Patterns

1. Introduction

In a famous puzzle , slightly modified for our purposes, 7 grains of wheat are placed on the first square of a chessboard and then 15,23, 31…. grains are placed on the second, third, fourth…squares of the board.

In other words the number of grains on each square exceeds the number on the previous square by a constant difference of 8. The puzzle asks, of course, for the total number of grains on the board!

The solution to this problem involves adding up the series
7 + 15 + 23 + 31 + 39 +....... until there are 64 terms in this sum.

The first step in solving is to find out the 64_{th} number to be added. We observe that finding this term involves adding the constant difference 8 to the first term 63 times. Therefore the last square of the board holds 7 + 63(8) = 511 grains of wheat.

Thus the total is S = 7 +15 + 23 + 31+ 39 +......+ 495 + 503 + 511.

Since these numbers are equally spaced their average (arithmetic mean) will always be exactly in the middle of the sequence.

The average value is thus
$\frac{7+511}{2}$ ($\frac{15+503}{2}$ or $\frac{23+495}{2}$ or $\cdots\cdots$) and the sum is 64($\frac{7+511}{2}$)= 16,576.

2. Patterns

A **pattern** is a group of numbers, shapes, or objects that follow a rule while repeating or changing.

To extend a pattern you can use a table or a pattern rule that relates the term number to the pattern rule.

A **term number** is the number that tells the position of an item in a pattern.
For example, the pattern 2, 4, 6, 8, 10, … can be shown in a table like this:

Term number	**Number in pattern**
1	2
2	4
3	6
4	8
5	10

A pattern rule to get any number in the pattern is multiply 2 by the term number.
10th term = 2 × 10 = 20

Patterns can be represented using a spreadsheet. A **spreadsheet** is a computer program that has columns of data that are related. Each number in a spreadsheet has its own cell.

To represent a pattern, enter information for the first term. Use one or more operations to get the rest of the terms in the pattern.

For example, the spreadsheet below shows a pattern. The first term is $2.

The formula to get the second term is B3 = 2*B2, the third term is B4 = 2*B3, and so on.

	A	**B**
1	**Term number**	**Cost**
2	1	$2
3	2	$4
4	3	$8
5	4	$16

1) General Number Patterns

These general number patterns encourage the children to look carefully at the differences between the numbers in the sequence, and use that information to work out what comes next, or to fill in the gaps.

1) A very simple pattern, involving repeated addition of one (one times table)
 1, 2, 3, 4, 5, 6, 7, ...

2) Another simple pattern, involving repeated addition of two (two times table)
 2 4 6 8 10 12 14

3) Repeated addition of five (five times table) ... 5 10 15 20 25 30 35

4) Doubling each time ... 1 2 4 8 16 32 64

5) Adding one more each time ... 1 2 4 7 11 16 22

 i.e. 1 + ***1*** = 2, 2 + ***2*** = 4, 4 + ***3*** = 7, 7 + ***4*** = 11, 11 + ***5*** = 16, 16 + ***6*** = 22

6) Halving each time ... 1600 800 400 200 100 50 25

7) Adding three each time ... 8 11 14 17 20 23 26

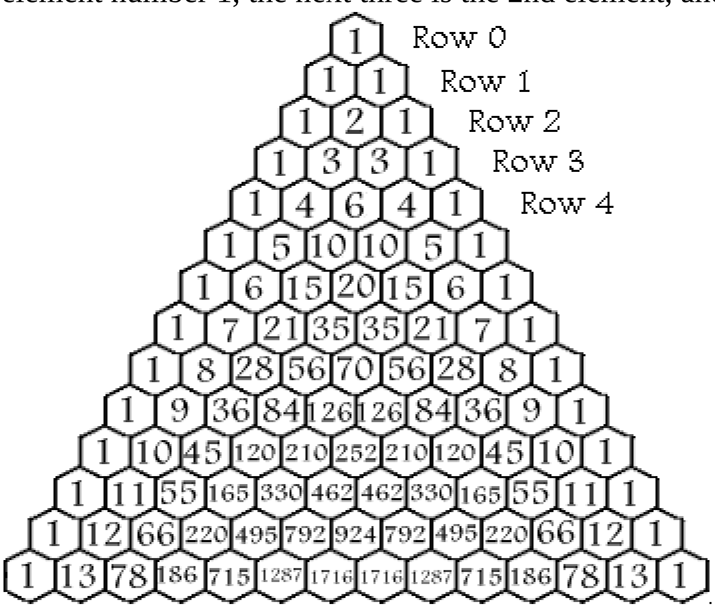
8) Halving each time ... 8 <u>4</u> 2 1 <u>1/2</u> 1/4 1/8

