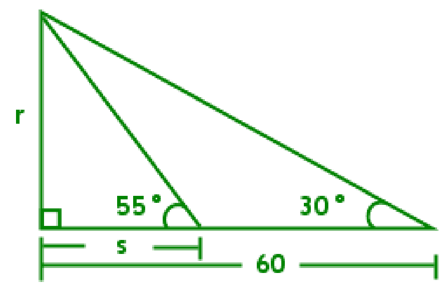
Chapter 6 Trigonometry 2

1. Trigonometry Involving Two Triangles

Example 1: Find the angles and sides indicated by the letters in the diagram. Give each answer correct to the nearest whole number.



At first, this looks fairly intimidating. But then I notice that, to find the length of the height *r*, I can use the base angle 30° and the full base length of 60, because *r*/60 is "opposite" over "adjacent", which is the tangent.

$r/60 = tan(30°)$
$r = 60tan(30°) = 34.64101615...$

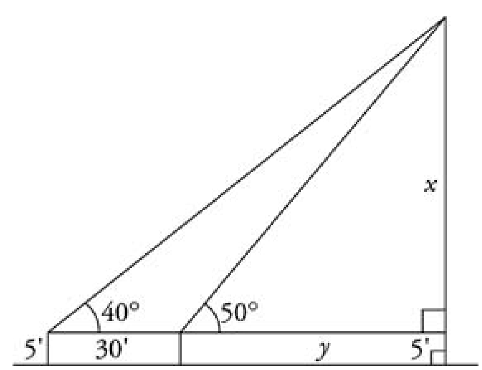
I'm supposed to the nearest whole number, so $r = 35$.

Now that I have the value of *r*, I can use *r* and the other base angle, 55°, to find the length of the other base, *s*, by using $r/s = tan(55°)$:

$35/s = tan(55°)$
$35/tan(55°) = s = 24.50726384...$

***r* = 35, *s* = 25**

Example 2: A woodcutter wants to determine the height of a tall tree. He stands at some distance from the tree and determines that the angle of elevation to the top of the tree is 40°. He moves 30′ closer to the tree, and now the angle of elevation is 50°. If the woodcutter's eyes are 5′ above the ground, how tall is the tree?



$$\tan 50 = \frac{x}{y} \qquad \tan 40° = \frac{x}{y+30'}$$

$$y = 0.8381x \qquad .8391 = \frac{x}{y+30'}$$

$$x = 0.8391\left(y+30'\right)$$

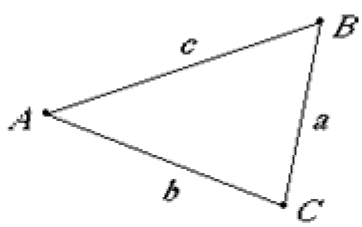
$$x = 0.8391(0.8391x + 30')$$
$$x = 0.7041x + 25.17'$$
$$.2959x = 25.17'$$
$$x = 85.06'$$

Note that 5′ must be added to the value of *x* to get the height of the tree, or 90.06′ tall.

2. The Laws of Sine and Cosine

If a triangle does not contain a right angle, SOH CAH TOA does not apply. Two trigonometric laws have been developed to be applied with acute and obtuse triangles, namely Sine law and Cosine law.

As usual, we'll use a standard notation for the angles and sides of a triangle. That means the side *a* is opposite the angle *A*, the side *b* is opposite the angle *B*, and the side *c* is opposite the angle *C* .



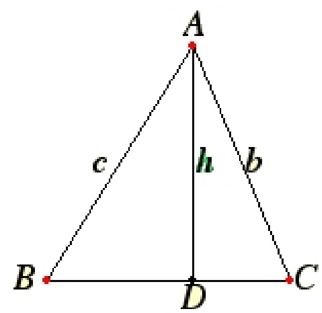
1) The Law of Sine

Now we induce the law of sine from below figure.

First, drop a perpendicular line *AD* from *A* down to the base *BC* of the triangle. The foot *D* of this perpendicular will lie on the edge *BC* of the triangle when both angles *B* and *C* are acute. Let *h* denote the length of this line *AD*, that is, the height (or altitude) of the triangle.

When angle *B* is acute, then sin *B* = *h/c*. at the same time, sin *C* = *h/b* in any case.

