Chapter 6 Trigonometry 1

1. Congruence and Similarity in Triangles

Definitions:
1. Ratio - a relationship between two quantities, usually expressed as a fraction.
2. Proportional - all ratios between corresponding sides are equal between two objects, and the ratio is called the scale factor.
3. Congruent - two objects are congruent if they have the same shape and size (the scale factor is 1)
4. Similar - objects are proportional, but not congruent. (the scale factor is not 1)

Two triangles may be congruent, similar, or neither. The order of the letters naming the vertices of congruent or similar triangles indicates the order in which the vertices, sides, and angles of one triangle correspond to the vertices, sides, and angles of the other.

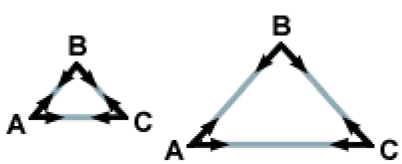
Two triangles are **similar**, then:

- three pairs of corresponding angles are congruent (therefore the third pair of corresponding angles are also congruent).
- three pairs of corresponding sides are proportional.

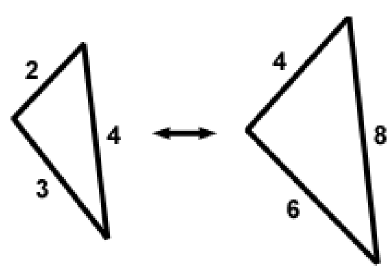
Proving Similarity of Triangles
There are three easy ways to prove similarity.

AA~ (Angle-Angle)

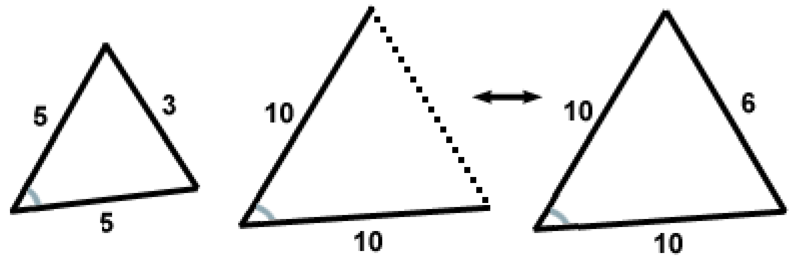
If two pairs of corresponding angles in a pair of triangles are congruent, then the triangles are similar.



SSS~ (Side-Side-Side)
Another way to prove triangles are similar is by SSS, side-side-side. If the measures of corresponding sides are known, then their proportionality can be calculated. If all three pairs are in proportion, then the triangles are similar.

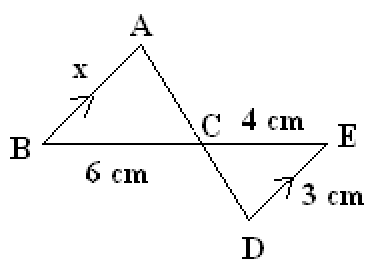


SAS~ (Side-Angle-Side)
If two pairs of corresponding sides are in proportion, and the included angle of each pair is equal, then the two triangles they form are similar. Any time two sides of a triangle and their included angle are fixed, then all three vertices of that triangle are fixed. With all three vertices fixed and two of the pairs of sides proportional, the third pair of sides must also be proportional.



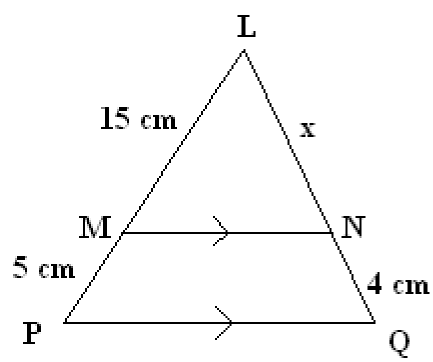
Example 1: Determine the unknown measures.

a)



