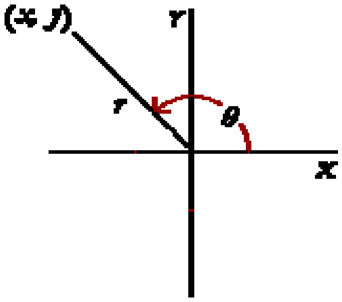
Trigonometric (2)

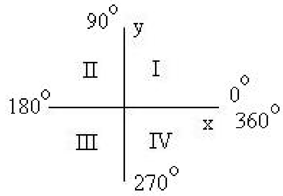
1. Trigonometry in Standard Position

- We can evaluate primary trig rations for angles between 0° and 360° (not just acute angles)
- We draw the angles in "**standard position**"

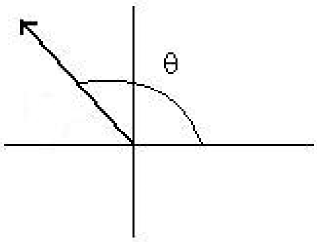


-vertex at origin
-**initial arm** along x-axis
-**terminal arm** rotates about origin
-θ is angle between initial and terminal arm
-r is length of terminal arm (can use Pythagorean)
-β is **related acute angle** (write at end), always the angle between the terminal arm and the x-axis.

The Cartesian plane is divided into four Quadrants: I, II, III, and IV.

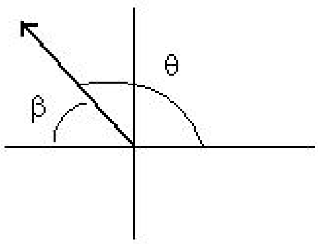


Principal Angle: $0° \le \theta \le 360°$
Measure from the positive x-axis to the terminal arm

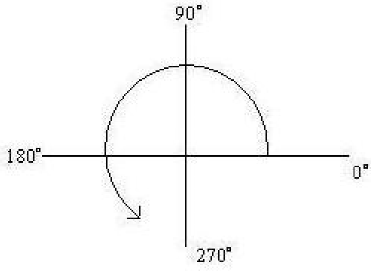


Related Acute Angle: $0° \le \beta \le 90°$

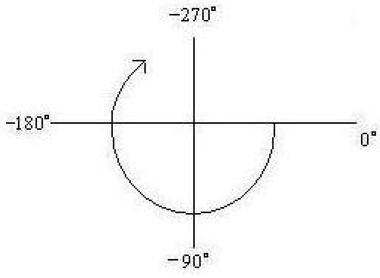
Measure from the closest x-axis to the terminal arm



Positive Angle (Counter Clockwise)

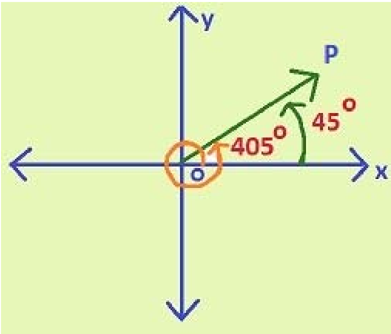


Negative Angle (Clockwise)



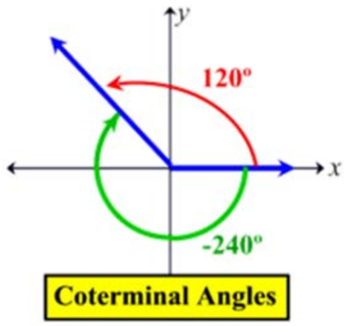
Coterminal Angles are angles which have the same terminal sides.

In standard position, sketch a 45° and 405° angle on Cartesian plane.



Another example: 120° and 120 – 360 = -240° are cotermianl angels.

