Tópicos en Matemáticas Discretas III - Online Algorithms and Scheduling
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Homework #1

P1. (Ice cream shop) Felipito owns a small ice cream shop with a machine that can produce two flavors of ice cream, vanilla and chocolate. At each point in time, the machine is in one of two modes: vanilla mode or chocolate mode. Before changing the mode, he has to run the cleaning program of the machine which is very long and the cost due to waiting of customers etc. is \$100. Producing a vanilla ice cream with the machine costs \$10, producing a chocolate ice cream costs \$20. Felipito can produce the ice cream manually as well, so he does not have to clean the machine, but then the costs are \$20 and \$40, respectively.

Suppose there is a queue outside the shop of unknown length in which each customer wants exactly one unit of ice cream. Felipito know the flavor that a customer wants only when she is the next in line. Suppose the starting mode of the machine is vanilla. What would you recommend Felipito to do in order to minimize his cost? Find a deterministic 2-competitive algorithm.

P2. (The lost pen problem) You lost your pen below the couch and you want to find it. Since there is no much light, the only way is to insert your hand in the space between the couch and the floor and start moving it until you touch the pen. ¿How could you minimize the total distance traveled?

Let us say that the couch is very long, so we model the position of the pen as a point $a \in \mathbb{R}$ and the starting point of the search as the origin. The *offline* optimal strategy costs $\textsc{opt}(a) = |a|$.

a) Is there an strictly competitive algorithm? This is, one such that for some constant c, $\textsc{alg}(a) \leq c \cdot \textsc{opt}(a)$, for every $a \in \mathbb{R}$ (without the additive constant).

b) Consider the following algorithm for fixed $\beta > 1$. You start moving to the right β, then you go back to the origin, then you move β^2 to the left, and continue going back to the origin and moving β^i. What is the asymptotic competitive ratio of this algorithm?

c) Calculate the value of β that minimizes the asymptotic competitive ratio.

P3. (Non-Additive Multislope Ski Rental) You have to buy equipment for a certain task that will take an unknown amount of time T. There are several types of equipment for performing the task, described by the pairs $(b_i, r_i)_{i=0}^{k}$, with $b_i \leq b_{i+1}$ and $r_i \geq r_{i+1}$. If you want to use equipment i, you need to pay b_i to purchase it and then you pay r_i per unit time in production cost. Assume that $b_0 = 0$ and $r_k = 0$, and that at any time (not necessarily integral) you can purchase equipment i by paying b_i, regardless of what you have bought so far.

a) Denote by $\textsc{opt}(T)$ the optimal value when the task lasts T units of time and find an expression for it in terms of T and $(b_i, r_i)_{i=0}^{k}$. Show that it is a concave function of T.

b) Denote by $y(t)$ what you have spent in total at time t. Assume you are at some timepoint such that $y(t) = 2\textsc{opt}(t)$. Let $u > t$ be a time such that $\textsc{opt}(u) = y(t)$ (assume it exists) and let i be the most expensive item such that $\textsc{opt}(u) = b_i + r_i u$. Prove that if you buy item i at time t and wait until time u without buying anything else, then $y(u) < 2\textsc{opt}(u)$. Find an upper bound for $y(t^+)$ (what you have spent right after purchasing i) to show that $y(t') \leq 4\textsc{opt}(t')$ for every $t' \in [t, u]$.

c) Use what you just showed to find a 4-competitive algorithm.