Chapter 6 Trigonometry 3

1. Sine Law and Cosine Law

Recall:
Sin Law can be used to solve triangles in

(1) Two angles and one side(ASA, AAS);

(2) Two sides and one angles not include in given sides(SSA)

Cos Law can be used to solve triangles in

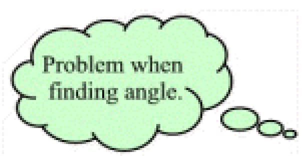
(1) Given three sides（SSS）;

(2) Given one angle and two arms（SAS）

In another word, cosine law can be used when sine law can't be used.

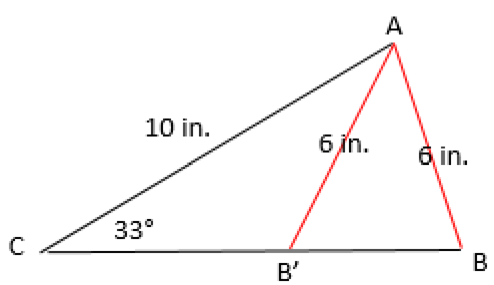
2. Ambiguous Case of Sine Law

By definition, the word **ambiguous** means *open to two or more interpretations.*
Such is the case for certain solutions when working with the Law of Sines.



- If you are given two angles and one side (ASA or AAS), the Law of Sines will nicely provide you with ONE solution for a missing side.
- Unfortunately, the Law of Sines has a problem dealing with SSA. If you are given two sides and one angle (where you must *find an angle*), the Law of Sines could possibly provide you with one or more solutions, or even no solution.

For example, take a look at this picture:



If you are told that $\angle C = 33°$, b = 10 in. and c= 6 in, there are two different triangles that match this criteria. As you can see in the picture, either an acute triangle or an obtuse triangle could be created because side c could swing either in or out along the unknown side a.

When you are given two sides and an angle not in between those sides, you need to be on the lookout for the ambiguous case.

Way 1: To determine if there is a 2$^{\text{nd}}$ valid angle:

1. See if you are given two sides and the angle not in between (SSA). This is the situation that may have 2 possible answers.
2. Find the value of the unknown angle.
3. Once you find the value of your angle, subtract it from 180° to find the possible second angle.
4. Add the new angle to the original angle. If their sum is less than 180°, you have two valid answers. If the sum is over 180°, then the second angle is not valid.

Example 1: Use the Law of Sines to find the measure of angle B from our example in which $\angle C = 33°$, b = 10 in. and c = 6 in.
First we know that this triangle is a candidate for the ambiguous case since we are given two sides and an angle **not** in between them.

We need to find the measure of angle B using the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{a}{\sin A} = \frac{10}{\sin B} = \frac{6}{\sin 33°}$$
$$\frac{10}{\sin B} = \frac{6}{\sin 33°}$$

Now we'll cross multiply and solve the equation:

$$10(\sin 33°) = 6 \cdot \sin B$$
$$\sin B = \frac{10(\sin 33°)}{6}$$
$$\mathrm{m}\measuredangle B = \arcsin \frac{10(\sin 33°)}{6}$$
$$\mathrm{m}\measuredangle B \approx 65.2°$$

We've found one value for angle B, but we need to see if there's another possible value. To do this, we'll subtract 65.2° from 180°.

180° - 65.2° = 114.8°

To see if 114.8° is a valid answer, we must add this to the other existing angle. If their sum is less than 180°, we know a triangle can exist. If, however, it is over 180°, then it is not a valid answer (because the three angles of a triangle must add up to 180°.)

114.8° + 33° = 147.8°
Since 147.8° is less than 180°, we know that 114.8° is a valid answer

Final answer: $\angle B = 65.2°$ and $\angle B = 114.8°$

Way 2: To see if there are 2 solutions:

1. See if you are given two sides and the angle not in between (SSA). This is the situation that may have 2 possible answers.
2. If given angle A, side a, and side b (SSA case), height of the triangle can be determined by h = b sin A. Then you need to check the following things:
   a) Angle A is acute.
   b) Side b > side a
   c) Height = b sin A < side a
3. Once you find the value of your angle, subtract it from 180° to find the possible second angle.

Use way 2 to determine if Example 1 has 2 solutions.

Given: $\angle C = 33°$, b = 10 in. and c = 6 in.

1. SSA – might be ambiguous
2. $\angle C = 33°$ is acute
3. b > c
4. b sin C = 10sin33 = 5.45 < 6

The givens satisfy all conditions, therefore, this triangle has 2 solutions.

