Lesson 10: Unit 5 - An introduction to vectors

We are going to look at the question: "What is a vector?"

Scalars and Vectors

Scalars (in Mathematics and Physics) are quantities described completely by a number and eventually a measurement unit.

Vectors are quantities described by a magnitude (length, intensity or size) and direction.

Scalar	Vector
Distance is a scalar quantity e.g., Keri lives 6 km from the University of Waterloo	**Displacement** is distance in a given direction e.g., Keri lives 6 km northeast of the University of Waterloo
Speed is a scalar e.g., the plane is travelling at a speed of 852 km/h	**Velocity** is speed in a given direction e.g., the velocity of the plane is 852 km/h west
Mass is a scalar e.g., Mr. Ford has a mass of 100 kg	**Weight** is force downwards due to gravity e.g., Mr. Ford has a weight of 980 N (downwards)

Ex . Classify each quantity as scalar or vector.

a) time ⇒ scalar

b) position ⇒ vector

c) temperature ⇒ scalar

d) electric charge ⇒ scalar

e) mass ⇒ scalar

f) force ⇒ vector

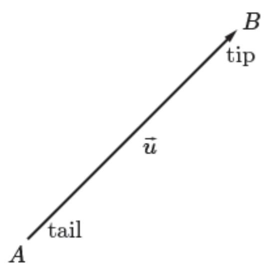
g) displacement ⇒ vector

Geometric Vectors

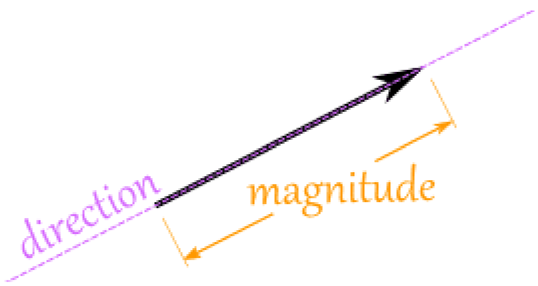
Geometric Vectors are vectors not related to any coordinate system.

To represent vectors we use **rays** (directed line segments).

For example, the directed line segment (ray) $\overrightarrow{AB}$:



where A is called the initial (start, tail) point and B is called the final (end, terminal, head or tip) point.



The length of the ray is a positive real number, which represents the **magnitude (**size, norm or intensity) of the vector.

The **magnitude** of the vector $\vec{u}$ is denoted by $|\vec{\boldsymbol{u}}|$**,** $||\vec{\boldsymbol{u}}||$ **or u.**

If the magnitude of a vector is zero, we call it the **zero vector** and denote it $\vec{0}$. This is a useful vector despite that its direction is undefined.

Equivalent or Equal Vectors

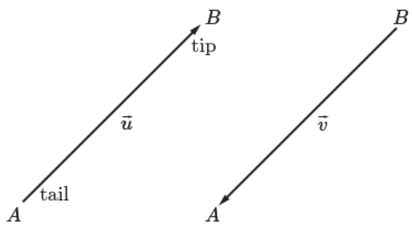
Two vectors are **equal (or equivalent)** if and only if they have the same **magnitude** and **direction**. They do not need to be in the same position.

For example, $\overrightarrow{AB} = \overrightarrow{PQ}$

Opposite Vectors

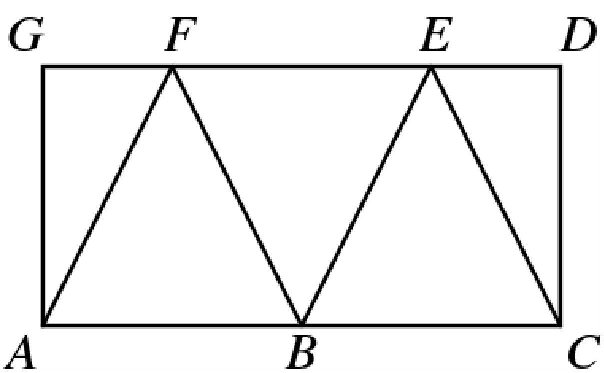
Two vectors are called opposite if they have the same magnitude and opposite direction.

The opposite vector of the vector $\vec{v}$ is denoted by -$\vec{v}$.



