Lesson 11: Unit 6 – Trigonometric Functions (2)

To sketch the reciprocal trigonometric functions, we could use a table of values approach as we did with primary trigonometric ratios in a previous module.

However, for the reciprocal function *y*=1/*x*, we provided a second approach(see Rational Functions).

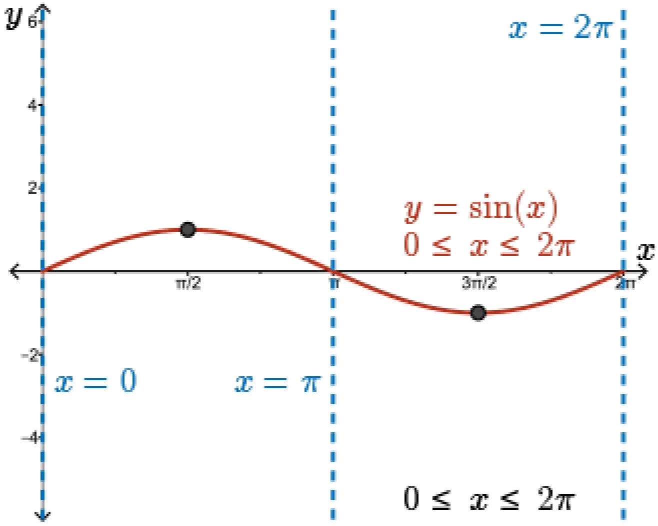
We will review this second approach and we will generalize it in this module to sketch the three reciprocal trigonometric functions.

Primary Trigonometric Functions	Reciprocal Trigonometric Functions
$y = \sin(x)$	$y = \csc(x) = \dfrac{1}{\sin(x)}$
$y = \cos(x)$	$y = \sec(x) = \dfrac{1}{\cos(x)}$
$y = \tan(x)$	$y = \cot(x) = \dfrac{1}{\tan(x)}$

General steps for sketching *f*(*x*)=1*g*(*x*)
1. Sketch the function *y*=*g*(*x*).
2. Identify the values of *x* where *g*(*x*)=1 or *g*(*x*)=−1. At these points *f*(*x*)=*g*(*x*). That is, these points are on both *f*(*x*) and *g*(*x*). These points are called **fixed points** or **static points**.
3. Identify the *x*-intercepts of *g*(*x*). At these points, *f*(*x*) is undefined. There will be vertical asymptotes for these values of *x*.
4. If required, determine what happens to the reciprocal function as *x* approaches the vertical asymptotes from the left and from the right.
5. If required, determine the end behaviour of *f*(*x*).

Sketch the graph of $y=\csc(x)$, $0 \le x \le 2\pi$.

1. Since $\csc(x) = 1/\sin(x)$, sketch $y = \sin(x)$, $0 \le x \le 2\pi$



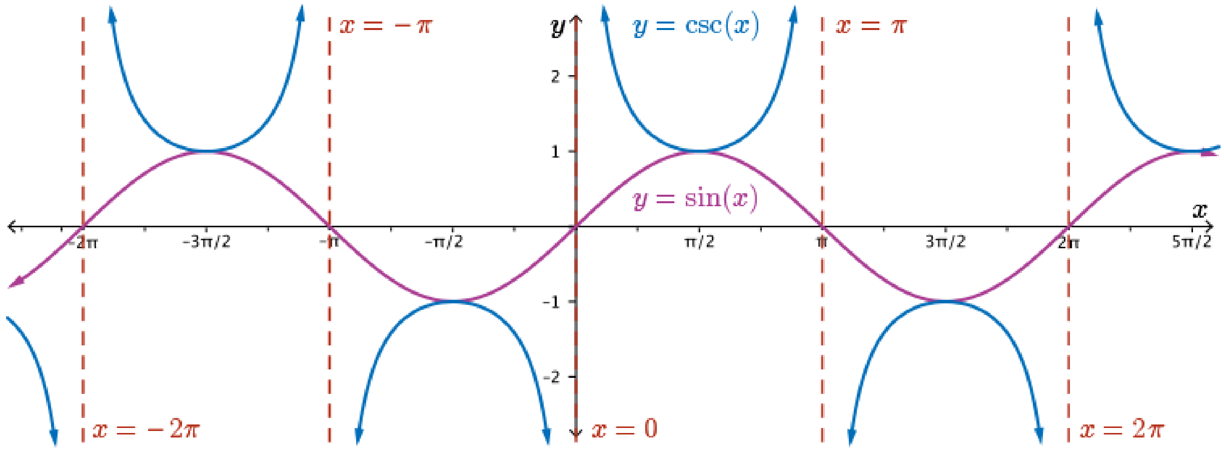
2. Identify the points which are on the graphs of both $y = \sin(x)$ and $y = \csc(x)$. We want the values of x for which $\sin(x) = 1$ or $\sin(x) = -1$. For $0 < x \le 2\pi$, $\sin(\pi/2) = 1$ and $\sin(3\pi/2) = -1$. The points (π 2, 1) and ($3\pi/2$, −1) are on both functions. Plot these points.

3. Identify the x-intercepts of $y=\sin(x)$. At these values of x, $y = \csc(x)$ is undefined and there will be vertical asymptotes at $x = 0$, $x = \pi$, and $x = 2\pi$.