***Simply add or subtract 360° to the original angle θ to get coterminal angles.**

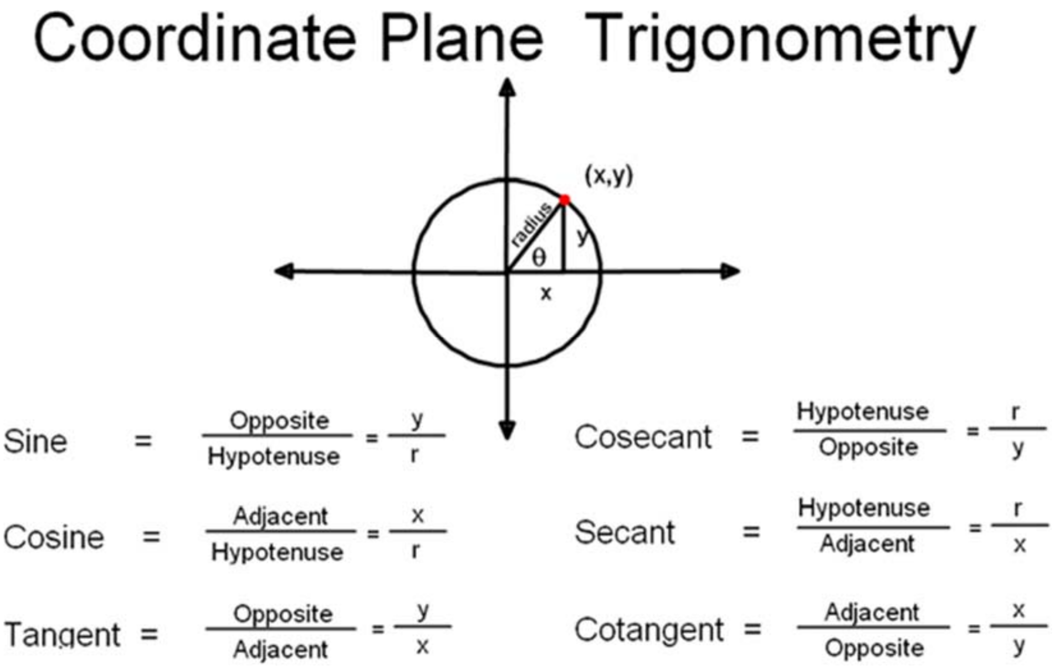


If given a point on the terminal arm, you can draw a sketch on a Cartesian plane. Trig. ratios can be determined using x, y and r rather than opposite, hypotenuse and adjacent.

ALWAYS DRAW THE PERPENDICULAR LINE SEGMENT TOWARDS THE **X-AXIS** TO FORM THE TRIANGLE!!!

To calculate the r value, we can rearrange the Pythagorean Theorem to get:

$$r = \sqrt{x^2 + y^2}$$



Example: Given the point P (3, -4) on the terminal arm of θ, find the all trig. ratios for θ in standard position if $0^\circ \le \theta \le 360^\circ$.

x = 3, y = -4

$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = 5$

sin θ = -4/5	cos θ = 3/5	tan θ = -4/3
csc θ = 5/-4	sec θ = 5/3	cot θ = 3/-4

2. Special Angles (0°, 30°, 45°, 60°, 90° **)**

1) Unit Circle - is a circle with a radius of one (a unit radius). In trigonometry, the unit circle is centered at the origin.

By the Pythagorean Theorem, we have $x^2 + y^2 = 1$.

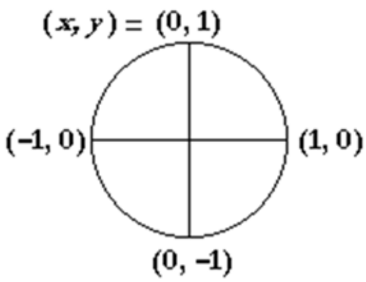
If we examine angle θ (in standard position) in this unit circle, we can see that

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{1} \quad \text{and} \quad \sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{1}$$

which show us that in a unit circle, $\cos\theta = x$ and $\sin\theta = y$

also creating $(x, y) = (\cos\theta, \sin\theta)$

2) Quadrantal angle - is an angle that terminates *on* the *x*- or *y*-axis. They are 0°, 90°, 180°, 270°, 360°; and angles coterminal with them.



Consider sin θ at each quadrantal angle. We just saw that the value of sin θ is the *y*-coordinate. Therefore at each quadrantal angle, we have the following:

At θ = 0° / 360°, sin θ = y = 0 **At θ = 90°, sin θ = y = 1**

At θ = 180°, sin θ = y = 0 **At θ = 270°, sin θ = y = -1**

Example 1: Find the coordinate on the unit circle if the principal angle is 30°.

$\cos\theta = \cos 30° = \frac{\sqrt{3}}{2}$ $\sin\theta = \sin 30° = \frac{1}{2}$

➔ $(x, y) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$

Without the restriction of θ, can you give me another angle with the same coordinate?

➔ 30 + 360 = 390°

➔ All coterminal angles of 30° can be another angle.

Example 2: Find ALL angles θ where $0° \le \theta \le 360°$ for $\cos\theta = 0$.

