Lesson 5: Unit 4- Rational Functions (1)

A **rational function** is a function of the form $f(x)=\frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials and $h(x)\neq 0$.

In the case where $h(x)=k, k\in R, k\neq 0$ (i.e., a constant polynomial of degree 0), the rational function reduces to the polynomial function $f(x)=\frac{1}{k}g(x)$.

Examples of rational functions include:

$y=\frac{1}{x^2-3x+2}$, $x\neq -1,-2$

$f(x)=\frac{x^2}{x-1}$, $x\neq 1$

$y=\frac{x^7+3x^2+10}{x^2-2x}$, $x\neq 0,2$

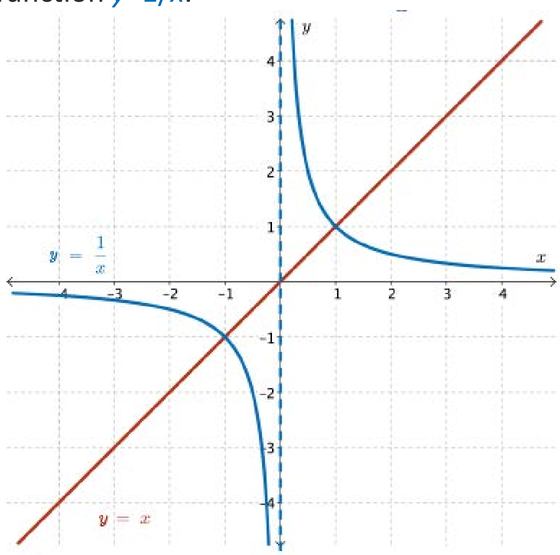
Each consists of a polynomial in the numerator and denominator. Restrictions are stated to ensure the denominator does not equal 0.

Functions such as $p(x)=\frac{\sqrt{x}}{x^2+7}$ and $r(x)=\frac{x}{|x-2|}$ are not rational functions, since the numerator in $p(x)$ and the denominator in $r(x)$ are not valid polynomials. The first rational function example, $y=\frac{1}{x^2-3x+2}$, is the reciprocal of the quadratic function $y=x^2-3x+2$.

In this module, we will investigate the behaviour of the graph of rational functions of the form $y=1/f(x)$, where $f(x)$ is a linear or quadratic polynomial function.

Example 1

We begin by considering the graph of the linear function $y=x$ and its related reciprocal function $y=1/x$.



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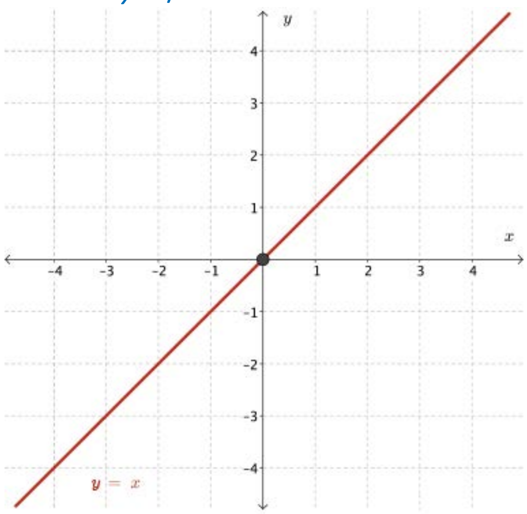
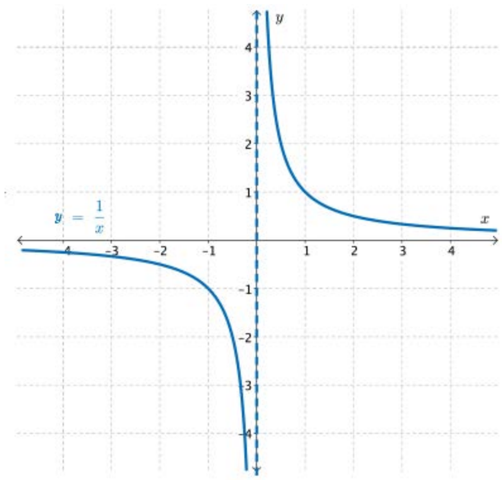


The function $y=x$ has an x-intercept at $x=0$.
The function $y=1/x$ is undefined at $x=0$.
The graph of $y=1/x$ has a vertical asymptote at $x=0$.
Recall that an **asymptote** is a line that the graph of a function approaches but does not touch, for some values of x in the domain of the function.

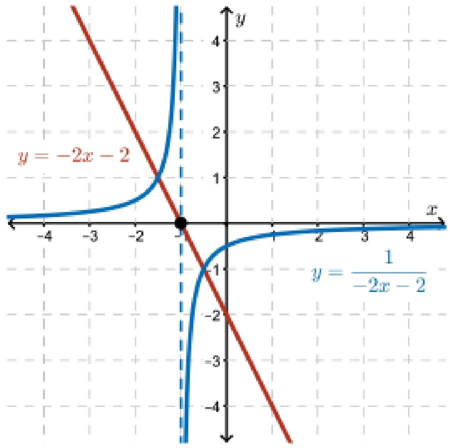


The function $y=x$ is an increasing function for all $x\in R$.
The function $y=1/x$ is decreasing on its domain (that is, for all $x\in R, x\neq 0$).
The graphs of both functions are found in the 1_{st} and 3_{rd}quadrants.
In other words, both functions are negative in the interval$x\in(-\infty,0)$ and positive in the interval $x\in(0,\infty)$.
The graphs of these functions intersect at $(-1,-1)$ and $(1,1)$.
The function $y=x$ has opposite end behaviours with $y\to-\infty$ as $x\to-\infty$ and $y\to\infty$ as $x\to\infty$.
The function $y=1/x$ has a horizontal asymptote of $y=0$, since $y\to 0$ as $x\to\pm\infty$.

