Lesson 14: Unit 8 – Rates of Change

In this lesson, we will develop the concepts of average and instantaneous rates of change. We will look at the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a specific point, both graphically and numerically, using a variety of applications.

Introduction

First, we will examine average rate of change and instantaneous rate of change, making connections to the slope of secants and tangents.

A **rate of change** is a measure of the change in the dependent variable, Δy, with respect to a change in the independent variable, Δx. When we calculate the slope of a line segment, we are calculating the rate of change of y with respect to x.

We are interested in examining two types of rates of change.
The first is average rate of change which is measured over an interval.
The second is instantaneous rate of change which is measured at a particular instant.

A **secant** is a line which passes through a curve in at least two distinct points.

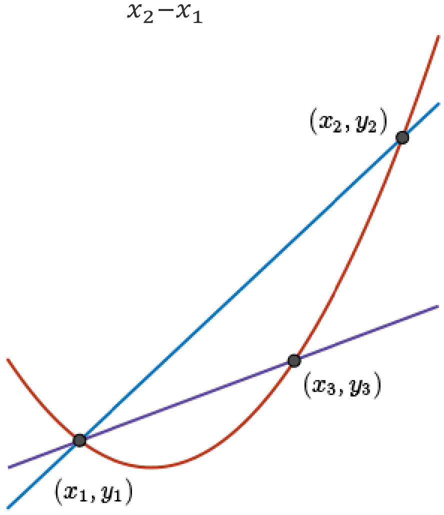
More than one secant can be drawn through a point on a curve.

A secant through a single point is not unique.
The **slope of the secant** between two points represents the **average rate of change** and can be found as follows:

m_{secant} = average rate of change

$$= \frac{\Delta y}{\Delta x}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$



The notation m_{secant} is read "*m* subscript secant" and is used to represent the slope of the secant.
For something like m_{PQ}, we read "*m* subscript *PQ*."
This notation denotes the slope of a secant through *PQ*.

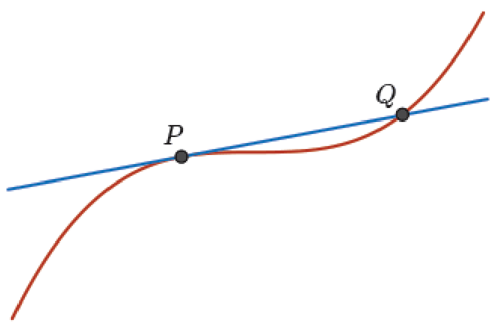
A **tangent** is a line which touches a curve at a point.

The point is called a point of tangency. At the point of tangency, the tangent (line) does not cross the curve but it may or may not cross the curve at some other point.
On the diagram, the line is tangent to the curve at point *P* and crosses the curve at point *Q*.
The line is not tangent to the curve at point *Q*.

A tangent most resembles the curve near that point.

The slope of the tangent to a specific point on the curve represents the instantaneous rate of change of the curve at that point.

This idea will be pursued later in the unit.

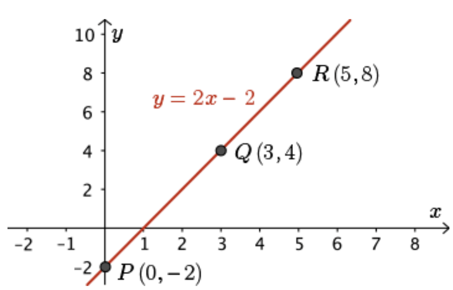


Consider the linear function $y=2x-2$.
A graph of $y=2x-2$ is shown with three specific points $P(0,-2)$, $Q(3,4)$, and $R(5,8)$ also plotted on the graph.

$m_{PQ} = \frac{4+2}{3-0} = 2$

$m_{PR} = \frac{8+2}{5-0} = 2$

$m_{PQ} = \frac{8-4}{5-3} = 2$



Again, the notation m_{PQ} is read "*m* subscript *PQ*" and is used to represent the slope of the secant through *PQ*.
The slope of every line segment contained on the line *y*=2*x*−2 is the same as the slope of the line.

In fact, the average rate of change for any two points on a linear function is constant.

The average rate of change for a linear function is the same as the slope of the line segment which, in turn, is the same as the slope of the linear function.

It follows that the instantaneous rate of change at any point on the linear function is also the same as the average rate of change between any two points on the line.

