Lesson 2: Unit 2- **Polynomial Functions**

In This Section

-We will review and introduce terminology that will be used throughout the Polynomial Functions unit.
-We will explore the behaviour of the graphs of $y = x^3$ and $y = x^4$.

Polynomials

A **polynomial** is a mathematical expression constructed by the sum and/or difference of algebraic terms. Each term consists of variable factors raised to non-negative integer exponents and multiplied by real numerical coefficients.

Example 1

$$-\frac{2}{5}x^7y^2z + \sqrt{3}x^2yz - x + 2\pi$$

This is a polynomial.

-The coefficients are all real values ($-\frac{2}{5}$, $\sqrt{3}$, -1, 2π).

-The exponents of the variable factors are all non-negative integers. Note that the last term is a constant which is allowed by the definition ($2\pi = 2\pi x^0$).

Example 2

$$3x^2y^{-3} + \frac{5}{x^2} + 10\sqrt{x}$$

is not a polynomial for several reasons.
-The variables have negative exponents ($3x^2y^{-3}$, $\frac{5}{x^2} = 5x^{-2}$).
-The variables have fractional exponents ($10\sqrt{x} = 10x^{\frac{1}{2}}$).

Degree of Polynomial

The degree of a term of a polynomial is determined by the number of variable factors in the term, and can be calculated by adding the variable exponents in the term.

$$-\frac{2}{5}x^7y^2z + \sqrt{3}x^2yz - x + 2\pi$$

Term 1	Term 2	Term 3	Term 4
$-\frac{2}{5}x^7y^2z$	$+\sqrt{3}x^2yz$	$-x$	$+2\pi$
Degree 10	Degree 4	Degree 1	Degree 0

Terms of degree 0 are called **constants**.

Polynomial Functions

A polynomial function is a function whose equation is defined by a polynomial in one variable,

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

where the numerical coefficients a_0, a_1, a_2,…, a_n are real numbers and the exponents of x given by n, $n-1$, $n-2$,… are whole numbers (non-negative integers).

Examples of Polynomial Functions

$$f(x) = x - 3x^8$$
$$y = -4x^{2016} - \sqrt[3]{5}x^2 - 7x + 2$$
$$g(x) = 8(2x + 7)^3 - 3$$

In the first two examples, all coefficients are real numbers and the exponents of x are non-negative integers.
We see that $g(x)$ is not in the form of the previous two.
However, by expanding and simplifying the right hand side of the equation, this function can be expressed by a polynomial function.

Examples of Non-Polynomial Functions

$h(x) = \frac{3}{x^2} = 3x^{-2}$

$y = 2\sqrt[3]{x} + x^{\frac{2}{5}}$

$x = y^4$

-In the first example, exponents of x cannot be negative in value.
-In the second example, exponents of x must be whole numbers.
-The final example is not a function.

Terminology

The numerical coefficient of the highest degree term in a polynomial is called the **leading coefficient.**

In the general function, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, *an* is the leading coefficient.

The terms of the polynomial are usually arranged in descending order of the degree of the term.
In this standard form, the leading coefficient is the coefficient of the first term.

In the polynomial function examples,

$$f(x) = x - 3x^8$$
$$y = -4x^{2016} - \sqrt[3]{5}x^2 - 7x + 2$$
$$g(x) = 8(2x+7)^3 - 3$$

the leading coefficients are −3,−4, and 64, respectively.

The domain of all polynomial functions is the set of all real numbers; that is, $D=\{x \mid x \in R\}$, since there is no restriction on the value of *x*.

The range of the polynomial function depends on the behaviour of its graph.

The **degree of the polynomial function** is given by the value of the highest exponent of the variable.

A **constant function**, $f(x) = a$, is a polynomial function of degree 0 since $f(x)=ax^0$.

A **linear function**, $f(x) = ax + b$, is a polynomial function of degree 1 or less. A constant function is also a linear function.

A **quadratic function**, $f(x) = ax^2 + bx + c$, $a \neq 0$, is a polynomial function of degree 2.

A 3*rd* degree polynomial function, $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, is called a **cubic function**.

A 4*th* degree polynomial function, $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$, is called a **quartic function**.

A 5*th* degree polynomial function, $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, $a \neq 0$, is called a **quintic function**.

Any polynomial function with degree *n*, where $n>5$, will be referred to as an n^{th} degree polynomial function.
Special names exist for some higher degree polynomial functions, but are less commonly used.

Different Forms of Polynomial Functions

Representing a polynomial in different forms can provide different types of information about the behaviour of the function.

For example, when working with quadratic functions, we have the following:
-In **standard form**, $f(x)=ax^2+bx+c$, the degree, 2; leading coefficient, a; and y-intercept, c, of the function are easily identified.
-In **factored form**, $y=a(x-p)(x-q)$, the zeros (or x- intercepts), p and q, are readily determined.
-In the form $y=a(x-h)^2+k$, the vertex of the function, (h, k), can be quickly identified. We can also identify the transformations applied to the parent function, $y=x^2$, which can help us graph the function. For a quadratic function, this form is referred to as **vertex form**.

Example

Consider the following polynomial function in factored form:

$$y=-5x(2x+1)(x-1)^2$$

To obtain the degree of the polynomial function, we must determine the exponent of the highest degree term of the polynomial.
Expanding to standard form will provide this information:

$y=-5x(2x+1)(x^2-2x+1)$

$=-5x(2x^3-4x^2+2x+x^2-2x+1)$

$=-5x(2x^3-3x^2+1)$

$=-10x^4+15x^3-5x$

However, simply considering the product of the highest degree terms in each linear factor will give the highest term of the polynomial. In $y=-5x(2x+1)(x-1)(x-1)$, calculating $-5x(2x)(x^2)$ will also produce $-10x^4$.
Thus, the function is a 4*th* degree polynomial, or a quartic, and the leading coefficient is −10.

In another situation, we may need to determine the zeros/roots/x-intercepts of a polynomial function given to us in standard form. We will need to factor the polynomial. Techniques used to factor polynomials will be discussed later half.

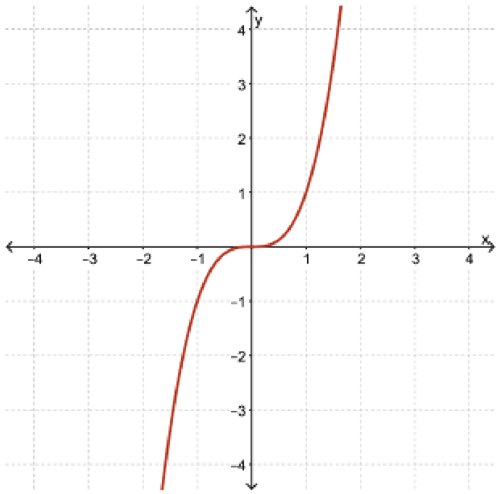
Simple Polynomial Functions: Power Functions

A **power function** is a function of the form $f(x)=ax^n$, where $a,n\in R$.

If the power function is also a polynomial function, then n is an integer and $n\geq 0$.

The Cubic Function $y=x^3$

Consider the function $y=x^3$.
Since there are no restrictions on the value of x, the domain of the function is the set of all real numbers (Domain: $\{x|x\in R\}$).
We can now create a table of values for the function and graph the curve.