Both functions are odd functions which are symmetrical about the origin.

Knowing the graph of y=f(x) and understanding the general relationship between a function and its reciprocal function y= $\frac{1}{f(x)}$ *can help when sketching the graph of the reciprocal function.*

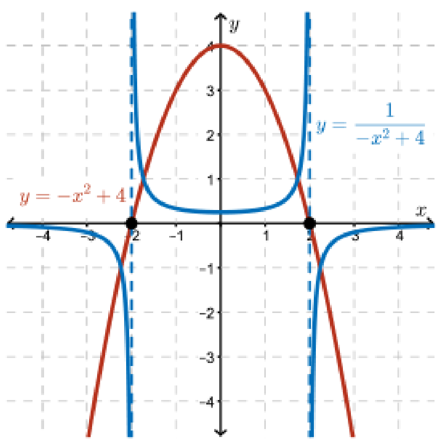
1.



$f(x)=-2x-2$

y= $\frac{1}{-2x-2}$

2.



f(x)= $-x^2+4$

y= $\frac{1}{-x^2+4}$

The vertical asymptotes of $y=1/f(x)$ occur at the zeros of the graph of $y=f(x)$.
If $x=a$ is a zero of $y=f(x)$, then $f(a)=0$ and 1$f(a)$ will be undefined; therefore, $x=a$ will be a vertical asymptote of $y=1/f(x)$. The domain of $y=1/f(x)$ is $\{x \mid f(x)\neq 0, x\in R\}$.

- A polynomial function and its related reciprocal function have the same positive and negative intervals.

When the function $y=f(x)$ is positive (above x-axis), then its reciprocal $y=1/f(x)$ is also positive.

When the function $y=f(x)$ is negative (below x-axis), then its reciprocal $y=1/f(x)$ is also negative.

- When the graph of the function $y=f(x)$ is decreasing, the graph of $y=1/f(x)$ is increasing and vice versa.

A local maximum point on $y=f(x)$ is a local minimum point on $y=1/f(x)$.
A local minimum point on $y=f(x)$ is a local maximum point on $y=1/f(x)$.
This follows from the fact that the intervals of increase of $f(x)$ are the intervals of decrease on the reciprocal, and vice versa.

- The end behaviour of all functions of the form $y=1/f(x)$ (where $f(x)$ is a polynomial) have $y\to 0$ in both directions, resulting in a horizontal asymptote of $y=0$.

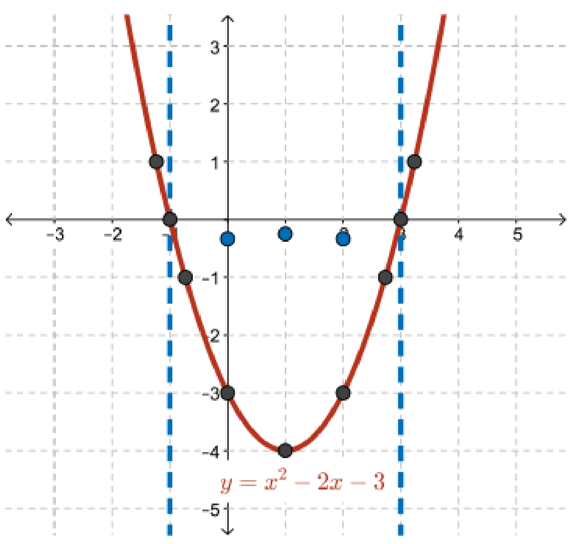
Since polynomials have end behaviour $y\to\pm\infty$, $1/f(x)\to 0$.

- If $y=f(x)$ is an even function, then its reciprocal is also even. If $y=f(x)$ is an odd function, then its reciprocal is also odd. If $y=f(x)$ is neither odd nor even, then its reciprocal is neither as well

Example 2

Using the graph of $y= x^2-2x-3$ and applying your understanding of the relationship between the graph of a function and its reciprocal, identify the key characteristics and sketch the graph of $y=\frac{1}{x^2-2x-3}$.

