Chapter 4 Quadratic Functions (1)

1. Quadratic Functions in Standard Form

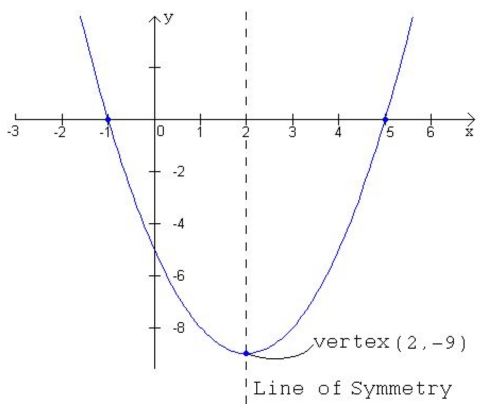
A relation defined by the rule **$y = ax^2 + bx + c$** (where a ≠ 0 and a, b and c are constants) is called a **quadratic relation in standard form**. Since the relations is one to one, it is also called **quadratic function.** Any relation described by a polynomial of **degree 2** is quadratic.

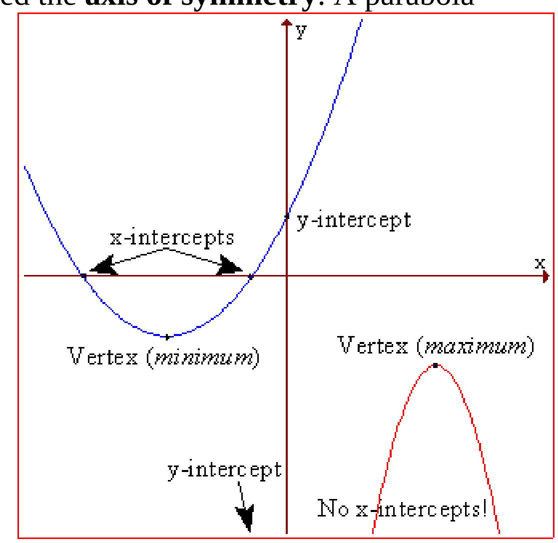
E.g. $y = x^2$, $y = x^2 + 4$, $y = 2x^2 + 3$, $y = x^2 – 6x$, $y = x^2 + 3x + 2$ etc. are quadratic functions.

In this section, we will learn the general characteristics of quadratic functions by plotting their graphs. Then these characteristics will be used to sketch other quadratic graphs.

The graph of a quadratic function is a curve called a **parabola**. Parabolas may open upward or downward and vary in "width" or "steepness", but they all have the same basic "U" shape. The picture below shows three graphs, and they are all parabolas.

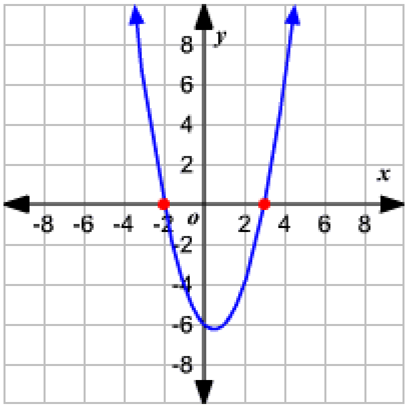
All parabolas are symmetric with respect to a line called the **axis of symmetry**. A parabola intersects its axis of symmetry at a point called the **vertex** of the parabola.



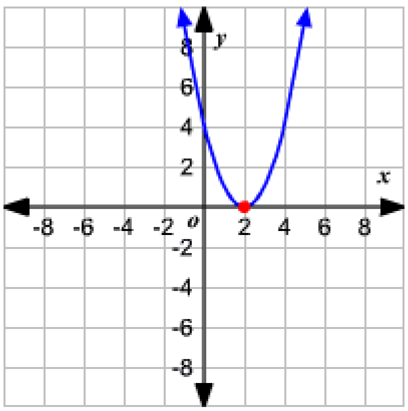


x-intercepts: x-value when y =0. The x-intercepts, if any, are also called **the roots** or **the zeroes** of the function. They are meaningful specifically as the zeroes of the function, but also represent the two roots for any value of the function.

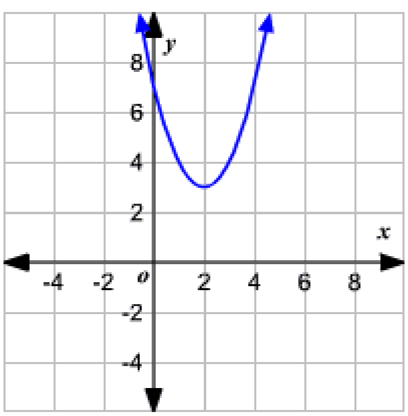
There are 3 cases for number of x-intercepts.

1) 

two x-intercepts

2) 

one x-intercept, which is the x-coordinate of the vertex.

