Lesson 2: Unit 2 - Applications of Vectors

The **dot product** of two vectors $\vec{u}$ and $\vec{v}$ is a scalar given by

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$$

where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$.

***Notice** that the dot product of two vectors is **NOT** a vector it is a scalar (number).*

Ex. Find the dot product of $\vec{u}$ and $\vec{v}$ given that $|\vec{u}|=8$, $|\vec{v}|=3$, and the angle between $\vec{u}$ and $\vec{v}$ is

a. $\theta=40^0$

b. $\theta=3\pi/4$

Solution

- **a.** $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta) = 8(3)\cos(40^0) \approx 18.385$
- **b.** $-12\sqrt{2}$

Ex. Prove that if the dot product of two non-zero vectors, $\vec{u}$ and $\vec{v}$, is equal to zero, then $\vec{u}$ and $\vec{v}$ must be perpendicular.

Solution

We are required to prove $\vec{u} \perp \vec{v}$ given that $\vec{u} \cdot \vec{v} = 0$.

Proof

$$\vec{u} \cdot \vec{v} = 0 \ \therefore |\vec{u}||\vec{v}|\cos(\theta) = 0 = 0$$

Since $|\vec{u}| \neq 0$ and $|\vec{v}| \neq 0$, it must be that $\cos(\theta)=0$.

Solving this equation, we get $\theta=\pm 90^\circ$.

Therefore, $\vec{u}$ is perpendicular to $\vec{v}$.

Is the converse true? That is, if non-zero vectors, $\vec{u}$ and $\vec{v}$, are perpendicular, then does $\vec{u} \cdot \vec{v} = 0$?

Yes.

This example proves an important **property of the dot product** of two vectors:

If $\vec{u} \cdot \vec{v} = 0$ where $\vec{u}, \vec{v} \neq \vec{0}$, then $\vec{u} \perp \vec{v}$. Conversely, if $\vec{u} \perp \vec{v}$, then $\vec{u} \cdot \vec{v} = 0$.

Two vectors are **orthogonal** if they are perpendicular, but includes the case where either (possibly both) vectors are $\vec{0}$.

Properties of Dot Product

1. $a(\vec{u} \cdot \vec{v}) = (a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v})$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
4. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

Proof of 3

$$\vec{u} \cdot \vec{u} = |\vec{u}||\vec{u}|\cos(\theta)$$

$$\text{since } \theta = 0^0$$

$$\cos(\theta) = 1$$

$$\therefore \vec{u} \cdot \vec{u} = |\vec{u}|^2$$

Ex. Evaluate each of the following dot products:

a. $\hat{\imath} \cdot \hat{\imath}$

b. $\hat{\imath} \cdot \hat{\jmath}$

Solution

$\hat{\imath} \cdot \hat{\imath} = |\hat{\imath}||\hat{\imath}|\cos(\theta)$ $(\theta = 0^0 \; and \; |\hat{\imath}| = 1)$ Thus, $\hat{\imath} \cdot \hat{\imath} = 1$

$\hat{\imath} \perp \hat{\jmath}$ therefore $\hat{\imath} \cdot \hat{\jmath} = 0$

Ex. Given $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$ prove that $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$.

Solution

$\vec{a} = (a_x, a_y)$

$= (a_x, 0) + (0, a_y)$

$=a_x(1,0) + a_y(0,1)$

$=a_x\hat{\imath} + a_y\hat{\jmath}$

Similarly,

$\vec{b} = a_x\hat{\imath} + a_y\hat{\jmath}$

$\vec{a} \cdot \vec{b} = (a_x\hat{\imath} + a_y\hat{\jmath})\cdot(b_x\hat{\imath} + b_y\hat{\jmath})$

$= a_x b_x(\hat{\imath}\cdot\hat{\imath}) + a_x b_y(\hat{\imath}\cdot\hat{\jmath}) + a_y b_x(\hat{\jmath},\hat{\imath}) + a_y b_y(\hat{\jmath}\cdot\hat{\jmath})$

$= a_x b_x + a_y b_y.$

This property extends to 3-dimensional algebraic vectors.