Solution



Using the zeros (−1,0),(3,0) and the vertex (1,−4), we will graph the parabola.

$$y=x^2-2x-3$$

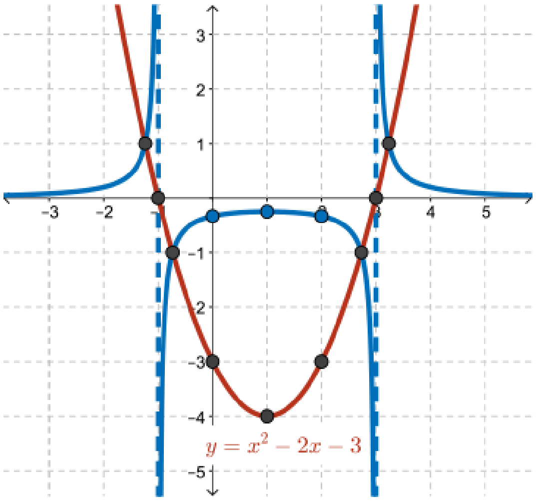
The zeros of the parabola correspond to the vertical asymptotes of the reciprocal function.
The points on the parabola where $y=\pm1$ will also be on the reciprocal function.
The vertex (1,−4) is a minimum point on the parabola, so (1,−1/4), on the reciprocal, is a local maximum point on the reciprocal function.
Other points (0,−3) and (2,−3) on the parabola will correspond to (0,−1/3) and (2,−1/3) on the reciprocal function.

When $x<-1$ or $x>3$ the quadratic function is positive, so the reciprocal is also positive in these intervals.

As $y\to\infty$,1$y\to0$. The reciprocal function has a horizontal asymptote of $y=0$.
As $y\to0$,1$y\to+\infty$, while approaching the vertical asymptotes.

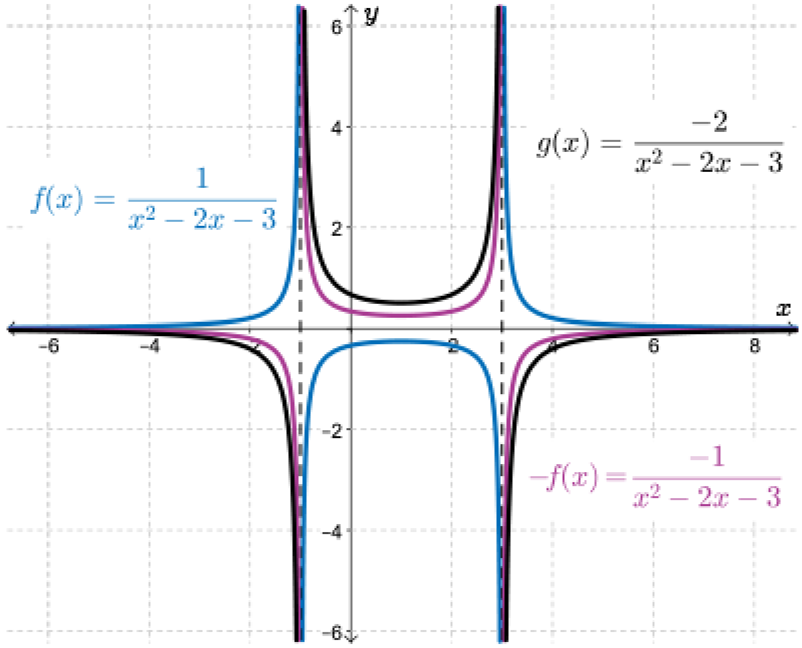
Note that for $x<-1$, the quadratic is decreasing, so its reciprocal is increasing.
For $x>3$, the quadratic is increasing, so the reciprocal is decreasing.
For $-1<x<3$, the quadratic function is negative, so the reciprocal function will also be negative in this interval.

As $y\to0$ for the parabola in this interval, the y-value of the reciprocal will approach $-\infty$.
In this interval, the quadratic function is decreasing until $x=1$ and then it is increasing. The reciprocal function is increasing until $x=1$ and then begins decreasing.
Note that the graphs of the quadratic function and its reciprocal function have neither even nor odd symmetry.



Example 2 - Extension

Use the graph of $y=\frac{1}{x^2-2x-3}$ sketched previously to obtain the sketch of the graph $y=\frac{-2}{x^2-2x-3}$.



Solution

The graph of $g(x)$=, $\frac{-2}{x^2-2x-3}$ which is $g(x)=-2f(x)$, can be obtained from the graph of $f(x)=\frac{1}{x^2-2x-3}$.
We apply a reflection in the x-axis.
We apply a vertical stretch from the x-axis by a factor of 2.

Example 3

Given the cubic function $y=x(x-2)(x+2)$, sketch the graph of $y=\frac{1}{x(x-2)(x+2)}$.

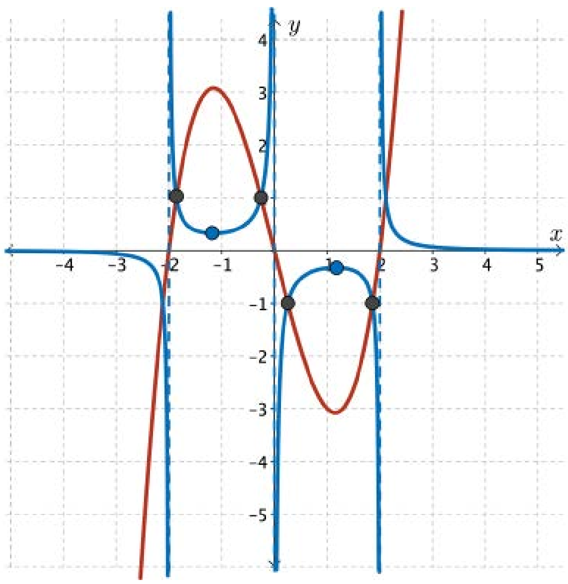
Solution

Vertical asymptotes of the reciprocal function occur at the zeros of the cubic function.

