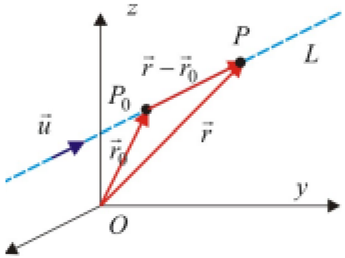
Lesson 13: Unit 7 – Equations of lines and planes

Vector Equation of a Line

In both 2-d and 3-d cases we can determine the equation of a line using its direction and any fixed point on the line.

Suppose a line passes through a fixed point $P_0(x_0,y_0,z_0)$ with position vector $\overrightarrow{r_0}$, and that the line is parallel to the vector $\vec{u}$. Consider *P* (*x*,*y*,*z*) be an arbitrary point on the line so that $\overrightarrow{OP}$ = $\vec{r}$.



Using the triangle law of vector addition,

$\overrightarrow{OP}$ = $\overrightarrow{OP_0}$ + $\overrightarrow{P_0P}$

(x,y,z) = (x_0,y_0,z_0) + t$\vec{u}$

or

$\boldsymbol{\vec{r}}$ **=** $\boldsymbol{\overrightarrow{r_0}}$ **+ t$\boldsymbol{\vec{u}}$, *t*∈R**

Where $\vec{u}$=(u_x,u_y,u_z) is a direction vector for the line.

This is the **vector equation of a line** in R^3.

Ex. Find two vector equations of the line L that passes through the points A(1,2,3) and B(2,−1,0)
.

Solution

If we use the direction vector $\vec{d}$ = $\overrightarrow{AB}$ = (1,−3,−3) and the point A(1,2,3)∈ L , then the vector equation of the line L is:

L : (x,y,z) = (1,2,3) + t(1,−3,−3), t ∈ R.

Ex. Find the vector equation of a line L_2 that passes through the origin and is parallel to the line

L_1 :$\vec{r}$ = (−2,0,3) + t(−1,0,2), t ∈ R.

Solution

L_2 :$\vec{r}$ = (0,0,0) + s(−1,0,2), s ∈ R.

L_2 :$\vec{r}$ = s(−1,0,2), s ∈ R.

Ex. Find the vector equation of a line that:

a) passes through A(3,−2,0) and is parallel to the y-axis

b) passes through M (−1,0,4) and is perpendicular to the yz-plane

Solution

a) $\vec{r}$ = (3,−2,0) + t(0,1,0), t ∈ R

b) $\vec{r}$ = (−1,0,4) + t(1,0,0), t ∈ R

Parametric Equations of a Line in R^3

Let's rewrite the vector equation of a line:

$\vec{r}$ = $\overrightarrow{r_0}$ + t$\vec{u}$, t∈R

as

(x,y,z) = (x_0,y_0,z_0) + t(u_x,u_y,u_z), t∈R where $\vec{u}$=(u_x,u_y,u_z).

The **parametric equations of a line** in R^3 are:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases}, \quad t \in R$$

Ex. Find the parametric equations of the line L that passes through the points A(0,−1,2) and B(1,−1,3) . Describe the line.

Solution

$\vec{u}= \overrightarrow{AB}$= (1,0,1);   A(0,-1,2)∈ L

L: $\vec{r}$= (0,-1,2)+ t(1,0,1),   t∈ R

(x,y,z)= (0,-1,2)+ t(1,0,1),   t∈ R

L: $\begin{cases} x = t \\ y = -1 \\ z = 2+t \end{cases}$, t∈ R

Symmetric Equation of a line

The parametric equations of a line may be written as:

$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases}$, t∈ R

From here, **the symmetric equations of the line** are:

$\boldsymbol{\frac{x-x_0}{u_x} = \frac{y-y_0}{u_y} = \frac{z-z_0}{u_z}}$

$u_x \neq 0, u_y \neq 0,\ u_z \neq 0$

Ex. Convert the vector equation of the line L: $\vec{r}$ = (0,1,−3) + t(−1,2,0), t ∈ R to the parametric and symmetric equations.