9) Adding 22 each time ... <u>12</u> 34 56 78 <u>100</u> 122 <u>144</u>

10) Repeated pattern ... 0 15 <u>30</u> 0 15 30 <u>0</u>

2) The Sums of the Rows in Pascal's Triangle

At the tip of Pascal's Triangle is the number 1, which makes up the zeroth row. The first row (1 & 1) contains two 1's, both formed by adding the two numbers above them to the left and the right, in this case 1 and 0 (all numbers outside the Triangle are 0's). Do the same to create the 2nd row: 0+1=**1**; 1+1=**2**; 1+0=**1**. And the third: 0+1=**1**; 1+2=**3**; 2+1=**3**; 1+0=**1**. In this way, the rows of the triangle go on infinitely. A number in the triangle can also be found by nCr (n Choose r) where n is the number of the row and r is the element in that row. For example, in row 3, 1 is the zeroth element, 3 is element number 1, the next three is the 2nd element, and the last 1 is the 3rd element.



The sum of the numbers in any row is equal to 2 to the n^{th} power or 2^n, when n is the number of the row.

For example:

$2^0 = 1$
$2^1 = 1+1 = 2$
$2^2 = 1+2+1 = 4$
$2^3 = 1+3+3+1 = 8$
$2^4 = 1+4+6+4+1 = 16$

3) Prime Numbers in Pascal's Triangle

If the first element in a row in Pascal's triangle is a prime number (remember, the 0th element of every row is 1), all the numbers in that row (excluding the 1's) are divisible by it.

For example, in row 7 (1 7 21 35 35 21 7 1) 7, 21, and 35 are all divisible by 7.

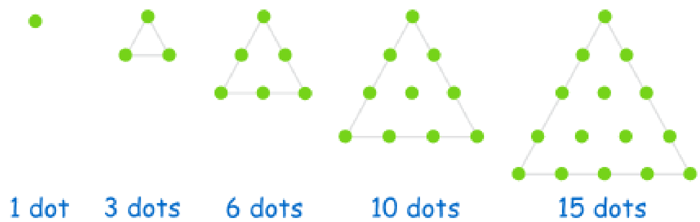
3. Special Numbers

1) Triangular Numbers

1, 3, 6, 10, 15, 21, 28, 36, 45, ...

This Triangular Number is generated from a pattern of dots which form a triangle.

By adding another row of dots and counting all the dots we can find the next number:



These numbers also appears on the third diagonal in Pascal's triangle.

2) Square Numbers

1, 4, 9, 16, 25, 36, 49, 64, 81, ...

The next number is made by squaring where it is in the pattern.

The second number is 2 squared (2^2 or 2×2)

The seventh number is 7 squared (7^2 or 7×7) etc

3) Cube Numbers

1, 8, 27, 64, 125, 216, 343, 512, 729, ...

The next number is made by cubing where it is in the pattern.

The second number is 2 cubed (2^3 or $2\times2\times2$)

The seventh number is 7 cubed (7^3 or $7\times7\times7$) etc

4) Fibonacci Numbers

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The Fibonacci sequence is found by adding the two numbers before it together.

The 2 is found by adding the two numbers before it (1+1)

The 21 is found by adding the two numbers before it (8+13)

The next number in the sequence above would be 55 (21+34)

4. Figure pattern

1) Polygonal Numbers

Polygonal Numbers are really just the number of vertexes in a figure formed by a certain polygon.

The first number in any group of Polygonal Numbers is always 1, or a point. The second number is equal to the number of vertexes of the polygon.

