Lesson 4: Unit 3- *Polynomial Equations and Inequalities (2)*

In this lesson we will look at a variety of methods used to factor higher degree polynomials.

When factoring a polynomial, $P(x)$, of degree 3 or greater, consider the following:

- First, look for a common factor.
- Next, check to see if the polynomial can be factored by grouping. Cubic polynomials, $ax3+bx2+cx+d$, can be factored by grouping if (and only if) $ad=bc$.
- The factor theorem can be used to determine the first factor of the polynomial. If $P(a)=0$, then $(x-a)$ is a factor. Long and synthetic division, or the “have and need” method, can be used to find the corresponding factor.
- Polynomials of the form $ax^{2n}+bx^n+c$ are essentially quadratic in form. Factor these polynomials as you would a quadratic (it may help to make the substitution $t=x^n$ to obtain at^2+bt+c).
- To factor the sum or difference of cubes, the following two formulas can be applied:

$$A^3+B^3= (A+B)(A^2-AB+B^2)$$
$$A^3-B^3= (A-B)(A^2+AB+B^2)$$

Note: Not all polynomials can be factored using these methods.

Factor Using the Factor Theorem

Example 1

Factor $x^4-2x^3-5x^2+6x$.

Solution

First look for a *common factor*. In this case is x.

$x^4-2x^3-5x^2+6x = x(x^3-2x^2-5x+6)$

Next, use the rational root theorem and factor theorem to determine a factor of the cubic factor. Test values include ±1,±2,±3,±6.

$$P(1)=(1)3-2(1)2-5(1)+6=0$$

Therefore, $(x-1)$ is a factor.

$x^4-2x^3-5x^2+6x = x(x-1)(x^2+ ?x -6)$

Then Use long division, synthetic division, or the “have and need” method to determine the quadratic factor for the next step.

$x^4 -2x^3-5x^2\}+6x = x(x-1)(x^2-x-6)$

The final step will be to factor the quadratic, if possible.

$x^4 -2x^3-5x^2\}+6x = x(x-1)(x+2)(x-3)$.

Factor by Grouping

Example 2

Factor $3x^3-2x^2+6x-4$.

Solution

Working in pairs, factor out the greatest common factor for each pair of terms.
Factor $(3x-2)$, which is common to both groupings.
Factoring is complete, since $x2+2$ cannot be factored further.

$$\underbrace{3x^3-2x^2}+\underbrace{6x-4}$$
$$=x^2(3x-2)+2(3x-2)$$
$$=(3x-2)(x^2+2)$$

This method can certainly save time, but it will not work for every polynomial.

In order to use the method of grouping to factor a polynomial, the remaining factor from each grouping must be the same after the initial factoring has been completed.
Consider

$$\underbrace{x^3-x^2}_{x^2(x-1)}-\underbrace{7x+14}_{-7(x-2)}$$

Here the remaining factors $(x-1)$ and $(x-2)$ are not the same, so the next common factor step cannot be carried out.
This cubic cannot be factored by the grouping method.

A cubic polynomial of the form $ax3+bx2+cx+d$ can be factored by grouping if $ad = bc$.

In our first example, $3x^3-2x^2+6x-4$,
$ad=(3)(-4)=-12$ and $bc=(-2)(6)=-12$, so $ad=bc$.
But in our second example, $x^3-x^2-7x+14$,
$ad=(1)(14)=14$ and $bc=(-1)(-7)=7$, so $ad\neq bc$.
This rule makes it easier to determine if the grouping method for factoring will work, otherwise you must carry out the first step by factoring each group of terms to see, if in fact, you can continue.

Example 3

Factor $x^5+4x^4-2x^3-8x^2+x+4$.
Solution
Start by pairing the terms and factoring out the greatest common factor from each pair.

$$\begin{aligned}&\underbrace{x^5+4x^4}-\underbrace{2x^3-8x^2}+\underbrace{x+4}\\&=x^4(x+4)-2x^2(x+4)+(x+4)\\&=x^4(x+4)-2x^2(x+4)+1(x+4)\\&=(x+4)(x^4-2x^2+1)\end{aligned}$$

When the second common factoring step is complete we are left with a quartic factor (x^4-2x^2+1).
Usually we need to apply the factor theorem to factor quartic polynomials, but in this case, the quartic is "quadratic in form",

$$(x^2)^2-2(x^2)+1$$

so we can factor it much like a quadratic.