Example 2: Use the Law of Sines to find measure of angle A in this scenario: $\angle C = 64°$, c = 10 ft., and a = 8 ft. If it helps, you can draw a rough sketch to view this triangle, but this is optional. We know that this triangle is a candidate for the ambiguous case since we are given two sides and an angle **not** in between them.

Way 1:

We need to find the measure of angle B using the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{8}{\sin A} = \frac{b}{\sin B} = \frac{10}{\sin 64°}$$
$$\frac{8}{\sin A} = \frac{10}{\sin 64°}$$

Now we'll cross multiply and solve the equation:

$$8(\sin 64°) = 10 \cdot \sin A$$
$$\sin A = \frac{8(\sin 64°)}{10}$$
$$\mathrm{m}\measuredangle A = \arcsin \frac{8(\sin 64°)}{10}$$
$$\mathrm{m}\measuredangle A \approx 46.0°$$

We've found one value for angle A, but we need to see if there's another possible value. To do this, we'll subtract 46.0° from 180°.

180° - 46° = 134°

To see if 134° is a valid answer, we must add this to the other existing angle. If their sum is less than 180°, we know a triangle can exist. If, however, it is over 180°, then it is not a valid answer.

134° + 64° = 198°

Since 198° is greater than 180°, we know the second measurement of 134° is not a valid answer.

Final answer: $\angle A = 46°$

Way 2:

1. Given $\angle C = 64°$, c = 10 ft., and a = 8 ft. → SSA
2. $\angle C = 64°$ is acute
3. c > a

It doesn't satisfy the condition, therefore, it doesn't have 2 solutions.

We can find the measure of angle B using the Law of Sines to see if there are any solution:

$$\frac{8}{\sin A} = \frac{10}{\sin 64°}$$
$$8(\sin 64°) = 10 \cdot \sin A$$

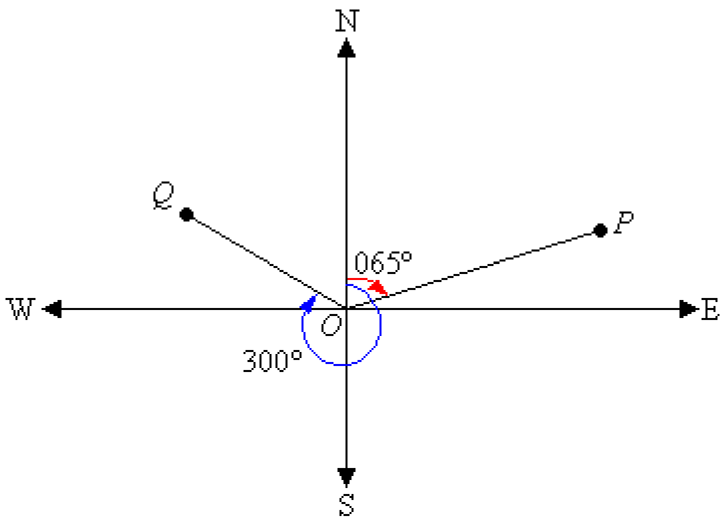
$$\sin A = \frac{8(\sin 64°)}{10}$$

$$m\measuredangle A \approx 46.0°$$

Therefore, it has 1 solution.

3. Compass Directions

The **true bearing** to a point is the angle measured in degrees in a clockwise direction from the north line. We will refer to the true bearing simply as the **bearing**.



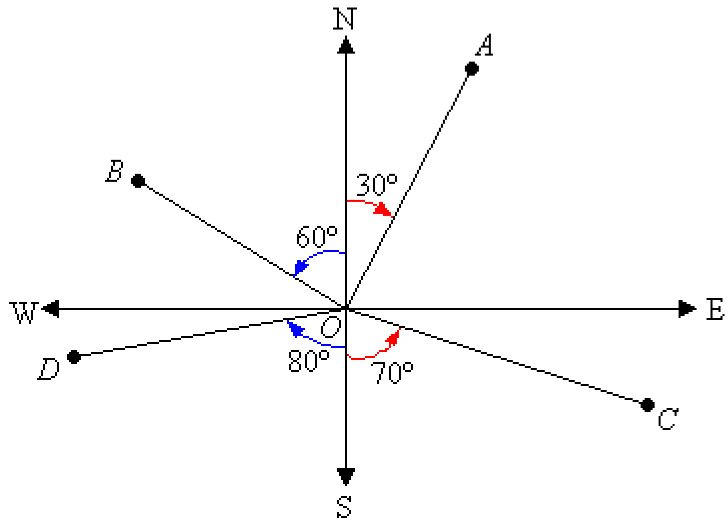
For example, the bearing of point *P* is 065° which is the number of degrees in the angle measured in a clockwise direction from the north line to the line joining the centre of the compass at *O* with the point *P* (i.e. *OP*).

