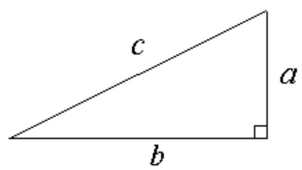
Coordinates (Analytical Geometry)

1. Distance Formula

Recall Pythagoras' Theorem:



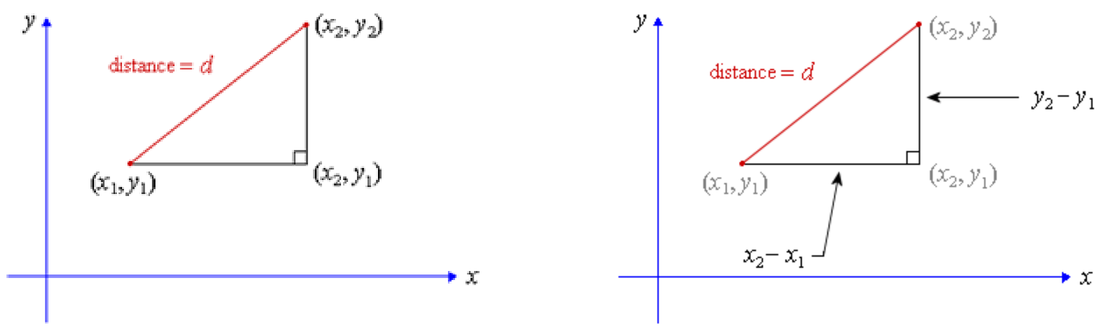
For a right-angled triangle with hypotenuse length c,

$$c = \sqrt{a^2 + b^2}$$

We use this to find the distance between any two points (x_1, y_1) and (x_2, y_2) on the **cartesian plane**:

The cartesian plane was named after Rene Descartes. It is also called the ***x* - *y* plane**.

See more about Descartes in Functions and Graphs.



The point (x_2, y_1) is at the right angle. We can see that:

- The distance between the points (x_1, y_1) and (x_2, y_1) is simply $x_2 - x_1$ and
- The distance between the points (x_2, y_2) and (x_2, y_1) is simply $y_2 - y_1$.

Using Pythagoras' Theorem we have the distance between (x_1, y_1) and (x_2, y_2) given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Find the distance between the points (3, -4) and (5, 7).
Here, $x_1 = 3$ and $y_1 = -4$; $x_2 = 5$ and $y_2 = 7$

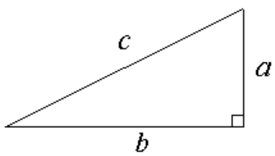
So the distance is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(5-3)^2 + (7-(-4))^2} = \sqrt{4+212} = 11.18$$

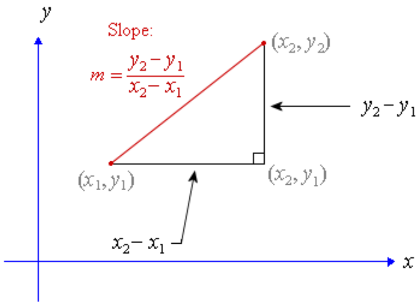
2. Gradient (or slope)

The **gradient** of a line is defined as $\dfrac{\textit{vertical rise}}{\textit{horizontal run}}$



In this triangle, the gradient of the line is given by: $\dfrac{a}{b}$

In general, for the line joining the points (x_1, y_1) and (x_2, y_2):

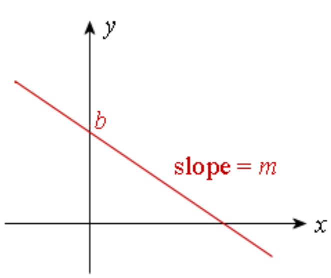


We see from the diagram above, that the **gradient** (usually written m) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

3. The Straight Line

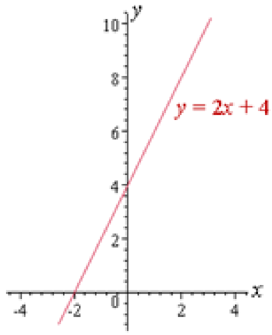
1) Slope-Intercept Form of a Straight Line



The slope-intercept form (otherwise known as "gradient, y-intercept" form) of a line is given by:

$$\boldsymbol{y = mx + b}$$

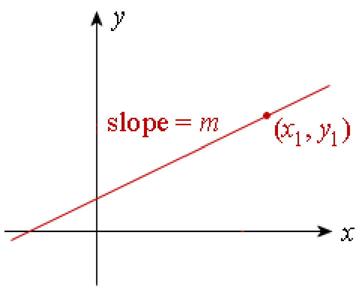
For example:
This tells us the slope of the line is m and the y-intercept of the line is b.



The line $y = 2x + 4$ has slope $m = 2$ and y-intercept $b = 4$.

