Lesson 15: Unit 9 – Operations on Functions

This unit extends our understanding of functions to include functions formed by the addition, subtraction, multiplication, and division of two functions, and from the composition of two functions. Through investigation, key characteristics and properties of these combined functions will be analyzed, and connections will be made between the algebraic, numeric, and graphical representations.

The sum and difference of two functions are defined as follows.
If f and g are two functions, the **sum function**, denoted by $f+g$, is defined by

$(f+g)(x) = f(x) + g(x)$

Similarly, the **difference function**, denoted by $f-g$, is defined by

$(f-g)(x) = f(x) - g(x)$

The sum or difference of two functions is a function whose domain is the set of all real numbers that are in the domain of both f and g ($D_{f\pm g} = D_f \cap D_g$).

Example

Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.
a. Determine $(f+g)(x)$ and $(f-g)(x)$.
b. Find $f(12)$, $g(12)$, $(f+g)(12)$, and $(f-g)(12)$.
c. Find $(f+g)(-6)$.
d. Sketch the graph of $y = (f+g)(x)$ and $y = (f-g)(x)$.

Solution

a. Since $(f+g)(x) = f(x)+g(x)$, then

$$(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$$

Similarly, $(f-g)(x) = f(x)-g(x)$. So,

$$(f-g)(x) = \frac{1}{2}x - \sqrt{x+4}$$

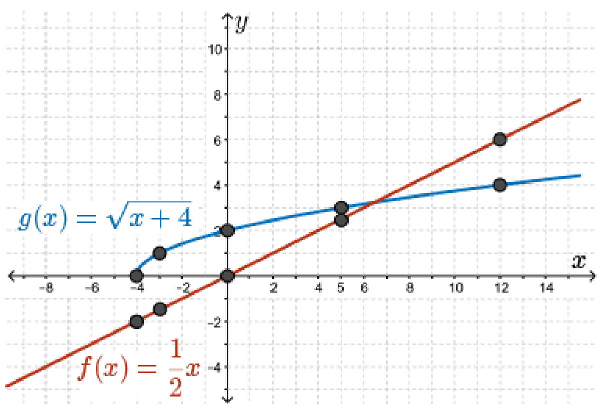
b. Using the equations for f and g and the results from part a),
f(12) = 6, g(12) = 4, (f + g)(12) = 10, and (f – g)(12) = 2.

c. The value of $(f+g)(-6)$ is undefined in the real numbers.
This is due to the fact that $g(x) = \sqrt{x+4}$ is undefined when $x<-4$.
The domain of $f(x) = \frac{1}{2}x$ is $\{x \mid x \in \mathbb{R}\}$. The domain of $g(x) = \sqrt{x+4}$ is $\{x \mid x \geq -4, x \in \mathbb{R}\}$.
The domain of $(f+g)(x)$ is the set of all real numbers that are in both the domain of $f(x)$ and $g(x)$.
That is, $\{x \mid x \in \mathbb{R}\} \cap \{x \mid x \geq -4, x \in \mathbb{R}\}$.

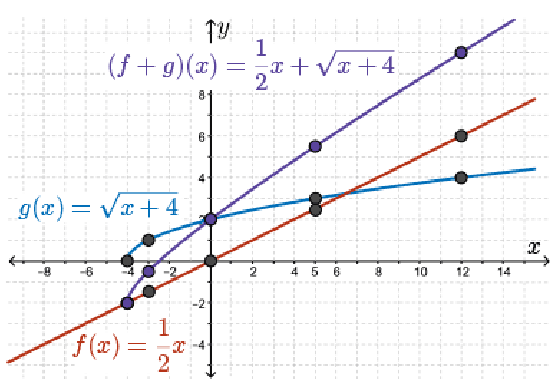
Thus, the domain of $(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$ is $\{x \mid x \geq -4, x \in \mathbb{R}\}$ or, using interval notation, $x \in [-4, \infty)$, $x \in \mathbb{R}$.
The function $(f-g)(x) = \frac{1}{2}x - \sqrt{x+4}$ has the same domain.
d.

