Gravitational Force and Motion in Space

Unit 3: Force Fields

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Files for You to Download

The "print version" the class slides for this unit are now downloadable from the school website:

- `Phys12-3a-gravField-print.pdf`–Gravitational field and motion in space
- `Phys12-3b-elecField-print.pdf`–Electric field
- `Phys12-3c-magField-print.pdf`–Magnetism and applications

As usual, if you wish to print out the slides, we recommend printing 4 slides per page. Please download/print the PDF file before the start of each unit. There is no point copying notes that are already printed out for you. Instead, take notes on things that are not necessarily on the slides.

Where Are We In the Course

1. Fundamentals of Dynamics
2. Momentum, Impulse and Energy
3. Gravitational, Electric and Magnetic Fields
4. Wave Nature of Light
5. Theory of Special Relativity
6. Introduction to Quantum Mechanics

Force Fields

When we think of a "force field", many of us start thinking about fantastic crazy technologies in science fiction movies

What We Are REALLY Talking About

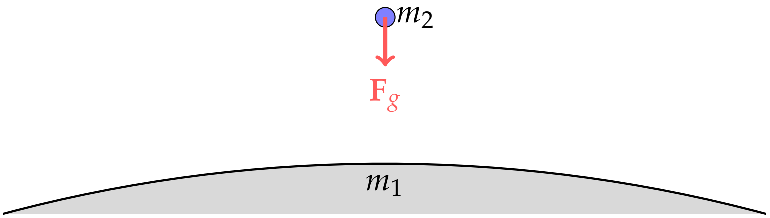
We're really dealing with is actually:

- Gravitational force ($\mathbf{F}_g$)
- Electrostatic force ($\mathbf{F}_q$)
- Magnetic force ($\mathbf{F}_m$)

Sorry if you thought otherwise

Newton's Law of Universal Gravitation

Gravity is the force of attraction between all objects that have mass.



$$\boxed{F_g = \frac{G m_1 m_2}{r^2}}$$

where $G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2 is the universal gravitation constant

Universal Gravitation

If m_1 exerts a force $\mathbf{F}_g$ on m_2, then m_2 also exerts a force $-\mathbf{F}_g$ on m_1.

- These two forces are equal in magnitude and opposite in direction (Newton's 3rd law)

Assumptions:

- m_1 and m_2 are *point masses* that do not occupy any space
- The two objects haven't collided into one another, i.e.:

$$r > (r_1 + r_2)$$

A Simple Example Problem

Example 1: A $65.0\,\mathrm{kg}$ astronaut is walking on the surface of the moon, which has a mean radius of $1.74 \times 10^3\,\mathrm{km}$ and a mass of $7.35 \times 10^{22}\,\mathrm{kg}$. What is the weight of the astronaut?

Example Problem

Example 2: How far apart would you have to place two $7.0\,\mathrm{kg}$ bowling balls so that the force of gravity between them would be 1.25×10^{-8} N?

Notice the magnitude of gravitational force between the two objects. In fact, gravitational force is the weakest of all fundamental forces.

Think Gravitational Field: What is g?

We are used to describe gravity using a simpler equation:

$$\mathbf{F}_g = m\mathbf{g}$$

To find g, we group the variables in Newton's universal gravitation equation:

$$F_g = \underbrace{\left[\frac{Gm_1}{r^2}\right]}_{=g} m_2 = m_2 g$$

On Earth's surface, we use $m_1 = m_{\mathrm{Earth}}$ and $r = r_{\mathrm{Earth}}$ to compute $g = 9.81\,\mathrm{m/s^2}$, or $g = 9.81\,\mathrm{N/kg}$ (both units are equivalent)

Gravitational Field

A closer look at g show's that it is actually a function of a source mass m_s and the distance r from it. This is called the **gravitational field**:

$$\boxed{g(m_s, r) = \frac{Gm_s}{r^2}}$$

It shows how source mass m_s influences the gravitational forces on other masses in its vicinity

Quantity	Symbol	SI Unit
Gravitational field intensity	g	N / kg
Universal gravitational constant	G	$N \cdot m^2 / kg^2$
Source mass (a point mass)	m_s	kg
Distance from centre of source	r	m

Relating Gravitational Field & Gravitational Force

$\mathbf{g}$ itself doesn't do anything unless there is another mass m. At which point, the other mass m experiences a gravitational force related to $\mathbf{g}$ by:

$$\boxed{\mathbf{F}_g = m\mathbf{g}}$$

Quantity	Symbol	SI Unit
Gravitational field	$\mathbf{g}$	N / kg
Gravitational force on a mass	$\mathbf{F}_g$	N
Mass inside the gravitational field	m	kg

$\mathbf{F}_g$ and $\mathbf{g}$ are vectors in the same direction (towards the centre of the source mass that created the field), therefore all vector operations apply.