$\vec{\boldsymbol{u}}$ **= -**$\vec{\boldsymbol{v}}$

Ex. In the diagram below, $\triangle AFB$ and $\triangle BEC$ are equilateral, and *ACDG* is a rectangle.



a. Write down two other vectors **equal** to $\overrightarrow{AB}$.
b. Write down three vectors which are **opposite** to $\overrightarrow{FE}$.
c. What vector is the **opposite** of $\overrightarrow{DC}$?
d. Write down 3 vectors which have the same magnitude as $\overrightarrow{BC}$, but different direction.

Possible Solutions. a. $\overrightarrow{FE}$ and $\overrightarrow{BC}$, b. $\overrightarrow{EF}$ and $\overrightarrow{CB}$, c. $\overrightarrow{AG}$, d. $\overrightarrow{AF}$, $\overrightarrow{EB}$ or $\overrightarrow{CE}$.

Parallel (Collinear) Vectors

Two vectors are **parallel** if their directions are either the same or opposite.

If $\vec{u}$ and $\vec{v}$ are parallel, then we write $\vec{u}$ | | $\vec{v}$.

Ex. *ABCDEF* is a regular hexagon. Give examples of vectors formed between pairs of vertices of hexagon *ABCDEF*:

a. parallel but with different magnitudes

b. equal in magnitude but opposite in direction

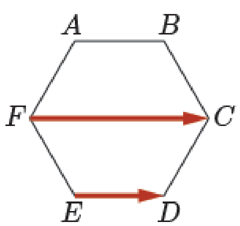
c. equal in magnitude but not parallel

d. different in both magnitude and direction

Solutions.

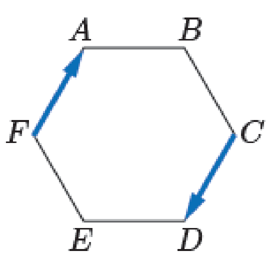
a. parallel but with different magnitudes

Possible Solution: $\overrightarrow{ED}$ and $\overrightarrow{FC}$



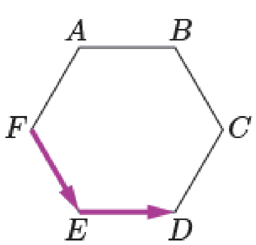
b. equal in magnitude but opposite in direction

Possible Solution: $\overrightarrow{FA}$ and $\overrightarrow{CD}$



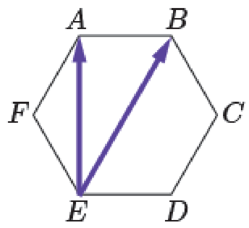
c. equal in magnitude but not parallel

Possible Solution: $\overrightarrow{FE}$ and $\overrightarrow{ED}$



d. different in both magnitude and direction

Possible Solution: $\overrightarrow{EA}$ and $\overrightarrow{EB}$



To express the **direction** of a vector in a horizontal plane, the following standards are used

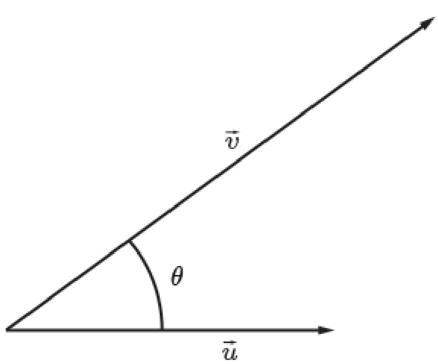
True (Azimuth) Bearing The direction of the vector is given by the angle between the North and the vector, measured in a clockwise direction.

Quadrant Bearing The direction is given by the angle between the North-South line and the vector

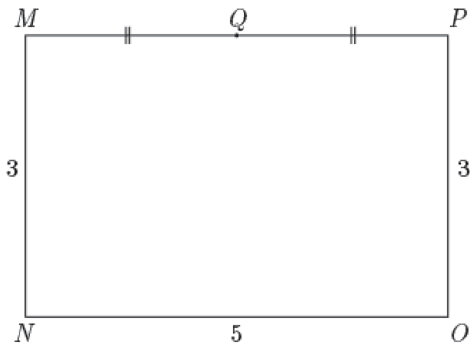
To specify direction, we state a compass direction (True Bearing) or the angle it makes with the horizontal or vertical(Quadrant Bearing) (e.g., 30∘ above the horizontal or N 20∘ W).
We can also state its *direction relative to another vector*.

