Special Relativity

Grade 12 Physics

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What Everyone Knows

$$\boxed{E = mc^2}$$

Most people know about *this* equation, but what does it actually mean?

Frame of Reference

A Quick Review

Think of a **frame of reference** (or **reference frame**, or just **frame**) as a hypothetical mobile "laboratory" an observer uses to make measurements of physical quantities (e.g. mass, lengths, time). At a minimum, it includes:

- Some rulers (coordinate system) to measure lengths
- A clock to measure the passage of time
- A scale to compare forces
- A balance to measure masses

Many high-school textbooks refer to the frame of reference as a "coordinate system". This incomplete definition can make it difficult to understand relativity.

Frame of Reference

A Quick Review

- We assume that the "hypothetical laboratory" is *perfect*
  - The hypothetical "instruments" have zero errors
  - There are perfect instruments to measure anything you want
- What matters is the *motion* (at rest, uniform motion, acceleration etc) of your laboratory, and how it affects the measurement that you make
- "From the point of view of..."

Inertial Frame of Reference

An **inertial frame of reference** (or **rest frame**) is one that is moving in uniform motion (i.e. constant velocity, zero acceleration)

- In an inertial frame, Newton's first and second laws of motion are valid
- Since all uniform motion are treated the same way, you may consider any inertial frames of reference to be *at rest*

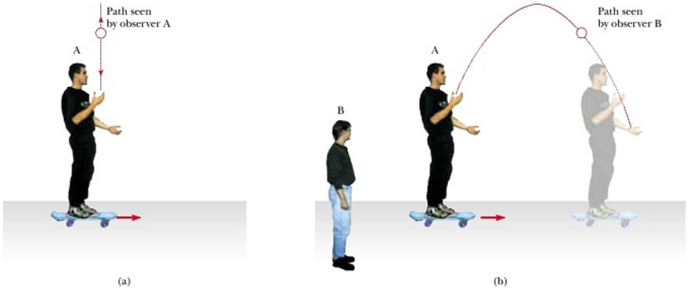
The Principle of Relativity

All laws of motion must apply equally in all inertial frames of reference.

Inertial Frame of Reference

A Quick Review

- Observer A is moving uniformly with the skateboard
    - It is perfectly valid for A to think that he is at rest, but B is moving
- Observer B is standing on the side of the road
    - It is perfectly valid for B to think that he is at rest, but A is moving
- A & B see different motion, but they agree on the equations that govern the motion



Newtonian (Classical) Relativity

When studying classical mechanics, we made some assumptions that are obvious:

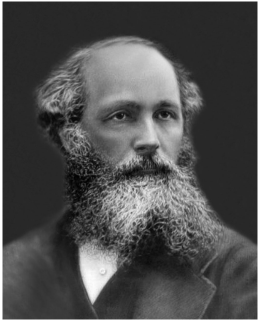
- $1\,\mathrm{m}$ is $1\,\mathrm{m}$ no matter where you are and how you are moving
- $1\,\mathrm{s}$ is $1\,\mathrm{m}$ no matter where you are and how you are moving
- Measurements of space and time are independent of motion of the observer

If space and time are absolute, then *all* velocities are relative

- Measured velocities depend on the motion of the observer
- In classical mechanics we use **Galilean rule for velocity addition** to calculate relative velocity:

$$\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{BC}$$

New Physics: Maxwell's Equations



James Maxwell

- For centuries, experimental results would always agree with this "Newtonian" framework. . .
- In the middle of 19th Century, instruments became accurate enough to measure behaviours that were slightly different from prediction
- **Maxwell's equations** in 1861 and 1862 are classical laws of electrodynamics that explains the relationship between
  - Electricity
  - Electric Circuits
  - Magnetism
  - Optics

Speed of Light

The solution Maxwell's equations[1] show that disturbances in electric and magnetic fields propagate though space as **electromagnetic waves**. In a vacuum, EM waves travel with a speed ("speed of light"):

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299\,792\,458\,\mathrm{m/s}$$

- But *what* is the speed of light relative to?
- All waves that physicists knew have to travel in some medium
  - Ocean waves $\longrightarrow$ water
  - Sound waves $\longrightarrow$ air, solids, liquids

[1] Solving Maxwell's equations is an exercise in university that only math, physics and some engineering students have the pleasure of doing

