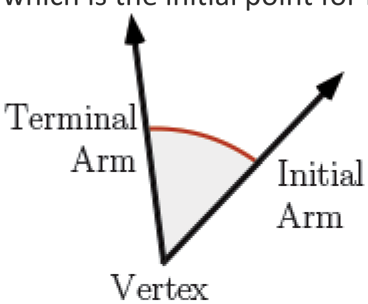
The Unit Circle

Next, we are going to

- *look over definitions,*
- *introduce new notation, and*
- *work with circles and trigonometric functions.*

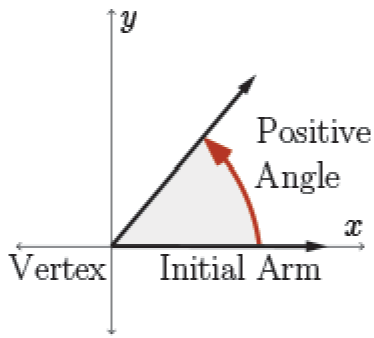
Angles in Standard Position

An angle has three parts: an initial arm and a terminal arm, both of which are rays, and a vertex which is the initial point for both rays.

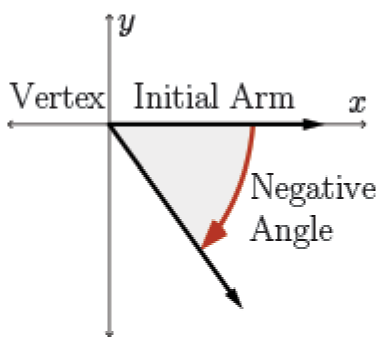


When the vertex of an angle is positioned at the origin and the initial ray lies along the positive *x*-axis, the angle is said to be in **standard position**.

A standard position angle is positive when the initial arm of the angle is rotated in a counterclockwise direction about the origin to the terminal arm.



A standard position angle is negative when the initial arm of the angle is rotated in a clockwise direction about the origin to the terminal arm.



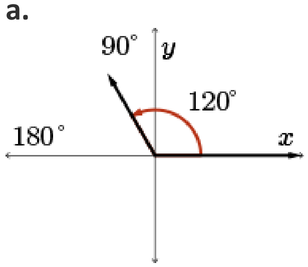
Example

Sketch the following angles in standard position

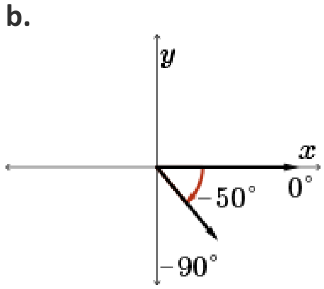
a. 120^0 **b.** -50^0 **c.** $\frac{\pi}{6}$ rad **d.** $5\pi/4$ rad

Solution

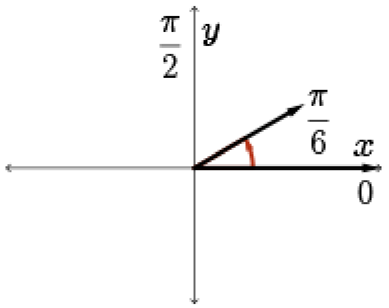
a.



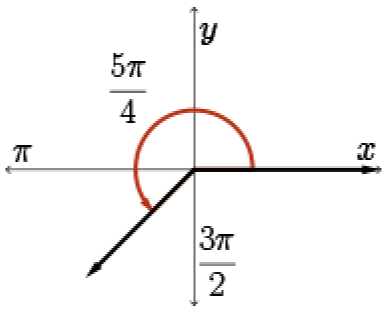
b.



c.



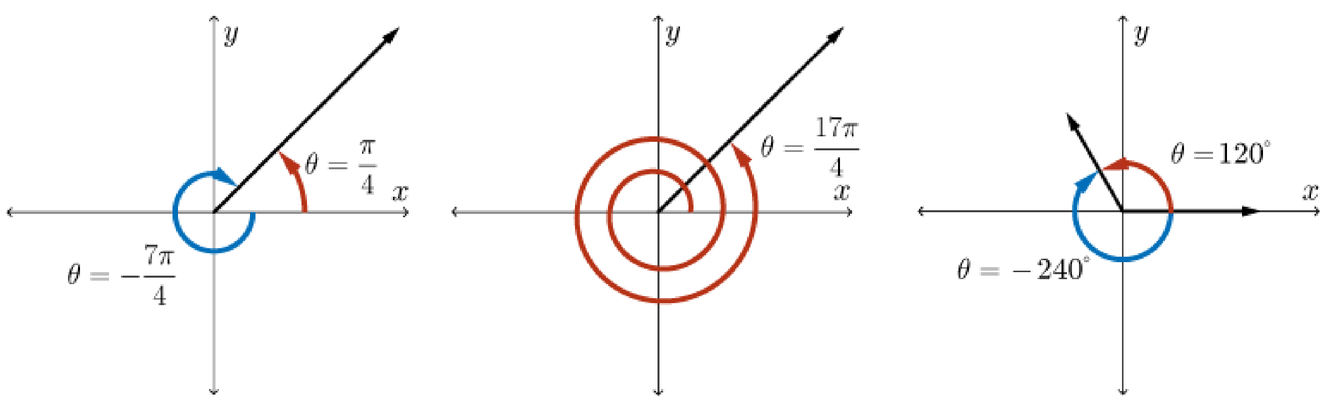
d.



Notice that the axes which form the boundary for the quadrant containing the terminal arm are marked with angle measures. These are provided for reference purposes only.

Coterminal Angles

Two angles in standard position are said to be **coterminal** if they share the same terminal arm.



The first diagram illustrates two coterminal angles: $\pi 4$ and $-7\pi.4$. The difference between the measure of the two angles is 2π radians. The angle $17\pi/4$, shown on the second diagram, is also coterminal with each angle shown on the first diagram. The difference between $17\pi/4$ and $\pi/4$ is 4π. The difference between $17\pi.4$ and $-7\pi.4$ is 6π.