For example, in 3-dimensional space the **dot product** of two algebraic vectors

$\boldsymbol{\vec{u} = (u_x,u_y,u_z)}$ and $\boldsymbol{\vec{v} = (v_x,v_y,v_z)}$ is given by

$\boldsymbol{\vec{u}\cdot\vec{v} = u_x v_x + u_y v_y + u_z v_z}$

Ex. Given $\vec{a}$ and $\vec{b}$, determine their dot product.

a. $\vec{a} = (2,-1)$ and $\vec{b} = (4,3)$

b. $\vec{a} = (1,0,3)$ and $\vec{b} = (-2,5,8)$

Solution

a.

$\vec{a} \cdot \vec{b} = (2,-1) \cdot (4,3)$

$= (2)(4)+(-1)(3)$

$= 5$

b.

$\vec{a} \cdot \vec{b} = (1,0,3) \cdot (-2,5,8)$

$= (1)(-2)+(0)(5)+(3)(8)$

We have now seen two definitions of dot product.

By comparing these two definitions, we see that

$\vec{u}\cdot\vec{v} = u_x v_x + u_y v_y + u_z v_z = |\vec{u}||\vec{v}|\cos(\theta)$

Ex. Find the angle θ between each of the following pairs of vectors.
a. $\vec{u} = (2,-1)$ and $\vec{v} = (3,-4)$

b. $\vec{u} = (2,5,-1)$ and $\vec{v} = (17,-7,-1)$

Solution

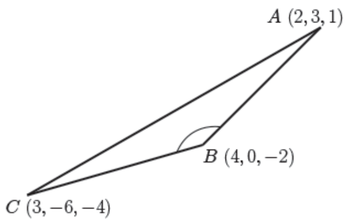
a.

$$\begin{aligned} \vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos(\theta) \\ \mathrm{u_x v_x + u_y v_y} &= |\vec{u}||\vec{v}|\cos(\theta) \\ 6+4 &= \sqrt{5}\sqrt{25}\cos(\theta) \\ 10/5\sqrt{5} &= \cos(\theta) \\ 2/\sqrt{5} &= \cos(\theta) \\ \theta &\approx 27^\circ \quad (\text{since } 0^\circ \le \theta \le 180^\circ) \end{aligned}$$

b.

$$\begin{aligned} \vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos(\theta) \\ \mathrm{u_x v_x + u_y v_y + u_z v_z} &= |\vec{u}||\vec{v}|\cos(\theta) \\ 34-35+1 &= |\vec{u}||\vec{v}|\cos(\theta) \\ 0 &= \cos(\theta) \\ \theta &= 90^\circ \quad (\text{since } 0^\circ \le \theta \le 180^0) \end{aligned}$$

Ex. Triangle *ABC* has vertices *A* (2,3,1), *B* (4,0,−2), and *C* (3,−6,−4). Calculate ∠*ABC*.



Solution (This question could be solved also using the cosine law after calculating the length of the sides of triangle ABC but using dot product is faster)

$$\overrightarrow{BA} = (-2,3,3)$$

$$\overrightarrow{BC} = (-1,-6,-2)$$

$$\overrightarrow{BA}\cdot\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos(\angle ABC)$$

$$-\frac{22}{\sqrt{22}\sqrt{41}} = \cos(\angle ABC)$$

$$\angle ABC \approx 137^\circ$$

Ex. Calculate the exact value of $(4\vec{x} - \vec{y}) \cdot (2\vec{x} + 3\vec{y})$ if $|\vec{x}|=3$, $|\vec{y}|=4$, and the angle between $\vec{x}$ and $\vec{y}$ is 150° .