$$\begin{aligned}&\underbrace{x^5+4x^4}-\underbrace{2x^3-8x^2}+\underbrace{x+4}\\&=x^4(x+4)-2x^2(x+4)+(x+4)\\&=x^4(x+4)-2x^2(x+4)+1(x+4)\\&=(x+4)(x^4-2x^2+1)\\&=(x+4)(x^2-1)(x^2-1)\end{aligned}$$

Factoring Sum and Difference of Cubes

Difference of Squares

If a quadratic polynomial is of the form $a^2x^2-b^2$, then

$$a^2x^2-b^2=(ax+b)(ax-b)$$

e.g.,

$$4x^2-25=(2x+5)(2x-5)$$

Can we factor the "difference of perfect cubes" using a similar rule?

Factoring Sum and Difference of Cubes

In general, an expression containing the sum of two perfect cubes can be factored using

$A^3+B^3=(A+B)(A^2-AB+B^2)$

An expression containing the difference of two perfect cubes can be factored using

$A^3-B^3=(A-B)(A^2+AB+B^2)$

Example 4

Factor $64x^3-1=(4x)^3-(1)^3$.

Solution

Note that $A=4x$ and $B=1$, so

$$64x^3-1=(4x-1)((4x)^2+(4x)(1)+(1)^2)$$

$$64x^3-1=(4x-1)(16x^2+4x+1)$$

Example 5

Factor $x4-10x3+25x2-25$ given there is no value $x=a$, $a\in R$ such that $P(a)=0$.

Solution

One approach is as follows (similar to factoring by grouping).

First, factor x^2 from the first three terms of the quartic, producing a quadratic factor.
We may recognize the three terms of the quadratic as terms of a perfect square with the form similar to

$$x^2\pm 2ax+a^2=(x\pm a)^2$$

By expressing this quadratic in factored form, we end up with a "difference of perfect squares" for the quartic.

$$\begin{aligned}
&x^4-10x^3+25x^2-25\\
&=x^2(x^2-10x+25)-25\\
&=x^2(x-5)^2-25\\
&=[x(x-5)-5][x(x-5)+5]\\
&=(x^2-5x-5)(x^2-5x+5)
\end{aligned}$$

Solving Higher Degree Equations

Example 1

Solve $2x^3+8x^2-3x-12=0$.

Solution

The first step in solving polynomial equations of degree 2 or higher is to ensure that all terms are to one side of the equation, leaving 0 on the other side.
Once factored, we can then rely on the premise that the product of factors is equal to zero when one of the factors is zero.

$$2x^3+8x^2-3x-12=0$$

Since (2)(−12)=(8)(−3), this cubic can be factored by grouping.
$2x^2(x+4) - 3(x+4) = 0$
$(x+4)(2x^2-3) = 0$
Thus, $x+4=0$ or $2x^2-3=0$.

Therefore, the roots are $x=-4$ and $\pm\frac{\sqrt{6}}{2}$.

Example 2

Solve $2x^2(x-5)=12x-48$.

Solution

First we need to expand and collect all terms to one side of the equation. Remove any common factors.
Let $P(x)=x^3-5x^2-6x+24$.
Use the rational root theorem and the factor theorem on the remaining factor, $P(x)$.
Test values for the factor theorem include the following:
±1,±2,±3,±4,±6,±8,±12,±24.
$P(2)=0$. Therefore, $(x-2)$ is a factor.
The corresponding quadratic factor can be found using long division, synthetic division, or the "have and need" method.

$$2(x-2)(x^2-3x-12)=0$$

Therefore, the 3 roots to the equation are $x=2$, $x = \frac{3\pm\sqrt{57}}{2}$.

If we consider the related cubic function of this polynomial,

$$y=2x^3-10x^2-12x+48$$

the x-intercepts of the function would be obtained by setting $y=0$ and solving the equation.
Hence, the zeros of the function would also be $x=2$, $x\approx5.27$, and $x\approx-2.27$.

The leading coefficient of the cubic is positive, so

$$x\to\infty, y\to\infty$$

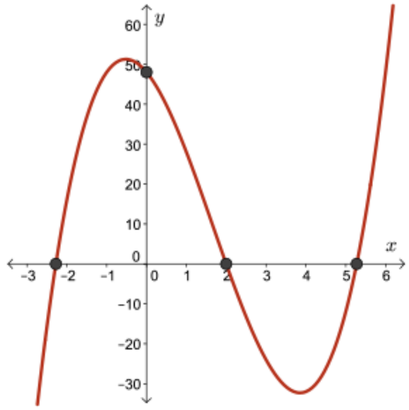
and

$$x\to-\infty, y\to-\infty$$

The y-intercept is 48 (the constant term of the polynomial), which is found by setting $x=0$.