Gravitational Potential Energy

Since g is not a constant, we instead use an equation consistent with the law of universal gravity (obtained using calculus, by integrating $\mathbf{F}_g$ by a distance r to find the work done):

$$\boxed{U_g = -\frac{GMm}{r}}$$

Quantity	Symbol	SI Unit
Gravitational potential energy	U_g	J
Masses	M, m	kg
Distance between centres of mass	r	m
Universal gravitational constant	G	$\mathrm{N \cdot m^2 / kg^2}$

The "reference level" is chosen at infinity (i.e. $U_g = 0$ at $r = \infty$) and *decrease* as r decreases

Relating Gravitational Potential Energy to Force

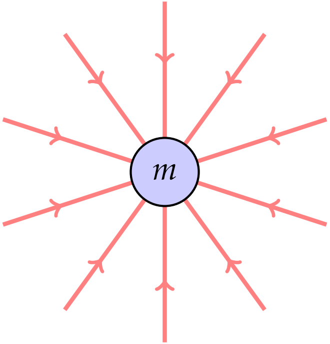
The work-kinetic theorem tells us that

- $\mathbf{F}_g$ always points in the direction from high to low potential energy, i.e.
- A falling object is always decreasing in U_g
- "Steepest descent": the direction of $\mathbf{F}_g$ is the shortest path to decrease U_g
- Objects travelling perpendicular to $\mathbf{F}_g$ has constant U_g

In vector calculus, we say that gravitational force ($\mathbf{F}_g$) is the negative gradient of the gravitational potential energy (U_g):

$$\mathbf{F}_g = -\nabla U_g = -\frac{dU_g}{dr}\hat{\mathbf{r}}$$

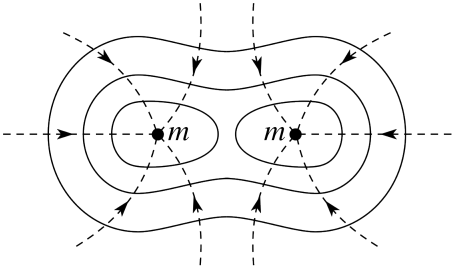
Gravitational Field Lines



- The direction of $\mathbf{g}$ is towards the centre of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\mathbf{g}$, only the direction

Gravitational Field Lines

When there are multiple masses, the total gravitational field (dotted line) is the vevtor sum of all the individual fields.



The solid lines are called **equipotential lines**, where the potential energy is constant. Equipotential lines are perpendicular to gravitational field lines.

We will now combine our knowledge of gravitational force and gravitational energy to understand motion of satellites, planets and stars.

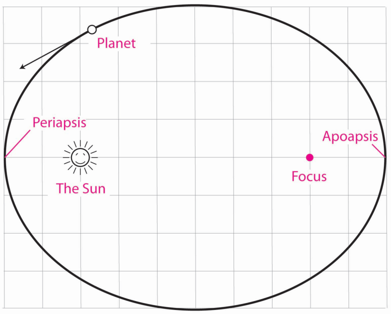
First, there was Kepler

- German mathematician, astronomer & astrologer
- Formulated his laws based on the observation of his teacher, Tycho Brahe
- Published his first 2 laws in 1609, third law in 1619
- His work was controversial because of another competing theory, but by 1670, most scientists have accepted his findings
- Kepler's laws are considered to be *empirical*:
    - Based purely on observed data
    - Kepler had no physical theory
    - "Fitting the curve" to the data

Kepler's Laws of Planetary Motion

Law of Ellipses

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.

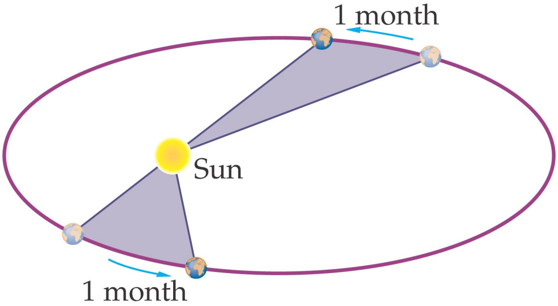


This is an extraordinary claim, because it is different from those of Capernicus, who claimed that the orbit of a planet is circular.

Kepler's Laws of Planetary Motion

Law of Equal Areas

2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.



That means that the planet moves faster when it is closer to the Sun, and slower when it farther.

