High Dimensionality and Double Machine Learning

Causal Inference using Machine Learning
Master in Economics, UNT

Andres Mena

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Content

High-Dimensional Linear Regression and LASSO

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$$\widehat{\beta}^{\text{LASSO}} = \arg\min_{b \in \mathbb{R}^p} \sum_{i=1}^{n} (Y_i - b' X_i)^2 + \lambda \sum_{j=1}^{p} |b_j|.$$

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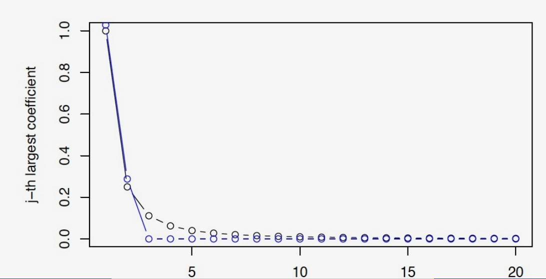

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- This condition ensures that only variables with sufficiently high marginal predictive contribution survive the penalty, inducing sparsity in the model.

Predictive Performance of LASSO: Theoretical Guarantees

Theorem (Predictive Performance of Lasso)

Under approximate sparsity and suitable regularity conditions, if we choose λ as recommended (e.g., $\lambda \propto \sigma\sqrt{n\log(\max\{p,n\})}$) and let s be the effective dimension (number of important parameters):

$$\sqrt{E_X[(\beta'X - \widehat{\beta}'X)^2]} \leq \text{const} \cdot \sqrt{E[\varepsilon^2]}\sqrt{\frac{s\log(\max\{p,n\})}{n}}.$$

Moreover, with high probability:

- The number of regressors selected by LASSO is of order s.
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Implication: LASSO adapts to unknown sparsity, suffering a modest $\sqrt{\log(\max\{p, n\})}$ factor over the ideal $\sqrt{s/n}$ rate.

Conclusions on LASSO's Properties

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- **Sparsity:** By setting some coefficients to zero, LASSO performs variable selection and yields parsimonious models.
- **Improved Predictive Performance for Y:** Despite the bias in $\widehat{\beta}$, the overall prediction $\widehat{\beta}'X$ often outperforms OLS predictions out-of-sample, especially when p is large and s is relatively small.

Bottom Line: LASSO trades off some bias in parameter estimates for substantial gains in predictive stability and interpretability. This is useful for high-dimensional econometric applications.

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Naive Estimator (1)

Naive Estimator Setup:

- Consider two separate LASSO regressions:

$$\widehat{\beta}_1 = \arg\min_{\beta_1} \sum_{i:D_i=1} (Y_i - W_i'\beta_1)^2 + \lambda_1 \sum_{j=1}^{p} |\beta_{1j}|,$$

and

$$\widehat{\beta}_0 = \arg\min_{\beta_0} \sum_{i:D_i=0} (Y_i - W_i'\beta_0)^2 + \lambda_0 \sum_{j=1}^{p} |\beta_{0j}|.$$

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- Then form the naive ATE estimator:

$$\widehat{\alpha} = \frac{1}{N} \sum_{i=1}^{N} W_i'(\widehat{\beta}_1 - \widehat{\beta}_0).$$

Naive Estimator (2)

Neyman Orthogonality and Bias:

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$$\frac{\partial}{\partial \beta_1} E[W'(\beta_1 - \beta_0) - \alpha] = E[W] \neq 0, \quad \frac{\partial}{\partial \beta_0} E[W'(\beta_1 - \beta_0) - \alpha] = -E[W]$$

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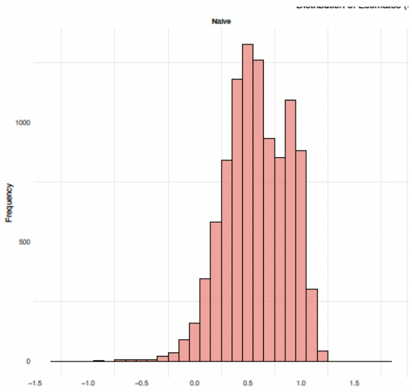
- **Not Neyman Orthogonal:** Since the derivatives are not zero, the moment condition is not orthogonal. Thus, the estimation error in $(\widehat{\beta}_1 - \widehat{\beta}_0)$ induces slow bias convergence for $\widehat{\alpha}$.
- **Consequence:** The bias converges at a slower than $\sqrt{n}$-rate, rendering standard inference methods unreliable for moderate sample sizes.

