Chapter 10 Factoring (1)

Factoring – change sum of terms into product of terms.

1. Factoring Expressions with Common Factors

A common factor is an expression a where every term can be written as a (something).

For example, $10x + 5$ can be written as $5(2x) + 5(1)$. Therefore, 5 is a common factor.
Then, we can use the distributive property to write this as $5(2x + 1)$, which is the factorization of $10x + 5$.

We want the greatest (biggest) common factor that goes in to every term. For example,
$30x^2 + 6x + 12 = 6(5x^2) + 6(x) + 6(2) = 6(5x^2 + x + 2)$

However, the greatest common factor is not always just a number. Sometimes, it has variables. In this case, it is important to recall the rule for exponents stating $a(x + y) = ax + ay$. So, if we have $3x^5y^3 + 6x^4y^4$, we see that 3 is a common factor. However, we have x and y in both terms.

The rule is to take out the smallest power of each variable that appears in any term. So, we have x^5 in the first term, and x^4 in the second. Since x^4 has the smaller exponent, this is the biggest power of x that is common to both terms. Similarly, y^3 is the biggest power of y that is common to both terms. So, we have 3, x^4 and y^3 all as factors. So, the greatest common factor is the product of all of these terms, $3x^4y^3$. Putting it all together, we have $3x^4y^3(x^2) + 3x^4y^3(2y)$.

Then, we can use the distributive property to get
$3x^5y^3 + 6x^4y^4 = 3x^4y^3(x^2) + 3x^4y^3(2y) = 3x^4y^3(x^2 + 2y)$

If every term is a power of *x*, as in this example, $x^7 + 3x^6 + 2x^5 + x^4$, then the *lowest* power is the highest common factor.

$x^7 + 3x^6 + 2x^5 + x^4 = x^4(x^3 + 3x^2 + 2x + 1)$.

For, lower powers are factors of higher powers.

$$x^7 = x^4 \cdot x^3 \;, x^6 = x^4 \cdot x^2 \;, x^5 = x^4 \cdot x$$

The lowest power, x^4 in this example, typically appears on the right. Again, when we write a polynomial, we begin with the highest exponent and go to the lowest. 7, 6, 5, 4.

Once more, to say that we have *factored* the polynomial on the left –

$x^7 + 3x^6 + 2x^5 + x^4 = x^4(x^3 + 3x^2 + 2x + 1)$

-- means that we will obtain that polynomial if we *multiply* the factors on the right.

Example. Pick out the highest common factor and factor these polynomials.

(How can you check your factoring? By multiplying!)

a) $x^8 + x^7 + x^6 + x^5 = x^5(x^3 + x^2 + x + 1)$
b) $5x^5 - 4x^4 + 3x^3 = x^3(5x^2 - 4x + 3)$
c) $x^3 + x^2 = x^2(x + 1)$
d) $6x^5 + 2x^3 = 2x^3(3x^2 + 1)$
e) $2x^3 - 4x^2 + x = x(2x^2 - 4x + 1)$
f) $3x^6 - 2x^5 + 4x^4 - 6 =$ *cannot be factored*

Factoring by Grouping

Sometimes, the entire expression has no greatest common factor that can be immediately factored out. However, it may be possible to find a common binomial factor. We play games to rewrite the expression in a way that we can see the common binomial term. This is done by looking at pairs of terms and trying to get them to all have a common binomial factor.

For example, what do we do with $x^3 + 2x^2 - 4x - 8$? We have to look at $x^3 + 2x^2$ and $-4x – 8$ as two groups. In the first expression, we see a common factor of x^2, so $x^3 + 2x^2 = x^2(x + 2)$. The second term has a common factor of -4, so $-4x - 8 = -4(x + 2)$. Then, we see that the terms have $(x + 2)$ as a common factor. Putting the pieces together, we have

$x^3 + 2x^2 - 4x - 8$
$= x^2(x + 2) + (-4)(x + 2)$
$= (x^2 - 4)(x + 2)$.
Thus, we have factored our expression.

Practice:

a) $2x^2 – 6x + 3xy – 9y$
$= 2x (x – 3) + 3y (x – 3)$
$= (2x + 3y) (x – 3)$

b) $3x^2 – 6x – x +2$
$= 3x (x – 2) – 1(x – 2)$
$= (3x – 1) (x – 2)$

2. Factoring trinomials

A Trinomial has the form of $ax^2 + bx + c$, where a, b and c are real numbers. We can no directly factor by grouping since there are an odd number of terms. This means we cannot look at pairs of terms, making factoring by grouping difficult. Luckily, we have the FOIL method for multiplying two binomials. This gives us insight into how we may be able to factor a polynomial of this type. For convenience, let’s start with a polynomial with a leading coefficient of 1. That is, a polynomial of the form $x^2 + ax + b$.