4. Determine what happens to the reciprocal function between the asymptotes.

- As the value of x moves from π2 towards 0, the value of $\sin(x)$ goes from 1 to 0. Then the value of the reciprocal goes from 1 to ∞.
- As the value of x moves from π2 towards 1, the value of $\sin(x)$ goes from 1 to 0. Then the value of the reciprocal goes from 1 to ∞.
- As the value of x moves from 3π2 towards π, the value of $\sin(x)$ goes from −1 to 0, approaching from below. Then the value of the reciprocal goes from −1 to −∞.
- As the value of x moves from 3π2 towards 2π, the value of $\sin(x)$ goes from −1 to 0, approaching from below. Then the value of the reciprocal goes from −1 to −∞.

It is straightforward to see that the cosecant function will be periodic and have the same period as the sine function. For the cosecant function, state the x and y-intercepts, the period and the equations of any asymptotes, and the domain and range.



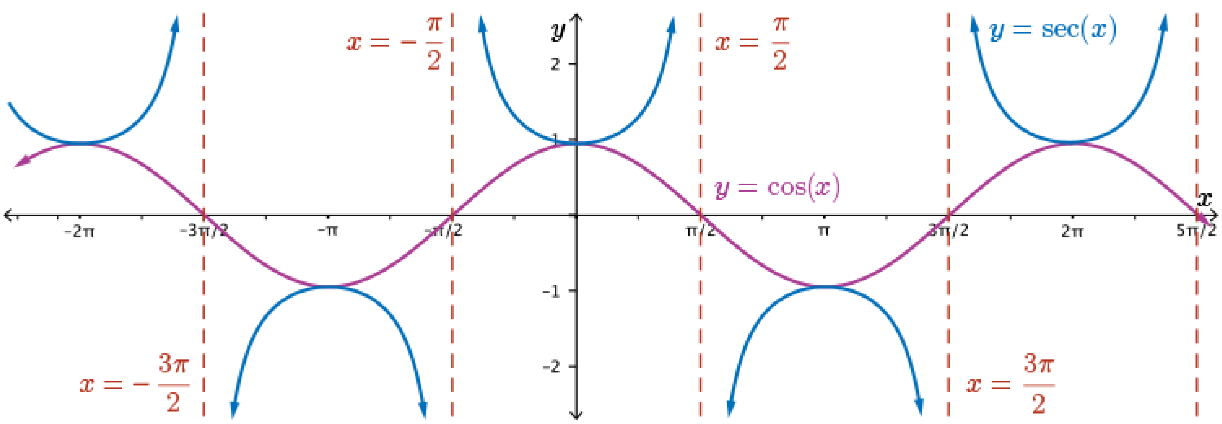
There are no *x*-intercepts. There is no *y*-intercept. The period of *y*=csc(*x*) is 2*π*.

The vertical asymptotes occur every *π* units with one of them at *x*=0. Therefore, the equations of the vertical asymptotes are *x*=*nπ*, *n* ∈Z. There are no horizontal asymptotes. The domain is {*x*| *x* ≠ *nπ*, *x* ∈R, *n* ∈Z}.

No values of *y* = csc(*x*) lie between −1 and 1. The range of *y*= csc(*x*) is {*y*| *y* ≤ −1 or *y* ≥ 1, *y* ∈R}. The range can also be written using absolute value notation as {*y*| |*y*| ≥ 1, *y* ∈R}.

Now, in a similar way, let's continue with the secant function.

For the secant function, state the *x* and *y*-intercepts, the period and the equations of any asymptotes, and the domain and range.



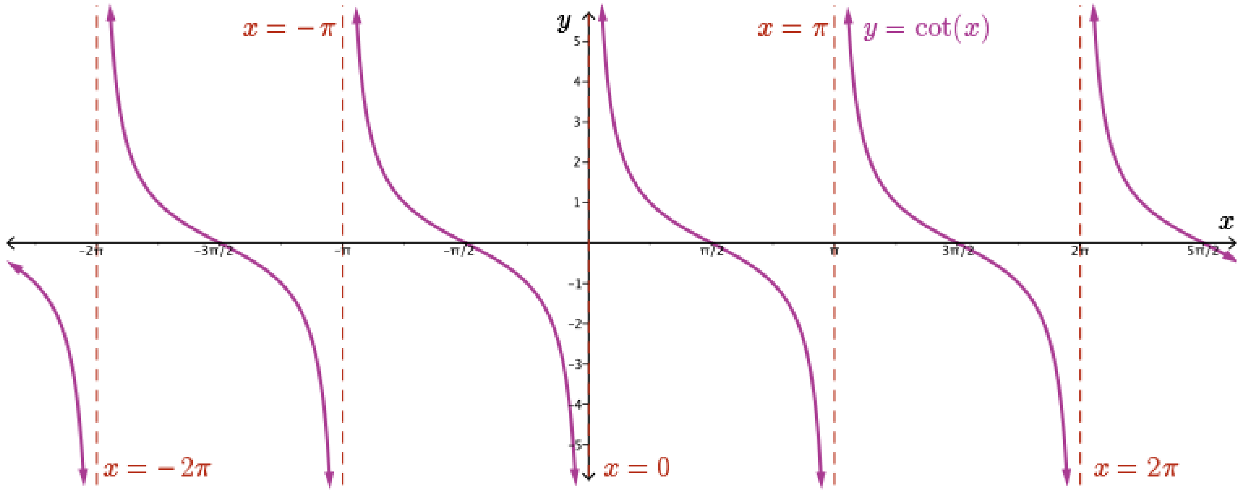
There are no *x*-intercepts. The *y*-intercept of the secant function is 1. The period of *y*=sec(*x*) is 2*π*.

