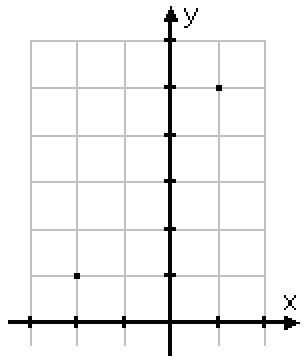
Chapter 2 Analytic Geometry (1)

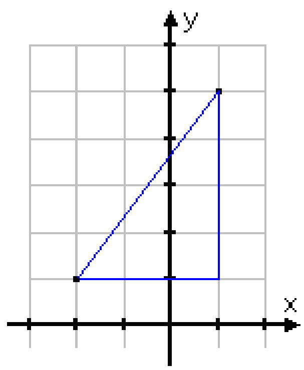
1. Length of a Line Segment

The Distance Formula is a variant of the Pythagorean Theorem that you used back in geometry. Here's how we get from the one to the other:

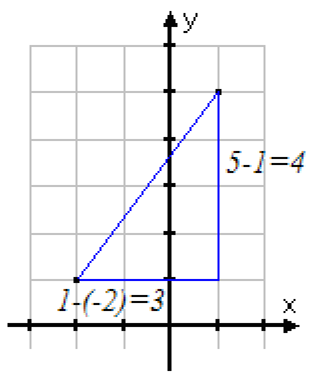
Suppose you're given the two points (–2, 1) and (1, 5), and they want you to find out how far apart they are. The points look like this:



You can draw in the lines that form a right-angled triangle, using these points as two of the corners:



It's easy to find the lengths of the horizontal and vertical sides of the right triangle: just subtract the *x*-values and the *y*-values:



Then use the Pythagorean Theorem to find the length of the third side (which is the hypotenuse of the right triangle):

$c^2 = a^2 + b^2$
so:

$$c^2 = (5-1)^2 + (1-(-2))^2$$

$$\begin{aligned} c &= \sqrt{(5-1)^2 + (1-(-2))^2} \\ &= \sqrt{(4)^2 + (3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5 \end{aligned}$$

This format always holds true. Given two points, you can always plot them, draw the right triangle, and then find the length of the hypotenuse.

The length of the hypotenuse is the distance between the two points. Since this format always works, it can be turned into a formula:

Distance Formula: Given the two points (x_1, y_1) and (x_2, y_2), the distance between these points is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Don't let the subscripts scare you. They only indicate that there is a "first" point and a "second" point; that is, that you have two points. Whichever one you call "first" or "second" is up to you. The distance will be the same, regardless.

Example 1
Find the distance between the points (–2, –3) and (–4, 4).

I just plug the coordinates into the Distance Formula:

$$\begin{aligned} d &= \sqrt{(-4-(-2))^2 + (4-(-3))^2} \\ &= \sqrt{(-4+2)^2 + (4+3)^2} \\ &= \sqrt{(-2)^2 + (7)^2} \\ &= \sqrt{4+49} = \sqrt{53} \approx 7.28 \end{aligned}$$

Then the distance is *sqrt*(53), or about 7.28, rounded to two decimal places.

The most common mistake made when using the Formula is to accidentally mismatch the x-values and y-values. Be careful you don't subtract an x from a y, or vice versa; make sure you've paired the numbers properly.

Also, don't get careless with the square-root symbol. If you get in the habit of omitting the square root and then "remembering" to put it back in when you check your answers in the back of the book, then you'll forget the square root on the test, and you'll miss easy points.

You also don't want to be careless with the squaring inside the Formula. Remember that you simplify inside the parentheses *before* you square, not after, and remember that the square is on everything inside the parentheses, *including the minus sign*, so the square of a negative is a positive.

By the way, it is almost always better to leave the answer in "exact" form (the square root "$\sqrt{53}$" above). Rounding is usually reserved for the last step of word problems. If you're not sure which format is preferred, do both, like this:

$d = \sqrt{53} \approx 7.28$

2. Midpoint of a Line Segment

Sometimes you need to find the point that is exactly between two other points. For instance, you might need to find a line that bisects (divides into equal halves) a given line segment. This middle point is called the "midpoint".

