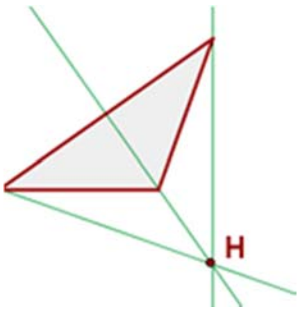
Geometry 2

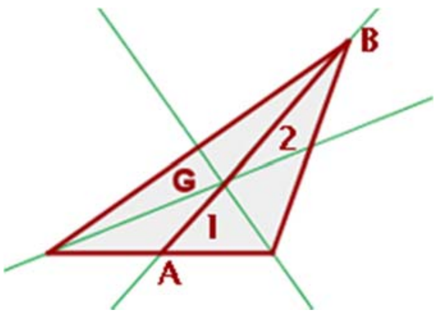
1. Lines in Triangles

Orthocenter



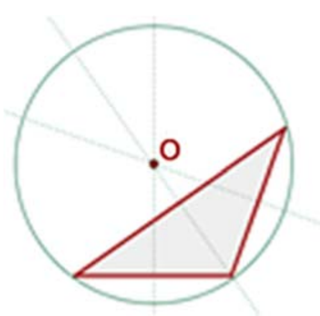
The **orthocenter** is the **point of intersection** of the **three heights** of a triangle. A **height** is each of the **perpendicular lines** drawn from one **vertex to the opposite side** (or its extension).

Centroid



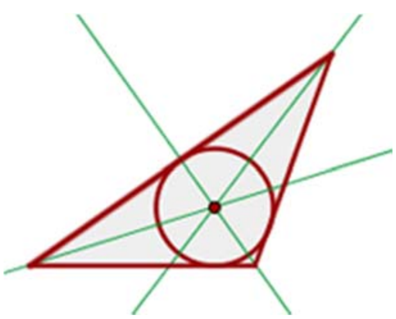
The **centroid** is the point of intersection of the **three medians**. A **median** is each of the **straight lines** that joins the **midpoint** of a side with the **opposite vertex.** The **centroid** divides each **median** into **two segments**, the segment joining the centroid to the vertex multiplied by two is equal to the length of the line segment joining the midpoint to the opposite side. **BG = 2GA**

Circumcenter



The **circumcenter** is the **point of intersection** of the **three perpendicular bisectors**. A **perpendicular bisectors** of a triangle is each line drawn perpendicularly from its midpoint.
The **circumcenter** is the **center** of the **circle inscribed** in the triangle.

Incenter



The **incenter** is the point of intersection of the three angle bisectors.The angle bisectors of a triangle are each one of the lines that divide an angle into two equal angles. The **incenter** is the center of the **circle inscribed** in the triangle.

2. Area of an Equilateral Triangle

The area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, where s is the side length of the triangle.

Proof

Dropping the altitude of our triangle splits it into two triangles. By HL congruence, these are congruent, so the "short side" is s/2.

Using the Pythagorean theorem, we get $s^2 = h^2 + \frac{s^2}{4}$, where h is the height of the triangle.

Solving, $h = \frac{s\sqrt{3}}{2}$. (note we could use 30-60-90 right triangles.)

We use the formula for the area of a triangle, bh/2 (note s is the length of a base), so the area is

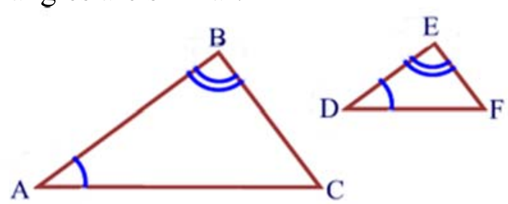
$$\boxed{\frac{s^2\sqrt{3}}{4}}$$

3. Similar Triangles

Triangles are **similar** if their corresponding (matching) angles are congruent (equal in measure) and the ratio of their corresponding sides are in proportion.

There are three accepted methods of proving triangles similar:

AA Similarity: If two angles of one triangle are congruent to two angles of another triangle, the triangles are similar.

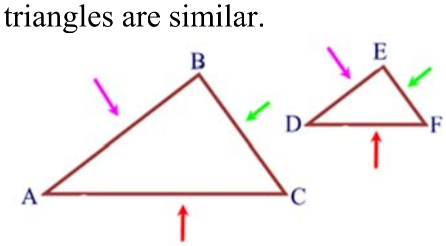


If: $\measuredangle A \cong \measuredangle D$

$\measuredangle B \cong \measuredangle E$

Then: $\Delta ABC \sim \Delta DEF$

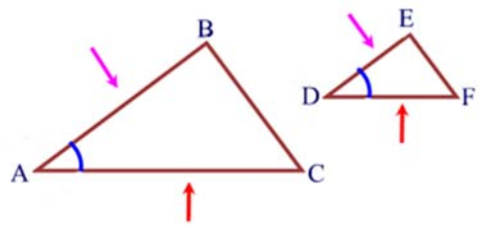
SSS Similarity: If the three sets of corresponding sides of two triangles are in proportion, the triangles are similar.