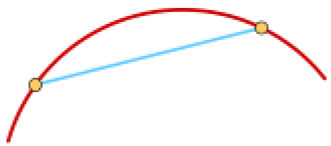


x	y
−3	−27
−2	−8
−1	−1
−12	−18
0	0
12	18
1	1
2	8
3	27

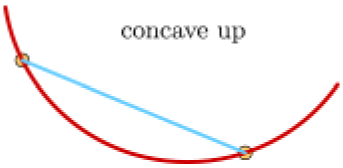
Since the function has opposite end behaviour with y approaching both negative and positive infinity, the range of the function is also the set of all real numbers (Range: $\{y|y\in R\}$).

Notice that this function becomes horizontal at $x=0$.
In this case, the function is said to be **stationary** at $x=0$ since it is neither increasing nor decreasing at this point.
The shape of the curve also changes at the origin.
Such a point on a curve is known as a type of **inflection point**.

Note: A **point of inflection** is defined as a point where a graph of a function changes concavity. Concavity is used to describe the way a curve bends.



If a line segment joining any two points on a curve is entirely below the curve, then the curve is said to be **concave down** between the two points.

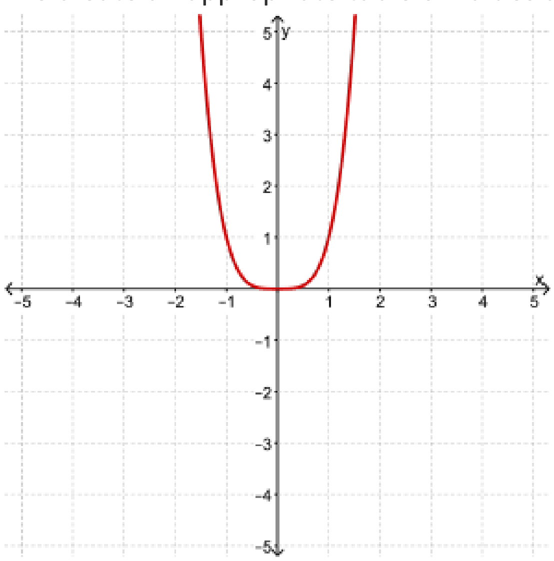


If a line segment is entirely above the curve, the curve is **concave up** between the two points.

The Quartic Function $y=x^4$

Moving on to the quartic function *y*=*x*4, we note that the domain is the set of all real values *(Domain: {x|x∈R}).*

We create an appropriate table of values and graph the function.



x	*y*
−3	81
−2	16
−1	1
−12	116
0	0
12	116
1	1

2	16
3	81

The range can now be identified as the set of all real values greater than or equal to zero
(Range: $\{y \mid y \geq 0, y \in R\}$*)*

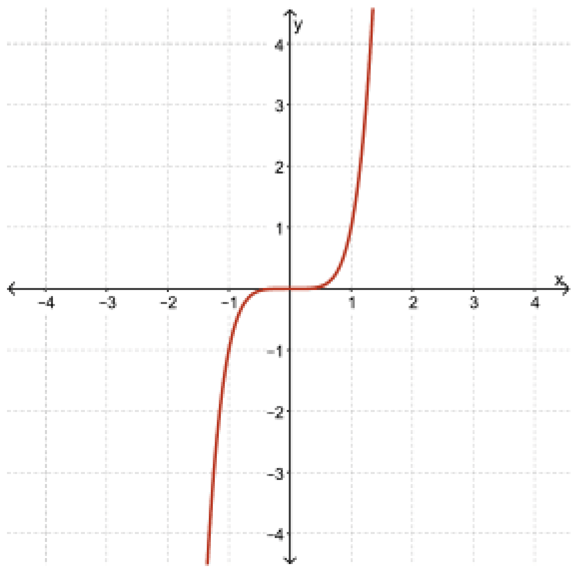
This quartic function's behaviour is similar to that of a quadratic function.
However, the curve lingers closer to the x-axis at the turning point (0,0) than it does in the quadratic,
and this gives the quartic function a broader, flatter appearance.

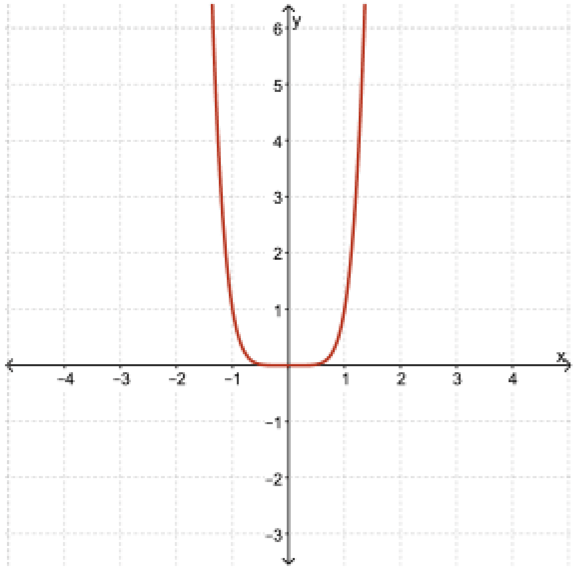
The graph is decreasing when $x<0$ and increasing when $x>0$.
The curve is much steeper than a quadratic when $x<-1$ or $x>1$, as can be seen by the y values in the table.
The graph of this quartic, like the quadratic $y=x^2$, is concave up.
Furthermore, the quartic is both an even degree function (x^4) and an even function.
That is, it is symmetrical about the y-axis.
Algebraically,

$$f(-x)=(-x)^4= x^4= f(x)$$

Next we are going to look to ***Higher Degree Power Functions***

The graphs of $y=x^5$ and $y=x^6$ have similar shapes to those of the parent cubic and quartic functions, respectively.





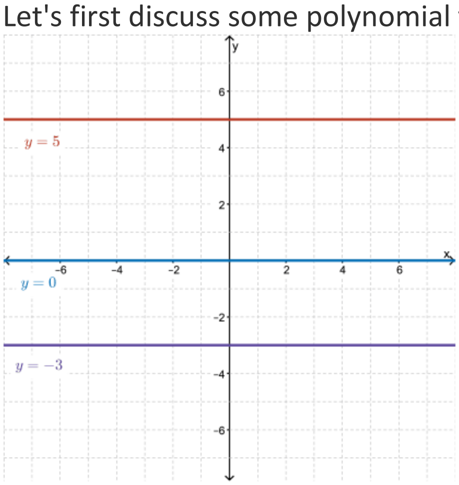
The curves will increase (or decrease) even more quickly when $x<-1$ or $x>1$, and linger more closely to the x-axis for $-1<x<1$.

In general, this behaviour repeats itself for $y=x^n$.
The function will have the shape of $y=x^3$ with a point of inflection if n is odd and it will have the shape of $y=x^2$ with a turning point if n is even.

Next we will review and explore the behaviour of the graphs of various polynomial functions ranging from degree 0 to degree 6.
The focus will be on determining the possible number of turning points, the possible number of x-intercepts, the end behaviour of the function's graph, and how the end behaviour is influenced by the leading coefficient of the function (the coefficient of the highest degree term).

Constant Functions

Let's first discuss some polynomial functions that are familiar to us.