The concept doesn't come up often, but the Formula is quite simple and obvious, so you should easily be able to remember it for later.

Think about it this way: If you are given two numbers, you can find the number exactly between them by averaging them, by adding them together and dividing by two.

For example, the number exactly halfway between 5 and 10 is $^{[5+10]}/_{2} = {}^{15}/_{2} = 7.5$.

The Midpoint Formula works exactly the same way. If you need to find the point that is exactly halfway between two given points, just average the *x*-values and the *y*-values.

Technically, the Midpoint Formula is the following:

$$\left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2}\right)$$

But as long as you remember that you're averaging the two points' *x*- and *y*-values, you'll do fine. It won't matter which point you pick to be the "first" point you plug in.

Example 2 Find the midpoint between (–1, 2) and (3, –6).

Apply the Midpoint Formula:

$$\left(\frac{-1+3}{2}, \frac{2+(-6)}{2}\right) = \left(\frac{2}{2}, \frac{-4}{2}\right) = (1, -2)$$

So the answer is *P* = (1, –2).

Example 3
Find the midpoint between (6.4, 3) and (–10.7, 4).

Apply the Midpoint Formula:

$$\left(\frac{6.4+(-10.7)}{2}, \frac{3+4}{2}\right) = \left(\frac{-4.3}{2}, \frac{7}{2}\right) = (-2.15,\ 3.5)$$

So the answer is *P* = (–2.15, 3.5)

Example 4

Find the value of *p* so that (–2, 2.5) is the midpoint between (*p*, 2) and (–1, 3).
I'll apply the Midpoint Formula:

$$\left(\frac{p+(-1)}{2}, \frac{2+3}{2}\right) = (-2,\ 2.5)$$

$$\left(\frac{p-1}{2}, \frac{5}{2}\right) = (-2,\ 2.5)$$

$$\left(\frac{p-1}{2}, 2.5\right) = (-2,\ 2.5)$$

This reduces to needing to figure out what *p* is, in order to make the *x*-values work:

$$\frac{p-1}{2} = -2 \qquad p-1 = -4 \qquad p = -3$$

So the answer is *p* = –3.

Example 5
Is *y* = 2*x* – 4.9 a bisector of the line segment with endpoints at (–1.8, 3.9) and (8.2, –1.1)?

If I just graph this, it's going to *look* like the answer is "yes". But I have to remember that a picture can *suggest* an answer, it can give me an *idea* of what is going on, but only the *algebra* can give me an exact answer.

So I'll need to find the midpoint, and then see if the midpoint is actually a point on the given line. First, I'll apply the Midpoint Formula:

$$P = \left(\frac{-1.8+8.2}{2}, \frac{3.9-1.1}{2}\right) = \left(\frac{6.4}{2}, \frac{2.8}{2}\right) = (3.2,\ 1.4)$$

Now I'll check to see if this point is on the line:
$y = 2x - 4.9$
$y = 2(3.2) - 4.9 = 6.4 - 4.9 = 1.5$

But I needed y to equal 1.4, so this line is *close* to being a bisector (as a picture would indicate), but it is not *exactly* a bisector (as the algebra proves). So the answer is "No, this is not a bisector."

Example 6

Find the perpendicular bisector of the line segment with endpoints at (–1.8, 3.9) and (8.2, –1.1).

This is a multi-part problem, and is actually typical of problems you will probably encounter at some point when you're learning about straight lines.

This is an example of a question where you'll be expected to remember the Midpoint Formula from however long ago you last saw it in class.

