Number Theory

1. Factors

A **factor** of a given number is a whole number that divides exactly into the given number.

For example, $12 \div 4 = 3$
So, 4 is a factor of 12 as it divides exactly into 12, and 3 is also a factor of 12.

Note: For example, $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$

If a number can be expressed as a product of two whole numbers, then the whole numbers are called **factors** of that number.

For example, $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$
So, the factors of 12 are 1, 2, 3, 4, 6 and 12.

2. Common Factors

Common factors are factors that are common to two or more numbers.

Example: Find the common factors of 22 and 33.

Solution:
$22 = 1 \times 22 = 2 \times 11$
$33 = 1 \times 33 = 3 \times 11$
So, the common factors of 22 and 33 are 1 and 11.

3. Prime Numbers

A **prime number** has only two different factors, 1 and itself.

For example, $13 = 1 \times 13$
So, 13 is a prime number since it has only two different factors, 1 and 13.
Clearly, $2=1\times 2$, $3=1\times 3$, $5=1\times 5$, $7=1\times 7$, $11=1\times 11$,
So, 2, 3, 5, 7, 11, … are prime numbers.

4. Composite Numbers

A **composite number** is a number that has more than two factors.

For example, $10 = 1 \times 10 = 2 \times 5$

So, 10 is a composite number as it has more than two factors.

Note: 1 is considered neither a prime number nor a composite number as it has 1 factor only. A prime number has 2 different factors.

Example: State which of the following numbers are a prime:
a. 6; b. 19

Solution:
a. $6 = 1 \times 6 = 2 \times 3$.
So, 6 is not a prime as it has more than two factors.
b. $19 = 1 \times 19$
So, 19 is a prime since it has only two different factors, 1 and 19.

5. Prime Factors

Prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, …

A **prime factor** is a factor of a number that is also a prime number.
To express a whole number as a **product of prime factors**, try prime numbers in order of their magnitude. That is, first try to find out whether 2 is a factor of the given whole number or not.

If 2 is a factor of the whole number, it will divide into it exactly and we can write the whole number as the product of 2 and another number. If 2 is a prime, try 2 again to see if we can further write the whole number as a product involving two 2s.

If not, try the next prime, 3, and so on. Stop when the number has been expressed as the product of prime numbers.

Example: Express 200 as a product of prime numbers.

Solution: $200 = 2 \times 100 = 2 \times 2 \times 50 = 2 \times 2 \times 2 \times 25 = 2^3 \times 5^2$

6. Divisibility

Problem: Is the number 621 prime or composite?

To determine if a number is prime or composite, follow these steps:
- Find all factors of the number.
- If the number has only two factors, 1 and itself, then it is prime.
- If the number has more than two factors, then it is composite.

The above procedure works very well for small numbers. However, it would be time-consuming to find all factors of 621. Thus we need a better method for determining if a large number is prime or composite.

Every number has one and itself as a factor. Thus, if we could find one factor of 621, other than 1 and itself, we could prove that 621 is composite.

One way to find factors of large numbers quickly is to use tests for divisibility.

Definition	**Example**
One whole number is divisible by another if, after dividing, the remainder is zero.	18 is divisible by 9 since 18 ÷ 9 = 2 with a remainder of 0.
If one whole number is divisible by another number, then the second number is a factor of the first number	Since 18 is divisible by 9, 9 is a factor of 18.
A divisibility test is a rule for determining whether one whole number is divisible by another. It is a quick way to find factors of large numbers.	Divisibility Test for 3: if the sum of the digits of a number is divisible by 3, then the number is divisible by 3

Let's look at some other tests for divisibility and examples of each.

A number is divisible by:	**If:**	**Example:**
2	The last digit is even (0,2,4,6,8)	128 is; 129 is not.
3	The sum of the digits is divisible by 3	381 (3+8+1=12, and 12÷3 = 4) Yes 217 (2+1+7=10, and 10÷3 = 3 $^{1}/_{3}$) No
4	The last 2 digits are divisible by 4	1312 is (12÷4=3) 7019 is not
5	The last digit is 0 or 5	175 is; 809 is not.
6	The number is divisible by both 2 *and* 3	114 (it is even, and 1+1+4=6 and 6÷3 = 2) Yes 308 (it is even, but 3+0+8=11 and 11÷3 = 3 $^{2}/_{3}$) No
7	If you double the last digit and subtract it from the rest of the number and the answer is: 0, or divisible by 7 (Note: you can apply this rule to that answer again if you want)	672 (Double 2 is 4, 67-4=63, and 63÷7=9) Yes 905 (Double 5 is 10, 90-10=80, and 80÷7=11 $^{3}/_{7}$) No
8	The last three digits are divisible by 8	109816 (816÷8=102) Yes 216302 (302÷8=37 $^{3}/_{4}$) No
9	The sum of the digits is divisible by 9	1629 (1+6+2+9=18, and again, 1+8=9) Yes 2013 (2+0+1+3=6) No

	(Note: you can apply this rule to that answer again if you want)	
10	The number ends in 0	220 is; 221 is not
11	If you sum every second digit and then subtract all other digits and the answer is: 0, or divisible by 11	1364 ((3+4) – (1+6) = 0) Yes 3729 ((7+9) – (3+2) = 11) Yes 25176 ((5+7) – (2+1+6) = 3) No
12	The number is divisible by both 3 *and* 4	648 (6+4+8=18 and 18÷3=6, also 48÷4=12) Yes 916 (9+1+6=16, 16÷3= $5^1/_3$) No

Let’s look at some examples in which we test the divisibility of a single whole number.