x	$f(x)$	$g(x)$	$(f+g)(x)$
-8	-4	undefined	undefined
-6	-3	undefined	undefined
-4	-2	0	$-2+0=-2$
-3	-1.5	1	$-1.5+1=-0.5$
-2	-1	$\sqrt{2}$	$-1+\sqrt{2} \approx 0.41$
0	0	2	$0+2=2$
2	1	$\sqrt{6}$	$1+\sqrt{6} \approx 3.45$
4	2	$\sqrt{8}$	$2+\sqrt{8} \approx 4.83$
5	2.5	3	$2.5+3=5.5$
6	3	$\sqrt{10}$	$3+\sqrt{10} \approx 6.16$
8	4	$\sqrt{12}$	$4+\sqrt{12} \approx 7.46$
12	6	4	$6+4=10$

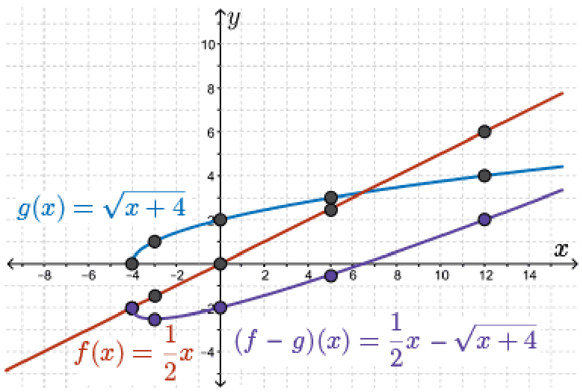
First, we graph $f(x)$ and $g(x)$.



Since $(f+g)(x) = f(x) + g(x)$, the graph of $(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$ can be obtained by adding the corresponding y-coordinates for values of x in the domain of both f and g.



Knowing the behaviour of each component graph helps in predicting the behaviour of the sum function.
Note that both *f* and *g* are increasing functions and *f*+ *g* is also an increasing function. Furthermore, the graph of the sum function has a shape similar to $g(x)=\sqrt{x+4}$, but its graph has also been influenced by the behaviour of $f(x)=\frac{1}{2}x$ since it increases more quickly than $g(x)$ as x increases in value.
Similarly, we can see that the graph of $(f-g)(x)=\frac{1}{2}x-\sqrt{x+4}$ has been influenced by the behaviour of the graphs of *f* and *g*. The graph is similar in shape to *g*.



In the interval where $f(x) < g(x)$, the difference function, $f-g$, is below the *x*-axis.
Where $f(x)>g(x)$, the difference function, $f-g$, is above the *x*-axis.
Wherever $f(x)$ and $g(x)$ intersect, $f-g$ has an *x*-intercept.
The component functions, *f* and *g*, are both increasing functions. However, $f-g$ appears to decrease from $x=-4$ to approximately $x=-3$ and then begins to increase.

Next we will investigate the behaviour and discuss the properties of functions formed by multiplying or dividing functions.
If *f* and *g* are two functions, the **product function**, denoted by $f \cdot g$ (or $f \times g$), is defined by

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

Similarly, the **quotient function**, denoted by *fg*, is defined by

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

The domain of the product or quotient function is the set of all real numbers, *x*, in the domain of both *f* and *g*, with possible added restrictions on *x* for $(f/g)(x)$ to ensure $g(x)\neq 0$.

Example

Consider the two functions $f(x)=x$ and $g(x)=x^2-4$.

a. Determine the equations of the functions $(f\cdot g)(x)$ and $\left(\frac{f}{g}\right)(x)$ and identify the domain of each.

b. Using the graphs of $f(x)=x$ and $g(x)=x^2-4$, sketch the graphs of $y=(f\cdot g)(x)$ and $y=(f/g)(x)$.