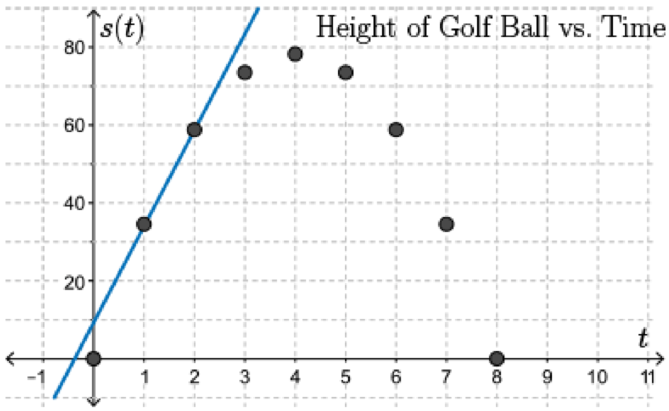
However, finding the average rate of change and the instantaneous rate of change of a curve presents a different challenge: the slope is not constant at every po int.

On a distance-time graph, the slope of the secant (or average rate of change) represents the average velocity.

$$m_{secant} = \text{average rate of change} = \text{average velocity}$$
$$= \frac{change\ in\ distance}{change\ in\ time} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$$

Consider data for the trajectory of a golf ball being struck off the ground.
The table summarizes the height of the ball (measured in metres) with respect to time (measured in seconds).

Time *t* (seconds)	**Height *s*(*t*) (metres)**
0	0
1	34.3
2	58.8
3	73.5
4	78.4
5	73.5
6	58.8
7	34.3
8	0



A secant which passes through (1,34.3) and (2,58.8) is shown on the graph of the data. Algebraically, the average rate of change or average velocity between 1 and 2 seconds can be calculated by finding the slope of the secant between the points (1,34.3) and (2,58.8).

m_{secant}= average rate of change =$\frac{change\ in\ distance}{change\ in\ time} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{58.8 - 34.3}{2-1}$ =24.5m/s

Example

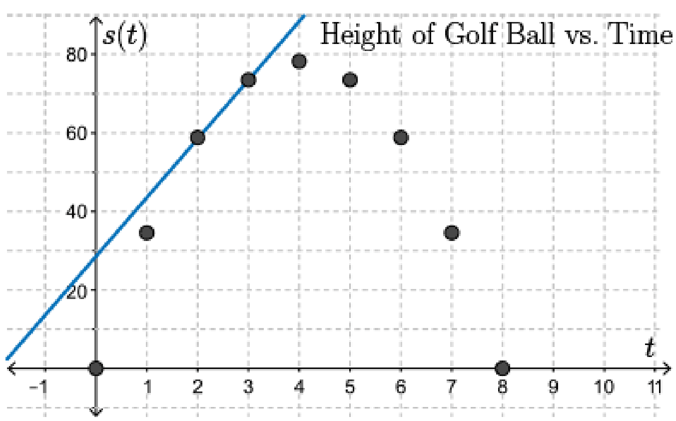
The table contains data relating the height of a golf ball, $s(t)$ in metres, with respect to the time, t in seconds, after the ball was struck.

Time *t* (seconds)	0	1	2	3	4	5	6	7	8
Height *s*(*t*) (metres)	0	34.3	58.8	73.5	78.4	73.5	58.8	34.3	0

a. Sketch the graph and then draw the secant between the points located at 2 and 3 seconds.

Solution

A secant through (2,58.8) and (3,73.5) is drawn on the graph.



b. Calculate the average rate of change, or the average velocity, between 2 and 3 seconds.

Solution

Using the points (2,58.8) and (3,73.5), we can calculate the slope of the secant.

This is the average rate of change (or average velocity) between these two points.

m_{secant}= average rate of change =$\frac{change\ in\ distance}{change\ in\ time}$ = $\frac{\Delta s}{\Delta t}$ = $\frac{73.5-58.8}{3-2}$ = 14.7 m/s

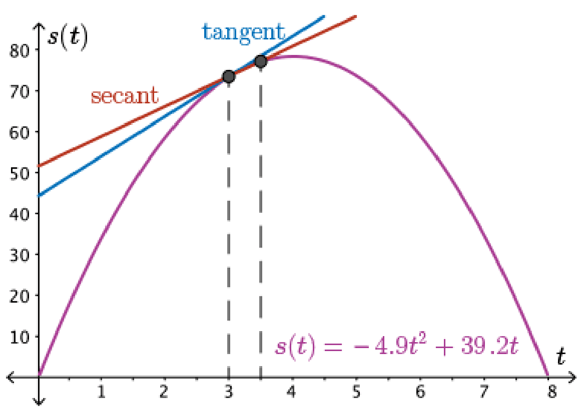
This average velocity is different from the average velocity between 1 and 2 seconds that was calculated earlier.

