Number Theory

Number Theory - The branch of mathematics that deals with the properties and relationships of numbers

Base 10 refers to the numbering system in common use. Take a number like 475, base ten refers to the position, the 5 is in the one's place, the 7 is in the ten's place and the 4 is in the hundred's place. Each number is 10 times the value to the right of it, hence the term base ten.

Example: $3217 = (3\times10^3) + (2\times10^2) + (1\times10^1) + (7\times10^0)$ noting that $10^0 = 1$.

If a number N is divided by a divisor d to give a quotient q and a remainder r, then $\boldsymbol{N=dq+r}$

Example: $N=58$, $d=7$, we obtain $q=8$ and $r=2$ $\rightarrow 58 = 7\times8+2$

1. Prime Number

A Prime Number can be divided evenly only by 1, or itself. (In other words, its factors are only 1 or itself)

Example 1: 13 can only be divided evenly by 1 or 13, so it is a prime number.

Example 2: 14 can be divided evenly by 1, 2, 7 and 14 so it is a composite number.

2. Composite Number

A Composite Number can be divided evenly by numbers other than 1 or itself. (In other words, it has more than the two factors of 1 and itself)

Example 1: 9 can be divided evenly by 1, 3 and 9, so 9 is a composite number.

Example 2: 7 can only be divided evenly by 1 and 7, so it is not composite. It must be a prime number.

3. Beprisque number

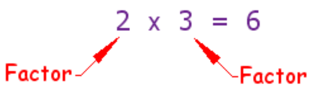
A *beprisque number n* is an integer which is either one more than a prime number and one less than a perfect square, or one more than a square and one less than a prime.

The only prime beprisque numbers are 2 and 3. For a prime number to be a beprisque number, it has to neighbor another prime number; the only two primes fitting that bill are of course 2 and 3.

The only odd beprisque numbers are 1 and 3. For an odd number to be a beprisque number, it has to neighbor an even prime, and since 2 is the only even prime, only 1 and 3 can be beprisque numbers.

4. Factors

You can make the same definitions using Factors.



"Factors" are the numbers you multiply together to get another number.

So here is just a different way of saying the same thing from above.

When the only factors of a number are 1 and the number, then it is a Prime Number

Examples:

$3 = 1 \times 3$ (the only factors are 1 and 3)	Prime
$6 = 1 \times 6$ or $6 = 2 \times 3$ (the factors are 1,2,3 and 6)	Composite

Examples from 1 to 14

Number	Can be Evenly Divided By	Prime, or Composite?
1	*(1 is not considered prime or composite)*	
2	1, 2	Prime
3	1, 3	Prime
4	1, 2, 4	Composite
5	1, 5	Prime
6	1, 2, 3, 6	Composite
7	1, 7	Prime
8	1, 2, 4, 8	Composite
9	1, 3, 9	Composite
10	1, 2, 5, 10	Composite
11	1, 11	Prime
12	1, 2, 3, 4, 6, 12	Composite
13	1, 13	Prime
14	1, 2, 7, 14	Composite
...	...	...

So when there are more factors than 1 or the number itself, the number is Composite.

5. Divisibility Rules

The divisibility rules make math easier. Did you ever wonder how people could tell if something was divisible by a number just by looking at it?
These rules are how they do it. Memorize a few simple rules and simplifying fractions and prime factorization will be so much easier.

A number is divisible by:

1: All numbers are divisible by 1.
2: If it is an even number (ends in 0, 2, 4, 6, or 8)
3: If the sum of its digits can be divided by 3. (Ex. 5625: 5 + 6+ 2 + 5 = 18 and 18 is divisible by 3. So is 5625.)
4: If its last two digits can be divided by 4. (Do not add them together.)(9136: 36, the last two digits, is divisible by 4. So is 9136.)
5: If the number ends with a 5 or a 0. (875 and 390 are both divisible by 5)

6: If the number is divisible by BOTH 2 and by 3. (See the rules for 2 and for 3.)

7: If when you take the last digit, double it and subtract from remaining digits, your answer is 0, 7, or can be divisible by 7.

8: If the last three digits can be divided by 8.

9: If the sum of its digits can be divided by 9. (Ex. 6813: 6 + 8 + 1 + 3 = 18 and 18 can be divided by 9. So can 6813.)

10: If the number ends with a 0.

11: If the difference between the sums of every other digit can be divided by 11. (Ex. 4191: 4 – 1 + 9 – 1 = 11) 11 can be divided by 11. So can 4191.

