Lesson 12: Unit 7 – Identities and Equations (1)

In this lesson

- *We will review and examine the basic fundamental relationships between trigonometric functions.*
- *We will identify equivalent trigonometric expressions, demonstrating equivalence algebraically and graphically.*
- *We will determine the non-permissible values of the variable involved in trigonometric expressions and equations.*

Definition of an Identity

A mathematical **identity** is an equation or statement that is true for all permissible values of the variables in the equation.

For example,

$$(x + y)^2 = x^2 + 2xy + y^2$$

is a mathematical statement that is true for all values of x and y. Thus, it is an identity.

Let's recall some fundamental trigonometric identities, you should be familiar with, from the previous unit on trigonometric functions.

The reciprocal identities, by definition of the functions, are

Cosecant: $\csc\theta = \dfrac{1}{\sin\theta}$

Secant: $\sec\theta = \dfrac{1}{\cos\theta}$

Cotangent: $\cot\theta = \dfrac{1}{\tan\theta}$

The quotient identities are

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

The Pythagorean identity is

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

The quotient and Pythagorean identities were developed previously, using the unit circle, in the lesson “Trigonometric Ratios and Special Triangles” of the Trigonometric Functions unit.

Fundamental Trigonometric Identities

Reciprocal identities

$\csc\theta = \dfrac{1}{\sin\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$

Quotient identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Pythagorean identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

Each of these fundamental identities, with the exception of , $sin^2\,\theta + cos^2\,\theta = 1$ have non-permissible values for the variable, θ.

First, the trigonometric functions within the equations have restrictions on their domain and secondly, the denominator of any rational trigonometric expression cannot equal zero.

Example

Simplify $\dfrac{1+\cot(x)}{\csc(x)}$, identifying any non-permissible values of the variable.

Solution

Express each trigonometric ratio in terms of sin(*x*) and cos(*x*).

$$\frac{1+\cot(x)}{\csc(x)} = \frac{1+\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}}$$

= sin(x) + cos(x).

To determine the non-permissible values of *x* in the expression $\frac{1+\cot(x)}{\csc(x)}$, we must identify when cot(*x*) is undefined and when csc(*x*) is zero or undefined.
Non-permissible values may be more easily identified from the non-simplified equivalent expression

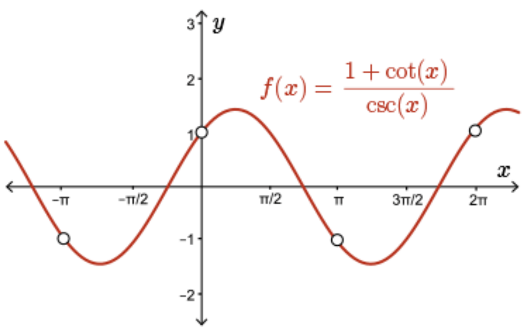
$$\frac{1+\cot(x)}{\csc(x)} = \frac{1+\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}}$$

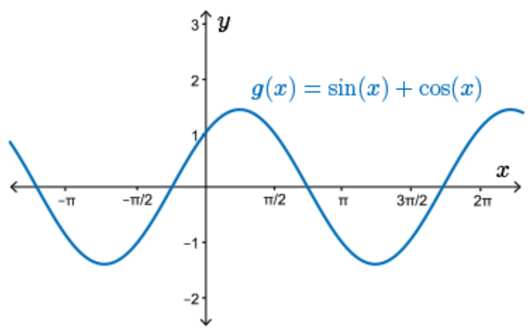
We see that $\sin(x) \neq 0$ so $x \neq n\pi$, $n \in Z$.
Note that $\csc(x)$, that is $1/\sin(x)$, will never equal zero.
Therefore, $\frac{1+\cot(x)}{\csc(x)} = \sin(x) + \cos(x)$ for all real values of x, $x \neq n\pi$, $n \in Z$.

This equivalence can be demonstrated by graphing the two functions

$$f(x) = \frac{1+\cot(x)}{\csc(x)} \quad \text{and} \quad g(x) = \sin(x) + \cos(x)$$

using graphing technology.