Here's how to answer it:
First, I need to find the midpoint, since any bisector, perpendicular or otherwise, must pass through the midpoint. I'll apply the Midpoint Formula:

$$P = \left(\frac{-1.8+8.2}{2}, \frac{3.9-1.1}{2}\right) = \left(\frac{6.4}{2}, \frac{2.8}{2}\right) = (3.2,\ 1.4)$$

Now I need to find the slope of the line segment. I need this slope in order to find the perpendicular slope. I'll apply the Slope Formula:

$$m = \frac{3.9-(-1.1)}{-1.8-8.2} = \frac{3.9+1.1}{-1.8-8.2} = \frac{5}{-10} = -\frac{1}{2}$$

The perpendicular slope (for my perpendicular bisector) is the negative reciprocal of the slope of the line segment. Remember that "negative reciprocal" means "flip it, and change the sign".

Then the slope of the perpendicular bisector will be $+\,{}^{2}/_{1} = 2$. With the slope and a point (the midpoint, in this case), I can find the equation of the line:

$y - 1.4 = 2(x - 3.2)$
$y - 1.4 = 2x - 6.4$
$y = 2x - 6.4 + 1.4$
$y = 2x - 5$

Example 7
Find the center of the circle with a diameter having endpoints at (–4, 3) and (0, 2).

Since the center is at the midpoint of any diameter, I need to find the midpoint of the two given endpoints:
$\left(\frac{-4 + 0}{2}, \frac{3 + 2}{2} \right) = \left(\frac{-4}{2}, \frac{5}{2} \right) = (-2, 2.5)$

These examples really are fairly typical. You will have some simple "plug-n-chug" problems when the concept is first introduced, and then later, out of the blue, they'll hit you with the concept again, except it will be buried in some other type of problem.

3. Verifying Properties of Geometric Figures

A) <u>TRIANGLES</u>:

Classifying Triangles By The Number Of Equal Sides And Equal Angles:

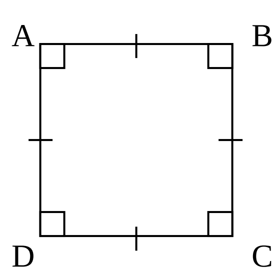
1) Equilateral Triangle 2) Isosceles Triangle 3) Scalene Triangle

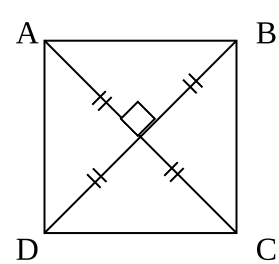
Classifying Triangles By The Measures Of The Angles:

1) Acute Triangle 2) Right Triangle 3) Obtuse Triangle

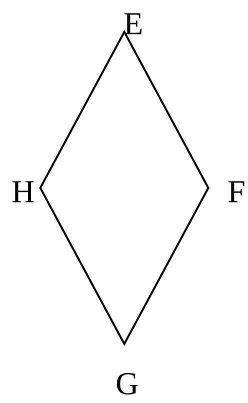
B) <u>QUADRILATERALS</u>: A quadrilateral is any figure enclosed by four sides.

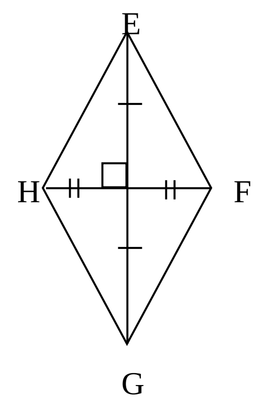
<u>Square</u>: A square is a quadrilateral with all sides of equal length and all angles measuring 90°. The diagonals of a square are ***perpendicular bisectors*** of each other. This means each diagonal cuts the other diagonal in half and the two diagonals meet each other at 90°. In a square, the diagonals are the same length.





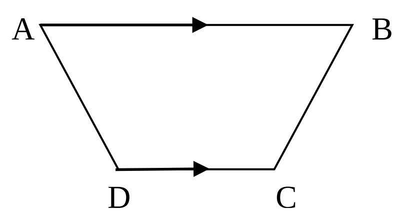
<u>Rhombus</u>: A rhombus is a quadrilateral with all sides of equal length but the angles do not have to measure 90°. The diagonals of a rhombus are also perpendicular bisectors of each other.





