Chapter 4 Quadratic Functions (3)

1. Quadratic Graphs by Transformations Review

Given the quadratic function $\mathbf{y = a(x - h)^2 + k}$

a - Vertical stretch by a factor of a if |a| > 1.
- Vertical compression by a factor of a if |a| < 1.
- Reflection about the x-axis if a < 0.

h – Horizontal translation (shift) to the right by h units if h > 0.
- Horizontal translation (shift) to the left by h units if h < 0.

k – Vertical translation (shift) upward by k units if k > 0.
Vertical translation (shift) downward by k units if k < 0.

The **order** to transform is:

1) Stretch / compression

2) Reflection

3) Translations

Vertex Form of a Quadratic Relation
The **Vertex Form** of a Quadratic Relation: $\mathbf{y = a(x – h)^2 + k}$ with vertex (h, k).

2. Graphing using the vertex and two symmetric points

The quadratic function $\mathbf{f(x) = a(x - h)^2 + k}$, a not equal to zero, is said to be in **vertex form**. If a is positive, the graph opens upward, and if a is negative, then it opens downward. The **line of symmetry** is the vertical line x = h, and the **vertex** is the point (h, k).

Two points determines a line, and three points determines a parabola.

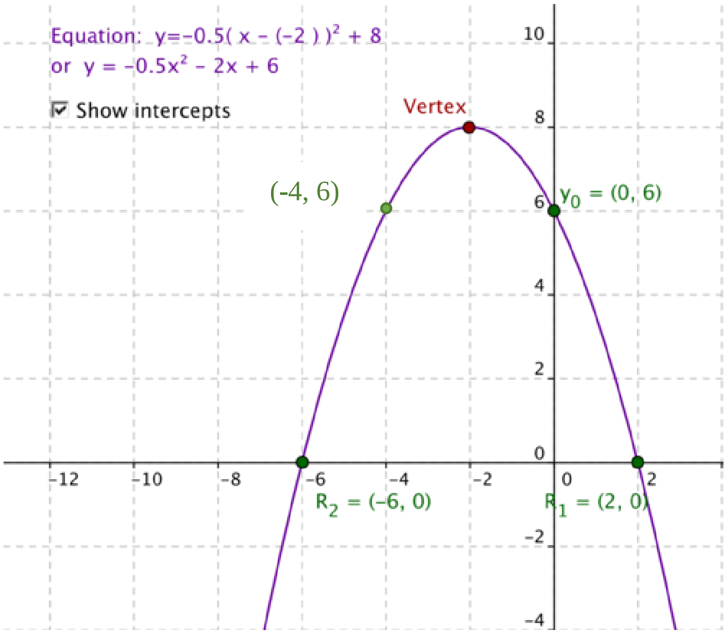
Example 1: graph $y = -0.5(x + 2)^2 + 8$

h = -2, k = 8 → Vertex: (-2, 8)

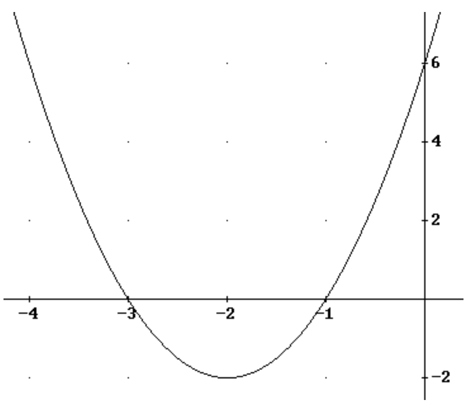
Find a point. → Choose x = 0, $y = -0.5(2)^2 + 8 = 6$

Find the symmetric point: (-4, 6)

The three points are: V (-2, 8), y-int (0, 6), symmetric to y-int (-4, 6)



Example 2: Find the equation of the quadratic function whose graph is shown below.



Solution: There are several methods to answer the above question but all of them have one idea in common: you need to understand and then select the right information from the graph.

<u>Method 1:</u> Using factored form

The above graph has two x intercepts at (-3, 0) and (-1, 0) and a y intercept at (0, 6). The x coordinates of the x intercepts can be used to write the equation of function f as follows:

$y = a(x + 3)(x + 1)$

We now use the y intercept: $6 = a(0 + 3)(0 + 1)$ and solve for “a” to find $a = 2$.