Angle between Vectors

The angle between two vectors is the angle ≤180∘ formed when the vectors are placed **tail to tail**, that is, starting at the same point.



Ex. *MNOP* is a rectangle with side lengths 3 and 5. *Q* is the midpoint of *MP*.



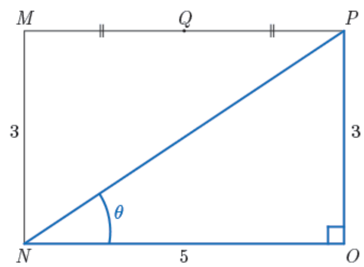
Find the angle between the following vectors:

a. $\overrightarrow{NP}$ and $\overrightarrow{NO}$

b. $\overrightarrow{NM}$ and $\overrightarrow{NQ}$

c. $\overrightarrow{NQ}$ and $\overrightarrow{QP}$

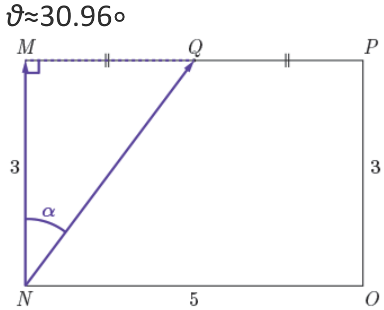
Solution

a. 

$\tan\vartheta=35$

$\vartheta=\tan-1(35)$

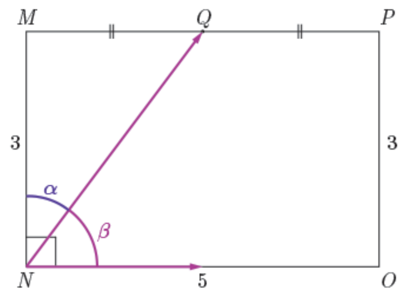
$\vartheta\approx30.96\circ$

b. 

$\tan\alpha=2.53$

$\alpha=\tan-1(2.53)$

$\alpha\approx39.81\circ$

c. 

$\beta = 90^\circ - \alpha$

$\beta \approx 90^\circ - 39.81^\circ$

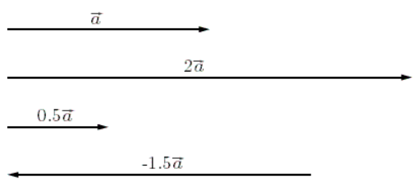
$\beta \approx 50.19^\circ$

Scalar Multiplication

In general, given some real number k, $k\vec{a}$ is a **vector** with the following attributes:

a. Its *magnitude* is $|k\vec{a}| = |k||\vec{a}|$.

ii. Its *direction* is the same as $\vec{a}$ if $k>0$ and opposite to $\vec{a}$ if $k<0$ (if $k=0$, then $k\vec{a} = \vec{0}$).



Important: *When two vectors are parallel, one of the vectors can be expressed in terms of the other using scalar multiplication.*

Ex. Given that $|\vec{v}| = 5$, find the magnitude of each of the following vectors:

a. $2\vec{v}$

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b. $-4\vec{v}$

c. $15\vec{v}$

Solution

a. $|2\vec{v}| = |2||\vec{v}| = (2)(5) = 10$

b. $|-4\vec{v}| = |-4||\vec{v}| = (4)(5) = 20$

c. $|15\vec{v}| = |15||\vec{v}| = 15(5) = 1$

Unit Vector

An **unit vector** is a vector having a magnitude of 1.

The question is how to create a unit vector in the direction of a non-zero u $\vec{}$*?*

Multiply $\vec{u}$ by the scalar equal to the reciprocal of the magnitude of $\vec{u}$.

That is, $\vec{v} = \frac{1}{|\vec{u}|}\vec{u}$ is a unit vector in the direction of $\vec{u}$.

Notational, this is written $\boldsymbol{\hat{v} = \frac{1}{|\vec{u}|}\vec{u}}$**.** (Read u cap or u hat).

Ex. If $|\vec{a}|$=12, state a unit vector in the direction of $\vec{a}$.

Solution

$\hat{a}=\frac{1}{12}\vec{a}$.

Now let's stop to recap.