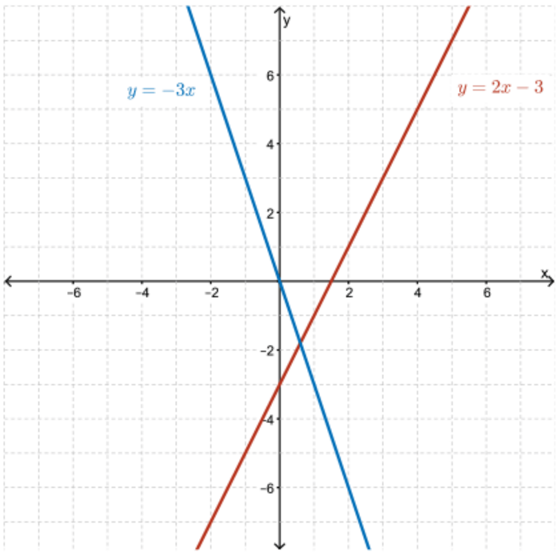


Polynomial functions of degree 0 are constant functions of the form $y=a, a\in\mathbb{R}$.
Their graphs are horizontal lines with a y-intercept at $(0,a)$.

The end behaviours of these graphs can be summarized with the statement, “as $x \rightarrow \pm\infty, y=a$.” Constant functions have no turning points and no zeros, except for the case of $y=0$, which is the equation of the x-axis and therefore has an infinite number of zeros.

Linear Functions

Polynomial functions of degree 1 are linear functions of the form $y=ax+b, a \neq 0$.
We know that a provides the slope of the line and b is the y-intercept.
We also know that lines do not have turning points.



If we consider the graphs of the lines $y=2x-3$and $y=-3x$, we see that a line will have one x-intercept if the line is not horizontal (as seen previously).

The end behaviours of a linear function are dependent on the slope of the line, which is given by the leading coefficient.

When $a>0$,

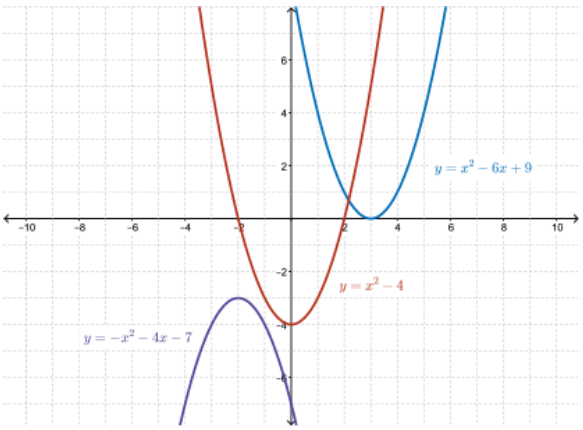
$$y \rightarrow -\infty \text{ as } x \rightarrow -\infty$$
$$y \rightarrow \infty \text{ as } x \rightarrow \infty$$

When $a<0$,

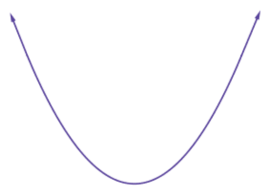
$$y \rightarrow \infty \text{ as } x \rightarrow -\infty$$
$$y \rightarrow -\infty \text{ as } x \rightarrow \infty$$

Quadratic Functions

The quadratic function, $y=ax^2+bx+c$, is a polynomial function of degree 2.
The graph of a quadratic function (a parabola) has one turning point, which is an absolute maximum or minimum point on the curve.



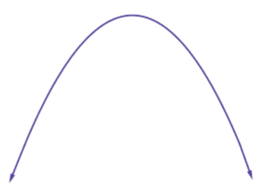
We can see from the graphs of the quadratic functions shown that a parabola may have no *x*-intercept, 1 *x*-intercept, or 2 *x*-intercepts depending on where the vertex is and the direction in which the parabola opens.
If the leading coefficient is positive ($a>0$), then the parabola will open upward.



When $a>0$,

$$y \to \infty \text{ as } x \to -\infty$$
$$y \to \infty \text{ as } x \to \infty$$

If the leading coefficient is negative ($a<0$), then the parabola will open downward.

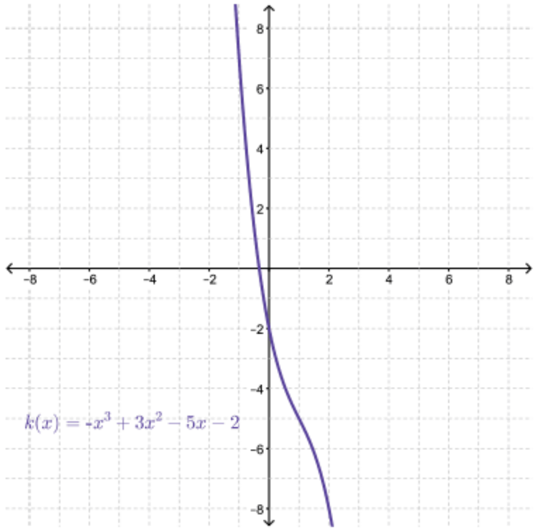


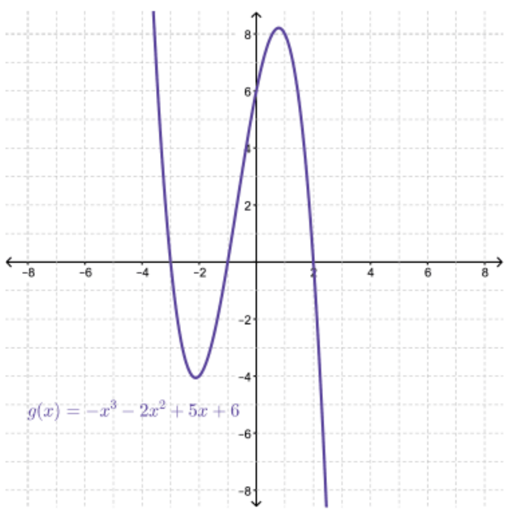
When $a<0$,

$$y \to -\infty \text{ as } x \to -\infty$$
$$y \to -\infty \text{ as } x \to \infty$$

The range of a quadratic function is dependent on the maximum or minimum value of the function and the direction of opening.

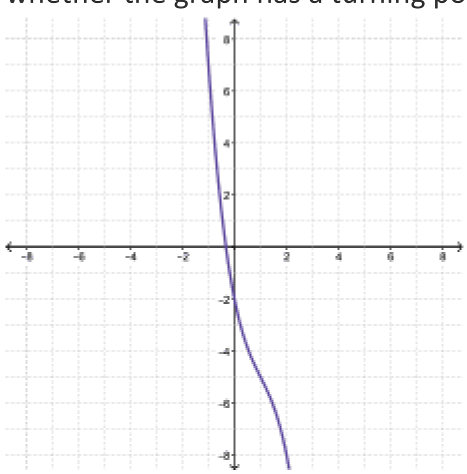
Cubic Functions: $y=ax^3+bx^2+cx+d, a\neq 0$
The graph of a cubic function has either no turning point or two turning points.

