# Chapter 10 Factoring (2)

**1. Continue with factoring trinomials -** Trinomial has the form of $ax^2 + bx + c$.

**Type 2: If a $\neq$ 1 → $ax^2 + bx + c$**

Not all trinomial expressions can be factored, but when a trinomial is factorable, there are several ways to factor this kind of trinomials.

## 1.1 Factoring Trinomials by Decomposition

**Investigation:** Expand $(2x + 3)(4x + 5)$

$(2x + 3)(4x + 5)$
$= (2x)(4x) + (2x)(5) + (3x)(4) + (3)(5)$
$= (2)(4)x^2 + [(2)(5) + (3)(4)]x + (3)(5)$
$= 8x^2 + 22x + 15$
$= ax^2 + bx + c$

Let's compare the coefficients.
$a = 2(4)$
$c = 3(5)$
$b = 2(5) + 3(4)$
$ac = (2)(4)(3)(5) = (2)(5)(3)(4)$

What is b? b is the sum of 2(5) and 3(4) which are the factors of ac.

To factor, we need to go backward. Remember that factoring is the opposite of expanding.
We notice that going from $8x^2 + 22x + 15$ back to $(2x + 3y)(4x + 5y)$, we need the middle step:
$(2x)(4x) + (2x)(5) + (3x)(4) + (3)(5) = 8x^2 + 10x + 12x + 15$

Notice that 22x is broken down into two terms of 10x and 12x.

The method is called "**decomposition**" because the coefficient of the middle ($x$) term of the trinomial is broken down (or "decomposed") into two smaller numbers.

The steps are shown below, using as an example the expression $9x^2 + 21x – 8$.
**Step 1:** If necessary, rearrange the expression so the terms are in standard form (i.e. exponents decrease from left to right). The example is already in the correct order.

**Step 2:** We look for two numbers that have a sum equal to the middle term coefficient, and a product equal to the product of the first and last coefficients.
In this case, we want two numbers with a sum of +21 and a product of $9 \times (–8) = –72$

**Step 3:** We list pairs of factors of the desired product in an orderly fashion; we don't list any pairs where the smaller factor is greater than the square root of the desired product.

We list the factors of 72 (we ignore the minus sign for now) in pairs; we stop when the smaller factor is 8 (since 9 is bigger than the square root of 72). We get this table:

| | |
|---|---|
| 1 | 72 |
| 2 | 36 |
| 3 | 24 |
| 4 | 18 |
| 6 | 12 |
| 8 | 9 |

**Step 4:** If the product we actually want is *positive*, look for a row in the table where the *sum* of the factors equals the middle term coefficient (ignoring its sign); if the product is *negative*, look for a row where the *difference* equals the middle term coefficient (ignoring its sign).

We have a *negative* product, so we look for a *difference* of 21. In the third row of the table, we find the numbers 3 and 24; these numbers have a difference of 21.

**Step 5:** Adjust the signs of the two numbers: if the product is *positive*, make the signs of *both* numbers the same as the sign of the middle term coefficient; if the product is *negative*, make the sign of the *bigger* number the same as that of the middle term coefficient, and the sign of the smaller number the *opposite* of that of the middle term coefficient.

The product is negative, so we make the sign of the bigger number (24) positive (since the middle term of the original expression has a positive coefficient) and the sign of the smaller number (3) negative. So our numbers are now –3 and 24.

**Step 6:** Rewrite the original expression, splitting the middle term into the two parts determined above.

We will split the middle term ($21x$) into the two parts determined by the numbers we obtained in the previous step, i.e. $-3x$ and $24x$:
$9x^2 + 21x - 8 = 9x^2 - 3x + 24 - 8$

**Step 7:** Factor the first two terms and the last two terms separately, by finding a common factor in each pair of terms. If there is no common factor, use 1 (or –1).

The first two terms have a common factor of $3x$, and the last two have a common factor of 8. So we get: $3x(3x - 1) + 8(3x - 1)$

**Step 8:** The second (bracketed) factor in each of the resulting terms *must* be the same; if not, we've made a mistake. Take out this common factor from each term, and we have the final result.