We have

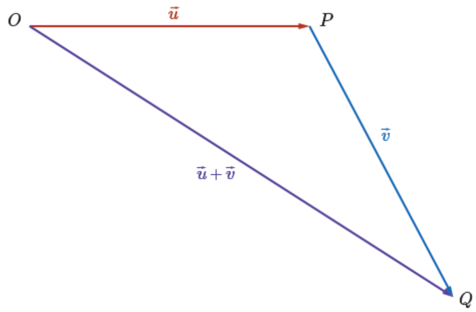
1. Distinguished the difference between a **vector** and a **scalar**,
2. Defined **opposite vectors**, **equal vectors, parallel vectors**, and the **magnitude of a vector**,
3. Looked at how to determine the **angle between two vectors**,
4. Talked about **scalar multiplication**, and
5. Talked about a **unit vector**.

Addition of two Vectors

The vector addition $\vec{s}$ of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} + \vec{b}$ and is called the sum or resultant of the two vectors.

In order to find the sum or resultant of to vectors we can use two rules:

a. *The Triangle Rule*(law)



In order to find the sum (resultant) of two geometric vectors:

i) Arrange the two vectors **tip to tail.**
ii) The **resultant vector** is the third side of the triangle.

Its direction is from the start (tail) of the first vector to the tip of the second vector.

$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$.

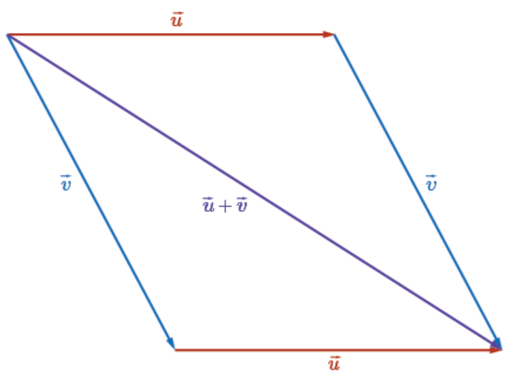
b) *Parallelogram Rule*

i) Arrange the two vectors so they share a common vertex (**tail to tail**).

ii) Complete the parallelogram.

The **resultant vector**, $\vec{u} + \vec{v}$, is the diagonal drawn from the common vertex.

Notice also that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.



Now again just to recap:

The **triangle law** was useful when arranging vectors tip to tail.
The **parallelogram law** was useful when arranging vectors tail to tail.

Notice also that the two rules are equivalent (you may apply triangle rule or parallelogram rule and the result is evidently the same).

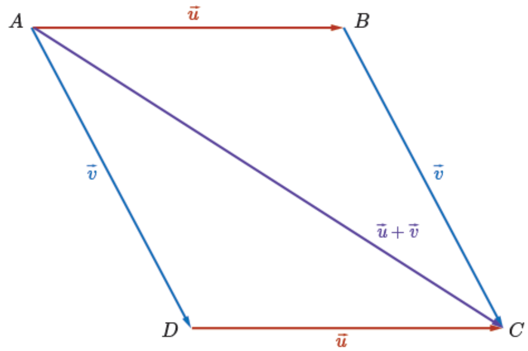
*Now how to **subtract** two vectors?*

To subtract vectors, we add the opposite.

Think

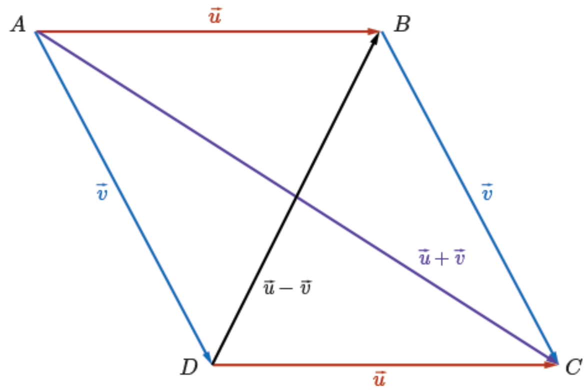
$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}).$$

Consider the parallelogram formed by the vectors $\vec{u}$ and $\vec{v}$.



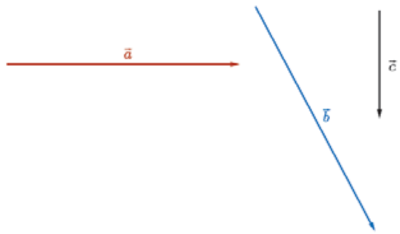
The sum $\vec{u} + \vec{v}$ is the vector created by the diagonal from the tail of the two vectors $\vec{u} + \vec{v} = \overrightarrow{AC}$.

