Operations and Calculations

1. Order of Operations - *BEDMAS*

"Operations" mean things like add, subtract, multiply, divide, squaring, etc. If it isn't a number it is probably an operation.

But, when you see something like $7 + (6 \times 5^2 + 3)$, what part should you calculate first?

What is mean for *BEDMAS*?

B	**B**rackets first
E	**E**xponents (ie Powers and Square Roots, etc.)
DM	**D**ivision and Multiplication (left-to-right)
AS	**A**ddition and Subtraction (left-to-right)

- Divide and Multiply rank equally (and go left to right).
- Add and Subtract rank equally (and go left to right)

In the US they say "Parenthesis" instead of Brackets, so they say "*PEMDAS*"

P	**P**arentheses first
E	**E**xponents (ie Powers and Square Roots, etc.)
MD	**M**ultiplication and **D**ivision (left-to-right)
AS	**A**ddition and **S**ubtraction (left-to-right)

- Divide and Multiply rank equally (and go left to right).
- Add and Subtract rank equally (and go left to right)

You can remember by saying "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally".

Note:
In the UK they say BODMAS (Brackets, Orders, Divide, Multiply, Add, Subtract), and in Canada they say BEDMAS (Brackets, Exponents, Divide, Multiply, Add, Subtract). It all means the same thing! It doesn't really matter how you remember it, just so long as you get it right.

Example: How do you work out $12 / 6 \times 3$?

Division and Multiplication rank equally, so just go left to right:
First $12 / 6 = 2$, then $2 \times 3 = 6$

Oh, yes, and what about $7 + (6 \times 5^2 + 3)$?

$7 + (6 \times 5^2 + 3)$	
$7 + (6 \times 25 + 3)$	Start inside Brackets, and then use "Orders" First
$7 + (150 + 3)$	Then Multiply
$7 + (153)$	Then Add
$7 + 153$	Brackets completed, last operation is add
160	DONE !

2. Multiplying Powers of the Same Base

An exponent is a number which tells us how many times to multiply a factor (either a numerical, or a variable factor) times itself. This "factor" is called the base.

Using the rule stated above the result would be
*for a numerical base; $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$
*for a variable base; $x^5 = x \cdot x \cdot x \cdot x \cdot x$

Now let's look at the special rule for exponents under the operation of multiplication.

We know that 3^2 means 3•3 and 3^3 means 3•3•3
therefore $3^2 \bullet 3^3 = \underline{3 \bullet 3} \bullet \underline{3 \bullet 3 \bullet 3}$ or 3^5

using a variable factor
$a^4 \bullet a^2 = \underline{a \bullet a \bullet a \bullet a} \bullet \underline{a \bullet a}$ or a^6

If you look closely you'll see that the exponent in the product is equal to the sum of the exponents in each of the factors.

In general the rule of exponents, when multiplying powers of the same base, is:

$$x^a \cdot x^b = x^{a+b}$$

*Note: This rule does NOT work if the bases are NOT the same.

$2^2 \bullet 3^3$ *cannot* be simplified using the rule of exponents stated above because;

$2^2 \bullet 3^3 = 2 \bullet 2 \bullet 3 \bullet 3 \bullet 3$
and not all of the factors (bases) are the same.

For examples:
$3^2 \bullet 3^5 = 3^{2+5} = 3^7$
$x^2 \bullet x^7 \bullet x^4 = x^{2+7+4} = x^{13}$

3. Dividing Powers of the Same Base

We know that 3^4 means 3 • 3 • 3 • 3 and 3^2 means 3 • 3
therefore $3^4 \div 3^2 = 81 \div 9 = 9$ or 3^2

If you look closely you'll see that the exponent in the quotient is equal to the difference of the exponents in each of the factors.

In general the rule of exponents, when dividing powers of the same base, is:

$$x^a \div x^b = x^{a-b}$$

*Note: This rule does NOT work if the bases are NOT the same.

$2^5 \div 4^2$ *cannot* be simplified using the rule of exponents.
To perform this division you just simplify the terms and perform the division:
$2^5 = 32$, and $4^2 = 16$
$32 \div 16 = 2$

For example:
$4^{16} \div 4^7 = 4^{16-7} = 4^9$
$m^5 \div m^4 = m^{5-4} = m^1$ or m
$x^{12} \div x^7 \div x^2 = x^{12-7-2} = x^3$

4. Powers of Different Bases

Caution! The rule above works only when multiplying powers of the same base. For instance,

$$(x^3)(y^4) = (x)(x)(x)(y)(y)(y)(y)$$

If you write out the powers, you see there's no way you can combine them.