Vertical asymptotes occur every *π* units with one at *x*=*π*/2. Therefore, the equations of the vertical asymptotes are *x* = *π*/2 + *nπ*, *n* ∈Z. There are no horizontal asymptotes.

The domain is {*x*| *x* ≠ *π*/2 + *nπ*, *x* ∈R, *n* ∈Z}. No values of *y* = sec(*x*) lie between −1 and 1.

The range of *y* = sec(*x*) is {*y*| *y* ≤ −1 or *y* ≥ 1, *y* ∈R}. The range can also be written using absolute value notation {*y*| |*y*|≥ 1, *y* ∈R}.

For the cotangent function, state the *x* and *y*-intercepts, the period and the equations for any asymptotes, and the domain and range.



There is no *y*-intercept. The period of $y=\cot(x)$ is π. There are no horizontal asymptotes. The vertical asymptotes occur every π units with one of them at $x=0$. Therefore, the equations of the vertical asymptotes can be written $x=n\pi, n \in Z$. The domain is $\{x |\ x \neq n\pi, x \in R, n \in Z\}$. The range is $\{y |\ y \in R\}$.

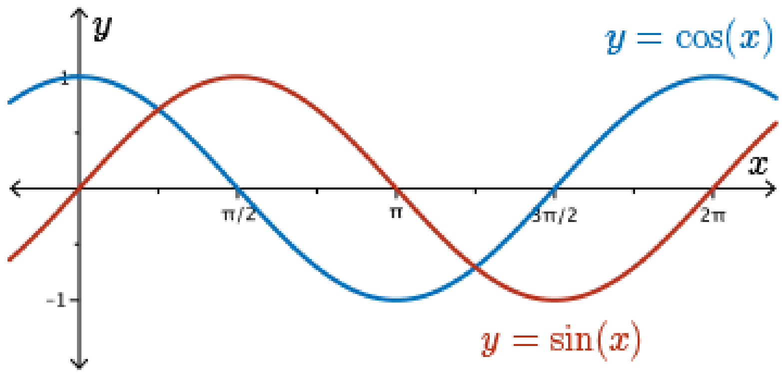
The sine function and the cosine function are referred to as **sinusoidal functions**.
Their graphs have the property that they oscillate above and below a central horizontal line.
For both $y = \sin(x)$ and $y = \cos(x)$, this central horizontal line or axis is $y = 0$.

The equation of the horizontal axis is $y = \dfrac{\text{maximum value} + \text{minimum value}}{2}$.

The sinusoidal functions are cyclic. That is, the function values repeat over regular intervals of the domain.
We say that these functions are **periodic**.
The horizontal length of one cycle is called the **period**.
The period of both $y = \sin(x)$ and $y = \cos(x)$ is 2π radians or 360^0.
The **amplitude** is the perpendicular distance from the central horizontal axis to either a maximum or minimum point on the curve.
We can calculate the amplitude with the formula

Amplitude = $\dfrac{\text{maximum value} - \text{minimum value}}{2}$.

For both functions, $y=\sin(x)$ and $y=\cos(x)$:
The domain is $\{x|\ x \in R\}$ and the range is $\{y|\ -1 \le y \le 1,\ y \in R\}$.
We will refer to $y=\sin(x)$ and $y=\cos(x)$ as the parent sinusoidal functions.



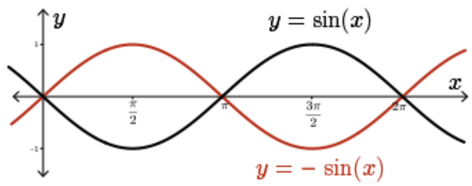
In an earlier unite, we looked at transformations. Transformations on a function $y = f(x)$ can be identified when the function is written in the form $y = af[b(x-h)] + k$.

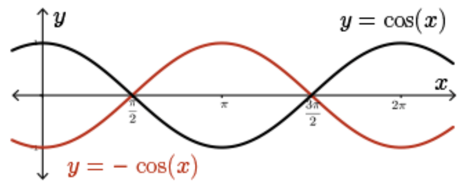
The Sine Function
$y = a\sin[b(x-h)] + k$

The Cosine Function
$y = a\cos[b(x-h)] + k$

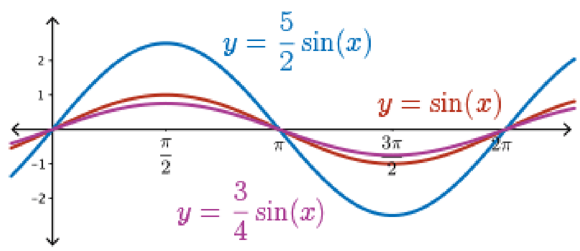
We will review the role of the parameters a, b, h and k in transforming the sinusoidal functions.