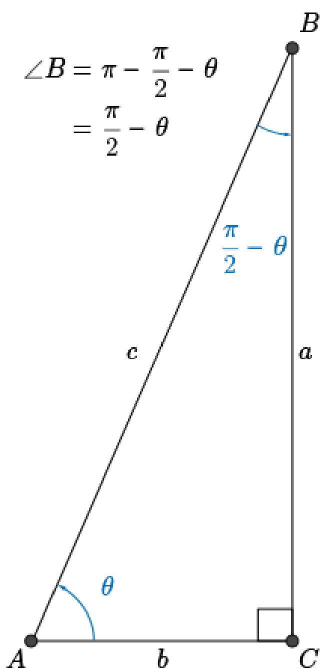
The graphs of these two functions appear to be identical; however, the domain of each function is not the same.
The domain of $g(x)=\sin(x)+\cos(x)$ is $\{x|\ x \in R\}$.
From our discussion of the non-permissible values, we know that the domain of
$f(x) = \frac{1+\cot(x)}{\csc(x)}$ must be $\{x|\ x \neq n\pi,\ x \in R,\ n \in Z\}$.
There are, in fact, holes in the graph of this function at these non-permissible values.

Cofunction Identities

In the right triangle *ABC*, if $\angle BAC = \theta$ then, by the sum of the angles of a triangle, $\angle ABC = \frac{\pi}{2} - \theta$.

Since $\sin(\theta) = a/c$ and $\cos(\frac{\pi}{2} - \theta) = a/c$, then $\sin(\theta) = \cos(\frac{\pi}{2} - \theta)$.
Similarly, $\cos(\theta) = b/c$ and $\sin(\frac{\pi}{2} - \theta) = b/c$.

Therefore, $\cos(\theta) = \sin(\frac{\pi}{2} - \theta)$



Also, $\tan(\theta) = a/b$ and $\cot(\frac{\pi}{2} - \theta) = a/b$.

Therefore, $\tan(\theta) = \cot(\frac{\pi}{2} - \theta)$.

Summary of Cofunction Identities

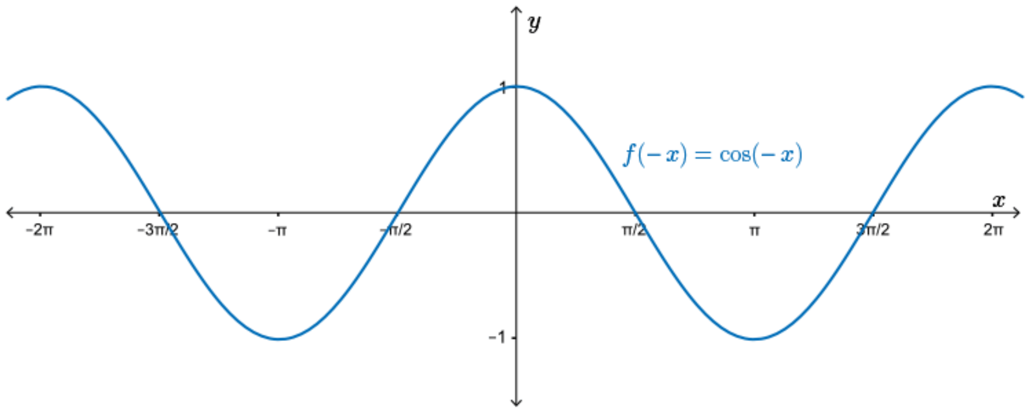
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$
$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$	$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

These identities are called cofunction identities since they show a relationship between sine and cosine and a relationship between tangent and cotangent.
The value of one function at an angle is equal to the value of the cofunction at the complement of the angle.
For example, $\sin(10^0) = \cos(80^0)$ and $\tan(\pi/3) = \cot(\pi/6)$.
This group of identities extends to include the reciprocal functions.