Except in one case: **If the bases are different but the exponents are the same**, then you can **combine them**. Example:

$$(x^3)(y^3) = (x)(x)(x)(y)(y)(y)$$

But you know that it doesn't matter what order you do your multiplications in or how you group them. Therefore,

$$(x)(x)(x)(y)(y)(y) = (x)(y)(x)(y)(x)(y) = (xy)(xy)(xy)$$

But from the very definition of powers, you know that's the same as $(xy)^3$. And it works for any common power of two different bases:

$$\mathbf{x^a y^a = (xy)^a}$$

It should go without saying, but I'll say it anyway: all the laws of exponents work in both directions. If you see $(4x)^3$ you can decompose it to $(4^3)(x^3)$, and if you see $(4^3)(x^3)$ you can combine it as $(4x)^3$.

$(4x)^3 = (4^3)(x^3) = 64x^3$

In the same way, **dividing different bases** can't be simplified unless the exponents are equal. $x^3 \div y^2$ can't be combined because it's just xxx/yy; But $x^3 \div y^3$ is xxx/yyy, which is (x/y)(x/y)(x/y), which is $(x/y)^3$.

$$\boldsymbol{\frac{x^a}{y^a} = \left(\frac{x}{y}\right)^a}$$

5. Powers of Powers

What do you do with an expression like $(x^5)^4$? There's no need to guess—work it out by counting.

$$(x^5)^4 = (x^5)(x^5)(x^5)(x^5)$$

Write this as an array:

$$\begin{array}{l} x^5 = (x)\ \ (x)\ \ (x)\ \ (x)\ \ (x) \\ x^5 = (x)\ \ (x)\ \ (x)\ \ (x)\ \ (x) \\ x^5 = (x)\ \ (x)\ \ (x)\ \ (x)\ \ (x) \\ x^5 = (x)\ \ (x)\ \ (x)\ \ (x)\ \ (x) \end{array}$$

How many factors of x are there? You see that there are 5 factors in each row from x^5 and 4 rows from $(\ )^4$, in all 5×4=20 factors. Therefore,

$$(x^5)^4 = x^{20}$$

As you might expect, this applies to any **power of a power: you multiply the exponents**. For instance, $(k^{-3})^{-2} = k^{(-3)(-2)} = k^6$. In general,

$$\mathbf{(x^a)^b = x^{ab}}$$

I can just hear you asking, "So when do I add exponents and when do I multiply exponents?" Don't try to remember a rule—work it out! When you have a **power of a power**, you'll always have a rectangular array of factors, like the example above. Remember the old rule of length×width, so the combined exponent is formed by multiplying. On the other hand, when you're only <u>multiplying two powers together</u>, like g^2g^3, that's just the same as stringing factors together,

$$g^2g^3 = (gg)(ggg) = (ggggg) = g^5$$

You can always refresh your memory by counting simple cases, like

$$x^2x^3 = (xx)(xxx) = x^5$$

versus

$$(x^2)^3 = (xx)^3 = (xx)(xx)(xx) = x^6$$

6. The Zero Exponent

You probably know that **anything to the 0 power** is 1. But now you can see why. Consider x^0.

By the <u>division rule</u>, you know that $x^3/x^3 = x^{(3-3)} = x^0$. But anything divided by itself is 1, so $x^3/x^3 = 1$. Things that are equal to the same thing are equal to each other: if x^3/x^3 is equal to both 1 and x^0, then 1 must equal x^0. Symbolically,

$$x^0 = x^{(3-3)} = x^3/x^3 = 1$$

There's one restriction. You saw that we had to create a fraction to figure out x^0. But division by 0 is not allowed, so our evaluation works for anything to the 0 power *except* zero itself:

$$\mathbf{x^0 = 1 \ (x \neq 0)}$$

7. Fractional Exponents

But what if the exponent is a fraction?

An exponent of $^1/_2$ is actually **square root**

And an exponent of $^1/_3$ is **cube root**

An exponent of $^1/_4$ is **4th root**

And so on!

$$4^{\frac{1}{2}} = \sqrt{4}$$

$$4^{\frac{1}{3}} = \sqrt[3]{4}$$

$$4^{\frac{1}{4}} = \sqrt[4]{4}$$

etc...