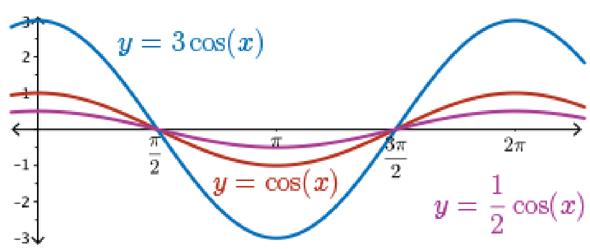
Effect of a: If $a<0$, the sinusoidal function is reflected in the x-axis.





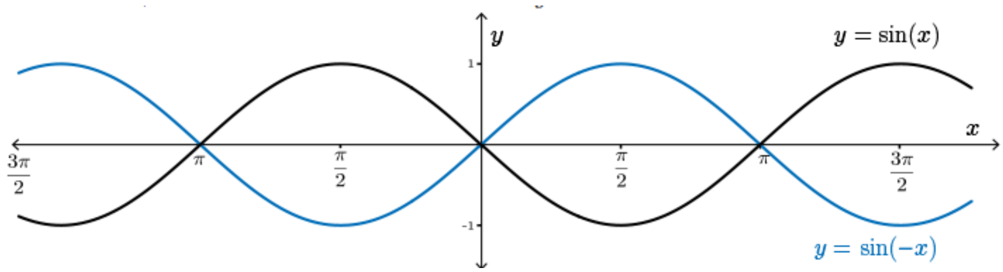
The sinusoidal function is stretched vertically from the x-axis by a factor of $|a|$.



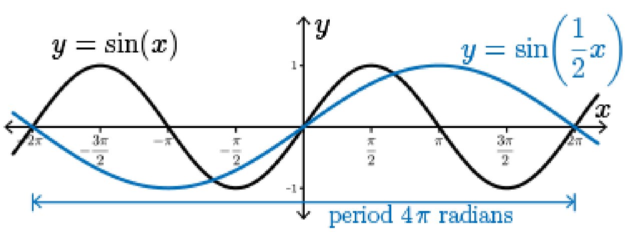


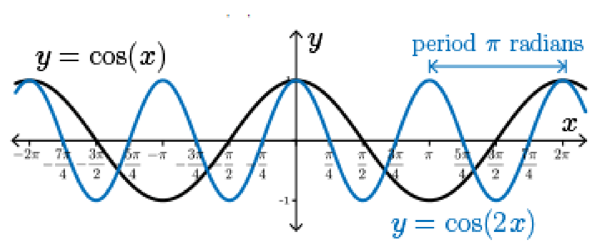
The amplitude of a sinusoidal function is affected by a vertical stretch.

Effect of *b*: If *b*<0, the sinusoidal function is reflected in the *y*-axis.



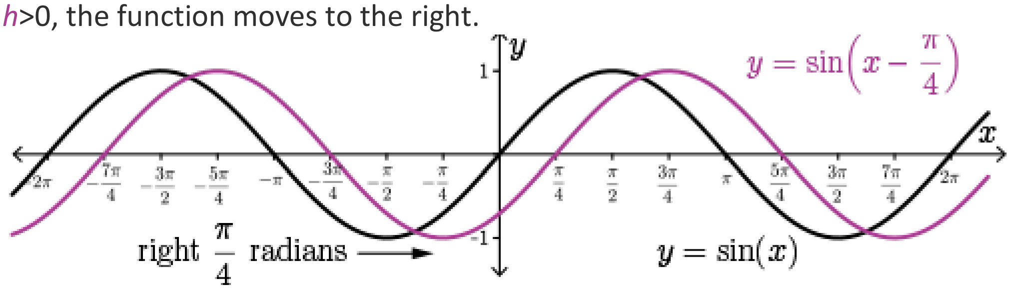
The graph of *y* = sin(−*x*) appears to be the same as the graph of *y*=−sin(*x*) from earlier in the unite.
Since *y* = cos(*x*) is symmetric about the *y*-axis, a reflection in the *y*-axis has no effect. That is, cos(*x*) = cos(−*x*).
The sinusoidal function is stretched horizontally from the *y*-axis by a factor of 1/|*b*|.



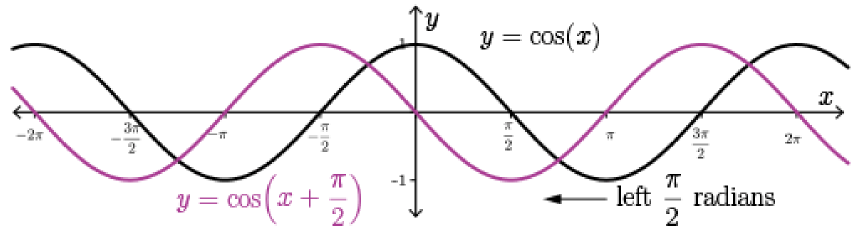


The period of a sinusoidal function is affected by a horizontal stretch and can be obtained by multiplying the original period by the horizontal stretch factor 1/|*b*|.
The length of one period of the horizontally stretched function is shown on each graph.
In radians, the period is $(1/|b|)(2\pi)= 2\pi/|b|$. In degrees, the period is $(1/|b|)(360^0)=360^0/|b|$.

Effect of *h*: The sinusoidal function is translated horizontally *h* units.
If *h*>0, the function moves to the right.