The three zeros are of multiplicity 1. Therefore, the curve will pass directly through the axis at each zero.

The exact location of the turning points of the curve can be determined using calculus techniques, but this is beyond the scope of this course.



Example 3

Sketch a possible graph of $f(x)=-x^3-4x+5$ using the zeros, y-intercept, and end behaviour of the function.

Solution

First, we find the zeros. Set $f(x)=0$, and then we determine the roots of the cubic equation.

$$0=-x^3-4x+5$$

It may be easy to factor this cubic by first factoring the leading coefficient, −1, from each term.

$$0=-(x^3+4x-5)$$

$f(1)=0$. Therefore, $(x-1)$ is a factor of $f(x)$.

We can then use long division, synthetic division, or the "have and need" method to find the corresponding factor.

$$0=-(x-1)(x^2+x+5)$$

The roots are x= 1 and other two non-real roots.

$f(x)= -(x-1)(x^2+x+5)$

Graphically, this means that there is only one x-intercept at (1,0). The y-intercept is (0,5). This is a cubic function with negative leading coefficient, −1, hence

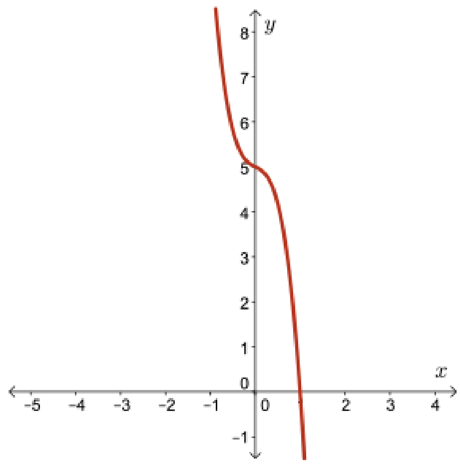
$$x\to\infty, y\to-\infty$$

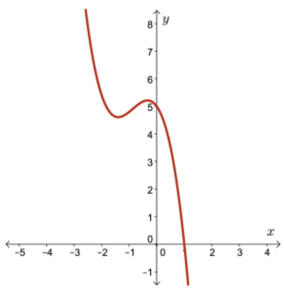
and

$$x\to-\infty, y\to\infty$$

With only this information, we are unsure of whether the cubic has two turning points or a point of inflection (no turning point).

The following two graphs satisfy the information we have for the function, but they may not accurately depict the graph of this specific function. More information is needed.





Summary

- If a polynomial equation, $P(x)=0$, is factorable, the roots of the equation are determined by factoring the polynomial, setting each factor to zero, and then solving each of these equations individually.
- An *nth* degree polynomial equation has at most *n* distinct roots.
- The solutions to a polynomial equation $P(x)=0$ are the zeros of the corresponding polynomial function $y=P(x)$.
- The x-intercepts of the graph of $y=P(x)$ are the real zeros of the polynomial function.
- If a polynomial equation of degree 3 or greater cannot be factored, the roots of the equation must be determined using technology or higher-level mathematical procedures.

Solving Polynomial Inequalities

Example

Solve $2x^2-x^3 \geq 2-x$, $x \in R$.

Algebraic Solution

We begin by arranging the terms on one side of the inequality.
$-x^3 + 2x^2 + x - 2 \geq 0$
Then factor,
$(x-2)(x-1)(x+1) \geq 0$.

The case approach takes too much time, so we need another method.

- We begin by identifying the zero values for the factors: $x=\pm1$ and $x=2$.

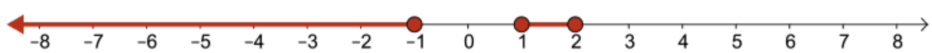
We will use these values to identify intervals where changes in the sign of the factors may occur.