Equivalent Expressions from Symmetry

***f*(*x*) = cos(*x*)**

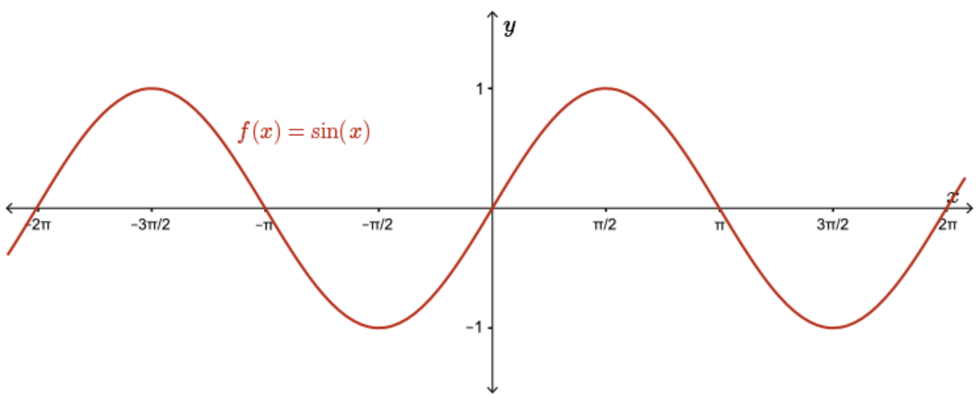
The function $f(x) = \cos(x)$ is an even function and is symmetric about the y-axis.



This means that $f(-x) = f(x)$; thus, $\cos(-x) = \cos(x)$.

$f(x) = \sin(x)$

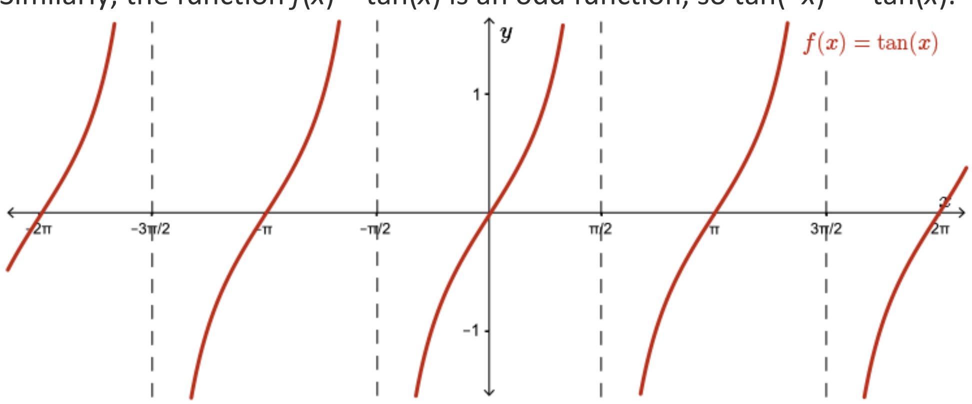
The function $f(x) = \sin(x)$ is an odd function, symmetric about the origin.



This means that $-f(-x) = f(x)$ or $f(-x) = -f(x)$; thus, $\sin(-x) = -\sin(x)$.

$f(x) = \tan(x)$

Similarly, the function $f(x) = \tan(x)$ is an odd function, so $\tan(-x) = -\tan(x)$.



Equivalent Expressions from Symmetry

Even Symmetry

cos(-x) = cos(x)

sec(-x) = sec(x)

Odd Symmetry

sin(-x) = -sin(x)

csc(-x) = -sin(x)

tan(-x) = -tan(x)

cot(-x) = -cot(x)

From our studies of reciprocal functions, we know $y = 1/f(x)$ is even if $y = f(x)$ is even and $y = 1/f(x)$ is odd if $y = f(x)$ is odd. We can extend this list of identities to include the reciprocal trigonometric functions.

Example

Express $\sec(\theta - \frac{\pi}{2})$ in terms of $\sin(\theta)$.

