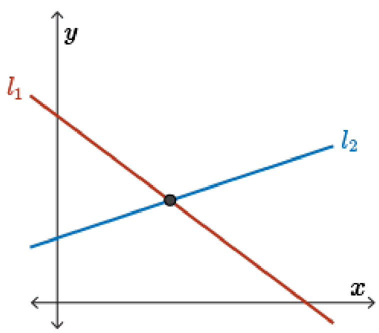
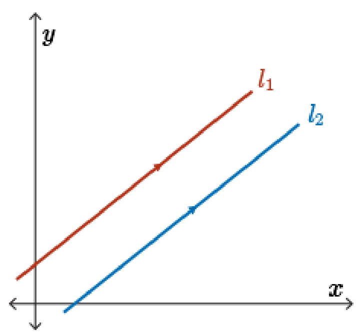
Lesson 14. Unit 8 – Relationships between points, lines and planes(1)

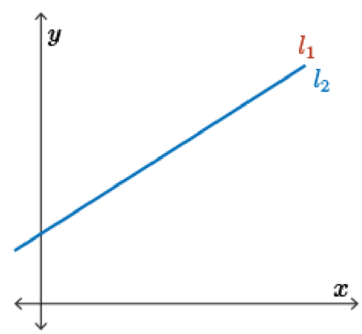
Intersection of Lines in R^2 and R^3

First let's take a look at 2-d case.

There are three cases to consider with respect to the way two lines may intersect in R^2.







1. Intersect in exactly one point 2. No intersection points – 3. Infinite number of intersection points -

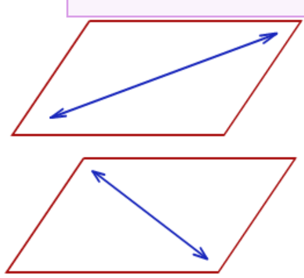
the lines are **parallel and distinct** the lines are **coincident**

In 3-d the three cases in which two lines may intersect in R^2 also exist in R^3. The two lines may

- intersect in exactly one point,
- be parallel and distinct and not intersect, or
- be coincident and intersect in an infinite number of points.

However, there is one additional possibility in R3 not found in R2: skew lines.

> Two lines in R3 are said to be **skew lines** if they are not parallel and do not intersect.
> Equivalently, they are lines that are not coplanar.



Ex. Find the points of intersection of the following lines.

a. l_1:2x−3y+8=0 l_2:2x−3y−1=0

Solution

Method 1

Since the normal vectors for the two lines are equal $\overrightarrow{n_1}$= (2,−3) =$\overrightarrow{n_2}$ and the constant terms are not equal (8≠−1), then l_1 and l_2 are parallel and distinct.

Thus, the two lines, l_1 and l_2, do not intersect.

Method 2

Using elimination, we subtract the two equations to get 9=0.

This statement is clearly not true and is independent of the values of *x* and *y*.

Hence, the two lines share no common point and thus do not intersect.

Since these are lines in R^2, they must be parallel and distinct.

If a linear system of equations has no solutions, such as the system in part **a**, then the system is said to be **inconsistent**.

If a linear system has at least 1 solution, then it is said to be **consistent**.

b. l_1:x−5y+6=0 l_2:3x+10y−7=0

Solution

Recognizing that the normal vectors are not scalar multiples of one another (l_1 and l_2 are not parallel), we use the method of elimination, and we get x=-1 and y=1.
Hence, the two lines intersect at the point (−1,1).

c. l_1:(x,y,z)=(9,3,4)+t (4,0,2), t∈R
l_2:x=3r−2, y=6−3r, z=r−1, r∈R

Solution

There are many different approaches for solving systems of this form.

Method 1

In this first method, we will solve by converting both lines into parametric equations and determining the values of the parameters *t* and *r*.

Converting $l1$ into parametric form gives $x=9+4t$,

$y=3$

$z=4+2t$.

Equating the parametric equations for x, we get

$$9+4t = 3r-2$$

$$\therefore 3r = 4t+11 \quad (1)$$

Equating the parametric equations for y, we get

$$3 = 6-3r$$

$$3r = 3$$

$$\therefore r = 1$$

Substituting $r=1$ into the equation for l_2 gives $x=3(1)-2=1, y=6-3(1)=3$, and $z=1-1=0$.

We must verify that this point (1,3,0) also lies on l_1.

By substituting $r=1$ into equation (1), we can find the value of the parameter t for l_1.

$$3r = 4t+11 \quad (1)$$

$$3(1) = 4t+11$$

$$4t = -8$$

$$t = -2$$

Substituting $t=-2$ into the equation for l_1 gives $(x,y,z)=(1,3,0)$.