Why the Naive Estimator Performs Poorly



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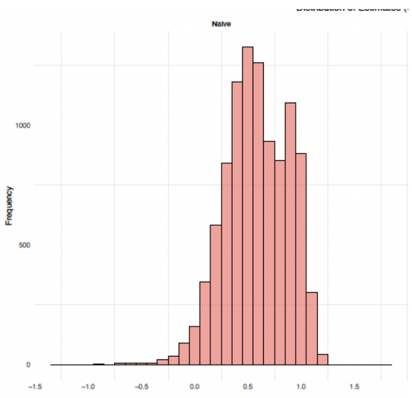

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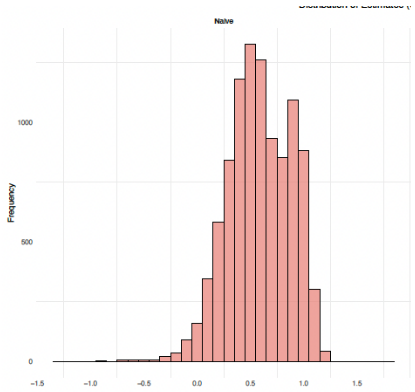

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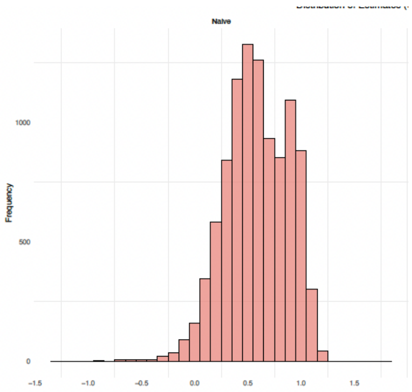

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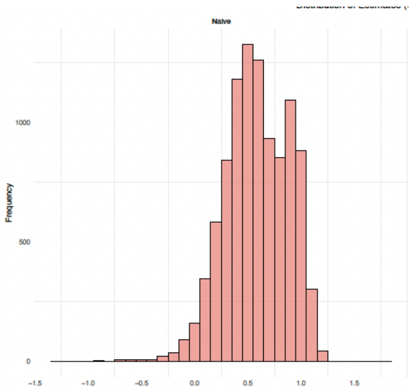

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Homework: Replicate CIML Example 4.3.1.

Double Lasso Procedure

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Double Lasso Steps:

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2. Run OLS of $\tilde{Y}$ on $\tilde{D}$:

$$\widehat{\alpha} = \frac{\sum \tilde{D}_i \tilde{Y}_i}{\sum \tilde{D}_i^2}.$$

Double Lasso Conditions

Approximate Sparsity: For good performance, we need the coefficients for Y-on-W and D-on-W to be approximately sparse:

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Without these conditions, and relying solely on cross-validation, performance may suffer in moderate samples.

Inference Under Double Lasso

Theorem 4.2.1 (Adaptive Inference with Double Lasso in High-Dimensional Regression)

Under the stated approximate sparsity, the conditions required for Theorem 3.2.1 (e.g. restricted isometry), and additional regularity conditions, the estimation error in D_i and Y_i has no first-order effect on $\widehat{\alpha}$, and

$$\sqrt{n}(\widehat{\alpha} - \alpha) \xrightarrow{d} N(0, V),$$

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A 95% confidence interval for α is:

$$\left[\widehat{\alpha} \pm 1.96 \cdot \sqrt{\frac{\widehat{V}}{n}}\right].$$

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Key Idea:

- The target parameter α depends on nuisance parameters $\eta = (\gamma_D, \gamma_Y)$, representing projection coefficients.
- The Double Lasso estimator $\widehat{\alpha}(\eta)$ satisfies:

$$\partial_\eta \alpha(\eta_0) = 0,$$

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Implication:

- Estimation errors in first-step Lasso regressions for nuisance components do not propagate into first-order bias for $\widehat{\alpha}$.
- Enables $\sqrt{n}$-consistent inference on α, even in high-dimensional settings.