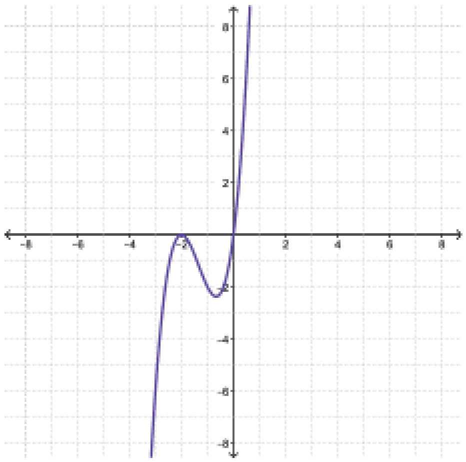


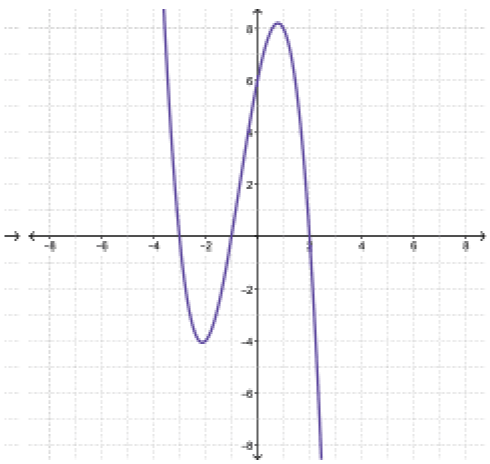


If the graph has no turning point, it will have a point of inflection similar to that of $y=x^3$.

It is possible for the graph of a cubic function to have 1, 2, or 3 x-intercepts depending on whether the graph has a turning point and the position of the graph on the axis.

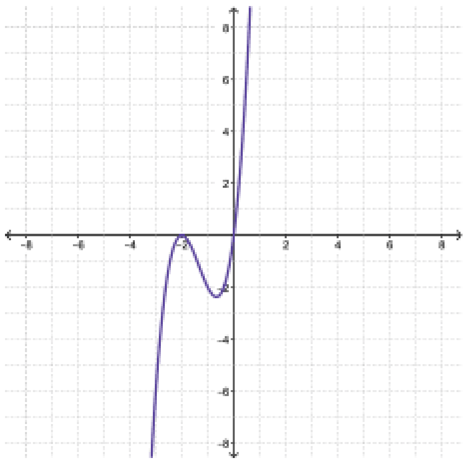






The graph of a cubic function has opposite end behaviours.
If the leading coefficient is positive, then

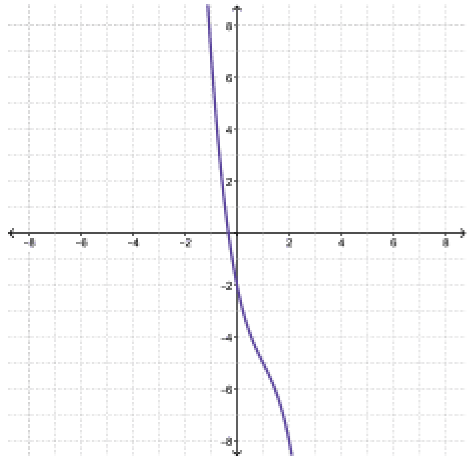
$$y \to -\infty \text{ as } x \to -\infty$$
$$y \to \infty \text{ as } x \to \infty$$



$$y=2x^3+8x^2+8x$$

The graph will begin in the third quadrant and end in the first quadrant.
If the leading coefficient is negative, then

$y \to \infty$ as $x \to -\infty$
$y \to -\infty$ as $x \to \infty$

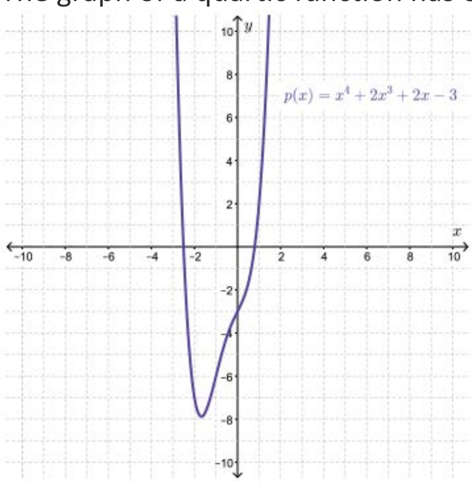


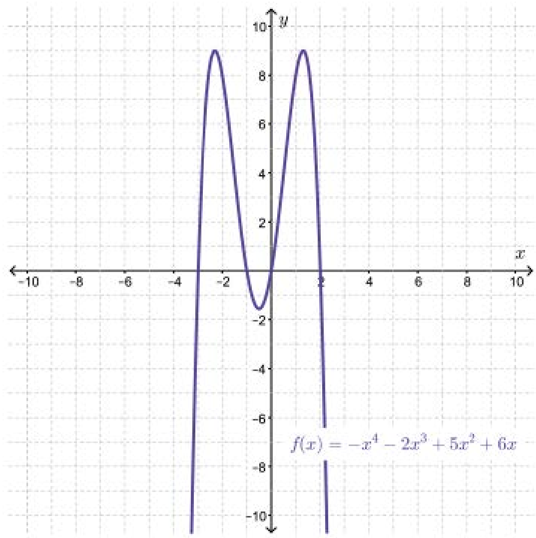
$y=-x^3+3x^2-5x-2$

The graph will start in the second quadrant and end in the fourth quadrant.

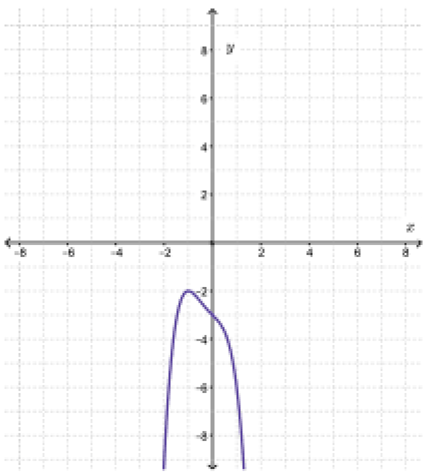
Quartic Functions: $y=ax^4+bx^3+cx^2+dx+e, a\neq 0$

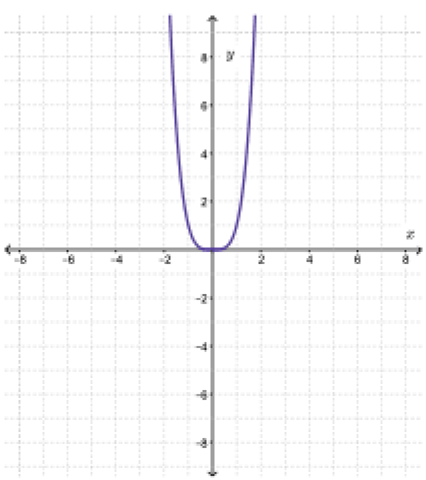
The graph of a quartic function has either one turning point or three turning points.