3) 

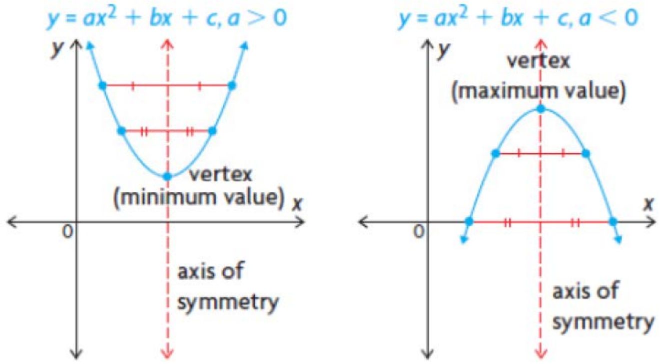
no x-intercept

y-intercept: y-value when x = 0. The importance of the y-intercept is usually as an *initial value* or *initial condition* for some state of an experiment, especially one where the independent variable represents time.

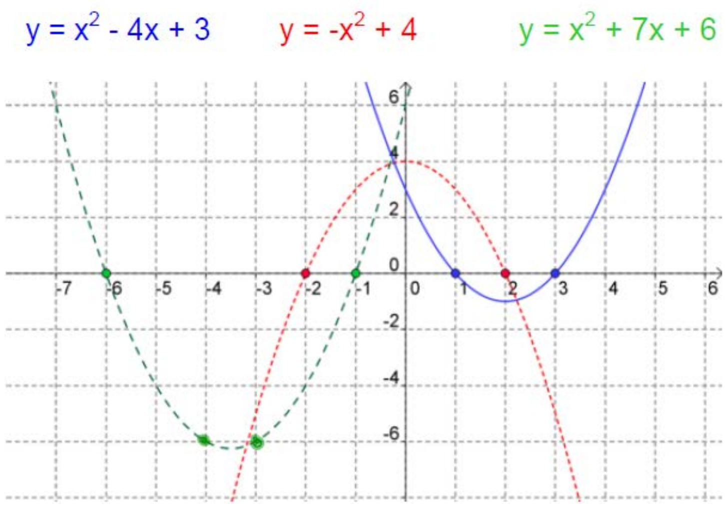
Direction of opening: depends on whether the coefficient of x^2 – "a" is positive or negative. The parabola opens upward if $a > 0$. The parabola opens downward if $a < 0$.

Vertex: The vertex represents the maximum (or minimum) value of the function.

Optimal Value: y-value of the vertex. It is a maximum value if the parabola opens down and a minimum value if the parabola opens up.



Example 1: From your graphs, determine key features of each.



	$y = x^2 - 4x + 3$	$y = -x^2 + 4$	$y = x^2 + 7x + 6$
Vertex	(2, -1)	(0, 4)	(-3.5, -6.25)
Opening	Up	Down	Up
Max/min	Min	Max	Min
Optimal value	-1	4	-6.25
y-intercept	3	4	6
Zeroes	1, 3	2, -2	-6, -1
Axis of symmetry (AOS)	x = 2	x = 0	x = -3.5

We can see that given the standard form of quadratic relation $y = ax^2 + bx + c$, the y-intercept is *c*.

2. Quadratic Functions in Factored Form

If $\mathbf{y = ax^2 + bx + c}$ can be factored into the form $y = a(x – s)(x – t)$, then we can find the x-intercepts by setting $y = 0$.

$0 = a(x – s)(x – t)$

$x – s = 0$ or $x – t = 0$

$x = 0$ or t

Therefore, the x-intercepts are s and t.

Then we can find the vertex (h, k). How to find the vertex after we have our x-intercepts?

We need to notice that the x-coordinate of the vertex is in the middle of the x-intercepts. Taking the mid-value of the x-intercept would be your x-coordinate of the vertex. The x-coordinate of the vertex h is the midpoint of the x-intercepts.

$\mathbf{h = \frac{s+t}{2}}$

To find the y-coordinate of the vertex k simply substitute h into the equation.

Example 1: Find the x-intercepts, y-intercept for $y = x^2 – 5x – 14$

Since the relation is in standard form, y-intercept is $c = -14$

To find the x-intercepts, we need the factored form.

$y = x^2 – 5x – 14 = (x – 7)(x + 2) = (x – 7)(x – (-2))$

There are two x-int: -2, and 7

Example 2: Given the parabola $y = x^2 – 2x – 15$.
a) Find the x-intercepts.

Change the standard form to factored form.
$y = x^2 – 2x – 15 = (x + 3)(x – 5)$
Therefore, the x-intercepts are -3 and 5.

b) What is the equation of the axis of symmetry?

AOS is located in the middle of the parabola since the parabola is always symmetrical.
Use the midpoint formula: x = $\frac{-3+5}{2}$ = 1
Therefore, the AOS is x = 1.

c) What are the coordinates of the vertex?

Now we need to find the y-value when x = 1.