Neyman Orthogonality

Derivation:

- The Double Lasso residuals are defined as:

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- Using the Implicit Function Theorem:

$$\partial_\eta \alpha(\eta_0) = -\left(\partial_a M(\alpha, \eta_0)\right)^{-1} \partial_\eta M(\alpha, \eta_0).$$

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- The derivative of $M(a, \eta)$ with respect to η is:

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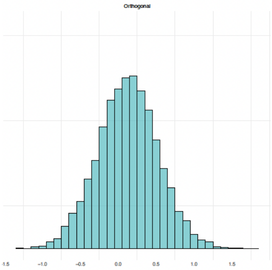

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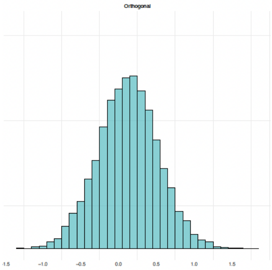

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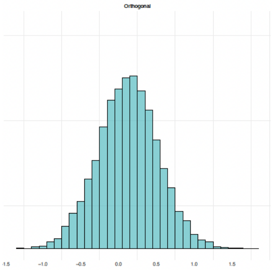

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- Double Lasso solves two separate prediction problems (Y on W and D on W) and uses residuals to form an orthogonal moment condition.
- Under approximate sparsity and proper tuning, Double Lasso achieves $\sqrt{n}$-rate inference and classical-style confidence intervals for the target parameter.

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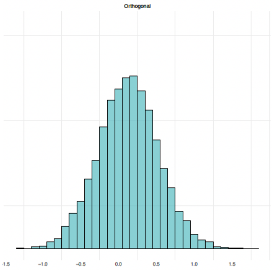

Key Takeaways:

- The naive LASSO-based estimator fails because it is not Neyman orthogonal, leading to slow bias convergence.
- Double Lasso solves two separate prediction problems (Y on W and D on W) and uses residuals to form an orthogonal moment condition.
- Under approximate sparsity and proper tuning, Double Lasso achieves $\sqrt{n}$-rate inference and classical-style confidence intervals for the target parameter.
- Neyman orthogonality is the key principle ensuring that errors in nuisance estimation do not propagate to first-order bias in the final estimate.

Content

Partially Linear Model (PLM)

Model Setup:

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$$Y = \beta D + g(X) + \epsilon, \quad \text{where } E[\epsilon|D, X] = 0.$$

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- β: Measures the predictive or causal effect of D on Y, controlling for X.

Residualized Vectors

Partialling-out X:

- Define residualized Y and D by removing the influence of X:

$$\tilde{Y} := Y - \ell(X), \quad \tilde{D} := D - m(X),$$

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Key Insight:

- Residualization isolates the effect of D on Y by removing confounding through X.
- The moment condition $E[\tilde{Y} - \beta\tilde{D}]\tilde{D} = 0$ identifies β.

FWL Partialling-Out for PLM

Theorem

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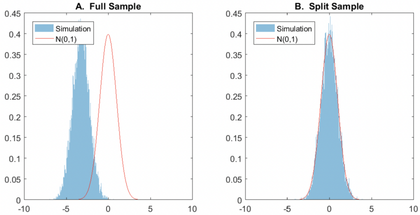
Interpretation:

- β is the regression coefficient of residualized Y on residualized D, generalizing the Frisch-Waugh-Lovell theorem to PLMs.
- Residuals ensure confounding from X is removed.

Cross-Fitting and Overfitting



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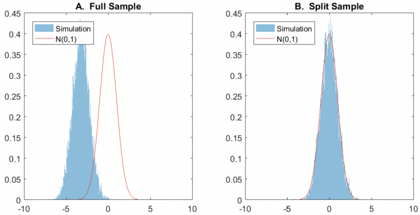

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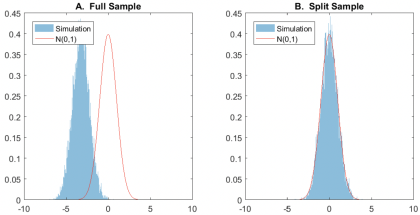

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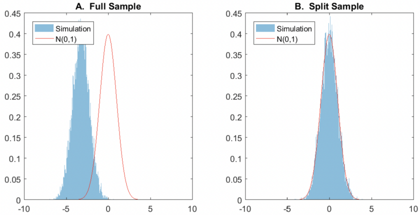

