Number Theory

1. Prime Number

A Prime Number can be divided evenly only by 1, or itself. (In other words, its factors are only 1 or itself)

Example 1: 13 can only be divided evenly by 1 or 13, so it is a prime number.

Example 2: 14 can be divided evenly by 1, 2, 7 and 14 so it is a composite number.

2. Composite Number

A Composite Number can be divided evenly by numbers other than 1 or itself. (In other words, it has more than the two factors of 1 and itself)

Example 1: 9 can be divided evenly by 1, 3 and 9, so 9 is a composite number.

Example 2: 7 can only be divided evenly by 1 and 7, so it is not composite. It must be a prime number.

3. Beprisque number

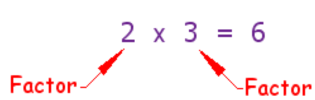
A *beprisque number* *n* is an integer which is either one more than a prime number and one less than a perfect square, or one more than a square and one less than a prime.

The only prime beprisque numbers are 2 and 3. For a prime number to be a beprisque number, it has to neighbor another prime number; the only two primes fitting that bill are of course 2 and 3.

The only odd beprisque numbers are 1 and 3. For an odd number to be a beprisque number, it has to neighbor an even prime, and since 2 is the only even prime, only 1 and 3 can be beprisque numbers.

4. Factors

You can make the same definitions using Factors.



"Factors" are the numbers you multiply together to get another number.

So here is just a different way of saying the same thing from above.

When the only factors of a number are 1 and the number, then it is a Prime Number

Examples:

$3 = 1 \times 3$ (the only factors are 1 and 3)	Prime
$6 = 1 \times 6$ or $6 = 2 \times 3$ (the factors are 1,2,3 and 6)	Composite

Examples from 1 to 14

We have highlighted any factors other than 1 or the number itself:

Number	Can be Evenly Divided By	Prime, or Composite?
1	*(1 is not considered prime or composite)*	
2	1, 2	Prime
3	1, 3	Prime
4	1, 2, 4	Composite
5	1, 5	Prime
6	1, 2, 3, 6	Composite
7	1, 7	Prime
8	1, 2, 4, 8	Composite
9	1, 3, 9	Composite
10	1, 2, 5, 10	Composite
11	1, 11	Prime
12	1, 2, 3, 4, 6, 12	Composite
13	1, 13	Prime
14	1, 2, 7, 14	Composite
...	...	...

So when there are more factors than 1 or the number itself, the number is Composite.

5. Prime Factorization

"Prime Factorization" is finding which prime numbers you need to multiply together to get the original number.

Example 1: What are the prime factors of 12?

It is best to start working from the smallest prime number, which is 2, so let's check:
$12 \div 2 = 6$
But 6 is not a prime number, so we need to factor it further:
$6 \div 2 = 3$
And 3 is a prime number, so:

$12 = 2 \times 2 \times 3$

As you can see, every factor is a prime number, so the answer must be right - the prime factorization of 12 is $2 \times 2 \times 3$, which can also be written as $2^2 \times 3$.

Example 2: What is the prime factorization of 147?

Can we divide 147 evenly by 2? No, so we should try the next prime number, 3:
$147 \div 3 = 49$
Then we try factoring 49, and find that 7 is the smallest prime number that works:
$49 \div 7 = 7$
And that is as far as we need to go, because all the factors are prime numbers.
$147 = 3 \times 7 \times 7 = 3 \times 7^2$

Another Method
We showed you how to do the factorization by starting at the smallest prime and working upwards, but sometimes it is easier to break a number down into any factors you can, then work those factor down to primes.

Example 3: What are the prime factors of 90?
Break 90 into 9×10
The prime factors of 9 are 3 and 3
The prime factors of 10 are 2 and 5
So the prime factors of 90 are 3, 3, 2 and 5

And here is another thing:
There is only one (unique!) set of prime factors for any number.

Example 4: The prime factors of 330 are 2, 3, 5 and 11:
$330 = 2 \times 3 \times 5 \times 11$
There is no other possible set of prime numbers that can be multiplied to make 330.

6. Least Common Multiple

The smallest (non-zero) number is a multiple of two or more numbers.

1) What is a "Multiple"?
The multiples of a number are what you get when you multiply it by other numbers (such as if you multiply it by 1,2,3,4,5, etc). Just like the multiplication table.

Here are some examples:

The multiples of 3 are 3, 6, 9, 12, 15, 18, 21, etc ...

The multiples of 12 are 12, 24, 36, 48, 60, 72, etc...

2) What is a "Common Multiple"?

When you list the multiples of two (or more) numbers, and find the same value in both lists, then that is a *common multiple* of those numbers.

For example, when you write down the multiples of 4 and 5, the *common* multiples are those that are found in both lists:

The multiples of 4 are 4,8,12,16,20,24,28,32,36,40,44,...

The multiples of 5 are 5,10,15,20,25,30,35,40,45,50,...

Notice that 20 and 40 appear in both lists?
So, the common multiples of 4 and 5 are: 20, 40, (and 60, 80, etc ..., too)

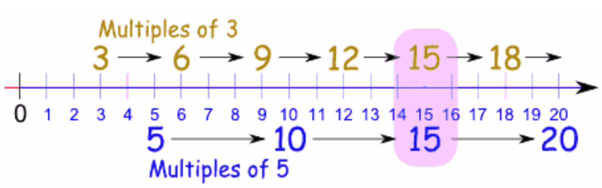
3) What is the "Least Common Multiple"?
It is simply the smallest of the common multiples.