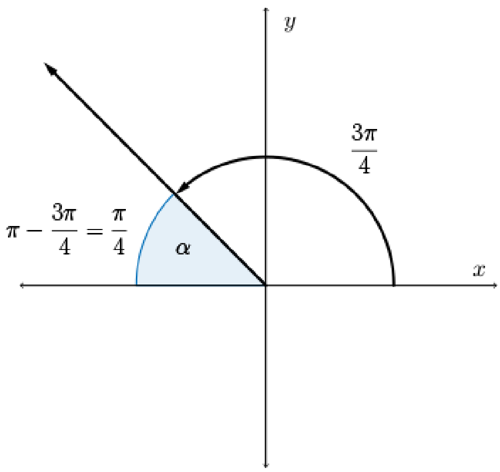
Coterminal angles measured in radians will differ by some non-zero integer multiple of 2π radians.

On the third diagram, the difference between the measure of the two angles is 360^0.

Coterminal angles measured in degrees will differ by some non-zero integer multiple of 360^0.

In general, two angles in standard position are coterminal if their difference is a non-zero integer multiple of 2π or 360^0.

Reference Angles

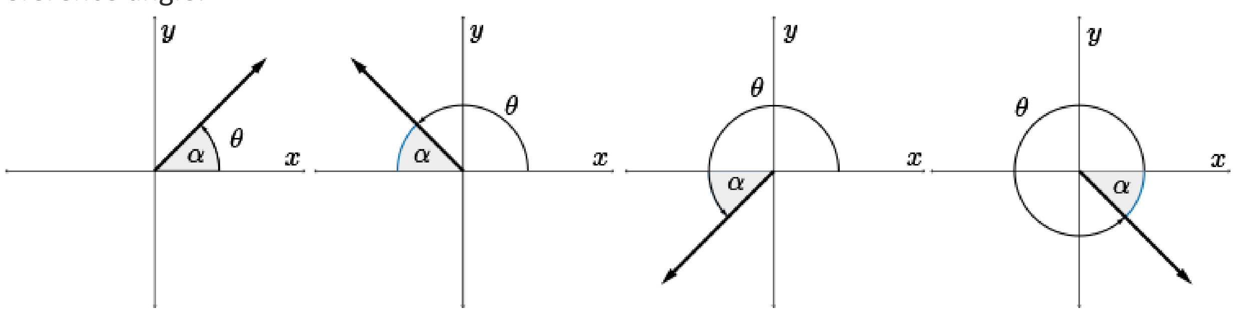


A **reference angle** is the acute angle formed between the terminal arm of a standard position angle and the *x*-axis. A reference angle is also referred to as a related acute angle.

In the diagram, the standard position angle, $3\pi/4$, has its terminal arm in quadrant 2. The reference angle is

$$\pi - 3\pi/4 = \pi/4$$

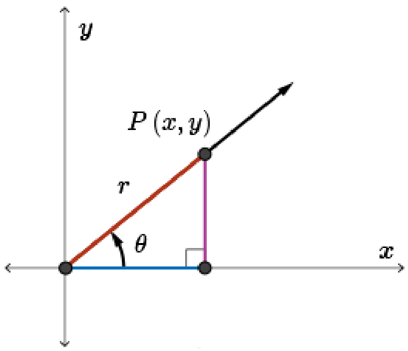
In each of the following diagrams, ϑ is a principal angle so $0 \le \vartheta \le 2\pi$ or $0^\circ \le \vartheta \le 360^\circ$ and α is the reference angle.



- If the terminal arm lies in quadrant 1, then $\alpha = \theta$.
- If the terminal arm lies in quadrant 2, then $\alpha = \pi - \theta$ or $\alpha = 180^\circ - \theta$.
- If the terminal arm lies in quadrant 3, then $\alpha = \theta - \pi$ or $\alpha = \theta - 180^\circ$.
- If the terminal arm lies in quadrant 4, then $\alpha = 2\pi - \theta$ or $\alpha = 360^\circ - \theta$.

The reference angle can be determined from a sketch. There is no need to memorize the computation for each quadrant.

Relating Trigonometric Ratios and the Unit Circle



For $P(x, y)$, any point on the terminal arm of some standard position angle θ, with r the distance from the origin to P, the definitions for the sine, cosine and tangent ratios are given in terms of x, y, r, and θ:

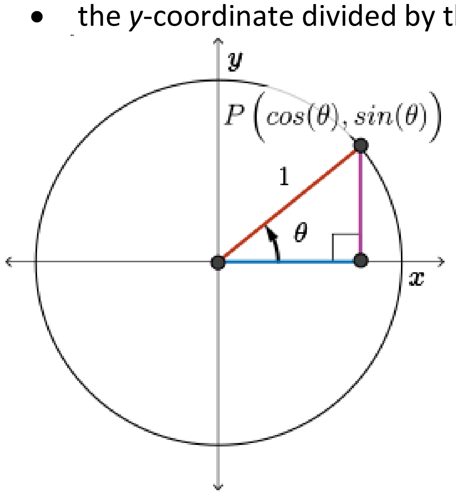
$$\sin(\theta)=\frac{y}{r} \qquad \cos(\theta)=\frac{x}{r} \qquad \tan(\theta)=\frac{x}{y}$$

These definitions were covered in earlier courses.
The unit circle is the circle with the centre the origin of the system of coordinates and radius 1(the equation of the unit circle is $x^2 + y^2 = 1$).
If this same point P (x, y) also lies on the unit circle, since r=1, then the definitions become

$$\sin(\theta)=\frac{y}{1} \qquad \cos(\theta)=\frac{x}{1} \qquad \tan(\theta)=\frac{x}{y}$$

It follows that $P(x, y)$ can be written $P(\cos(\theta), \sin(\theta))$.
That is, for any point on both the unit circle and the terminal arm of some standard position angle θ,

- the x-coordinate is $\cos(\theta)$,
- the y-coordinate is $\sin(\theta)$, and
- the y-coordinate divided by the x-coordinate is $\tan(\theta)$.