12: If the number is divisible by BOTH 3 and by 4. (See the rules for 3 and for 4).

6. Prime Factorization

"Prime Factorization" is finding which prime numbers you need to multiply together to get the original number.

Example 1: What are the prime factors of 12?

It is best to start working from the smallest prime number, which is 2, so let's check: $12 \div 2 = 6$
But 6 is not a prime number, so we need to factor it further: $6 \div 2 = 3$
And 3 is a prime number, so: $12 = 2 \times 2 \times 3$

As you can see, every factor is a prime number, so the answer must be right - the prime factorization of 12 is $2 \times 2 \times 3$, which can also be written as $2^2 \times 3$.

Example 2: What is the prime factorization of 147?

Can we divide 147 evenly by 2? No, so we should try the next prime number, 3: $147 \div 3 = 49$
Then we try factoring 49, and find that 7 is the smallest prime number that works: $49 \div 7 = 7$
And that is as far as we need to go, because all the factors are prime numbers.
$147 = 3 \times 7 \times 7 = 3 \times 7^2$

Another Method
We showed you how to do the factorization by starting at the smallest prime and working upwards, but sometimes it is easier to break a number down into any factors you can, then work those factor down to primes.

Example 3: What are the prime factors of 90?
Break 90 into 9×10
The prime factors of 9 are 3 and 3. The prime factors of 10 are 2 and 5
So the prime factors of 90 are 3, 3, 2 and 5

And here is another thing: There is only one (unique!) set of prime factors for any number.

Example 4: The prime factors of 330 are 2, 3, 5 and 11: $330 = 2 \times 3 \times 5 \times 11$
There is no other possible set of prime numbers that can be multiplied to make 330.

7. Least Common Multiple

The smallest (non-zero) number is a multiple of two or more numbers.

1) What is a "Multiple"?
The multiples of a number are what you get when you multiply it by other numbers (such as if you multiply it by 1, 2,3,4,5, etc.). Just like the multiplication table.

Here are some examples:

The multiples of 3 are 3, 6, 9, 12, 15, 18, 21, etc ...

The multiples of 12 are 12, 24, 36, 48, 60, 72, etc...

2) What is a "Common Multiple"?
When you list the multiples of two (or more) numbers, and find the same value in both lists, then that is a *common multiple* of those numbers.

For example, when you write down the multiples of 4 and 5, the *common* multiples are those that are found in both lists:

The multiples of 4 are 4,8,12,16,20,24,28,32,36,40,44,...

The multiples of 5 are 5,10,15,20,25,30,35,40,45,50,...

Notice that 20 and 40 appear in both lists?
So, the common multiples of 4 and 5 are: 20, 40, (and 60, 80, etc ..., too)

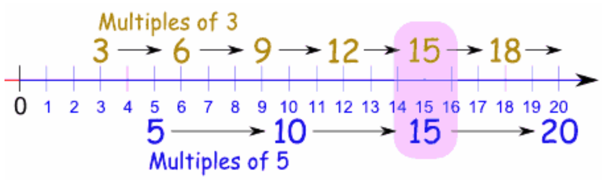
3) What is the "Least Common Multiple"?
It is simply the smallest of the common multiples.

In our previous example, the smallest of the common multiples is 20, so the *Least* Common Multiple of 4 and 5 is 20.

4) Finding the Least Common Multiple
It is a really easy thing to do. Just start listing the multiples of the numbers until you get a match.

Example 1: Find the least common multiple for 3 and 5:

The multiples of 3 are *3, 6, 9, 15...* and the multiples of 5 are *5, 10, 15, 20, ...*, like this:



As you can see on this number line, the first time that multiples match up is 15. Answer: 15
More than 2 Numbers. You can also find the least common multiple of 3 (or more) numbers.

Example 2: Find the least common multiple for 4, 6, and 8

Multiples of 4 are: 4, 8, 12, 16, 20, 24, 28, 32, 36...
Multiples of 6 are: 6, 12, 18, 24, 30, 36...
Multiples of 8 are: 8, 16, 24, 32, 40 ...
So, 24 is the least common multiple (I can't find a smaller one!)

8. Properties of Real Numbers

1) Commutative Property - interchange or switch the elements
For example shows commutative property for addition:

$$X + Y = Y + X$$

2) Associative Property- regroups the elements
For example shows associative property for addition:

$$(X + Y) + Z = X + (Y + Z)$$

3) Identity Property- What returns the input unchanged?

$X + 0 = X$ Additive Identity
$X \cdot 1 = X$ Multiplicative Identity

4) Inverse Property- What brings you back to the identity element using that operation?