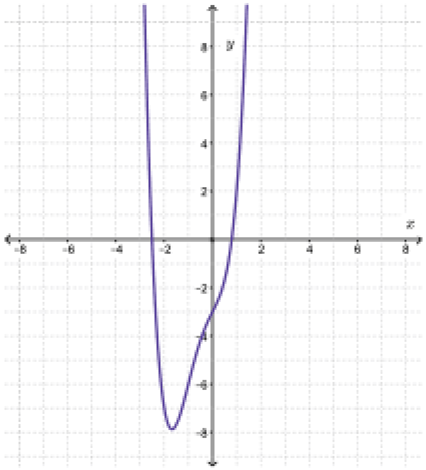


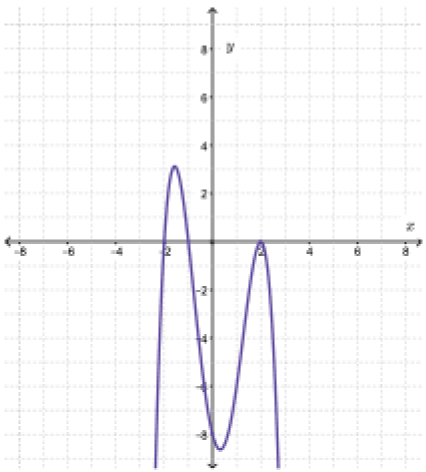


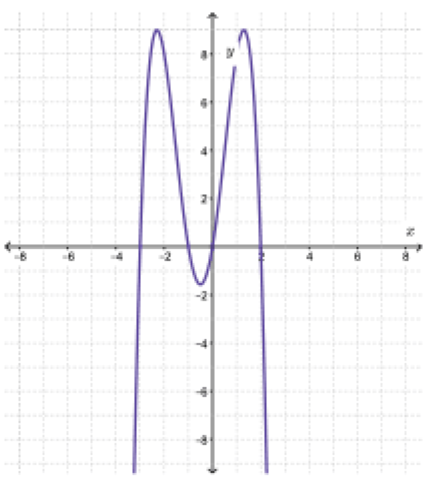
It is possible for the graph of a quartic function to have 0, 1, 2, 3, or 4 *x*-intercepts.





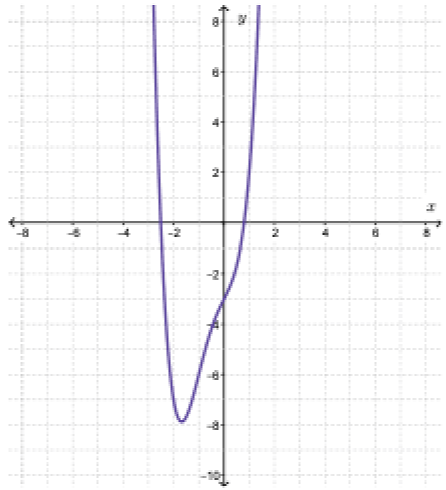






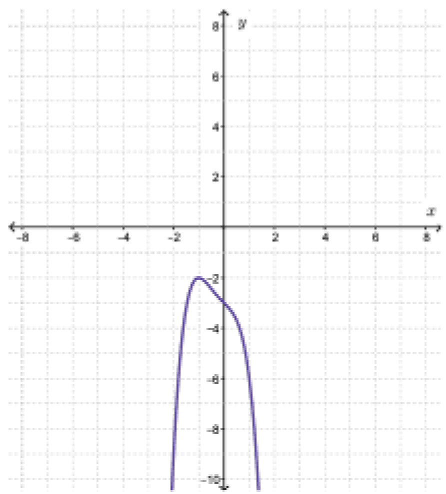
The graph of a quartic function has "same" end behaviours, similar to that of a quadratic function.
If the leading coefficient is positive, then

$$y \to \infty \text{ as } x \to \pm\infty$$



If the leading coefficient is negative, then

$y \to -\infty$ as $x \to \pm\infty$



This end behaviour is similar to that of a parabola.

This pattern continue for 5th and 6th degree polynomial functions.

Observations

The following are characteristics of the graphs of *nth* degree polynomial functions where ***n*** is **odd**:

- The graph will have end behaviours similar to that of a linear function.
  If the leading coefficient is positive, then $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$.
  If the leading coefficient is negative, then $y \to \infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.
- The graph will have an even number of turning points to a maximum of $n-1$ turning points.
  For example, a 5*th* degree polynomial function may have 0, 2, or 4 turning points.
- The graph will have at least one x-intercept to a maximum of n x-intercepts.

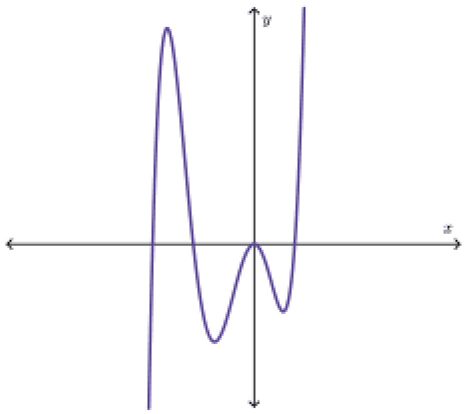
If we consider a 5*th* degree polynomial function, it must have at least 1 x-intercept and a maximum of 5 x-intercepts.

The following are characteristics of the graphs of *nth* degree polynomial functions where ***n*** is **even**:

- The graph will have end behaviours similar to that of a parabola, often described as same end behaviours.
  If the leading coefficient is positive, then $y \to \infty$ as $x \to \pm\infty$.
  If the leading coefficient is negative, then $y \to -\infty$ as $x \to \pm\infty$.
- The graph will have an odd number of turning points to a maximum of $n-1$ turning points.
  A 6*th* degree polynomial function will have a possible 1, 3, or 5 turning points.
- The graph will have an absolute maximum or minimum point due to the nature of the end behaviour.
  The range of these functions will depend on the absolute maximum or minimum value and the direction of the end behaviours.
- The graph will have a minimum of 0 to a maximum of n x-intercepts.
  For example, a 6*th* degree polynomial function will have a minimum of 0 x-intercepts and a maximum of 6 x-intercepts.

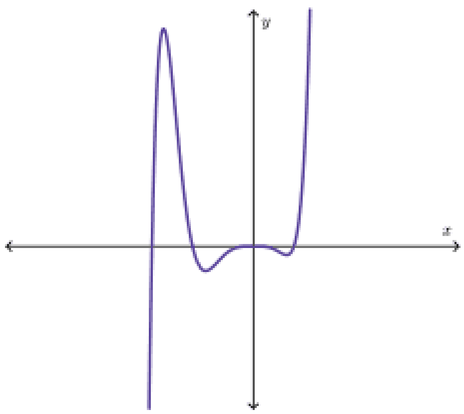
Example

Given the shape of a graph of the polynomial function, determine the least possible degree of the function and state the sign of the leading coefficient.



Note: It is possible for a higher odd degree polynomial function to have a similar shape.
The actual function is a 5*th* degree polynomial.

$$y=x^2(x-2)(x+3)(x+5)$$



Here is a graph of a 7*th* degree polynomial with a similar shape.