Therefore, l_1 and l_2 intersect at the point (1,3,0).

Method 2

Another way to solve this system is to write the equation of l_1 in symmetric form, and then substitute the parametric equations of l_2 into l_1.

To this effect, the symmetric equations of l_1 are

$$l_1 : \ \frac{x-9}{4} = \frac{z-4}{2}, \ y=3$$

Substituting l_2 (x=3r−2 and z=r−1) into l_1 and solving for r, we get

$\frac{(3r-2)-9}{4} = \frac{(r-1)-4}{2}$

r =1

When r=1, the x and z coordinates of $l2$ are x=3(1)−2=1 and z=1−1=0.
These (x=1,z=0) also satisfy the symmetric equations of $l1$, since $\frac{1-9}{4}=\frac{0-4}{2}$.
We must still verify that the y coordinate of l_1 y=3, also satisfies l_2 when r=1.
Substituting, we get y=6−3r=6−3(1)=3.
Therefore, the point of intersection is (1,3,0).

d. l_1: $\vec{r}$ =(3,1,−1)+t (2,1,−3), t∈R and l_2: $\frac{x-7}{2} = \frac{y-12}{-4} = \frac{z-17}{5}$

Solution

Method 1
3t+5r=−18

For the system of equations to be consistent (have a solution), r=9/5 and t=19/5 must also satisfy equation above.
Substituting, we get

3*t*+5*r*=3(19/5)+5(9/5) = 102/5 ≠−18

Therefore, this system of equations has no solution and hence these two lines do not intersect.
By inspecting the parametric equations of both lines, we see that the direction vectors of the two lines are not scalar multiples of each other, so the lines are not parallel.
These lines are in R3, are not parallel, and do not intersect, and so l_1 and l_2 are skew lines.
Method 2

As before, an alternative solution is to derive the parametric equations for l_1, and then substitute them into the symmetric equation for l_2 and verify consistency.

Substituting the parametric equations x=3+2t and y=1+t of $l_1$into the symmetric equations of l_2,

$\frac{x-7}{2} = \frac{y-12}{-4}$

$\frac{(3+2t)-7}{2} = \frac{(1+t)-12}{-4}$

t =19/5

Repeating the process by using the parametric equations y=1+t and z=−1−3t, we get

$$\frac{y-12}{-4} = \frac{z-17}{5}$$

And we get

t =−127/7≠19/5

In last unit, we have learned three common ways to express the equation of a plane in R^3. These are vector, parametric, and cartesian (scalar) equations of the plane.

Next we will consider how to determine if a given line intersects a given plane.

Given a line and a plane in R^3, there are three possibilities for the intersection of the line with the plane.

1. The line and the plane intersect at a single point. There is exactly one solution.
2. The line is parallel to the plane. The line and the plane do not intersect. There are no solutions.
3. The line lies on the plane, so every point on the line intersects the plane. There are an infinite number of solutions.

To determine algebraically whether or not a line intersects a plane, we may substitute the parametric equations of the line into the scalar equation of the plane. The resulting linear equation reduces to one of three possibilities, with each determining the type of intersection between the line and the plane.

Ex. Determine all points of intersection between the line *l* described by

$$x=5+2t$$

$$y=1+6t$$

$$z=-5t,\ t\in \mathrm{R}$$

and the plane π:2x+3y+5z−1=0.

Solution

Substitute the appropriate parametric equation of the line for the appropriate variable in the scalar equation of the plane.

$2(5+2t) + 3(1+6t) + 5(-5t)-1=0$

$10 + 4t + 3 + 18t - 25t - 1=0$

$-3t + 12=0$

$t=4$

Ex. Determine if the line *l* described by the symmetric equations

$\frac{x-2}{5} = \frac{y+3}{-1} = \frac{z-1}{3}$

intersects the plane $\pi: 3x-5z=6$.

Solution

Recall that we may set each expression of the symmetric equations equal to a parameter, *t*, and solve for the variables *x*,*y*, and *z* to determine the parametric equations of the line.

$x = 2 + 5t$

$y = -3 - t$

$z = 1 + 3t,\ t \in R$

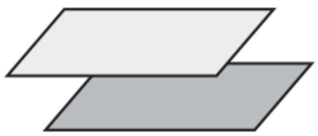
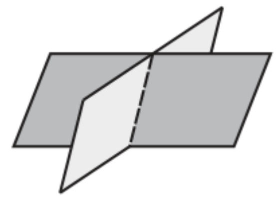
Substituting the parametric equations into the scalar equation for π, we get

$3(2+5t) - 5(1+3t) = 6$
$6+15t-5-15t = 6$
$0t = 5$
However, $0t \neq 5$ for all $t \in R$ and so no value of *t* exists such that the line shares a point in common with the plane.