Since AB // ED
∴ ∠ A = ∠ D and ∠ B = ∠ E
(Alternate Interior Angles)
∴ ΔABC ~ ΔDEC (AA~)
∴ $\frac{AB}{DE} = \frac{BC}{EC}$
∴ $\frac{x}{3} = \frac{6}{4}$
x = 9/2

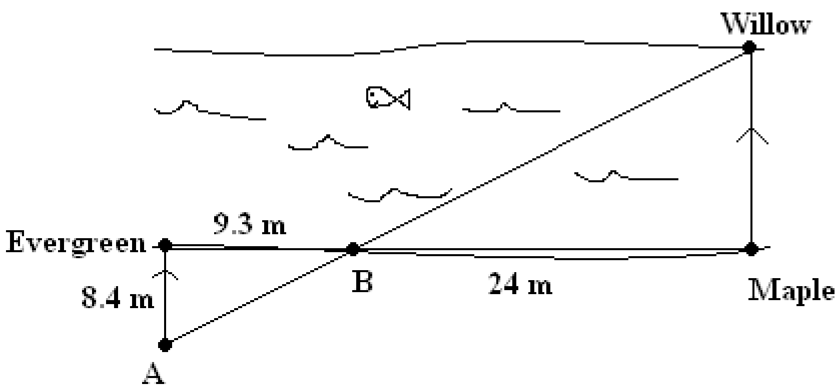
b)



Since MN // PQ
∴ ∠ LMN = ∠ P and ∠ LNM = ∠ Q
(Corresponding Angles)
∴ ΔLMN ~ ΔLPQ (AA~)
∴ $\frac{LM}{LP} = \frac{LN}{LQ}$
∴ $\frac{15}{15+5} = \frac{x}{x+4}$
15(x + 4) = 20x
15x + 60 = 20x
5x = 60
x = 12

Example 2: Solve for an Unknown Side

To determine the width of a river, Naomi finds a willow tree and a maple tree that are directly across from each other on opposite shores. Using a third tree on the shoreline, Naomi plants two stakes, A and B, and measures the distances shown. Find the width of the river.



Solution:

ΔABE ~ ΔWBM (AA~)

$\therefore \frac{EB}{MB} = \frac{AE}{WM}$

$\therefore \frac{9.3}{24} = \frac{8.4}{width}$

9.3width = 201.6

x = 21.7 m

Congruent triangles are a special type of similar triangles. Congruent triangles have the same shape (similar triangles) and size.

Two triangles are congruent if all 6 pairs of corresponding angles and sides are congruent.

Proving Congruence of Triangles

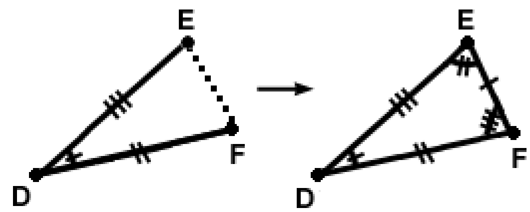
When proving that triangles are congruent, it is not necessary to prove that all three pairs of corresponding angles and all three pairs of corresponding sides are congruent.

The following three methods are shortcuts for determining congruence between triangles without having to prove the congruence of all six corresponding parts. They are called SSS, SAS, and ASA.

SSS (Side-Side-Side)
The simplest way to prove that triangles are congruent is to prove that all three sides of the triangle are congruent. When all the sides of two triangles are congruent, the angles of those triangles must also be congruent. This method is called side-side-side, or SSS for short. To use it, you must know the lengths of all three sides of both triangles, or at least know that they are equal.

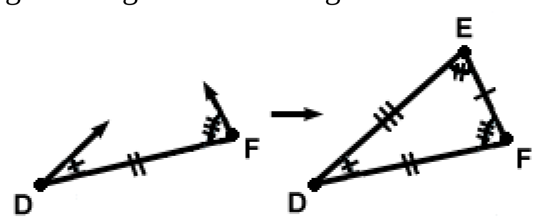
SAS (Side-Angle-Side)

A second way to prove the congruence of triangles is to show that two sides and their included angle are congruent. This method is called side-angle-side. It is important to remember that the angle must be the included angle--otherwise you can't be sure of congruence. When two sides of a triangle and the angle between them are the same as the corresponding parts of another triangle there is no way that the triangles aren't congruent. When two sides and their included angle are fixed, all three vertices of the triangle are fixed. Therefore, two sides and their included angle is all it takes to define a triangle; by showing the congruence of these corresponding parts, the congruence of each whole triangle follows.