Where is $\vec{u}$ - $\vec{v}$?



We see that $\vec{u}$ - $\vec{v}$ is the diagonal, $\overrightarrow{DB}$.

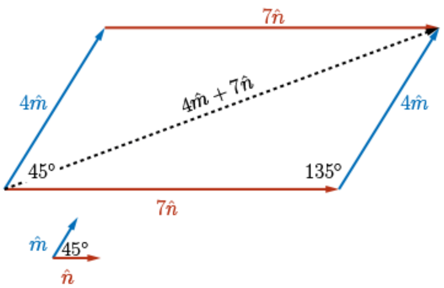
Ex. Determine $\vec{a} + \vec{b} - \vec{c}$.



Ex. If $\hat{m}$ and $\hat{n}$ are unit vectors that make a 45^0 angle with each other, calculate $|4\hat{m} + 7\hat{n}|$.

Solution

Using the parallelogram law (not triangle rule because we have the angle between the vectors) and a diagram, the vector $4\hat{m}+7\hat{n}$ can be visualized as shown.



To calculate $|4\hat{m} + 7\hat{n}|$, we must determine the length of the diagonal in the parallelogram.

The side lengths of the parallelogram are the magnitudes of the vectors $4\hat{m}$ and $7\hat{n}$, so 4 and 7, respectively.

Apply the cosine law to find the length of the diagonal.

$$|4\hat{m} + 7\hat{n}|^2 = |4\hat{m}|^2 + |7\hat{n}|^2 - 2|4\hat{m}||7\hat{n}|\cos(135^0)$$

$$= 4^2 + 7^2 - 2(4)(7)\left(-\frac{1}{\sqrt{2}}\right)$$

$$= 65 + 28\sqrt{2}$$

$$\therefore |4\hat{m} + 7\hat{n}| = \sqrt{65 + 28\sqrt{2}} \text{ (since } |4\hat{m} + 7\hat{n}| \geq 0)$$

$$\approx 10.23$$

Properties of Vectors

For all $\vec{a}$, $\vec{b}$, $\vec{c}$ and $m \in R$, the following properties hold:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ commutative
2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ associative
3. $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$ distributive
4. $\vec{a} + \vec{0} = \vec{a}$ existence of a zero vector
5. $\vec{a} + (-\vec{a}) = \vec{0}$ existence of an additive inverse

What is a zero vector?

We can see $\vec{0}$ as a vector with magnitude 0 and arbitrary direction.

Ex. Given that $\vec{m} = \vec{a} + 3\vec{b}$ and $\vec{n} = 2\vec{a} - 4\vec{b}$, simplify $2\vec{m} - \vec{n}$.

Solution $2\vec{m} - \vec{n} = 2(\vec{a} + 3\vec{b}) - (2\vec{a} - 4\vec{b})$

$= 2\vec{a} + 6\vec{b} - 2\vec{a} + 4\vec{b}$ distributive

$= 2\vec{a} - 2\vec{a} + 6\vec{b} + 4\vec{b}$ commutative

$= 2\vec{a} + (-2\vec{a}) + 10\vec{b}$ associative

$= \vec{0} + 10\vec{b}$ existence of an additive inverse

$= 10\vec{b}$ existence of a zero vector

We can notice that this work is very similar with the one done in algebra.

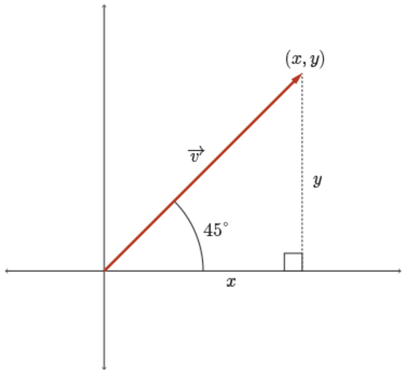
Algebraic Vectors

In the first part we learnt about *geometric vectors*(basically we looked at vectors having magnitude and direction represented as a directed line segment or arrow) and now we will introduce vectors in a Cartesian coordinate system referred as *algebraic vectors* or vectors in component form.