Solution: $\cos\theta = x$ in a unit circle, $x = 0$ when $\theta = 90°$ or $270°$.

3. Radians and Angle Measure

Any real number θ may be interpreted as the radian measure of an angle as follows:

If $\theta \ge 0$, think of wrapping a length θ of string around the standard unit circle *C* in the plane, with initial point *P*(1,0), and proceeding counterclockwise around the circle; do the same if $\theta < 0$, but wrap the string clockwise around the circle. This process is described in Figure. 1 below.

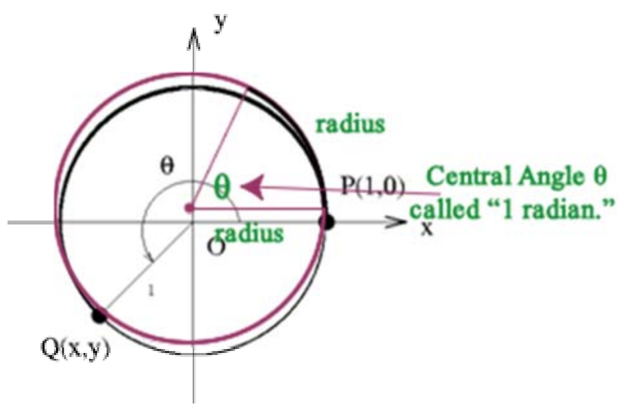


Figure.1

If $Q(x, y)$ is the point on the circle where the string ends, we may think of θ as being an angle by associating to it the central angle with vertex O (0, 0) and sides passing through the points P and Q. If instead of wrapping a length s of string around the unit circle, we decide to wrap it around a circle of radius r, the angle θ (in radians) generated in the process will satisfy the following relation:

$$\boldsymbol{a} = \mathbf{r}\theta \qquad \textbf{(1)}$$

$\theta = \dfrac{a}{r}$ **or number of radians = arc length / radius**

Observe that the length a of string gives the measure of the angle θ only when r =1.

As a matter of common practice and convenience, it is useful to measure angles in *degrees*, which are defined by partitioning one whole revolution into 360 equal parts, each of which is then called one degree. In this way, one whole revolution around the unit circle measures 2π radians and also 360 degrees (or 360^0), that is:

$$360^0 = 2\pi \ \textbf{ radians, or } \ 180^0 = \pi \ \textbf{ radians} \qquad \textbf{(2)}$$

So $\boxed{1^0 = (\frac{\pi}{180}) \text{ radian}}$ and $\boxed{1 \text{ radian} = (\frac{180}{\pi})^0}$

Each degree may be further subdivided into 60 parts, called *minutes*, and in turn each minute may be subdivided into another 60 parts, called *seconds*:

1 degree = 60 minutes = 60' **(3)**

1 minute = 60 seconds = 60'' **(4)**

Example 1: Express the angle 236.345^0 in Degree-Minute-Second (DMS) notation.

Solution: We use Equation (3) to convert a fraction of a degree into minutes and a fraction of a minute into seconds:

$$\begin{aligned} 236.345^\circ &= 236^\circ + 0.345^\circ = 236^\circ + 0.345^\circ \cdot \frac{60'}{1^\circ} \\ &= 236^\circ + 20.7' = 236^\circ + 20' + 0.7' \cdot \frac{60''}{1'} \\ &= 236^\circ 20' 42'' \end{aligned}$$

Therefore, $236.345^\circ = 236^\circ 20' 42''$.

Example 2: Express the angle 236.345^0 in radians.

Solution: From Equation (2), we see that

$$236.345^\circ = 236.345^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = 4.125 \text{ radians (to 3 decimal places)}.$$

Example 3: Express the angle 60 degrees in exact radians.

$$60 \times \left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

Example 4: Express each radian measure to degree measure.

a) $\frac{\pi}{4} = \frac{\pi}{4} \times \frac{180}{\pi} = 45^\circ$

b) $2.2 = 2.2 \times \frac{180}{\pi} = 126^\circ$

Example 5: Find the length of an arc on a circle of radius 75 inches that spans a central angle of measure 126°.