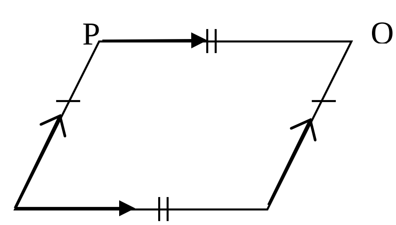
EF = FG = GH = HE

Trapezoid: A trapezoid is a quadrilateral with one pair of parallel sides.

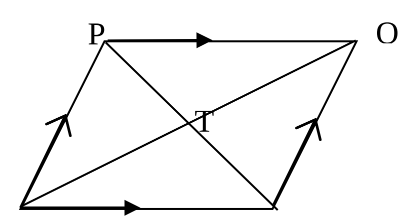


AB || CD

Parallelogram: A parallelogram is a quadrilateral with two pairs of parallel sides. In a parallelogram, the opposite sides are equal in length and the opposite angles are equal in measure. The diagonals of a parallelogram bisect each other but they do not always meet at 90°. In addition, the diagonals of a parallelogram may not be the same length.

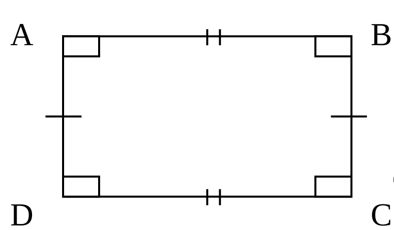


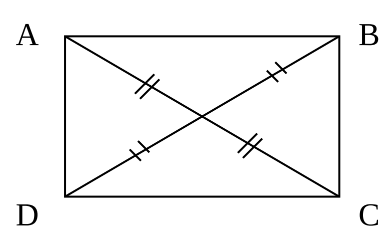
PQ || SR and SP || RQ



PT = TR and ST = TQ

Rectangle: A rectangle is a parallelogram in which all the angles have a measure of 90°. In a rectangle the diagonals are the same length and they bisect each other but the diagonals do not always meet at 90°.

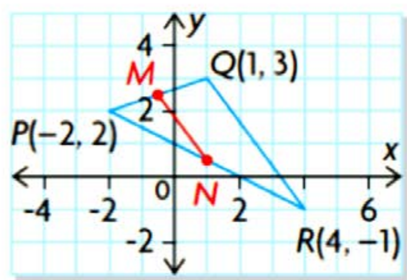




NAME	SIDES	ANGLES	DIAGONALS
Quadrilateral	• No equal sides	• No equal angles	-
Square	• Four equal sides • Opposite sides are parallel	• Four equal angles (90°)	• Diagonals are the same length • Diagonals meet at 90°
Rhombus	• Four equal sides • Opposite sides are parallel	• Opposite angles are equal	• Diagonals are not the same length • Diagonals meet at 90°
Trapezoid	• No equal sides • One pair of parallel sides	• Interior angles are supplemental	
Parallelogram	• Opposite sides are equal • Opposite sides are parallel	• Opposite angles are equal	• Diagonals are not the same length • Diagonals do not meet at 90°
Rectangle	• Opposite sides are equal • Opposite sides are parallel	• Four equal angles (90°)	• Diagonals are the same length • Diagonals do not meet at 90°

Example 1

A triangle has vertices at $P(-2,2)$, $Q(1,3)$, and $R(4,-1)$. Show that the midsegment joining the midpoints of *PQ* and *PR* is parallel to *QR* and half its length.



$$M\left(\frac{-2+1}{2},\frac{2+3}{2}\right) = (-0.5, 2.5)$$

$$N\left(\frac{-2+4}{2},\frac{2-1}{2}\right) = (1, 0.5)$$

$$m_{MN} = \frac{0.5-2.5}{1+0.5} = \frac{-2}{1.5} = -\frac{4}{3}$$

$$m_{QR} = \frac{-1-3}{4-1} = \frac{-4}{3} = -\frac{4}{3}$$

Since the slope of MN and QR are the same, therefore, MN is parallel to QR.

