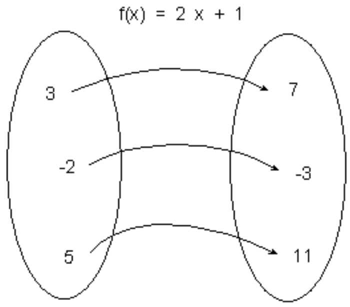
Transformations of Functions (2)

1. Definition of Inverse Function

Before defining the inverse of a function we need to have the right mental image of function.

Consider the function $f(x) = 2x + 1$. We know how to evaluate f at 3, $f(3) = 2*3 + 1 = 7$. In this section it helps to think of f as transforming a 3 into a 7, and f transforms a 5 into an 11, etc.



Now that we think of f as "acting on" numbers and transforming them, we can define the inverse of f as the function that "**undoes**" what f did. In other words, the inverse of f needs to take 7 back to 3, and take -3 back to -2, etc.

Let $g(x) = (x - 1)/2$. Then $g(7) = 3$, $g(-3) = -2$, and $g(11) = 5$, so g seems to be undoing what f did, at least for these three values. To prove that g is the inverse of f we must show that this is true for any value of x in the domain of f. In other words, g must take f(x) back to x for all values of x in the domain of f. So, $g(f(x)) = x$ must hold for all x in the domain of f. The way to check this condition is to see that the formula for g(f(x)) simplifies to x.

$g(f(x)) = g(2x + 1) = (2x + 1 -1)/2 = 2x/2 = x$.

This simplification shows that if we choose any number and let f act it, then applying g to the result recovers our original number. We also need to see that this process works in reverse, or that f also undoes what g does.

$f(g(x)) = f((x - 1)/2) = 2(x - 1)/2 + 1 = x - 1 + 1 = x$.

If the inverse of a function f(x) is also a function, it is called the inverse function of f(x).

This inverse of a function f(x) is denoted by $f^{-1}(x)$, as $f^{-1}(x) = g(x)$.

Letting f^{-1} denote the inverse of f, we have just shown that $g = f^{-1}$.

Definition:

Let f and g be two functions. If **f(g(x)) = x and g(f(x)) = x,** then g is the inverse of f and f is the inverse of g.

Note: the -1 in f^{-1} is not an exponents, so $\mathbf{f^{-1}} \neq \frac{1}{f}$

Given a set of points, inverse can be found by switching domain and range (switching x and y).

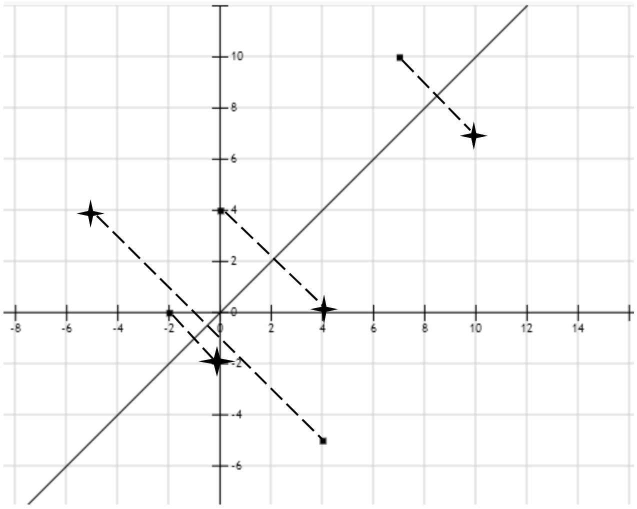
Example 1

Original (•)

x	**y**
-2	0
0	4
4	-5
7	10

Inverse (✦)

x	**y**
0	-2
4	0
-5	4
10	7



Example 2: Find the inverse of f(x) = 2x + 3 algebraically. Graph both relations on the same grid paper. Is the inverse also a function?