If: $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

Then: $\Delta ABC \sim \Delta DEF$

SAS Similarity: If an angle of one triangle is congruent to the corresponding angle of another triangle and the lengths of the sides including these angles are in proportion, the triangles are similar.

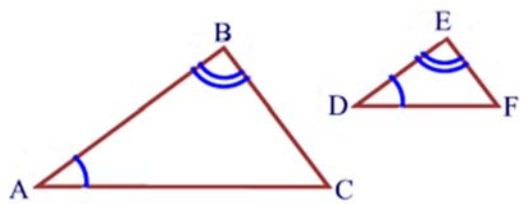


If: $\measuredangle A \cong \measuredangle D$

$$\frac{AB}{DE} = \frac{AC}{DF}$$

Then: $\Delta ABC \sim \Delta DEF$

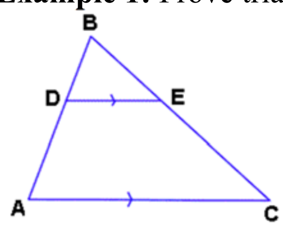
Once the triangles are similar: The corresponding sides of similar triangles are in proportion.



If : $\Delta ABC \sim \Delta DEF$

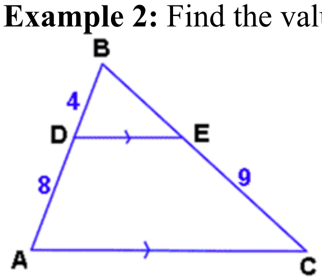
Then : $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

Example 1: Prove triangle BDE is similar to triangle BAC.



Since $\overline{DE} \,//\, \overline{AC}$, we know that we have $<BDE$ congruent to $<DAC$ (by corresponding angles). $<B$ is shared by both triangles, so the two triangles are similar by AA similarity.

Example 2: Find the value of BE.



From example 1, we know that triangle BDE is similar to triangle BAC. We can form the following ratio:

Let x be BE.

$$\frac{4}{12} = \frac{x}{x+9}$$

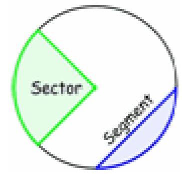
$4x + 36 = 12x$

$36 = 8x$

$x = 4.5$

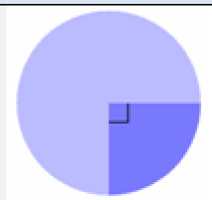
4. Circle Sector and Segment

Slices
There are two main "slices" of a circle:
The "pizza" slice is called a **Sector**.
And the slice made by a chord is called a **Segment**.



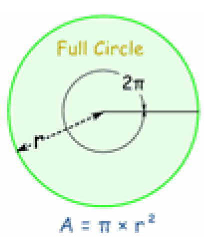
1) Common Sectors

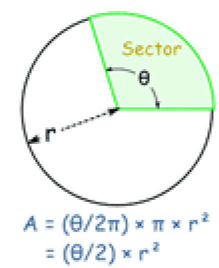
The Quadrant and Semicircle are two special types of Sector:

Quarter of a circle is called a **Quadrant**	Half a circle is called a **Semicircle.**
	

2) Area of a Sector

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.
Note: I am using radians for the angles.





This is the reasoning:

- A circle has an angle of 2π and an Area of: πr^2
- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

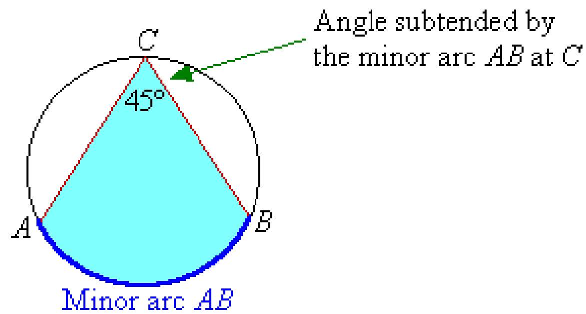
Area of Sector = $\frac{1}{2} \times \theta \times r^2$ *(when θ is in radians)*
Area of Sector = $\frac{1}{2} \times (\theta \times \pi/180) \times r^2$ *(when θ is in degrees)*

5. Angle at the Circumference

If the end points of an arc are joined to a third point on the circumference of a circle, then an angle is formed. It is called the **inscribed angle**.