Where a local maximum occurs on the cubic function, a local minimum occurs for the reciprocal function.
Similarly, where a local minimum occurs on the cubic function, a local maximum occurs for the reciprocal function.
Points where $y=\pm1$ are common to both graphs.

Both functions are positive in the same intervals and negative in the same intervals.

When the cubic function is increasing the reciprocal function is decreasing and vice versa.
The graph of the reciprocal function will approach the x-axis, that is $y \to 0$, as $x \to \pm\infty$.



Next we will identify the vertical asymptotes and/or point(s) of discontinuity of a rational function.
We are going to study the behaviour of the rational function to the left and right side of the discontinuity and we will introduce the language of limits to assist us in communicating our understanding of the behaviour of the function close to its vertical asymptotes.

Vertical Asymptotes

A rational function $y = \frac{g(x)}{h(x)}$, $h(x) \neq 0$ will have a **vertical asymptote** at $x=a$ if $h(a)=0$ and $g(a) \neq 0$, when the function is in simplest form.

The Language of Limits

The concept of a limit, which is fundamental to calculus, is used when mathematicians are concerned with the behaviour of a function near a particular value of x (or as $x \to \pm\infty$).

If the limit of a function, $y=f(x)$, as x approaches a is equal to L, then we write

$$\lim_{x \to a} f(x) = L$$

This means that $f(x)$ gets closer and closer to the value L, as x gets closer and closer to the value a. That is, $y \to L$ as $x \to a$.

A limit provides information about how a function behaves **near**, not **at**, a specific value of x. It should also be mentioned that the $\lim_{x \to a} f(x)$ exists only when the left and right side limits exist and are equal.

Example

Determine the vertical asymptotes, if any, for the function $f(x)=\frac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

Solution

We start by factoring the denominator of the function to identify any restrictions on the value of *x*. By factoring the denominator, we can determine the restrictions on *x*.

$$f(x)=\frac{-2x+4}{x^2-x-2}$$

The domain of the function is $\{x|x\neq -1,2,\ x\in \mathrm{R}\}$.
$f(-1)=6/0$, an undefined value. This indicates that there is a vertical asymptote at $x=-1$.

What happens to the value of $f(x)$ as $x\to -1^+$?

x	$f(x)$
0	−2
−0.5	−4
−0.9	−20
−0.99	−200
−0.999	−2000

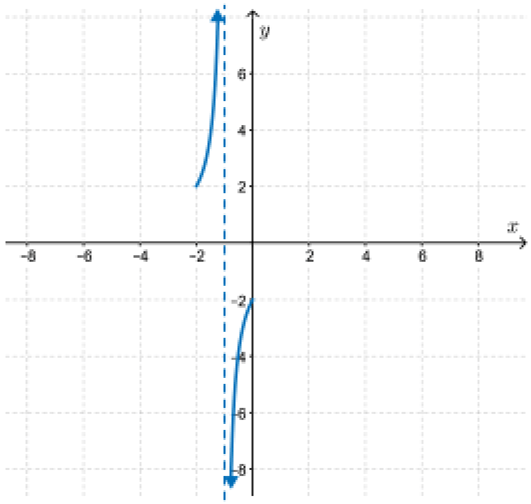
Similarly $y\to -\infty$ as $x\to -1^+$, so

$$\lim_{x\to -1^+} f(x)=-\infty$$

Thus,

$$\lim_{x\to -1^-} f(x)=\infty \text{ and } \lim_{x\to -1^+} f(x)=-\infty$$

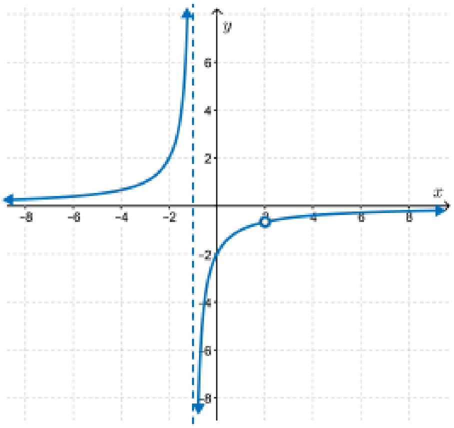
So the limit $\lim_{x\to -1} f(x)$ does not exist.



What happens at x=2?
Algebraically, the equation of the function simplifies.

$$f(x)=\frac{-2(x-2)}{(x+1)(x-2)} \rightarrow \text{f(x)}=\frac{-2}{x+1}\ x\neq -1,2$$

The graph of $f(x)$ is the same as the graph $y=\frac{-2}{x+1}$ everywhere except at the point (2, −2/3), where $f(x)$ has hole.
Note that using the simplified equation and ignoring the restriction $x\neq 2$, $f(2)=-2/3$, the y-coordinate of the point of discontinuity.



Example

Determine, with support, an equation for each rational function of the form $y=g(x)/h(x)$ that satisfy the given conditions.
a. A hole exists at (4,−2) and a vertical asymptote occurs at $x=0$.
b. $g(-3)=0$ and $h(-3)=0$, but a vertical asymptote occurs at $x=-3$.