Solution

(x,y,z)= (0,1,−3) + t(−1,2,0),   t ∈ R

$$\therefore \begin{cases} x = -t \\ y = 1 + 2t \\ z = -3 \end{cases}, \quad t \in R$$

$\therefore \frac{x}{-1} = \frac{y-1}{2}$, z= -3

Ex. Convert the symmetric equations for a line: $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{4}$ to the parametric and vector equations.

Solution

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{4} = t \rightarrow \begin{cases} x - 2 = 3t \\ y + 1 = -2t \\ z = 4t \end{cases}, \quad t \in R$$

$$\therefore \begin{cases} x = 2 + 3t \\ y = -1 - 2t \\ z = 4t \end{cases}, \quad t \in R$$

$\therefore \vec{r}$= (2,-1,0)+ t(3,-2,4), t∈ R

Ex. For each case, find if the given point lies on the given line.

a) L: $\vec{r}$= (1,2,-3)+ t(0,1,-2); P(1,4,-7)

b) L: $\begin{cases} x = -2 + 3t \\ y = -t \\ z = 5 \end{cases}$; P(0,1,5)

c) L: : $\frac{x+1}{-2} = \frac{y-2}{1} = \frac{z}{-3}$; P(-2,3,-3)

Solution

a) (1,4,-7)= (1,2,-3)+ t(0,1,-2)

(1,4,-7)- (1,2,-3)= t(0,1,-2)

(0,2,-4)= t(0,1,-2) ⇨ t= 2 ∴ P∈ L

b) $\begin{cases} 0 = -2 + 3t & \Rightarrow t = \frac{2}{3} \\ 1 = -t & \Rightarrow t = -1 \\ 5 = 5 & \Rightarrow true \end{cases}$ ∴ P∉ L

c) $\frac{-3+1}{-2} = \frac{3-2}{1} = \frac{-3}{-3}$

1=1=1 ∴ P∈ L

Ex. Consider the line L: $\vec{r}$ = (3,−2,3) + t(−1,2,−3), t ∈ R. Find the intersection points between this line and the coordinates axes and planes.

Solution

L: $\begin{cases} x = 3 - t \\ y = -2 + 2t \\ z = 3 - 3t \end{cases}$

x= 0 → t= 3 → y= -2+ 2(3)= 4, z= 3- 3(3)= -6

∴ yz-int is A(0,4,-6)= L ∩ yz-plane

y= 0 → t= 1 → x= 2, z= 0

∴ xz-int is B(2,0,0)= L ∩ xz-plane

z= 0 → t= 1 →x= 2, y= 0

∴ xy-int is B(2,0,0)= L ∩ xy-plane

∴ x-int is B(2,0,0)= L ∩ x-axis

Note that y-intercept and z-intercept do not exist.

Now let's recap

Vector and Parametric Equations of a Line in R^2 in R^3

$\vec{r} = \vec{r_0} + t\vec{d}$, t∈R

$\vec{r} = \vec{r_0} + t\vec{d}$, t∈R

where $\vec{d}$=(d_x,d_y).

$$\begin{cases} x = x_0 + td_x \\ y = y_0 + td_y \end{cases} \quad t \in R$$

where $\vec{d}$=(d_x,d_y,d_z)

$$\begin{cases} x = x_0 + td_x \\ y = y_0 + td_y \\ z = z_0 + td_z \end{cases} \quad t \in R$$

Note. *The vector equation of a line is not unique. It depends on the specific point P_0 and on the direction vector u r that are used.*

Parallel lines

Two lines L_1 and L_2 with direction vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are parallel if $\overrightarrow{u_1} \parallel \overrightarrow{u_2}$ (collinear) or, there exist k∈R such that: $\overrightarrow{u_2}$= k$\overrightarrow{u_1}$

or: $\overrightarrow{u_1} \times \overrightarrow{u_2} = \vec{0}$

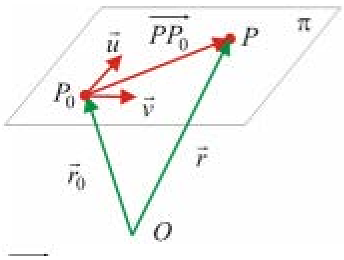
Perpendicular lines

Two lines L_1 and L_2 with direction vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are perpendicular if $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$.