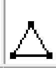
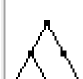
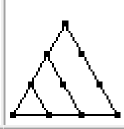
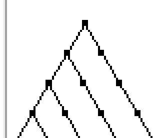
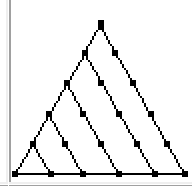
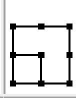
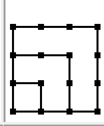
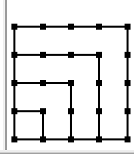
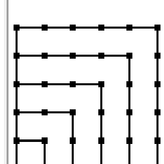
For example, the second Pentagonal Number is 5, since pentagons have 5 vertexes (and sides).

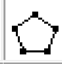
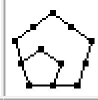
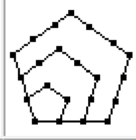
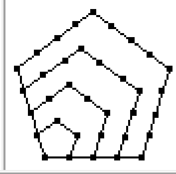
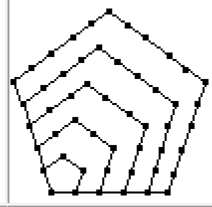
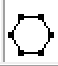
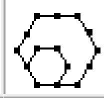
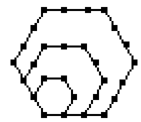
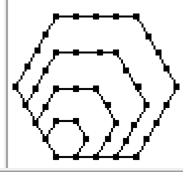
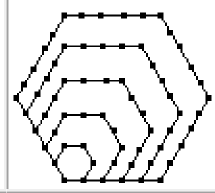
The third Polygonal Number is made by extending two of the sides of the polygon from the second Polygonal Number, completing the larger polygon, and placing vertexes *and other points where necessary*.

The third Polygonal Number is found by adding all the vertexes *and points* in the resulting figure. (Look at the table below for a clearer explanation).

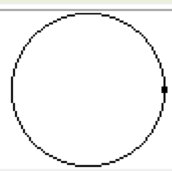
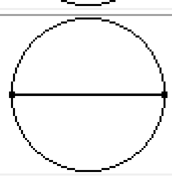
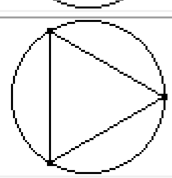
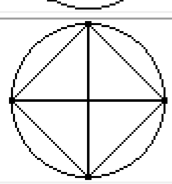
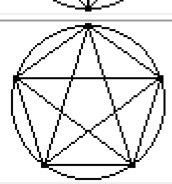
A formula that will generate the n^{th} x-gonal number (for example: the 2^{nd} 3-gonal, or triangular number) is:

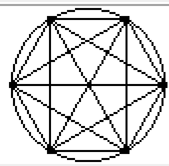
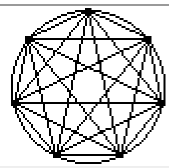
$$\frac{n^2 - n}{2} \times (x - 2) + n$$

Type	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**
Triangular						
Value	1	3	6	10	15	21
Square						
Value	1	4	9	16	25	36

Pentagonal						
Value	1	5	12	22	35	51
Hexagonal						
Value	1	6	15	28	45	66

2) Points on a Circle

Image	Points	Segments	Triangles	Quadrilaterals	Pentagons	Hexagons	Heptagons
	1						
	2	1					
	3	3	1				
	4	6	4	1			
	5	10	10	5	1		

	6	15	20	15	6	1	
	7	21	35	35	21	7	1

As you may have noticed, the numbers in the chart above are actually the tip of the right-angled form of Pascal's Triangle, except the preceding 1's in each row are missing. The circular figures are formed

►Questions in class

1. For the set of numbers {1, 10, 100} we can obtain 7 *distinct* numbers as totals of one or more elements of the set.

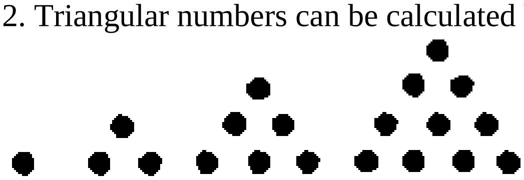
These totals are 1, 10, 100, 1+10=11, 1+100 =101, 10 +100 =110, and 1+10 +100 =111. The "power-sum" of this set is the *sum* of these totals, in this case, 444.