The ratios sine, cosine, and tangent are referred to as the primary trigonometric ratios.
A second set of ratios exist that are called the reciprocal trigonometric ratios.
The reciprocal trigonometric ratios are cosecant, (**csc**); secant, (**sec**); and cotangent, (**cot**).

The reciprocal trigonometric ratios are defined as follows:

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

On the unit circle, since *r*= 1, the definitions become

$\csc\theta = \frac{1}{y}$ $\sec\theta = \frac{1}{x}$ $\cot\theta = \frac{x}{y}$

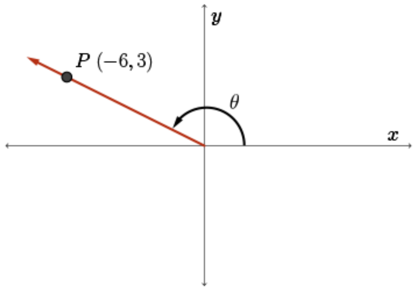
That is, for any point on both the unit circle and the terminal arm of some standard position angle θ, the reciprocal of the *x*-coordinate is sec(θ), the reciprocal of the *y*-coordinate is csc(θ) and cot(θ) is the *x*-coordinate divided by the *y*-coordinate.

Example

The point *P*(−6,3) is on the terminal arm of an angle θ in standard position where $0 \le \theta \le 2\pi$. Determine the exact values of the six trigonometric ratios.

Solution

A sketch is generally helpful in visualizing the problem.



We know that $x^2 + y^2 = r^2$ with *x*=−6 and *y*= 3.
Substituting, $r^2 = 45$ and $r = 3\sqrt{5}$ follows since *r*>0.

We can now calculate the six trigonometric ratios: $\sin(\theta) = \frac{y}{r} = \frac{3}{3\sqrt{5}} = \frac{\sqrt{5}}{5}$

$\csc\theta = \frac{1}{\sin\theta} = \sqrt{5}.$

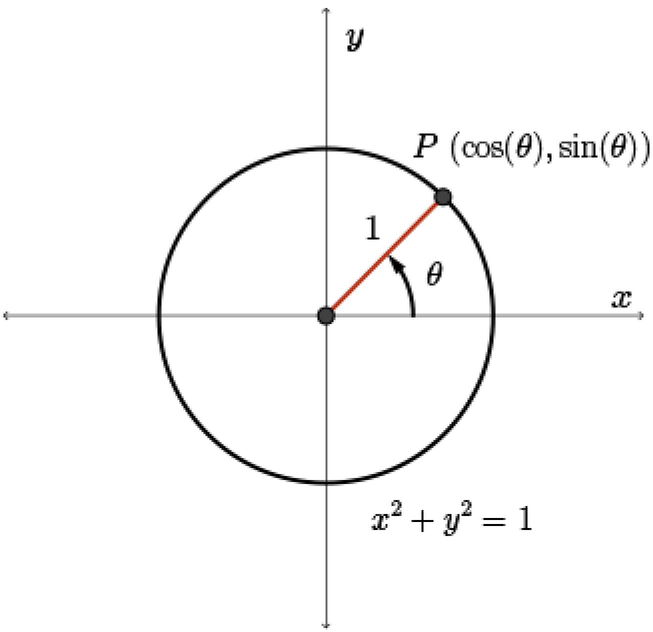
$\cos\theta = \frac{x}{r} = -\frac{2\sqrt{5}}{5}$ $\sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{5}}{2}$

$\tan(\theta) = \frac{x}{y} = -\frac{1}{2}$ $\cot\theta = \frac{1}{\tan\theta} = -2$

So far, we

- defined and sketched standard position angles,
- defined coterminal angles and reference angles,
- developed expressions for general coterminal angles and found coterminal angles in a specified domain,
- developed an equation for a circle with centre (0,0) and radius *r*,
- introduced the unit circle,
- reviewed the definitions of the primary trigonometric functions,
- introduced reciprocal trigonometric functions with definitions, and
- applied the definitions to the unit circle.

The Unit Circle and Trigonometric Ratios



Recall that a circle centred at the origin with radius 1 is called a unit circle and has equation $x^2 + y^2 = 1$.

Any point on both the unit circle and the terminal arm of some standard position angle θ can be written $P(\cos(\theta), \sin(\theta))$.

Two observations can be made at this point.

We know that $\tan(\theta) = y/x$, and, on the unit circle, $x = \cos(\theta)$ and $y = \sin(\theta)$, so $\tan(\theta) = \sin(\theta)/\cos(\theta)$.

This is true for all values of θ in the domain of $\tan(\theta)$, and is called a **quotient identity**.

Also, since every point is on the unit circle $x^2 + y^2 = 1$, we can substitute $x = \cos(\theta)$ and $y = \sin(\theta)$.

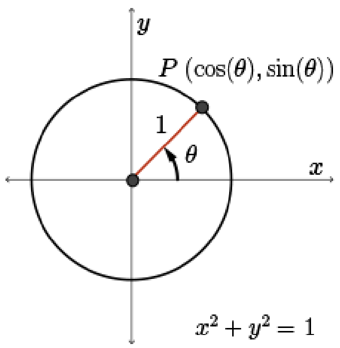
Therefore, $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. This is usually written $\cos^2(\theta) + \sin^2(\theta) = 1$.

This relationship is true for all values of θ and is called the **Pythagorean identity**.

We will do more with identities later in this unit.