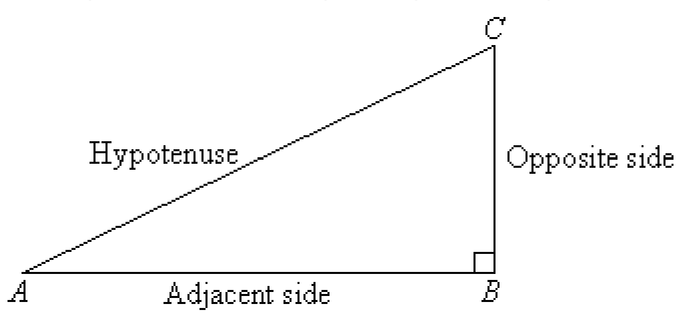
ASA (Angle-Side-Angle)

The third major way to prove congruence between triangles is called ASA, for angle-side-angle. If two angles of a triangle and their included side are congruent, then the pair of triangles is congruent. When the side of a triangle is determined, and the two angles from which the other two sides point, the whole triangle is already determined, there is only one point, the third vertex, where those other sides could possibly meet. For this reason, ASA is also a valid shortcut/technique for proving the congruence of triangles.



2. Right-Angled Triangle

Naming the Sides of a Right-Angled Triangle



The side opposite the right angle is called the **hypotenuse**. It is the largest side of a right-angled triangle.

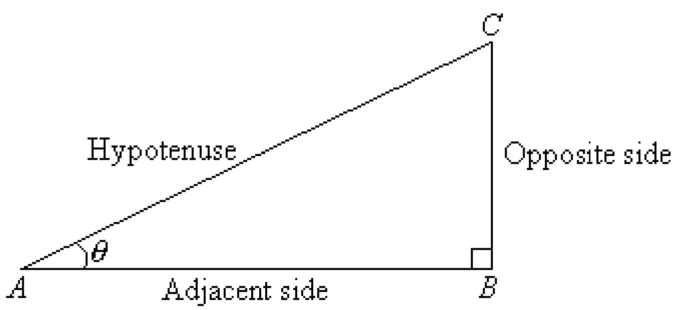
If you stand at *A* in the triangle *ABC*, the side *BC* is opposite to you and the side *AB* is next to you. We therefore say that *BC* is the **opposite side** to angle *A* and *AB* is the **adjacent side** to angle *A*.

Notation

The Greek letter θ, theta (rhymes with 'neater'), is often used to represent the measure of an angle in degrees.

3. Trigonometric Ratios

The trigonometric ratios – sine, cosine, and tangent – for an angle θ in a right-angled triangle are defined as follows:



$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \{\text{SOH}\}$$

$$\cos = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \{\text{CAH}\}$$

$$\tan = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \{\text{TOA}\}$$

Note that the sine of the angle θ is written as $\sin\theta$ ('sin' rhymes with 'line'); and similarly, cosine is written as cos (pronounced 'coz') and tangent is written as tan.

If you have trouble remembering the definitions, just remember **SOH CAH TOA**.

4. Using a Graphics/Scientific Calculator

A graphics calculator can be used to find the values of the trigonometric ratios sin, cos and tan for the given angle θ, when θ is measured in degrees.