**$9x^2 + 21x - 8 = (3x - 1)(3x + 8)$**

**Example 2:** Factor $2x^2 + x - 6$.

I have a = 2, b = 1, and c = –6, so ac = (2)(–6) = –12. So I need to find factors of –12 that add up to +1. The pairs of factors for 12 are 1 and 12, 2 and 6, and 3and 4. Since –12 is negative, I need one factor to be positive and the other to be negative (because positive times negative is negative). This means that I'll want to use the pair "3 and 4", and I'll want the 3 to be negative, because –3 + 4 = +1.

| | | |
|---|---|---|
| $\mathbf{2x^2 + x - 6}$ | As we can see, it can be grouped into two ways where both will produce the same answer. | $\mathbf{2x^2 + x - 6}$ |
| $\mathbf{= 2x^2 - 3x + 4x - 6}$ | | $\mathbf{= 2x^2 + 4x - 3x - 6}$ |
| $\mathbf{= x(2x - 3) + 2(2x - 3)}$ | | $\mathbf{= 2x(x + 2) - 3(x + 2)}$ |
| $\mathbf{= (2x - 3)(x + 2)}$ | | $\mathbf{= (x + 2)(2x - 3)}$ |

Nevertheless, can you correctly factor the following?
$2x^2 - 5x + 3$ $\mathbf{= (2x - 3)(x - 1)}$

**Trick of Factoring Using Decomposition Without Grouping**

Recall the investigation: Expand (2x + 3)(4x + 5)
(2x + 3)(4x + 5)
= (2x)(4x) + (2x)(5) + (3x)(4) + (3)(5)
= $8x^2$ + 10x + 12x + 15
= (2)(4)$x^2$ + [(2)(5) + (3)(4)]x + (3)(5)
= $8x^2$ + 22x + 15

Now given $8x^2$ + 22x + 15, we have learned how to find the 2 possible numbers to decompose the middle term: $8x^2$ + 10x + 12x + 15

Without grouping, how can we find the numbers are (**2**x + **3**)(**4**x + **5**)?

Notice that if we take the ratio of 8 and the 2 numbers, we get:
8 : 10 = 4 : 5 and 8 : 12 = 2 : 3.
These are exactly what we want.

We can also take the ratio of the 2 numbers and 15, we get:
10 : 15 = 2 : 3 and 12 : 15 = 4 : 5
These are also exactly what we want.

Therefore, after finding the 2 numbers to decompose the middle term, take the ratio of “a” and the 2 numbers or the ratio of 2 numbers and “c”, the numbers are the coefficients in the brackets.

### 1.2 Factoring Trinomials by Criss-Cross

Recall the investigation: Expand (2x + 3)(4x + 5)
(2x + 3)(4x + 5)
= (2x)(4x) + (2x)(5) + (3x)(4) + (3)(5)
= $8x^2$ + 10x + 12x + 15
= (2)(4)$x^2$ + [(2)(5) + (3)(4)]x + (3)(5)
= $8x^2$ + 22x + 15

a = 8 = 2(4) → 2 and 4 are the factors of a
c = 15 = 3(5) → 3 and 5 are the factors of c
b = 22 = 2(5) + 3(4) → Cross multiply the factors, the sum is b.

We can put it in the following table:

| **Factors of a** | **Factors of c** |
|---|---|
| 2x | 3 |
| 4x | 5 |

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10x + 12x = 22x

You may argue that 1 and 8 are also factors of 8, and 1 and 15 are also factors of 15. How do we know to use 2 and 4 as well as 3 and 5 as our correct factors? Well, that is when we need to middle term to check! After they cross multiply, the sum must be 22. If not, those are the wrong factors.
To choose the appropriate factors, we may need to try several times. The more practice you do, you will know what to choose. For this question, we can see that 22 is not that big comparing to 8 and 15. Therefore, it is safer to choose the factors that are close to each other instead of the extremes.

Let's start with an easy example.

**Example**: As for a quadratic trinomial $2x^2 + 9x - 5$, we know that it will be factored as a product of binomials: (?x ?)(?x ?)

