Ratio, Fraction and Percent

1. Concepts

1) A ratio is a comparison of two or more quantities expressed in the some unit
2) A rate is a comparison of two quantities with different units
3) Equivalent ratio represent the comparison among the terms
4) A proportion is a statement of the equality of two ratios.
5) A ratio or fraction in which the second term on denominator is always 100.

Formulae

$\frac{a}{b}\pm\frac{c}{b}=\frac{a\pm c}{b}$, $\frac{a}{b}+\frac{c}{d}=\frac{ad\pm bc}{bd}$, $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$, $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$

If $\frac{a}{b}=\frac{c}{d}$ (or a:b = c:d), a, b, c, d cannot be 0,

then $ad=bc$ $\frac{b}{a}=\frac{d}{c}$ $\frac{a}{c}=\frac{b}{d}$ $\frac{a+b}{b}=\frac{c+d}{d}$

2. Simple & Compound Fractions

A **simple fraction** is given by a pair of whole or natural numbers M/N, where N denotes the denominator and M denotes the numerator.

A ratio of simple fractions is a compound fraction A/B in which the numerator and denominator are both decimal fractions.

If $A=\frac{C}{D}$ and $B=\frac{E}{F}$, then $\frac{A}{B}=\frac{C}{D}\cdot\frac{F}{E}$

In other words, the **compound fraction** $\frac{\frac{C}{D}}{\frac{E}{F}}=\frac{C}{D}\cdot\frac{F}{E}$

There is a rule in developing mathematics: Saying how to compute a quantity defines it. The latter expression may be taken as the definition of the compound fraction $\frac{\frac{C}{D}}{\frac{E}{F}}$

We see that a compound fraction formed from a pair of fractions gives a fraction.

Special Case: $\frac{1}{\frac{E}{F}}=1\cdot\frac{F}{E}$

That is 1 divided by a fraction equals the reciprocal of the fraction.

3. Multiplying by the Reciprocal

If $A = \frac{C}{D}$ and $B = \frac{F}{E}$, then $\frac{A}{B} = A \cdot \frac{1}{B}$

This is another way of saying division of a first fraction by a second fraction has the same result as multiplying the first fraction by the reciprocal of the second. The case where A and B are decimal fractions will appear below.

Theorem: If M and N are given by decimal fractions

$$M = \frac{P}{10^a} \quad \text{and} \quad N = \frac{Q}{10^b}$$

then M has a or fewer places after the decimal point, N has b or fewer places after the decimal point, and

$$\frac{M}{N} = \frac{P}{10^a} * \frac{10^b}{Q}$$

Moreover if a = b then

$$\frac{M}{N} = \frac{P}{Q}$$

Note: We can extend the decimal expansion of M or N with zeroes after the decimal point to make a = b. The replacement of the decimal compound fraction M/N by P/Q where P and Q are both whole numbers provides a rule for moving the decimal point or eliminating it in fractions involving decimals.

Note: The rule

$$\frac{M}{N} = M * \frac{10^b}{Q} = \frac{M*10^b}{Q}$$

provides a method for shifting the decimal point in the dividend M and divisor N to obtain a whole number divisor Q. Then the long division algorithm can be applied as is, or in continued form.

Division of One Fraction by Another

A number or fraction T such that

$$T * \frac{E}{F} = \frac{C}{D}$$

has a unique solution $T = \frac{\frac{C}{D}}{\frac{E}{F}} = \frac{C}{D} * \frac{F}{E}$

Check:

$$T * \frac{E}{F} = \left(\frac{C}{D} * \frac{F}{E}\right) * \frac{E}{F} = \frac{C}{D} * \left(\frac{F}{E} * \frac{E}{F}\right) = \frac{C}{D} * \mathbf{1} = \frac{C}{D}$$

Here

$$T = \frac{\frac{C}{D}}{\frac{E}{F}} = \frac{\text{C}}{\text{D}} \text{ divided by } \frac{\text{E}}{\text{F}}$$

is the fractional number of times that the denominator $\frac{E}{F}$ goes into the numerator $\frac{C}{D}$ of the compound fraction $\frac{\frac{C}{D}}{\frac{E}{F}}$

Note: Compound fractions of the form M/N where M and N are real numbers will be treated later.

4. Ratio of Decimal Fractions

A ratio of decimal fractions is a compound fraction $\frac{M}{N}$ in which the numerator and denominator are both decimals - proper or improper fractions expressible as whole number over a power of 10.

5. Shifting the Decimal Point

In general if M and N are given by decimal fractions then

$$\text{M} = \frac{\text{P}}{10^{\text{a}}} \quad \text{and} \quad \text{N} = \frac{\text{Q}}{10^{\text{b}}}$$

for some whole or natural numbers P, Q, a and b. Here we assume a and b give the number of digits after the decimal point in the finite decimal notation for M and N respectively, and we may also assume the ones digit in both P and Q are nonzero. Now

$$\frac{M}{N} = \frac{\frac{P}{10^a}}{\frac{Q}{10^b}} = \frac{P}{10^a} \cdot \frac{10^b}{Q} = \frac{P \cdot 10^b}{10^a} \cdot \frac{1}{Q} = \frac{\frac{P \cdot 10^b}{10^a}}{Q}$$

The foregoing justifies the long division method of shifting the decimal point in the dividend and divisor by number of decimal places in the divisor to obtain an integral (whole number) divisor Q

Another Twist: $\frac{M}{N} = \frac{\frac{P}{10^a}}{\frac{Q}{10^b}} = \frac{P}{10^a} \cdot \frac{10^b}{Q} = \frac{P \cdot 10^b}{Q \cdot 10^a}$