Solution: We use Equation (1), $\mathbf{a} = \mathbf{r}\theta$, with r = 75 inches and

$$\theta = 126^\circ = 126^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = 2.199\ldots \text{ radians}$$

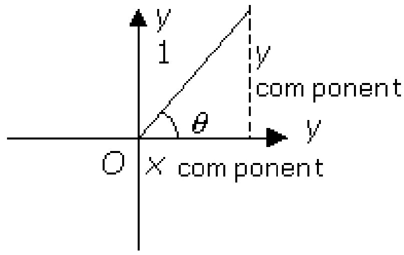
$a = r\theta = (75)(2.199) = 164.934$ (to 3 decimal places)

4. Trigonometric Ratios of Any Angle

1st Quadrant

We define the trigonometric ratios as follows:
These definitions agree with those you have already used in dealing with right-angled triangle.

$$\sin\theta = \frac{y}{1} = y$$
$$\cos\theta = \frac{x}{1} = x$$
$$\tan\theta = \frac{y}{x}$$

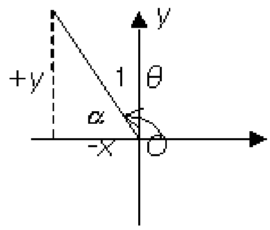


2nd Quadrant

Here x is negative, y is positive. θ is obtuse and the basic angle ($180° - \theta$) is acute.

$a = 180° - \theta$

$$\sin\theta = y = \sin(180° - \theta)$$
$$\cos\theta = -x = -\cos(180° - \theta)$$
$$\tan\theta = \frac{y}{-x} = -\tan(180° - \theta)$$

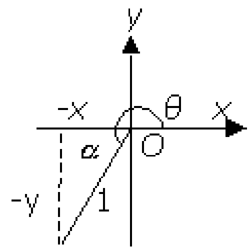


3rd Quadrant

When the angle lies in the third quadrant, both components x and y are negative.

$a = \theta - 180°$

$$\sin\theta = -y = -\sin(\theta - 180°)$$
$$\cos\theta = -x = -\cos(\theta - 180°)$$
$$\tan\theta = \frac{-y}{-x} = \tan(\theta - 180°)$$

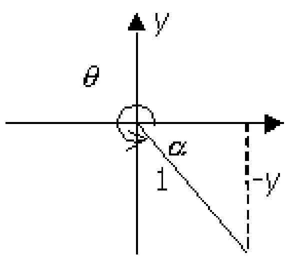


4th Quadrant

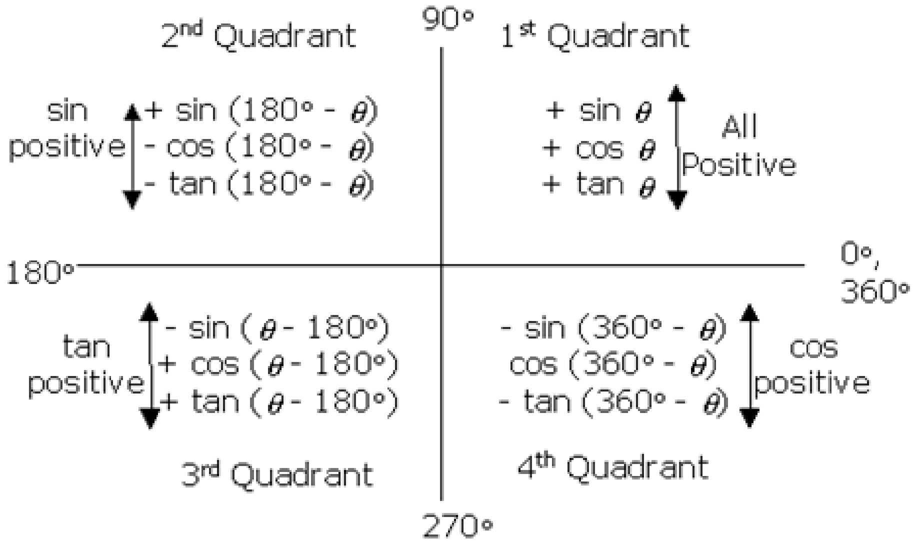
In this quadrant, the x component is positive, the y component is negative.

$a = 360° - \theta$.

$$\sin\theta = -y = -\sin(360° - \theta)$$
$$\cos\theta = x = \cos(360° - \theta)$$
$$\tan\theta = \frac{-y}{x} = -\tan(360° - \theta)$$



Connecting any angle and a basic angle are summarized in the diagram below.



<u>**C.A.S.T. Rule**</u> - tells us where the positive values are located.

Sin	All
Tan	Cos

Example 1: Express sin 323°, tan 98° and cos 236° in terms of the ratios of their basic angles.