Now suppose that the height of the golf ball at any time, *t* in seconds, is given by the formula $s(t) = -4.9t^2 + 39.2t$, where $s(t)$ is measured in metres.
For example, the height after 2 seconds is equal to $s(2) = -4.9(2)^2 + 39.2(2) = 58.8$ m.
This is the value we had in our table when *t*=2 seconds.
Using this algebraic model, we are able to find the average velocity over any time interval.
In particular, we can find the average velocity over the interval from *t*=3 to *t*=3.5.
The height of the golf ball at *t*=3 seconds is $s(3) = 73.5$ m and the height of the golf ball at *t*=3.5 seconds is $s(3.5)=77.175$ m.

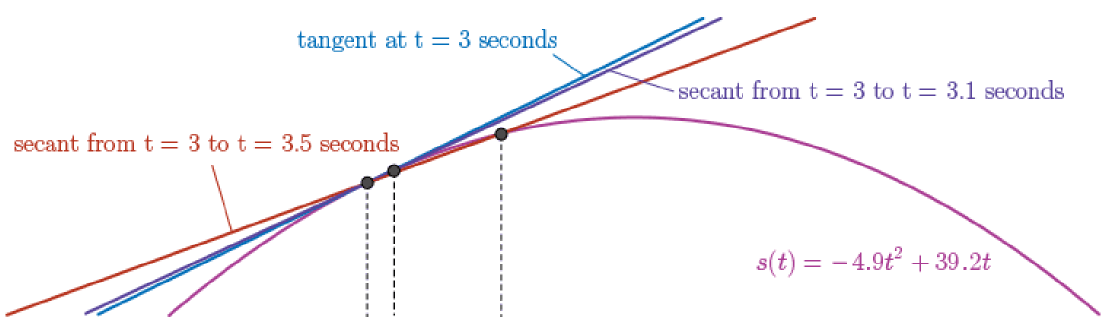
We can now calculate the average velocity over this time interval as follows:

$$\text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{77.175-73.5}{3.5-3} = 7.35 \text{ m/s}$$

Notice that we picked a reasonably small time interval: Δt = 3.5−3=0.5 seconds.
The sketch shows the function $s(t)=-4.9t^2 + 39.2t$.
The tangent to the point (3, 73.5) is shown on the graph.
The sketch also includes a secant drawn for the interval from *t*=3 to *t*=3.5 seconds.



If we draw a secant for the time interval from *t*=3 to *t*=3.1 seconds, the slope of the secant will more closely resemble the slope of the tangent at *t*=3 seconds.



Using the time interval from 3 to 3.1 seconds gives Δ*t*=3.1−3=0.1.
Since *s*(3.1)=−4.9(3.1)^2 + 39.2(3.1) = 74.431 m,
m_{secant}= average velocity =$\frac{\Delta s}{\Delta t}$ = $\frac{s(3.1)-s(3)}{3.1-3}$ = 9.31m/s

Now, let's try it with a smaller time interval from 3 to 3.01 seconds.
This gives Δ*t* = 3.01−3=0.01.
Since *s*(3.01)=−4.9(3.01)2+39.2(3.01)=73.59751 m
m_{secant}= average velocity =$\frac{\Delta s}{\Delta t}$ = $\frac{s(3.01)-s(3)}{3.01-3}$ =9.751m/s

And lastly, let's try our calculation with an even smaller time interval from 3 to 3.001 seconds.
This gives Δ*t*=3.001−3=0.001.
Since *s*(3.001)=−4.9(3.01)2+39.2(3.01)=73.5097951 m,
m_{secant}= average velocity =$\frac{\Delta s}{\Delta t}$ = $\frac{s(3.001)-s(3)}{3.001-3}$ =9.7951m/s

The table shows the results of our previous calculations and one other calculation that has been done in the same way as the previous results.