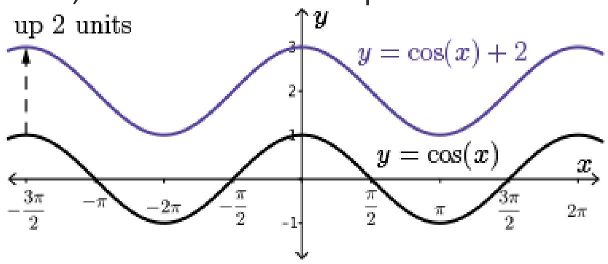


If *h*<0, the function moves to the left.

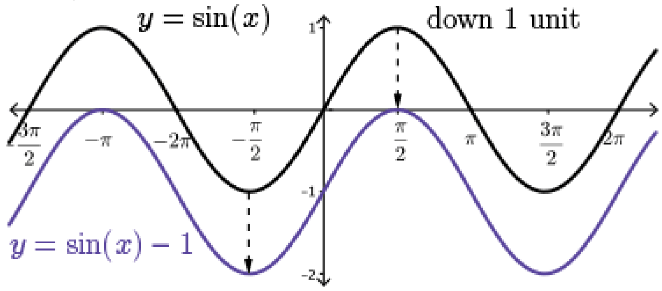


A horizontal translation affects the *x*-coordinate of every point on a sinusoidal function.
The *y*-coordinates stay the same.
When sketching sinusoidal functions, the horizontal translation is called the **phase shift**.

Effect of *k*: The sinusoidal function is translated vertically *k* units.
If *k*>0, the function moves up.



If *k*<0, the function moves down.



A vertical translation affects the *y*-coordinate of every point on a sinusoidal function.
The *x*-coordinates stay the same.
The central horizontal axis is translated up or down depending on the value of *k*.
This vertical movement is often referred to as **vertical displacement**.

Mapping Notation

As discussed in earlier modules, the image of each point after the transformation is applied is defined by the mapping:

$$(x, y) \quad \rightarrow \quad \left(\frac{1}{b}x + h,\ ay + k\right)$$

There are several ways in which we could sketch sinusoidal functions.
No matter which method of sketching we use, we should first identify the amplitude, the period, the phase shift, and the vertical displacement.

Then, apply the reflections, stretches and translations to sketch the image.

Or, using our knowledge of the general parent function, we could determine the location of five key points in one period to sketch the graph.

Or, apply the general mapping to each of five key points in one period of the parent function to find their respective images in order to sketch the curve.

Example

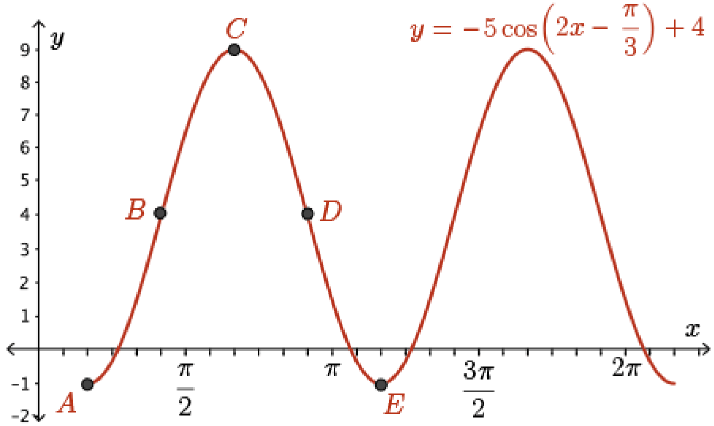
Sketch two periods of the function $y = -5\cos(2x - \frac{\pi}{3}) + 4$.

Solution

The equation must be written in the form $y = a\cos[b(x-h)] + k$.
Rewriting the equation, we obtain $y = -5\cos[2(x - \frac{\pi}{6})] + 4$.
From the equation, we see that $a = -5$, $b = 2$, $h = \pi/6$, and $k = 4$.
There is a reflection about the x-axis since $a < 0$.
We can list attributes of the image: amplitude = 5, period $= 2\pi/2 = \pi$, phase shift $= \pi/6$, and vertical displacement = 4.
The minimum and maximum values of y are −1 and 9, respectively.
In the five-point sketch the x-coordinates of the first and last points are $\pi/6$ and $\pi/6 + \pi = 7\pi/6$.
The horizontal distance between each of the points on the sketch is $\pi \div 4 = \pi/4$.
Five points on the sketch are

$$A\ (\pi/6, -1),\ B\ (5\pi/12, 4),\ C\ (2\pi/3, 9),$$
$$D\ (11\pi/12, 4),\ \text{and}\ E\ (7\pi/6, -1)$$

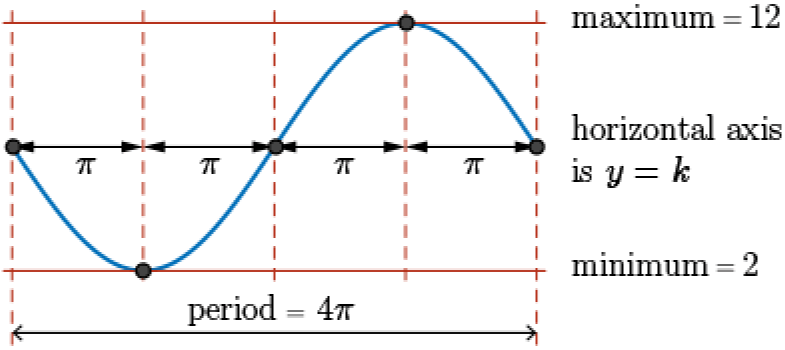
The sketch is shown. The horizontal scale is $\pi/12$ radians.