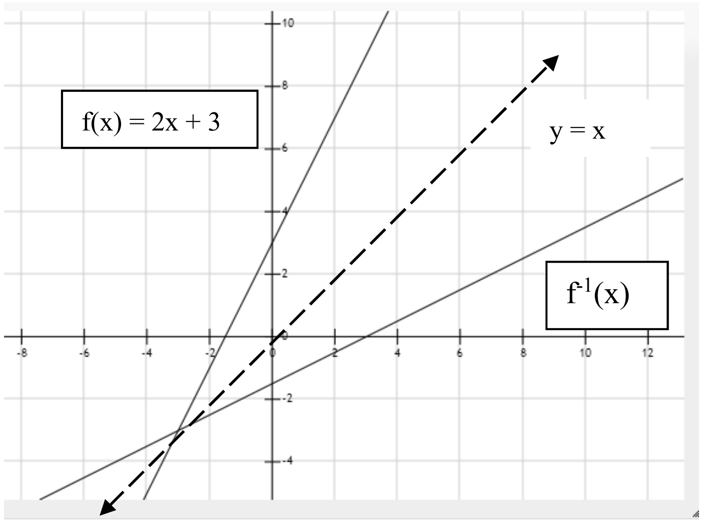
Solution:

x = 2y + 3

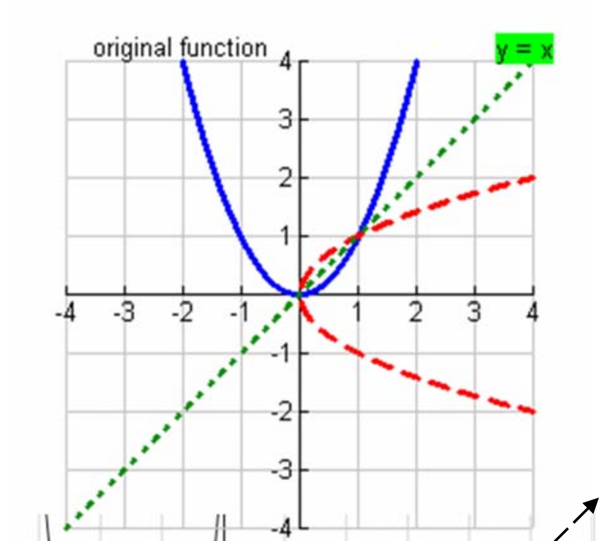
2y = x – 3

$y = \frac{x-3}{2}$

Therefore, $f^{-1}(x) = \frac{x-3}{2} = \frac{1}{2}x - \frac{3}{2}$, which is also linear. The inverse is a function.



Example 3: Find the inverse of $f(x) = x^2$ algebraically. Graph both relations on the same grid paper. Is the inverse also a function?



Solution:

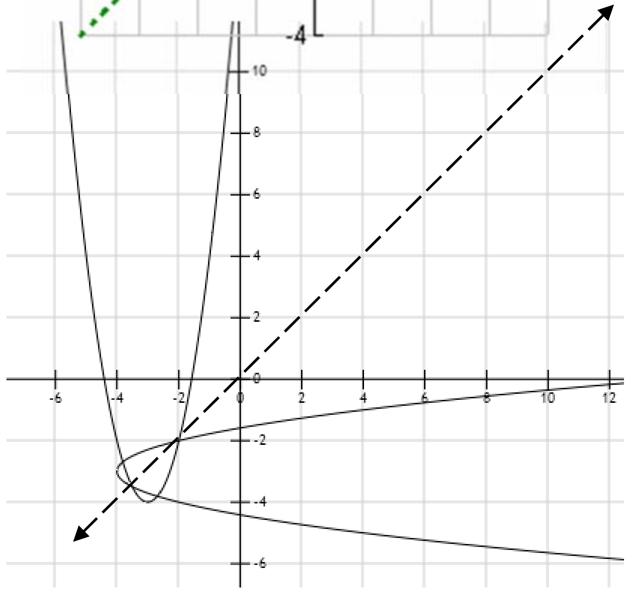
$x = y^2$

$y^2 = x$

$y = \pm\sqrt{x}$

Therefore, $f^{-1}(x) = \pm\sqrt{x}$.