(a) How many distinct numbers may be obtained as a sum of one or more different numbers from the set {1, 10, 100, 1000}? Calculate the power-sum for this set.

(b) Determine the power-sum of the set {1, 10, 100, 1000, 10 000, 100 000, 1 000 000}.
Extension: Consider the set {1, 2, 3, 6, 12, 24, 48, 96}. How many different totals are now possible if a total is defined as the sum of 1 or more elements of a set?

(c) Consider the set {1, 2, 3, 6, 12, 24, 48, 96}. How many different totals are now possible if a total is defined as the sum of 1 or more elements of a set?

2. Triangular numbers can be calculated by counting the dots in the following triangular shapes:



The first triangular number is 1, the second is 3, the third is 6, the fourth is 10, and the nth triangular number equals $1 + 2 + 3 +\ldots+ (n-1) + n$.

(a) Calculate the 10th and 24th triangular numbers.

(b) Prove that the sum of any three consecutive triangular numbers is always 1 more than three times the middle of these three triangular numbers.

(c) The 3rd, 6th and 8th triangular numbers (6, 21 and 36) are said to be in arithmetic sequence because the second minus the first equals the third minus the second, ie.
$21 - 6 = 36 - 21$. Also, the 8th, 12th and 15th triangular numbers (36, 78 and 120) are in arithmetic sequence. Find three other triangular numbers, each larger than 2004, which are in arithmetic sequence.

3. Points B, C, and D lie on a line segment AE, as shown.

The line segment AE has 4 basic sub-segments AB, BC, CD, and DE, and 10 sub-segments:

AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE.
The super-sum of AE is the sum of the lengths of all of its sub-segments.

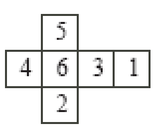
(a) If AB = 3, BC = 6, CD = 9, and DE = 7, determine the lengths of the 10 sub-segments, and calculate the super-sum of AE.

(b) Explain why it is impossible for the line segment AE to have 10 sub-segments of lengths 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.

(c) When the super-sum of a new line segment AJ with 9 basic sub-segments of lengths from left to right of 1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9 is calculated, the answer is 45.

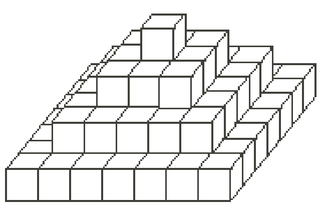
Determine the super-sum of a line segment AP with 15 basic sub-segments of lengths from left to right of 1, 1/2, 1/3, . . ., 1/15.

4. Dmitri has a collection of identical cubes. Each cube is labelled with the integers 1 to 6 as shown in the following net:



(This net can be folded to make a cube.)

He forms a pyramid by stacking layers of the cubes on a table, as shown, with the bottom layer being a 7 by 7 square of cubes.

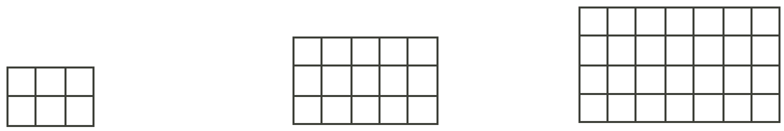


(a) Determine the total number of cubes used to build the pyramid. Explain how you got your answer.

(b) How many faces are visible after the pyramid is built and sitting on the table?

(c) Explain in detail how he should position the cubes so that if all of the visible numbers are added up, the total is as large as possible. What is this total?

5. Squares measuring 1 by 1 are arranged to form the following sequence of rectangles:



Rectangle 1 Rectangle 2 Rectangle 3

Many more rectangles are drawn, with each rectangle having one more row and two more columns than the previous rectangle.

(a) How many 1 by 1 squares would there be in Rectangle 4? Explain how you obtained your answer.

(b) Determine the perimeter of Rectangle 4. Explain how you obtained your answer.

(c) Determine the perimeter of Rectangle 7. Explain how you obtained your answer.

(d) Rectangle n has a perimeter of 178. Determine n. Explain how you obtained your answer.