We have also defined three **reciprocal trigonometric ratios**:

$\csc(\theta) = 1/\sin(\theta)$ $\quad$ $\sec(\theta) = 1\cos(\theta)$ $\quad$ $\cot(\theta) = 1/\tan(\theta)$



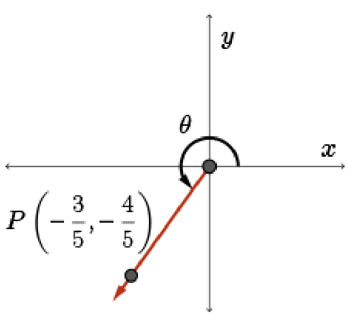
Given any point $P(x, y)$ on the terminal arm of a standard position angle, we can determine the value of any or all of the six trigonometric ratios.
If that point is also on the unit circle, the process of determining the value of any or all of the six trigonometric ratios is easier.

Example

Point $P(-3/5,-4/5)$ is on the terminal arm of standard position angle θ. Determine the exact values of each of the six trigonometric ratios.

Solution

A sketch is generally helpful.



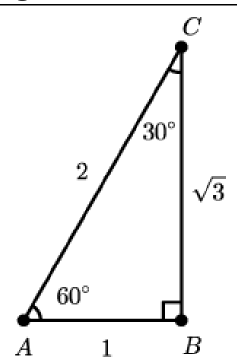
First, we calculate the value of r.
$r^2 = x^2 + y^2 = (-3/5)^2 + (-4/5)^2 = 1$
Since $r = 1$, the point $P(-3/5, -4/5)$ is on the unit circle and $(x, y) = (\cos(\theta), \sin(\theta))$. Then,

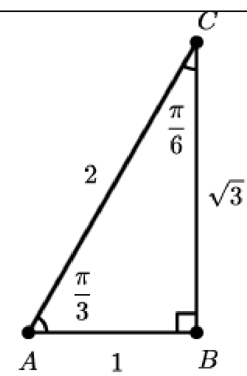
$\sin(\theta) = -\frac{4}{5}$	$\cos(\theta) = -\frac{3}{5}$	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{4}{3}$
$\csc(\theta) = \frac{1}{\sin(\theta)} = -\frac{5}{4}$	$\sec(\theta) = \frac{1}{\cos(\theta)} = -\frac{5}{3}$	$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{3}{4}$

Special Triangles

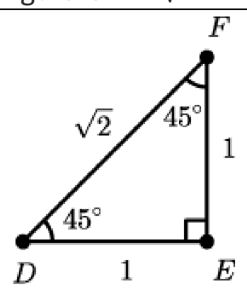
In a previous course, two special triangles were introduced.

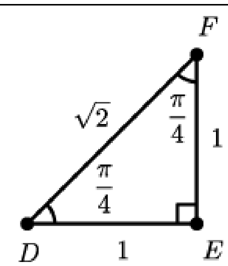
The first of the triangles is a 30^0-60^0-90^0 triangle. The ratio of the corresponding opposite side lengths is 1: $\sqrt{3}$: 2.





The second triangle is a 45^0 -45^0 -90^0 triangle. The ratio of the corresponding opposite side lengths is 1:1:$\sqrt{2}$.





Using radian measure, the first triangle is a $\pi/6$-$\pi/3$-$\pi/2$ triangle and the second triangle is a $\pi/4$-$\pi/4$-$\pi/2$ triangle.
The ratio of the side lengths remains the same.
Using these triangles, we are able to determine the exact value of each of the six trigonometric ratios corresponding to angles $\pi/6$, $\pi/4$, and $\pi/3$ or 30^0, 45^0, and 60^0 .
Determine the exact values of the six trigonometric ratios for each of the angles $\pi/6$, $\pi/4$, and $\pi/3$.

Angle (θ)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{\pi}{6}$ or $30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$ or $45°$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$ or $60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

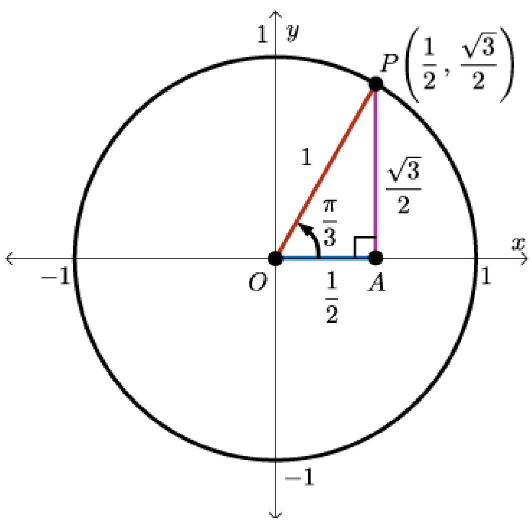
Angle (θ)	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
$\frac{\pi}{6}$ or $30°$	2	$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$ or $45°$	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$ or $60°$	$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

Example

A unit circle is shown with *OP* on the terminal arm of a *π*3radian standard position angle. Determine the coordinates of *P*.

Solution

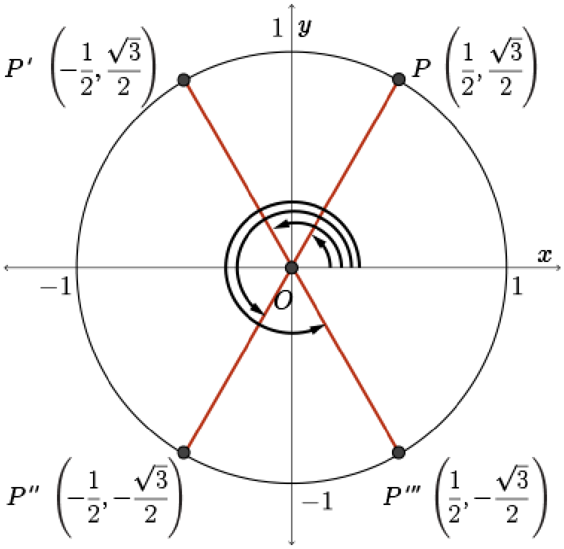
Drop a perpendicular from *P* to the *x*-axis at *A*.