The formula for the quadratic function is given by:

$$\mathbf{y = 2(x + 3)(x + 1) = 2x^2 + 8x + 6}$$

Method 2: Using vertex form

The above parabola has a vertex at (-2, -2) and a y intercept at (0, 6). The vertex form of a quadratic function can be written

$y = a(x + 2)^2 - 2$

We use the y intercept $6 = a(0 + 2)^2 - 2$.

Solve for “a” to find $a = 2$.

The formula for the quadratic function is given by:

$$\mathbf{y = 2(x + 2)^2 - 2 = 2x^2 + 8x + 6}$$

Method 3: Using standard form

Since a quadratic function has the general form

$y = ax^2 + bx + c$

we need 3 points on the graph in order to write 3 equations and solve for a, b and c.

The following points are on the graph

(-3 , 0) , (-1 , 0) and (0 , 6)

point (0 , 6) gives $6 = a(0)^2 + b(0) + c = c$, solve for c to obtain $c = 6$

The two other points gives two more equations

(-3 , 0) gives $0 = a(-3)^2 + b(-3) + 6$

which leads to $9a - 3b + 6 = 0$

and (-1 , 0) gives $0 = a(-1)^2 + b(-1) + 6$

which becomes $a - b + 6 = 0$

Solve the last two equations in a and b to obtain

a = 2 and b = 8 and gives $\mathbf{y = 2x^2 + 8x + 6}$

3. Sketch Graphs by Completing the Squares

Given the vertex form $y = a(x - h)^2 + k$, we can get the standard form by expanding it. Given the standard form $y = Ax^2 + Bx + C$, we can change it to vertex form by **completing the squares.**

COMPLETING THE SQUARE TECHNIQUE

The process of finding the correct number to add to an expression of the form $x^2 + bx$ to form a perfect square trinomial is called ***completing the square***.

The correct number to add is $\left(\frac{b}{2}\right)^2$.

That is, take the coefficient of the x term, divide it by 2, and then square the result.

Then:

$$\overbrace{x^2 + bx}^{\text{start with this}} + \overbrace{\left(\frac{b}{2}\right)^2}^{\text{and add this number}} = \overbrace{\left(x + \frac{b}{2}\right)^2}^{\text{to get a perfect square}}$$

Remember to subtract it to make the equation true!

Example 1: $x^2 + 6x + 7$

("b" is 6 in this case)

$$x^2 + 6x + \boxed{} + 7\ \boxed{}$$

Complete the Square: $+\left(\frac{6}{2}\right)^2 \qquad -\left(\frac{6}{2}\right)^2$

Also **subtract** the new term

Simplify it and we are done.

$$\underbrace{x^2+6x+\left(\frac{6}{2}\right)^2}_{\left(x+\frac{6}{2}\right)^2} \; + \; \underbrace{7-\left(\frac{6}{2}\right)^2}_{7-9} = (x+3)^2-2$$

The result: $x^2 + 6x + 7 \;=\; (x + 3)^2 - 2$

If "a" is not 1, you need to factor it out.

ax^2+bx+c	(original expression)
$=(ax^2+bx)+c$	(group first two terms)
$=a(x^2+\frac{b}{a}x)+c$	(factor $a \neq 0$ out of the first two terms)
$=a\left(x^2+\frac{b}{a}x+(\frac{b}{2a})^2-(\frac{b}{2a})^2\right)+c$	(add zero in an appropriate form inside the parent note that $\frac{b}{a} \div 2 = \frac{b}{a}\cdot\frac{1}{2} = \frac{b}{2a}$)
$=a\left(x^2+\frac{b}{a}x+(\frac{b}{2a})^2\right)-a(\frac{b}{2a})^2+c$	(distributive law)
$=a(x+\frac{b}{2a})^2+\text{stuff}$	(rename as a perfect square)

Notice that the x-value of the vertex is -b/2a. This is worth recording and memorizing. Don't bother memorizing the yucky formula for the y-value of the vertex; once you have the x-value, it's easy to compute the corresponding y-value.

Example 2: Write $y = 5x^2 + 3x - 1$ in vertex form and give the coordinates of the vertex.

Use the technique of completing the square to put the function in vertex form:

$$\begin{aligned} y &= 5x^2+3x-1 \\ &= 5(x^2+\frac{3}{5}x)-1 \\ &= 5\left(x^2+\frac{3}{5}x+(\frac{3}{10})^2-(\frac{3}{10})^2\right)-1 \\ &= 5(x+\frac{3}{10})^2-5\cdot\frac{9}{100}-1 \\ &= 5(x+\frac{3}{10})^2-\frac{29}{20} \end{aligned}$$

Thus, the vertex is $(-\frac{3}{10}, -\frac{29}{20})$.

