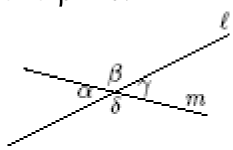
Geometry 1

1. Lines and angles

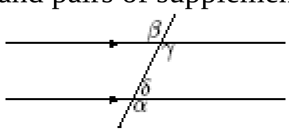
We take as an axiom that the angle at a point on a straight line is a constant regardless of the point or line. Such an angle is called a **straight angle** and its measure is 180^o .
When two lines intersect, four angles are formed. Two such angles are called **vertically opposite** (or just opposite) if they are not formed on the same side of one of the lines. The straight angle axiom (postulate) implies the following theorem.

Theorem 1. The opposite angles formed by intersecting straight lines are equal. In the diagram, $\alpha = \gamma$ and $\beta = \delta$.



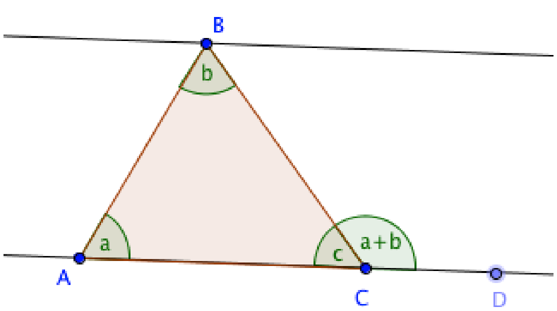
Parallel lines

In the diagram the two horizontal lines are parallel. The line cutting the parallel lines is called a **transversal**. Angles α and β are called **alternate exterior angles**, α and γ are **corresponding angles,** and angles γ and δ are **supplementary angles**. Alternate angles and corresponding angles are equal, and pairs of supplementary angles sum to 180^o.



Exterior Angle Theorem

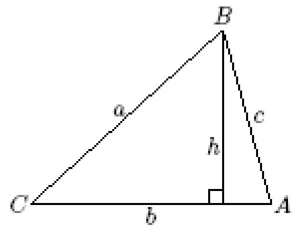
Any triangle has a property that an exterior angle of a triangle equals the sum of the two non-adjacent interior angles.



2. Areas and perimeters

Theorem 1. The area of a parallelogram is equal to bh where b is the length of its base and h is its height (the perpendicular distance from the base to the parallel side opposite).

Theorem 2. The area of a triangle is equal to $\frac{1}{2}$bh where b is the length of its base and h is its height (the perpendicular distance from the base to the vertex opposite).



Theorem 3. (Heron's Theorem). If the lengths of the sides of a triangle are a, b and c, so that the semi perimeter $s = \left(a + b + c\right) = 2$ then the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$

3. Congruence of Triangles

Two polygons are called **congruent** if their corresponding sides and corresponding angles are equal. Triangles may be determined to be congruent by any of the following rules.

- **SAS Rule** If two sides and the included angle of one triangle are equal to the two sides and the included angle of another, then the triangles are congruent.

- **SSS Rule** If three sides of one triangle are equal to the three sides of another, then the triangles are congruent.

- **ASA Rule** If two angles and the included side of one triangle are equal to the two angles and the included side of another, then the triangles are congruent.

- **RHS/HL Rule** If the hypotenuse and one other side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle, then the triangles are congruent.

Note that when we say two triangles ABC and XYZ are congruent we mean that the correspondence of vertex A to X, B to Y and C to Z determines the congruence. We denote that two triangles ABC and XYZ are congruent by writing $\Delta ABC \cong \Delta XYZ$.

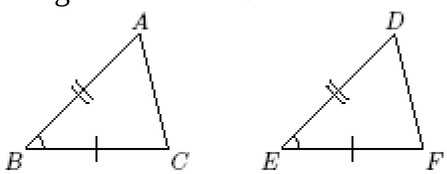
A triangle is isosceles if two of its sides are equal. By convention, the common vertex of the two equal sides of an isosceles triangle is written between the other two vertices, i.e. to say ΔXYZ is isosceles we imply that YX = YZ.

Proof of SAS Rule.

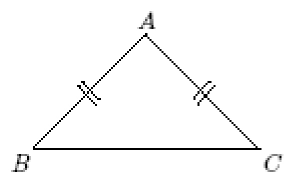
In the triangles ABC and DEF we have side AB = DE, included angle $\angle ABC = \angle DEF$, and side BC = EF. We must show $\Delta ABC \cong \Delta DEF$.

Place triangle ABC over triangle DEF so that B falls on E and edge BC runs along line EF. Since BC = EF, C falls on F. Since ∠ABC = ∠DEF, line BA falls on ED, and since AB = DE, A fall on D. Since A falls on D and C falls on F, line segment AC falls on DF.
Hence ΔABC ≅ ΔDEF.

Theorem 2. If a triangle is isosceles then the angles opposite the equal sides are equal. Conversely, if two angles of a triangle are equal then the two sides opposite the equal angles are equal, so that the triangle is isosceles.

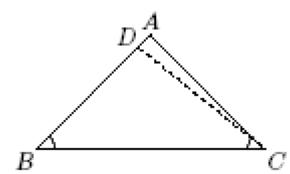


Proof of Theorem 2.



Assume in triangle ABC that AB = AC. Then	
AB = AC,	given (1)
∠BAC = ∠CAB,	same angle
AC = AB,	equivalent to (1)
ΔABC ≅ ΔACB,	by SAS Rule
∠ABC = ∠ACB	

So we have shown that an isosceles triangle has the angles opposite its equal sides equal.



Now assume in ΔABC that ∠ABC = ∠ACB.

