**Tópicos en Matemáticas Discretas III - Online Algorithms and Scheduling**
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# Homework #2

**P1.**
a) Formulate the Ski Rental Problem as a Metrical Task System.
b) Formulate the list accessing problem as a Metrical Task System.
c) Consider the greedy algorithm for an MTS, this is, one that from state $s_i$ moves to $s_{i+1} \in \arg\text{mín}_x\, d(s_i, x) + \tau_{i+1}(x)$ to process task $\tau_{i+1}$. Show that this algorithm has an unbounded competitive ratio.

**P2.** Using Yao's Principle find a lower bound of at least 1,1 for the competitivity ratio of any randomized algorithm for the list accessing problem.

**P3.** A stack is a Last-in-First-Out memory/buffer $S$, on which the following operations are defined:

- Push$(S, x)$ adds the object $x$ on top of the stack, with cost 1.
- Pop$(S)$ returns the topmost object of the stack and removes it, with cost 1.
- Multipop$(S, k)$ returns the topmost $k$ objects with cost $k$.

We analyze the cost of processing a sequence of $n$ stack operations.

a) Apply the usual worst case analysis to obtain a bound of $O(n^2)$ in the processing time, when begining with an empty stack.
b) Let $c_i$ be the cost of the $i$-th operation and $S_i$ the stack after the $i$-th operation. For some potential function $\Phi$, define the amortized cost

$$a_i = c_i + \Phi(S_i) - \Phi(S_{i-1})$$

and show that the total cost is at most $\Phi(S_0) + \sum_{i=1}^{n} a_i$.
c) Use the potential $\Phi(S) = |S|$ to show a bound of $O(n)$ in the processing time (assume that whenever a Pop or Multipop operation is done there are sufficiently many elements on the stack).

**P4.** A *tight example* for a $c$-competitive algorithm for a minimization problem is a family of instances $(I_\varepsilon)_{\varepsilon>0}$ such that

$$\text{ALG}(I_\varepsilon) \geq (c - \varepsilon)\text{OPT}(I_\varepsilon).$$

Find a *tight example* for the traversal algorithm for Metrical Task Systems with $N$ states.