Sub x = 1 into the equation: $y = x^2 - 2x - 15 = (1)^2 - 2(1) - 15 = -16$

Therefore, the vertex is (1, -16).

Example 3: Sketch the graph of the relation $y = x^2 - 6x$.

To sketch we need the vertex and 2 other symmetric points – in this case, x-intercepts would be the easy points to find since the relation can be factored.

$y = x^2 - 6x = (x)(x - 6)$

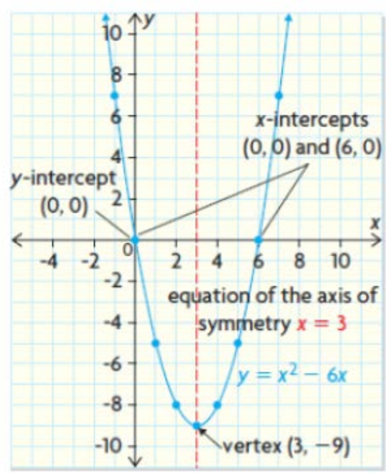
Zeroes: s = 0, t = 6

Vertex: h = $\frac{0+6}{2}$ = 3

x = 3 is the axis of symmetry.

$k = (3)^2 - 6(3) = -9$

The coordinate of the vertex is (3,−9).



Example 4: Grace hits a golf ball out of a sand trap, from a position that is level with the green. The path of the ball is approximated by the equation $\mathbf{y = -x^2 + 5x}$, where x represents the horizontal distance travelled by the ball in meters and y represents the height of the ball in meters. Determine the greatest height of the ball and the distance away that it lands.

To find the greatest height of the ball, we need the vertex.

$y = -x^2 + 5x = (-x)(x - 5)$

s = 0, t = 5

h = $\frac{0+5}{2}$ = 2.5

$k = -(2.5)^2 + 5(2.5) = 6.25$

The greatest height of the ball is 6.25 m and the distance away that it lands is 5 m.

Example 5: Find the standard form algebraic model that represents a parabola with zeros at 2 and -4 and a y-intercept of 16.

Since we know the zeros, we can write out the factored form of the quadratic relation.

$y = a(x - 2)(x + 4)$

Now we have to find the "a" by substituting the y-intercept (0, 16)

$y = a(x - 2)(x + 4)$

$16 = a(0 - 2)(0 + 4)$

$16 = -8a$

$a = -2$

$y = -2(x - 2)(x + 4)$ – Factored form

Apply the distributive property to convert the equation to standard form.

$y = -2(x - 2)(x + 4) = -2x^2 - 4x + 16$

Example 6: Graph the relation $y = x^2 + 4x + 4$.

First, we need to factor it $y = x^2 + 4x + 4 = (x + 2)^2$.

s = -2, t = -2, there is only 1 x-intercept. How do we find the vertex then?

The vertex is the x-intercept → (-2, 0). How do we graph it with only 1 point? We can't. We would need one more point with its symmetric point.

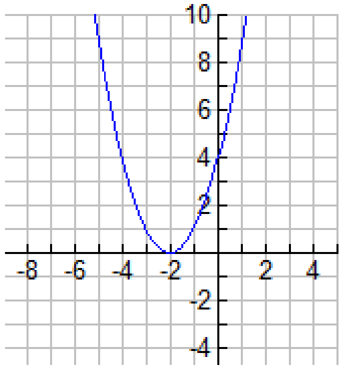
The easiest point to use is the y-intercept, c = 4 → (0, 4)

To find its symmetric point, we need the midpoint formula again.

$\frac{x+0}{2} = -2, x = -4$

The symmetric point is (-4, 4)

Now, we can graph it.



Class Practice: Find the x-intercepts, y-intercept and the vertex for the following quadratic function $y = 2x^2 - 2x - 12$.

3. Finite Differences

Finite differences are differences found from the y-values in tables with evenly spaced x-values. First differences are calculated by subtracting consecutive y-values. Second differences are calculated by subtracting consecutive first differences.

If the first differences are constant, we can say that the relation is **linear**. If the second differences are constant, we can say that the relation is **quadratic.** If the first and second differences are not constant we can say the graph is **non-linear / neither linear nor quadratic.**

Example:
Complete the table and determine whether the relation is **linear, quadratic or neither.**

x	y	1^{st} diff	2^{nd} diff
1	-10	0 – (-10) = 10	18 – 10 = 8
3	0	18 – 0 = 18	26 – 18 = 8
5	18	44 – 18 = 26	34 – 26 = 8
7	44	78 – 44 = 34	
9	78		

Since the 2^{nd} differences are constant, the relation is **quadratic.**

x	y	1^{st} diff
-2	-1	1 – (-1) = 2
0	1	3 – 1 = 2
2	3	5 – 3 = 2
4	5	7 – 5 = 2
6	7	

Since the 1^{st} differences are constant, the relation is **linear.**