Randomized Experiment

Causal Inference using Machine Learning
Master in Economics, UNT

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The first RCT

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- **Peirce (1885)**: Used random sequencing in psychology to prevent bias from expectations, anticipating randomization principles.
- **Gossett and Fisher (1920s)**: Gossett mentioned random plot placement; Fisher formalized randomization as essential for causal inference.

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- Developed notation for potential yields in agricultural experiments, allowing estimation across different treatment groups.
- Emphasized the role of assignment mechanisms in calculating causal effects.
- Proposed an estimator for the Variance of the Average Treatment Effect (ATE) in randomized experiments.

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- Emphasized the need for physical randomization to eliminate confounding variables.
- Proposed methods for testing hypotheses in a controlled experimental setup.

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- Together, they laid the groundwork for randomized experiments and causal inference.

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Definition of Assignment Vector D

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- For two treatment groups, D is a binary vector with 2^N possible values.

Assignment Mechanism

Assignment Mechanism: Given a population of N units, the assignment mechanism is a row-exchangeable function, denoted as $\Pr(D|X, Y(0), Y(1))$, which takes values in the interval $[0, 1]$ and satisfies:

$$\sum_{D \in \{0,1\}^N} \Pr(D|X, Y(0), Y(1)) = 1$$

for all possible values of X (covariates), $Y(0)$, and $Y(1)$ (potential outcomes). (Row-exchangeability implies that the order of units within vectors or matrices is irrelevant to the function $\Pr(\cdot)$.)

Example: Assignment Mechanism with Two Units

Define the **treatment effect** for unit i as: $\tau_i = Y_i(1) - Y_i(0)$

$$\Pr(D|X, Y(0), Y(1)) = \begin{cases} 1 & \text{if } \tau_2 > \tau_1 \text{ and } D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \text{if } \tau_2 < \tau_1 \text{ and } D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{1}{2} & \text{if } \tau_2 = \tau_1 \text{ and } D \in \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \\ 0 & \text{if } D \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \\ 0 & \text{if } \tau_2 < \tau_1 \text{ and } D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 & \text{if } \tau_2 > \tau_1 \text{ and } D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

Unit Assignment Probability

The **unit-level assignment probability** for unit i is defined as:

$$p_i(X, Y(0), Y(1)) = \sum_{D:D_i=1} \Pr(D|X, Y(0), Y(1)),$$

Propensity Score

The **propensity score** at x is the average unit assignment probability for units with $X_i = x$. It is defined as:

$$e(x) = \frac{1}{N(x)} \sum_{i:X_i=x} p_i(X, Y(0), Y(1)),$$

Individualistic Assignment Mechanism

Individualistic Assignment Mechanism: In this mechanism, the probability of each unit's treatment assignment depends only on the unit's covariates and potential outcomes, independent of the assignments of other units.

$$\Pr(D_i|X, Y(0), Y(1)) = f(X_i, Y_i(0), Y_i(1))$$

Probabilistic Assignment Mechanism

Probabilistic Assignment Mechanism: Under this mechanism, each unit has a non-zero probability of being assigned to either treatment or control, ensuring randomness in the assignment process.

$$0 < \Pr(D_i = 1 | X, Y(0), Y(1)) < 1 \quad \text{for all units } i$$

Unconfounded Assignment Mechanism

Unconfounded Assignment Mechanism: This mechanism assumes that assignment to treatment is independent of the potential outcomes, given the covariates. In other words, the assignment is "as good as random" conditional on covariates.

$$\Pr(D|X, Y(0), Y(1)) = \Pr(D|X)$$

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Randomized Experiment

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- (i) **Individualistic**: Each unit's treatment assignment depends only on its own covariates and potential outcomes, independent of other units.
- (ii) **Unconfounded**: Assignment to treatment is independent of potential outcomes given covariates, meaning assignment is "as good as random" conditional on covariates.

Bernoulli Trials

A **Bernoulli trial** is a classical randomized experiment where each unit is independently assigned to treatment or control, often based on a coin toss.

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- Each unit's assignment is independent of others, meaning the assignment for one unit does not affect the assignment for another.
- The assignment mechanism is:
  - **Individualistic**: Each unit's assignment depends only on its own characteristics.
  - **Probabilistic**: Each unit has a non-zero chance of receiving either treatment or control.
  - **Unconfounded**: Given covariates, assignment does not depend on potential outcomes.
  - **Controlled by the Researcher**: The probability q is specified by the researcher.