The twist yields

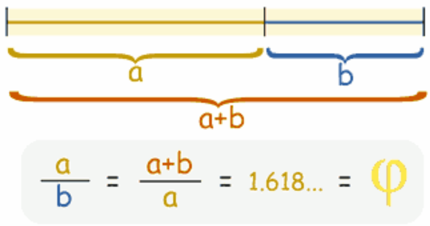
$$\frac{456.89}{34.567} = \frac{456.89 \times 1000}{34.567 \times 100} = \frac{456890}{3456.7} = \frac{4568900}{34567}$$

6. Golden Ratio

The golden ratio *(symbol is the Greek letter "phi" shown at left)* is a special number approximately equal to 1.618
It appears many times in geometry, art, architecture and other areas.

The Idea Behind It

If you divide a line into two parts so that:
the longer part divided by the smaller part
is also equal to the whole length divided by the longer part then you will have the golden ratio



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1) Guessing It

There is only one value that would make a/b equal to (a+b)/a. Let us try out some possibilities to see if we can discover it:

Let us try a=7 and b=3, so a+b=10:

7/3 = 2.333..., but 10/7 = 1.429..., *so that won't work*

Let us try a = 6 and b = 4, so a + b = 10:

6/4 = 1.5, but 10/6 = 1.666..., *closer but not there yet!*

Let us try a=6.18 and b=3.82, so a+b=10:

6.18/3.82 = 1.6178..., and 10/6.18 = 1.6181..., *getting very close!*

In fact the value is:
1.61803398874989484820... (keeps going, without any pattern)
The digits just keep on going, with no pattern. In fact the Golden Ratio is known to be an Irrational Number, and I will tell you more about it later.

2) Calculating It

You can calculate it yourself by starting with any number and following these steps:
A) divide 1 by your number (1/number)
B) add 1
C) that is your new number, start again at A

With a calculator, just keep pressing "1/x", "+", "1", "=", around and around. I started with 2 and got this:

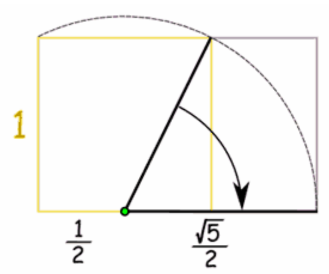
Number	1/Number	Add 1
2	1/2=0.5	0.5+1=1.5
1.5	1/1.5 = 0.666...	0.666... + 1 = 1.666...
1.666...	1/1.666... = 0.6	0.6 + 1 = 1.6

1.6	1/1.6 = 0.625	0.625 + 1 = 1.625
1.625	1/1.625 = 0.6154...	0.6154... + 1 = 1.6154...
1.6154...		

It is getting closer and closer!
But it would take a long time to get there, however there are better ways and it can be calculated to thousands of decimal places quite quickly.

3) Drawing It

Here is one way to draw a rectangle with the Golden Ratio:
Draw a square (of size "1")
Place a dot half way along one side
Draw a line from that point to an opposite corner (it will be √5/2 in length)
Turn that line so that it runs along the square's side
Then you can extend the square to be a rectangle with the Golden Ratio.

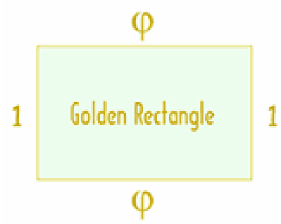


4) The Formula

Looking at the rectangle we just drew, you can see that there is a simple formula for it. If one side is 1, the other side will be:

$$\varphi = \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$

The square root of 5 is approximately 2.236068, so The Golden Ratio is approximately (1+2.236068)/2 = 3.236068/2 = 1.618034. This is an easy way to calculate it when you need it.



Some artists and architects believe the Golden Ratio makes the most pleasing and beautiful shape.
This rectangle has been made using the Golden Ratio, Looks like a typical frame for a painting, doesn't it?

Do you think it is the "most pleasing rectangle"? Maybe you do or don't, that is up to you!

► Questions in class

1. If $\dfrac{4a+2}{5b+3}=\dfrac{2}{3}$, find the ratio of *a : b*.

2. If $\dfrac{4+\frac{3}{a}}{5+\frac{2}{b}}=\dfrac{4}{5}$, find the ratio of *a : b*.

3. If $3(4x+5\pi)=P$, then $6(8x+10\pi)=?$

4. A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

5. What is the ratio $\dfrac{2^{2001}\cdot 3^{2003}}{6^{2002}}$?

6. The ratio $\dfrac{10^{2000}+10^{2002}}{10^{2001}+10^{2001}}$ is closest to which of the following numbers?

(A) 0.1 (B) 0.2 (C) 1 (D) 5 (E) 10

7. If an arc of 45^{o} on circle A has the same length as an arc of 30^{o} on circle B, then what is the ratio of the area of circle A to the area of circle B?

8. The scale on a map states that 1 cm represents 6 miles. How many square miles would be represented by an area on the map of 240 cm^2?

9. A, B, and C are thermometers with different scales. When A reads 10^{o} and 34^{o}, B reads 15^{o} and 31^{o} respectively. When B reads 30^{o} and 42^{o}, C reads 5^{o} and 77^{o} respectively. If the temperature drops 18^{o} using A's scale, how many degrees does it drop using C’s Scale?

10. If the repeating decimal $0.26767676767\cdots$ is represented as a ratio of positive integers in lowest terms, then what is the numerator of this ratio?