Solution

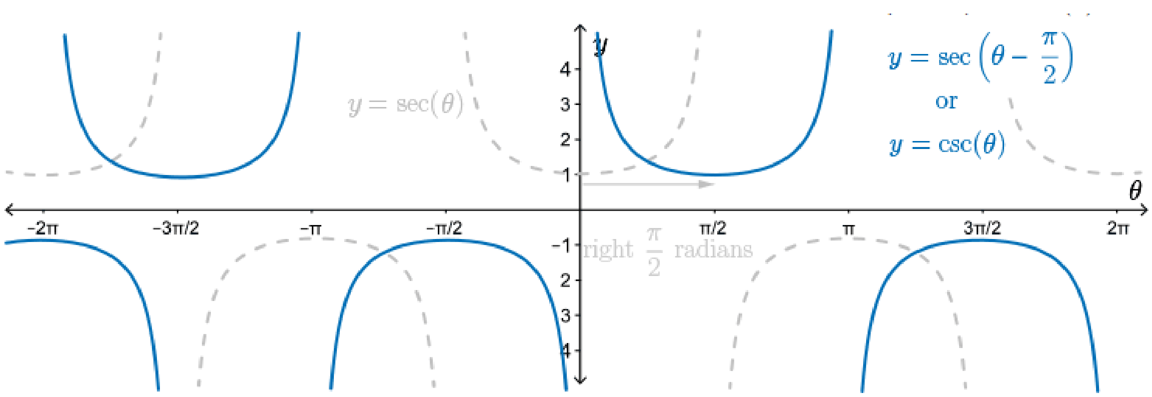
$$\sec(\theta - \tfrac{\pi}{2}) = \frac{1}{\cos(\theta - \frac{\pi}{2})}$$

$$= \frac{1}{\cos(\frac{\pi}{2} - \theta)}$$

$$= \frac{1}{\sin(\frac{\pi}{2} - \theta)}$$

$$= \frac{1}{\sin(\theta)}$$

Notice that the graph of $y = \sec(\theta - \frac{\pi}{2})$ can be obtained by translating the graph of $y = \sec(\theta)$ to the right by $\pi/2$ units.



Due to the symmetry and periodic nature of trigonometric functions, equivalent trigonometric expressions can be identified or derived using transformations (particularly through reflections and horizontal translations).

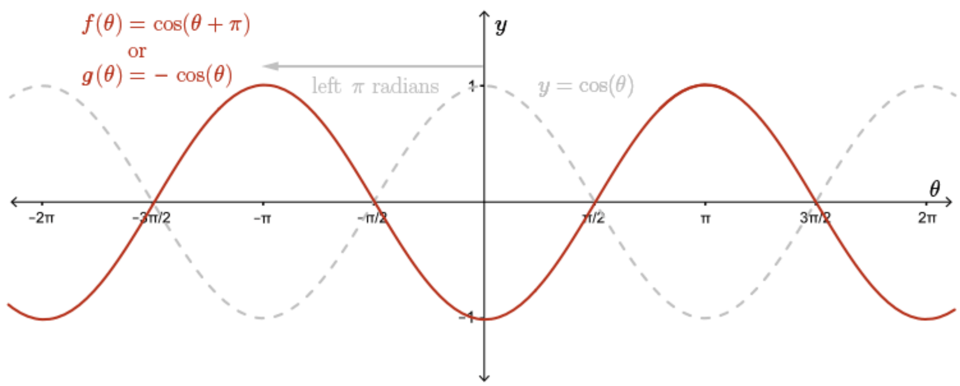
Example

Show graphically that $\cos(\theta+\pi) = -\cos(\theta)$.

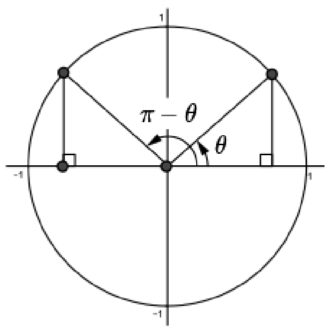
Solution

Consider $f(\theta) = \cos(\theta+\pi)$ and $g(\theta) = -\cos(\theta)$.

The graph of $y = f(\theta)$ can be obtained by translating the graph of $y = \cos(\theta)$ to the left π units. The graph of this function is the reflection of the cosine function in the x-axis and can therefore be defined by $g(\theta) =- \cos(\theta)$. Thus confirming $\cos(\theta+\pi) = -\cos(\theta)$.



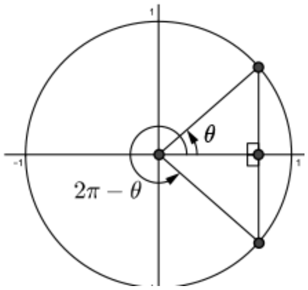
We can see that this identity, $\cos(\theta+\pi) = -\cos(\theta)$, supports our understanding of the relationship between a principal angle and its reference or related acute angle.
For $0 < \theta < \pi/2$, an angle, $\pi+\theta$, in standard position, has a terminal arm in the third quadrant with a reference angle of θ.
Thus, $\cos(\theta+\pi) = -\cos(\theta)$.