To graph the inverse, it is a reflection about the line y = x. By using the vertical line test, the inverse is NOT a function.



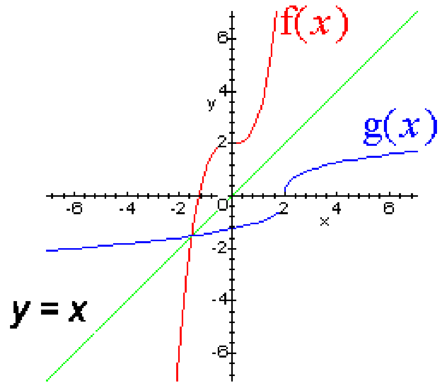
Example 4: Set restrictions on the domain of $f(x) = 2(x + 3)^2 - 4$, so that its inverse is a function as well. Sketch both f and f^{-1}.

Solution: Restricted domain is $x \geq -3$.

Cubic Function (Optional)

1) Find the inverse of $f(x) = x^3 + 2$
2) graph f(x) and its inverse.
3) is the inverse of f(x) a function?
4) determine the domain and the range of f(x) and its inverse.

Solution:



1) $f^{-1}(x) = g(x) = \sqrt[3]{x-2}$

2) Let $f(x) = x^3 + 2$. Then f(2) = 10 and the point (2,10) is on the graph of f. The inverse of f must take 10 back to 2, i.e. $f^{-1}(10)=2$, so the point (10,2) is on the graph of f^{-1}. The point (10,2) is the reflection in the line y = x of the point (2,10). The same argument can be made for all points on the graphs of f and f^{-1}.

The graph of f^{-1} is the reflection about the line y = x of the graph of f.

3) For the inverse $f^{-1}(x)$, there is a value of y for each value of x.

So the inverse $f^{-1}(x)$ is a function (If there are two values of y for each value of x. So the inverse $f^{-1}(x)$ is not a function).

4) For $f(x) = x^3 + 2$ the domain is the set of real number. The range is the set of real number, y is the set of real number

2. Stretch or Compress Functions

1) Horizontal Stretch or Compress

$f(ax)$ stretches/compresses $f(x)$ horizontally

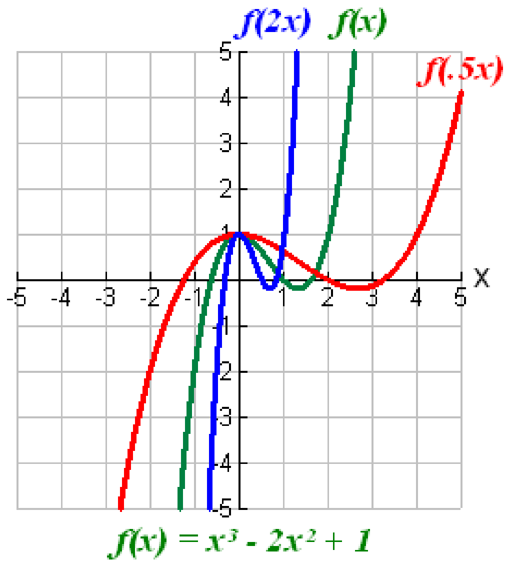
A horizontal stretching is the stretching of the graph away from the y-axis.
A horizontal compression is the squeezing of the graph towards the y-axis.

If the original (parent) function is $y = f(x)$, the horizontal stretching or compressing of the function is the function $f(ax)$.

1) If $0 < a < 1$ (a fraction), the graph is stretched horizontally by a factor of $1/a$ units.

2) If $a > 1$, the graph is compressed horizontally by a factor of $1/a$ units.