Solution

a. For a hole to exist at $x=4$, we need $g(4)=0$ and $h(4)=0$, so $x-4$ is a factor of both the numerator and the denominator.
For a vertical asymptote to exist at $x=0$, then $h(0)=0$ and $g(0)\neq 0$, so x is a factor of the denominator, but not the numerator.
A function which satisfies these conditions is of the form

$$f(x)=\frac{k(x-4)}{x(x-4)},\ \text{where } k\neq 0,\ k\in\mathrm{R},\ \text{and } x\neq 0,\ 4.$$

We must determine the value of k such that the hole is located at (4,−2)

$$f(x)=\frac{k(x-4)}{x(x-4)}=\frac{k}{x},\ x\neq 0,4,\ k\in\mathrm{R}$$

The graph of the function, $g(x)=k/x$, $x\neq 0$ is identical to the graph of $f(x)$, with the exception of the hole at (4,−2).
Using $g(4)=-2$, we can determine the value of k that will place the hole at the correct location.
Here, $k4=-2$ and $k=-8$. Thus,

$$f(x)=\frac{-8(x-4)}{x(x-4)},\ x\neq 0,4$$

Therefore, a function with a hole at (4,−2) and vertical asymptote of x=0 is $f(x)=\frac{-8(x-4)}{x(x-4)}$.

b. Given $g(-3)=0$ and $h(-3)=0$, then $x+3$ must be a factor of the numerator and denominator. However, a vertical asymptote exists at $x=-3$.

This means that $h(-3)=0$ and $g(-3)\neq 0$ when the function is in simplest form.

For this to happen, $x+3$ must be a factor of multiplicity 2 or greater in the denominator (at least two factors of $x+3$), and of lesser multiplicity (but at least 1) in the numerator.

One such function is given by $f(x)=\frac{x+3}{(x+3)^2}=\frac{1}{x+3}$, $x\neq -3$.

When this equation is simplified to $y=\frac{1}{x+3}$,$x\neq -3$, the indeterminate form $(\frac{0}{0})$ of the equation at $x=-3$ is lost, but the graph of the function remains the same.

This simplified equation is not a valid solution to this problem as it does not satisfy the first condition.

Therefore, the function $f(x)$=, $\frac{x+3}{(x+3)^2}$,$x\neq 3$ is indeterminate in form at $x=-3$, but has a vertical asymptote at $x=-3$.

In both situations, other solutions can be generated by using $y=kf(x)$, $k\in R$, $k\neq 0$.
As well, there are other, more complicated, rational functions that would satisfy the given conditions.

Summary

For a rational function $y= g(x)/h(x)$, $h(x)\neq 0$:

- The function will be discontinuous at $x=a$ if $h(a)=0$.
- The function has a vertical asymptote at $x=a$ if $h(a)=0$ and $g(a)\neq 0$, when the function is in simplest form.
- If $h(a)=0$ and $g(a)=0$ for some value of $a\in R$, then $x-a$ is a factor of the numerator and denominator of the function, and a point of discontinuity (a hole) may occur at $x=a$. To verify this, express the function in simplified form and then determine if it generates a single point of discontinuity, or a vertical asymptote.
- Since the value of a limit provides information near, but not at, a specific value of x, limits are often used to analyze a function near its asymptotes.

Next we will study the end behaviour of the graph of a rational function and identify any horizontal asymptote, if it exists.

We will identify the conditions when a rational function does not have a horizontal asymptote. In this case, we will determine a slant (or oblique) asymptote for the function, if one exists. We will continue to use the language of limits to assist us in communicating our understanding of the end behaviour of the function.

Horizontal Asymptotes

A horizontal asymptote is a guideline for the end behaviour of the function.
To identify a horizontal asymptote of a rational function, if it exists, we must study the end behaviours of the function.
Using the language of limits, this means that we must determine

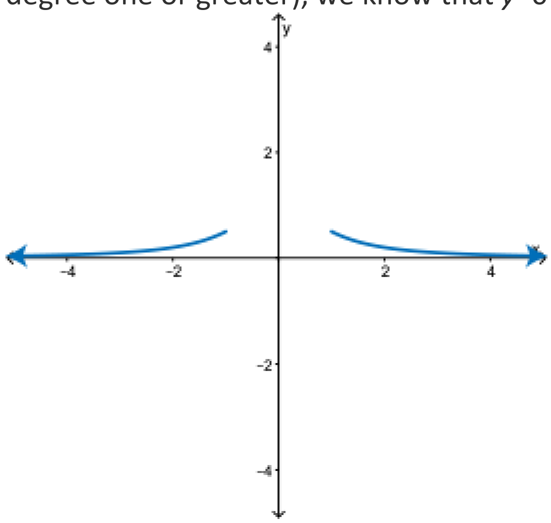
$$\lim_{x\to+\infty} f(x) \text{ and } \lim_{x\to-\infty} f(x)$$

Example 1

What is the horizontal asymptote of the function $y=\frac{1}{x^2+1}$? How does the graph of the function behave near the horizontal asymptote?