Peculiar Features of Maxwell's Equations

- No mention of the *medium* in which EM waves travels
- When applying *Galilean transformation* (classical equation for relative motion), the law for magnetism breaks down: magnetic field lines appear to have beginnings/ends
- On first glance:
  - In *some* inertial frames of reference, Maxwell's equations are simple and elegant; but
  - In *other* inertial frames that are supposedly equal, they are ugly and complex
- Some physicists theorized that there may be a *preferred* inertial frame
  - This violate the principle of relativity

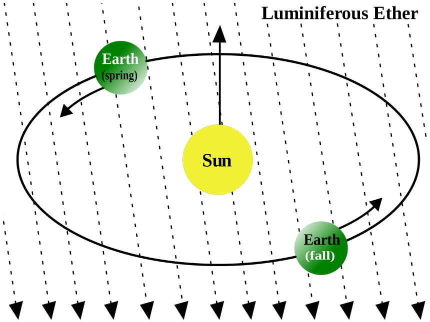
The Illusive Aether

- Maxwell's hypothesis: the speed of light c is relative to a hypothetical **luminiferous aether**, or just **ether**
- In order for aether to exist, it must have some fantastic (as in "a fantasy", too good to be true) properties:
  - *All* space is filled with aether
  - Massless
  - Zero viscosity
  - Non-dispersive
  - Incompressible
  - Continuous at a very small (sub-atomic) scale

Great! But how do we find it?

The Michelson-Morley Experiment

If aether exists, at different times of the year, Earth should have different relative velocities with respect to it:

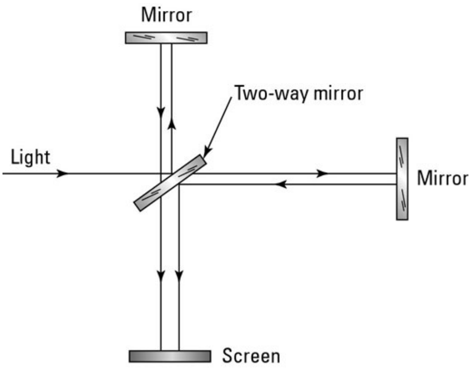


And aether should cause the light to either speed up, or slow down.

The Michelson-Morley Experiment

- German-American physicist Albert Michelson, an expert in wave interference, designed an experiment to measure the effects of ether on the speed of light
- The experiment is ingenious but very difficult
- Used an “interferometer” designed by Michelson
- Later collaborated with American chemist Edward Morley
- Together performed the experiments over several years

The Michelson Interferometer



- A beam of light is split into two using a two-way mirror
- The two beams are reflected and finally arriving at the screen
- The path are the same length; if the *speed* of the light changes, we should see an interference pattern.
- **Except none were ever found!**

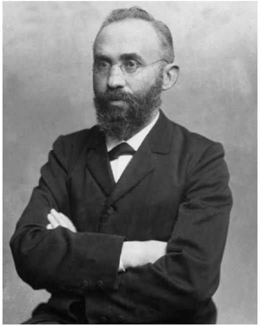
Michelson-Morley Experiment

What To Do with “Null Result”

The Michelson-Morley experiment failed to detect the presence of ether, even after many refinements. What does this mean?

- Majority view:
  - **The experiment was flawed!**
  - Keep improving the experiment (or design a better experiment) and the ether will eventually be found
- Minority view:
  - **The hypothesis is wrong!**
  - The experiment showed it for what it is: ether cannot be found because it’s a bad hypothesis
- A few physicists: The must be *another explanation*

Hendrik Lorentz: Length Contraction



Hendrik Antoon Lorentz

- Considered the Michelson-Morley experiment to be significant
- Hypothesis: objects travelling in the direction of ether contracts in length, nullifying the experimental results
- Lorentz Factor:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

- No known physical phenomenon can cause anything to contract like this!
- Lorentz was on to something, but his thinking was wrong

Strange Behavior in Absolute Space Time

French mathematician Henri Poincaré also hypothesized that ether affects the flow of time the direction of flow. The equation is similar to the hypothesis by Lorentz:

$$\boxed{t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

But *no known physical phenomenon* can alter the flow of time!