The bearing of point *Q* is 300° which is the number of degrees in the angle measured in a clockwise direction from the north line to the line joining the centre of the compass at *O* with the point *Q* (i.e. *OQ*).

The **conventional bearing** of a point is stated as the number of degrees east or west of the north-south line. We will refer to the conventional bearing simply as the **direction**.

To state the direction of a point, write:

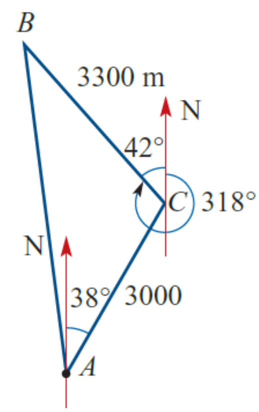
- N or S which is determined by the angle being measured
- the angle between the north or south line and the point, measured in degrees
- E or W which is determined by the location of the point relative to the north-south line



E.g. In the above diagram, the direction of:

- *A* from *O* is N30ºE. N30ºE means the direction is 30º east of north.
- *B* from *O* is N60ºW.
- *C* from *O* is S70ºE.
- *D* from *O* is S80ºW.

Example 1: A yacht starts from a point *A* and sails on a bearing of 38° for 3000 m. It then alters its course to a bearing of 318°, and after sailing for 3300 m it reaches a point *B*.
a) Find the distance *AB* correct to the nearest metre.
b) Find the bearing of *B* from *A* correct to the nearest degree



Solution:

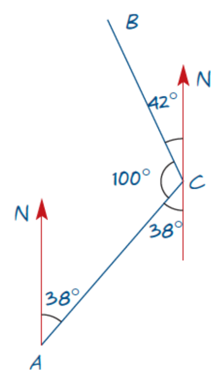
a. $180 - 38 - 42 = 100^\circ$

$AB^2 = 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \times \cos 100^\circ$

$= 23\,328\,233.92$

$\therefore AB = 4829.93104\ldots$

The distance of B from A is 4830 m (to the nearest metre).



b. $\dfrac{3300}{\sin A} = \dfrac{AB}{\sin 100}$

$\therefore \sin A = 0.6728\ldots$

$\therefore A = 42.29^\circ$

$\therefore$ The bearing of B from A

$= 360^\circ - (42.29^\circ - 38^\circ)$

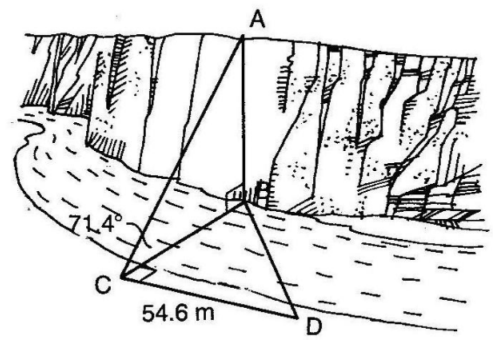
$= 355.71^\circ$

The bearing of B from A is 356° to the nearest degree.

4. 3D Trigonometry

Example 1: A cliff is inaccessible due to the swift current of the river. In order to find the height of the cliff, AB, a surveyor sets up the transit on the opposite side of the river at C, and lays off a base line CD, perpendicular to CB, then measures ∠ACB. The following data are recorded:

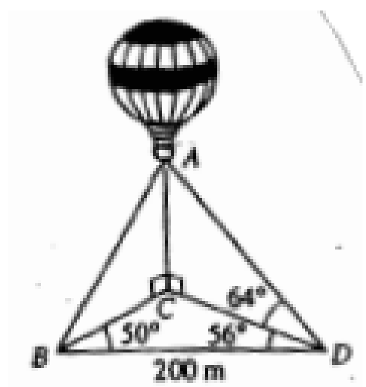
∠LACB = 71.4°
CD = 54.6 m
∠CDB = 37.7°



Find the height of the cliff.

Solution:
tan 37.7 = CB / 54.6
CB = 42.2 m
AB /42.2 = tan 71.4
AB = 125.4 m
The height of the cliff is 125.4 m.

Example 2: From the information in the diagram, determine the height of the hot air balloon in the diagram, to the nearest metre.



Solution:
180 – 50 – 56 = 74°

$$\frac{sin74}{200} = \frac{sin50}{CD}$$

CD = 159.3831

$$tan64 = \frac{AC}{159.3831}$$

AC = 326.78 m
The height of the hot air balloon is about 327 m.