The first term of each binomial will be the factors of $2x^2$, and the second term will be the factors of 5.
Now, how can we produce $2x^2$? There is only one way: $2x \cdot x$ : (2*x* ?)(*x* ?)
And how can we produce 5? Again, there is only one way: 1· 5.
But does the 5 go with 2*x* -- (2*x* 5)(*x* 1) or with *x* -- (2*x* 1)(*x* 5) ?

How shall we decide between these two possibilities?
It is the combination that when you cross multiply and add, the result will correctly give the middle term, 9*x*.

Notice: We have not yet placed any signs! Since we want the middle term to be positive, the smaller product would have the negative sign.

$$\begin{array}{ccc} 2x & & 5 \\ & \times & \\ 1x & & -1 \end{array}$$

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-2x + 5x = 3x

$$\begin{array}{ccc} 2x & & -1 \\ & \times & \\ 1x & & 5 \end{array}$$

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10x – x = 9x

Yes $10x - x = 9x$ -- we choose +5 and −1: $(2x - 1)(x + 5)$
Therefore, $2x^2 + 9x - 5 = (2x - 1)(x + 5)$.

Skill in factoring depends on skill in multiplying -- particularly in picking out the middle term!

**Practice in class.** Place the correct *signs* to give the middle term.

a) $2x^2 + 7x - 15 = (2x \underline{\;\;-\;\;} 3)(x \underline{\;\;+\;\;} 5)$
b) $2x^2 - 7x - 15 = (2x \underline{\;\;+\;\;} 3)(x \underline{\;\;-\;\;} 5)$

**Note:** When the constant term is negative, then the <u>signs</u> in each factor will be different. But when the constant term is positive, the signs will be the same.

**Practice in class. Factor these trinomials.**

a) $3x^2 + 8x + 5 = (3x + 5)(x + 1)$
b) $3x^2 + 16x + 5 = (3x + 1)(x + 5)$

**Practice in class. Factor these trinomials.**

a) $2x^2 - 7x + 5 = (2x - 5)(x - 1)$
b) $2x^2 - 11x + 5 = (2x - 1)(x - 5)$
c) $3x^2 + x - 10 = (3x - 5)(x + 2)$

**Example 4:** Factor completely $6x^8 + 30x^7 + 36x^6$.

To factor completely means that the polynomial may be factored more than once.
First, remove any common factors

$6x^8 + 30x^7 + 36x^6 = 6x^6(x^2 + 5x + 6)$.

Now continue by factoring the trinomial:

$= 6x^6(x + 2)(x + 3)$.

**Practice in class.** Factor completely. First remove any common factors.

a) $x^3 + 6x^2 + 5x$ = $\mathbf{x(x^2 + 6x + 5) = x(x + 5)(x + 1)}$

b) $x^5 + 4x^4 + 3x^3$ = $\mathbf{x^3(x^2 + 4x + 3) = x^3(x + 1)(x + 3)}$

**Example 5:** Factor by making the <u>leading term</u> positive.

$-x^2 + 5x - 6 = \mathbf{-(x^2 - 5x + 6) = -(x - 2)(x - 3).}$

**Practice in class.** Factor $-x^2 - 2x + 3$.

$-x^2 - 2x + 3$ = $\mathbf{-(x^2 + 2x - 3) = -(x + 3)(x - 1)}$

**2. Factor Special Trinomials**

**Type 1: $(a \pm b)^2 = a^2 \pm 2ab + b^2$ → $a^2 \pm 2ab + b^2 = (a \pm b)^2$ (Perfect Square Formula)**

**Example 1.** Is this a perfect square trinomial: $x^2 + 14x + 49$?
(Recall: a perfect square trinomial is the trinomial that can be written as a square of binomial)

**Answer**. Yes. It is the square of $(x + 7)$.
$x^2$ is the square of $x$. 49 is the square of 7. And $14x$ is twice the product of $x \cdot 7$.