Interval	Δ*t*	**Average Velocity**
t=3 to *t*=3.5 seconds	0.5 s	7.35 m/s
t=3 to *t*=3.1 seconds	0.1 s	9.31 m/s
t=3 to *t*=3.01 seconds	0.01 s	9.75 m/s
t=3 to *t*=3.001 seconds	0.001 s	9.7951 m/s
t=3 to *t*=3.0001 seconds	0.0001 s	9.79951 m/s

As Δ*t* becomes smaller and smaller, the average velocity appears to be approaching a certain value.
Graphically, each successive secant is getting closer and closer to the tangent at *t*=3 seconds.
More specifically, as Δ*t* becomes smaller and smaller,

- the secant more closely resembles the tangent to the curve at the point (3,73.5),
- the average rate of change becomes a better approximation for the instantaneous rate of change at *t*=3seconds, and
- the average velocity becomes a better approximation for the instantaneous velocity at *t*=3 seconds.

One final note is appropriate here. We could also approach *t*=3 from below.
That is, we could look at time intervals from *t*=2.9 to *t*=3 seconds, *t*=2.99 to *t*=3 seconds, and *t*=2.999to *t*=3 seconds.
We would obtain similar results.

It is reasonable to *approximate* the **instantaneous rate of change** at $x=a$ by adding a small increment, such as 0.001, to the independent variable.
That is, using $\Delta x=0.001$.
Then, the slope of the secant between the points $(a,f(a))$ and $(a+0.001,f(a+0.001))$ can be found using our slope calculation.

$$\text{Instantaneous Rate of Change}=\frac{df}{dx}=\frac{f(a+0.001)-f(a)}{0.001}$$

The slope of the secant between the points $(a,f(a))$ and $(a+0.001,f(a+0.001))$ is a reasonable approximation of the slope of the tangent at $(a,f(a))$.

Example

A kettle is used to heat water. The equation $T(t)=\frac{110t+800}{t+40}$, $0 \le t \le 320$, expresses the temperature, T, in degrees Celsius, as a function of time, t, in seconds.
a. Calculate the average rate of change in temperature from $t=40$ to $t=50$ seconds.

Solution

Calculating the temperature at $t=40$ and $t=50$,

$$T(40)=110(4)+80040+40=65^0C$$

and

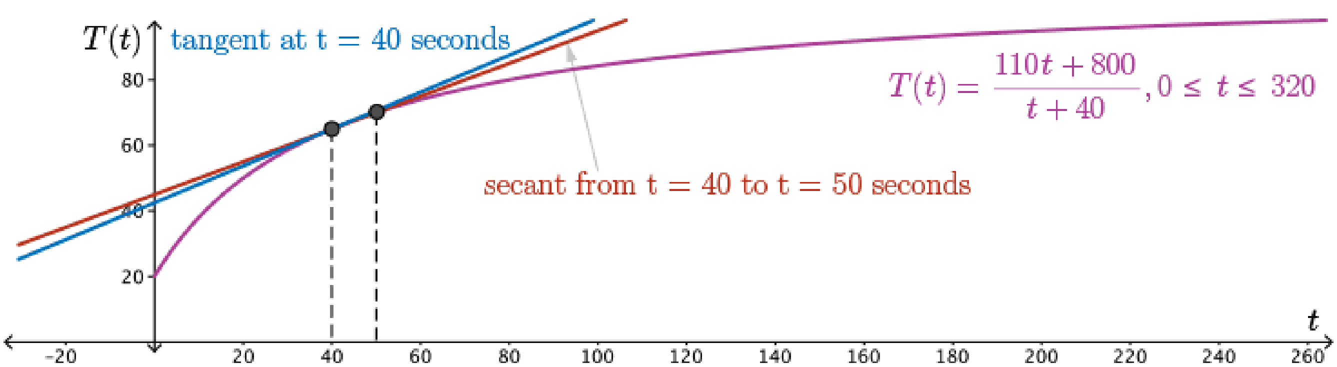
$$T(50)=110(50)+80050+40=70^0C$$

Then, m_{secant}= average rate of change $=\Delta T/\Delta t=(T(50)-T(40))/(50-40) = 0.5^0C/s$.

b. Approximate the instantaneous rate of change in temperature at 40 seconds, rounded to three decimal places.

Solution

Using an increment of $\Delta t=0.001$, $T(40.001)=110(40.001)+80040.001+40\approx65.0006^0C$.
The approximated instantaneous rate of change = $\Delta T/\Delta t$ = $0.562^0C/s$.