Example 1

Evaluate the following to 4 decimal places: Solution:

a. $\sin 25°$

b. $\cos 40°$

c. $\tan 54°$

a. To find the value of $\sin 25°$, press:

$\sin 25$ ENTER

to obtain $\sin 25° = 0.4226$

b. To find the value of $\cos 40°$, press:

$\cos 40$ ENTER

to obtain $\cos 40° = 0.7660$

c. To find the value of $\tan 54°$, press:

$\tan 54$ ENTER

to obtain $\tan 54° = 1.3764$

Recall that:

1 degree = 60 minutes

1 minute = 60 seconds

This is abbreviated as follows:

$1° = 60'$ $\quad 1' = 60''$

Example 2

Evaluate the following to 4 decimal places:

a. $\sin 42°30'$

b. $\cos 30°48'$

c. $\tan 55°25'$

Note:

To find the value of $\sin 42°30'$, the angle must be converted to degrees.

Solution:

a. To find the value of $\sin 42°30'$, press:

$\sin(42 + 30/60)$ ENTER

to obtain $\sin 42°30' = 0.6756$

b. To find the value of $\cos 30°48'$, press:

$\cos(30 + 48/60)$ ENTER

to obtain $\cos 30°48' = 0.8590$

c. To find the value of $\tan 55°25'$, press:

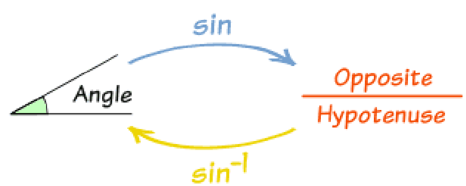
$\tan(55 + 25/60)$ ENTER

to obtain $\tan 55°25' = 1.4505$

Finding the Angles – Use $\sin^{-1}$

What is $\sin^{-1}$?

Well, the Sine function ***"sin"*** takes an angle and gives us the **ratio** "opposite/hypotenuse",



But in this case we know the ratio “opposite/hypotenuse” but want to know the **angle**.
So we want to go **backwards**. That is why we use ***sin^{-1}***, which means “inverse sine” or “arcsin”.

Example: Sine Function: sin(**30°**) = **0.5** Inverse Sine Function: $\sin^{-1}$(**0.5**) = **30°**

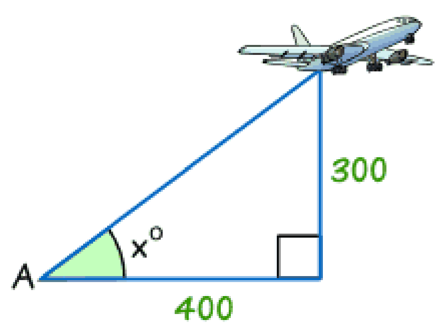
On the calculator you would press one of the following (depending on your brand of calculator): either '2ndF sin' or 'shift sin'.

Step By Step

These are the four steps we need to follow:

- **Step 1** Decide which two sides we know – out of Opposite, Adjacent and Hypotenuse.
- **Step 2** Use SOHCAHTOA to decide which one of Sine, Cosine **or** Tangent to use in this question.
- **Step 3** Use your calculator to calculate the fraction Opposite/Hypotenuse, Adjacent/Hypotenuse **or** Opposite/Adjacent (whichever is appropriate).
- **Step 4** Find the angle from your calculator, using one of $\sin^{-1}$, $\cos^{-1}$ **or** $\tan^{-1}$

Example 1: Find the size of the angle of elevation of the plane from point A on the ground.

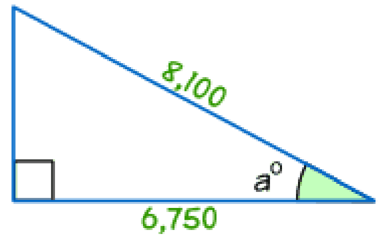


- **Step 1** The two sides we know are **O**pposite (300) and **A**djacent (400).
- **Step 2** SOHCAH**TOA** tells us we must use **T**angent.
- **Step 3** Use your calculator to calculate **Opposite/Adjacent** = 300/400 = **0.75**
- **Step 4** Find the angle from your calculator using **$\tan^{-1}$**

tan x° = opposite/adjacent = 300/400 = 0.75

$\tan^{-1}$ of 0.75 = **36.9°** (correct to 1 decimal place)