Thus, there is no point of intersection between the line and the plane.
This line is parallel to the plane and does not lie on the plane.

Now let's look at ***the intersection of two planes.***

There are also three possibilities for the intersection of two planes:

		
The two planes are parallel and distinct.	The two planes are coincident.	The two planes intersect in a line.
The normal vectors of the planes are scalar multiples of each other, but the equations of the planes in scalar form are not multiples of each other.	The normal vectors of the planes are scalar multiples of each other, and the equations in scalar form are multiples of each other.	The normal vectors of the planes are not scalar multiples of each other.
There are no points that satisfy the equations of both planes.	Any point on either plane satisfies the equation of the other plane.	Solving the system of two equations (the equations of the two planes) in three variables will give the equation of the line of intersection.

Ex. Determine the intersection of the two planes:

π_1 : 3x -2y + 4z – 17 = 0

π_2 : (x,y,z) = (5,-1,0) + t(2,1,-1) + s(-10,-3,6), s, t ∈ R

Solution

A normal vector of π_1 is $\overrightarrow{n_1}$ = (3,-2,4).

From π_2: $\overrightarrow{d_2}$ = (2,1,-1) and $\overrightarrow{e_2}$ = (-10,-3,6) are two direction vectors of the plane.

A normal vector for π_2 is $\overrightarrow{n_1} = \overrightarrow{d_2} \times \overrightarrow{e_2}$ = (3,-2,4).

Since the two normal vectors collinear (they are in fact equals), the planes are parallel.

We need to determine if the planes are coincident or if they are parallel and distinct.

The point P_2(5,-1,0) ∈ $\boldsymbol{\pi_2}$ **.** We need to check if P_2(5,-1,0) ∈ π_1.

3(5) -2(-1) + 4(0) – 17 = 0 *true*

The planes are coincident.

We have shown that the two planes are parallel and have a point in common, so the planes are in fact coincident.

The solution is the set of all points in either plane.

Note:

If P_2(5,−1,0) did not lie on π_1, the planes would have been parallel and distinct.
In this case, there would be no intersection between the two planes.

Ex. Find the intersection of the two planes:

π_1 : x - y - z – 12 = 0

π_2 : 3x -2y - 4z – 8 = 0

Solution

Normal vectors for the plane are $\overrightarrow{n_1}$ = (1,-1,-1) and $\overrightarrow{n_1}$ = (3,-2,-4) and they are not parallel, so the planes must intersect in a line.

To determine the equation of the line of intersection of these two planes we need to solve the linear system

x - y - z – 12 = 0 (1)

3x -2y - 4z – 8 = 0 (2)

Chose z = t and then solve for x and y ⇨ x = 2t – 16 and y = t – 28.

$\pi_1 \cap \pi_2$ = L : $\begin{cases} x = -16 + 2t \\ y = -28 + t \\ z = t \end{cases}$, t ∈ R.

So far, we have discussed the possible ways that two lines, a line and a plane, and two planes can intersect one another in 3-space.
*Next we are going to look at the different ways that **three planes can intersect in $\boldsymbol{R^3}$**.*

Ex. Find all points of intersection of the following three planes:

x + 2y – 4z = 3 **(1)**

-2x + y + 3z = 4 **(2)**

4x -3y – z = -2 **(3)**

Solution

As we done in before we begin by examining the normal vectors

$\overrightarrow{n_1}$ = (1,2,-4), $\overrightarrow{n_2}$ = (-2,1,3), and $\overrightarrow{n_3}$ = (4,-3,-1).

Since they are not parallel to another, we conclude that the three planes are non-parallel.

The approach we will take to finding points of intersection, is to eliminate variables until we can solve for one variable, and then substitute this value back into the previous equations to solve for the other two.

2**(1) + (2)** ⇨ 5y – 5z = 10 ⇨ y – z = 2   **(4)**

-4**(1)** + **(3)** ⇨ -11y + 15z = -14   **(5)**

We now use equations **(4)** and **(5)** to eliminate y and solve for z.

Solving, we get z =2.

We back substitute z = 2 into **(5)** to get y = 4.

Substitute y = 4 and z = 2 into **(1), (2)** or **(3)** to get x = 3.

∴ (x,y,z) = (3,4,2) a point.

This can be geometrically interpreted as three planes intersecting in a single point.