Calculations and Operations

► Concepts

1. The Commutative Properties of Addition and Multiplication

a + b = b + a ***and*** ***ab = ba***

The Commutative Property, in general, states that changing the ORDER of two numbers either being added or multiplied, does NOT change the value of it. The two sides are called equivalent expressions because they look different but have the same value.

Example 1: Use the commutative property to write an equivalent expression to 2.5x + 3y.

Using the commutative property of addition (where changing the order of a sum does not change the value of it) we get
2.5x + 3y = 3y + 2.5x.

Example 2: Use the commutative property to write an equivalent expression to $\frac{1}{5}\cdot\frac{x}{7}$.

Using the communicative property of multiplication (where changing the order of a product does not change the value of it), we get

$$\frac{1}{5}\cdot\frac{x}{7}=\frac{x}{7}\cdot\frac{1}{5}$$

2. The Associative Properties of Addition and Multiplication

a + (b + c) = (a + b) + c ***and*** ***a(bc) = (ab)c***

The Associative property, in general, states that changing the GROUPING of numbers that are either being added or multiplied does NOT change the value of it. Again, the two sides are equivalent to each other.

Example 1: Use the associative property to write an equivalent expression to (a + 5b) + 2c.

Using the associative property of addition (where changing the grouping of a sum does not change the value of it) we get
(a + 5b) + 2c = a + (5b + 2c).

Example 2: Use the associative property to write an equivalent expression to (1.5x)y.

Using the associative property of multiplication (where changing the grouping of a product does not change the value of it) we get
(1.5x)y = 1.5(xy)

3. Distributive Properties

a(b + c) = ab + ac* or *(b + c)a = ba + ca
a(b - c) = ab - ac* or *(b - c)a = ba - ca

In other words, when you have a term being multiplied times two or more terms that are being added (or subtracted) in a ( ), multiply the outside term times EVERY term on the inside.

Remember terms are separated by + and -.

Example 1: Use the distributive property to write 2(x - y) without parenthesis.

Multiplying every term on the inside of the ( ) by 2 we get:
$2(x-y)=2(x)-2(y)=2x-2y$

Example 2: Use the distributive property to write - (5x + 7) without parenthesis.
$-(5x+7)=(-1)(5x)+(-1)(7)=-5x-7$

Basically, when you have a negative sign in front of a ( ), like this example, you can think of it as taking a -1 times the ( ). What you end up doing in the end is taking the opposite of every term in the ( ).

Example 3: Use the distributive property to find the product 3(2a + 3b + 4c).

As mentioned above, you can extend the distributive property to as many terms as are inside the ( ). The basic idea is that you multiply the outside term times EVERY term on the inside.
$3(2a+3b+4c)=3(2a)+3(3b)+3(4c)=6a+9b+12c$

4. Identity Properties

1) Addition

The additive identity is 0
a + 0 = 0 + a = a

In other words, when you add 0 to any number, you end up with that number as a result.

2) Multiplication

Multiplication identity is 1
a(1) = 1(a) = a

And when you multiply any number by 1, you wind up with that number as your answer.

5. The Inverse Properties

1) Additive Inverse (or negative)

For each real number a, there is a unique real number, denoted -a,
such that a + (-a) = 0.

In other words, when you add a number to its additive inverse, the result is 0. Other terms that are synonymous with additive inverse are negative and opposite.

2) Multiplicative Inverse (or reciprocal)

For each real number a, except 0, there is a unique real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

In other words, when you multiply a number by its multiplicative inverse the result is 1. A more common term used to indicate a multiplicative inverse is the **reciprocal**.

A multiplicative inverse or reciprocal of a real number *a* (except 0) is found by "flipping" *a* upside down. The numerator of *a* becomes the denominator of the reciprocal of *a* and the denominator of *a* becomes the numerator of the reciprocal of *a*.

Example 1: Write the opposite (or additive inverse) and reciprocal (or multiplicative inverse) of -3.

The opposite of -3 is 3, since -3 + 3 = 0.
The reciprocal of -3 is -1/3, since -3(-1/3) = 1.

When you take the reciprocal, the sign of the original number stays intact. Remember that you need a number that when you multiply times the given number you get 1. If you change the sign when you take the reciprocal, you would get a -1, instead of 1, and that is a no.