Bernoulli Trials - Probability of an Assignment Vector

For a Bernoulli trial, the probability of an assignment vector D for N units is given by:

$$\Pr(D|X, Y(0), Y(1)) = \prod_{i=1}^{N} \left(q^{D_i} \cdot (1-q)^{1-D_i} \right)$$

where:

- $D_i = 1$ if unit i is assigned to treatment, $D_i = 0$ otherwise.
- q: Probability of treatment assignment.

If $q = 0.5$, then $\Pr(D|X, Y(0), Y(1)) = 0.5^N$.

Completely Randomized Experiment - Definition

A **completely randomized experiment** assigns a fixed number N_t of units to treatment, and the remaining $N - N_t$ units to control.

- The assignment is achieved by randomly selecting N_t units from a pool of N units.
- Ensures a balanced distribution of treated and control units, with exactly N_t in treatment and $N - N_t$ in control.
- Each unit's assignment is NOT independent of others, but the total number of treated units is fixed by design.
- The assignment mechanism is:
    - **Individualistic**: Each unit's assignment depends only on its own characteristics.
    - **Probabilistic**: Each unit has a positive probability of being selected for treatment or control.
    - **Unconfounded**: Given covariates, assignment does not depend on potential outcomes.
    - **Controlled by the Researcher**: The number N_t of treated units is specified by the researcher.

Completely Randomized Experiment - Probability of an Assignment Vector

In a completely randomized experiment, the probability of an assignment vector D is:

$$\Pr(D|X, Y(0), Y(1)) = \begin{cases} \frac{1}{\binom{N}{N_t}} & \text{if } \sum_{i=1}^{N} D_i = N_t \\ 0 & \text{otherwise} \end{cases}$$

where N_t is the predetermined number of units assigned to treatment.

Stratified Randomized Experiment - Definition

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- Reduces variability and improves the precision of causal inference estimates.
- The goal is to reduce variance in the estimator and increase the power of statistical tests, enhancing the study's ability to detect treatment effects.

Stratified Randomized Experiment - Probability of an Assignment Vector

For a stratified randomized experiment with J blocks, the probability of an assignment vector D is:

$$\Pr(D|X, Y(0), Y(1)) = \prod_{j=1}^{J} \frac{1}{\binom{N(j)}{N_t(j)}}$$

where:

- $N(j)$: Number of units in block j,
- $N_t(j)$: Number of treated units in block j.

Paired Randomized Experiment - Definition

A **paired randomized experiment** is an extreme form of stratified randomization, where each block (or stratum) contains exactly two units.

- Within each pair, one unit is randomly assigned to treatment and the other to control, ensuring direct comparison between similar units.

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- Minimizes differences between treated and control units on covariates, reducing bias in the estimated treatment effect.
- Reduces variance in the estimator by closely aligning treatment and control units.
- Increases statistical power by enhancing the precision of the causal inference, making it easier to detect treatment effects.

Paired Randomized Experiment - Probability of an Assignment Vector

For a paired randomized experiment with $N/2$ pairs, the probability of an assignment vector D is:

$$\Pr(D|X, Y(0), Y(1)) = 2^{-\frac{N}{2}}$$

- Each unit within a pair has an equal probability of being assigned to treatment or control.

Number of Possible Values for the Assignment Vector by Design and Sample Size

Type of Experiment and Design	Number of Possible Assignments	Number of Units (N) in Sample			
		4	**8**	**16**	**32**
Bernoulli trial	2^N	16	256	65,536	4.2×10^9
Completely randomized experiment	$\binom{N}{N/2}$	6	70	12,870	0.6×10^9
Stratified randomized experiment	$\binom{N/2}{N/4}^2$	4	36	4,900	0.2×10^9
Paired randomized experiment	$2^{N/2}$	4	16	256	65,536

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References I

Fisher, R. A. (1935). *The Design of Experiments*. Oliver and Boyd, Edinburgh, UK.

Neyman, J. (1923). On the application of probability theory to agricultural experiments. essay on principles. section 9. *Statistical Science*, 5(4):465–472.