$L_1 \perp L_2$ if $\overrightarrow{u_1} \perp \overrightarrow{u_2}$ or $\overrightarrow{u_1} \cdot \overrightarrow{u_2}$= 0.

Vector Equations of a Plane

Let consider a plane π. Two vectors $\vec{u}$ and $\vec{v}$, parallel to the plane π but not parallel between them(not collinear), are called *direction vectors* of the plane π.



The vector $\overrightarrow{P_0P}$ from a specific point $P_0(x_0, y_0, z_0)$ to a generic point P(x,y,z) of the plane is a linear combination of direction vectors $\vec{u}$ and $\vec{v}$:

$\overrightarrow{P_0P}$= s$\vec{u}$ + t$\vec{v}$; s,t ∈ R

The vector equation of the plane is:

: $\boldsymbol{\vec{r}}$ **=** $\boldsymbol{\overrightarrow{r_0}}$ **+ s**$\boldsymbol{\vec{u}}$ **+ t**$\boldsymbol{\vec{v}}$**;** s,t ∈ R

Ex. A plane π is given by the following vector equation:

: $\boldsymbol{\vec{r}}$ **= (-1,0,2) + s(0,0,1) + t(1,0,-1);** s,t ∈ R

a) Find two points on this plane.

b) Find one line on this plane.

c) Find the vector equation of a line $L_\perp$ that passes through the origin and is perpendicular to this plane.

Solution

a) If s=0 and t=0, then $\boldsymbol{\vec{r}}$ **= (-1,0,2)** ⇨ P_0(-1, 0, 2) ∈ **π**

If s=1 and t=2, then $\boldsymbol{\vec{r}}$ **= (-1,0,2) + 1(0,0,1) + 2(1,0,-1)= (1,0,1)** ⇨ A(1,0,1) ∈ **π**

b) Let L: $\boldsymbol{\vec{r}}$ **= (-1,0,2) + s(0,0,1);** s ∈ R ⇨ L⊂ $\boldsymbol{\pi}$

c) A director vector for the line $L_\perp$ is:

$\vec{u} \times \vec{v}$ = (0,1,0)

$L_\perp$: $\vec{r}$ = q(0,1,0); q∈ R.

Parametric Equations of a Plane

The vector equation of a plane given by

: $\boldsymbol{\vec{r}}$ **=** $\boldsymbol{\overrightarrow{r_0}}$ **+ s**$\boldsymbol{\vec{u}}$ **+ t**$\boldsymbol{\vec{v}}$**;** s,t ∈ R

(x,y,z) = (x_0,y_0,z_0) + t(u_x,u_y,u_z) + s(u_x,u_y,u_z)

can be written also as

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} ; \quad \text{s,t} \in \text{R}$$

These are **the parametric equations of a plane**.

Ex. Convert the vector equation to the parametric equations.

: $\vec{r}$ = (-1,0,2) + s(0,1,-1) + t(1,-2,0); s,t ∈ R

Solution

(x,y,z) = (-1,0,2) + s(0,1,-1) + t(1,-2,0); s,t ∈ R

$$\begin{cases} x = -1 + t \\ y = s - 2t \\ z = 2 - s \end{cases} ; \quad \text{s,t} \in \text{R}$$

Ex. Convert the parametric equations to the vector equation.