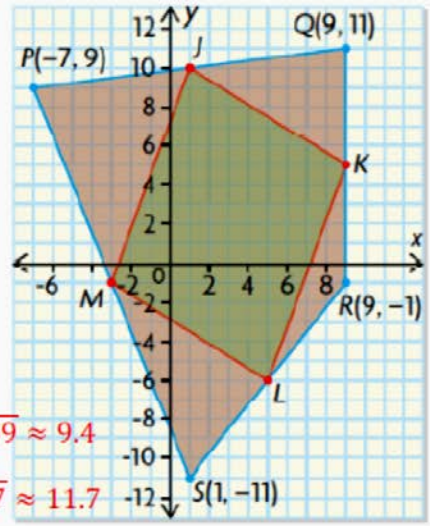
$$d_{MN} = \sqrt{(1+0.5)^2+(0.5-2.5)^2} = \sqrt{1.5^2+(-2)^2} = \sqrt{2.25+4} = \sqrt{6.25} = 2.5$$

$$d_{QR} = \sqrt{(4-1)^2+(-1-3)^2} = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Since the length of MN is 2.5 and the length of QR is 5, therefore, MN is half of QR.

Example 2

Carlos has hired a landscape designer to give him some ideas for improving his backyard, which is a quadrilateral. The designer's plan on a coordinate grid shows a lawn area that is formed by joining the midpoints of the adjacent sides in the quadrilateral. The four triangular areas will be gardens. Show that the midsegments of the quadrilateral, with vertices at $P(-7,9)$, $Q(9,11)$, $R(9,-1)$), and $S(1,-11)$, form a parallelogram.



$$J\left(\frac{-7+9}{2},\frac{9+11}{2}\right)=\left(\frac{2}{2},\frac{20}{2}\right)=(1,10)$$

$$K\left(\frac{9+9}{2},\frac{11-1}{2}\right)=\left(\frac{18}{2},\frac{10}{2}\right)=(9,5)$$

$$L\left(\frac{1+9}{2},\frac{-11-1}{2}\right)=\left(\frac{10}{2},\frac{-12}{2}\right)=(5,-6)$$

$$M\left(\frac{1-7}{2},\frac{-11+9}{2}\right)=\left(\frac{-6}{2},\frac{-2}{2}\right)=(-3,-1)$$

$$d_{JK}=\sqrt{(9-1)^2+(5-10)^2}=\sqrt{8^2+(-5)^2}=\sqrt{64+25}=\sqrt{89}\approx 9.4$$

$$d_{KL}=\sqrt{(9-5)^2+(5+6)^2}=\sqrt{4^2+11^2}=\sqrt{16+121}=\sqrt{137}\approx 11.7$$

$$d_{LM}=\sqrt{(5+3)^2+(-6+1)^2}=\sqrt{8+(-5)^2}=\sqrt{64+25}=\sqrt{89}\approx 9.4$$

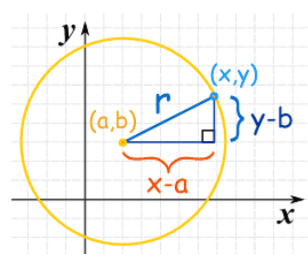
$$d_{MJ}=\sqrt{(1+3)^2+(10+1)^2}=\sqrt{4^2+11^2}=\sqrt{16+121}=\sqrt{137}\approx 11.7$$

$$m_{JK}=\frac{5-10}{9-1}=\frac{-5}{8}=-\frac{5}{8} \qquad m_{KL}=\frac{-6-5}{5-9}=\frac{-11}{-4}=\frac{11}{4}$$

$$m_{LM}=\frac{-1+6}{-3-5}=\frac{5}{-8}=-\frac{5}{8} \qquad m_{MJ}=\frac{10+1}{1+3}=\frac{11}{4}$$

Since the opposite sides are equal in the length and with the same slope, therefore, JKLM is a parallelogram.

Equation of Circles



Circle - The set of all points on a plane that are a fixed distance from a center.

To find the equation of the circle, we simply make a right-angled triangle and then use Pythagoras Theorem ($a^2 + b^2 = c^2$):

$(x-a)^2 + (y-b)^2 = r^2$ with centre (a, b) and radius r