Both Poincaré and Lorentz depended their hypothesis on

- Absolute time and space
- Existence of aether

Making Maxwell’s Equations Work

Albert Einstein in 1905, Age 26

Albert Einstein

- Einstein was working as a patent clerk in Switzerland
  - Believed in the principle of relativity, and therefore
  - Rejected the concept of a preferred frame of reference
- Viewed the failure of the Michelson-Morley experiment to find the flow of ether as proof that it does not exist
- In order to make Maxwell’s equations to work again, Einstein revisited two most fundamental concepts in physics: *space* and *time*

Special Relativity

Published in the journal *Annalen der Physik* on September 26, 1905 in the article *On the Electrodynamics of Moving Bodies*

- Submitted on June 30, 1905 and passed for publication
- Mentions 5 other scientists: Newton, Maxwell, Hertz, Doppler and Lorentz
- No references to any publications, and
- Provided no experimental results
- Initially ignored by most physicists, until Max Planck took interests
- It describes a “special case” without effects of gravity
- Later published the theory of “general relativity” in 1915 which included the effects of gravity (much more complicated)

Postulates of Special Relativity

The Principle of Relativity

All laws of physics must apply equally in all inertial frames of reference.

- Reaffirms the principle in which *all* physics is based on
- Extend the principle to include Maxwell's equations

The Principle of Invariant Light Speed

As measured in any inertial frame of reference, light always propagates in empty space with a definite velocity c, independent of the motion of the emitting body.

- Reaffirms the results from Michelson-Morley experiment
- The speed of light predicted in Maxwell's equations must apply to all inertial frames of reference

What About Ether?

As for the hypothetical aether, forget it, it just does not exist!

Postulates of Special Relativity

The two postulates are unremarkable by themselves, but Einstein was able to show that when combined, the consequences are profound

What's so Special About Special Relativity?

Classical (Newtonian) relativity:

- Space and time are absolute
- Therefore the speed of light must be relative
- The speed of light from any light source should depend on the motion of the source, and whether it is moving towards/away from me!

Einstein says: Impossible! This violates Maxwell's equations and the principle of relativity.

What's so Special About Special Relativity?

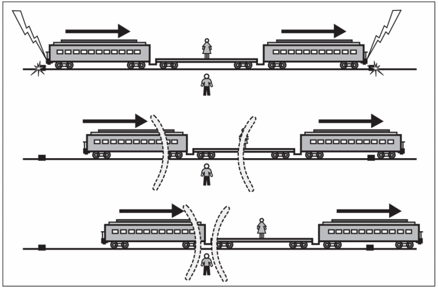
Einstein's special relativity:

- Speed of light is absolute
- Space and time must be relative
- We must modify our traditional concepts:
    - Measurement of space (our ruler in the x-, y- and z-directions)
    - Measurement of time (our clock)
    - Concept of simultaneity (whether two events happens at the same time)

Relativity of Simultaneity

A Thought Experiment

Lightning bolt strikes the ends of a high-speed moving train.



- Two *independent* events: lightning striking the front, and lightning striking the back of the train
- The man on the ground sees the lightning bolt striking at the same time
- The woman on the train sees the lightning bolt on the front first

Relativity of Simultaneity

A Thought Experiment

From the man's perspective:

- He is stationary; the train and the woman are moving
- When the lightnings strike, he is at equal distance from the front and the back of the train
- The flash from the two lightning bolts arrive at his eyes at the same time

Therefore, his conclusions are:

- The two lightnings must have happened at the same time
- The woman in the train made the wrong observation: she only thinks that the lightning struck the front first because she's moving towards the light from the front

Relativity of Simultaneity

A Thought Experiment

Now, from the woman's perspective:

- *She* and the train are stationary; the man and the world are moving
- When the lightnings strike, she is equal distance from the two ends of the train
- The flash from the front arrive first, then the back

Therefore, her conclusions are:

- Lightnings must have struck the front first
- The man on the road made the wrong observation: he *thinks* that the lightning struck at the same time because he's moving towards the light from the back

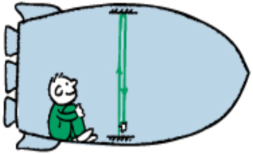
Relativity of Simultaneity

A Thought Experiment

- The two observers disagree on the result, but
  - Neither person is wrong
  - Neither person is misinformed
- Both observers are valid *inertial* frames of reference
- This means that simultaneity depends on your motion

Events that are simultaneous in one inertial frame of reference are not simultaneous in another.