Solution

$(4\vec{x} - \vec{y}) \cdot (2\vec{x} + 3\vec{y}) = 8|\vec{x}|^2 + 10\vec{x}\cdot\vec{y} - 3|\vec{y}|^2$

$= 8|\vec{x}|^2 + 10|\vec{x}||\vec{y}|\cos(150^0) - 3|\vec{y}|^2$

$= 24 - 60\sqrt{3}$

Ex. The magnitude of the sum of vectors $\vec{a}$ and $\vec{b}$ is equal to the magnitude of their difference. Determine the angle between $\vec{a}$ and $\vec{b}$.

Solution(could be solved also using the parallelogram rule maybe faster)

$$|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

$$\vec{a}\cdot\vec{a}+2(\vec{a}\cdot\vec{b})+\vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a}-2(\vec{a}\cdot\vec{b})+\vec{b}\cdot\vec{b}$$

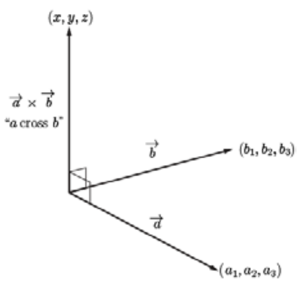
$$4(\vec{a}\cdot\vec{b}) = 0$$

Therefore, $\vec{a}\cdot\vec{b}=0$ and so $\vec{a}$ and $\vec{b}$ are orthogonal vectors. If we assume they are both non-zero vectors, then we may say that they are perpendicular.

The angle between non-zero vectors $\vec{a}$ and $\vec{b}$ is 90°.

Orthogonal vectors in 3-dimensions-The Cross Product of two vectors

Given any two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, find a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$.



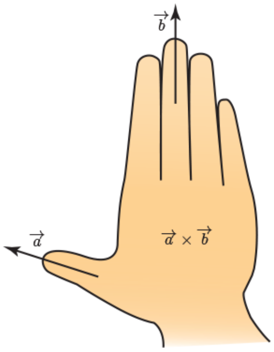
The **cross product** between two vectors $\vec{a}$ and $\vec{b}$ is a **vector** quantity denoted by $\vec{a} \times \vec{b}$ having the following properties:

a) $|\vec{a} \text{ x } \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\alpha$ where $\alpha = \angle(\vec{a},\vec{b})$

b) $\vec{a}$ **x** $\vec{b}$ is **perpendicular** to both $\vec{a}$ and $\vec{b}$ (is perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$)

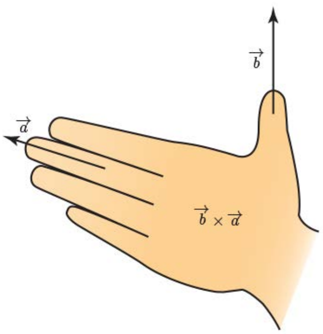
c) the vectors $\vec{a}$, $\vec{b}$, and $\vec{a}$ x $\vec{b}$ form a right-handed system

The Right-handed system is used to determine the direction of the cross product vector.



- Align the hand so that the thumb points in the direction of the first vector listed, in this case $\vec{a}$, and the fingers point in the direction of the second vector listed, $\vec{b}$.
- Holding your hand in this position, the direction your palm is facing is the direction of $\vec{a} \times \vec{b}$.
- In the diagram, the direction of $\vec{a} \times \vec{b}$ will be into the screen.

Notice the difference if you want to find the direction of $\vec{b} \times \vec{a}$.



- Similarly, the direction of $\vec{b} \times \vec{a}$ is determined by aligning the thumb of your right hand with the first vector listed, $\vec{b}$, and your fingers with the second vector listed, $\vec{a}$.
- The direction of $\vec{b} \times \vec{a}$ is the direction that the palm of the hand faces.
- In this case, it will be out of the screen.