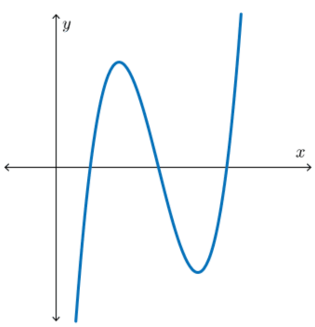
$$y=x^4(x-2)(x+3)(x+5)$$

Example

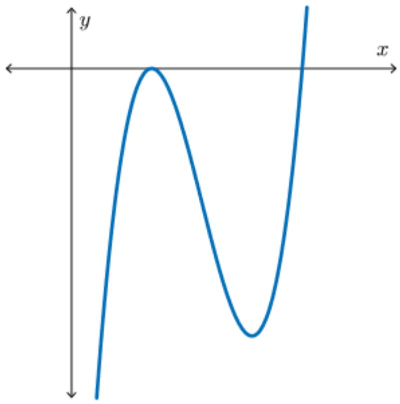
Sketch a possible graph of a 3*rd* degree polynomial function with a positive leading coefficient that has:
a. three *x*-intercepts
b. two *x*-intercepts
c. one *x*-intercept

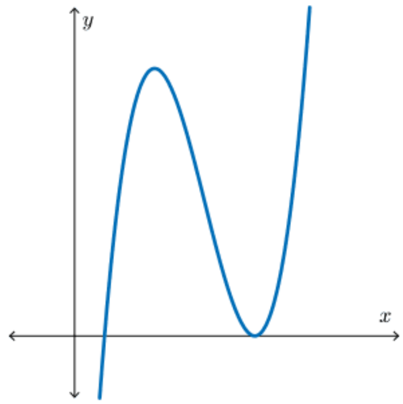
Solution

a. A cubic function can have zero or two turning points.
To create a function with three *x*-intercepts, we will need to work with two turning points.
A positive leading coefficient means that $y\to-\infty$ as $x\to-\infty$ and $y\to\infty$ as $x\to\infty$, similar to a line with a positive slope.
Start with the shape and then place the *x*-axis such that the curve crosses the *x*-axis three times.

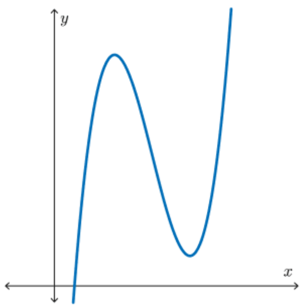


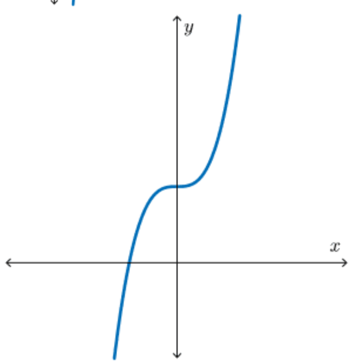
b. To create the cubic with two *x*-intercepts, we would need to move the axis such that a turning point is at one of the zeros.





c. To create the cubic with one *x*-intercept, we can simply move the *x*-axis again, or work with a cubic that has no turning point.





Families of Functions

A **family of functions** is a group of functions that share common characteristics.

A **family of *nth* degree polynomial functions** that share the same *x*-intercepts can be defined by

$$f(x)=k(x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n)$$

where k is the leading coefficient, $k\neq 0$, $k\in R$, and $a_1,a_2,a_3,\ldots,a_n$ are the zeros (*x*-intercepts) of the function.

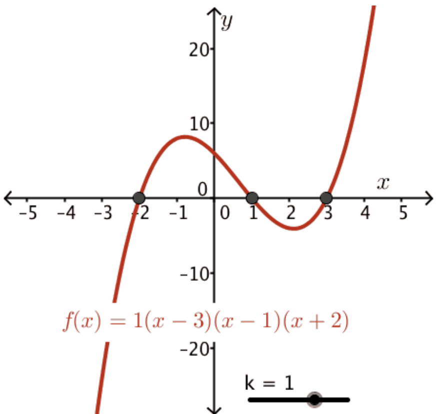
To express $y=2x^2+7x-15$ in this form, the leading coefficient, 2, can be divided out of the first factor, leaving a fractional value as a term in the factor.

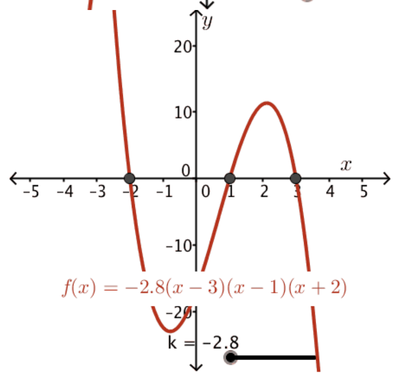
$$y=2(x-\tfrac{3}{2})(x+5)$$

Normally, we do not try to divide out a value that is not a factor of all the terms in the expression we are factoring, but it can be done.

Consider the family of cubic functions defined by

$$y=k(x-3)(x-1)(x+2)$$





The zeros are at $x=-2,1,$ and 3.
By varying the value of the parameter k, the shape of the function can be changed.
A vertical stretch or compression is applied to the graph.
When $k<0$, the function is reflected in the x-axis and the end behaviour changes.

Multiplicity

The quintic function $f(x)=k(x-s)(x-t)(x-u)(x-v)(x-w)$ has zeros $x=s,t,u,v,$ and w.
These zeros can be any real number and may sometimes be equal in value.
For example,

$$f(x)=k(x-s)(x-s)(x-t)(x-t)(x-t)=k\,(x-s)^2\,(x-t)^3$$

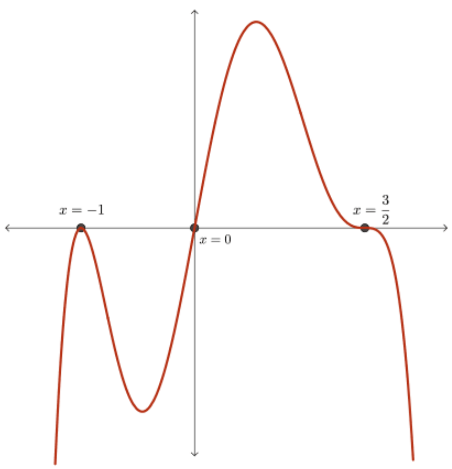
The zero $x=s$ has **multiplicity** 2 and is called a **double zero** or **zero of order** 2.
The zero $x=t$ has **multiplicity** 3 and is called a **triple zero** or **zero of order** 3.
The **multiplicity** (or **order**) of a zero is the number of times the zero is repeated. If a polynomial function in factored form has a factor $(x-a)$ repeated n times, then the zero, $x=a$, is said to have multiplicity n (or order n), for $n\in\mathbb{N}$.

How does the behaviour of the graph of a polynomial function change at its zero as the multiplicity of the zero changes?

Use the worksheet provided to investigate the behaviour of a polynomial function at its zeros. In particular, note the behaviour of the graph of the function at the zero when the order (or multiplicity) of the zero changes.

Sketching a Graph

$y=2x(x+1)^2(3-2x)^3 \quad \rightarrow \quad y=-16x(x+1)^2(x-3/2)^3$



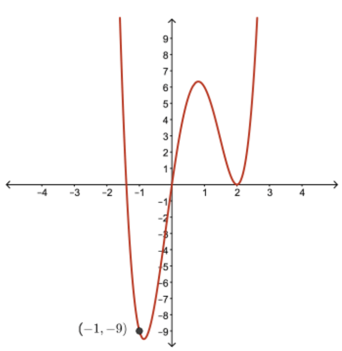
Key Properties

- A negative leading coefficient.
- The zeros are at $x=-1$ (multiplicity 2), $x=0$ (multiplicity 1), and $x=3/2$ (multiplicity 3).

Example

Determine the equation of the quartic function with zeros at $x=-7/5$, 0, and 2. The zero at $x=2$ is a turning point on the graph and the function passes through $(-1,-9)$.

Solution



$y=kx(5x+7)(x-2)2$

Substitute the point $(-1,-9)$ into the equation for x and y to determine the value of k.

