Tópicos en Matemáticas Discretas III - Online Algorithms and Scheduling
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Homework #3

P1. a) Show that the greedy algorithm for the k-server problem is not c-competitive for any constant c.

b) A space (X, d) is called asymmetric if $d : X^2 \to [0, +\infty]$ satisfies the conditions

I. $d(x, x) = 0, \forall x \in X$

II. $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$

but it is not symmetric. Show that the competitive ratio for k-server for such spaces cannot be bounded by a function of k.

P2. Consider a uniform MTS on n states, i.e. a space (X, d) such that $d(x, y) = 1$ for every pair $x, y \in X$; in which the tasks are functions $\tau : X \to \{0, 1\}$.

a) Find a random instance on k tasks such that the expected cost of any deterministic algorithm is k/n.

b) Let $c(k)$ be the expected cost of the optimal offline algorithm. Prove that $\text{lím}_k \frac{c(k)}{k} = \frac{1}{nH_n}$, where H_n is the n-th harmonic number. It will be helpful to remember the following result: let $(X_i)_{i\in\mathbb{N}}$ be a sequence of positive and iid random varibles with finite expectation, and define $S_n = \sum_{i=1}^{n} X_i$. We call $N(t) = \text{mín}\{n \in \mathbb{N} : S_n \geq t\}$ a renewal process, and the following limit holds $\text{lím}_{t\to\infty} \frac{\mathbb{E}\big(N(t)\big)}{t} = \frac{1}{\mathbb{E}(X_1)}$.

c) Conclude that no random algorithm can be better than H_n-competitive.

d) For a state s and a sequence of tasks $\hat{\tau}$, define $\rho(s, \hat{\tau}) = w(s, \hat{\tau}) - \text{mín}_x\, w(x, \hat{\tau})$, where w is the work function of the MTS. Consider $\rho(\cdot, \hat{\tau})$ as a point in $\mathbb{R}^X$, and describe its possible values.

e) Consider now ρ as a description of the system after a secuence of tasks. Find a reasonable random algorithm that moves to a random state given by a distribution that depends only on ρ, that is H_n-competitive. **Hint:** Consider the potential $\Phi = H_m$, where $m = |\arg\text{mín}_{x\in X} \rho(x, \hat{\tau})|$.

P3. Use the DC-TREE algorithm to construct an algorithm for paging problem with competitive ratio k. What is the interpretation of this algorithm in paging, ignoring k-sever?

P4. Consider the class of metric spaces $(\mathcal{M}, d)$, where $\mathcal{M}$ is a convex subset of a vector space and d satisfies that for every three different points $x, y, r \in \mathcal{M}$ such that $y = \lambda x + (1 - \lambda)r$, for some $\lambda \in (0, 1)$, the following implication is true:

$$\big[d(p, x) \leq d(p, r) \Rightarrow d(p, y) \leq d(p, r)\big], \forall p \in X.$$

a) Prove that every euclidean space is in this class of spaces.

b) Prove that the algorithm $\text{SC}_{1/2}$ we studied in the TA class is 3-competitive for every space in this class.

P5. Consider the following 2-server algorithm. After serving each request, label the server at the request as s_1 and the other as s_2 (if both are in the same point, label arbitrarily). Consider the next request r and set $b = d(s_1, r)$. If $d(s_2, r) < 3b$, serve r with s_2. Otherwise, serve it with s_1 and move s_2 a distance $3b$ towards r. Prove that this algorithm is $O(1)$-competitive in any Euclidean space.