Multivariable Calculus

Andres Duarte

2018

Contents

1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Definition 1.1. A *linear equation is an equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Definition 1.2. A *linear system* is a set of linear equations involving like variables.

Definition 1.3. A *solution* to a linear system is an ordered set that makes the linear system true.

Definition 1.4. A *solution set* is the set of all possible solutions to the linear system.

Remark. Two linear systems with like solution sets are *equivalent*.

Remark. A linear system is *consistent* if it has at least one solution, and *inconsistent* if it has no solutions.

Definition 1.5. A *coefficient matrix* is a matrix that consists of the coefficients of the variables of a linear system.

Remark. Each column of the coefficient matrix corresponds to a variable in the linear system.

Definition 1.6. An *augmented matrix* consists of the coefficient matrix with an added column containing the constants of the RHS of the linear system.

Definition 1.7. An $m \times n$ *matrix* is a rectangular array of elements with m rows and n columns.

1.1.1 Elementary Row Operations

- add the multiple of one row to another
- switch two rows
- scale a row by a nonzero constant

Remark. Row operations are reversible.

Definition 1.8. Two matrices are *row equivalent* if a sequence of row operations can transform one into the other.

Remark. All row equivalent augmented matrices have the same solution set.

1.1.2 Questions

- does a solution to the linear system exist?
- If it does, is it unique?

1.2 Row Reduction and Echelon Forms

Definition 1.9. The *leading entry* of a row is its left-most non-zero entry.

Definition 1.10. A matrix is in *echelon form* if:

- all non-zero rows are above any all-zero rows
- the leading entry of each row is in a column to the right of the leading entry of the row above it
- all entries in a column below a leading entry are zeros

Definition 1.11. A matrix is in *reduced row echelon form* if:

- it's in echelon form
- all leading entries are 1
- all leading entries are the only non-zero entries in their columns

Remark. A matrix can be row equivalent with many echelon forms but only one reduced echelon form.

Definition 1.12. A *pivot position* corresponds to the position of one of the leading entries of the reduced echelon form of a matrix.

Definition 1.13. A column of the coefficient matrix is a *free column* if it doesn't contain a pivot position.

Definition 1.14. A column of the augmented matrix is a *pivot column* if it contains a pivot position.

Remark. Variables corresponding to free columns are *free variables.* Variables corresponding to pivot columns of the coefficient matrix are *basic variables.*

Remark. The solution set of a consistent linear system has a *parametric representation* in which by convention free variables act as parameters. The solution set of an incosistent linear system is empty and has **no** parametric representation.

Remark. Solving a system amounts to finding a parametric representation of the solution set or determing that the solution set is empty.

Remark. A linear system is consistent iff the right-most column of the augmented matrix is **not** a pivot column.

Theorem 1.1 (Existence and uniqueness theorem)**.** *A linear system is consistent iff the right-most column of the augmented matrix is **not** a pivot column. A consistent linear system has either a unique solution, if it has no free variables, or infinitely many solutions if it has at least one free variable.*

1.3 Vector Equations

$$
\begin{aligned}
\mathbb{R}^n &:= \text{ Set of ordered n-tuples } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_1, \ldots, x_n \in \mathbb{R} \\
\mathbb{R}^1 &:= \mathbb{R} = \text{Set of real numbers} = \text{Number line} \\
\mathbb{R}^2 &:= \text{ Plane} \\
\mathbb{R}^3 &:= \text{ Space}
\end{aligned}
$$

Definition 1.15. A *vector* in $\mathbb{R}^n$ is an element of $\mathbb{R}^n$.

Remark. Two vectors are equal iff their corresponding entries are equal.

1.3.1 Algebraic Properties of $\mathbb{R}^n$

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \text{ and } c, d \in \mathbb{R} \tag{1}$$

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + 0 = 0 + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = 0$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Definition 1.16. If $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$ and $c_1, \cdots, c_p \in \mathbb{R}$, then

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

Definition 1.17. Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the set of all linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

1.4 The Matrix Equation $Ax = b$

Definition 1.18. If $A^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$ then $A\mathbf{x} = \mathbf{b}$ is the linear combination of the columns of A using the corresponding entries of x as weights

$$A\mathbf{x} = [\,\mathbf{a}_1 \;\cdots\; \mathbf{a}_n\,] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

Remark. $A\mathbf{x}$ is defined iff the number of columns of A equals the number of entries in $\mathbf{x}$.

Theorem 1.2. *If $A^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, then the matrix equation*

$$A\boldsymbol{x} = \boldsymbol{b}$$

has the same solution set as the vector equation

$$x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n = b$$

which in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\,\boldsymbol{a}_1 \;\cdots\; \boldsymbol{a}_n \;\boldsymbol{b}\,]$$

Remark. The equation $A\mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b}$ is a linear combination of the columns of A.