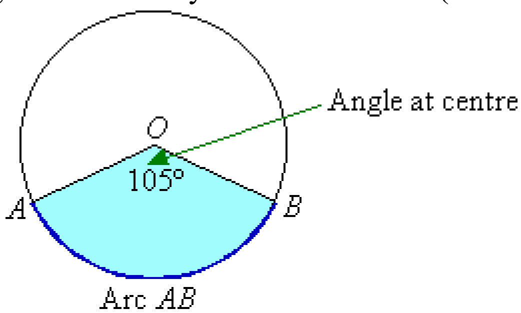
For example, the minor arc *AB* subtends an angle of 45º at *C*. The angle *ACB* is said to be the angle subtended by the minor arc *AB* (or simply arc *AB*) at *C*.

The angle *ACB* is an angle at the circumference standing on the arc *AB*.



6. Angle at the Centre

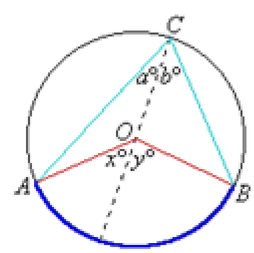
If the end points of an arc are joined to the centre of a circle, then an angle is formed.
For example, the minor arc *AB* subtends an angle of 105º at *O*. The angle *AOB* is said to be the angle subtended by the minor arc *AB* (or simply arc *AB*) at the centre *O*.



The angle *AOB* is an angle at the centre *O* standing on the arc *AB,* it is also called the **central angle**.

Angle at Centre Theorem

Use the information given in the diagram to prove that the angle at the centre of a circle is twice the angle at the circumference if both angles stand on the same arc.



Given:
$\angle AOB$ and $\angle ACB$ stand on the same arc; and O is the centre of the circle.
To prove:
$\angle AOB = 2\angle ACB$

Proof:

From ΔOAC, $x = a + a$ {Exterior angle of a triangle}

$\therefore x = 2a$ ... (1)

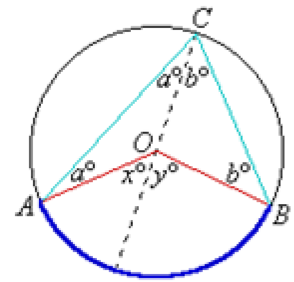
From ΔOBC, $y = b + b$ {Exterior angle of a triangle}

$\therefore y = 2b$ ... (2)

Adding (1) and (2) gives:

$x + y = 2a + 2b = 2(a + b)$ $\therefore \angle AOB = 2\angle ACB$

As required.



In general: The angle at the centre of a circle is twice the angle at the circumference if both angles stand on the same arc. This is called the **Angle at Centre Theorem**.

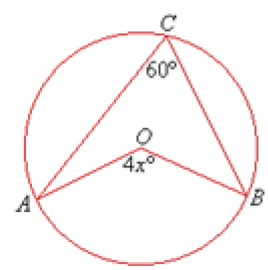
Example 1: Find the value of unknown in the following circle centred at *O*.

Solution:

$4x = 2 \times 60$ {Angle at Centre Theorem}

$4x = 120$

$\frac{4x}{4} = \frac{120}{4}$ $x = 30$



7. Angle in a Semi-Circle

Let $\angle AOB = 180°$.

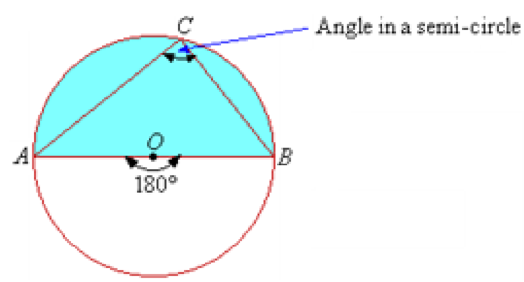
Then $\angle ACB$ is in a semi-circle.

By the Angle at Centre Theorem, we have:

$\angle AOB = 2\angle C$ $2\angle C = 180°$

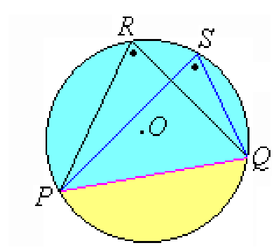
$\therefore \angle C = 90°$

In general: The angle in a semi-circle is a right angle.



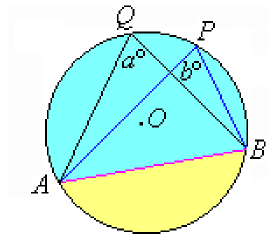
8. Angles in the Same Segment

In the left diagram, $\angle PRQ$ and $\angle PSQ$ are in the major segment. So, we say that angle *PRQ* and angle *PSQ* are in same segments.