Kepler's Laws of Planetary Motion

Law of Periods

3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

It means this:

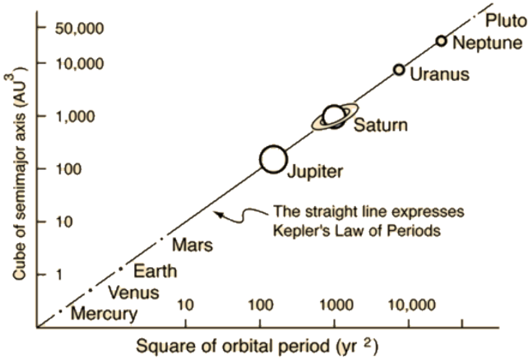
$$\boxed{\frac{T^2}{r^3} = \text{constant}} \quad \text{or} \quad \boxed{\frac{T_A^2}{r_A^3} = \frac{T_B^2}{r_B^3}}$$

where A and B are two different planets circulating the same sun

Kepler's Laws of Planetary Motion

Orbital Radii and Periods of Different Planets

Planet	R (AU)	T (days)
Mercury	0.389	87.77
Venus	0.742	224.70
Earth	1.000	365.25
Mars	1.524	686.98
Jupiter	5.200	4332.62
Saturn	9.150	10,579.20



Astronomical unit (AU) is defined as the average radius of Earth's orbit

Kepler's Law of Planetary Motion

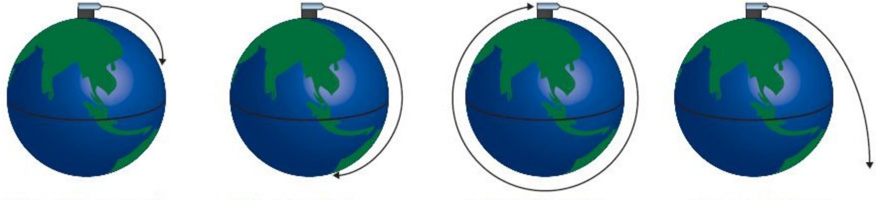
The elliptical orbits of most of the planets in the solar system have very small eccentricity (i.e. their orbits are close to being circular) but comets are much more eccentric

Object	e
Mercury	0.206
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0485
Saturn	0.0556
Uranus	0.0472
Neptune	0.0086
Pluto	0.25
Halley's Comet	0.9671
Comet Hale-Bopp	0.9951
Comet Ikeya-Seki	0.9999

Orbital Velocity

Newton's Thought Experiment

In *Treatise of the System of the World*, the third book in *Principia*, Newton presented this thought experiment:



- How fast does the cannonball have to travel before it goes around Earth without falling? (i.e. goes into orbit)
- How fast does the cannonball have to travel before it never comes back?

Orbital Velocity

Orbital velocity (or more accurately, **orbital speed**) is the speed required for an object to stay in a circular path without falling back onto the surface. For example:

- A spy satellite orbiting around the Earth
- The moon orbiting around the Earth
- Planets of the solar system orbiting around the Sun

Staying in orbit requires a centripetal force that is provided by gravitational force:

$$F_g = F_c \quad \rightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$$

Orbital Velocity

Cancelling m and r in both sides of the equation, and solving for v, we get the equation for orbital velocity v_{orbit}:

$$\boxed{v_{\text{orbit}} = \sqrt{\frac{GM}{r}}}$$

- v_{orbit} does not depend on the mass of the object that is in orbit
  - A 1.5×10^{-13} kg speck of cosmic dust and the 4.2×10^{5} kg International Space Station both have the same v_{orbit} around Earth at the same altitude.
- The equation is only valid for *perfectly circular* orbits
- Applicable when for a small object orbiting a much more massive object

Example Problem

Example 3: What is the orbital velocity of a satellite at a height of $300\,\text{km}$ above the surface of Earth? (The mass of Earth is $5.97 \times 10^{24}\,\text{kg}$ and the radius of Earth is $6370\,\text{km}$)

1. $5.42 \times 10^{1}\,\text{m/s}$
2. $1.15 \times 10^{6}\,\text{m/s}$
3. $7.7 \times 10^{3}\,\text{m/s}$
4. $6.0 \times 10^{6}\,\text{m/s}$
5. $3.0 \times 10^{8}\,\text{m/s}$

Make sure you use the right r!

Example Problem

Example 4: At what velocity and altitude must a satellite orbit in order to be geostationary?

Hint: "Geostationary" means that the position of the satellite is over the same position on earth all the time. This means that period of the satellite must be the same as Earth's rotation (1 day).

Escape Velocity

An object can leave the surface of Earth at any velocity. But when all the kinetic energy of that object is converted into gravitational potential energy, it will return back to the surface of Earth. However, there is a *minimum* velocity at which the object would not fall back to Earth because of gravity, called the **escape velocity** (or **escape speed**).

Escape Velocity

- The gravitational potential energy of an object with mass m on the surface of a planet (with mass M and radius r) is

$$U_g = -\frac{GMm}{r}$$

- The most amount of work that you can do is to bring it to the other side of the universe $r = \infty$, where $U_g = 0$.
- Because gravitational force is *conservative*, the work done by gravity converts kinetic energy into gravitational potential energy
- If you start with *more* kinetic energy than required to do all the work, then the object has *escaped* the gravitational pull of the planet.