$X + (-X) = 0$ Additive Inverse

$X \cdot 1/X = 1$ Multiplicative Inverse

5) Distributive Property - *multiply* across the parentheses.

Each element inside the parentheses is multiplied by the element outside the parentheses.

$$a(b+c) = a \cdot b + a \cdot c$$

9. Greatest Common Factor

The highest number divides exactly into two or more numbers. It is the "greatest" thing for simplifying fractions!

1) What is the "Greatest Common Factor"?

It is simply the largest of the common factors. In our previous example, the largest of the common factors is 15, so the Greatest Common Factor of 15, 30 and 105 is 15

The "Greatest Common Factor" is the largest of the common factors of two or more numbers.

2) Why is this Useful?

One of the most useful things is when we want to simplify a fraction:

Example: How could we simplify $^{12}/_{30}$?

At the top we found that the Common Factors of 12 and 30 were 1, 2, 3 and 6, and so the Greatest Common Factor is 6.

This means that the **largest** number we can divide both 12 and 30 evenly by is 6.

The Greatest Common Factor of 12 and 30 is **6**. And so $^{12}/_{30}$ can be simplified to $^{2}/_{5}$

3) Finding the Greatest Common Factor

Here are three ways:

a. You can find all **factors** of both numbers, then select the ones that are **common** to both, and then choose the **greatest**.

Example:

Two Numbers	**All Factors**	**Common Factors**	**Greatest Common Factor**	***Example Simplified Fraction***
9 and 12	**9**: 1,3,9 **12**: 1,2,3,4,6,12	1,3	**3**	$^{9}/_{12}$ » $^{3}/_{4}$

And another example:

Two Numbers	**All Factors**	**Common Factors**	**Greatest Common Factor**	***Example Simplified Fraction***
6 and 18	**6**: 1,2,3,6 **18**: 1,2,3,6,9,18	1,2,3,6	**6**	$^{6}/_{18}$ » $^{1}/_{3}$

b. You can find the prime factors and combine the common ones together:

Two Numbers	**Thinking ...**	**Greatest Common Factor**	***Example Simplified Fraction***
24 and 108	**2** × **2** × 2 × **3** = 24, and **2** × **2** × **3** × 3 × 3 = 108	2 × 2 × 3 = **12**	$^{24}/_{108}$ » $^{2}/_{9}$

c. And sometimes you can just **play around** with the factors until you discover it:

Two Numbers	**Thinking ...**	**Greatest Common Factor**	***Example Simplified Fraction***
9 and 12	**3** × 3 = 9 and **3** × 4 = 12	**3**	$^{9}/_{12}$ » $^{3}/_{4}$

But in that case you had better be careful you have found the **greatest** common factor.

10. Powers

Recall that a perfect square is a number that can be represented as a product of two equal factors, for example $25=5\times5=5^2$. A perfect cube is a number that can be represented as a product of three equal factors, for example $8=2\times2\times2=2^3$. A product of six equal factors can be written as a power, for example $2^6 = 2\times2\times2\times2\times2\times2$.

5^2 means that there are 5 perfect square numbers from 1 to 5^2 namely 1^2, 2^2, 3^2, 4^2, 5^2.
4^3 means that there are 4 perfect cube numbers from 1 to 4^3 namely 1^3, 2^3, 3^3, 4^3.

Questions in class

1. Is $121^{13} - 101^4$ divisible by 2?

2. Is $326^2 - 325^2$ divisible by 3?

3. Is 65,314,638,792 divisible by 24?

4. How many 3 digit numbers are divisible by 5?

5. If $1998 = p^s q^t r^u$, where p, q and r are prime numbers, what is the value of $p + q + r$?

6. A number is *Beprisque* if it is the only natural number between a prime number and a perfect square (e.g. 10 is Beprisque but 12 is not). What is the number of *two-digit* Beprisque numbers (including 10)?

7. What is the least value of x which makes $\frac{24}{x - 4}$ an integer?

8. What is the number of ordered pairs (a, b) of *integers* which satisfy the equation $a^b = 64$?

9. If $a^2 + 3b = 33$, where a and b are positive integers, what is the value of ab?

10. When 14 is divided by 5, the remainder is 4. When 14 is divided by a positive integer n, the remainder is 2. For how many different values of n is this possible?

11. If $496 = 2^m - 2^n$, where m and n are integers, then $m + n$ is equal to what?

12. The product of the digits of a four-digit number is 810. If none of the digits is repeated, what is the sum of the digits?