Solution

a. $(f{\cdot}g)(x)=f(x){\cdot}g(x) = x^3 - 4x$

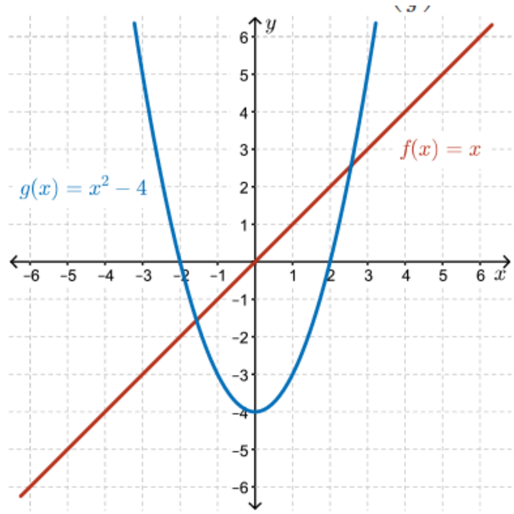
$(\frac{f}{g})(x)= \frac{f(x)}{g(x)} = \frac{x}{x^2-4}$

$f{\cdot}g$ is a polynomial function, as are f and g, so the domain is $\{x|\ x \in R\}$.

$\frac{f}{g}$ is a rational function which is undefined when $x^2-4=0$.

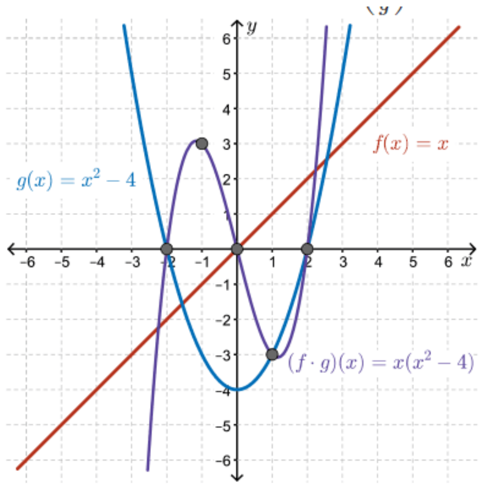
Thus, the domain of f/g is $\{x|\ x \neq \pm 2, x \in R\}$.

b. First, graph $f(x)=x$ and $g(x)=x^2-4$ on the same axes. Note that the domain of both functions is the set of all real numbers.



By multiplying the corresponding y-coordinates of points on f and g, we can determine points on the graph of $f{\cdot}g$.

x	$f(x)$	$g(x)$	$(f \cdot g)(x)$
-3	-3	5	$-3 \times 5 = -15$
-2	-2	0	0
-1	-1	-3	3
0	0	1	0
1	1	-3	-3
2	2	0	0
3	3	5	15



The product function, $(f\cdot g)(x)=x^3-4x$, is a cubic polynomial function.
The leading coefficient is positive, so $y \to -\infty$ as $x \to -\infty$, and $y \to \infty$ as $x \to \infty$.
From the factored form of the equation, we can verify that the zeros are −2, 0, and 2.

$(f\cdot g)(x) = x^3-4x$

$0 = x^3-4x$

$0 = x(x^2-4)$

$0 = x(x-2)(x+2)$

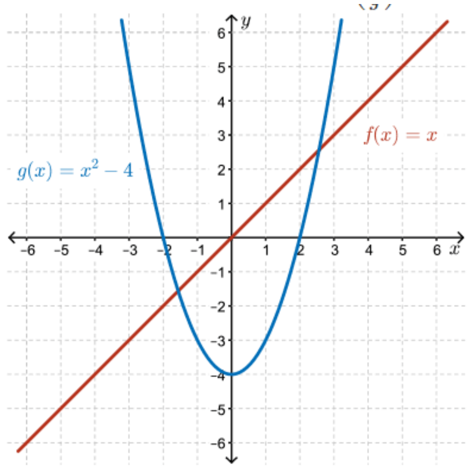
$0 = -2, 0, 2$

All the zeros are of order (multiplicity) 1, so the graph will pass directly through the x-axis at each zero.

Some general observations:

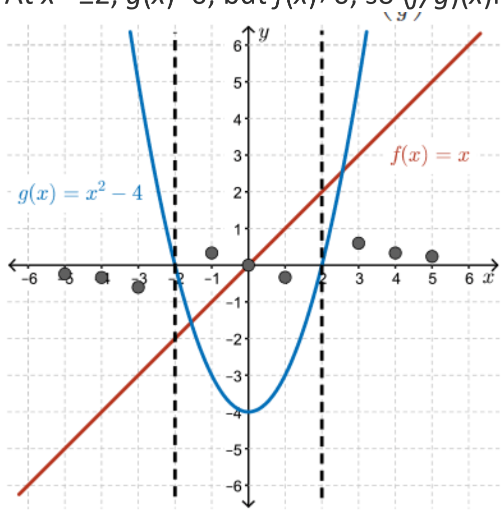
- *The zeros of f·g consist of all the zeros of the two functions f and g.*
- *When the graphs of f and g are both above the x-axis, or both below the x-axis, then the graph of f·g is above the x-axis.*
- *When the graphs of f and g lie on opposite sides of the x-axis, then the graph of f·g is below the x-axis.*
- *When f(x)=1, (f·g)(x) intersects g(x) since*
  $$(f\cdot g)(x)= f(x)\cdot g(x)= (1)g(x)= g(x)$$
- *When g(x)=1, (f·g)(x) intersects f(x) since* $(f\cdot g)(x)= f(x)\cdot g(x)= f(x)(1)= f(x)$

To determine the graph of $(f/g)(x)= \frac{x}{x^2-4}$, we start with the graph $f(x)=x$ and $g(x)=x^2-4$.
The domain of f/g is $\{x|\ x \neq \pm 2,\ x \in \mathbb{R}\}$ since $g(x) \neq 0$.
By dividing the y-coordinates of f by the corresponding y-coordinates of g, we can determine points on the graph of f/g.



x	$f(x)$	$g(x)$	$\left(\frac{f}{g}\right)(x)$
-5	-5	21	$-\frac{5}{21}$
-4	-4	12	$-\frac{1}{3}$
-3	-3	5	$-\frac{3}{5}$
-2	-2	0	undefined
-1	-1	-3	$\frac{1}{3}$
0	0	1	0
1	1	-3	$-\frac{1}{3}$
2	2	0	undefined
3	3	5	$\frac{3}{5}$
4	4	12	$\frac{1}{3}$
5	5	21	$\frac{5}{21}$

At $x = \pm 2$, $g(x)=0$, but $f(x) \neq 0$, so $(f/g)(x)$has vertical asymptotes $x=2$ and $x=-2$.

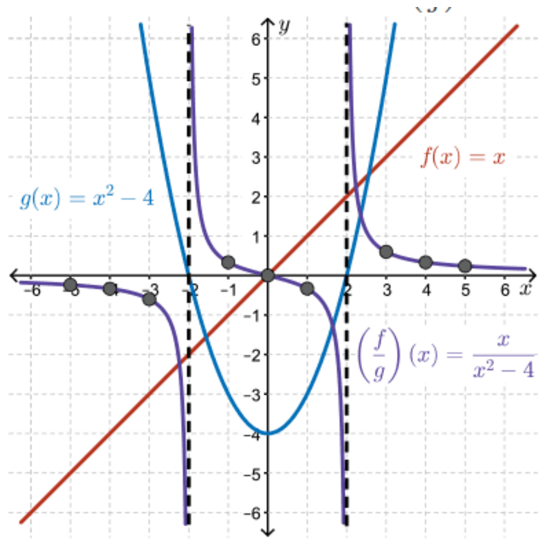


Since the degree of $g(x)$ is greater than the degree of $f(x)$, then $(f/g)(x) \to 0$ as $x \to \pm\infty$.

When $x<-2$, $f<0$ (below the x-axis) and $g>0$(above the x-axis), so $f/g<0$ and its graph will approach the x-axis from below as $x \to -\infty$ and approach $-\infty$ as $x \to -2^-$.

When $-2<x<0$, both $f<0$ and $g<0$, so $f/g>0$ and the graph approaches $+\infty$ as $x \to -2^+$.

In a similar way, we can determine the behaviour of the graph to the right of the y-axis. Notice that $(f/g)(x)$has odd symmetry.



Composition of Functions

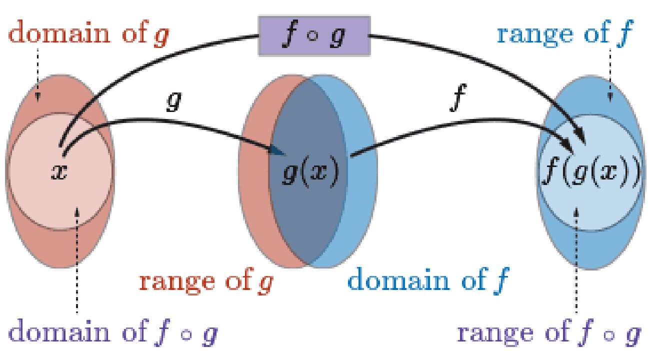
The function, f composed with g, is denoted by

$$(f \circ g)(x)$$

and defined by $(f \circ g)(x)=f(g(x))$.