Next we need to find the cross product of the unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

It's easy to see that:

$\hat{\imath} \times \hat{\imath} = \vec{0}$ $\quad$ $\hat{\jmath} \times \hat{\jmath} = \vec{0}$ $\quad$ $\hat{k} \times \hat{k} = \vec{0}$

$\hat{\imath} \times \hat{\jmath} = \hat{k}$ $\quad$ $\hat{\jmath} \times \hat{k} = \hat{\imath}$ $\quad$ $\hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}$ $\quad$ $\hat{k} \times \hat{\jmath} = -\hat{\imath}$ $\quad$ $\hat{\imath} \times \hat{k} = -\hat{\jmath}$

Now we can find the cross product of two algebraic vectors.

The **cross product** of two algebraic vectors $\vec{a} = (a_1,a_2,a_3)$ and $\vec{b} = (b_1,b_2,b_3)$ is given by

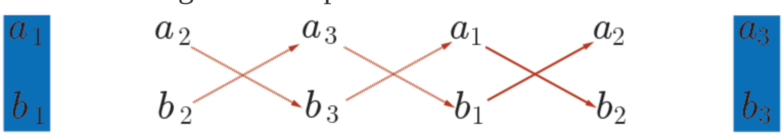
$\boldsymbol{\vec{a} \times \vec{b} = (a_2b_3\text{-}a_3b_2\ ,\ a_3b_1\text{-}a_1b_3\ ,\ a_1b_2\text{-}a_2b_1\ )}$

This formula is not easy to memorize but there is another way to derive the same formula.

The alternative method for computing the cross product involves writing the coordinates of $\vec{a} =(a_1,a_2,a_3)$ and $\vec{b} =(b_1,b_2,b_3)$ in an array as shown.

- Cross out the first and last columns.
- Proceed using the "down minus up" method. First draw a "down" arrow from a_2 to b_3 and determine the "down" product a_2b_3.
- Then draw an "up" arrow from b_2 to a_3 and compute the "up" product a_3b_2.
  The *x* component of the cross product is the value of the down product minus the up product, $a_2b_3\text{-}a_3b_2$.
- To determine the *y* component of the cross product, repeat with a_3b_1 and a_1b_3.
- Finally, to determine the *z* component of the cross product repeat using a_3b_2 and a_2b_1.

This is the original cross product formula!



$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Ex. Find two vectors orthogonal to both $\vec{a} =(1,-3,1)$ and $\vec{b} =(0,4,-1)$.

Solution

Setting up the array to compute the cross product gives

$$\vec{a} \times \vec{b} = (3-4,0+1,4+0)=(-1,1,4)$$

Check:

$(-1,1,4) \cdot (1,-3,1) = -1-3+4=0$

$(-1,1,4) \cdot (0,4,-1) = 0+4-4=0$

Therefore, one vector orthogonal to both $\vec{a}$ and $\vec{b}$ is $(-1,1,4)$.

Taking a scalar multiple of this vector, (say, its opposite vector) $(1,-1,-4)$ is sufficient for finding a second.

Properties of the Cross Product

The cross product possesses certain algebraic properties similar, but not equal to, that of the dot product.

1. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ anti-commutative
2. $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ distributive
3. $(\vec{b} + \vec{c}) \times \vec{a} = (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a})$ distributive
4. $k(\vec{a} \times \vec{b}) = (k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b})$ associative

The proofs are left as student exercises.

Property 1 illustrates the fact that the cross product is **not** commutative, and hence the order in which the vectors are written makes a difference.

This also affects the geometric interpretation of the cross product; the cross product of $\vec{a}$ with $\vec{b}$ and the cross product of $\vec{b}$ with $\vec{a}$ are different vectors.

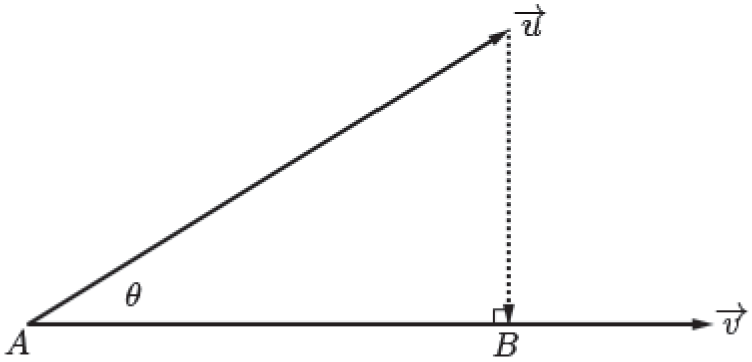
As was discussed earlier, these vectors are opposites of each other.