Escape Speed from Circular Orbits

Setting K to equal to $-U_g$ and solving for v, we find the escape velocity:

$$\frac{1}{2}mv^2 = \frac{GMm}{r} \quad \longrightarrow \quad \boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

- Any object with $v > v_{\text{esc}}$ can break free of a planet's gravitational pull.
- Assumptions:
    - Mass of the object remains contant
    - All the work is done by gravity

This is not the case for launching a rocket.

Example Problem

Example 5: Determine the escape velocity and energy for a 1.60×10^4 kg rocket leaving the surface of Earth.

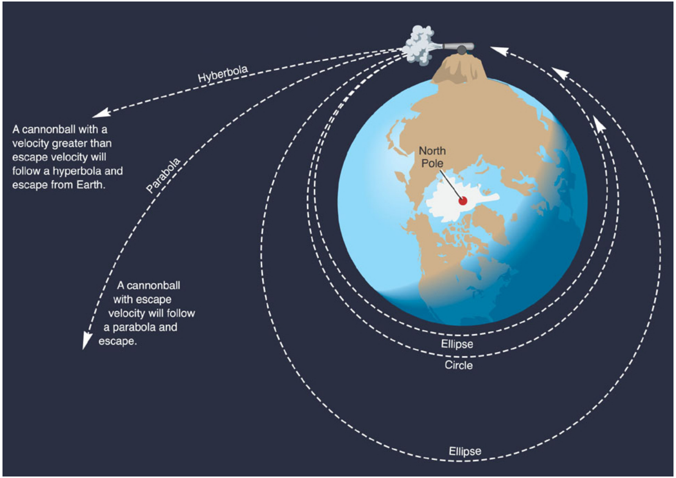
Comparing Orbital Velocity to Escape Velocity

Orbital and escape velocities differ by a factor of $\sqrt{2}$:

$$v_{\text{orbit}} = \sqrt{\frac{GM}{r}} \qquad v_{\text{esc}} = \sqrt{\frac{2GM}{r}} = \sqrt{2}v_{\text{orbit}}$$

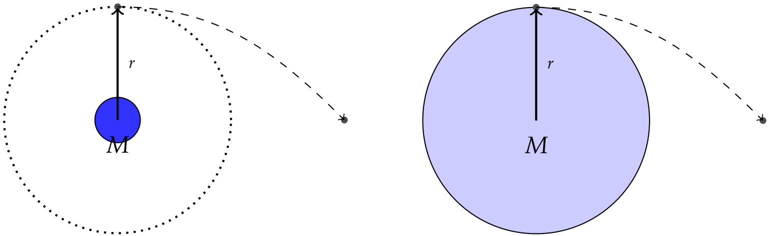
- What happens if $v_{\text{orbit}} < v < v_{\text{esc}}$? What kind of an orbit does it have?
- What happens if $v < v_{\text{orbit}}$? It doesn't necessarily mean that the object will crash into the planet.

Non-Circular Orbits



Escaping from Orbit

These two objects have the same escape velocity:



Both are a distance r from the centre of the planet, but the one in orbit (left) already has orbital velocity v_{orbit}, so escaping from orbit only requires an additional speed of

$$\Delta v = v_{\text{esc}} - v_{\text{orbit}} = (\sqrt{2} - 1)v_{\text{orbit}}$$

Orbital Kinetic Energy

We can obtain the **orbital kinetic energy** by applying the orbital speed in our expression of kinetic energy:

$$K_{\text{orbit}} = \frac{1}{2}mv_{\text{orbit}}^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \boxed{\frac{GMm}{2r}}$$

This equation uses the expression for orbital velocity v_{orbit}, it relies on the same assumption, i.e. a perfectly circular orbit

Total Orbital Energy

Total energy is the sum of kinetic and gravitational potential energies:

$$E_{\text{tot}} = K + U_g$$

Substituting the expressions for K and U_g derived for objects in orbit:

$$E_{\text{tot}} = \frac{GMm}{2r} + \left(-\frac{GMm}{r}\right) = \boxed{-\frac{GMm}{2r}}$$

Orbital Energies

Simple relationships between K, U_g and E_{tot}

Orbital kinetic energy:

$$K_{\text{orbit}} = \frac{GMm}{2r}$$

Gravitational potential energy:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orbit}}$$

Total orbital energy:

$$E_T = K + U_g = -\frac{GMm}{2r} = -K_{\text{orbit}}$$

Example Problem

Example 7: On March 6, 2001, the Mir space station was deliberately crashed into Earth. At the time, its mass was $1.39 \times 10^{3}\,\mathrm{kg}$ and its altitude was $220\,\mathrm{km}$.

- Prior to the crash, what was its binding energy to Earth?
- How much energy was released in the crash? Assume that its orbit was circular