Randomized Experiment

Causal Inference using Machine Learning
Master in Economics, UNT

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The first RCT

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- **Peirce (1885)**: Used random sequencing in psychology to prevent bias from expectations, anticipating randomization principles.
- **Gossett and Fisher (1920s)**: Gossett mentioned random plot placement; Fisher formalized randomization as essential for causal inference.

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- Emphasized the role of assignment mechanisms in calculating causal effects.
- Proposed an estimator for the Variance of the Average Treatment Effect (ATE) in randomized experiments.

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- Emphasized the need for physical randomization to eliminate confounding variables.
- Proposed methods for testing hypotheses in a controlled experimental setup.

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- Together, they laid the groundwork for randomized experiments and causal inference.

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Definition of Assignment Vector D

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- For N units, D is an N-vector where $D_i = d$ if unit i receives the treatment d.
- For two treatment groups, D is a binary vector with 2^N possible values.

Assignment Mechanism

Assignment Mechanism: Given a population of N units, the assignment mechanism is a row-exchangeable function, denoted as $\Pr(D|X, Y(0), Y(1))$, which takes values in the interval $[0, 1]$ and satisfies:

$$\sum_{D \in \{0,1\}^N} \Pr(D|X, Y(0), Y(1)) = 1$$

for all possible values of X (covariates), $Y(0)$, and $Y(1)$ (potential outcomes). (Row-exchangeability implies that the order of units within vectors or matrices is irrelevant to the function $\Pr(\cdot)$.)

Example: Assignment Mechanism with Two Units

Define the **treatment effect** for unit i as: $\tau_i = Y_i(1) - Y_i(0)$

$$\Pr(D|X, Y(0), Y(1)) = \begin{cases} 1 & \text{if } \tau_2 > \tau_1 \text{ and } D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \text{if } \tau_2 < \tau_1 \text{ and } D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{1}{2} & \text{if } \tau_2 = \tau_1 \text{ and } D \in \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \\ 0 & \text{if } D \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \\ 0 & \text{if } \tau_2 < \tau_1 \text{ and } D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 & \text{if } \tau_2 > \tau_1 \text{ and } D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

Unit Assignment Probability

The **unit-level assignment probability** for unit i is defined as:

$$p_i(X, Y(0), Y(1)) = \sum_{D:D_i=1} \Pr(D|X, Y(0), Y(1)),$$

Propensity Score

The **propensity score** at x is the average unit assignment probability for units with $X_i = x$. It is defined as:

$$e(x) = \frac{1}{N(x)} \sum_{i:X_i=x} p_i(X, Y(0), Y(1)),$$

Example: Propensity Score

1. $D = (0, 0, 0, 0)$ $P(D = 1) = 0$
2. $D = (1, 0, 0, 0)$ $P(D = 2) = \frac{3}{16}$
3. $D = (0, 1, 0, 0)$ $P(D = 3) = \frac{2}{16}$
4. $D = (0, 0, 1, 0)$ $P(D = 4) = 0$
5. $D = (0, 0, 0, 1)$ $P(D = 5) = 0$
6. $D = (1, 1, 0, 0)$ $P(D = 6) = \frac{1}{16}$
7. $D = (0, 1, 1, 0)$ $P(D = 7) = \frac{2}{16}$
8. $D = (0, 0, 1, 1)$ $P(D = 8) = \frac{1}{16}$
9. $D = (1, 0, 1, 0)$ $P(D = 9) = \frac{3}{16}$
10. $D = (1, 0, 0, 1)$ $P(D = 10) = \frac{2}{16}$
11. $D = (0, 1, 0, 1)$ $P(D = 11) = \frac{2}{16}$
12. $D = (1, 1, 1, 0)$ $P(D = 12) = 0$
13. $D = (1, 0, 1, 1)$ $P(D = 13) = 0$
14. $D = (0, 1, 1, 1)$ $P(D = 14) = 0$
15. $D = (1, 1, 0, 1)$ $P(D = 15) = 0$
16. $D = (1, 1, 1, 1)$ $P(D = 16) = 0$

Content

Individualistic Assignment

Definition 3.4 (Individualistic Assignment): For some function $q(\cdot) \in [0, 1]$:

$$p_i(X, Y(0), Y(1)) = q(X_i, Y_i(0), Y_i(1)), \quad \text{for all } i = 1, \ldots, N,$$

and

$$\Pr(D|X, Y(0), Y(1)) = c \cdot \prod_{i=1}^{N} q(X_i, Y_i(0), Y_i(1))^{D_i} (1 - q(X_i, Y_i(0), Y_i(1)))^{1-D_i}$$

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- The constant c ensures that the probabilities sum to unity.