Along the ray BA, construct (by compass) the point D such that DB = AC. Now we have

DB = AC,	by construction
∠ABC = ∠DBC = ∠ACB,	given
BC = CB,	same line segment
ΔDBC ≅ ΔACB,	by SAS Rule

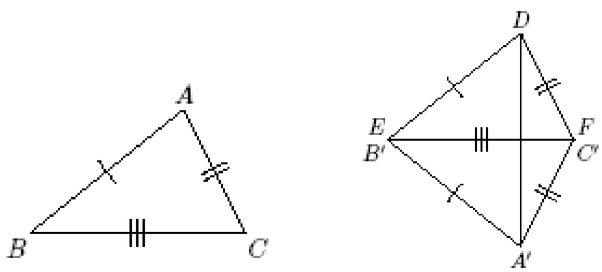
∴ ∠DCB = ∠ABC = ∠ACB

Thus line DC coincides with line AC. Hence D = A, and AB = DB = AC. So we have shown that a triangle with two angles equal has the sides opposite the equal angles equal.

Proof of SSS Rule.

Assume in triangles ABC and DEF that AB = DE, BC = EF and CA = FD. Transport triangle ABC so that B falls on E and line BC runs along EF. Since BC = EF, C falls on F.

Now let triangle ABC fall on the opposite side of line EF to triangle DEF so that A falls on A0. The transported copy of ΔABC is ΔA’B’C’ in the diagram.



By construction, ΔABC ≅ ΔA'B'C', where E = B' and F = C'. In particular,
A'E = AB = DE and A'F = AC = DF, so that triangles DEA' and DFA' are isosceles. So by Theorem 2, we have
∠EDA' = ∠EA'D and ∠FDA' = ∠FA'D:

Hence,
∠EDF = ∠EDA' + ∠FDA' = ∠EA'D + ∠FA'D = ∠EA'F = ∠BAC:

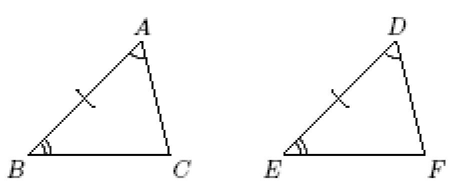
So now ΔABC ≅ ΔDEF by the SAS Rule.

Proof of ASA Rule.

In the triangles ABC and DEF we have ∠ABC = ∠DEF, included side AB = DE and ∠BAC = ∠EDF. Place ΔABC over ΔDEF so that A falls on D and AB runs along DE.

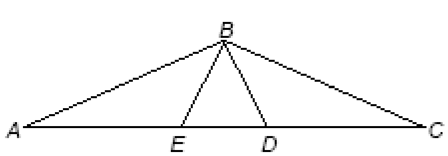
Since AB = DE, B falls on E. Also AC runs along DF because ∠BAC = ∠EDF. Similarly, BC runs along EF because ∠ABC = ∠DEF.

Thus the intersection point C of AC and BC must fall on the intersection point F of EF and DF. So ΔABC is exactly superimposed over ΔDEF, and hence ΔABC ≅ ΔDEF.

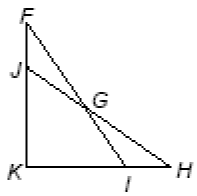


►Questions in class

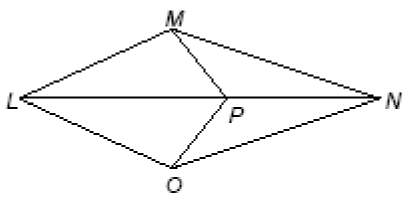
1. Given: *∠ABD ≅ ∠EBC, ∠A ≅ ∠C, BE ≅ BD.* **Prove:** *BA ≅ BC.*



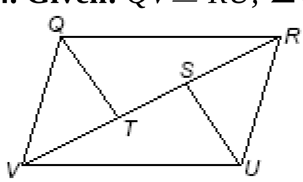
2. Given: $JK \cong KI$, $FI \cong JH$, $\angle K$ is a right angle. **Prove:** $\angle F \cong \angle H$.



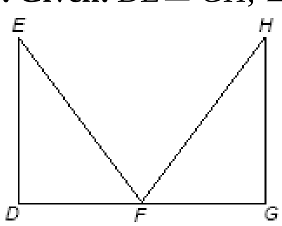
3. Given: $MP \cong PO$, $\angle MPN \cong \angle OPN$. **Prove:** $\angle PMN \cong \angle PON$.



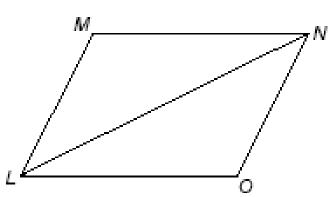
4. Given: $QV \cong RU$, $\angle QVT \cong \angle URS$. **Prove:** $QR \cong VU$.



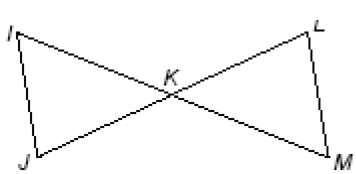
5. Given: $DE \cong GH$, $\angle D \cong \angle G$, F is the midpoint of DG. **Prove:** $DEF \cong GHF$.



6. Given: $LM \cong NO$, $MN \cong LO$. **Prove:** $\angle M \cong \angle O$



7. Given: K bisects IM, $\angle J \cong \angle L$. **Prove:** K bisects JL



8. Given: $AF \cong FC$, $EF \cong DF$. **Prove:** $\angle A \cong \angle C$.