Example 3: Sketch the graph of $y = x^2 + 4x$.

Solution:

$y = x^2 + 4x$ {Add and subtract half the coefficient of x, squared}

$= x^2 + 4x + 4 - 4$

$= (x^2 + 4x + 4) - 4$ {Associative Law}

$= (x+2)^2 - 4$

So, the graph is a parabola opening upwards with axis of symmetry $x = -2$ and vertex at (-2, -4).

x-intercepts:

When $y = 0$, $x^2 + 4x = 0$

$x(x+4) = 0$

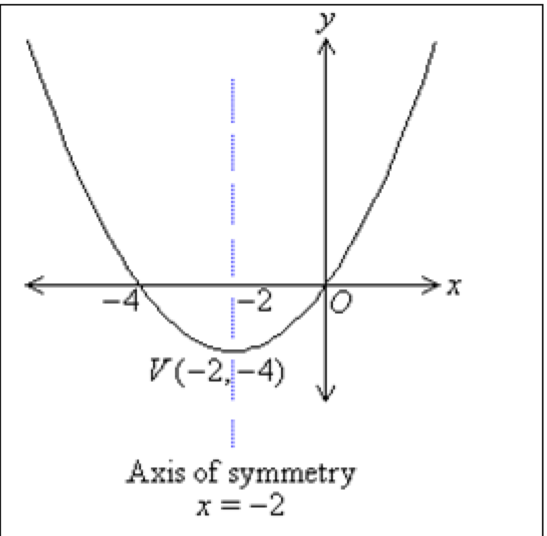
$\therefore$ $x = 0$ or $x + 4 = 0$

$x = 0$ or $x = -4$

y-intercept: c = 0

When $x = 0$, $y = 0$

The minimum value of *y* is – 4.



Example 4: Sketch the graph of y = x² – 6x + 5.

Solution:

$y = x^2 - 6x + 5$ {Add and subtract half the coefficient of x, squared}

$= x^2 - 6x + 9 - 9 + 5$

$= (x^2 - 6x + 9) - 4$ {Associative Law}

$= (x-3)^2 - 4$

So, the graph is a parabola opening upwards with axis of symmetry x = 3 and vertex at (3, -4).

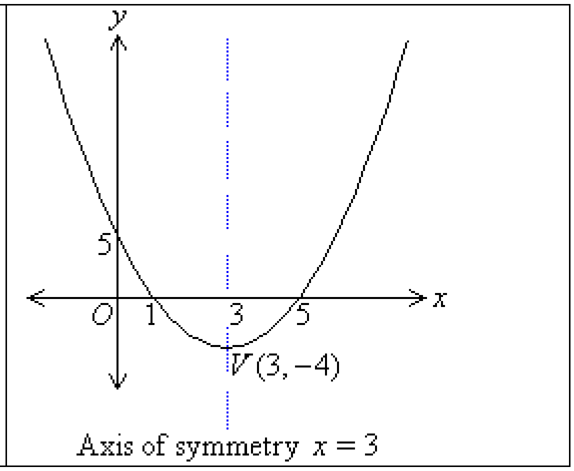
y-intercept: c = 5

When $x = 0$, $y = 0 - 0 + 5 = 5$

Reflection point of y-intercept:
(6, 5)

Given 3 points – vertex, y-int, and reflection of y-int, we can graph.

The minimum value of *y* is – 4.



Example 3: Sketch the graph of $y = -x^2 + 4x - 3$.

Solution:

1) Completing the square method:

$$y = -x^2 + 4x - 3 = -(x^2 - 4x + 3) = -(x^2 - 4x + 4 - 4 + 3) = -[(x-2)^2 - 1] = -(x-2)^2 + 1$$

So, the graph is a parabola opening downwards with axis of symmetry x =2 and vertex at (2, 1).

2) Solve with the formula of finding the vertex from a standard form y = ax^2 + bx + c.

V($-\frac{b}{2a}$, k)

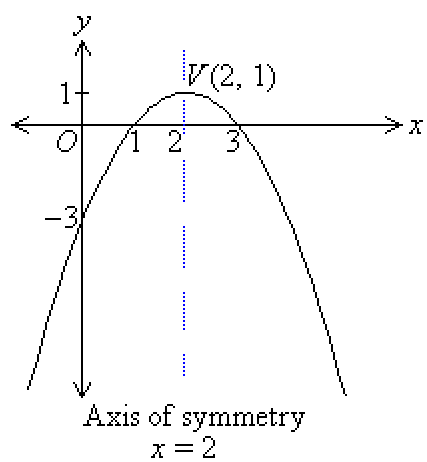
h = $-\frac{b}{2a} = -\frac{4}{-2} = 2$

k = -(2)^2 + 4(2) – 3 = 1

Vertex is (2, 1)

y-intercept: c = -3

Reflection of y-int: (4, -3)



4. Problem Solving

The graph of a quadratic function has the shape of a parabola. Parabolas have many applications in engineering and science. For example, radar, antennas and the paths of projectiles are parabolic.