We know from FOIL that we are looking for factors that look like: $(x + p)(x + q)$
So, we know what we want the answer to look like. How do we determine what p and q are?
Well, let’s FOIL $(x + p)(x + q)$ and see what we get.
$(x + p)(x + q) = x^2 + qx + px + pq = x^2 + (p + q)x + pq$

Comparing this to the expression $x^2 + ax + b$ we see that we need a 'p' and a 'q' such that pq = b and p + q = a. To see this a little more clearly, let's use an example. Suppose we want to factor $x^2 + 5x - 6$. We need to find p and q so that pq = 6 and p + q = 5.

Let's consider all of the possible factorizations of 6: 1, -6; 2, -3; -3, 2; -1, 6

Now, we want the sum of the factors to equal 5. 1 - 6 = 5, 2 - 3 = 1, - 3 + 2 = - 1 and -1 + 6 = 5.

Since -1 x 6 = 6 and -1 + 6 = 5, we have the numbers p = -1 and q = 6 we were looking for.
So, the factorization is $x^2 + 5x - 6 = \mathbf{(x - 1)(x + 6)}$

Example 1. Factor $x^2 + 3x - 10$.

What are the factors of -10 → c? (p, q) could be (1, -10), (-1, 10), (2, -5), or (-2, 5)
Choose (5, −2) because 5 + (-2) = 3 → b

Therefore, $x^2 + 3x - 10 = \mathbf{(x + 5)\ (x - 2)}$.

Note: The order of the factors does not matter. $(x - 2)(x + 5) = (x + 5)\ (x - 2)$.

Example 2: Factor $x^2 - 5x + 6$.

The constant term is 6, but the middle coefficient this time is negative. Since I multiplied to a positive six, then the factors must have the same sign. (Remember that two negatives multiply to a positive.) Since I'm adding to a negative (–5), then both factors must be negative. So rather than using 2 and 3, as in the first example, this time I will use –2 and –3:

$$x^2 - 5x + 6 = \mathbf{(x - 2)(x - 3)}$$

Example 3: Factor $x^2 - x - 12$.

We must find factors of -12 whose algebraic sum will be the coefficient of x: −1. Since the constant term is -12 which is negative, the factors must be opposite sign. Since we want the sum to be -1 which is negative, the bigger factor must be negative.

Choose −4 and + 3: $x^2 - x - 12 = \mathbf{(x - 4)\ (x + 3)}$.

Example 4: Factor $x^2 + x - 6$.

Since I am multiplying to a negative six, I need factors of opposite signs; that is, one factor will be positive and the other will be negative. The larger factor (in absolute value) will get the "plus" sign, because I am adding to a positive 1. Since these opposite-signed numbers will be adding to1, I need the two factors to be one unit apart. The factor pairs for six are 1 and 6, and 2 and 3.

The second pair are one apart, so I want to use 2 and 3, with the 3 getting the "plus" sign (so the 2 gets the "minus" sign).

$$x^2 + x - 6 = \mathbf{(x - 2)(x + 3)}.$$

Example 5: Factor $x^2 + 7x - 6$.

Since the constant term is negative, I'll be needing a positive and a negative number such that, when I multiply them together, I get 6, but when I add them, I get 7. The factor pairs for 6 are 1 and 6, and 2 and 3. You may think that I should use 1 and 6, but —

One of the factors has to be negative in order to multiply to get a "minus" six! Trying the first factor pair of 1 and 6, the sum would be either $(-1) + 6 = 5$ or else $1 + (-6) = -5$. And the other factor pair, 2 and 3, won't work, either, because $(-2) + 3 = 1$ and $2 + (-3) = -1$.

In other words, there is no pair of factors of -6 that will add to $+7$. And if something isn't factorable, it's prime. Then $x^2 + 7x - 6$ is "**prime**", or "**unfactorable over the integers**" (because I couldn't find integers that would work).

Note that you can use clues from the signs to determine which factors to use, as I did in this last example above:

- If c is positive, then the factors you're looking for are either both positive or else both negative.
  If b is positive, then the factors are positive
  If b is negative, then the factors are negative.
  In either case, you're looking for factors that add to b.

- If c is negative, then the factors you're looking for are of alternating signs;
  that is, one is negative and one is positive.
  If b is positive, then the larger factor is positive.
  If b is negative, then the larger factor is negative.
  In either case, you're looking for factors that are b units apart.

Practice in class. Factor. Again, the order of the factors does not matter.

a) $x^2 + 5x + 6 = (x + 2)(x + 3)$
b) $x^2 - x - 6 = (x - 3)(x + 2)$
c) $x^2 + x - 20 = (x + 5)(x - 4)$

Practice in class. Factor.

a) $x^2 - 10x + 9 = (x - 1)(x - 9)$
b) $x^2 + x - 12 = (x + 4)(x - 3)$
c) $x^2 - 6x - 16 = (x - 8)(x + 2)$