Relativity of Time: Time Dilation

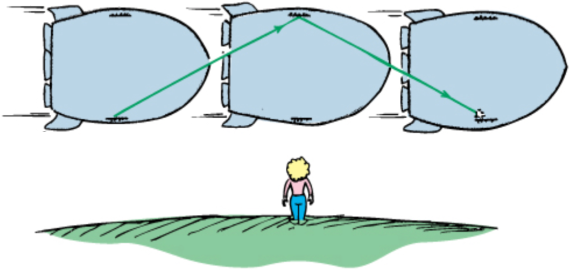


I'm on a spaceship travelling in deep space, and I shine a light from A to B. The distance between A and B is:

$$|AB| = c\Delta t_0$$

I know the speed of light c, and I know how long it took for the light pulse to reach B. (The reason I used Δt_0 will be obvious later.)

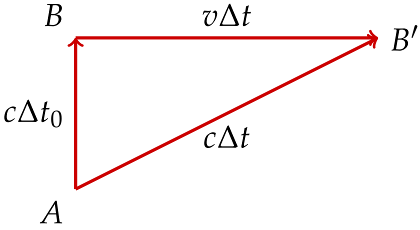
Relativity of Time: Time Dilation



You are on a small planet watching my spaceship go past you at speed v. You would see that same beam of light travel from A to B' instead.

Relativity of Time: Time Dilation

We can relate the time interval observed by me on the spaceship (Δt_0) and your time interval on the small planet (Δt) using simple geometry and the Pythagorean theorem:



$$c^2\Delta t^2 = v^2\Delta t^2 + c^2\Delta t_0^2$$

$$\left(c^2 - v^2\right)\Delta t^2 = c^2\Delta t_0^2$$

$$\left(1 - \frac{v^2}{c^2}\right)\Delta t^2 = \Delta t_0^2$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

The **relativity of time**: the passage of time as measured by two observers in two different inertial references are different!

Relativity of Time: Time Dilation

The passage of time as measured by two observers in two different inertial frames of reference are related by:

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

Variable	**Symbol**	**SI Unit**
Proper time (ordinary time)	Δt_0	s
Dilated time (expanded time)	Δt	s
Speed	v	m/s
Speed of light	c	m/s

Relativity of Time: Time Dilation

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

- **Proper time** (Δt_0) is the time interval measured by a person *at rest* relative to the events
- **Dilated time** (Δt) is the time interval measured by a moving observer in another inertial frame of reference.
- Since $\sqrt{1 - \left(\frac{v}{c}\right)^2} < 1$, $\Delta t > \Delta t_0$.

Example Problem

A Simple Problem That is Not So Simple

Example 1a: A rocket speeds past an asteroid at $0.800c$. If an observer in the rocket sees $10.0\,\mathrm{s}$ pass on her watch, how long would that time interval be as seen by an observer on the asteroid?

Example 1b: A rocket speeds past an asteroid at $0.800c$. If an observer in the *asteroid* sees $10.0\,\mathrm{s}$ pass on his watch, how long would that time interval be as seen by an observer on the *rocket*?

How can that be?

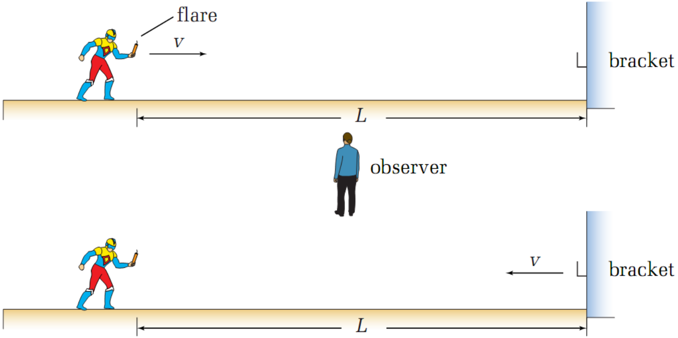
Relativity of Space: Length Contraction

Another Thought Example

Captain Quick is a comic book hero who can run at nearly the speed of light. In his hand, he is carrying a flare with a lit fuse set to explode in $1.50\,\mu s$. The flare must be placed into its bracket before this happens. The distance (L) between the flare and the bracket is $402\,m$.

Relativity of Space: Length Contraction

Another Example



Both Captain Quick and the observers on the side of the road agree that he is moving at $v = 2.00 \times 10^8$ m/s relative to the bracket.

Relativity of Space: Length Contraction

Another Example

If Captain Quick runs at 2.00×10^{8} m/s, according to classical mechanics, he does not make it in time, and the flare will explode:

$$\Delta t = \frac{L}{v} = \frac{402\,\mathrm{m}}{2.00 \times 10^{8}\,\mathrm{m/s}} = 2.01 \times 10^{-6}\,\mathrm{s} = 2.01\,\mu\mathrm{s}$$

But according to relativistic mechanics, he makes it just in time...