3) If a is negative, the horizontal compression or horizontal stretching of the graph is followed by a reflection of the graph across the y-axis.

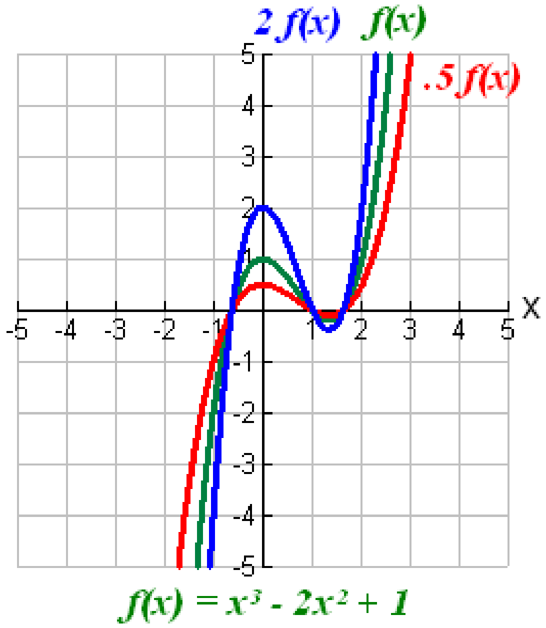


2) Vertical Stretch or Compress

$af(x)$ stretches/compresses $f(x)$ vertically

A vertical stretching is the stretching of the graph away from the x-axis.
A vertical compression is the squeezing of the graph towards the x-axis.
If the original (parent) function is $y = f(x)$, the vertical stretching or compressing of the function is the function $af(x)$.

1) If $0 < a < 1$ (a fraction), the graph is compressed vertically by a factor of a units.
2) If $a > 1$, the graph is stretched vertically by a factor of a units.
3) If a is negative, then the vertical compression or vertical stretching of the graph is followed by a reflection across the x-axis.



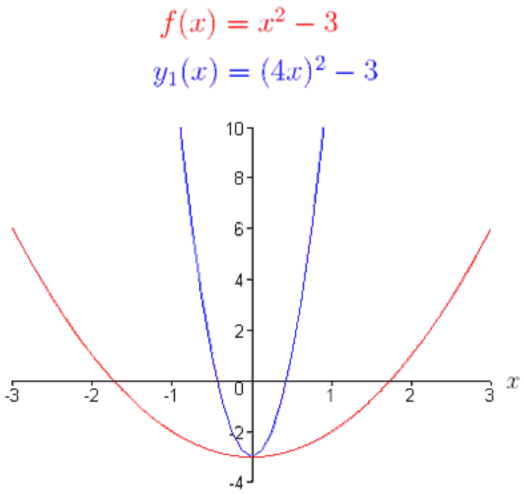
Examples of Horizontal Stretches and Shrinks

Consider the following base functions,
(1) $f(x) = x^2 - 3$, (2) $g(x) = \cos(x)$.

The graphical representation of function (1), $f(x)$, is a parabola. What do you suppose the graph of $\boldsymbol{y_1(x) = f(4x)}$ looks like? Using the definition of $f(x)$, we can write $y_1(x)$ as,

$$y_1(x) = f(4x) = (4x)^2 - 3 = 16x^2 - 3$$

Based on the definition of horizontal shrink, the graph of $y_1(x)$ should look like the graph of $f(x)$, shrunk horizontally by a factor of 1/4. Take a look at the graphs of $f(x)$ and $y_1(x)$.