Similarly, by considering a principal angle in the other two quadrants, related to the reference angle, θ, we have

$\cos(\pi\text{-}\theta) = -\cos(\theta)$ and $\cos(2\pi\text{-}\theta) = \cos(\theta)$

These statements are true for all values of ϑ. We can validate each statement using transformations.



Using the relationship between a principal angle and its reference or related acute angle, other identities using sine and tangent ratios can be identified. This list can be extended to include the reciprocal trigonometric ratios.

$\sin(\pi-\theta)=\sin(\theta)$	$\sin(\pi+\theta)=-\sin(\theta)$	$\sin(2\pi-\theta)=-\sin(\theta)$
$\cos(\pi-\theta)=-\cos(\theta)$	$\cos(\pi+\theta)=-\cos(\theta)$	$\cos(2\pi-\theta)=\cos(\theta)$
$\tan(\pi-\theta)=-\tan(\theta)$	$\tan(\pi+\theta)=\tan(\theta)$	$\tan(2\pi-\theta)=-\tan(\theta)$
$\csc(\pi-\theta)=\csc(\theta)$	$\csc(\pi+\theta)=-\csc(\theta)$	$\csc(2\pi-\theta)=-\csc(\theta)$
$\sec(\pi-\theta)=-\sec(\theta)$	$\sec(\pi+\theta)=-\sec(\theta)$	$\sec(2\pi-\theta)=-\sec(\theta)$
$\cot(\pi-\theta)=-\cot(\theta)$	$\cot(\pi+\theta)=\cot(\theta)$	$\cot(2\pi-\theta)=-\cot(\theta)$

Due to the periodic nature of trigonometric functions, these functions can be expressed in many equivalent forms.

Example

Express $\dfrac{\sin(\pi-x)\cos(\pi+x)\tan(2\pi+x)}{\sec(\frac{\pi}{2}+x)\csc(\frac{3\pi}{2}-x)\cot(\frac{\pi}{2}-x)}$ in terms of $\sin(x)$.

Solution

$$\frac{\sin(\pi-x)\cos(\pi+x)\tan(2\pi+x)}{\sec(\frac{\pi}{2}+x)\csc(\frac{3\pi}{2}-x)\cot(\frac{\pi}{2}-x)} = \frac{\sin(\pi-x)\cos(\pi+x)\tan(2\pi+x)}{\frac{1}{\cos(\frac{\pi}{2}+x)}\frac{1}{\sin(\frac{3\pi}{2}-x)}\frac{1}{\tan(\frac{\pi}{2}-x)}}$$

$$= \sin(\pi-x)\cos(\pi+x)\tan(2\pi+x)\cos(\tfrac{\pi}{2}+x)\sin(\tfrac{3\pi}{2}-x)\tan(\tfrac{\pi}{2}-x)$$

$= \sin(x)(-\cos(x))\tan(x)(-\sin(x))\sin(-(x-\frac{3\pi}{2}))\tan(\frac{\pi}{2}-x)$

$= -\sin^2(x)\cos^2(x)$

$= -\sin^2(x)(1 - \sin^2(x))$

$= \sin^4(x) - \sin^2(x)$

Therefore,

$\dfrac{\sin(\pi-x)\cos(\pi+x)\tan(2\pi+x)}{\sec(\frac{\pi}{2}+x)\csc(\frac{3\pi}{2}-x)\cot(\frac{\pi}{2}-x)} = \sin^4(x) - \sin^2(x)$, $x \neq \dfrac{n\pi}{2}$, $n \in \mathbb{Z}$.