Let's consider an example.

Vector $\vec{v}$ is defined to have magnitude 10 and a direction of north 45∘ east. This vector, $\vec{v}$, is an example of a **geometric vector**.

This was a typical representation of the concept throughout the first vector unit.



Can we represent the same vector, $\vec{v}$, in a different way?

Yes, by using the notion of **algebraic vectors**.

To represent a geometric vector algebraically, position the tail of the vector $\vec{v}$, at the origin of a Cartesian (*x*,*y*)-plane.

In our case, $|\vec{v}|=10$ and the direction is north 45^0 east, which corresponds to an angle of 45^0 CCW (counter-clockwise) from the positive *x*-axis (this is defined as **standard position**).

We can now represent vector $\vec{v}$, using coordinates *x* and *y*, where (*x*,*y*) is the location of the tip of vector $\vec{v}$, when its tail is positioned at the origin.

This raises the question: How do we find the coordinates *x* and *y*?

$\cos(45^0) = \frac{x}{|\vec{v}|}$

$x = 5\sqrt{2}$

Similarly,

$y = 5\sqrt{2}$

Therefore, vector $\vec{v}$ can be written algebraically as an ordered pair $(5\sqrt{2}, 5\sqrt{2})$.

The ordered pair $(5\sqrt{2}, 5\sqrt{2})$ is an example of an **algebraic vector**.

In the ordered pair, *x* = $5\sqrt{2}$ and *y* =$5\sqrt{2}$ are referred to as the *x* and *y* **components** of the vector $\vec{v}$. As a result, $\vec{v}$ = $(5\sqrt{2}, 5\sqrt{2})$ may be said to be in **component form**.

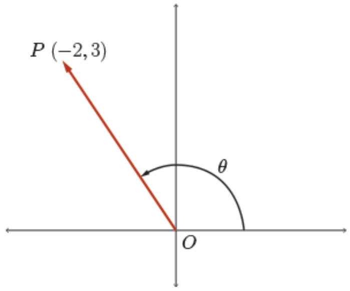
So, this ordered pair represent a vector but could also represent a point. How do we know the difference? From context.

Position Vector

The **position vector** of a point *P* is the vector $\overrightarrow{OP}$ with tail at the origin *O* and tip at the point *P*.

Ex. Draw vector $\overrightarrow{OP}$= (−2,3) and determine its magnitude and direction.

Solution



$\left|\overrightarrow{OP}\right|$ = $\sqrt{(-2)^2 + 3^2}$

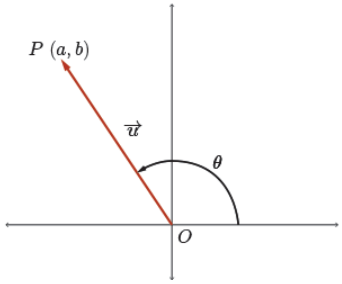
= $\sqrt{13}$

$\tan\theta$ = $\frac{3}{-2}$

θ = $\tan^{-1}\frac{3}{-2}$

θ ≈ 124^0.

In general, any vector, $\vec{u}$, can be written as an ordered pair (*a*,*b*)



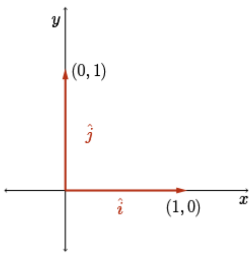
where $|\vec{u}| = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(\frac{b}{a})$.

Note that ϑ is an angle measured CCW from the positive *x*-axis.

When solving for ϑ=tan−1(*b/a*), there are two possible solutions for 0∘≤ϑ≤360∘. The quadrant that (*a*,*b*) lies in will determine the correct ϑ.

Consider also what happens when a=0 or b=0.

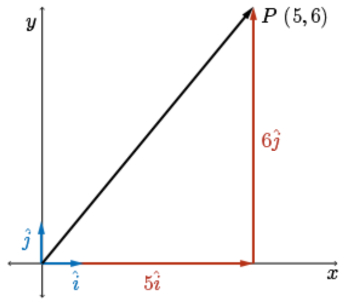
Let's look to unit vectors again.



The special unit vectors which point in the direction of the positive *x*-axis and positive *y*-axis are given the names $\hat{\imath}$ and $\hat{\jmath}$ respectively, where $\hat{\imath}$ = (1,0) and $\hat{\jmath}$ = (0,1).