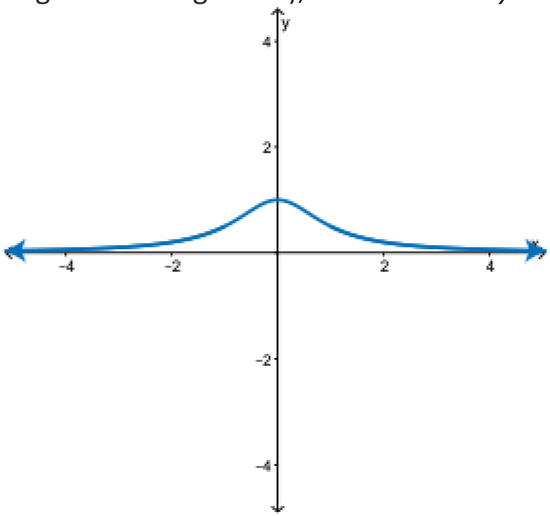
Solution

This function is the reciprocal of the quadratic $y=x^2+1$.
From previous studies of functions of the form $y=1/f(x)$, where $f(x)$ is a polynomial function (of degree one or greater), we know that $y=0$ is the horizontal asymptote.



As $x\to\infty$, $x^2+1\to\infty$ and thus $\frac{1}{x^2+1}\to 0$.
Since $y=\frac{1}{x^2+1}>0$ for all $x\in R$, the curve lies above the x-axis and will approach $y=0$ from above as x increases in value.
Similarly, as $x\to-\infty$, $x^2+1\to\infty$ and $\frac{1}{x^2+1}\to 0$.
So, again, the graph of the function will approach $y=0$ from above as x decreases in value.
This function is the reciprocal of the quadratic $y=x^2+1$.

From previous studies of functions of the form $y=1/f(x)$, where $f(x)$ is a polynomial function (of degree one or greater), we know that $y=0$ is the horizontal asymptote.



We can say that $\lim_{x\to\pm\infty} \frac{1}{x^2+1} = 0$

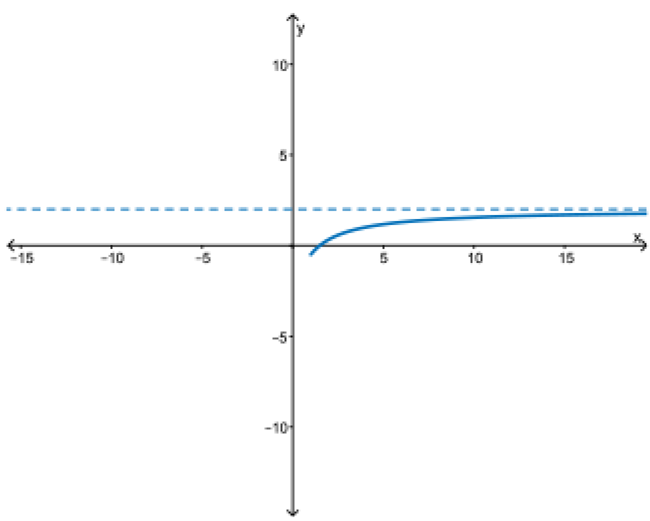
This function has a horizontal asymptote of $y=0$.

There is no vertical asymptote since $x2+1\neq 0$ for $x\in R$.

Example

Determine the horizontal asymptote of $g(x) = \frac{2x-3}{x+1}$ and discuss the behaviour of the graph about this asymptote.

Solution



Let's study the behaviour of the function as x approaches a large positive value (i.e., $x\to\infty$).

x	$g(x)$
10	1.545454545
100	1.95049505
1000	1.995004995
10000	1.99950005

From the table, $y \to 2$ as $x \to \infty$

We observe that y approaches 2 from below.

This table studies the opposite end behaviour of $g(x)$ (as $x \to -\infty$).

x	$g(x)$
−10	2.555555556
−100	2.050505051
−1000	2.005005005
−10000	2.00050005

From the table, $y \to 2$ as $x \to -\infty$

We observe that y approaches 2 from above.

Using limits to describe this end behaviour, we have

$$\lim_{x \to \pm\infty} \frac{2x-3}{x+1} = 2$$

The horizontal asymptote is $y=2$.
The function has a vertical asymptote at $x=-1$.

Alternate Approach

Without using a table of values, we can determine the equation of the horizontal asymptote of the function by focusing on the term in the numerator and denominator that has the most influence on the value of the numerator and denominator as $x \to \pm\infty$.

Let's return to the function $g(x)=\frac{2x-3}{x+1}$.

As x approaches a large positive or negative value, the -3 in the numerator and $+1$ in the denominator become insignificant relative to the value of $2x$ and x.

$$g(10000)=1.99950005$$

If we ignore the insignificant terms of $g(x)=\frac{2x-3}{x+1}$, then $g(x) \to \frac{2x}{x}$, that is $g(x) \to 2$ as $x \to \pm\infty$.

Thus, $y=2$ is the horizontal asymptote.

Example

Determine the horizontal asymptote, if any, of $f(x)=\frac{x^2-1}{3x^2-2x+5}$. Does the curve approach the horizontal asymptote from above or below?