Example

Determine an equation for a sine function that has been reflected about the x-axis, has a maximum value 12, a minimum value 2, a period 4π, and a phase shift $-\pi/3$.

Solution



A quick sketch of one period is shown.
The equation is of the form

$$y = a\sin[b(x{-}h)] + k$$

Since there is a reflection about the x-axis, $a < 0$.
To determine the amplitude, we subtract the minimum value from the maximum value and divide the result by 2.
The amplitude is (12−2)/2= 10/2= 5 and it follows that $a = -5$.
We know that k is the average of the maximum and minimum values so k = (12−2)/2= 14/2= 7.
(It follows that the equation of the central horizontal axis is y= 7.)
Since the phase shift is given, we know that $h = -\pi/3$.
We know that the period = $2\pi/|b|$.
Rearranging, $|b| = 2\pi$ period= $2\pi/4\pi = 1/2$.
Since there is no reflection about the y-axis, b>0 and it follows that b=12. Combining the information, we have a= −5, b= 1/2, h= $-\pi/3$ and k= 7.
A possible equation is $y = -5\sin[\frac{1}{2}(x + \frac{\pi}{3})] + 7$.

Example

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, R, to the extreme left point, L, and back to R again. The horizontal distance between the two extreme points R and L is 36 cm.

a. Determine the period and amplitude of this motion.

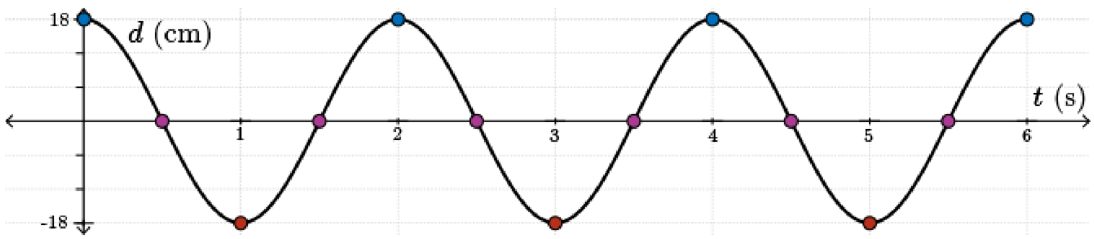
Solution

Since there are 30 complete swings in 60 seconds (1 minute), 1 period is 60÷30=2 seconds.
The horizontal distance between the two extreme points is 36 cm.
The amplitude is half of this value so the amplitude is 18.

b. Draw a sketch to model the pendulum swing for the first 6 seconds. Assume the pendulum starts at *R*.

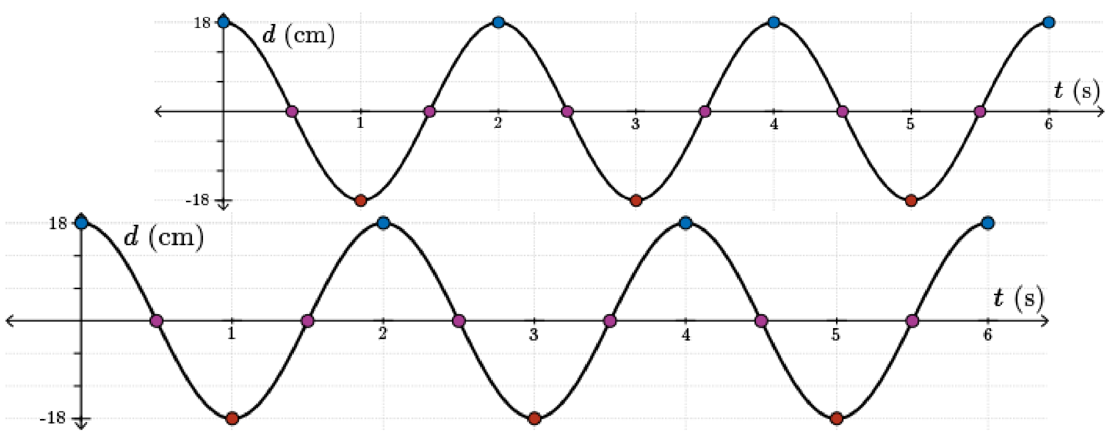
Solution

Let *d* represent the horizontal distance from *M*, with positive distance to the right and negative distance to the left.
When the pendulum is at *R*, d=18. When the pendulum is at *L*, d=−18.
Since the period is 2 seconds, the pendulum will be at *R*, a maximum, at 0, 2, 4, and 6 seconds.
The minimum will occur at the halfway point of each period so the pendulum will be at *L* at 1, 3, and 5 seconds.
And halfway between the maximum and minimum, the pendulum will be at *M*.
The pendulum will be at *M* at 0.5, 1.5, 2.5, 3.5, 4.5, and 5.5 seconds.
A sketch is shown.



c. Determine an equation to model the path of the pendulum.