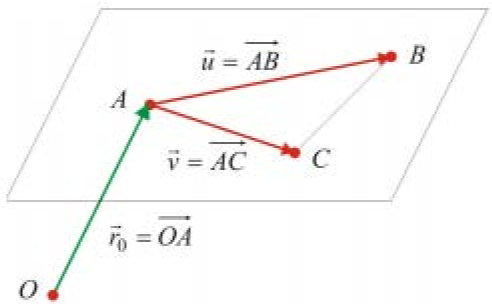
$$\begin{cases} x = 1 + s - 2t \\ y = 3t \\ z = 4 - s \end{cases} ; \quad \text{s,t} \in \text{R}$$

Solution

: $\vec{r}$ = (1,0,4) + s(1,0,-1) + t(-2,3,0); s,t ∈ R

Ex. Find the vector equation of the plane π that passes through the points A(0,1,−1) , B(2,−1,0) , and C(0,0,1) .

Solution



Let $\overrightarrow{r_0}$ = $\overrightarrow{OA}$ = (0,1,-1), $\vec{u}$ = $\overrightarrow{AB}$ = (2,-2,1), and $\vec{v}$ = $\overrightarrow{AC}$ = (0,-1,2).

Then:

: $\vec{r}$ = (0,1,-1) + s(2,-2,1) + t(0,-1,2); s,t ∈ R

Ex. Find the vector and parametric equations of the plane π that contains the following parallel and distinct lines:

L_1 : $\vec{r}$ = (1,2,1) + s(0,-1,-2); s∈R

L_1 : $\vec{r}$ = (3,4,0) + t(0,1,2); t∈R

Solution

A(1,2,2), B(3,4,0) ⇨$\overrightarrow{AB}$ = (3-1,4-2,0-2) = (2,2,-2)

Take $\vec{u}$ = $\overrightarrow{AB}$ and $\vec{v}$ = (0,1,2) and P_0 = A ⇨ **π** : $\vec{r}$ = (1,2,2) + s(0,1,2) + t(2,2,-2); s,t ∈ R

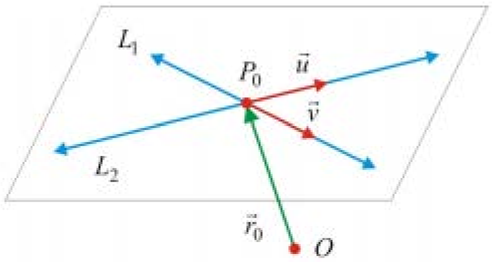
$$:\begin{cases} x = 1 + 2t \\ y = 2 + s + 2t \\ z = 2 + 2s - 2t \end{cases} ; \quad s,t \in R$$

Ex. Find the vector equation of the plane π determined by the following intersecting lines:

L_1: $\vec{r}$= (0,0,1)+ s(-1,0,0); s∈R

L_2: $\vec{r}$= (-3,0,1)+ t(0,0,2); t∈R

Solution



First let's find the point of intersection.

$$\begin{cases} -s = -3 \\ 0 = 0 \\ 1 = 1 + 2t \end{cases} \Rightarrow \text{s=3 and t=0}$$

$P_0 = L_1 \cap L_2 \Rightarrow P_0(-3,0,1)$

Let $\overrightarrow{r_0} = \overrightarrow{OP_0}$ = (-3,0,1), $\vec{u} = \overrightarrow{u_2}$ = (0,0,2), and $\vec{v} = \overrightarrow{u_1}$ = (-1,0,0).

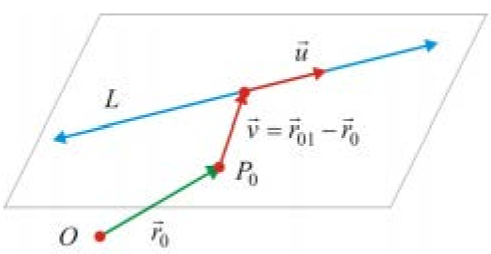
Then:

π : $\vec{r}$ = (-3,0,1) + s(0,0,2) + t(-1,0,0); s,t ∈ R

Ex. Find the vector equation of the plane π that passes through the origin and contains the line