Example: Determine whether 35,120 is divisible by 2, 3, 4, 5, 6, 8, 9 and 10.
Solution: 35,120 is divisible by 2, 4, 5, 8 and 10.

7. Application of factoring (the *sieve of Eratosthenes).*

To find all the prime numbers less than 100 we can use the *sieve of Eratosthenes.*

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Cross out every other number starting at 2.

Cross out every third number starting at 3.
(You don't have to check every fourth number. Why?)

Cross out every fifth number starting at 5
(You don't have to check every sixth number. Why?)

Cross out every seventh number starting at 7
(You don't have to check every eighth number. Why?)
(You don't have to check every ninth number. Why?)
(You don't have to check every tenth number. Why?)

The explanation why you don't have to check numbers bigger than 10 is beyond this text.

The numbers that are not crossed out are primes.

We can use divisibility rules in factoring bigger numbers, too. Look at the examples.

```
5850
 /    \
10  × 585
```

5850 ends in 0, so it is divisible by 10.

```
 5850
 /     \
  10  ×   585
  /  \       /  \
5 × 2 × 5 × ?
```

585 is divisible by 5 since it ends in 5.
10 we know is 5 × 2.

```
5850
 /     \
10   ×   585
  /  \        /   \
5 × 2 × 5  × 117
```

Dividing 585 by 5 we find that 585 = 5 × 117.

```
5850
 /      \
10     ×    585
  /   \       /    \
5 × 2  × 5  × 117
|      |      |       /  \
5 × 2 × 5  × 9 × ?
```

117 is divisible by 9 since the sum of its digits is 1 + 1 + 7 = 9.

```
5850
 /      \
10    ×    585
    /   \       /  \
5 × 2 × 5  × 117
 /     /      /      /  \
5 × 2 × 5  × 9 × 13
```

The division tells us that 265 = 5 × 53.

```
5850
 /      \
  10   ×   585
  /  \        /  \
5 × 2 × 5  × 117
     /     /     /       /  \
5 × 2 × 5  × 9 × 13
  /     /     /     / \       \
5 × 2 × 5  × 3 × 3 × 13
```

8. Exponents

1) Power: A **power** has two parts: base$^{\text{exponent}}$ = 3^4

2) Base: The number being multiplied in a power.
For example, In 3^4, the base is 3.

3) Exponents: The number of times a number is being multiplied by itself in a power
For example, In 3^4, the exponent is 4.

Study the pattern for powers of 10.

Exponential Notation (Powers of 10)		
Exponential Form (Power)	**Expanded Form/Repeated Multiplication**	**Standard Name**
10^6	10x10x10x10x10x10	1000000
10^5	10x10x10x10x10	100000

10^4	10x10x10x10	10000
10^3	10x10x10	1000
10^2	10x10	100
10^1	10	10
10^0	1	1
10^{-1}	1/10	0.1
10^{-2}	1/10 x 1/10	0.01
10^{-3}	1/10 x 1/10 x 1/10	0.001

The following rules apply to numbers with exponents of 0, 1, 2 and 3:

Rule	**Example**
Any number (except 0) raised to the zero power is equal to 1.	$149^0 = 1$
Any number raised to the first power is always equal to itself.	$8^1 = 8$
If a number is raised to the second power, we say it is *squared*.	3^2 is read as *three squared*
If a number is raised to the third power, we say it is *cubed*.	4^3 is read as *four cubed*

9. Order of Operations

Problem: Evaluate this arithmetic expression: $18 + 36 \div 3^2$

How to evaluate an arithmetic expression with more than one operation according to the following rules:

Rule 1: Simplify all operations inside parentheses.

Rule 2: Perform all multiplications and divisions, working from left to right.

Rule 3: Perform all additions and subtractions, working from left to right.

However, the problem above includes an exponent, so we cannot solve it without revising our rules.

Rule 1: Simplify all operations inside parentheses.

Rule 2: Simplify all exponents, working from left to right.

Rule 3: Perform all multiplications and divisions, working from left to right.

Rule 4: Perform all additions and subtractions, working from left to right.

We can solve the problem above using our revised order of operations.

Example: An interior decorator charges $15 per square foot to lay a carpet, and an installation fee of $150. If the room is square and each side measures 12 feet, how much will it cost to carpet it?