In our previous example, the smallest of the common multiples is 20, so the *Least* Common Multiple of 4 and 5 is 20.

4) Finding the Least Common Multiple
It is a really easy thing to do. Just start listing the multiples of the numbers until you get a match.

Example 1: Find the least common multiple for 3 and 5:

The multiples of 3 are *3, 6, 9, 15, ...,*
and the multiples of 5 are *5, 10, 15, 20, ...,* like this:



As you can see on this number line, the first time the multiples match up is 15.
The LCM is 15.

Example 2: Find the least common multiple for 4, 6, and 8

Multiples of 4 are: 4, 8, 12, 16, 20, 24, 28, 32, 36, ...
Multiples of 6 are: 6, 12, 18, 24, 30, 36, ...
Multiples of 8 are: 8, 16, 24, 32, 40, ...
So, 24 is the least common multiple (I can't find a smaller one !)

7. Properties of Real Numbers

1) Commutative Property - interchange or switch the elements
For example shows commutative property for addition:

$$X + Y = Y + X$$

2) Associative Property- regroups the elements

For example shows associative property for addition:

$$(X + Y) + Z = X + (Y + Z)$$

3) Identity Property- What returns the input unchanged?

$X + 0 = X$ Additive Identity

$X \bullet 1 = X$ Multiplicative Identity

4) Inverse Property- What brings you back to the identity element using that operation?

$X + (-X) = 0$ Additive Inverse

$X \bullet 1/X = 1$ Multiplicative Inverse

5) Distributive Property - *multiply* across the parentheses.

Each element inside the parentheses is multiplied by the element outside the parentheses.

$$a(b+c) = a \cdot b + a \cdot c$$

8. Greatest Common Factor

The highest number divides exactly into two or more numbers. It is the "greatest" thing for simplifying fractions!

1) What is the "Greatest Common Factor"?

It is simply the largest of the common factors. In our previous example, the largest of the common factors is 15, so the Greatest Common Factor of 15, 30 and 105 is 15

The "Greatest Common Factor" is the largest of the common factors of two or more numbers.

2) Why is this Useful?

One of the most useful things is when we want to simplify a fraction:

Example: How could we simplify $^{12}/_{30}$?

At the top we found that the Common Factors of 12 and 30 were 1, 2, 3 and 6, and so the Greatest Common Factor is 6.

This means that the **largest** number we can divide both 12 and 30 evenly by is 6.

The Greatest Common Factor of 12 and 30 is **6**. And so $^{12}/_{30}$ can be simplified to $^{2}/_{5}$

3) Finding the Greatest Common Factor

Here are three ways:

a. You can find all **factors** of both numbers, then select the ones that are **common** to both, and then choose the **greatest**.

Example:

Two Numbers	**All Factors**	**Common Factors**	**Greatest Common Factor**	***Example Simplified Fraction***
9 and 12	**9**: 1,3,9 **12**: 1,2,3,4,6,12	1,3	**3**	$^9/_{12}$ » $^3/_4$

And another example:

Two Numbers	**All Factors**	**Common Factors**	**Greatest Common Factor**	***Example Simplified Fraction***
6 and 18	**6**: 1,2,3,6 **18**: 1,2,3,6,9,18	1,2,3,6	**6**	$^6/_{18}$ » $^1/_3$

b. You can find the prime factors and combine the common ones together:

Two Numbers	**Thinking ...**	**Greatest Common Factor**	***Example Simplified Fraction***
24 and 108	$\mathbf{2} \times \mathbf{2} \times 2 \times \mathbf{3} = 24$, and $\mathbf{2} \times \mathbf{2} \times \mathbf{3} \times 3 \times 3 = 108$	$2 \times 2 \times 3 = \mathbf{12}$	$^{24}/_{108}$ » $^2/_9$

c. And sometimes you can just **play around** with the factors until you discover it:

Two Numbers	**Thinking ...**	**Greatest Common Factor**	***Example Simplified Fraction***
9 and 12	$\mathbf{3} \times 3 = 9$ and $\mathbf{3} \times 4 = 12$	**3**	$^9/_{12}$ » $^3/_4$

But in that case you had better be careful you have found the **greatest** common factor.

►Questions in class

1. What is the tens digit in the sum 7! + 8! + 9! + …+ 2006! ?

2. How many sets of two or more consecutive positive integers have a sum of 15?

3. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

4. How many numbers between 1 and 2005 are integer multiples of 3 or 4 but not 12?

5. For how many ordered pairs of positive integers (x, y) is x + 2y = 100?

6. A standard six-sided die is rolled, and P is the product of the five numbers that are visible. What is the largest number that is certain to divide P?

7. The sum of the two 5-digit numbers AMC10 and AMC12 is 123422. What is A+M+C?

8. For how many positive integers n is $n^2 - 3n + 2$ a prime number?

9. For how many positive integers m does there exist at least one positive integer n such that $m \cdot n \le m + n$?

10. In the year 2001, the United States will host the International Mathematical Olympiad. Let I, M, and *O* be distinct positive integers such that the product $I \cdot M \cdot O = 2001$. What is the largest possible value of the sum I + M + *O?*