Example 2: Write the opposite (or additive inverse) and reciprocal (or multiplicative inverse) of 1/5.

The opposite of 1/5 is -1/5, since 1/5 + (-1/5) = 0.
The reciprocal of 1/5 is 5, since 5(1/5) = 1.

6. Difference of Squares

This expression $a^2 - b^2$ is called a difference of two squares.
(Notice the subtraction sign between the terms.)

We have the difference of squares formula: $(a+b)(a-b) = a^2 - b^2$

The factors of $a^2 - b^2$ are $(a+b)$ and $(a-b)$.

Example: Evaluate $90^2 – 89^2$ by using difference of squares.

$90^2 – 89^2 = (90 + 89)(90 – 89) = 179(1) = 179$

7. Negative Exponents

A negative exponent just means that the base is on the wrong side of the fraction line, so you need to flip the base to the other side. For instance, "x^{-2}" (x to the minus two) just means "x^2, but underneath, as in $1/(x^2)$".

Example 1: Write x^{-4} using only positive exponents.

$$x^{-4} = \frac{1x^{-4}}{1} = \frac{1}{x^4}$$

Example 2: Write x^2 / x^{-3} using only positive exponents.

$$\frac{x^2}{x^{-3}} = \frac{1x^2}{1x^{-3}} = \frac{1x^2x^3}{1} = x^5$$

Example 3: Write $2x^{-1}$ using only positive exponents.

$$2x^{-1} = \frac{2x^{-1}}{1} = \frac{2}{x^1} = \frac{2}{x}$$

Note that the "2" above does not move with the variable; the exponent is only on the "x".

Example 4: Write $(3x)^{-2}$ using only positive exponents.

$$(3x)^{-2} = \frac{(3x)^{-2}}{1} = \frac{1}{(3x)^2} = \frac{1}{9x^2}$$

Unlike the previous exercise, the parentheses meant that the negative power did indeed apply to the three as well as the variable.

Example 5: Write $(x^{-2} / y^{-3})^{-2}$ using only positive exponents.

$$\left(\frac{x^{-2}}{y^{-3}}\right)^{-2} = \frac{(x^{-2})^{-2}}{(y^{-3})^{-2}} = \frac{(y^{-3})^2}{(x^{-2})^2} = \frac{y^{-6}}{x^{-4}} = \frac{x^4}{y^6}$$

This one can also be done as:

$$\left(\frac{x^{-2}}{y^{-3}}\right)^{-2} = \frac{(x^{-2})^{-2}}{(y^{-3})^{-2}} = \frac{x^4}{y^6}$$

Since exponents indicate multiplication, and since order doesn't matter in multiplication, there will often be more than one sequence of steps that will lead to a valid simplification of a given exercise. Don't worry if the steps in your homework look quite different from the steps in a classmate's homework. As long as your steps were correct, you should both end up with the same answer.

► Questions in class:

1. Suppose $a * b$ equals the sum of the digits in the product of a and b. Thus, $4 * 7 = 10$. What is $(15 * 10) * (15 \times 10)$?

2. If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, then solve for x in $\begin{vmatrix} 2x & -4 \\ x & 1 \end{vmatrix} = 18$.

3. If $a * b = ab + a - b$, then $(7 * p) - (p * 7)$ equals what?

4. What is the units digit in $(2127)^{753}$?

5. If $a^* = \frac{1}{a^2}$, for any positive number a, then what is the $(2^* + 1^*)^*$?

6. What is the value of $53634^2 - 53633^2$?

7. What is the last digit in $(7^5)^3$?

8. If $a * b = \frac{a}{b} + \frac{b}{a}$ find the value of (1*2)*3.

9. We define the operation "o" as follows: a o $b = a \times b + a - b$. What is the value of the expression (2 o 5) o (5 o 2)?

10. A certain calculator has only two keys [+1] and [×2]. When you press one of the keys, the calculator automatically displays the result. For instance, if the calculator originally displayed "9" and you press [+1], it would display "10". If you then pressed [×2], it would display "20". Starting with the display "1", what is the fewest number of keystrokes you would need to reach "200"?