We may express any vector in the *xy*-plane as a sum of scalar multiples of the vectors $\hat{\imath}$ and $\hat{\jmath}$.

For example, the vector $\overrightarrow{OP}$ = (5,6) can be expressed as a vector sum of scalar multiples of $\hat{I}$ and $\hat{J}$ in the following manner:



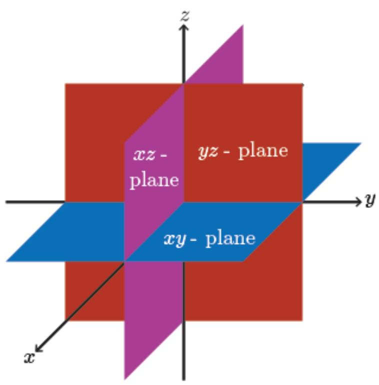
In general, ordered pair notation and unit vector notation are equivalent.

Any algebraic vector can be written in either form:

$$\boldsymbol{\vec{u}} = \overrightarrow{\boldsymbol{OP}} = (\boldsymbol{a},\boldsymbol{b}) \quad \text{or} \quad \boldsymbol{\vec{u}} = \overrightarrow{\boldsymbol{OP}} = \boldsymbol{a}\hat{\boldsymbol{\imath}} + \boldsymbol{b}\hat{\boldsymbol{\jmath}}.$$

Previously, we had considered geometric and algebraic vectors in the 2-dimensional (Cartesian) plane.

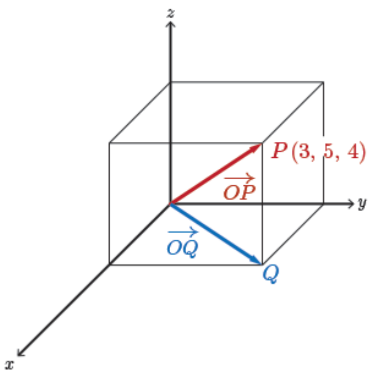
Now let's extend to vectors in 3 dimensions.



To plot the 3-dimensional point with coordinates (*a*,*b*,*c*), move *a* units from the origin in the *x*-direction, *b* units in the *y*-direction, and *c* units in the *z*-direction.

Ex. Locate the point *P* (3,5,4) and sketch the position vector

$\overrightarrow{OP}$ = 3$\hat{\imath}$ + 5$\hat{\jmath}$ + 4$\hat{k}$, where $\hat{\imath}$=(1,0,0), $\hat{\jmath}$=(0,1,0), and $\hat{k}$=(0,0,1) are the special unit vectors pointing in the direction of the positive *x*-, *y*-, and *z*-axes, respectively. Calculate the magnitude of $\overrightarrow{OP}$.



Solution

We see that $|\overrightarrow{OQ}| = \sqrt{3^2 + 5^2} = \sqrt{34}$.

Thus, by the Pythagorean theorem

$|\overrightarrow{OP}|^2 = |\overrightarrow{OQ}|^2 + |\overrightarrow{QP}|^2$

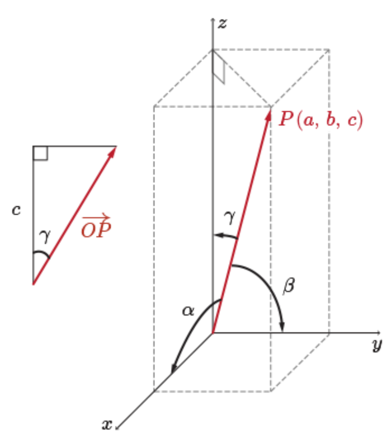
$|\overrightarrow{OP}|^2 = \sqrt{50} = 5\sqrt{2}$

In general, if $\vec{u}$ **= (a,b,c),** then $|\vec{u}|^2 = \sqrt{\boldsymbol{a^2 + b^2 + c^2}}$.

Direction Angles

In 2 dimensions, the direction of a vector is given by a single angle, as we previously saw.

In 3 dimensions, the direction of a vector is given by 3 angles, called ***direction angles***.



The direction angles of vector (a,b,c) are α, β, and γ, which are the angles the vector makes with the positive x, y, and z axes, respectively.