$-9 = k(-1)(5(-1)+7)(-1-2)2$

$-9 = k(-1)(2)(-3)2$

$-9 = k(-18)$

$k = 12$

Therefore, $f(x)=12x(5x+7)(x-2)^2$.

Example

Determine the equation of the family of cubic functions with zeros at $x=-1,\ 3\pm2\sqrt{3}$.

Solution

The factors in the equation are $(x+1)$, $(x-(3+2\sqrt{3}))$, and $(x-(3-2\sqrt{3}))$.

$$y = k(x+1)(x-(3+2\sqrt{3}))(x-(3-2\sqrt{3}))$$

Therefore,

$$y=k(x+1)(x^2-6x-3), k\neq0, k\in R$$

Example

Determine the equation of the polynomial function with two real zeros $x=-5\pm\sqrt{2}$, two non-real zeros $x=1\pm i\sqrt{3}$, and a y-intercept of -46.

Solution

The equation of the polynomial function is $y=-12(x^2+10x+23)(x^2-2x+4)$.

Investigating Finite Differences of Polynomial Functions

A line has a constant rate of change, in other words, a constant slope.

Consider the table of values for the linear function $y=3x-2$.

x	$y=3x-2$	1*st* Difference Δy
-2	-8	$-5-(-8)=3$
-1	-5	$-2-(-5)=3$
0	-2	$1-(-2)=3$
1	1	$4-1=3$
2	4	

The x values in this table are in increments of 1, that is $\Delta x=1$.

To calculate the first differences, denoted by Δy, we will compute the changes or differences in the y values of the function.
The first differences are equal, with with a constant value of 3. Therefore, $\Delta y=3$ and $\Delta x=1$.
So, the slope of the line $\Delta y/\Delta x$ is 3.

For a quadratic function, the rate of change of y as x changes is variable.
The parabola does not have a constant slope.

x	$y=-x2+3x+1$	1*st* Difference Δy	2*nd* Difference $\Delta^2 y$
−2	−9	−3−(−9)=6	4−6=−2
−1	−3	1−(−3)=4	2−4=−2
0	1	3−1=2	0−2=−2
1	3	3−3=0	
2	3		

With quadratic functions, the first differences, Δy, are variable.
But the difference in the first differences, that is, the second differences, denoted by $\Delta^2 y$, are constant.

Consider the following finite difference tables for four cubic functions.

x	$y=-3x^3+2x^2$	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	32	-27	22	-18
-1	5	-5	4	-18
0	0	-1	-14	
1	-1	-15		
2	-16			

x	$y=2x^3+x$	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	-18	15	-12	12
-1	-3	3	0	12
0	0	3	12	
1	3	15		
2	18			

Consider the following finite difference table for a quartic functions.

x	$y = 2x^4 - x^2 - 1$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	27	-27	26	-24	48
-1	0	-1	2	24	48
0	-1	1	26	72	
1	0	27	98		
2	27	125			
3	152				

From our observation, it appears that the *nth* differences are constant for a polynomial of degree *n*; the value of the constant difference is given by $\mathbf{\Delta^n y = a \times n!}$ where *a* is the leading coefficient of the function and *n* is the degree of the polynomial.
This last observation works only if the change in *x* in the table of values is 1, that is $\Delta x=1$, which is the case for all the tables presented previously.

In actual fact, if $f(x)$ is an *nth* degree polynomial function, then

$$(\Delta^n y)/(\Delta x)^n = a \times n!$$

where $\Delta^n y$ is the *nth* constant difference and Δx is the difference in *x*-values.
So, the *nth* differences of the polynomial are given by

$$\mathbf{\Delta^n y = a \times n! \times (\Delta x)^n}$$

When $\Delta x=1$, then $\mathbf{\Delta^n y = a \times n!}$

Finite differences provide a means for identifying polynomial functions from a table of values. Knowing the relationship between the value of the constant difference and the leading coefficient of the function can also be useful.

Example

Determine the equation of the polynomial function that models the data found in the table.

x	y
-2	-29
-1	-26
0	-5
1	16
2	19
3	-14

Solution

First, determine the degree of the polynomial function represented by the data by considering finite differences.

x	y	Δy	$\Delta 2y$	$\Delta 3y$

−2	−29	3	18	−18
−1	−26	21	0	−18
0	−5	21	−18	−18
1	16	3	−36	
2	19	−33		
3	−14			

Since the third differences are constant, the polynomial function is a cubic.
We can now find the equation using the general cubic function, $y=ax^3+bx^2+cx+d$, and determining the values of a, b, c, and d.
We can find the value of the leading coefficient, a, by using our constant difference formula.
From the table, $\Delta x=1$. The constant difference $\Delta^3 y=-18$.
$-18=a\times 3!\times(1)3=a\times 3\times 2\times 1$
Solving gives $a=-18/6=-3$.
We can now use 3 of the points from the table to create 3 equations and solve for the values of b, c, and d.
A good point to start with is the y-intercept $(0,-5)$ which will provide the value of d.

$$-5=-3(0)3+b(0)2+c(0)+d$$

Therefore, $d=-5$.
So far, we have $y=-3x3+bx2+cx-5$.
Using $(1,16)$ gives

$$16=-3(1)3+b(1)2+c(1)-5$$
$$24=b+c$$

Using $(-1,-26)$ gives

$$-26=-3(-1)3+b(-1)2+c(-1)-5$$
$$-24=b-c$$

This gives a system of two equations with two unknowns.

$$24=b+c$$
$$-24=b-c$$

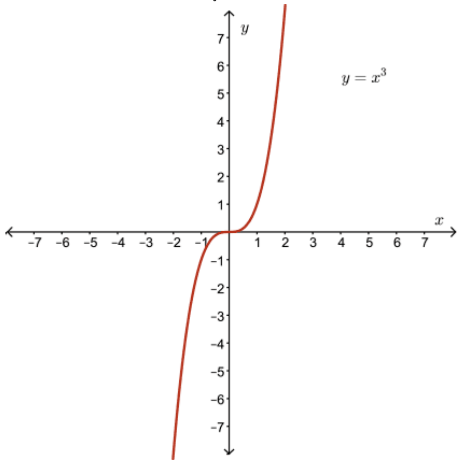
Adding the two equations, we obtain $2b=0$ and so, $b=0$.
Subtracting the two equations, we obtain $2c=48$ and so, $c=24$.
Therefore, $y=-3x^3+24x-5$ is the equation of the function.

Next we will investigate the symmetry of higher degree polynomial functions and we will generalize a rule that will assist us in recognizing even and odd symmetry, when it occurs in a polynomial function.

Symmetry in Polynomials

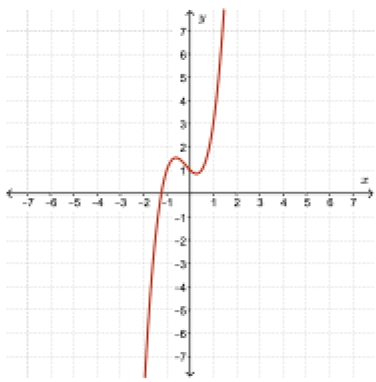
Recall, a function can be even, odd, or neither depending on its symmetry.
If a function is symmetric about the y-axis, then the function is an **even function** and $f(-x)=f(x)$.