Solution

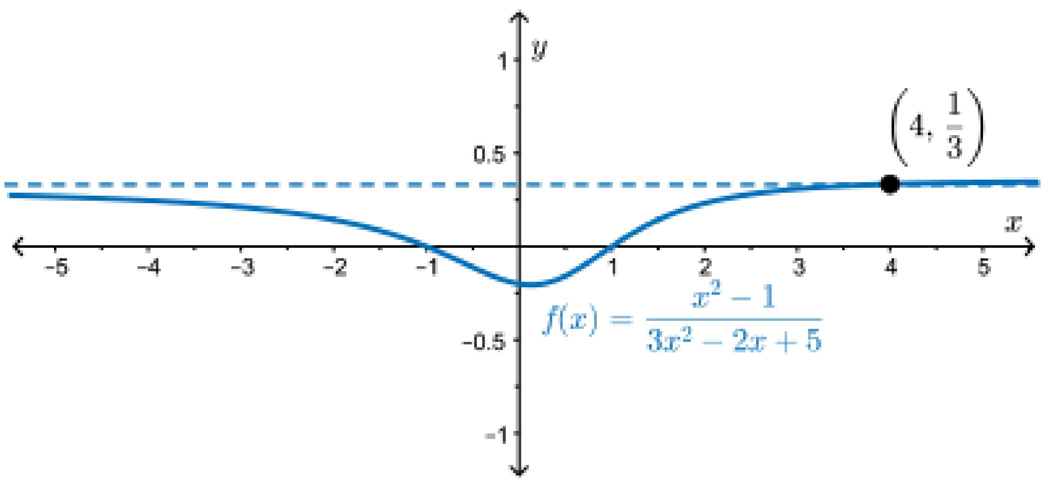
As *x* becomes larger and larger, the linear and constant terms become less significant.
We can argue that $f(x)\to\frac{3x^2}{x^2}$; that is, $f(x)\to\frac{1}{3}$ as $x\to\pm\infty$.
Thus,
$\lim_{x\to\pm\infty}\frac{x^2-1}{3x^2-2x+5}$ = 1/3
and y=1/3 is the horizontal asymptote for $y=f(x)$.

The function is continuous with no vertical asymptotes since the equation $3x^2-2x+5=0$ has no real roots (the discriminant is negative).

This means the graph of the function must cross the horizontal asymptote. This is possible since horizontal asymptotes are "end behaviour" asymptotes.

The graph of this function will intersect the horizontal asymptote at x=4.

$f(4)$=1/3



Example

Determine the horizontal asymptote, if any, of $y=\frac{x^2+2x+3}{x-1}$.

Solution

As $x\to\pm\infty$, $y\to x/2x$.

Therefore, $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$.
The function has unbounded end behaviours.
$\lim\limits_{x \to +\infty} \frac{x^2+2x+3}{x-1} = +\infty$ and $\lim\limits_{x \to -\infty} \frac{x^2+2x+3}{x-1} = -\infty$
So, $\lim\limits_{x \to \pm\infty} \frac{x^2+2x+3}{x-1}$ do not exist.
This function does not have a horizontal asymptote.

In general, a rational function will not have a horizontal asymptote when

the degree of the numerator > the degree of the denominator

since the numerator will grow faster than the denominator, in size, as $x \to \pm\infty$.

Oblique Asymptotes

An **oblique asymptote**, often called a **slant asymptote**, is a linear asymptote that is neither horizontal nor vertical.
A rational function will have an oblique asymptote when the degree of the polynomial in the numerator of the function is one greater than the degree of the polynomial in the denominator.

That is,

the degree of the numerator = the degree of the denominator +1

Example

Determine the equation of the oblique asymptote of $y=\frac{x^2+2x+3}{x-1}$.

Solution

The equation of this function can be written in the form $y=q(x)+\frac{r(x)}{x-1}$, where $q(x)$ is the quotient and $r(x)$ is the remainder when the numerator, x^2+2x+3, is divided by the denominator, $x-1$.

$$
\begin{array}{r|rrrrr}
 & & & x & + & 3 \\
\cline{2-6}
x-1 & x2 & + & 2x & + & 3 \\
 & x2 & - & x & & \\
\cline{2-6}
 & & & 3x & + & 3 \\
 & & & 3x & - & 3 \\
\cline{4-6}
 & & & & & 6
\end{array}
$$

Therefore,

$$y= x+3+\frac{6}{x-1}$$

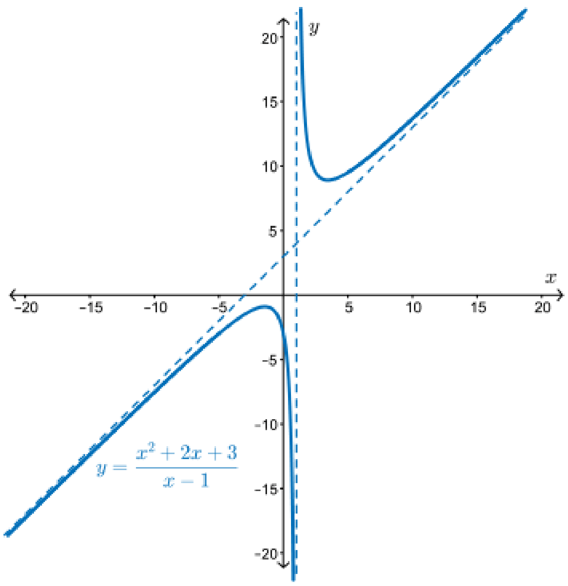
Now, as $x \to \pm\infty$, $6x-1 \to 0$ so $y \to x+3$.
The oblique asymptote is $y=x+3$.
The end behaviours of the function can still be described by

$$y \to -\infty \text{ as } x \to -\infty$$

and

$$y \to \infty \text{ as } x \to \infty$$

However, the graph of the function, in fact, moves closer and closer to the line $y=x+3$ as $x \to \pm\infty$.
Note: Since an oblique asymptote is an "end behaviour" asymptote, the graph of a function may cross its oblique asymptote; but this is not the case for this example.