These angles are restricted to be

$$0^0 \le \alpha, \beta, \gamma \le 180^0$$

For example, in the diagram

$$\cos(\gamma) = \frac{c}{|\overrightarrow{OP}|}$$

Similarly,

$$\cos(\alpha) = \frac{a}{|\overrightarrow{OP}|} \quad \cos(\beta) = \frac{b}{|\overrightarrow{OP}|}$$

Ex. Find the direction angles of the vector $\vec{u} = (-2,3,0)$.

Solution

Using the definition of the direction angles,

$\cos(\alpha) = \frac{a}{|\vec{u}|}$

$= \frac{-2}{\sqrt{(-2)^2+3^2+0^2}}$

$\therefore \alpha \approx 124^0$

Similarly,

$\therefore \beta \approx 34^0$

$\therefore \gamma = 90^0$

Ex. Find a unit vector in the direction of $\vec{v} = (a,b,c)$ and prove that $cos^2(\alpha)+cos^2(\beta)+cos^2(\gamma)=1$, where α,β,γ are the direction angles of $\vec{v}$.

Solution

Since $\hat{v} = \frac{1}{|\vec{v}|}(\vec{v}) = \frac{1}{|\vec{v}|}(a,b,c) = (\frac{a}{|\vec{v}|}, \frac{b}{|\vec{v}|}, \frac{c}{|\vec{v}|})$ is indeed a unit vector, then

$\sqrt{\left(\frac{a}{|\vec{v}|}\right)^2 + \left(\frac{b}{|\vec{v}|}\right)^2 + \left(\frac{c}{|\vec{v}|}\right)^2} = 1$, $\left(\frac{a}{|\vec{v}|}\right)^2 + \left(\frac{b}{|\vec{v}|}\right)^2 + \left(\frac{c}{|\vec{v}|}\right)^2 = 1$ **(1)**

But recall that the direction angles are given by

$$\cos(\alpha) = \frac{a}{|\vec{v}|}, \cos(\beta) = \frac{b}{|\vec{v}|}, \cos(\gamma) = \frac{c}{|\vec{v}|}$$

Substituting these into **(1),** we get $cos^2(\alpha)+cos^2(\beta)+cos^2(\gamma)=1$ where α,β,γ are the direction angles of $\vec{v}$, as required.

Now let's look again to addition, subtraction, and scalar multiplication of vectors but given in algebraic form.

Given two algebraic vectors $\vec{u} = (u_x,u_y,u_z)$ and $\vec{v} = (v_x,v_y,v_z)$ then we have

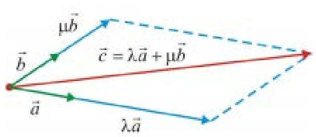
$\vec{u} \pm \vec{v} = (u_x \pm v_x, u_y \pm v_y, u_z \pm v_z)$

$k\vec{u} = (ku_x, u_y, ku_z)$ where k∈R.

Linear Dependency

Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linear dependent if there exist λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$.

Note. In order to be linear dependant the vectors must be coplanar (must belong to the same plan).



Ex. Prove that the vectors $\vec{a} = (-1,2,-3)$, $\vec{b} = (2,0,-1)$, and $\vec{c} = (-7,6,-7)$ are linear dependant.

Solution

We need to find two scalars λ and μ such that

$\vec{c} = \lambda\vec{a} + \mu\vec{b}$

(-7, 6, -7)= λ(-1,2, -3) + μ (2,0, -1) $\Rightarrow \begin{cases} -7 = -\lambda + 2\mu & (1) \\ 6 = 2\lambda & (2) \\ -7 = -3\lambda - \mu & (3) \end{cases}$

(2) ⇨ $\lambda = 3$ (4)

(4) ⇨ (1): $-7 = -3 + 2\mu$ ⇨ $\mu = -2$ (5)

(4) and (5) ⇨ (3): $-7 = -3(3) - (-2)$ *true*

$\vec{c} = 3\vec{a} - 2\vec{b}$

Therefore, the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linear dependant.

So far, addition, subtraction, and scalar multiplication of vectors have been introduced and explore.

While each of these operations has an intuitive geometrical interpretation, they alone are unable to handle certain applications of vectors. In the next unit we will develop two new operations that will allow us to extend our ability to apply vectors to physical and geometrical situations.