However, both vectors are mutually orthogonal to $\vec{a}$ and $\vec{b}$.

To distinguish between the two in a geometric sense, the right-hand rule may be adopted.

While the dot product and cross product may seem to be simply abstract mathematical concepts, they have a wide range of interesting geometrical applications, which have been very useful in fields such as physics.

Projections



Given two vectors, $\vec{u}$ and $\vec{v}$, placed tail to tail with angle θ between them, drop a perpendicular from the tip of $\vec{u}$ to the line containing $\vec{v}$.

The vector lying along the line containing $\vec{v}$, which has magnitude equal to the component of $\vec{u}$ in the direction of $\vec{v}$ (i.e., AB in our diagram), is called the **vector projection** of $\vec{u}$ onto $\vec{v}$.

To find the magnitude of this new vector, **proj** ($\vec{u}$ onto $\vec{v}$), we can use simple trigonometric ratios.

$$|\cos(\theta)| = |\textbf{proj}\ (\vec{u} \text{ onto } \vec{v})|/|\vec{u}|$$

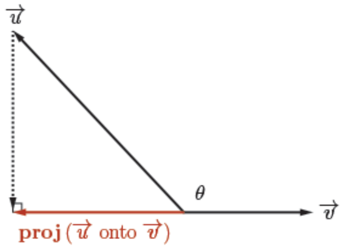
$$|\textbf{proj}\ (\vec{u} \text{ onto } \vec{v})| = |\vec{u}||\cos(\theta)|$$

$$= |\vec{u}||\vec{v}||\cos(\theta)|/|\vec{v}|$$

$$= ||\vec{u}||\vec{v}|\cos(\theta)|/|\vec{v}|$$

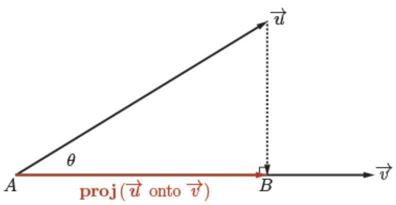
$$= |\vec{u} \cdot \vec{v}|/|\vec{v}|$$

Note: If $90^\circ < \theta < 180^\circ$, then $\cos(\theta) < 0$.



In this case, the projection vector would have direction opposite to $\vec{v}$.
We need magnitude (absolute value bars) around $\cos(\theta)$in the derivation to ensure that the **magnitude** of the vector is positive when $90^\circ < \theta < 180^\circ$.

Thus the vector projection of $\vec{u}$ onto $\vec{v}$ is a vector with magnitude $||\vec{u} \cdot \vec{v}|||\vec{v}|$ and direction along the line containing $\vec{v}$.



$|$proj ($\vec{u}$ onto $\vec{v}$)$|$ = $\frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}$

OK. So we know the magnitude and the direction of the projection vector but what is the vector?

$$\therefore \textbf{proj}\ (\vec{u} \text{ onto } \vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\hat{v}$$

$$=(\vec{u} \cdot \vec{v} / |\vec{v}|)(\vec{v} / |\vec{v}|)$$

$$=\frac{(\vec{u}\cdot\vec{v})}{|\mathbf{v}|^2}\vec{\mathbf{v}}$$

where we drop the magnitude bars on $\vec{u} \cdot \vec{v}$ to allow for the possibility that the projection vector is in a direction opposite to $\vec{v}$.

At this point, it is worth defining the **scalar projection** of $\vec{u}$ onto $\vec{v}$ as the signed (positive or negative) magnitude of the vector projection of $\vec{u}$ onto $\vec{v}$.