These two equations tell us that *h* equals both *c* sin *B* and *b* sin *C*.
From the equation: *c* sin *B* = *b* sin *C*, we can easily get the law of sine:



Similarly, we get $\frac{\sin B}{b} = \frac{\sin C}{c}$ and $\frac{\sin A}{a} = \frac{\sin B}{b}$

So, the law of sine is $\boldsymbol{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$

This can also be interpreted as three equations:

$$\frac{\sin A}{a} = \frac{\sin B}{b}, \frac{\sin B}{b} = \frac{\sin C}{c}, \text{ and } \frac{\sin A}{a} = \frac{\sin C}{c}$$

Since the three versions differ only in the labeling of the triangle, it is enough to verify one just one of them, so we'll just consider the version stated first.

Example 1: Angle A = 35.849 degrees, side a = 9.288, side c = 14.387. Find angles B, C, and side b.

Solution: Notice that sides a, and c are given, and angle A. This is SSA. We have a pair of corresponding side a and angle A. We will begin with the law of sine.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\sin C = \sin^{-1}\left(\frac{c \sin A}{a}\right)$$

$$c = 65.117^o$$

Now we can find angle B. B = 180 - A - C = 79.035 degrees.

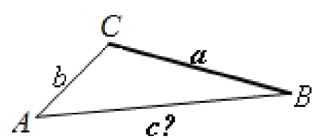
Finally, to get b:

$$\frac{b}{\sin B} = \frac{a}{\sin A}, \; b = \frac{a \sin B}{\sin A} = 15.569$$

Example 2: Find the length of side c when A = 40°, C = 110°, and a = 25 mm.

Since we are solving for *c*, and have the information about side *a* and angle *A*, we use the parts with *a* and *c*.

$$\frac{c}{\sin C} = \frac{a}{\sin A},$$



$$\frac{c}{\sin 110°} = \frac{25\,mm}{\sin 40°}, \; \frac{c}{0.9397} = \frac{25\,mm}{0.6428}$$

$$0.9397 \cdot \frac{c}{0.9397} = \frac{25\,mm}{0.6428} \cdot 0.9397, \; c = 36.55 \text{ mm}$$

Example 3: Find the size of angle A in the illustrated triangle: C = 79°, a = 36 mm, and c = 50mm.

$$\frac{c}{\sin C} = \frac{a}{\sin A}, \; \frac{50}{\sin 79°} = \frac{36}{\sin A}, \; \frac{50}{\sin 79°} = \frac{36}{\sin A}$$

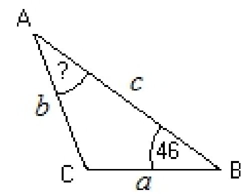
$$36 \cdot \sin 79° = 50 \cdot \sin A, \; \frac{50 \cdot \sin A}{50} = \frac{36 \cdot \sin 79°}{50}$$

sin A = 0.7067716, $A = \sin^{-1}(0.7067716)$,

A = 44.972845°

Considering significant digits, the final answer is A = 45°

Example 4: For this triangle, the sides a = 1.5, b = 1.7, and angle B = 46°. Use the law of sine to find angle A.



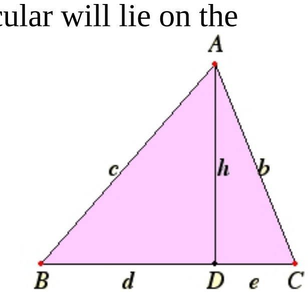
$\frac{a}{\sin A} = \frac{b}{\sin B}, \frac{1.5}{\sin A} = \frac{1.7}{\sin 46°}$

1.7 sinA = 1.5 sin46°, sinA = 1.5 sin46° /1.7

sin A = 0.6347, A = $\sin^{-1}$ 0.6347, ∠A = 39°

2) The Law of Cosine

In the figure below, we assume that the angle C is an acute triangle. Drop a perpendicular line AD from A down to the base BC of the triangle. The foot D of the perpendicular will lie on the edge BC if angle B is acute.



Let h denote the height of the triangle, let d denote BD, and e denote CD.

Then we can read the following relationships from the diagram:

$$c^2 = d^2 + h^2,\ b^2 = e^2 + h^2$$
$$\cos C = e/b, \quad d^2 = (a - e)^2$$

That last equation requires explanation. If the point D lies on the side BC, then $d = a - e$, but if D lies on BC extended, then $d = e - a$. In either case, $d^2 = (a - e)^2$.