General Rule: It worked for ½, it worked with ¼, in fact it works generally:

$$\mathbf{x^{1/n} = \text{The n-th Root of } x}$$

8. Associative, Commutative and Distributive Laws

1) Commutative Laws
The "Commutative Laws" just mean that you can swap numbers over and still get the same answer when you add, or when you multiply.
$a + b = b + a$
$a \times b = b \times a$

Examples:

You can swap when you add: $3 + 6 = 6 + 3$
You can swap when you multiply: $2 \times 4 = 4 \times 2$

2) Associative Laws
The "Associative Laws" mean that it doesn't matter how you group the numbers (ie which you calculate first) when you add:

$(a + b) + c = a + (b + c)$ or when you multiply:
$(a \times b) \times c = a \times (b \times c)$

Examples:

This: $(2 + 4) + 5 = 6 + 5 = 11$

Has the same answer as this:	$2 + (4 + 5) = 2 + 9 = 11$
This:	$(3 \times 4) \times 5 = 12 \times 5 = 60$
Has the same answer as this:	$3 \times (4 \times 5) = 3 \times 20 = 60$

3) Distributive Law

The "Distributive Law" is the BEST one of all, but needs careful attention.

It means you get the same answer when you:
add up some numbers then do a multiply, or do each multiply separately then add them
Like this:
$(a + b) \times c = a \times c + b \times c$

Examples:

This:	$(2 + 4) \times 5 = 6 \times 5 = 30$
Has the same answer as this:	$2\times5 + 4\times5 = 10 + 20 = 30$
This:	$(6 - 4) \times 3 = 2 \times 3 = 6$
Has the same answer as this:	$6\times3 - 4\times3 = 18 - 12 = 6$

Uses:
Sometimes it is easier to break up a difficult multiplication:

What is 204×6?

$204 \times 6 = 200 \times 6 + 4 \times 6 = 1{,}200 + 24 = 1{,}224$

Or to combine:

What is $6 \times 16 + 4 \times 16$?

$6 \times 16 + 4 \times 16 = (6+4) \times 16 = 10 \times 16 = 160$

Conclusion

Commutative Laws:	$\mathbf{a + b = b + a}$ $\mathbf{a \times b = b \times a}$
Associative Laws:	$\mathbf{(a + b) + c = a + (b + c)}$ $\mathbf{(a \times b) \times c = a \times (b \times c)}$
Distributive Law:	$\mathbf{(a + b) \times c = a \times c + b \times c}$

Questions in class

1. In base four, one counts 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, 100, etc. What is the base ten equivalent of the base four number 123?

2. In base four, what would be the value of $12_{four} + 13_{four}$?

3. When the base ten number 99 is converted to base five, what is the digit in the units place?

4. Which of the following terms is equal to $\sqrt{\sqrt{\sqrt{2}}}$?

a. $2^{\frac{3}{2}}$ b. $2^{\frac{1}{8}}$ c. $2^{\frac{1}{3}}$ d. $2^{\frac{2}{3}}$ e. $2^{\frac{3}{8}}$

5. Which of the following numbers is the least?

a. $\left(-\frac{1}{8}\right)^{\frac{2}{3}}$ b $\left(\frac{1}{8}\right)^{-\frac{2}{3}}$ c $\left(\frac{1}{8}\right)^{-\frac{4}{3}}$ d $\left(-\frac{1}{8}\right)^{-\frac{2}{3}}$ e $(-8)^{-\frac{4}{3}}$

6. In the following number written in scientific notation, what is the actual value of the "9"?
$2.71972164 \times 10^{-11}$

7. Pat has a collection of stickers for numbering pages in a scrapbook. Each sticker has 1 digit on it (0, 1, 2, 3, 4, 5, 6, 7, 8, or 9). Pat has twenty-two stickers that have "2" on them. Pat has plenty of the rest of the digits. If Pat starts at 1, how many pages can he number?

8. Mersenne primes are prime numbers that can be written in the form 2^n-1. Which of the following numbers is NOT a Mersenne prime.

a.3 b.7 c.31 d.127 e.513

9. What is the value of $\frac{11! - 10!}{9!}$?

10. For what value of x does $10^x \cdot 100^{2x} = 1000^5$?