Then as we previously observed, the scalar projection of $\vec{u}$ onto $\vec{v}$ is equal to $|\vec{u}|\cos(\theta) = (\vec{u} \cdot \vec{v}) / |\vec{v}|$.

The applications of vector projections are many and varied. They exist in areas such as engineering, quantum mechanics to name only a few. In fact there is an entire branch of geometry called Projective Geometry.

To recap

The **magnitude of the vector projection** of $\vec{u}$ onto $\vec{v}$ is $\frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}$

The **vector projection** of $\vec{u}$ onto $\vec{v}$ is **proj** ($\vec{u}$ onto $\vec{v}$)= $\frac{(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})}{|\mathbf{v}|^2}\vec{\mathbf{v}}$

Ex. Calculate the projection of $\vec{u}$ = (1,–2,3) onto $\vec{v}$ = (3,0,4) and determine its magnitude.

Solution

Vector:

proj ($\vec{u}$ onto $\vec{v}$) $=(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|})\hat{v}$

$=\frac{15}{25}(3,0,4)$

$=(\frac{9}{5},0,\frac{12}{5})$

Magnitude:

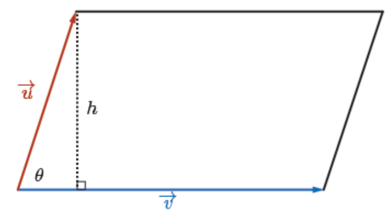
|**proj** ($\vec{u}$ onto $\vec{v}$)|= $\frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}$ =15/5=3

Or alternatively,

|**proj** ($\vec{u}$ onto $\vec{v}$)|=|(9/5,0,12/5)|=$\sqrt{(\frac{9}{5})^2+0^2+(\frac{12}{5})^2}$=3

The magnitude of the cross product is connected with geometry.

It is equal to the area of the completed parallelogram formed by two vectors, $\vec{u}$ and $\vec{v}$.



Recall:

Area of a parallelogram = base × perpendicular height

From the diagram, the length of the base is $|\vec{v}|$ and since $\sin(\theta)=h/|\vec{u}|$, then $h=|\vec{u}|\sin(\theta)$.

Substituting these gives $A=|\vec{v}||\vec{u}|\sin(\theta)=|\vec{u}\times\vec{v}|$.

6. The area of a parallelogram is

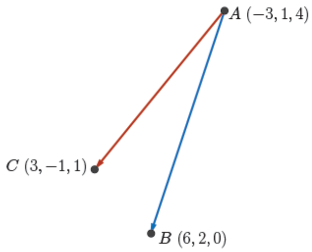
$$\boldsymbol{A=|\vec{u}\times\vec{v}|}$$

where $\vec{u}$ and $\vec{v}$ are adjacent sides of a parallelogram

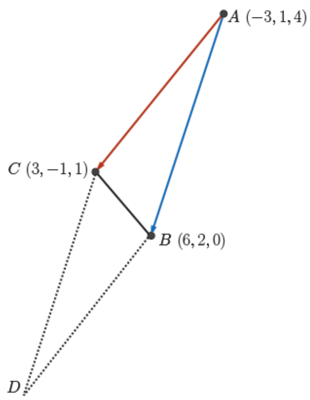
Ex. Find the area of a triangle with vertices $A\,(-3,1,4)$, $B\,(6,2,0)$, and $C\,(3,-1,1)$.

Solution

Draw the diagram and label the vertices.



Next, we determine the vectors between the vertices, and then apply the formula for the area of the parallelogram.



Since a diagonal of a parallelogram bisects its area, the area of $\triangle ABC$ is one half the area of parallelogram $ABDC$.

$\overrightarrow{AC}$=(6,−2,−3) and $\overrightarrow{AB}$=(9,1,−4).