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- The constant c ensures that the probabilities sum to unity. **Homework:** Compute the value of c for a generic assignment mechanism with two units and a binary treatment.

Probabilistic Assignment Mechanism

Probabilistic Assignment Mechanism: Under this mechanism, each unit has a non-zero probability of being assigned to either treatment or control, ensuring randomness in the assignment process.

$$0 < \Pr(D_i = 1 | X, Y(0), Y(1)) < 1 \quad \text{for all units } i$$

Unconfounded Assignment Mechanism

Unconfounded Assignment Mechanism: This mechanism assumes that assignment to treatment is independent of the potential outcomes, given the covariates. In other words, the assignment is "as good as random" conditional on covariates.

$$\Pr(D|X, Y(0), Y(1)) = \Pr(D|X)$$

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$$\Pr(D|X, Y(0), Y(1)) = \Pr(D|X)$$

- Given individualistic assignment and unconfoundedness

$$\Pr(D|X, Y(0), Y(1)) = c \cdot \prod_{i=1}^{N} q(X_i)^{D_i} \cdot (1 - q(X_i))^{1-D_i}$$

so that

$$e(x) = q(x)$$

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Randomized Experiment

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- (i) **Individualistic**: Each unit's treatment assignment depends only on its own covariates and potential outcomes, independent of other units.
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Bernoulli Trials

A **Bernoulli trial** is a classical randomized experiment where each unit is independently assigned to treatment or control, often based on a coin toss.

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- Each unit has a probability q of being assigned to treatment and $1 - q$ of being assigned to control.
- Each unit's assignment is independent of others, meaning the assignment for one unit does not affect the assignment for another.
- The assignment mechanism is:
  - **Individualistic**: Each unit's assignment depends only on its own characteristics.
  - **Probabilistic**: Each unit has a non-zero chance of receiving either treatment or control.
  - **Unconfounded**: Given covariates, assignment does not depend on potential outcomes.
  - **Controlled by the Researcher**: The probability q is specified by the researcher.

Bernoulli Trials - Probability of an Assignment Vector

For a Bernoulli trial, the probability of an assignment vector D for N units is given by:

$$\Pr(D|X, Y(0), Y(1)) = \prod_{i=1}^{N} \left(e(X_i)^{D_i} \cdot (1 - e(X_i))^{1-D_i} \right)$$

where:

- $D_i = 1$ if unit i is assigned to treatment, $D_i = 0$ otherwise.
- $e(X_i)$: Propensity Score for unit i.

If $e(X_i) = q = 0.5$, then $\Pr(D|X, Y(0), Y(1)) = 0.5^N$.

Completely Randomized Experiment - Definition

A **completely randomized experiment** assigns a fixed number N_t of units to treatment, and the remaining $N - N_t$ units to control.

- The assignment is achieved by randomly selecting N_t units from a pool of N units.
- Ensures a balanced distribution of treated and control units, with exactly N_t in treatment and $N - N_t$ in control.
- Each unit's assignment is NOT independent of others, but the total number of treated units is fixed by design.
- The assignment mechanism is:
  - **Probabilistic**: Each unit has a positive probability of being selected for treatment or control.
  - **Unconfounded**: Given covariates, assignment does not depend on potential outcomes.
  - **Controlled by the Researcher**: The number N_t of treated units is specified by the researcher.

Completely Randomized Experiment - Probability of an Assignment Vector

In a completely randomized experiment, the probability of an assignment vector D is:

$$\Pr(D|X, Y(0), Y(1)) = \begin{cases} \frac{1}{\binom{N}{N_t}} & \text{if } \sum_{i=1}^{N} D_i = N_t \\ 0 & \text{otherwise} \end{cases}$$

where N_t is the predetermined number of units assigned to treatment.