Summary

Consider a rational function $y=g(x)/h(x)$, $h(x)\neq 0$ and the degree of $h(x)\geq 1$.

- If the degree of $g(x)$ is less than the degree of $h(x)$, then $y=0$ is the horizontal asymptote of the function.
- If the degree of $g(x)$ equals the degree of $h(x)$, then $y=ab$ is the horizontal asymptote of the function, where a and b are the coefficients of the highest degree term in the numerator and denominator, respectively.
- If the degree of $g(x)$ is greater than the degree of $h(x)$, there is no horizontal asymptote.
  If the degree of $g(x)$ is exactly one greater than the degree of $h(x)$, then the function has an oblique asymptote.
  Use long division to express the equation of the function in the form $y=ax+b+\frac{r(x)}{h(x)}$,
  where $ax+b$, $a\neq 0$, is the quotient and $r(x)$ is the remainder.
  The oblique (or slant) asymptote is given by $y=ax+b$.
  A rational function may intersect its horizontal or oblique (or slant) asymptote but will never cross its vertical asymptotes.
- Since the value of a limit provides information near, but not at, a specific value of x, limits are often used when analyzing a function near its asymptotes.

Next

- We will study and compare a variety of rational functions involving linear and/or quadratic expressions in the numerator and denominator.
- We will identify distinguishing features that will allow us to predict the behaviour of the function and recognize its graph.
- We will apply our knowledge of asymptotes (vertical, horizontal, and oblique), points of discontinuity, *x*- and *y*-intercepts, positive and negative intervals, and consider the symmetry of the function.

Example

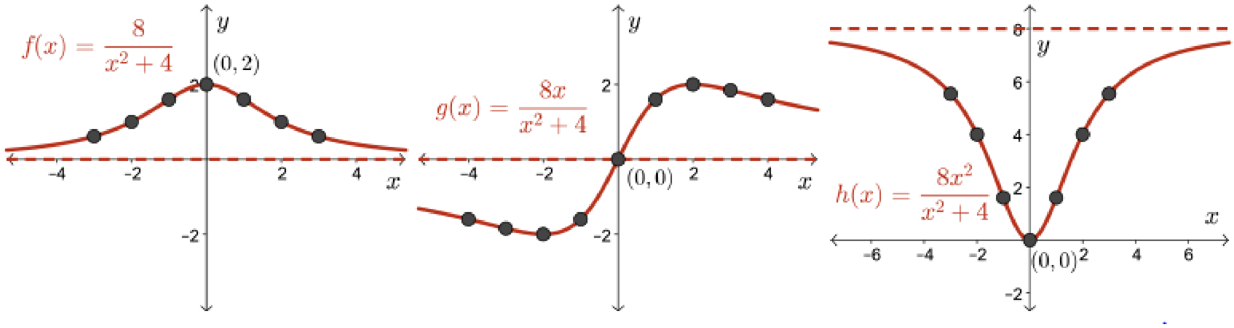
Compare the behaviour of the three functions defined by

$f(x)=\frac{8}{x^2+4}$ $g(x)=\frac{8x}{x^2+4}$ $h(x)=\frac{8x^2}{x^2+4}$

Based on your analysis, predict the shape of each graph. Use technology to verify your prediction.

Solution

Function	$f(x)=\frac{8}{x^2+4}$	$g(x)=\frac{8x}{x^2+4}$	$h(x)=\frac{8x^2}{x^2+4}$
Domain	$\{x \mid x \in R\}$	$\{x \mid x \in R\}$	$\{x \mid x \in R\}$
Vertical Asymptote(s) and/or Points of Discontinuity	None	None	None
Horizontal Asymptote(s)	$y=0$; approaches from above as $x \to \pm\infty$	$y=0$; approaches from below as $x \to -\infty$ and from above as $x \to \infty$	$y=8$; approaches from below as $x \to \pm\infty$
Positive/Negative Intervals	$f(x)>0$ for all $x \in R$	$g(x)<0$ when $x<0$, $g(x)>0$ when $x>0$	$h(x)>0$ for all $x \in R$, $x \neq 0$
x-intercepts	None	(0, 0)	(0, 0)
y-intercepts	(0, 2)	(0, 0)	(0, 0)
Symmetry (Even/Odd)	$f(-x)=f(x)$ (even function)	$g(-x)=-g(x)$ (odd function)	$h(-x)=h(x)$ (even function)



It is worth noting that we may not find the exact position of the local maximum or local minimum point on $y=g(x)$, or inflection points in the curves of the graphs, as this requires calculus.

However, by finding additional points, we can create a fairly reasonable sketch.

It is for this reason that analyzing and graphing rational functions consisting of higher degree polynomials is covered more fully in calculus.