Spaces with Bounded Curvature

Andrés Ahumada Gómez

(Ahumada Gómez) Departamanto de Matemáticas, CUNEF Universidad
Email address: andres.ahumada@cunef.edu

Contents

Introduction

In this notes we try to do a compilation of all the information about some objects called **spaces with bounded curvature** and also we include the tools of metric geometry to understand them.

CHAPTER 1

Basic information

The **space form** M_k^2 is the complete and simply connected Riemannian manifold of dimension 2 with constant curvature $k \in \mathbb{R}$. The diameter of M_k^2 is denoted by D_k.

A **curve** in a metric space (X, d) is continuous map $\alpha : [a, b] \to X$, where $[a, b] \subset \mathbb{R}$ is an interval. The **space of curves from** $[a, b]$ **to** X is denoted by $\mathcal{C}([a, b]; X)$. A metric space induces a length functional

$$L_d \colon \mathcal{C}([a, b]; X) \to \mathbb{R}$$

given by

$$L_d(\gamma) = \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \right\},$$

where we take the supremum over

$$\{\mathcal{P} = \{a = t_0 t_1 \cdots t_n = b\}\}.$$

This functional induces a metric $\hat{d}$ on X given by

$$\hat{d}(x, y) = \inf \{L_d(\gamma) \mid \gamma \in \mathcal{C}([a, b]; X),\ \gamma(a) = x \text{ y } \gamma(b) = y\}.$$

A **length space** is a metric space such that $\hat{d} = d$.

Remark 1. A length space is not necesarily complete, for example $\mathbb{R}^2 \setminus \{0\}$ is an incomplete length space.

A **(geodesic) segment** (or a **minimizing curve**) is a curve $\gamma \colon [a, b] \to X$ such that if β is another curve joining $\gamma(a)$ with $\gamma(b)$, then $L_d(\gamma) L_d(\beta)$. A segment joining x with y on X is denoted by $[x, y]$. The image of $[x, y]$ is also called a **geodesic segment**.

Any curve $\gamma \colon [a, b] \to X$ can be reparameterized over the interval $[0, 1]$ using the function $\rho(t) = tb + (1 - t)a$. A **rectifiable curve** (a finite length curve) has a **constant speed parameterization** if $\gamma \colon [0, 1] \to X$ and $L_d(\gamma, 0, t) := L_d(\gamma \mid_{[0,t]}) = L_d(\gamma)\, t$. Moreover, we say that it is a **parameterization by arc length** if $\gamma \colon [0, L_d(\gamma)] \to X$ and $L_d(\gamma, 0, t) = t$. Unless we explicitly state the opposite, every geodesic segment is parameterized by arc length. A **geodesic** is a curve $\gamma \colon I \to X$ such that for every $t \in I$ there is an interval $[a, b]$ containing t, $[a, b] \subset I$, and $\gamma \mid_{[a,b]}$ is a minimizing segment.

A **(geodesic) triangle** in X consists of three distinct points $p, q, r \in X$ and three geodesic segments $[p, q]$, $[q, r]$ and $[r, p]$, these could be colineal. We denote such triangle by $\triangle(p, q, r)$ or $\triangle([p, q], [q, r], [r, p])$. A **comparison triangle** for $\triangle(p, q, r)$ is a triangle in M_k^2 with vertices $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ such that $d(p, q) = d(\tilde{p}, \tilde{q})$, $d(q, r) = d(\tilde{q}, \tilde{r})$ and $d(r, p) = d(\tilde{r}, \tilde{p})$. We denote this triangle by $\triangle^k(p, q, r)$, $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ or $\triangle([\tilde{p}, \tilde{q}], [\tilde{q}, \tilde{r}], [\tilde{r}, \tilde{p}])$. We illustrate these concepts in Figure 1. A **comparison point** for $x \in [p, q]$ is a point $\bar{x} \in [\bar{p}, \bar{q}]$ with $d(p, x) = d(\bar{x}, \bar{p})$. The angle of $\triangle(\bar{p}, \bar{q}, \bar{r})$ at $\bar{p}$ is called a **comparison angle** between $[p, q]$ and $[p, r]$ at p and it is denoted by $\sphericalangle^k(q, p, r)$. In particular, for the angle in the Euclidean plane, we use the notation $\sphericalangle^0(q, p, r) = \bar{\sphericalangle}(q, p, r)$.

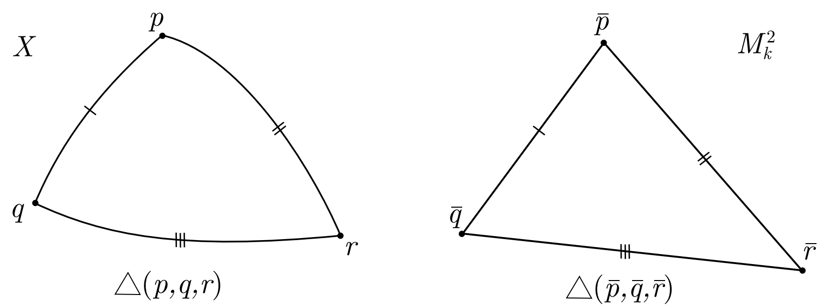

FIGURE 1. Comparison triangles

REMARK 2. Instead of denoting differently the metrics we distiguish the metric spaces involved taking notice which set the points belong to.

REMARK 3. In general comparison triangles exist and are unique up to isometry if the perimeter is less than $2D_k$ (see [**?**]). Thus for $k > 0$ comparison triangles always exist.

Bibliography

[1] A. D. Alexandrov. Über eine Verallgemeinerung der Riemannschen Geometrie. Begriff des Raumes in der Geometrie. Ber. Riemann-Tagung Forsch.-Inst. Math. 33-84 (1957)., 1957.

[2] V. N. Berestovskij. Introduction of a Riemann structure into certain metric spaces. *Sib. Math. J.*, 16:499–507, 1976.

[3] Georges de Rham. *Differentiable manifolds. Forms, currents, harmonic forms. Transl. from the French by F. R. Smith. Introduction to the English ed. by S. S. Chern*, volume 266. Springer, Berlin, 1984.

[4] Kenji Fukaya and Takao Yamaguchi. Isometry groups of singular spaces. *Math. Z.*, 216(1):31–44, 1994.

[5] S. B. Myers and N. E. Steenrod. The group of isometries of a Riemannian manifold. *Ann. of Math. (2)*, 40(2):400–416, 1939.

[6] I. G. Nikolaev. On the parallel displacement of vectors in spaces with two-sided bounded curvature in the sense of A. D. Aleksandrov. *Sib. Math. J.*, 24:106–119, 1983.

[7] I. G. Nikolaev. Smoothness of the metric of spaces with two-sided bounded Aleksandrov curvature. *Sib. Math. J.*, 24:247–263, 1983.

[8] I. G. Nikolaev. Closure of the set of classical Riemannian spaces. *Journal of Mathematical Sciences*, 55(6):2100–2115, 1991.

[9] Peter Petersen. *Riemannian geometry*, volume 171 of *Grad. Texts Math.* Cham: Springer, 3rd edition edition, 2016.