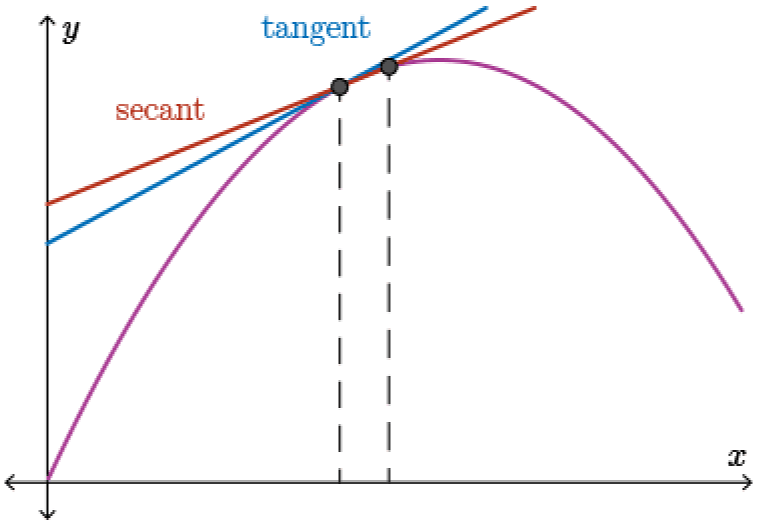
The slope of the secant is $0.5\circ C/s$ and the approximate slope of the tangent is $0.562^0C/s$.

c. What is the significance of the difference between the average rate of change in part a) and the approximation of the instantaneous rate of change at $t=40$ seconds in part b)?

Solution

As Δt becomes smaller, the slope of the secant (or average rate of change) more closely resembles the slope of the tangent (or instantaneous rate of change).

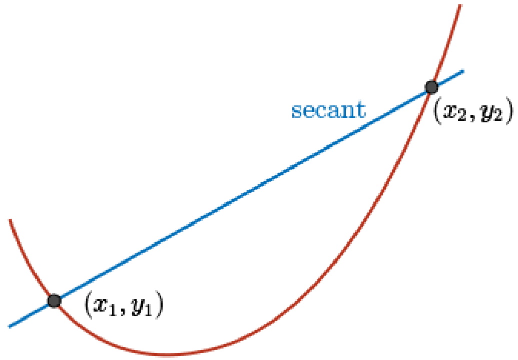
Summary



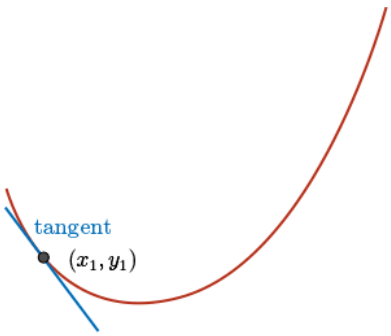
So far, we have
-looked at secants and tangents to curves,
-discussed average rates of change and instantaneous rates of change,
-calculated the average rate of change,
-approximated instantaneous rate of change, and
-applied rates of change to several applications.

Moving from Secants to Tangents

A secant intersects a curve in at least two distinct points.



The slope of the secant represents the average rate of change of the function over an interval.
A tangent touches a curve at one point.



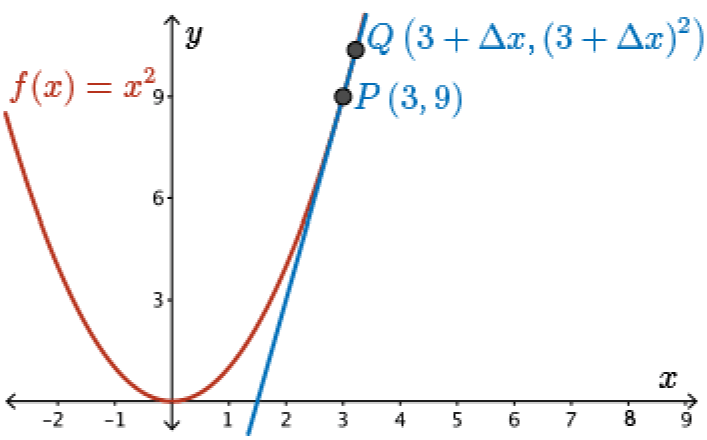
It does not cross the curve at that point if extended.
A tangent most resembles the curve near that point.
The slope of the tangent represents the instantaneous rate of change of the function at that point.

In this module, we will develop an algebraic method using limits to find the slope of a tangent to a curve.
That is, we will develop a method for determining the instantaneous rate of change of a function at a point.

Example 1

Find the slope of the tangent to the parabola *f*(*x*)=*x*2 at the point *P*(3,9).