Example 2: Find the size of angle a°



- **Step 1** The two sides we know are **A**djacent (6,750) and **H**ypotenuse (8,100).
- **Step 2** SOH**CAH**TOA tells us we must use **C**osine.
- **Step 3** Use your calculator to calculate Adjacent / Hypotenuse = 6,750/8,100 = 0.8333
- **Step 4** Find the angle from your calculator using $\mathbf{cos^{-1}}$ of 0.8333:

cos a° = 6,750/8,100 = 0.8333

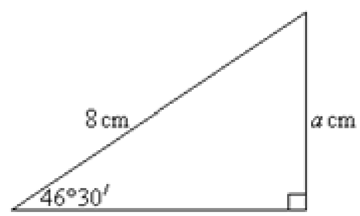
$\mathbf{cos^{-1}}$ of 0.8333 = **33.6°** (to 1 decimal place)

5. Finding Side-Lengths

We can use the trigonometric ratios sine, cosine and tangent to find any side-length of a right-angled triangle if we know another angle and one side-length.

1) Using the Sine Ratio

Example 1: Find the value of the pronumeral, correct to 2 decimal places, in the given diagram.



Solution:

$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \{\text{SOH}\}$$

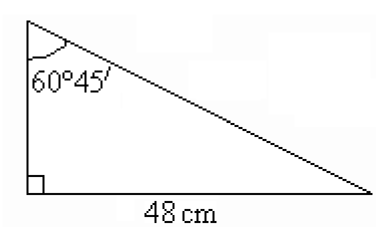
$$\sin 46°30' = \frac{a}{8} \qquad \{\text{Make } a \text{ the subject}\}$$

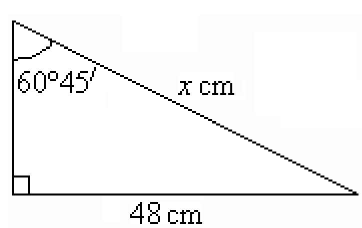
$$8 \times \sin 46°30' = \frac{a}{8} \times 8$$

$$\therefore a = 5.8030$$

$$= 5.80 \qquad \{\text{Correct to 2 decimal places}\}$$

Example 2: Find the length of the hypotenuse, correct to 2 decimal places, in the given diagram.





***Solution**:* Let the length of the hypotenuse be *x* cm.

$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \{\text{SOH}\}$$

$$\sin 60°45' = \frac{48}{x} \qquad \{\text{Make } x \text{ the subject}\}$$

$$x \times \sin 60°45' = x \times \frac{48}{x}$$

$$x \sin 60°45' = 48$$

$$\therefore x = \frac{48}{\sin 60°45'}$$

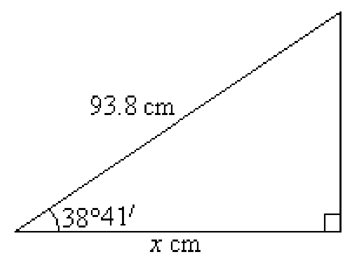
$$= 55.0146$$

$$= 55.01 \qquad \{\text{Correct to 2 decimal places}\}$$

So, the length of the hypotenuse is 55.01 cm.

2) Using the Cosine Ratio

Example 1: Find the value of the pronumeral in the following diagram, correct to 2 decimal places.



Solution:

$$\cos = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \{\text{CAH}\}$$

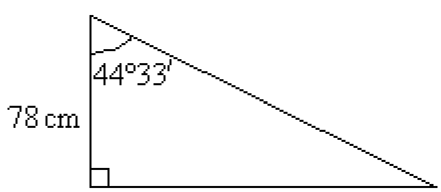
$$\cos 38°41' = \frac{x}{93.8} \qquad \{\text{Make } x \text{ the subject}\}$$

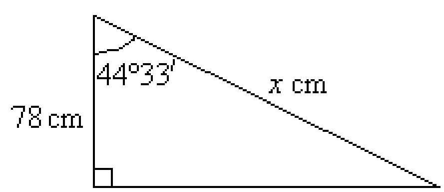
$$93.8 \times \cos 38°41' = \frac{x}{93.8} \times 93.8$$

$$\therefore x = 73.2214$$

$$= 73.22 \qquad \{\text{Correct to 2 decimal places}\}$$

Example 2: Find the length of the hypotenuse in the following diagram, correct to 2 decimal places.