Theorem 1.3. *If $A^{m\times n}$, then the following statements are logically equivalent.*

- $A\boldsymbol{x} = \boldsymbol{b}$ *has a solution* $\forall \boldsymbol{b} \in \mathbb{R}^m$
- *Each* $\boldsymbol{b} \in \mathbb{R}^m$ *is a linear combination of the columns of* A
- *Span* $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} = \mathbb{R}^m$
- A *has a pivot position in every row*

Remark. The ith entry of $A\mathbf{x}$ is the sum of the products of the entries of the ith row of A with the corresponding entries of $\mathbf{x}$

Definition 1.19. The *identity matrix*, denoted I_n, is an $n \times n$ matrix with 1's on the diagonal and 0's elsewhere.

$$I_n\,\mathbf{x} = \mathbf{x}\;\forall \mathbf{x} \in \mathbb{R}^n$$

Theorem 1.4. *If $A^{m\times n}$, $\boldsymbol{u}$ and $\boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then*

$A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$

$A(c\boldsymbol{u}) = c(A\boldsymbol{u})$

1.5 Solution Sets of Linear Systems

Definition 1.20. A system of linear equations is *homogeneous* if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

where $A^{m \times n}$ and $\mathbf{0} \in \mathbb{R}^m$

Remark. All homogeneous linear systems, $A\mathbf{x} = \mathbf{0}$, have a trivial solution.

It follows. From the existence and uniqueness theorem.
A homogeneous linear system has a non-trivial solution iff it has at least one free variable.

Definition 1.21. The parametric vector form of the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ with one free variable is

$$\mathbf{x} = t\mathbf{v} \qquad \text{where } t \in \mathbb{R}$$

Remark. The solution set of a homogeneous linear system, $A\mathbf{x} = \mathbf{0}$, with only one free variable is a line through the origin.

Definition 1.22. The parametric vector form of the solution set of the non-homogeneous linear system $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \qquad \text{where } t \in \mathbb{R}$$

Theorem 1.5. *The solution set of a consistent non-homogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ is obtained by translating the solution set of its corresponding homogeneous linear system $A\boldsymbol{x} = \boldsymbol{0}$ by any particular solution $\boldsymbol{p}$ of $A\boldsymbol{x} = \boldsymbol{b}$.*

Definition 1.23. An indexed set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\} \in \mathbb{R}^n$ is *linearly independent* if the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

Definition 1.24. An indexed set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\} \in \mathbb{R}^n$ is *linearly dependent* if $\exists$ weights $c_1, \cdots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Remark. The columns of A are linearly independent iff the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Remark. An indexed set of vectors is linearly dependent iff at least one of the vectors in the set is a linear combination of the others.

Remark. A set containing a single vector is linearly independent iff its not the zero vector. It follows: the zero vector is linearly dependent.

Theorem 1.6. *Any set $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$*

Theorem 1.7. *Any set $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\} \in \mathbb{R}^n$ that contains the zero vector is linearly dependent.*

1.6 Introduction to Linear Transformations

Definition 1.25. A *transformation* T from $\mathbb{R}^n$ to $\mathbb{R}^m$, written $T : \mathbb{R}^n \to \mathbb{R}^m$, is a rule that assigns to each vector $\mathbf{x} \in \mathbb{R}^n$ a vector $T(\mathbf{x}) \in \mathbb{R}^m$.

- $\mathbb{R}^n$ is the *domain* of T and $\mathbb{R}^m$ is its *codomain.*
- $\forall\ \mathbf{x} \in \mathbb{R}^n$ the vector $T(\mathbf{x}) \in \mathbb{R}^m$ is the *image* of x.
- The set of all images $T(\mathbf{x})$ is the *range* of T.

Remark. The codomain is all the places the transformation could take you, the range is all the places it does.

Remark. $\forall\ \mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$ and written as $\mathbf{x} \mapsto A\mathbf{x}$.

Remark. The range of T is the set of all linear combinations of the columns of A.

Definition 1.26. A transformation T is *linear* if:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad \forall\ \mathbf{u}, \mathbf{v}$ in the domain of T.
- $T(c\mathbf{u}) = c\,T(\mathbf{u}) \qquad \forall\ c, \mathbf{u}$ in the domain of T.

Remark. Every matrix transformation is a linear transformation.

Remark. Linear Transformations preserve the operations of vector addition and scalar multiplication.

It follows. From linearity.

$$T(\mathbf{0}) = 0$$