

# Multivariable Calculus

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## Contents

<b>1</b>	<b>Linear Equations in Linear Algebra</b>	<b>2</b>
1.1	Systems of Linear Equations . . . . .	2
1.1.1	Elementary Row Operations . . . . .	2
1.1.2	Questions . . . . .	2
1.2	Row Reduction and Echelon Forms . . . . .	3
1.3	Vector Equations . . . . .	4
1.3.1	Algebraic Properties of $\mathbb{R}^n$ . . . . .	4
1.4	The Matrix Equation $Ax = b$ . . . . .	5
1.5	Solution Sets of Linear Systems . . . . .	6

# 1 Linear Equations in Linear Algebra

## 1.1 Systems of Linear Equations

**Definition 1.1.** A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

**Definition 1.2.** A *linear system* is a set of linear equations involving like variables.

**Definition 1.3.** A *solution* to a linear system is an ordered set that makes the linear system true.

**Definition 1.4.** A *solution set* is the set of all possible solutions to the linear system.

*Remark.* Two linear systems with like solution sets are *equivalent*.

*Remark.* A linear system is *consistent* if it has at least one solution, and *inconsistent* if it has no solutions.

**Definition 1.5.** A *coefficient matrix* is a matrix that consists of the coefficients of the variables of a linear system.

*Remark.* Each column of the coefficient matrix corresponds to a variable in the linear system.

**Definition 1.6.** An *augmented matrix* consists of the coefficient matrix with an added column containing the constants of the RHS of the linear system.

**Definition 1.7.** An  $m \times n$  *matrix* is a rectangular array of elements with  $m$  rows and  $n$  columns.

### 1.1.1 Elementary Row Operations

- add the multiple of one row to another
- switch two rows
- scale a row by a nonzero constant

*Remark.* Row operations are reversible.

**Definition 1.8.** Two matrices are *row equivalent* if a sequence of row operations can transform one into the other.

*Remark.* All row equivalent augmented matrices have the same solution set.

### 1.1.2 Questions

- does a solution to the linear system exist?
- If it does, is it unique?

## 1.2 Row Reduction and Echelon Forms

**Definition 1.9.** The *leading entry* of a row is its left-most non-zero entry.

**Definition 1.10.** A matrix is in *echelon form* if:

- all non-zero rows are above any all-zero rows
- the leading entry of each row is in a column to the right of the leading entry of the row above it
- all entries in a column below a leading entry are zeros

**Definition 1.11.** A matrix is in *reduced row echelon form* if:

- it's in echelon form
- all leading entries are 1
- all leading entries are the only non-zero entries in their columns

*Remark.* A matrix can be row equivalent with many echelon forms but only one reduced echelon form.

**Definition 1.12.** A *pivot position* corresponds to the position of one of the leading entries of the reduced echelon form of a matrix.

**Definition 1.13.** A column of the coefficient matrix is a *free column* if it doesn't contain a pivot position.

**Definition 1.14.** A column of the augmented matrix is a *pivot column* if it contains a pivot position.

*Remark.* Variables corresponding to free columns are *free variables*. Variables corresponding to pivot columns of the coefficient matrix are *basic variables*.

*Remark.* The solution set of a consistent linear system has a *parametric representation* in which by convention free variables act as parameters. The solution set of an inconsistent linear system is empty and has **no** parametric representation.

*Remark.* Solving a system amounts to finding a parametric representation of the solution set or determining that the solution set is empty.

*Remark.* A linear system is consistent iff the right-most column of the augmented matrix is **not** a pivot column.

**Theorem 1.1** (Existence and uniqueness theorem). *A linear system is consistent iff the right-most column of the augmented matrix is **not** a pivot column. A consistent linear system has either a unique solution, if it has no free variables, or infinitely many solutions if it has at least one free variable.*

### 1.3 Vector Equations

$$\mathbb{R}^n := \text{Set of ordered n-tuples } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_1, \dots, x_n \in \mathbb{R}$$

$$\mathbb{R}^1 := \mathbb{R} = \text{Set of real numbers} = \text{Number line}$$

$$\mathbb{R}^2 := \text{Plane}$$

$$\mathbb{R}^3 := \text{Space}$$

**Definition 1.15.** A *vector* in  $\mathbb{R}^n$  is an element of  $\mathbb{R}^n$ .

*Remark.* Two vectors are equal iff their corresponding entries are equal.