The function $f \circ g$ is obtained by substituting $g(x)$ for all occurrences of x in $f(x)$.

The domain of $(f \circ g)(x)$ is the set of all values of x in the domain of g that produce an output, $g(x)$, which is a value in the domain of f.



In This part of the Lesson

- *We will expand our understanding of composite functions, combining a variety of functions studied in this course.*
- *We will determine the composition of two functions numerically, algebraically, and graphically.*
- *We will identify when the value of a composite function exists by identifying its domain.*
- *We will look at ways of expressing a given function as a composition of two or more functions.*

Example

Given the table of values for $f(x)$ and $g(x)$, find:

a. $(g\circ f)(3)$

b. $f(g(2))$

c. $(f\circ f)(4)$

x	$f(x)$	$g(x)$
1	5	6
2	3	5
3	1	4
4	3	3
5	5	2

Solution

a. From the table, we know that $f(3)=1$, so

$(g\circ f)(3) \quad =g(f(3))$

$=g(1)$

$=6$

b. Since $g(2)=5$,

$(f\circ g)(2) =f(g(2))$

$=f(5)$

$=5$

c. Since $f(4)=3$,

$(f\circ f)(4) =f(f(4))$

$=f(3)$

$=1$

Consider $f(g(1))$. Since $g(1)=6$, then $f(g(1))=f(6)$. However, f is not defined when $x=6$.

That is, x=6 is not in the domain of f.
Thus, $f(g(1))$ is not defined. We can therefore conclude that x=1 is not in the domain of $f∘g$.
Recall: The domain of $(f∘g)(x)$ is the set of all values of x in the domain of g that produce an output, $g(x)$, which is in the domain of f.
The domain of $(f∘g)(x)$ is {2,3,4,5}.

Example

For $f(x) = \frac{x}{x+2}$, show that $f(f^{-1}(x)) = f^{-1}(f(x))$ for all values of x in the domain of f and f^{-1}.

Solution

First, note that the domain of $f(x)= \frac{x}{x+2}$ is $\{x|\ x\neq -2, x\in R\}$.
To find f^{-1}, we interchange the x and y in the equation of f and solve for y.
$x = \frac{y}{y+2}$, $y \neq -2$
$y = \frac{-2x}{x-1}$, $x \neq 1$
Thus, $f^{-1}(x) = -\frac{2x}{x-1}$, $x \neq 1$.

Example

Express each function as a composition of two or more functions. Verify your choice of functions.

a. $f(x)= 2^{3x-1}$

b. $g(x)= \frac{3}{2\sin(x)-1}$

Solution

a. If $p(x)=3x-1$ and $q(x)=2^x$, then

$q(p(x)) = q(3x-1)$

$= 2^{3x-1}$

Thus, $f(x)=(q∘p)(x)$ when $p(x)=3x-1$ and $q(x)=2^x$.
An alternate solution has $p(x)=3x$ and $q(x)=2^{x-1}$.

In this case,

$q(p(x)) = q(3x)$

$= 2^{3x-1}$

Thus, $f(x)=(q∘p)(x)$ when $p(x)=3x$ and $q(x)=2^{x-1}$.
b. Approach 1

If $p(x)= 2\sin(x)-1$ and $q(x)=\frac{3}{x}$, then

$q(p(x)) = q(2\sin(x)-1)$

$= \frac{3}{2\sin(x)-1}$

Thus, $g(x)= (q \circ p)(x)$ when $p(x)=2\sin(x)-1$ and $q(x)= \frac{3}{x}$.

b. Approach 2

If $p(x)= \sin(x)$ and $q(x)= \frac{3}{2x-1}$, then

$q(p(x)) = q(\sin(x))$

$= \frac{3}{2\sin(x)-1}$

Thus, $g(x)=(q \circ p)(x)$ when $p(x)=\sin(x)$ and $q(x)= \frac{3}{2x-1}$.