Relativity of Space: Length Contraction

Another Example

To the stationary observers, the time on the flare is slowed:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1.50 \times 10^{-6}\,\mathrm{s}}{\sqrt{1-\left(\frac{2.00}{3.00}\right)^2}} = \frac{1.50 \times 10^{-6}\,\mathrm{s}}{0.745} = 2.01 \times 10^{-6}\,\mathrm{s}$$

The stationary observers see a passage of time of $\Delta t = 2.01\,\mu\mathrm{s}$, but Captain Quick, who is in the same reference frame as the flare, experiences a passage of time of $\Delta t_0 = 1.50\,\mu\mathrm{s}$, precisely the time for the flare to explode.

Relativity of Space: Length Contraction

Another Example

- So, if Captain Quick sees only $\Delta t_0 = 1.50\,\mu\text{s}$, then how far did he travel?
- He isn't travelling any faster, so he only other possibility is that *the distance actually got shorter* (in his frame of reference).
- How much did the distance contract?

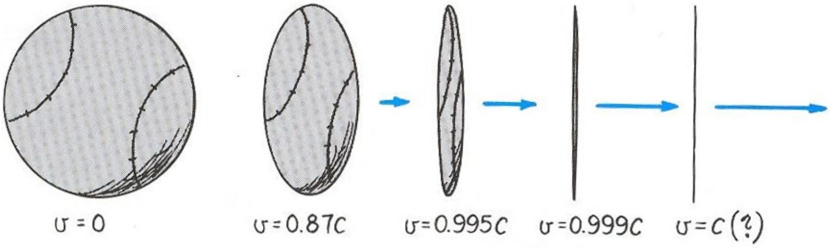
$$\boxed{L = L_0\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

For this example:

$$L = L_0\sqrt{1 - \left(\frac{v}{c}\right)^2} = (402\,\text{m}) \cdot \sqrt{1 - \left(\frac{2}{3}\right)^2} = 300\,\text{m}$$

Relativity of Space: Length Contraction

Length contraction only occurs in the direction of motion



Lorentz Factor

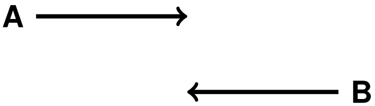
The **Lorentz factor** γ is a short-hand for writing length contraction, time dilation and relativistic mass:

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

Then time dilation and length contraction can be written simply as:

$$\boxed{\Delta t = \gamma \Delta t_o} \qquad \boxed{L = \frac{L_o}{\gamma}}$$

Let’s Summarize



If A and B are moving uniformly with respect to one another (doesn’t matter if they’re moving towards, or away from each other)

- They do not agree whether any events happens at the same time or not
- Each sees the other’s clock running slow
- Each sees the other “contracted” in length along the direction of motion

Relative Velocities

The Galilean velocity addition rule violates the postulates of relativity, so Einstein replaces it with **Einstein velocity addition rule**:

$$\boxed{v_{AC} = \frac{v_{AB} + v_{BC}}{1 + \frac{v_{AB} v_{BC}}{c^2}}}$$

We can see that when velocities are slow ($v_{AB} \leq c$, and $v_{BC} \leq c$) we recover the Galilean velocity addition rule.

Example Problem

Example 2: A spacecraft passes Earth at a speed of $2.00 \times 10^{8}\ \mathrm{m/s}$. If observers on Earth measure the length of the spacecraft to be $554\ \mathrm{m}$, how long would it be according to its passengers?

But What About. . .

We've studied the most important principles in special relativity already, but what about:

$$\boxed{E = mc^2}$$

We seem to be no closer to learning about it!

Relativistic Momentum

Recall that momentum is mass times velocity, and velocity is displacement over time:

$$\mathbf{p} = m\mathbf{v} = m\frac{\Delta\mathbf{x}}{\Delta t}$$

But since $\Delta\mathbf{x}$ and Δt depend on the motion, the expression needs to be revised:

$$\mathbf{p} = m_0\frac{\Delta\mathbf{x}}{\Delta t_0} = \frac{m_0\Delta\mathbf{x}}{\Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{m_0\mathbf{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

We have not changed the *definition* of momentum, only the underlying assumptions about space and time.