L : $\vec{r}$= (0,1,2)+ t(-1,0,3); t∈R

Solution



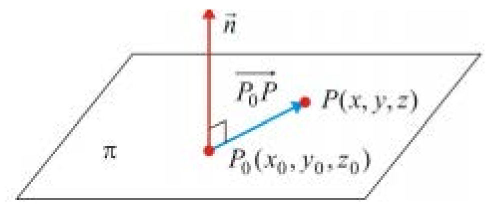
Let $\overrightarrow{r_0}$ = (0,0,0), $\vec{u}$ = (-1,0,3), and $\vec{v}$ = (0,1,2) – (0,0,0) = (0,1,2). Then the vector equation of the plane is:

π : $\vec{r}$ = (0,0,0) + s(-1,0,3) + t(0,1,2); s,t ∈ R

∴ : $\vec{r}$ = s(-1,0,3) + t(0,1,2); s,t ∈ R

Normal Equation of a Plane

A plane may be determined by a point $P_0(x_0,y_0,z_0)$ and a vector perpendicular to the plane $\vec{n}$ called the normal vector.



If P(x, y, z) is a generic point on the plane, then:

$\overrightarrow{P_0P} \perp \vec{n}$ and

$\boldsymbol{\overrightarrow{P_0P} \cdot \vec{n}}$ **= 0**

This is **the normal equation of a plane**.

10. Cartesian Equation of a Plane

Let's write the normal vector of a plane in the form:

$\vec{n}$ = (A, B, C)

Then, the normal equation may be written as:

(x-x_0, y-y_0, z-z_0) · (A,B,C) = 0

Ax + By + Cz - Ax_0 - By_0 - Cz_0 = 0

or

Ax + By + Cz + D = 0

equation which is called **the Cartesian equation of a plane**.

Note. A normal vector to the plane is:

$\vec{n} = \vec{u} \times \vec{v}$

where $\vec{u}$ and $\vec{v}$ are the direction vectors of the plane.

Ex. Consider the plane π defined the Cartesian equation π : 2x – 3y + 6z +12 = 0 .

a) Find a normal vector to this plane.

b) Find two points on this plane

c) Find if the point P(1,2,3) is a point on this plane.

Solution

a) $\vec{n}$ = (2,-3, 6)

b) If x=0 and y=0 then z=-2. So (0, 0,-2) ∈**π**.

If x=0 and z=0 then y=4. So (0, 4, 0) ∈ ***π***.

c) 2(1) – 3(2) + 6(3) + 12 = 26 ≠ 0 ⇨ P∉**π**.

Ex. Find the Cartesian equation of a plane π that passes through the points A(1,−1,0) , B(0,0,1) , and C(0,−2,1) .

Solution

$\vec{u} = \overrightarrow{AB}$ = (-1,1,1); $\vec{u} = \overrightarrow{AC}$ = (-1,-1,1)

$\vec{n} = \vec{u} \times \vec{v}$ = (2,0,2) = (A,B,C)

π: 2x + 2z +D =0

A ∈ **π** ⇨ 2(1) +2(0) + D = 0 ⇨ D = -2

∴ **π**: 2x + 2z -2 = 0 or x + z -1 = 0

Ex. Find parametric and vector equations for the plane: π : x – 2y + 3z – 6 = 0.

Solution

Take y=s and z=t then x=6+2s-3t.

$$:\begin{cases} x = 6 + 2s - 3t \\ y = s \\ z = t \end{cases} \quad ; \quad s,t \in R$$

π : $\vec{r}$ = (6,0,0) + s(2,1,0) + t(-3,0,1); s,t ∈ R

Ex. Find the intersections with the coordinate axes for the plane π : 3x + 2y + z – 6 = 0 . Represent the plane graphically.

Solution

Let A = **π** ∩ x-axis ⇨ y_A = z_A = 0 ⇨ x_A = 2.

x-int is A(2,0,0)

Similarly,

y-int is B(0,3,0)

z-int is C(0,0,6)