We know that $\sin(\frac{\pi}{3})= \frac{\sqrt{3}}{2}$, $\cos(\frac{\pi}{3})= \frac{1}{2}$, , and $\tan(\frac{\pi}{3})= \sqrt{3}$.

We also know that, on the unit circle, each point has coordinates (cos(θ), sin(θ)).Therefore, *P* has coordinates ($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$) and it follows that *OA*= $\frac{1}{2}$, *AP*=$\frac{\sqrt{3}}{2}$, and *OP*=1.

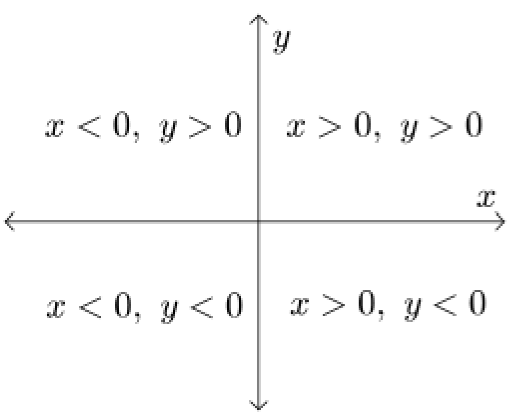
Finding More Points on the Unit Circle



Each of the four points, *P*,*P*,*P*′′, and *P*′′′ lie on the unit circle.
The standard position angles drawn to *P*, *P*′, *P*′′, and *P*′′′ all have the same reference angle: $\frac{\pi}{3}$ in this case.
If we disregard signs, all of the four points have the same *x* and *y*-coordinates.

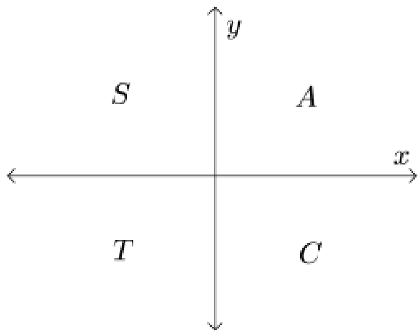
So for each of the four related standard position angles, each trigonometric ratio, without regard for the sign, will be equal.
The sign of the trigonometric ratio depends on the quadrant in which the terminal arm lies.

Determining the Sign of a Trigonometric Ratio



In previous courses, you may have determined the sign of a particular trigonometric ratio by analyzing the signs of *x* and *y* in the quadrant containing the terminal arm of the standard position angle.

Others may have used a memory device known as the **CAST rule**.



Each letter in the rule tells which trigonometric ratio is positive in a particular quadrant.
In quadrant 1, **a**ll trigonometric ratios are positive.
In quadrant 2, **s**in(θ) is positive. (csc(θ) is also positive.)
In quadrant 3, **t**an()θ is positive. (cot(θ) is also positive.)
In quadrant 4, **c**os(θ) is positive. (sec(θ) is also positive.)

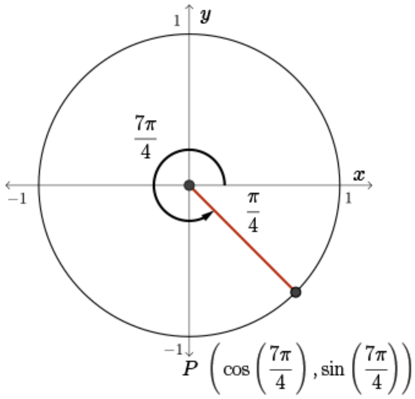
Note: The trigonometric ratios associated with any reference angle are positive. (The terms related acute angle and reference angle mean the same thing. In this course, both terms are used.)

Example

Determine the exact value of sin($\frac{7\pi}{4}$).

Solution

The standard position angle $\frac{7\pi}{4}$ has its terminal arm in quadrant 4.
The reference angle is $\pi/4$ and we know that sin($\pi/4$)= $\frac{1}{\sqrt{2}}$.



The sine function is negative in quadrant 4.

Therefore, $\sin(\frac{7\pi}{4})=-\sin(\pi/4)= -\frac{1}{\sqrt{2}}$.

This is the exact value of $\sin(\frac{7\pi}{4})$.

Most calculators will be able to give the approximate value of a trig ratio but not the exact value.

When $-\frac{1}{\sqrt{2}}$ is converted to a decimal, it is −0.707, rounded to three decimals.

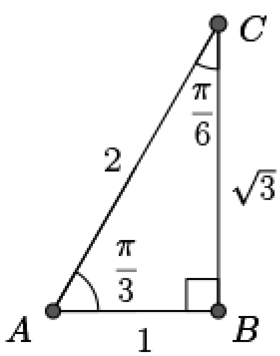
This second answer is an approximate answer.

Example

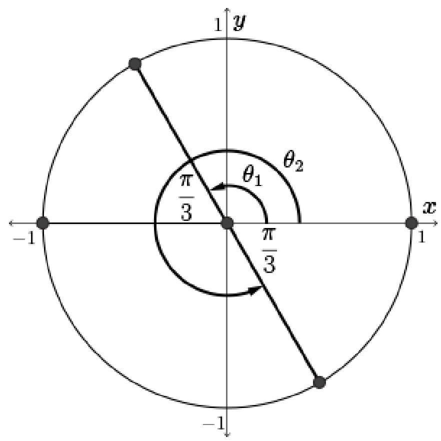
If $\tan(\theta)= -\sqrt{3}$, determine the possible values of θ such that $0 \le \theta \le 2\pi$.

Solution

Some observations are helpful here. First, the ratio $\sqrt{3}$ is familiar. We know that $\tan(\pi/3)= \sqrt{3}$. It follows that $\pi/3$ is the reference angle.