Solution: If one side of the square-shaped room is 12 feet, then the area of the room is (12 feet)^2.

15 x 12^2 + 150	= 15 x 144 + 150	Simplify all exponents (Rule 2)
15 x 144 + 150	= 2,160 + 150	Multiplication (Rule 3)
2,160 + 150	= 2,310	Addition (Rule 4)

Answer: It will cost $2,310 to carpet this room.

Summary:
The order of operations:
Parentheses, Exponents, Multiplication & Division, Addition & Subtraction

Note that although there are six words, they correspond to four rules.

10. Greatest Common Factor

The greatest common factor (GCF) of two (or more) numbers is the largest common factor.

Example: Find the highest common factor of 16 and 32.

Solution:

$16 = 1 \times 16 = 2 \times 8 = 4 \times 4$

$32 = 1 \times 32 = 2 \times 16 = 4 \times 8$

So, GCF = 16

11. Lowest Common Multiple (LCM) by Prime Factors

The **LCM** of two (or more) numbers is calculated as follows:

- Express the numbers as a product of prime factors.
- Circle all of the prime factors of the smaller of the two numbers.
- Circle any prime factors of a larger number that have not already been circled for the smaller number (or smaller numbers if you are looking for the LCM of more than two numbers).

The LCM is the product of the circled prime factors.
These steps are better understood by reading the following examples.

Example: Find the LCM of 300 and 375.

Solution:

$300 = 2\times150$ $= 2\times2\times75$ $= 2\times2\times3\times25$ $= 2\times2\times3\times5\times5$	$375 = 3\times125$ $= 3\times5\times25$ $= 3\times5\times5\times5$

Organize the above information as shown below and circle all of the prime factors of the smaller number. Then circle any prime factors of the larger number that have not already been circled in the smaller number. The LCM is the product of the circled prime factors.

$300 = 2\times2\times3\times5\times5 \qquad 375 = 3\times5\times5\times5$

{Circle any prime factors that haven't already been circled}

$$\begin{aligned}\therefore \text{ LCM of 300 and 375} &= (2\times2\times3\times5\times5)\times5\\ &= 300\times5\\ &= 1500\end{aligned}$$

Note: The product of prime factors of 375 has three 5s but the product of prime factors of 300 has only two 5s. Since we circled two 5s for 300, we must circle the extra 5 for 375 as shown above.

1500 is a multiple of 300 as 5 × 300 = 1500.
1500 is a multiple of 375 as 4 × 375 = 1500.

In general: The lowest common multiple of two (or more) numbers can be computed as follows: Express each number as a **product** of its **prime factors** using powers.

Then circle the prime factors to the highest power from the given numbers.

Find the product of each prime factor to its highest power from the given numbers. This product represents the LCM of the numbers.

Example: Find the LCM of 9, 40 and 48.

Solution:

$9 = 3\times3$ $= 3^2$	$40 = 2\times20$ $= 2\times2\times10$ $= 2\times2\times2\times5$ $= 2^3\times5$	$48 = 2\times24$ $= 2\times2\times12$ $= 2\times2\times2\times6$ $= 2\times2\times2\times2\times3$ $= 2^4\times3$

Organize the above information and circle each prime factor with its highest power as shown below.

The highest power of 2 is 2^4, the highest power of 3 is 3^2 and the highest power of 5 is 5^1.

So, circle 2^4, 3^2 and 5^1. The LCM is the product of the prime factors to the highest powers.

$9 = 3^2 \qquad 40 = 2^3 \times 5 \qquad 48 = 2^4 \times 3$

$\therefore$ LCM of 9, 40 and 48 $= 2^4 \times 3^2 \times 5 = 16 \times 9 \times 5 = 720$

{Product of each prime factor to its highest power}

► Questions in class

1. Of the whole numbers 1 to 1000 inclusive, how many are multiples of 3 but not multiples of 5?

2. What is the sum of ALL the prime factors of 100?

3. A three digit number is between 130 and 200. It is divisible by 6 and 8. The tens digit is greater than the ones digit. What is the number?

4. When the 171st positive even integer is subtracted from the 220th positive odd integer the result is Z. Determine Z.

5. How many integers between 17 and 2628 are evenly divisible by 11?

6. Let A = $6a^3$ and B = $2b^5$ be two 3 digit numbers. If 9 divides A + B, then what is the value of a + b?

7. If the four digit integer 5ab4 is a perfect square, then a + b equals what?

8. The perimeter of a rectangle is equal to 36 cm. What is the largest possible area of this rectangle?

9. How many zeroes does the result of 35! end in?

10. If a and b are positive integers and $3^a + 2^b = 25$, then a + b = ?

11. If n is a natural number, we can state with absolute certainty that the product n(n+1)(n+2) is
a) Odd only when n is odd b) always divisible by 4 c) never divisible by 3
d) divisible by 4 only when n is odd e) always divisible by 6

12. If a = 2b and b = 3c and c = 4d, then a + b + c equals what in terms of d?