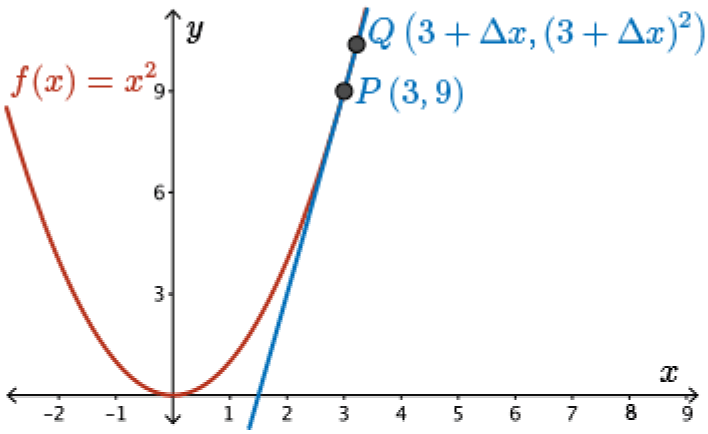
Solution



When using the slope formula $m=\frac{y_2-y_1}{x_2-x_1}$, two ordered pairs are required.

The tangent only has one known ordered pair: the point of tangency.
We need to determine another point on $f(x)=x^2$ “close” to the point of tangency.
Let Q be a point on $f(x)= x^2$, close to P, with x-coordinate $3+\Delta x$.
Then, $f(3+\Delta x)=(3+\Delta x)^2$.
We will let the values of Δx get closer and closer to zero so that the point Q gets closer and closer to the point of tangency, P.

Solution



Part 1: For Δ*x*=0.1, determine the coordinates of *Q* and the slope of the secant *PQ*.
For Δ*x*=0.1, the *x*-coordinate of point *Q* is *x*=3+0.1=3.1, and *f*(3.1)= 9.61.
So, the point *Q* is (3.1,9.61).
We can calculate the slope of the secant *PQ*:

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(3.1)-f(3)}{3.1-3} = 6.1$$

Part 2: For Δ*x*=0.001, determine the coordinates of *Q* and the slope of the secant *PQ*.
For Δ*x*=0.001, the *x*-coordinate of point *Q* is *x*=3+0.001=3.001, and $f(3.001)=3.001^2=9.006001$.
So, the point *Q* is (3.001,9.006001).
We can calculate the slope of the secant *PQ*:

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(3.001)-f(3)}{3.001-3} = 6.001$$

Part 3: For Δ*x*=−0.001, determine the coordinates of *Q* and the slope of the secant *PQ*.
For Δ*x*=−0.001, the *x*-coordinate of point *Q* is *x*= 3−0.001=2.999, and $f(2.999)= 2.999^2=8.994001$.
So, the point *Q* is (2.999, 8.994001).
We can calculate slope of the secant *PQ*:

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(2.999)-f(3)}{2.999-3} = 5.999$$

Part 4: The following table contains results found by using different values of Δ*x*. Three of the slopes were calculated in this example.

Δ*x*	**Coordinates of *Q***	**Slope of *PQ***	Δ*x*	**Coordinates of *Q***	**Slope of *PQ***
0.1	(3.1,9.61)	6.1	−0.1	(2.9,8.41)	5.9
0.01	(3.01,9.0601)	6.01	−0.01	(2.99,8.9401)	5.99
0.001	(3.001,9.006001)	6.001	−0.001	(2.999,8.994001)	5.999

Using the values in the table, predict the slope of the tangent to the parabola *f*(*x*)=*x*2 at the point *P*(3,9).
As *Q* gets closer and closer to *P*, approaching from the left, the slope of the secant gets closer and closer to 6, approaching from below.
As *Q* gets closer and closer to *P*, approaching from the right, the slope of the secant gets closer and closer to 6, approaching from above.
It would be reasonable to say that the slope of the tangent at *P*(3,9) is 6.

Using Limits to Determine the Slope of a Tangent to a Curve

In the last example, we estimated the slope of the tangent at x=3 to be 6 by determining what the slope of the secant approaches as Δx approaches 0.
That is, as $\Delta x \to 0$, the slope of the secant $\frac{f(3+\Delta x)-f(3)}{(3+\Delta x)-3} = \frac{f(3+\Delta x)-f(3)}{\Delta x} \to 6$.

The concept of a limit was introduced earlier in the Rational Functions unit, and it was used to describe the behaviour of the graph of a function about its asymptotes as well as its end behaviour.

We can use limits in this situation to communicate our findings concerning the slope of the tangent to a function at a point.
A limit provides information about how a function behaves *near*, not *at*, a specific value of x.