Summary: *The ability to simplify trigonometric expressions or identify equivalent forms allows for more efficient problem solving when studying trigonometric relations.*

In this module, we have identified many trigonometric identities, some by their definition and others by relationships that exist in the ratios or the graphs of the functions. Some fundamental identities include:

Reciprocal Identities			Quotient Identities	
$\csc(\theta) = \frac{1}{\sin(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities		
$\sin^2(\theta) + \cos^2(\theta) = 1$	$1 + \cot^2(\theta) = \csc^2(\theta)$	$\tan^2(\theta) + 1 = \sec^2(\theta)$
Cofunction Identities		
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$
Symmetry		
$\sin(-\theta) = -\sin(\theta)$	$\cos(-\theta) = \cos(\theta)$	$\tan(-\theta) = -\tan(\theta)$
Translations and Reflections		
$\sin(\pi - \theta) = \sin(\theta)$	$\sin(\pi + \theta) = -\sin(\theta)$	$\sin(2\pi - \theta) = -\sin(\theta)$
$\cos(\pi - \theta) = -\cos(\theta)$	$\cos(\pi + \theta) = -\cos(\theta)$	$\cos(2\pi - \theta) = \cos(\theta)$
$\tan(\pi - \theta) = -\tan(\theta)$	$\tan(\pi + \theta) = \tan(\theta)$	$\tan(2\pi - \theta) = -\tan(\theta)$

Next,

- *We will analyze trigonometric identities numerically and graphically.*
- *We will discuss techniques used to manipulate and simplify expressions in order to prove trigonometric identities algebraically.*

Example

Is the statement $\dfrac{1}{1+\sin(x)} + \dfrac{1}{1-\sin(x)} = 2\sec^2(x)$ an identity?

Solution

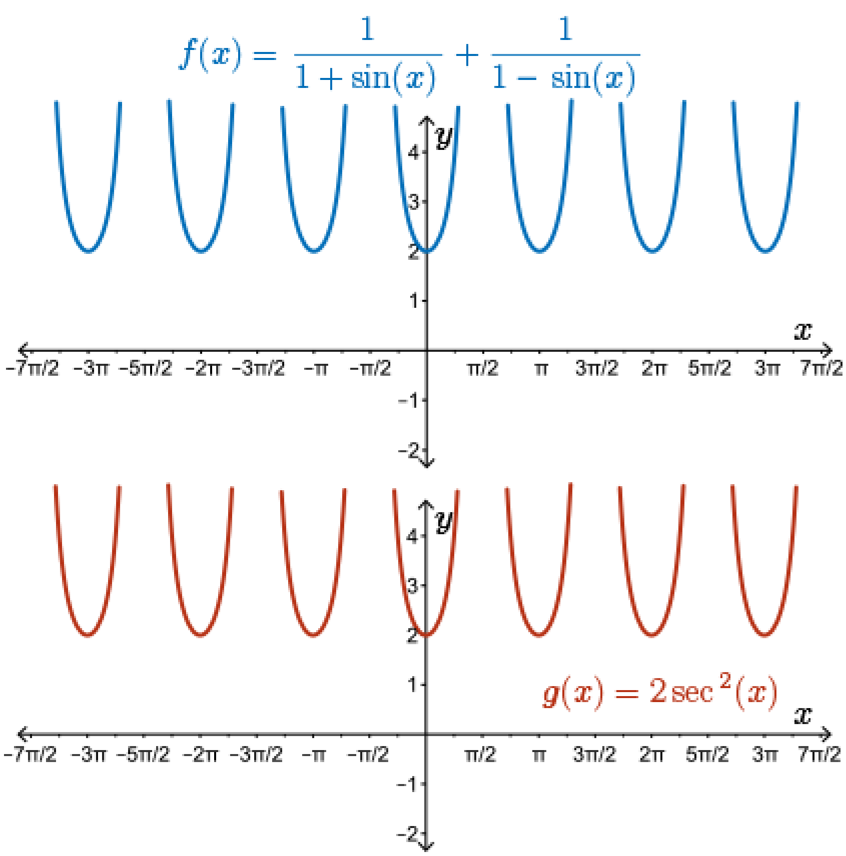
Let's first consider the graphs of the two functions

$$f(x) = \frac{1}{1+\sin(x)} + \frac{1}{1-\sin(x)}$$

and

$$g(x) = 2\sec^2(x)$$

The graphs of $f(x)$ and $g(x)$ appear identical.
Each function shares the same restrictions on x.