Example

Find all possible functions f such that $f(x)= ax + b$ and $f(f(f(x)))=bx-a^2$ with $a\neq 0$, $b\neq 0$, a, $b \in \mathrm{R}$.

Solution

First, using the appropriate substitutions, determine the function $f(f(f(x)))$ given that $f(x)= ax + b$:

$f(f(f(x))) = f(f(ax + b))$

$= f(a(ax + b)+b)$

$= f(a^2x+ab+b)$

$= a(a^2x+ab+b)+b$

$= a^3x+a^2b+ab+b$

We have $f(f(f(x)))= a^3x + a^2b + ab + b$ and $f(f(f(x)))= bx - a^2$, a, $b\neq 0$, a, $b \in \mathrm{R}$, so

$$a^3x + a^2b+ ab + b = bx-a^2$$

Equating coefficients, we have

$$a^3 = b \qquad (1)$$

$$a^2b+ab+b = -a^2 \qquad (2)$$

Substituting (1) into (2) and simplifying,

$a^2(a^3) + a(a^3) + a^3 = -a^2$

$a^5 + a^4 + a^3+ a^2 = 0$

This polynomial equation can be solved by factoring.

$a^5 + a^4+ a^3+ a^2 = 0$

$a^2(a^3+ a^2+ a + 1) = 0$

We can use the factor theorem or we can factor by grouping.

$a^5 + a^4 + a^3 + a^2 = 0$

$a^4(a+1) + a^2(a+1) = 0$

$(a^4 + a^2)(a+1) = 0$

$a^2(a^2 + 1)(a+1) = 0$

The solutions to the equation

$a^2(a^2 + 1)(a+1) = 0$

are $a=0$ or $a=-1$. Note that $a^2+1 \neq 0$ for $a \in \mathbb{R}$.
However, $a \neq 0$, so $a=-1$.
Substituting $a=-1$ into (1), we get

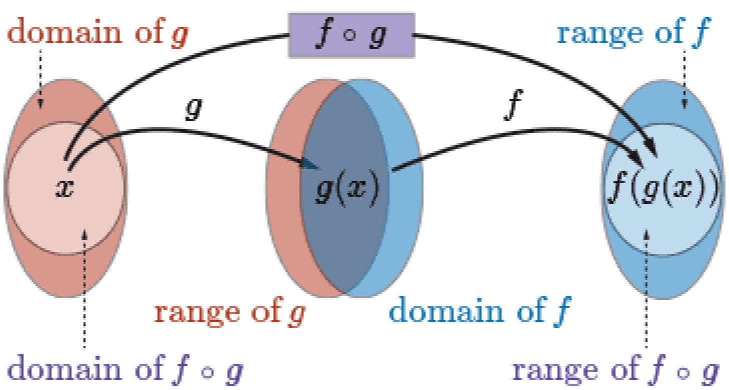
$b = a3$

$b = (-1)3$

$b = -1$

Therefore, the only possible function f, which satisfies the conditions, is $f(x) = -x-1$.

Summary



- The composition of functions applies one function to the result of another function. For two functions f and g, applying f to the output of g results in the composite function defined by $(f \circ g)(x) = f(g(x))$.
- To determine $(f \circ g)(x)$, substitute the expression for $g(x)$ into $f(x)$, for every occurrence of x in the expression for $f(x)$.
- To determine $(f \circ g)(a)$, where $a \in \mathbb{R}$, substitute the value of $g(a)$ into $f(x)$ for x, or substitute $x=a$ into the expression for $(f \circ g)(x)$.
- The domain of $(f \circ g)(x)$ is the set of all values, x, in the domain of g that produce an output, $g(x)$, that lies in the domain of f.
- Different approaches can be used to graph composite functions depending on the functions involved.
- The decomposition of a function involves identifying two or more (often simpler) functions such that the composition of these functions results in the given function.

- The composition of functions is generally not commutative. For most functions *f* and *g*,

$$(f \circ g)(x) \neq (g \circ f)(x)$$

- For any function *f* that has an inverse function f^{-1},

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

for all values of *x* in the domain of *f* and f^{-1}.