Solution



Since a maximum point on the sketch is on the *y*-axis (or in this case, the *d*-axis), there is no horizontal translation and so h=0.
The central horizontal axis is on the *x*-axis (or in this case, the *t*-axis) so there is no vertical translation and k=0.
The amplitude is 18 cm. Since a maximum point is on the positive *y*-axis, there has been no reflection about the *x*-axis and a=18.
The period is 2 seconds. Thus, $b = 2\pi/\text{period} = 2\pi/2 = \pi$.
Substituting for a=18, b=π, h=0, and k=0 into $d = a\cos[b(x-h)] + k$, we obtain $d=18\cos(\pi x)$.
If we use *t* instead of *x*, an equation of the function graphed becomes $d=18\cos(\pi t)$, $0 \le t \le 6$.

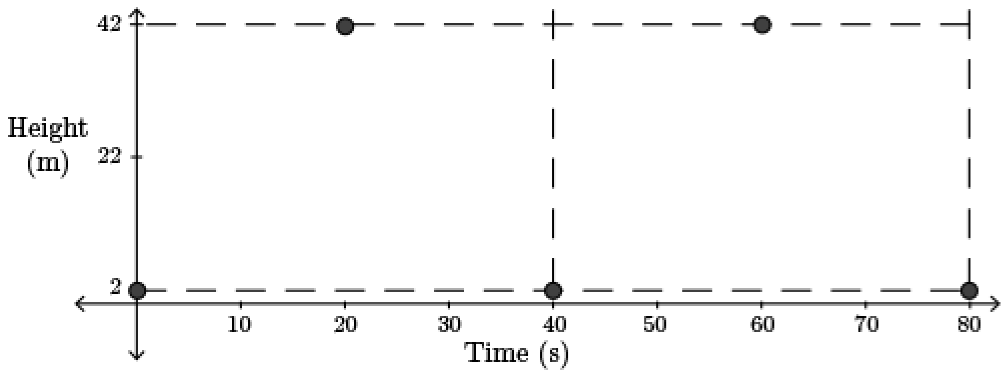
- When $d<0$, the pendulum is closer to L than it is to R.
- When $d>0$, the pendulum is closer to R than it is to L.
- When $d=0$, the pendulum is at M.

Example

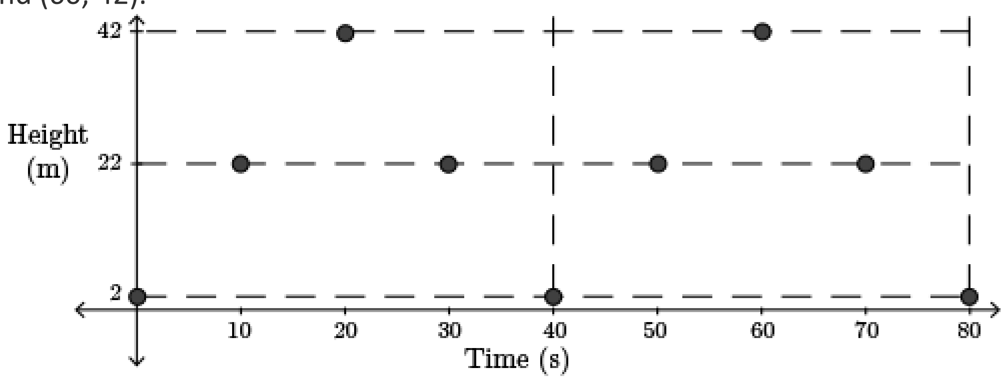
A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider's initial position on the ride is at the bottom of the wheel, 2 m above the ground.

a. Draw a graph showing a rider's height above the ground during the first two revolutions of the wheel.

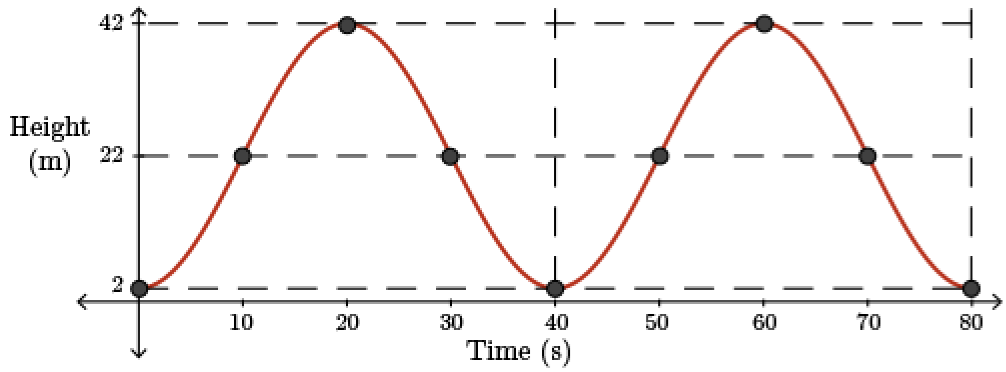
Solution



Since one revolution takes 40 s, the period length is 40 s.
The rider gets on at the lowest point so a minimum occurs at (0, 2). Minimums also occur at (40, 2) and (80, 2) since each complete revolution takes 40 s.
The rider gets on at a point 2 m above the ground. The Ferris wheel has a radius of 20 m or a diameter of 40 m.
The maximum height above the ground is 2 + 40 or 42 m.
Maximums occur half of a period after each minimum. Therefore, maximums occur at (20, 42) and (60, 42).