That is, there are two possible answers in the required domain.



In quadrant 2, $\theta_1 = \pi - \frac{\pi}{3} = 2\pi/3$.
In quadrant 4, $\theta_2 = 2\pi - \pi/3 = 5\pi/3$.
If there had been no restriction on the domain, we would have to add integer multiples of 2π to the answers.
That is, $\theta = \{\frac{2\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n, n \in Z\}$.

Check to see if any coterminal angles are in the given domain.
In this case, there are no additional coterminal angles.

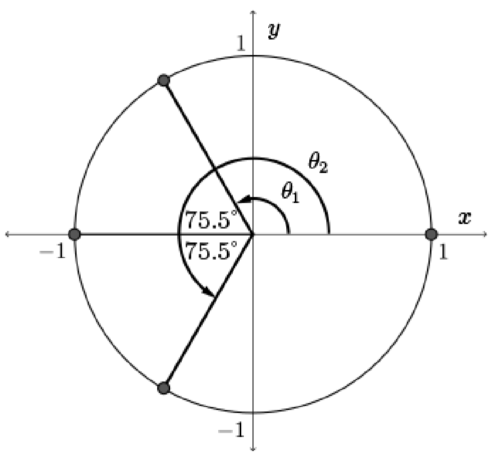
Example

If $\cos(\theta) = -1/4$, determine the possible values of θ such that $-180^0 \le \theta \le 180^0$.

Solution

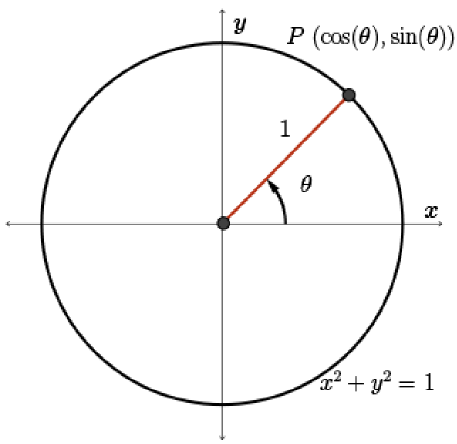
Some observations are helpful here.
First, $\cos(\theta) < 0$. Therefore, the terminal arm can be in quadrant 2 or 3.

Second, the ratio 1/4 is unfamiliar.
We can use a calculator to determine the approximate value of the reference angle.



Let α be the reference angle. Then cos(α) = +1/4.
(The trig ratio of any angle between 0^0 and 90^0 is greater than zero.)
Using your calculator, $\alpha = \cos^{-1}(1/4) \approx 75.5^0$.
Be sure your calculator is set in degree mode for this example.
In quadrant 2, $\theta_1 = 180^0 - \alpha \approx 104.5^0$.
In quadrant 3, $\theta_2 = 180^0 + \alpha \approx 255.5^0$.
An angle coterminal with 255.5^0 in the required domain is $255.5^0 - 360^0$ or -104.5^0. Therefore, $\theta \approx \{-104.5^0, 104.5^0\}$.

Previously, the unit circle was introduced and used to determine the exact trigonometric ratios for multiples of $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$ radians and multiples of 30^0, 45^0, 60^0, and 90^0.
We discovered that any point, P, on the unit circle and also on the terminal arm of a standard position angle, θ, can be written $P(\cos(\theta), \sin(\theta))$.



We will use this background to help us draw the graphs for the three primary trigonometric functions.
New terminology will be introduced as we examine properties associated with these graphs.

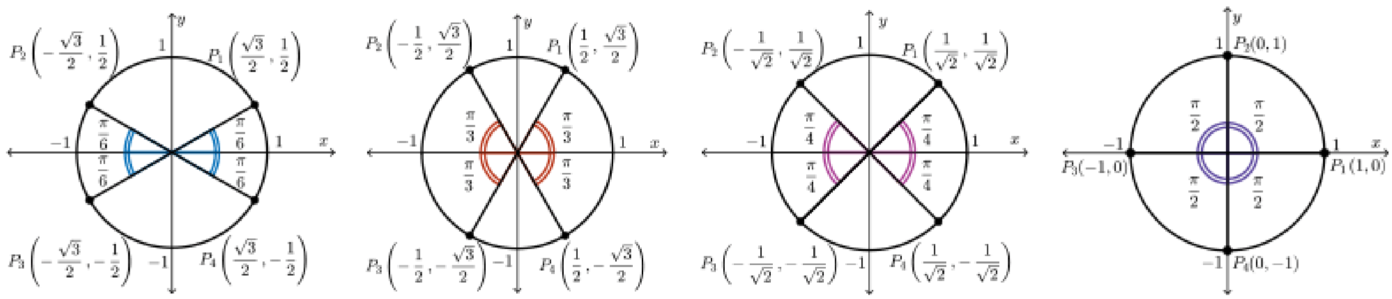
Primary Trigonometric Functions:

$y=\sin(\theta)$, $\quad y=\cos(\theta)$, $\quad y=\tan(\theta)$

Throughout this module, and in fact for the rest of the unit, you will also see $y=\sin(x)$, $y=\cos(x)$, and $y=\tan(x)$ where x is used instead of θ.

Graphing the Sine Function $y=\sin(x)$

Construct a table of values relating x and sin(x) for $0 \le x \le 2\pi$. Use increments of $\pi/12$ radians (this corresponds to 15^0 intervals). Fill in the exact values for sin(x) by referring to the appropriate unit circle diagram. We have not determined the exact values for sin(x) for which the reference angle is either $\pi/12$ or $5\pi/12$ so we will calculate approximate values later. In the table, these values are shown as “?”. Some have been filled in for you.