3. Combinations of Transformations

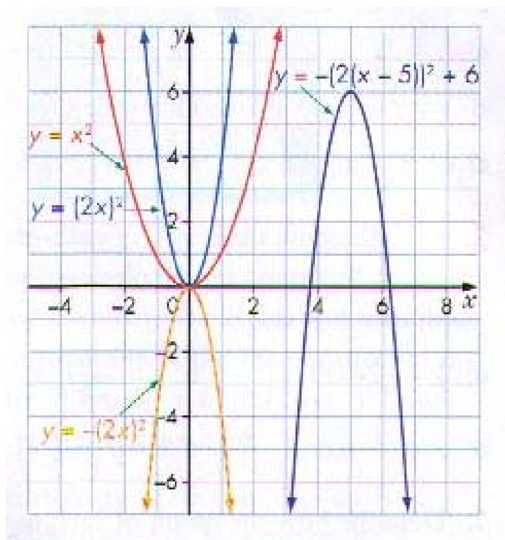
In this section, a combination of translations, expansions, compressions and reflections will be used to perform transformations on functions. We always start with the base function y = f(x), then transform to **y = af(k(x - d)) + c.**

Tips and Tricks:

- Vertical transformations are always on the outside of the function
- Horizontal transformations are always on the inside of the function
- Stretches/compressions are the only transformations that change the shape of the function, and always comes first
- Use key points on the original functions instead of trying to look at the whole thing at once

Example: Given **f(x) = x^2**, sketch the graph of **y = f(x)** and the graph of **y = - f (2(x-5)) + 6.**

Solution 1:



Given the base function: **y = x^2**

The graph of **y = - f (2(x-5)) + 6** is the graph of **y = - $(2(x-5))^2$ + 6.**

To sketch the graph of **y = - $(2(x-5))^2$ + 6**, first sketch the graph **y = $(2x)^2$.** This graph is a horizontal compression of the graph of **y = x^2** by a factor of **1/2.**

Then, sketch the graph of **y = - $(2x)^2$,** which is a reflection the graph **y = $(2x)^2$** in the x-axis.

Then, apply the horizontal translation of 5 units to the right and the vertical translation of 6 units upward.

The result is the graph of **y = - f (2(x-5)) + 6** or **$y = -(2(x-5))^2 + 6$**.

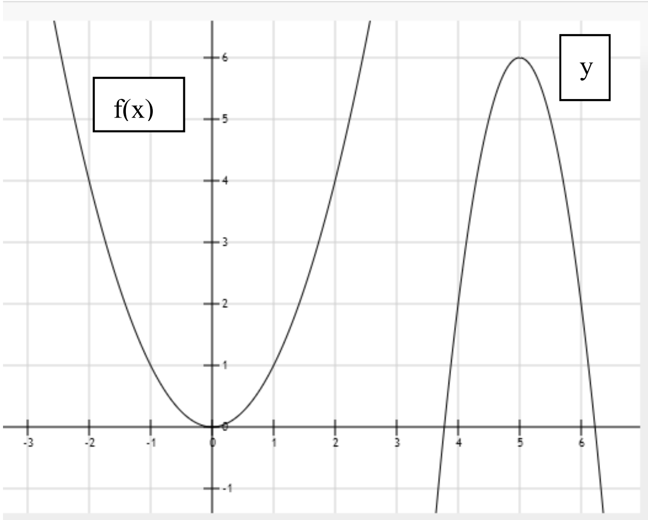
Solution 2:

5 key points for the base function $y = x^2$: (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)

Horizontally, the function is compressed by ½ and shift right by 5, so x → ½ x + 5

Vertically, the function is reflected and shift up by 6, so y → -y + 6

Old points	**New points**
(-2, 4)	(-2/2 + 5, -4 + 6) = (4, 2)
(-1, 1)	(-1/2 + 5, -1 + 6) = (4.5, 5)
(0, 0)	(0/2 + 5, - 0 + 6) = (5, 6)
(1, 1)	(1/2 + 5, -1 + 6) = (5.5, 5)
(2, 4)	(2/2 + 5, -4 + 6) = (6, 2)



Practice:

1. Graph y = 3|1/2 (x – 3)|

2. Graph y = $\frac{1}{-2x+2} - 3$