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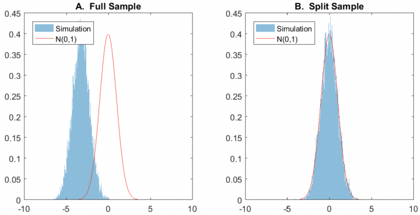

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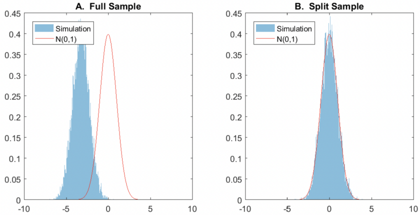

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Practical Implication:

- Cross-fitting ensures robust inference by preventing overfitting from contaminating estimates of the parameter of interest.
- Especially crucial when using complex machine learning methods.

Double Machine Learning (DML) Procedure

Steps:

1. **Cross-Fitting:**
   - Partition data into K folds: $\{1, ..., n\} = \cup_{k=1}^{K} I_k$.
   - For each fold k, estimate $\ell(X)$ and $m(X)$ using data outside I_k.
   - Compute cross-fitted residuals for $i \in I_k$:

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3. **Inference:**
   - Construct standard errors using:

$$\hat{V} = \left(\frac{1}{n}\sum \tilde{D}_i^2\right)^{-1} \left(\frac{1}{n}\sum \tilde{D}_i^2 \epsilon_i^2\right) \left(\frac{1}{n}\sum \tilde{D}_i^2\right)^{-1}.$$

   - Confidence Interval:

$$\left[\hat{\beta} \pm 1.96 \cdot \sqrt{\frac{\hat{V}}{n}}\right].$$

Summary and Extensions

Key Takeaways:

- The PLM framework combines flexibility (nonlinearity via $g(X)$) with interpretability (linear effect of D).
- DML ensures valid inference on β despite high-dimensional X.
- Cross-fitting mitigates overfitting from machine learning-based nuisance parameter estimation.

Summary and Extensions

Key Takeaways:

- The PLM framework combines flexibility (nonlinearity via $g(X)$) with interpretability (linear effect of D).
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When PLM Fails:

- DML estimates the best linear predictor (BLP) of $\tilde{Y}$ in terms of $\tilde{D}$, even if PLM does not hold.
- In this case, β captures an approximation of the effect of D on Y.

Content

Defining the Moment Condition

Key Ingredients

- DML is based on the **method-of-moments** framework, targeting a low-dimensional parameter of interest, θ_0, defined via the moment condition:

$$E[\psi(W; \theta_0, \eta_0)] = 0,$$

  where:
  - ψ: Score function.
  - W: Data vector.
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- **Interpretation:** θ_0 is identified when the above equation holds.

Neyman Orthogonality

Key Concept

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- **Importance:** Eliminates first-order bias from errors in the nuisance parameter estimates, $\hat{\eta}$.

Gateaux Derivative

Definition

- The **Gateaux derivative** formalizes sensitivity to small perturbations:

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- **Admissible Directions:** Δ is admissible if $\eta_0 + t\Delta$ stays in the parameter space for small t.

Good Learners for Nuisance Functions

Requirements for High-Quality Learners

- Learners must approximate the true nuisance parameters η_0 well:

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- Cross-validation and careful tuning are critical for robust performance.

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- **Outcome:** Avoids biases arising from overfitting complex machine learning methods.

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Moment Condition for PLM

$$\psi(W;\theta,\eta) = (Y - \ell(X) - \theta(D - m(X)))\,(D - m(X)).$$

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Interpretation: $\psi(W;\theta,\eta)$ generalizes residualization in linear models, enabling robust inference.

Example 2: Doubly Robust IPW

Doubly Robust IPW Estimator

$$\psi(W;\theta,\eta) = (g(1,X) - g(0,X)) + H(D,X)(Y - g(D,X)) - \theta,$$

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4. **Variance and Confidence Intervals:**
   - Estimate the asymptotic variance of $\hat{\theta}$ as:

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   - Confidence interval:

$$[\hat{\theta} \pm z_{1-\alpha/2}\sqrt{\hat{V}/n}].$$