If a function is symmetric about the origin, that is $f(-x)=-f(x)$, then it is an **odd function**.

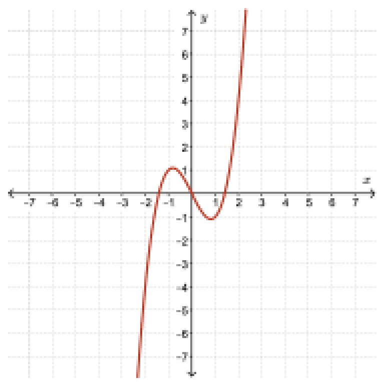


The cubic function, $y=x^3$, an odd degree polynomial function, is an odd function.
That is, the function is symmetric about the origin.
Consider the following cubic functions and their graphs.

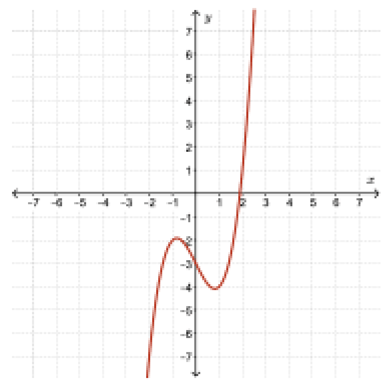
(1) $y=2x^3+x^2-x+1$



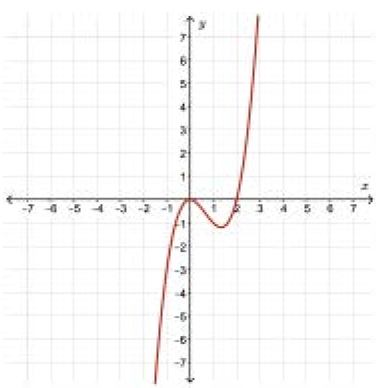
(2) $y=x^3-2x$



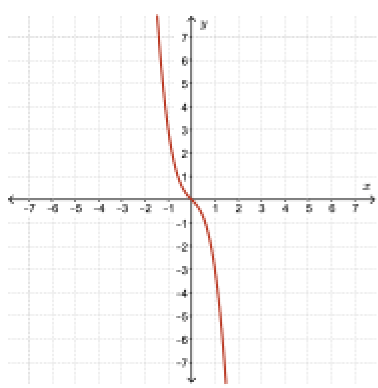
(3) $y=x^3-2x-3$



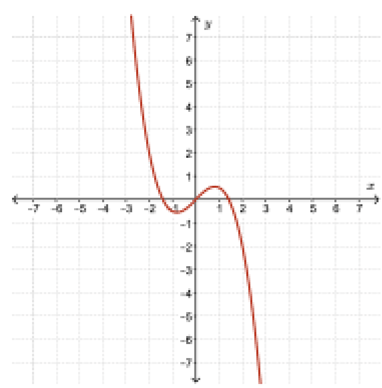
(4) $y=x^3-2x^2$



(5) $y=-2x^3-x$



(6) $y=-12x^3+x$



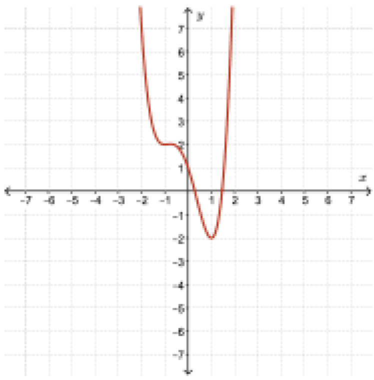
Notice the functions (2), (5), and (6). What do these functions have in common?
They have a term of degree 3 and a term of degree 1; that is, an *x*3 term and an *x* term.
The other 3 functions defined by

$$y=2x^3+x^2-x+1 \quad y=x^3-2x-3 \quad y=x^3-2x^2$$

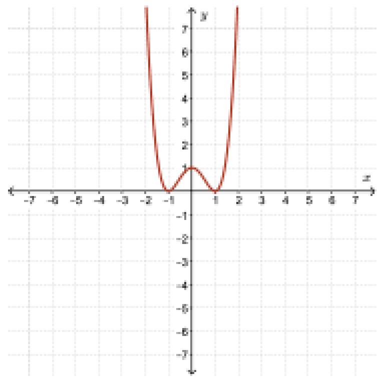
are neither even nor odd.
Along with an odd degree term, x^3, these functions also have terms of even degree; that is, an x^2 term and/or a constant term of degree 0.
It appears an odd polynomial must have only odd degree terms.

Similarly, it should follow that even polynomial functions would have only even degree terms.
To illustrate the results graphically, we compare the following functions:

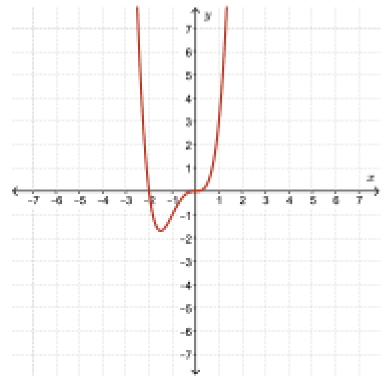
(1) $y=x^4+x^3-2x^2-3x+1$



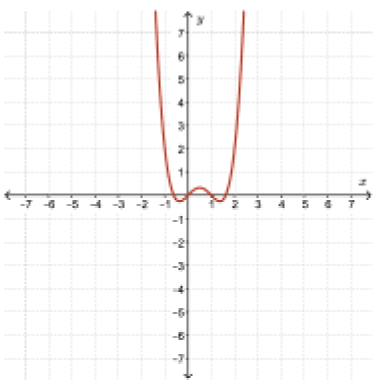
(2) $y=x^4-2x^2+1$



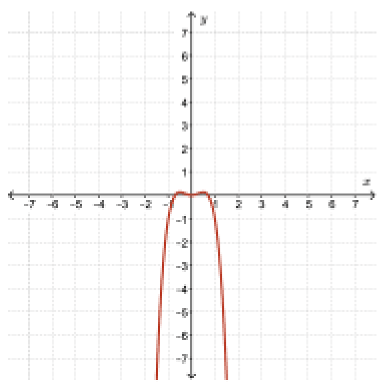
(3) $y=x^4+2x^3$



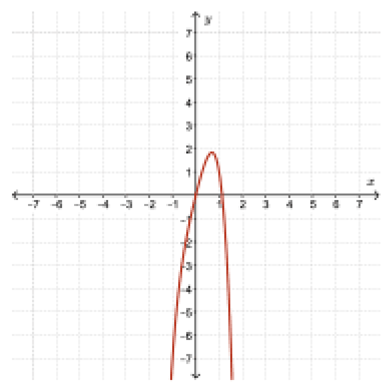
(4) $y=x^4-2x^3+x$



(5) $y=-2x^4+x^2$



(6) $y=-2x^4-x^2+4x$



In Summary

- A polynomial function is an **even function** if and only if each of the terms of the function is of an even degree.
- A polynomial function is an **odd function** if and only if each of the terms of the function is of an odd degree.
- The graphs of even degree polynomial functions will never have odd symmetry.
- The graphs of odd degree polynomial functions will never have even symmetry.

Note: The polynomial function $f(x)=0$ is the one exception to the above set of rules. This function is both an even function (symmetrical about the y-axis) and an odd function (symmetrical about the origin).