We do not need to set up a table of values to sketch this line. Starting at the y-intercept ($y = 4$), we sketch our line by going up 2 units for each unit we go to the right (since the slope is 2 in this example). To find the x-intercept, we let $y = 0$.
$2x + 4 = 0$
$x = -2$

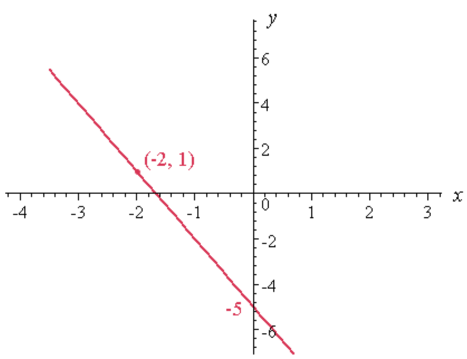
2) Point-Slope Form of a Straight Line



We need other forms of the straight line as well. A useful form is the **point-slope form** (or point - gradient form). We use this form when we need to find the equation of a line passing through a point (x_1, y_1) with slope m:

$$y_2 - y_1 = m(x_2 - x_1)$$

For example:
Find the equation of the line that passes through (-2, 1) with slope of -3.



We use: $y_2 - y_1 = m(x_2 - x_1)$
Here, $x_1 = -2$, $y_1 = 1$, $m = -3$

So the required equation is:
$y - 1 = -3(x - (-2)) = -3x - 6$
$y = -3x - 5$
We have left it in slope-intercept form. We can see the slope is -3 and the y-intercept is -5.

3) General Form

Another form of the straight line which we come across is **general form:**
$Ax + By + C = 0$

It can be useful for drawing lines by finding the y-intercept (put $x = 0$) and the x-intercept (put $y = 0$).

Conclusion:

Problems involving the use of coordinates in 2 dimensions are commonplace on many mathematics contests.

For some contests, such as our FGH contests, which require full solutions, such problems are generally quite approachable since they lend themselves to step-by-step development of the solutions rather than requiring major insights.

In fact the insightful aspects in the use of a coordinate or analytic approach usually come with the decision of whether or not to attempt a solution using these techniques.

Some useful formulae include:

1. $y = mx + b$: The equation of the line with slope m and y intercept b.

2. The coordinates of the midpoint M of the segment with endpoints A(x_1, y_1) and B(x_2, y_2) are: $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$

3. The distance D= $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ between points $A(x_1, y_1)$ and $B(x_2, y_2)$.

4. $Ax + By + C = 0$: The standard form for a line with slope $-\frac{A}{B}$ and intercepts $-\frac{C}{A}$ *and* $-\frac{C}{B}$

5. $(y - y_0) = m(x - x_0)$: The point-slope equation of the line with slope m through the point P (x_0, y_0).

► Questions in class (Please do the questions again)

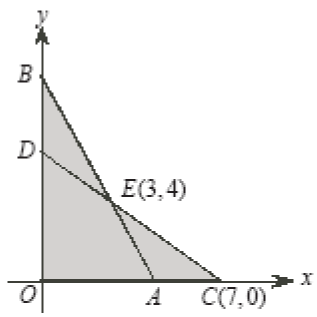
1. A point P is chosen on the line $y = 2x + 3$ and a point Q is chosen on $y = -x + 2$. If the midpoint M of the line segment PQ is (2, 5) calculate the coordinates of P and Q.

2. A triangle ABC has vertices A(-4, 0), B(9, 0) and C(0, 6). If *A'* is the reflection of A in BC, *B*'s the reflection of B in AC, and *C* is the reflection of C in AB, determine the area of triangle *A'B'C'*.

3. In triangle ABC the vertices are A(-3, 0), B(4, 0) and C(0, 5). Squares ACXY and BCPQ are drawn on the sides of the triangle facing outwards. Show that extension of the median CM (M is the midpoint of AB) of triangle CAB is also the altitude of triangle CPX.

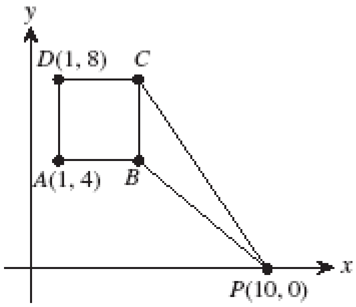
4. In triangle ABC the vertices are A(-3, 0), B(4, 0) and C(0, 5). Squares ACXY and BCPQ are drawn on the sides of the triangle facing outwards. Show that extension of the median CM (M is the midpoint of AB) of triangle CAB is also the altitude of triangle CPX . Prove that this result is true for any triangle ABC.

5. A line with slope −3 intersects the positive x-axis at A and the positive y-axis at B.
A second line intersects the x-axis at C(7, 0) and the y-axis at D. The lines intersect at E(3, 4).



(a) Find the slope of the line through C and E.
(b) Find the equation of the line through C and E, and the coordinates of the point D.
(c) Find the equation of the line through A and B, and the coordinates of the point B.
(d) Determine the area of the shaded region.

6. In the diagram, *ABCD* is a square and the coordinates of *A* and *D* are as shown.



(a) The point *E*(a, 0) is on the *x*-axis so that the triangles *CBE* and *ABE* lie entirely outside the square *ABCD*. For what value of a is the sum of the areas of triangles *CBE* and *ABE* equal to the area of square *ABCD*?

(b) The point *F* is on the line passing through the points *M*(6 , –1) and *N*(12, 2) so that the triangles *CBF* and *ABF* lie entirely outside the square *ABCD*. Determine the coordinates of the point *F* if the sum of the areas of triangle *CBF* and *ABF* equals the area of square *ABCD*.

(c) Show that there is no point *F* on the *x*-axis for which the area of triangle *ABF* is equal to the area of square *ABCD*.