Example 1: The rate of a chemical reaction, *v*, involving the transformation of one compound into another compound is given by *v = x(6 – x),* where *x* is the concentration of the new compound. What value of *x* results in the maximum value of *v*, and what is the maximum value?

Solution: To find the vertex, we need to complete the square.

$$\begin{aligned}
v &= x(6-x) && \{\text{Expand}\}\\
&= 6x - x^2 \\
&= -x^2 + 6x && \{\text{Make the coefficient of } x^2 = 1\}\\
&= -1(x^2 - 6x) && \{\text{Add and subtract half the coefficient of } x\text{, squared}\}\\
&= -(x^2 - 6x + 9 - 9) && \{\text{Associative Law}\}\\
&= -[(x-3)^2 - 9] \\
&= -(x-3)^2 + 9
\end{aligned}$$

So, the maximum value of v is 9 and it occurs when $x = 3$.

Example 2: A farmer wants to build a fence around a rectangular paddock using the straight edge of a canal as one side. She has 400 m of fencing and wants to enclose the maximum area.

a. If the width of the paddock is x m, express the:

(i) length, l, of the paddock in terms of x
(ii) area, A, of the paddock in terms of x

b. Express A in square completion form.

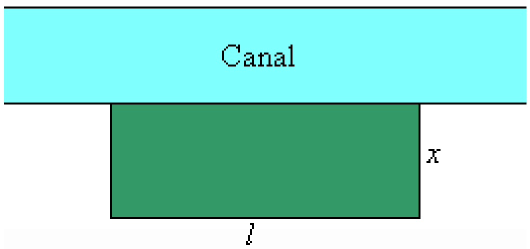
c. Sketch the graph of A showing all key features.

d. Determine the length and width of the paddock for its area to be maximized.

Solution:

a. (i)

$$\begin{aligned}
P &= x + x + l \\
400 &= 2x + l \\
\therefore l &= 400 - 2x
\end{aligned}$$



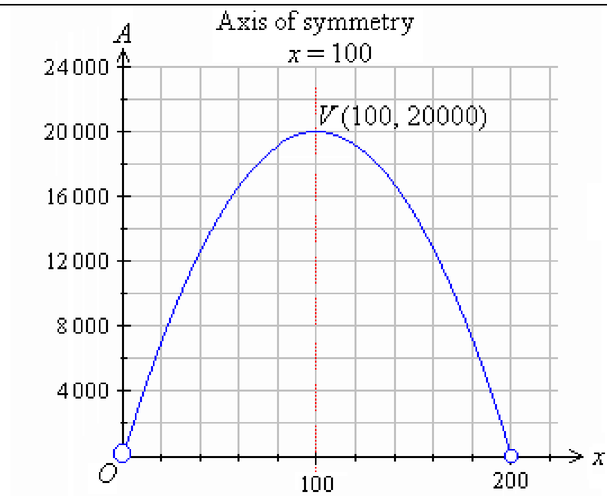
(ii)

$$\begin{aligned}
A &= lw \\
&= x(400-2x) \\
&= 400x - 2x^2 \\
&= -2x^2 + 400x
\end{aligned}$$

b. $A = -2x^2 + 400x$

$$= -2(x^2 - 200x)$$
$$= -2(x^2 - 200x + 100^2 - 100^2)$$
$$= -2[(x-100)^2 - 10000]$$
$$= -2(x-100)^2 + 20000$$

c. The graph is a parabola opening downwards with axis of symmetry x = 100 and vertex at (100, 20000).

x-intercepts: When $A = 0$, $x(400-2x) = 0$ $x = 0$ or $400 - 2x = 0$ $x = 0$ or $-2x = -400$ $x = 0$ or $x = 200$ *A*-intercept: When $x = 0$, $A = 0$ Note that x = 0 and x = 200 are excluded from the graph as 0 < x < 200	

d. The maximum area of the paddock is 20 000 m^2 and it occurs when width = 100 m and length = 200 m.