For $f(x) = \frac{1}{1+\sin(x)} + \frac{1}{1-\sin(x)}$, $\sin(x) \neq \pm 1$ and $g(x) = 2\sec^2(x)$ is undefined when $\cos(x)=0$.

Therefore, for both functions, $x \neq \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

Vertical asymptotes are $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$, for both functions.

The fact that the graphs of these two functions appear identical suggests that

$$\frac{1}{1+\sin(x)} + \frac{1}{1-\sin(x)} = 2\sec^2(x)$$

may be an identity; however, these graphs show the behaviour of the functions within a specific set of values and not for all values in the domain of each function. To prove this statement is an identity, we must take an algebraic approach.

When proving an identity we do not assume equality.
For this reason, we work with each side of the equation independently.

L.S. = $\frac{1}{1+\sin(x)} + \frac{1}{1-\sin(x)}$

= $\frac{2}{(1+\sin(x))(1-\sin(x))}$

= $\frac{2}{1-\sin^2(x)}$

= $\frac{2}{\cos^2(x)}$

= $2\sec^2(x)$

= R.S.

Thus, $\frac{1}{1+\sin(x)} + \frac{1}{1-\sin(x)} = 2\sec^2(x)$ is an identity.

Summary

Keep in mind the following strategies:

- It is usually helpful to begin with simplifying the more complicated side to the equation first, keeping in mind the expression on the other side.
- Make necessary substitutions using the fundamental identities such as the reciprocal, quotient, and Pythagorean identities.
- It is sometimes beneficial to express each side of the equation in terms of the sine and cosine ratios, or, if possible, in terms of one trigonometric ratio.
- Add and subtract rational expressions by obtaining a common denominator.
- Factor to simplify rational trigonometric expressions.
- Multiply the numerator and denominator of the rational trigonometric expression by the conjugate of either the numerator or denominator to create a difference of squares when simplified forms cannot be manipulated further.

Next, we will extend our knowledge of the fundamental trigonometric identities to include the compound angle identities shown here.

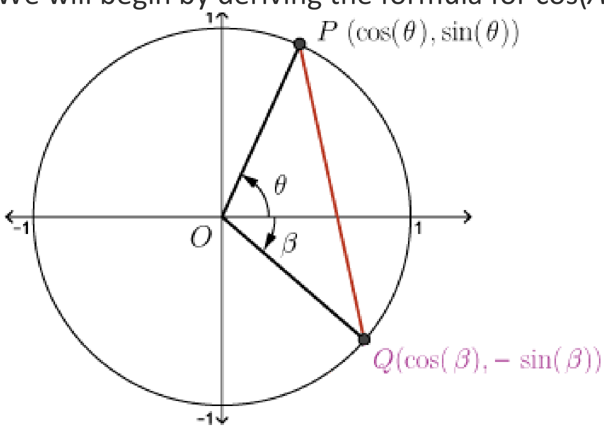
$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$
$\tan(A+B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$

These identities involve the sum and difference of two angles and are often referred to as formulas as they provide a means of

- simplifying trigonometric expressions and proving other identities,
- determining exact values for angles related to the acute angles $\pi/12$ or $5\pi/12$ (15^0 or 75^0), and
- solving certain trigonometric equations.

Derivation

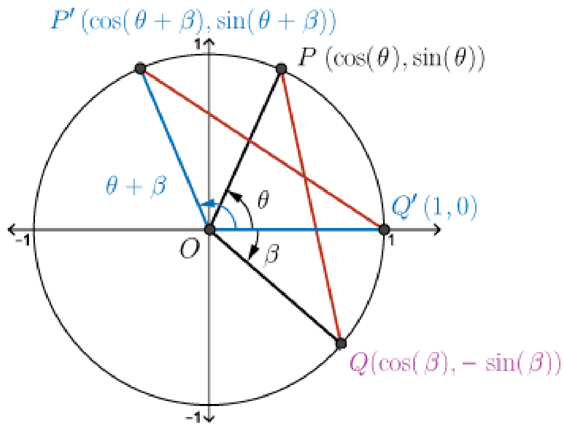
We will begin by deriving the formula for cos(*A*+*B*) using the unit circle.