Solution: Let the length of the hypotenuse be *x* cm.

$$\cos = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \{\text{CAH}\}$$

$$\cos 44°33' = \frac{78}{x} \qquad \{\text{Make } x \text{ the subject}\}$$

$$x\cos 44°33' = \frac{78}{x} \times x$$

$$x\cos 44°33' = 78$$

$$\therefore x = \frac{78}{\cos 44°33'}$$

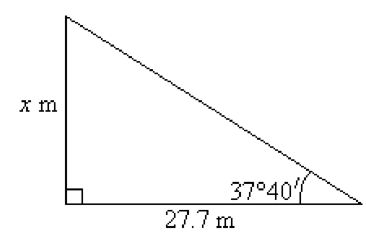
$$= 109.4524$$

$$= 109.45 \qquad \{\text{Correct to 2 decimal places}\}$$

So, the length of the hypotenuse is 109.45 cm.

3) Using the Tangent Ratio

Example 1: Find the value of the pronumeral in the following diagram, correct to 2 decimal places.



Solution:

$$\tan = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \{\text{TOA}\}$$

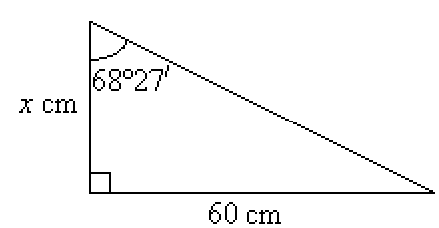
$$\tan 37°40' = \frac{x}{27.7} \qquad \{\text{Make } x \text{ the subject}\}$$

$$27.7 \times \tan 37°40' = 27.7 \times \frac{x}{27.7}$$

$$\therefore x = 21.3833$$

$$= 21.38 \qquad \{\text{Correct to 2 decimal places}\}$$

Example 2: Find the value of the pronumeral in the following diagram, correct to 2 decimal places.



Solution:

$$\tan = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \{\text{TOA}\}$$

$$\tan 68°27' = \frac{60}{x} \qquad \{\text{Make } x \text{ the subject}\}$$

$$x \tan 68°27' = x \times \frac{60}{x}$$

$$= 60$$

$$\therefore x = \frac{60}{\tan 68°27'}$$

$$= 23.6951$$

$$= 23.70 \qquad \{\text{Correct to 2 decimal places}\}$$

6. Application

Example 1: A surveyor measures the angle to the peak of a hill from point *A*, as shown in the diagram. Calculate the height, *h*, of the hill rounded to 2 decimal places.

Solution:

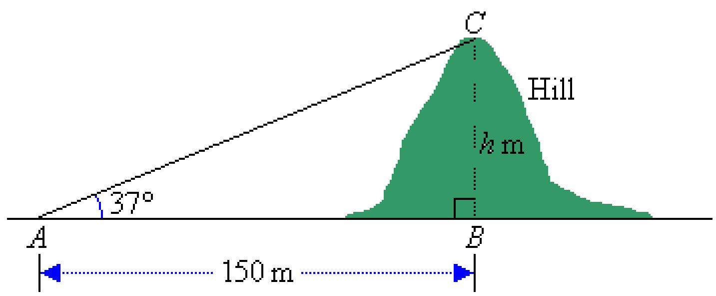
In the right-angled triangle *ABC*,

$$\tan = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \{\text{TOA}\}$$

$$\tan 37° = \frac{h}{150} \qquad \{\text{Make } h \text{ the subject}\}$$

$$150 \times \tan 37° = 150 \times \frac{h}{150}$$

$$\therefore h = 113.03$$



So, the height of the hill is 113.03 m.

Example 2: A ladder 5 m long has its end just resting on the top of a fence 3 m high. What angle, to the nearest minute, does the ladder make with the ground?