If the limit of a function $y=f(x)$ as x approaches a is equal to L, then we write

$$\lim_{x\to a} f(x) = L$$

We read this as, "the limit as x approaches a of $f(x)$ equals L."
This means that $f(x)$ gets closer and closer to the value L as x gets closer and closer to the value a.
That is, $y \to L$ as $x \to a$.

In this situation, we determined that

$$\frac{f(3+\Delta x)-f(3)}{\Delta x} \to 6 \text{ as } \Delta x \to 0$$

Thus, using limits and limit notation, the slope of the tangent at x=3 is given by

$$m_{tangent} = \lim_{\Delta x\to 0} \frac{f(3+\Delta x)-f(3)}{\Delta x} = 6$$

Generalizing the example, let Q be a point on $f(x)=x^2$ close to $P(3,9)$.
The coordinates for Q are $(3+h, f(3+h))$, where h is extremely close to zero.

The slope of the tangent P is given by the general formula

$$\begin{aligned}
m_{tangent} &= \lim_{\Delta x\to 0} \frac{dy}{dx} \\
&= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{(3+h)^2-9}{h} \\
&= \lim_{h\to 0} \frac{h^2+6h}{h} \\
&= \lim_{h\to 0} \frac{h(h+6)}{h} \\
&= \lim_{h\to 0} (h+6) \\
&= 6
\end{aligned}$$

In general, the slope of a tangent to a function (instantaneous rate of change) is defined by
$m_{tangent} = \lim\limits_{\Delta x \to 0} \frac{dy}{dx}$

To determine the slope of the tangent to *y*=*f*(*x*) at *x*=*a*, we can use
$\boldsymbol{m_{tangent}\big|_{x=a} = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}}$, Δx=h.

You will learn much more about limits in the study of calculus.

For now, we will use limit notation to communicate the procedure used when determining the instantaneous rate of change or the slope of a tangent. Limit notation will help distinguish this procedure from the method used to determine average rate of change or slope of a secant.

Example

Given the function $f(x)= x^2 + 3x$:
a. Find the slope of the tangent at *P*(2,10).

Solution

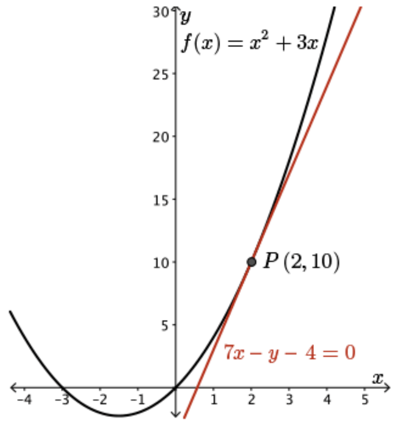
Find the *y*-coordinates of the points on the curve with *x*-coordinates 2 and 2+*h*.

$$f(2)= 10, f(2+h)= h^2 + 7h + 10$$

Now, find the slope of the tangent at point *P*.
$m_{tangent}\big|_{x=2} = \lim\limits_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim\limits_{h \to 0} \frac{h^2+7h}{h} = \lim\limits_{h \to 0} \frac{h(h+7)}{h} = \lim\limits_{h \to 0}(h+7) = 7$

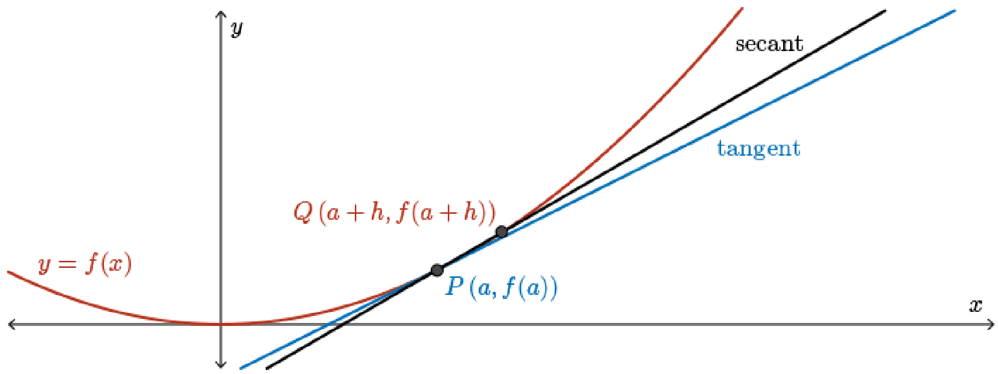
b. Find the equation of the tangent (in standard form) to the function $f(x)=x^2 + 3x$ at *P*(2,10).
Solution
From a), the slope of the tangent to $f(x)=x^2 + 3x$ at *P*(2,10) is *m*=7.