Consider the two points *P* and *Q* on the unit circle, where *P* is defined by ($\cos(\theta)$, $\sin(\theta)$) for some angle θ, $\theta > 0$ and *Q* is given by ($\cos(-\beta)$,$\sin(-\beta)$) for some angle β, $\beta > 0$. The coordinates of *Q* can be simplified to ($\cos(\beta)$, $-\sin(\beta)$).

The measure of $\angle POQ$, with *O* at the origin, is $\theta + \beta$.
The length of the line segment *PQ* can be found using
$|PQ| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$
Thus,
$|PQ| = \sqrt{2 - 2\cos(\theta)\cos(\beta) + 2\sin(\theta)\sin(\beta)}$

Now rotate the points *P* and *Q* counterclockwise about the origin by angle of β to obtain *Q*'(1,0) and *P*' ($\cos(\theta+\beta)$, $\sin(\theta+\beta)$) on the unit circle, as shown in the diagram.



As a condition of the rotation, $\angle P'OQ' = \theta+\beta$ and $|P'Q'| = |PQ|$.
The length of $P'Q'$ is given by

$$|P'Q'| = \sqrt{2 - 2\cos(\theta+\beta)}$$

Since $|P'Q'|=|PQ|$, we have

$$\sqrt{2 - 2\cos(\theta)\cos(\beta) + 2\sin(\theta)\sin(\beta)} = \sqrt{2 - 2\cos(\theta+\beta)}$$

$$\cos(\theta+\beta) = \cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta)$$

To obtain the formula for $\cos(\theta-\beta)$, we can express it as $\cos(\theta+(-\beta))$ and apply the angle sum formula: $\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$.

By setting $A=\theta$ and $B=-\beta$, we have

$$\cos(\theta-\beta) = \cos(\theta+(-\beta))$$

$$= \cos(\theta)\cos(-\beta) - \sin(\theta)\sin(-\beta)$$

$$= \cos(\theta)\cos(\beta) + \sin(\theta)\sin(\beta)$$

To obtain the formula for $\sin(\theta+\beta)$ we can use the cofunction identities,

$$\sin(x)=\cos(\tfrac{\pi}{2}-x) \text{ and } \cos(x)=\sin(\tfrac{\pi}{2}-x)$$

and apply the angle difference formula for cosine, $\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$.

$$\sin(\theta+\beta) = \cos(\tfrac{\pi}{2} - (\theta+\beta))$$
$$= \cos((\tfrac{\pi}{2} - \theta) - \beta)$$
$$= \cos(\tfrac{\pi}{2} - \theta)\cos(\beta) + \sin(\tfrac{\pi}{2} - \theta)\sin(\beta)$$
$$= \sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta)$$

Example

Determine the exact value of each using a compound angle formula.

a. $\sin(\frac{13\pi}{12})$

b. $\cos(195^0)$

Solution

a. $\sin(13\pi12)$

To determine the exact value of $\sin(\frac{13\pi}{12})$, we express $13\pi/12$ as a sum or difference of two angles corresponding to the related acute angles $\pi/6$, $\pi/4$, or $\pi/3$. For example,

$\frac{13\pi}{12} = \frac{9\pi}{12} + \frac{4\pi}{12}$

So, $\sin(\frac{13\pi}{12}) = \sin(\frac{9\pi}{12} + \frac{4\pi}{12})$

$= \sin(\frac{3\pi}{4} + \frac{\pi}{3})$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

b. $\cos(195^0)$

Since $195^0 = 225^0 - 30^0$

$\cos(195^0) = \cos(225^0 - 30^0)$

$= \frac{-\sqrt{2}-\sqrt{6}}{4}$

Example

Solve $\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}$, $0 \le x \le 2\pi$.

Solution

$(\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}\ ) \div \sqrt{3}$

$\sin(x) + \sqrt{3}\cos(x) = -\sqrt{2}$ $\quad$ ($\sqrt{3} = \tan(\frac{2\pi}{3})$)

$\sin(x + \frac{2\pi}{3}) = -\frac{\sqrt{2}}{2}$

Therefore, the solutions to $\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}$, for $0 \le x \le 2\pi$ are $11\pi/12$, $17\pi/12$.