- We set up a table as shown:

$x=-1$ $\quad$ $x=1$ $\quad$ $x=2$

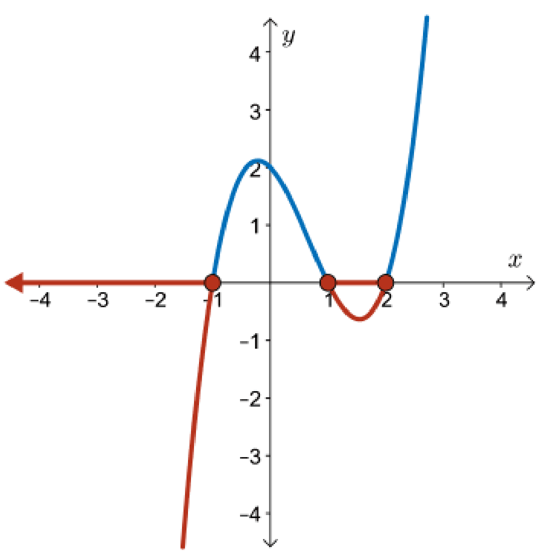
	$x<-1$	$-1<x<1$	$1<x<2$	$x>2$
$x-2$	$-$	$-$	$-$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$(x-2)(x-1)(x+1)$	$-$	$+$	$-$	$+$

- We choose a test value from within each interval, substitute this value into each factor, and identify if the factor is positive or negative in that interval.
  For the interval $x<-1$, we use $x=-2$ as a test value.
  For the interval $-1<x<1$, we use $x=0$ as a test value.
  For the interval $1<x<2$, we use $x=1.5$ as a test value.
  For the interval $x>2$, we use $x=3$ as a test value.
- We determine the sign of $(x-2)(x-1)(x+1)$ by determining the sign of the product of the factors.
- Therefore, the solution is $\{x \mid x\le -1 \text{ or } 1\le x\le 2,\ x\in R\}$.



- $x\in(-\infty,-1]\cup[1,2],\ x\in R$

Graphical Solution



We begin our sketch in the third quadrant, passing through each of the zeros and ending in the first quadrant.

From the graph of $f(x)=(x-2)(x-1)(x+1)$, we see that $f(x) \le 0$ when

$$x \le -1 \text{ or } 1 \le x \le 2$$

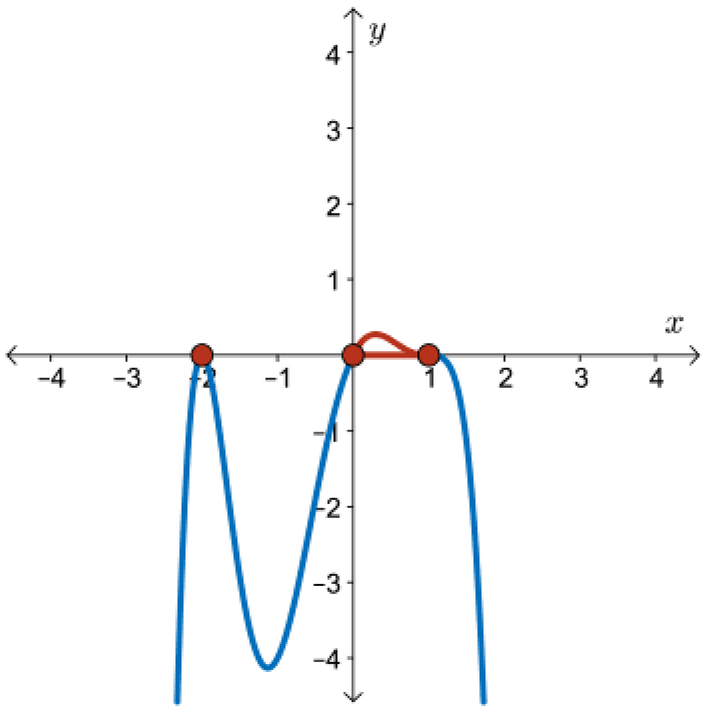
Therefore, the solution is

$$\{x \mid x \le -1 \text{ or } 1 \le x \le 2, x \in R\}$$

Example

Solve $-0.5x(x+2)^2(x-1)^3 \ge 0$.

Graphical Solution



$y=-0.5x(x+2)^2(x-1)^3$

Using the graph, we see that $y \ge 0$ (that is, the curve is on or above the x-axis) when

$$x=-2 \text{ or } 0 \le x \le 1$$

Therefore,

$$-0.5x(x+2)^2(x-1)^3 \ge 0$$

when

$$x \in \{-2\} \cup [0,1], x \in R$$

Algebraic Solution

	$x<-2$	$x=-2$	$-2<x<0$	$x=0$	$0<x<1$	$x=1$	$x>1$
$-0.5x$	$+$		$+$		$-$		$-$
$(x+2)^2$	$+$		$+$		$+$		$+$
$(x-1)^3$	$-$		$-$		$-$		$+$
$-0.5x(x+2)^2(x-1)^3$	$-$		$-$		$+$		$-$

We determine the sign of the −0.5x(x+2)2(x−1)3 by determining the product of the factors in each interval.