#### 1.3.1 Algebraic Properties of $\mathbb{R}^n$

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \text{ and } c, d \in \mathbb{R} \quad (1)$$

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

**Definition 1.16.** If  $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$  and  $c_1, \dots, c_p \in \mathbb{R}$ , then

$$\mathbf{y} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_p$ .

**Definition 1.17.**  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is the set of all linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_p$ .

## 1.4 The Matrix Equation $Ax = b$

**Definition 1.18.** If  $A^{m \times n}$  and  $\mathbf{x} \in \mathbb{R}^n$  then  $A\mathbf{x} = \mathbf{b}$  is the linear combination of the columns of  $A$  using the corresponding entries of  $\mathbf{x}$  as weights

$$A\mathbf{x} = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

*Remark.*  $A\mathbf{x}$  is defined iff the number of columns of  $A$  equals the number of entries in  $\mathbf{x}$ .

**Theorem 1.2.** If  $A^{m \times n}$  and  $\mathbf{b} \in \mathbb{R}^m$ , then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \ \cdots \ \mathbf{a}_n \ \mathbf{b}]$$

*Remark.* The equation  $A\mathbf{x} = \mathbf{b}$  has a solution iff  $\mathbf{b}$  is a linear combination of the columns of  $A$ .

**Theorem 1.3.** If  $A^{m \times n}$ , then the following statements are logically equivalent.

- $A\mathbf{x} = \mathbf{b}$  has a solution  $\forall \mathbf{b} \in \mathbb{R}^m$
- Each  $\mathbf{b} \in \mathbb{R}^m$  is a linear combination of the columns of  $A$
- $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \mathbb{R}^m$
- $A$  has a pivot position in every row

*Remark.* The  $i$ th entry of  $A\mathbf{x}$  is the sum of the products of the entries of the  $i$ th row of  $A$  with the corresponding entries of  $\mathbf{x}$

**Definition 1.19.** The *identity matrix*, denoted  $I_n$ , is an  $n \times n$  matrix with 1's on the diagonal and 0's elsewhere.

$$I_n \mathbf{x} = \mathbf{x} \ \forall \mathbf{x} \in \mathbb{R}^n$$

**Theorem 1.4.** If  $A^{m \times n}$ ,  $\mathbf{u}$  and  $\mathbf{v} \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ , then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$$

$$A(c\mathbf{u}) = c(A\mathbf{u})$$

## 1.5 Solution Sets of Linear Systems

**Definition 1.20.** A system of linear equations is *homogeneous* if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

where  $A^{m \times n}$  and  $\mathbf{0} \in \mathbb{R}^m$

*Remark.* All homogeneous linear systems,  $A\mathbf{x} = \mathbf{0}$ , have a trivial solution.

*It follows.* From the existence and uniqueness theorem.

A homogeneous linear system has a non-trivial solution iff it has at least one free variable.

**Definition 1.21.** The parametric vector form of the solution set of the homogeneous linear system  $A\mathbf{x} = \mathbf{0}$  with one free variable is

$$\mathbf{x} = t\mathbf{v} \quad \text{where } t \in \mathbb{R}$$

*Remark.* The solution set of a homogeneous linear system,  $A\mathbf{x} = \mathbf{0}$ , with only one free variable is a line through the origin.

**Definition 1.22.** The parametric vector form of the solution set of the non-homogeneous linear system  $A\mathbf{x} = \mathbf{b}$  is

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad \text{where } t \in \mathbb{R}$$

**Theorem 1.5.** *The solution set of a consistent non-homogeneous linear system  $A\mathbf{x} = \mathbf{b}$  is obtained by translating the solution set of its corresponding homogeneous linear system  $A\mathbf{x} = \mathbf{0}$  by any particular solution  $\mathbf{p}$  of  $A\mathbf{x} = \mathbf{b}$ .*