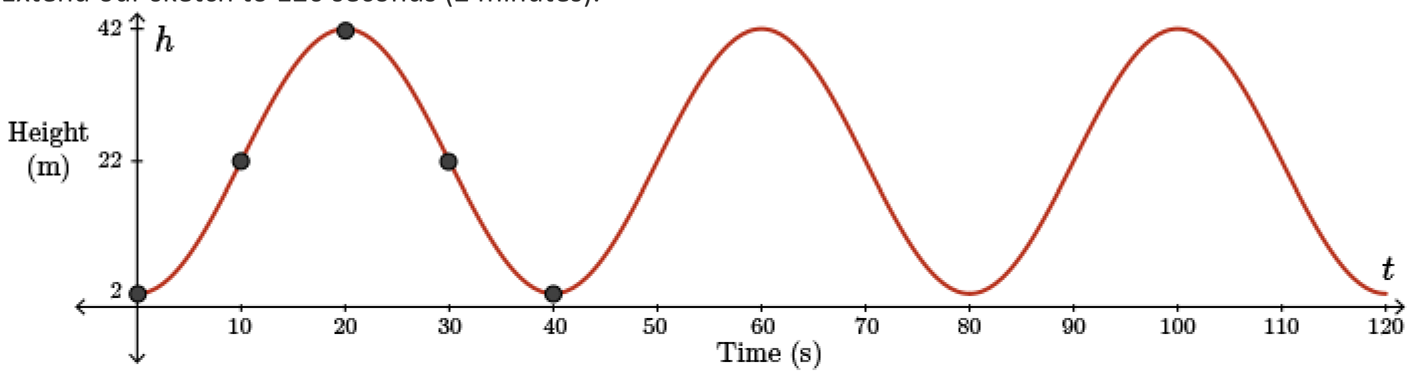
The centre of the wheel is 2+20=22 m above the ground. The rider is at this height at any time that is halfway between the times when adjacent maximum and minimum heights occur. Therefore, the rider is 22 m above the ground at 10 s, 30 s, 50 s, and 70 s.
A final sketch is shown.



b. Determine an equation modelling the rider's height above the ground during the first two minutes of the ride.

Solution

Extend our sketch to 120 seconds (2 minutes).



To keep things simpler, we will model with the negative cosine function since the phase shift will be 0. Five points are shown on the graph.
Since the amplitude is 20, let $a=-20$.
The middle height is 22 m so the central horizontal axis is at $y=22$ and it follows that $k=22$.
The period is 40 so $b = 2\pi/40 = \pi/20$.
Therefore, an equation to model this situation is $y = -20\cos(\frac{\pi}{20}x) + 22$.

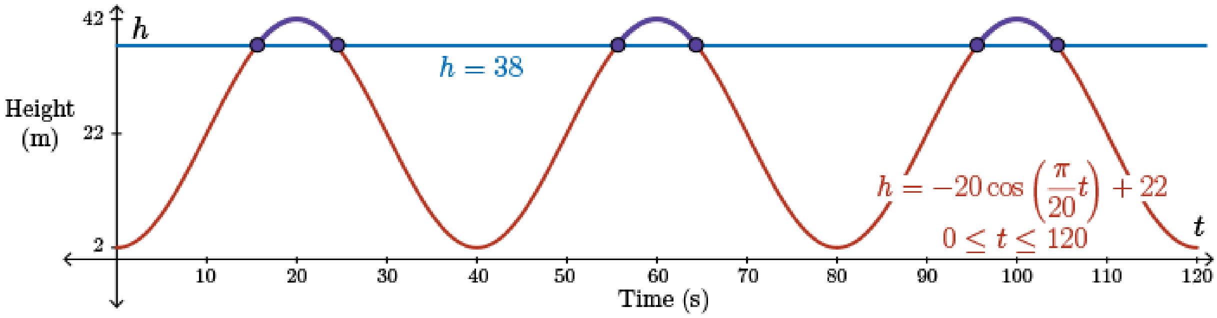
Using variables, which are more representative of the situation,

$$y = -20\cos(\frac{\pi}{20}t) + 22$$

for $0 \le t \le 120$, where h is the rider's height, in metres, above the ground at time t seconds.

c. At what times during the first two minutes will the rider's height above the ground be 38 m or higher?

Solution



To visualize this, draw the horizontal line h=38 on the graph.

We want to determine the times when the sinusoidal function intersects the line and when it is above the line.

Let $A=\frac{\pi t}{20}$ and h=38. Then,

38 =−20cos(A)+22

-4/5 = cos(A).

The reference angle, in radians, is $\cos^{-1}(0.8) \approx 0.644$, rounded to three decimal places.

Since cos(A)<0, A is in quadrant 2 or quadrant 3.

In quadrant 2, $A = \pi - \cos^{-1}(0.8) \approx 2.498$ and in quadrant 3, $A = \pi + \cos^{-1}(0.8) \approx 3.785$.

But $A=\frac{\pi t}{20}$, thus

$\frac{\pi t}{20} = 2.498$ or $\frac{\pi t}{20} = 3.785$

t = 15.9 s or t = 24.1 s.

For $15.9 \le t \le 24.1$, the rider is 38 m or higher above the ground.

Looking back at the graph, this is the first section where the sinusoidal function is on or above the line h=38.

To get the other two intervals, add multiples of the period, 40 seconds, to each endpoint of the first interval.

In the first 120 seconds, the rider is 38 m or higher above the ground from 15.9 s to 24.1 s, from 55.9 s to 64.1 s, and from 95.9 s to 104.1 s.