Therefore, the solution to −0.5x(x+2)2(x−1)3≥0 is {x|x=−2 or 0≤x≤1, x∈R}

Example

A cubic function, $y=f(x)$, has a turning point at (−2,0), an x-intercept at x=1, and $f(-1)=4$. Determine all values of x such that $0 < f(x) < 8$.

Solution

To determine the equation, we can use the zeros and an additional point on the curve.

- There is a zero at $x=-2$ of multiplicity two. Therefore, $(x+2)^2$ is a factor of the function.
- There is a zero at $x=1$ of multiplicity one. Therefore, $(x-1)$ is a factor of the function.

Let $f(x)=a(x-1)(x+2)^2$.

Since $f(-1)=4$,

$$44a=a(-1-1)(-1+2)^2 = a(-2)(1)^2 = -2$$

Therefore,

$$f(x)=-2(x-1)(x+2)^2$$

To determine all values of x such that $0<f(x)<8$, solve $0<-2(x-1)(x+2)^2 < 8$. This translates into solving the inequalities,

$$0<-2(x-1)(x+2)^2 \text{ and } -2(x-1)(x+2)^2 < 8$$

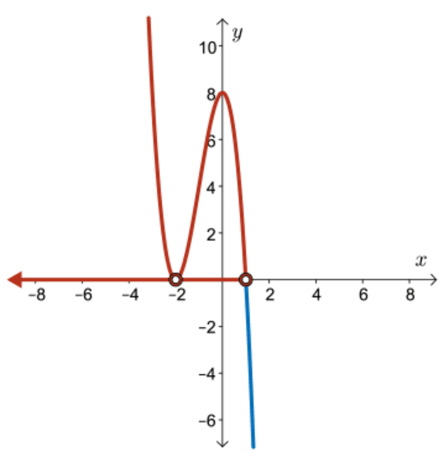
individually.

We solve $-2(x-1)(x+2)^2 > 0$ graphically.

Using the zeros and end behaviour, we can sketch the graph of $f(x)=-2(x-1)(x+2)^2$.

$f(x) > 0$ when $x < -2$ or $-2 < x < 1$

Therefore, {x|x<1,x≠−2, x∈R}.

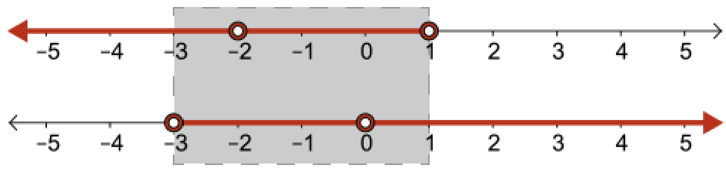


Now, solve $-2(x-1)(x+2)^2 < 8$ algebraically.
$-2(x-1)(x+2)^2 < 8 \leftrightarrow x^2(x\text{-}3) > 0$
We can solve this inequality using an interval table.

	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
x^2	$+$		$+$		$+$
$x+3$	$-$		$+$		$+$
$x^2(x+3)$	$-$		$+$		$+$

Therefore, $x^2(x+3)>0$ when $-3<x<0$ or $x>0$.

The solution to $0<f(x)<8$ is all $x \in \mathbb{R}$ such that $x<1, x\neq -2$ and $x>-3, x\neq 0$.
To determine which values of x satisfy both sets of conditions, it may be helpful to look at the solution of each inequality on a number line, as shown:



Since both sets of conditions must be satisfied, the grey area highlighting the overlap in the two sets of conditions helps us identify the solution.
Therefore, the solution is $\{x \mid -3<x<1, x\neq -2, x\neq 0, x\in \mathbb{R}\}$.

Summary

- When multiplying or dividing both sides of the inequality by a negative value, the inequality condition must be reversed.
- When solving a factorable polynomial inequality of degree 2 or greater, arrange the terms to one side of the inequality condition, with 0 on the other side. Factor the polynomial expression to identify when the expression is equal to 0.
- An interval table can be created using the zeros and factors of the polynomial to help identify the intervals when the polynomial expression is positive or negative in value, thus determining the solution to the inequality.
- An alternate approach is to graph the corresponding polynomial function, using the factored expression, and identify the intervals when the graph is above (positive) or below (negative) the x-axis.
- Number lines can also be used to provide a visual aid when solving inequalities.