Determining the cross product:

$$\overrightarrow{AC} \times \overrightarrow{AB} = (11,-3,24)$$

Quickly check that

$$(6,-2,-3) \cdot (11,-3,24)\ 66+6-72=0$$

$$(9,1,-4) \cdot (11,-3,24)=99-3-96=0$$

Thus, Area$\triangle ABC = \frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{AB}| = \frac{1}{2}\sqrt{706}$

Triple Scalar Product

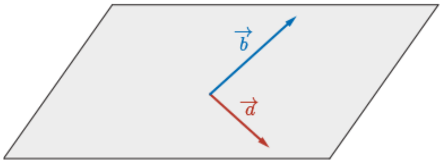
Another interesting connection between algebraic operations on vectors and geometry is the **triple scalar product** of three vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, which is defined as

$$\vec{c} \cdot (\vec{a} \times \vec{b})$$

Note that this is a scalar quantity.

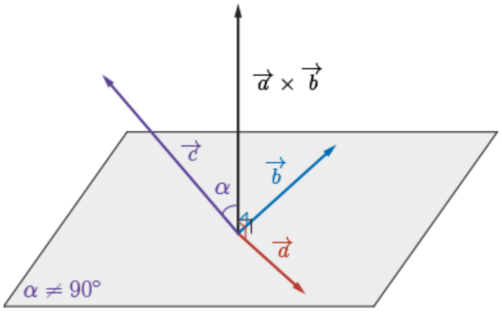
In addition to other applications, the triple scalar product is often used to determine if three vectors are coplanar (lie in the same plane).

How so?



Consider that any two vectors, $\vec{a}$ and $\vec{b}$, are coplanar.

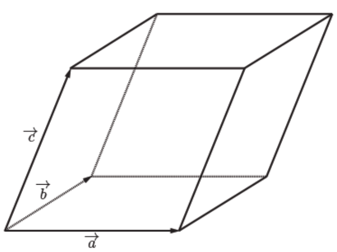
Assume that these two vectors, $\vec{a}$ and $\vec{b}$, are not collinear.
Therefore, their cross product, $\vec{a} \times \vec{b}$, is orthogonal to the plane in which $\vec{a}$ and $\vec{b}$ lie.



So, ***if*** $\vec{c} \cdot (\vec{a} \times \vec{b}) = 0$***, then*** $\vec{c}$ ***is orthogonal to*** $\vec{a} \times \vec{b}$***, and so*** $\vec{c}$ ***must lie in the same plane as*** $\vec{a}$ ***and*** $\vec{b}$***.***
Similarly, if $\vec{c} \cdot (\vec{a} \times \vec{b}) \neq 0$, then $\vec{c}$ is not orthogonal to $\vec{a} \times \vec{b}$, and so $\vec{a}$, $\vec{b}$, and $\vec{c}$ are not coplanar.

Volume of a Parallelepiped

A **parallelepiped** is a box-like solid, where the opposite faces of which are parallel and congruent parallelograms.



Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors whose tails meet at one vertex of the parallelepiped.

The absolute value of the triple scalar product of these three vectors gives the volume of the parallelepiped.

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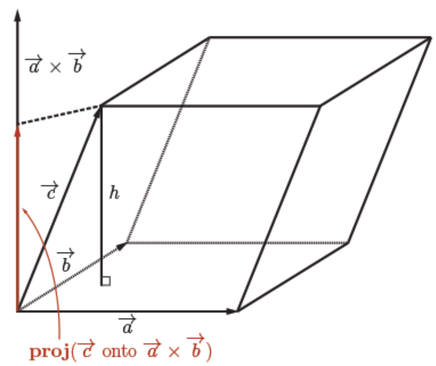
From the diagram, we see

Volume $=(\text{Area of base})\times h$

Area of base $=|\vec{a}\times\vec{b}|$



$h=|\mathbf{proj}\,(\vec{c}\text{ onto }(\vec{a}\times\vec{b}))|$

$=|\vec{c}\cdot(\vec{a}\times\vec{b})|/|\vec{a}\times\vec{b}|$

$\therefore$ Volume $=|\vec{c}\cdot(\vec{a}\times\vec{b})|$