These equations and a little algebra finish the proof as follows:

$$c^2 = d^2 + h^2 = d^2 - e^2 + b^2 = (d + e)(d - e) + b^2$$
$$= (a - 2e)a + b^2 = a^2 + b^2 - 2ae = a^2 + b^2 - 2ab \cos C$$

Thus, we now know that the law of cosines is valid when angle C is acute.

$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C}$$

There are two other versions of the law of cosines,

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}$$

and

$$\mathbf{b^2 = a^2 + c^2 - 2ac \cos B}$$

Since the three versions differ only in the labelling of the triangle, it is enough to verify one just one of them.

3) When to use sine law and cosine law?

The Law of Sine states that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \text{ or } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use the law of sine when you are given ASA, SSA, or AAS. An example of ASA is when you are given the measure of angles A, and C, and the length of side b.

An example of SSA is when you are given the sides c, and a, and angle C. An example of AAS is when you are given angles C and A, and side c.

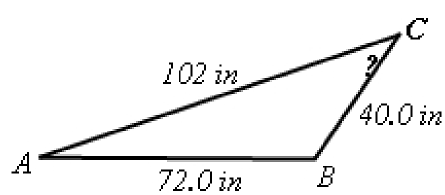
The Law of Cosines states that:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$

Use the law of cosines when you are given SAS, or SSS quantities. For example: If you were given the lengths of side b and c, and the measure of angle A, this would be SAS. SSS is when we know the lengths of the three sides a, b, and c.

In other word, if you are **given a pair of angle and its corresponding side**, then you can use **sine law** to solve the unknown. Otherwise, you can use cosine law to solve the unknown.

Example 1: Find the size of the unknown angle.



$c^2 = a^2 + b^2 - 2ab\cos C$
$72^2 = 40^2 + 102^2 - 240 \times 102 \times \cos(C)$
$5184 = 1600 + 10404 - 8160 \times \cos(C)$

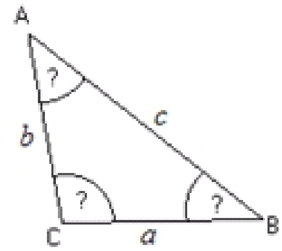
$5184 - 10404 - 1600 = 1600 + 10404 - 10404 - 1600 - 8160 \times \cos(C)$
$5184 - 10404 - 1600 = -8160 \times \cos(C)$

$\cos(C) = \frac{5184-10404-1600}{-8160}$, $\cos(C) \approx 0.8358$

$C \approx \cos^{-1}(0.8358) \approx 33.3°$

Example 2: sin *a* = 65.995, side b= 50.225, side c = 38.834. Find angles A, B, and C.

Solution: Notice that sides a, b, and c are given. This is SSS. We will need to use the law of cosines.



$a^2 = b^2 + c^2 - 2bc \cos A$

Solving for A

$\cos A = \dfrac{a^2 - b^2 - c^2}{-2bc}$

$A = \cos^{-1}\left(\dfrac{a^2 - b^2 - c^2}{-2bc}\right) = 94.775°$

Now let's use the law of sines to get angle B.

$\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$, $\sin B = \dfrac{b \sin A}{a}$, $B = \sin^{-1}\left(\dfrac{b \sin A}{a}\right) = 49.323$

Finally, angle C = 180 - A - B = 35.902 degrees.

Practice: Given side a = 5, side b = 6, and side c = 8, find all unknown angles.

Knowing all sides means that you must use the Law of Cosines to start.

$a^2 = b^2 + c^2 - 2bc \cos A.$

$5^2 = 6^2 + 8^2 - 2\,(8)(6) \cos A$

$25 = 36 + 64 - 2\,(48) \cos A$

$25 = 100 - 96 \cos A$

$-75 = -96 \cos A, \qquad 0.78125 = \cos A$

$Cos^{-1}\ 0.78125 = A, \angle A = 39°$

Now, we have a pair ∠A and a, we can use Sine Law to find another angle.

$$\frac{sinB}{6} = \frac{sin39}{5}$$

$\angle B = 49^{o}$

$\angle C = 180 - 39 - 49 = 92^{o}$

4) Application

Example 1: Tess was flying from Toronto to Hamilton at night. She noticed that her heading indicator as malfunctioning and decided to check her position. She called Hamilton Tower and St. Catharines Radio, knowing that the stations were 50 km apart. Hamilton reported that the position of her plane formed an angle of 65° with the line joining the two stations. St. Catharines reported that her position made an angle of 48°. How far was the plane from Hamilton, to the nearest kilometre?

Solution: Let x be the distance from Hamilton.