Solution:

Let θ be the angle made by the ladder with the ground.

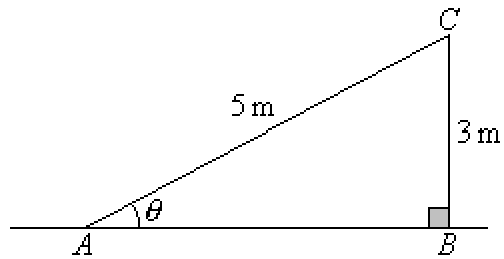
From right-angled triangle ABC,

$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \{SOH\}$$

$$\therefore \sin\theta = \frac{3}{5}$$

$$= 0.6$$

$$\theta = 36°52'$$

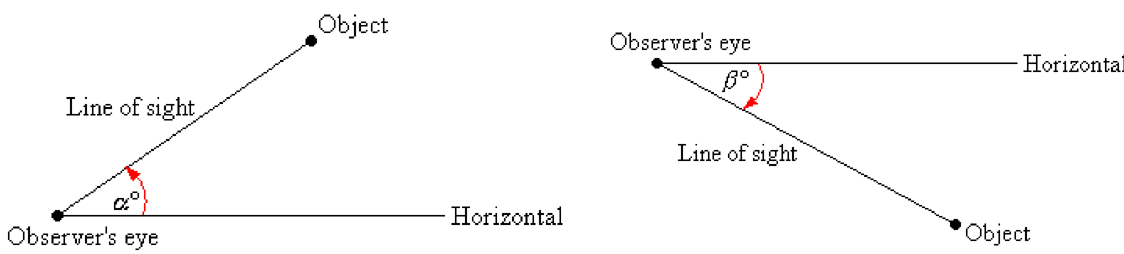


So, the angle made by the ladder with the ground is $36°52'$.

Angles of Elevation and Depression

The **angle of elevation** of an object as seen by an observer is the angle between the horizontal and the line from the object to the observer's eye (the line of sight).

If the object is below the level of the observer, then the angle between the horizontal and the observer's line of sight is called the **angle of depression**.



The angle of elevation of the object from the observer is $\alpha°$.

The angle of depression of the object from the observer is $\beta°$.

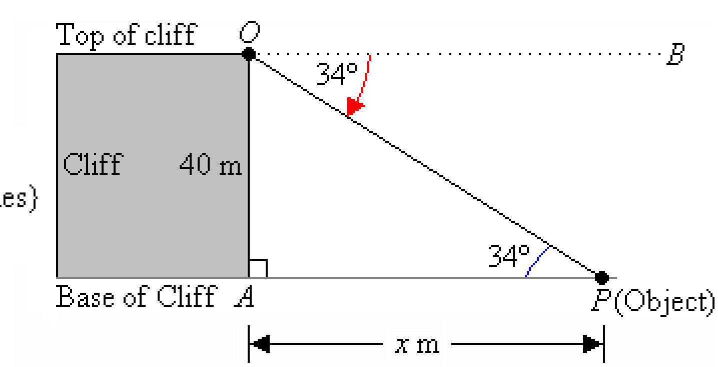
Example 2: From the top of a vertical cliff 40 m high, the angle of depression of an object that is level with the base of the cliff is 34°. How far is the object from the base of the cliff?

Solution: Let *x* m be the distance of the object from the base of the cliff.

Angle of depression $= 34°$

But $\angle APO = \angle BOP$ (Alternate angles)

$\therefore \angle APO = 34°$



From ΔAPO, we have:

$$\tan 34^\circ = \frac{40}{x} \qquad \{\text{TOA}\}$$

$$\therefore\ x\tan 34^\circ = \frac{40}{x}\times x \qquad \{\text{Multiply both sides by } x\}$$

$$0.6745x = 40 \qquad \{\text{Divide both sides by } 0.6745\}$$

$$\frac{0.6745x}{0.6745} = \frac{40}{0.6745}$$

$$x = 59.30$$

So, the object is 59.30 m from the base of the cliff.