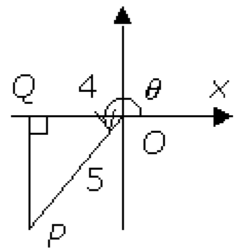
sin 323° = - sin(360° - 323°) = - sin 37°

cos 236° = cos(236° - 180°) = - cos 56°

tan 98° = tan(180° - 98°) = - tan 82°

Example 2: Given that $\cos\theta = -4/5$ and $180° \le \theta \le 270°$, evaluate $\tan\theta$ and $\sin\theta$.

Because $180° \le \theta \le 270°$, θ is in the third quadrant. Draw θ in standard position.

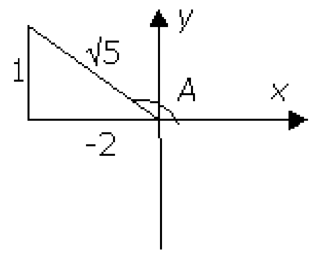


Let *OQ* = 4 units and *OP* = 5 units. By Pythagoras's theorem, PQ = 3
$\tan\theta = 3/4$
$\sin\theta = -3/5$

cos β = 4/5 where β is the related acute angle of θ.
β = 37°
$\theta = 180 + 37 = 217°$

Example 3: Given tan A = -½, find the values of cos A, sin A, and angle A.

Case 1: Quadrant II



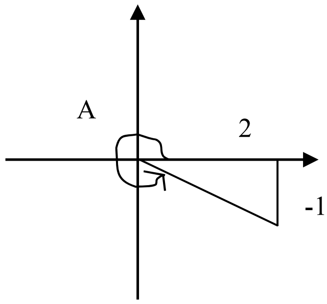
Let $x = -2$, $y = 1$, → r = $\sqrt{5}$

$\sin A = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$ $\cos A = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$

To find the angle A, we must first find the related acute angle β:

$\sin \beta = \frac{\sqrt{5}}{5}$, → β = 27° → A = 180 – 27 = 153°

Case 2: Quadrant IV



Let $x = 2$, $y = -1$, → r = $\sqrt{5}$

$\sin A = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$ $\cos A = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

$\sin \beta = \frac{\sqrt{5}}{5}$, → β = 27° → A = 360 – 27 = 333°

5. Negative Angles

In general, $\mathbf{\sin(-\theta) = -\sin\theta}$ $\mathbf{\cos(-\theta) = \cos\theta}$ $\mathbf{\tan(-\theta) = -\tan\theta}$

Example: Express the trigonometric ratios of -40° in terms of its basic angles.

The basic angle is in 4th quadrant
sin(-40°) = sin 320° = -sin (360° - 320°) = -sin 40°

cos(-40°) = cos 320° = cos (360° - 320°) = cos 40°

tan(-40°) = tan 320° = -tan (360° - 320°) = -tan 40°

6. Trigonometric Ratios of Special Angles

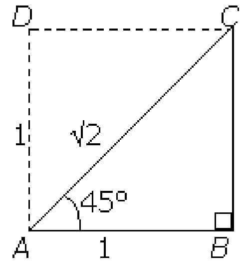
1) 45°

A square *ABCD*, is drawn, with sides of a unit length. *AC*, a diagonal, is drawn.

$$\sin 45° = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{1}{\sqrt{2}}$$

$$\tan 45° = 1$$



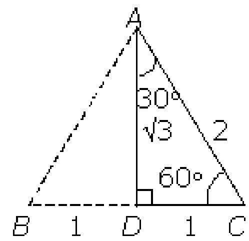
2) 60° & 30°

An equilateral triangle, *ABC*, of sides 2 units in length, is drawn. *AD* is drawn from A perpendicular to *BC*.

$$\sin 30° = \frac{1}{2} \qquad \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}} \qquad \tan 60° = \frac{\sqrt{3}}{1}$$



degrees	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	0	1	0	–	1	–
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	–	1	–	0

Example: Without using a calculator, evaluate

a) $\tan\frac{\pi}{4} + \tan\frac{\pi}{6}\ \tan\frac{\pi}{3}$

b) $\dfrac{\cos\frac{\pi}{4}}{\sin\frac{2\pi}{3}\tan\frac{11\pi}{6}}$

Solution:

a) $1+\dfrac{1}{\sqrt{3}}\times\dfrac{\sqrt{3}}{1}=2$

b) $\dfrac{\dfrac{1}{\sqrt{2}}}{\dfrac{\sqrt{3}}{2}\times(-\dfrac{1}{\sqrt{3}})}=-\dfrac{1}{\sqrt{2}}\times 2=-\sqrt{2}$