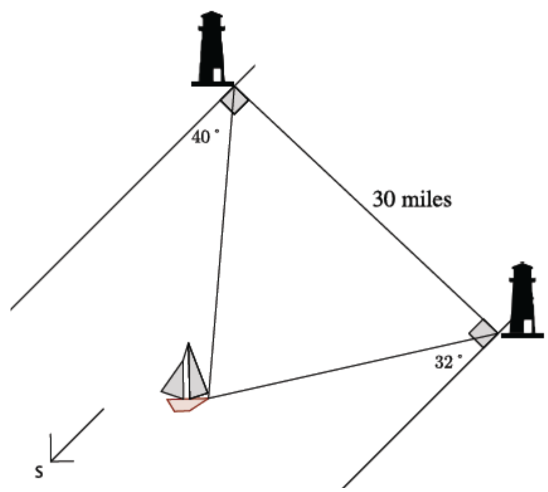
$180 - 48 - 65 = 67^o$

$$\frac{\sin 67}{50} = \frac{\sin 48}{x}$$

x = 40 km

Therefore, the plane is 40 km from Hamilton.

Example 2: Two lighthouses are $\mathbf{30}$ mi. apart on each side of shorelines running north and south, as shown. Each lighthouse keeper spots a boat in the distance. One lighthouse keeper notes the location of the boat as $\mathbf{40}^{\circ}$ east of south, and the other lighthouse keeper marks the boat as $\mathbf{32}^{\circ}$ west of south. What is the distance from the boat to each of the lighthouses at the time it was spotted? Round your answers to the nearest mile.



Let $\boldsymbol{x}$ be the distance from the southern lighthouse to the boat.

$90 - 40 = 50^o$ and $90 - 32 = 58^o$

$180 - 50 - 58 = 72^o$ which is the angle opposite of 30 miles, therefore, we can use sine law.

$$\frac{\sin 72}{30} = \frac{\sin 50}{x}$$

$\boldsymbol{x} = \mathbf{30\sin50/\sin72}$
$\boldsymbol{x} = \mathbf{24.16400382}\ldots$

The southern lighthouse is approximately $\mathbf{24}$ mi. from the boat.

With this information, you may choose to use the law of sines or the law of cosines to find the other distance. Shown below are both options.

Way 1: Let y be the distance from the northern lighthouse to the boat.

$$\frac{\sin 72}{30} = \frac{\sin 58}{y}$$

$y = \mathbf{30\sin58/\sin72}$
$y = \mathbf{26.75071612\ldots}$

Way 2: Let a be the distance from the northern lighthouse to the boat.

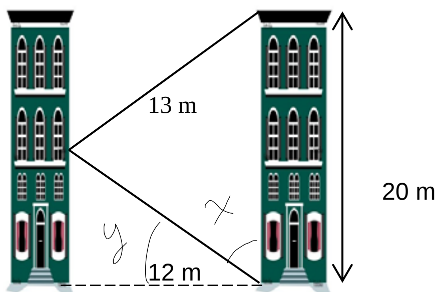
$a^2 = \mathbf{30^2+24^2-2(30)(24)\cdot\cos58}$

$a = \sqrt{(\mathbf{30^2+24^2-2(30)(24)\cdot\cos58})}$

$a = \mathbf{26.70049175\ldots}$

The northern lighthouse is approximately **27** mi. from the boat.

Example 3: Find the angle of elevation to the 2^{nd} floor.



Solution:

$13^2 = 20^2 + 12^2 - 2(20)(12)\cos(x)$

$2(20)(12)\cos(x) = 400 + 144 - 169$

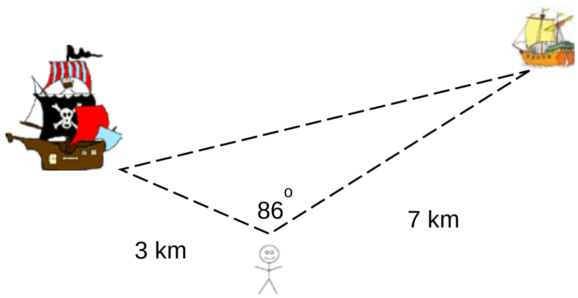
$\cos(x) = \frac{375}{480}$

$x = 38.6^o$

$y = 90 - 38.6 = 51.4^o$

Practice:

1. Justin is standing on a dock and observes two ships in the water as shown. How far apart are the ships?



2. Find the perimeter of the following triangle.