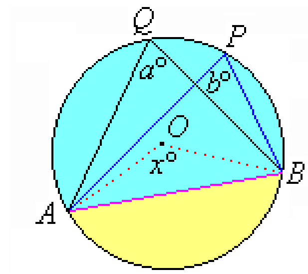
Theorem

Use the information given in the diagram to prove that the angles in the same segment of a circle are equal. That is, *a* = *b*.



Given: $\angle APB$ and $\angle AQB$ are in the same segment; and *O* is the center of the circle.
To prove: $\angle APB = \angle AQB$.

Construction: Join *O* to *A* and *B*.
Proof:



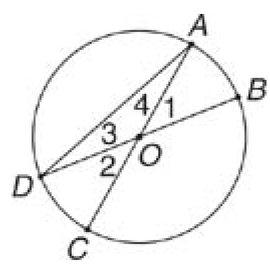
Let $\angle AOB = x°$.

Clearly, $x = 2a$	{Angle at Centre Theorem}
$x = 2b$	{Angle at Centre Theorem}
$\therefore\ 2a = 2b$	{Transitive}
$\therefore\ a = b$	As required.

In general: Angles in the same segment of a circle are equal.

Also, it is true that for all equal inscribed angles in the same circle, subtending arcs must be equal.

Given angle 3 = angle 4, then we can conclude that arc AB = arc CD

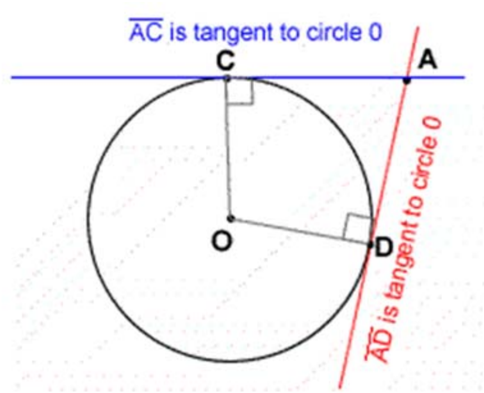


9. Tangent to a Circle

A tangent to a circle is perpendicular to the radius at the point of tangency.

- This is a very useful property when the radius that connects to the point of tangency is part of a right angle, because the trigonometry and the Pythagorean Theorem apply to right triangles.

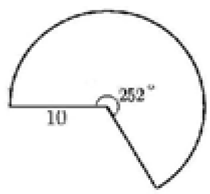
- A **tangent** intersects a circle at one point.
    - C and **D** are the **points of tangency** to circle O
    - AC and AD are tangent to circle O.
- **Perpendicular** means at right angles (meet at 90º).
    - OC and OD are radii of the circle O.
    - OC is perpendicular to AC.
    - OD is perpendicular to AD.

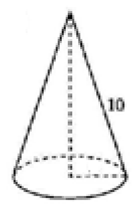


►Questions in class

1. A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance, m feet, between the stripes.

2. Determine the radius of the cones formed from a $252°$ sector of a circle of radius 10 by aligning the two straight sides?



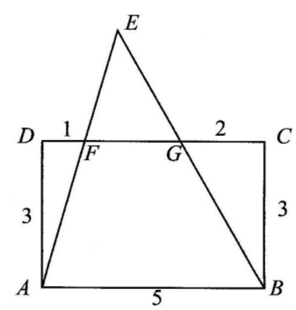


3. A regular octagon is formed by cutting an isosceles right triangle from each of the corners of a square with sides of length 2000. What is the length of each side of the octagon?

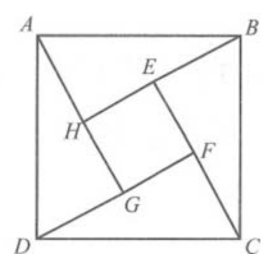
4. A regular octagon ABCDEFGH has sides of length two. Find the area of Δ ADG.

5. Let Δ XOY be a right-angled triangle with $m\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that XN = 19 and YM = 22, find XY.

6. In rectangle ABCD, AB = 5 and BC = 3. Points F and G are on CD so that DF = 1 and GC = 2. Lines AF and BG intersect at E. Find the area of Δ AEB.



7. In the figure, the length of side *AB* of square *ABCD* is $\sqrt{50}$, *E* is between *B* and *H,* and *BE* = 1. What is the area of the inner square *EFGH*?



8. A rectangle with a diagonal of length *x* that is twice as long as it is wide. What is the area of the rectangle?