In other words, $x^2 + 14x + 49$ could be *factored* as $x^2 + 14x + 49 = (x + 7)^2$

**Note:** If the coefficient of $x$ had been any number but 14, this would not have been a perfect square trinomial.

**Example 2** Is this a perfect square trinomial: $x^2 + 50x + 100$? Factor if it is.

**Answer**. No, it is not. $x^2$ is the square of $x$, and 100 is the square of 10. BUT, $50x$ is not twice the product of $x \cdot 10$. (Twice their product is $20x$.)

**Example 3** Is this a perfect square trinomial: $x^8 - 16x^4 + 64$? Factor if it is.

**Answer**. Yes. It is the perfect square of $x^4 - 8$.
$x^8 - 16x^4 + 64 = (x^4 - 8)^2$

**Practice in class.** Factor: $p^2 + 2pq + q^2$.
$p^2 + 2pq + q^2 = (p + q)^2$

**Practice in class.** Factor as a perfect square trinomial -- if possible.

a) $x^2 - 4x + 4 = \mathbf{(x - 2)^2}$  b) $x^2 + 6x + 9 = \mathbf{(x + 3)^2}$

**Practice in class.** Factor as a perfect square trinomial, if possible.

a) $25x^2 + 30x + 9 = \mathbf{(5x + 3)^2}$  b) $4x^2 - 28x + 49 = \mathbf{(2x - 7)^2}$

When the sum of two numbers multiplies their difference -- $(a + b)(a - b)$, then the product is the difference of their squares: $\boldsymbol{a^2 - b^2}$

**Type 2: $\boldsymbol{(a + b)(a - b) = a^2 - b^2 \rightarrow a^2 - b^2 = (a + b)(a - b)}$ (Difference of Squares Formula)**

**Example 1.** Multiply $(x + 2)(x - 2)$.

$(x + 2)(x - 2) = x^2 - 2x + 2x - 4 = x^2 - 4$

Recognize the form: $(a + b)(a - b)$. The product will be the difference of two squares:

When confronted with the form $(a + b)(a - b)$, the student should *not* do the FOIL method. The student should recognize immediately that the product will be $a^2 - b^2$.

Also, the order of factors never matters: $(a + b)(a - b) = (a - b)(a + b) = a^2 - b^2$.

**Practice in class.** Write only final product.

a) $(x + 9)(x - 9) = \mathbf{x^2 - 81}$  b) $(x^3 + 2)(x^3 - 2) = \mathbf{x^6 - 4}$.  c) $(y + z)(y - z) = \boldsymbol{y^2 - z^2}$

**Practice in class.** Factor.

a) $x^2 - 100 = \mathbf{(x + 10)(x - 10)}$  b) $y^2 - 1 = \boldsymbol{(y + 1)(y - 1)}$

**Practice in class.** Factor completely.

a) $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$

$= (x^2 + y^2)(x + y)(x - y)$

**Practice in class.** $xy^2 - xz^2 = \boldsymbol{x(y^2 - z^2) = x(y + z)(y - z)}$

**3. Summary of Multiplying/Factoring**

In summary, here are the four forms of Multiplying/Factoring that characterize algebra.

| | Multiplying $\longrightarrow$ | | |
|---|---|---|---|
| 1. Common Factor | $2(a + b)$ | $=$ | $2a + 2b$ |
| 2. Quadratic Trinomial | $(x + 2)(x + 3)$ | $=$ | $x^2 + 5x + 6$ |

3. Perfect Square Trinomial $(x-5)^2 = x^2 - 10x + 25$
4. The Difference of Two Squares $(x+5)(x-5) = x^2 - 25$

$\longleftarrow$ Factoring ——

**Practice in class.** Distinguish each form, and write only the final product.

a) $(x-3)^2$ **= $x^2 - 6x + 9$. Perfect square trinomial.**

b) $(x+3)(x-3)$ **= $x^2 - 9$. The difference of two squares.**

**Practice in class.** Factor.

a) $6x - 18$ **= $6(x-3)$. Common factor.**

b) $x^6 + x^5 + x^4 + x^3$ **= $x^3(x^3 + x^2 + x + 1)$. Common factor.**