Relativistic Mass

The relativistic momentum expression shows that mass is also relativistic:

$$\boxed{m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma m_0}$$

Variable	Symbol	SI Unit
Relativistic mass (measured in moving frame)	m	kg
Rest mass (measured in stationary frame)	m_0	kg
Speed	v	m/s
Speed of light	c	m/s
Lorentz Factor	γ	(no unit)

The rest mass does not change, but as an object moves, an observer will note that it behaves as if it is more massive. As $v \to c$, $m \to \infty$.

Example Problem

Example 3: An electron has a rest mass of 9.11×10^{-31} kg. In a detector, it behaves as if it has a mass of 12.55×10^{-31} kg. How fast is that electron moving relative to the detector?

Force and Work

Recall that force is the rate of change in momentum:

$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{d\mathbf{p}}{dt}$$

and work is force times displacement:

$$W = \mathbf{F} \cdot \Delta \mathbf{d} = \int \mathbf{F} \cdot d\mathbf{x}$$

We can substitute the impulse expression for force, then substitute expression for relativistic momentum, and after some calculus. . .

Work and Kinetic Energy

$$W = \frac{m_0 c^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - m_o c^2 = \Delta K$$

We know from the work-kinetic energy theorem that the work W done by a force $\mathbf{F}$ is equal to the change in kinetic energy, ΔK

Relativistic Energy

From the work-kinetic energy theorem, we obtain the expression for kinetic energy:

$$\boxed{K = mc^2 - m_0c^2}$$

Variable	Symbol	SI Unit
Kinetic energy of an object	K	J
Relativistic mass (measured in moving frame)	m	kg
Rest mass (measured in stationary frame)	m_0	kg
Speed of light	c	m/s

Relativistic Energy

What This All Means

$$\boxed{K = mc^2 - m_0c^2 = (\gamma - 1)m_0c^2}$$

The minimal energy that any object has, regardless of it's motion (or lack of) is its **rest energy**:

$$E_0 = m_0c^2$$

The **total energy** of an object has is

$$E_T = mc^2 = \gamma m_0c^2$$

The difference between total energy and rest energy is the kinetic energy:

$$K = E_T - E_0$$

Mass-Energy Equivalence

What This All Means

$$\boxed{E = mc^2}$$

- Whenever there is a change of energy, there is also a change of mass
- "Conservation of mass" and "conservation of energy" must be combined into a single concept of **conservation of mass-energy**
- Mass-energy equivalence doesn't just mean that mass can be converted into energy and vice versa (although this is true), but rather, one can be converted into the other *because they are fundamentally equivalent*

A General Rule for Scientific Discoveries

When a physicist “discovers” a new physical law, it is not enough that the new law explains what could not be explained before, but it also has to be consistent with what is already explained.

In the case of kinetic energy, that means that Einstein’s relativistic equation must agree with Newtonian (classical) mechanics for $v \leq c$

Kinetic Energy: Classical vs. Relativistic

Relativistic:

$$K = (\gamma - 1)m_0c^2$$

Newtonian:

$$K = \frac{1}{2}mv^2$$

Are they really that different? If we do a series expansion of the square-root term, we get:

$$K = m_0c^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right) - m_0c^2 \approx \frac{1}{2}mv^2 + \cdots$$

when v is small compared to c, we get back the $\frac{1}{2}mv^2$ expression that we know so well!

Comparing Classical and Relativistic Energy

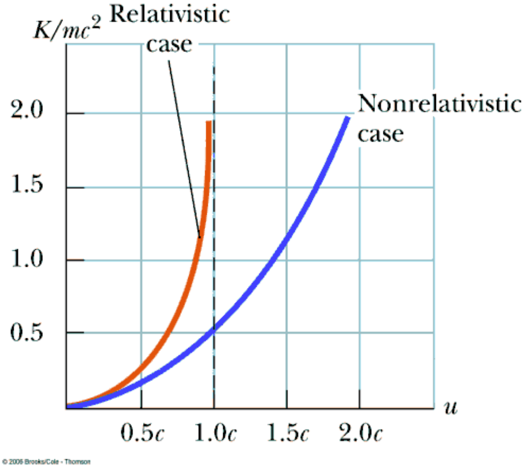
In classical mechanics:

$$K = \frac{1}{2}mv^2$$

In relativistic mechanics:

$$K = mc^2 - m_o c^2$$

Generally we see relativistic effects at around $v = 0.3c$



Example Problem

Example 4: A rocket car with a mass of 2.00×10^3 kg is accelerated from rest to 1.00×10^8 m/s. Calculate its kinetic energy:

1. Using the classical equation
2. Using the relativistic equation