Using the general slope formula,
$m= \frac{y-y_1}{x-x_1}$
$7 = \frac{y-10}{x-2}$
Therefore, $7x-y-4=0$ is the equation of the tangent to the curve $f(x)=x^2+3x$ at the point *P*(2,10).



The sketch shows the parabola $f(x)=x^2+3x$ and the tangent $7x-y-4=0$ to the parabola at the point $P(2,10)$.
In this section, we have presented an algebraic approach for determining the slope of the tangent to a function at a particular point on the function.

This is equivalent to determining the instantaneous rate of change at a particular point.



To determine the slope of the tangent to $y=f(x)$ at $x=a$, we use
$m_{tangent}\Big|_{x=a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, Δx=h.

Next, we will develop a second algebraic approach and consider more examples.

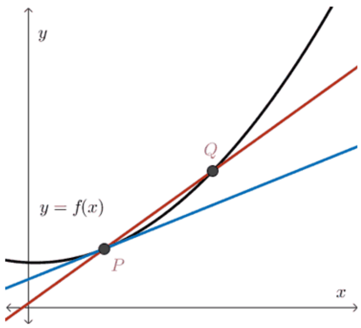
This method for finding the slope of the tangent is referred to as a first principles approach.

Basically, in a first principles approach, we return to the slope definition each time as we find the slope of the tangent.
In this specific approach to finding the slope of the tangent, we use two points $P(a,f(a))$ and $Q(a+h,f(a+h))$.

First Principles Formula with *h*

$$m_{tangent} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

This first method was developed completely in a previous module.
In a second first principles approach to finding the slope of the tangent, we will use two points $P(a,f(a))$ and $Q(x,f(x))$.



As $Q \to P$, $\Delta x \to 0$ and $x \to a$.

Slope of tangent (at x=a)= $\lim\limits_{\Delta x \to 0} \frac{dy}{dx}$ = $\lim\limits_{h \to 0} \frac{f(x)-f(a)}{x-a}$

First Principles Formula with *x*

$m_{tangent}$ = $\lim\limits_{x \to 3} \frac{f(x)-f(a)}{x-a}$

We have two first principles approaches which are similar.

Both methods will be used in each of the following examples.

You may prefer one over the other.

Example

Find the slope of the tangent to y=x^2 + 3x + 4 at x=3. Try both first principles approaches.

Solution

Let $f(x)$=x^2+ 3x + 4.
Then, $f(3)=(3)^2 + 3(3) + 4= 22$ and $f(3+h)= h^2 + 9h + 22$.

$\begin{aligned} m_{tangent} &= \lim\limits_{h \to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim\limits_{h \to 0} \frac{h^2+9h}{h} \\ &= \lim\limits_{h \to 0} \frac{h(h+9)}{h} \\ &= \lim\limits_{h \to 0}(h+9) \\ &= 9 \end{aligned}$	$\begin{aligned} m_{tangent} &= \lim\limits_{x \to 3} \frac{f(x)-f(3)}{x-3} \\ &= \lim\limits_{x \to 3} \frac{x^2+3x-18}{x-3} \\ &= \lim\limits_{x \to 3} \frac{(x-3)(x+6)}{x-3} \\ &= \lim\limits_{x \to 3}(x+6) \\ &= 9 \end{aligned}$

Example

For the function $f(x)=\frac{1}{x}$:

a. Find the slope of the tangent at x=3. Try both first principles approaches.

Then, $f(3)=\frac{1}{3}$ and $f(3+h)=\frac{1}{3+h}$.

$m_{tangent} = \lim_{h \to 0} \frac{f(3+h)-f(3)}{h}$ $= \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$ $= \lim_{h \to 0} \frac{-1}{3(3+h)}$ $= -\frac{1}{9}$	$m_{tangent} = \lim_{x \to 3} \frac{f(x)-f(3)}{x-3}$ $= \lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3}$ $= \lim_{x \to 3} \frac{-1}{3x}$ $= -\frac{1}{9}$

b. Find the equation of the tangent to the function $f(x)=1/x$ at x=3.

Solution

From part a), $m=-1/9$.
Using $f(x)$, the coordinates of point P are (3,1/3).
Using the slope y intercept formula, y = mx + b we get x + 9y – 6 = 0.