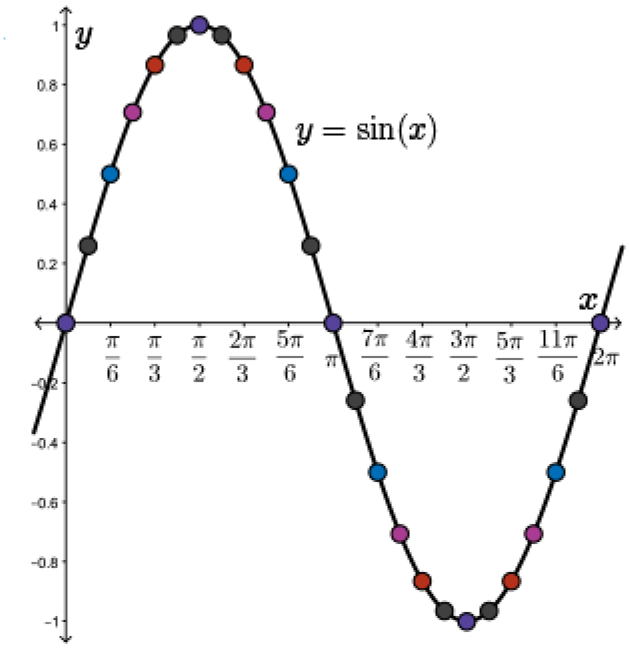


x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\sin(x)$	0	?	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	?	1	?	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	?	0
x	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$\frac{23\pi}{12}$	2π
$\sin(x)$	0	?	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	?	-1	?	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	?	0

From the table of values, we can plot points to determine the shape of the function *y*=sin(*x*). The exact values of sin(*x*) have been converted to approximate values in the following table, correct to two decimals, where necessary.
A calculator was used to determine the approximate values for multiples of $\pi/12$.
These were marked with a "?" mark in the previous chart.

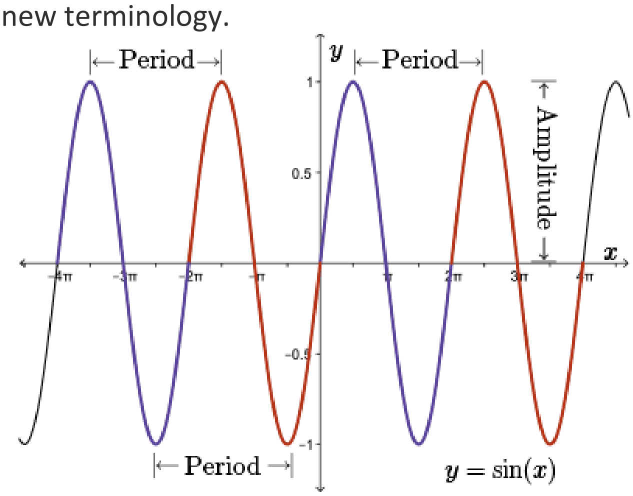
x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\sin(x)$	0	0.26	0.5	0.71	0.87	0.97	1	0.97	0.87	0.71	0.5	0.26	0
x	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$\frac{23\pi}{12}$	2π
$\sin(x)$	0	-0.26	-0.5	-0.71	-0.87	-0.97	-1	-0.97	-0.87	-0.71	-0.5	-0.26	0

Plot the points on a graph. The *x*-axis uses intervals of $\pi/12$ and the *y*-axis uses intervals of 0.2.
Connect the points on the graph with a smooth curve.
What happens if we continue our table from 2π to 4π?
It is relatively easy to see that the values in the table will repeat since we are passing through the same points on the unit circle each time we complete a full rotation.



What happens if we continue our table from 2π to 4π?
It is relatively easy to see that the values in the table will repeat since we are passing through the same points on the unit circle each time we complete a full rotation.
Another way to discuss this is through the use of coterminal angles.
For example, $25\pi/12$ is coterminal with $\pi/12$.
The terminal arm will intersect the unit circle at the same point for both $25\pi/12$ and $\pi/12$.
It follows that $\sin(25\pi/12)= \sin(\pi/12)$.

At this point, we will make some observations about the function $y=\sin(x)$ and introduce some new terminology.



The domain of the function $y=\sin(x)$ is $\{x|\ x \in R\}$.
The maximum value of the sine function is 1 and the minimum value is −1.
Therefore, the range of $y=\sin(x)$ is $\{y|\ -1 \le y \le 1,\ y \in R\}$.
The sine function cycles, that is it repeats over regular intervals of its domain.

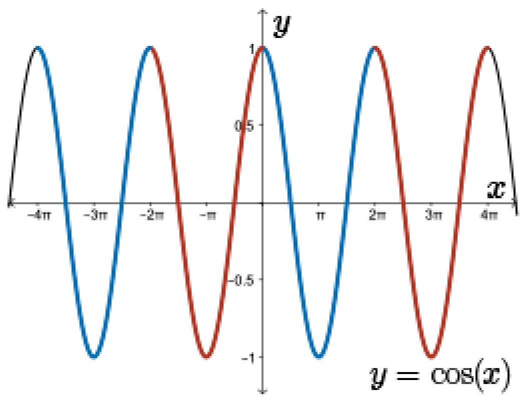
We say that sine function is **periodic**.
The horizontal length of one cycle is called the **period**.
The period of y=sin(x) is 2π or 360^0.
The y-intercept of the sine function is y=0.
There are many x-intercepts. On the graph, the x-intercepts are

$$\{\ldots-4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi\ldots\}$$

In general, the x-intercepts are $x=n\pi$, $n \in Z$ for x in radians and $x= n(180^0)$, $n \in Z$ for x measured in degrees.

A horizontal central line can be drawn through the sine curve so that the perpendicular distance from this line to a maximum point is the same as the perpendicular distance to a minimum point.
This distance is called the **amplitude**.
In this case, the horizontal central line is y=0, the x-axis, and the amplitude of the sine function is 1.
We used radian measure for the angles. The process of graphing y=sin(x) does not change if the angle measure is in degrees.

In the same way we can sketch y= cos(x).



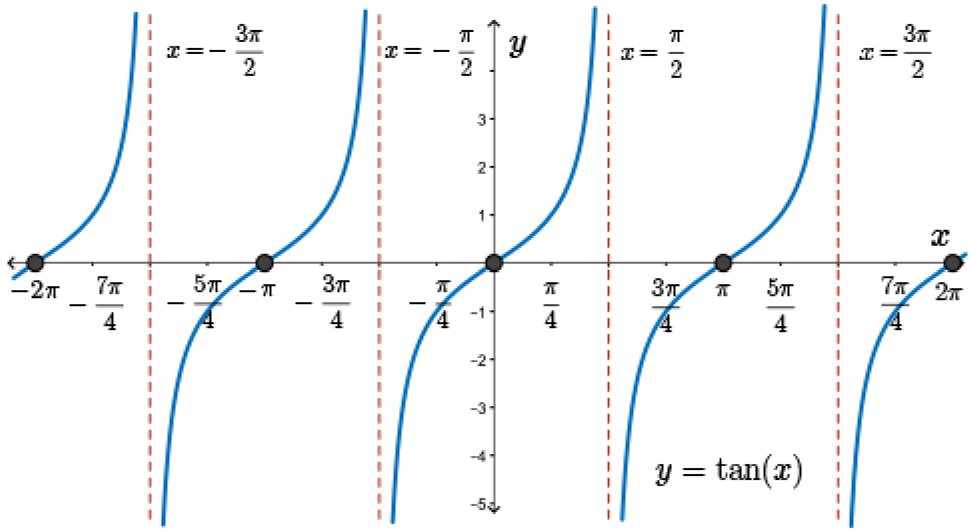
The domain of the function y= cos(x) is $\{x| \ x \in R\}$.
The maximum value of the cosine function is 1 and the minimum value is −1. Therefore, the range of y=cos(x) is $\{y| -1 \le y \le 1, y \in R\}$.
The cosine function has a period of 2π or 360∘.
The y-intercept of the cosine function is y=1.
There are many x-intercepts. On the graph, the x-intercepts are

$$\{\ldots-7\pi/2, -5\pi/2, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, 5\pi/2, 7\pi/2\ldots\}$$

In general, the x-intercepts are $x=\frac{\pi}{2} + n\pi$, $n \in Z$ for x measured in radians and = $90^0 + n(180^0)$, $n \in Z$ for x measured in degrees.
A horizontal central line is y=0, the x-axis, and the amplitude of the cosine function is 1.

The sine function and the cosine function are referred to as **sinusoidal functions**. Their graphs have the property that they oscillate above and below a central horizontal line.

Now we can continue with the function y=tan(x).



The function cycles every π radians or every 180^0, so the period of the tangent function is π radians or 180^0.

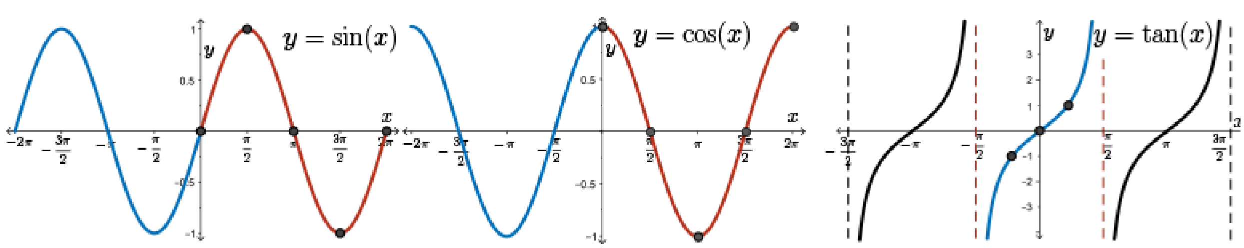
Since the function has no minimum or maximum value, the amplitude of the tangent function is not defined.

That is, the vertical asymptotes are $x = \frac{\pi}{2} + n\pi$, $n \in Z$ for x measured in radians and $x = 90^0 + n(180^0)$, $n \in Z$ for x measured in degrees. There are no horizontal asymptotes.

Since there are vertical asymptotes, it follows that the domain is $\{x|\ x \neq \pi/2 + n\pi,\ n \in Z,\ x \in R\}$ for x measured in radians and $\{x|\ x \neq 90^0 + n(180^0),\ n \in Z,\ x \in R\}$ for x measured in degrees.

The range is $\{y|\ y \in R\}$.

Summary



The results are summarized for radian measure in the following table.

	$y=\sin(x)$	$y=\cos(x)$	$y=\tan(x)$
Domain	$\{x \mid x\in\mathbb{R}\}$	$\{x \mid x\in\mathbb{R}\}$	$\left\{x \mid x\neq\frac{\pi}{2}+n\pi, n\in\mathbb{Z}, x\in\mathbb{R}\right\}$
Range	$\{y \mid -1\leq y\leq 1, y\in\mathbb{R}\}$	$\{y \mid -1\leq y\leq 1, y\in\mathbb{R}\}$	$\{y \mid y\in\mathbb{R}\}$
Maximum	$y=1$	$y=1$	none
Minimum	$y=-1$	$y=-1$	none
Period	2π	2π	π
Amplitude	1	1	not defined
Vertical Asymptotes	none	none	$x=\frac{\pi}{2}+n\pi, n\in\mathbb{Z}$
y-intercept	0	1	0
x-intercepts	$x=n\pi, \in\mathbb{Z}$	$x=\frac{\pi}{2}+n\pi, n\in\mathbb{Z}$	$x=n\pi, n\in\mathbb{Z}$

In a future lesson, we will sketch 3 reciprocal trigonometric functions: y=csc(x), y=sec(x), and y=cot(x).