Stratified Randomized Experiment - Definition

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- Reduces variability and improves the precision of causal inference estimates.
- The goal is to reduce variance in the estimator and increase the power of statistical tests, enhancing the study's ability to detect treatment effects.

Stratified Randomized Experiment - Probability of an Assignment Vector

For a stratified randomized experiment with J blocks, the probability of an assignment vector D is:

$$\Pr(D|X, Y(0), Y(1)) = \prod_{j=1}^{J} \frac{1}{\binom{N(j)}{N_t(j)}}$$

where:

- $N(j)$: Number of units in block j,
- $N_t(j)$: Number of treated units in block j.

Paired Randomized Experiment - Definition

A **paired randomized experiment** is an extreme form of stratified randomization, where each block (or stratum) contains exactly two units.

- Within each pair, one unit is randomly assigned to treatment and the other to control, ensuring direct comparison between similar units.

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- Reduces variance in the estimator by closely aligning treatment and control units.
- Increases statistical power by enhancing the precision of the causal inference, making it easier to detect treatment effects.

Paired Randomized Experiment - Probability of an Assignment Vector

For a paired randomized experiment with $N/2$ pairs, the probability of an assignment vector D is:

$$\Pr(D|X, Y(0), Y(1)) = 2^{-\frac{N}{2}}$$

- Each unit within a pair has an equal probability of being assigned to treatment or control.

Number of Possible Values for the Assignment Vector by Design and Sample Size

Type of Experiment and Design	Number of Possible Assignments	Number of Units (N) in Sample			
		4	**8**	**16**	**32**
Bernoulli trial	2^N	16	256	65,536	4.2×10^9
Completely randomized experiment	$\binom{N}{N/2}$	6	70	12,870	0.6×10^9
Stratified randomized experiment	$\binom{N/2}{N/4}^2$	4	36	4,900	0.2×10^9
Paired randomized experiment	$2^{N/2}$	4	16	256	65,536

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$$Y(1)_i = Y(1,1)_i = Y(1,0)_i \quad \text{for } i = 1, 2$$

Example: Under SUTVA, the potential outcomes for two units (1 and 2) would be consistent regardless of others' treatment status:

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- **Nonparametric Approach:** This method makes no assumptions about the distribution of the test statistic under the null hypothesis.(as t-test or ANOVA)
- **Flexibility** Both $H0$ and test statistic can be defined in various ways, making the method widely applicable.

Table 5.3: Cough Frequency for the First Six Units from the Honey Study

Unit	**Potential Outcomes**		**Observed Variables**		
	$Y_i(0)$	$Y_i(1)$	D_i	X_i	$Y_{\mathbf{obs}}$
1		3	1	4	3
2		5	1	6	5
3		0	1	4	0
4	4		0	4	4
5	0		0	1	0
6	1		0	5	1

Steps for the Exact Test

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3. Compute the observed $T_{\text{diff, obs}}$ from the data.
4. Compute $T_{\text{diff},k}$ for all possible assignment vectors D_k.
5. Compute the p-value: approximate the p-value by the fraction of these K statistics that are as extreme as, or more extreme than, the observed $T_{\text{diff, obs}}$:

$$p = \frac{1}{K} \sum_{k=1}^{K} 1\left\{ T_{\text{diff},k} \geq T_{\text{diff, obs}} \right\}$$

Homework: Fisher Exact P-Value

1. Take the class and compute the Fisher Exact P-Value as before.
2. Modify the test statistic and compute the rank statistic instead. Compare results
3. Modify the null hypothesis and test the null hypothesis that the unit treatment effect is 10% Compare results
4. Compute Fisher Exact P-value using covariates (pp78 CIS)

Content

Definition of Average Treatment Effect (ATE)

The **Average Treatment Effect (ATE)** is defined as:

$$\tau_{fs} = \frac{1}{N}\sum_{i=1}^{N}[Y_i(1) - Y_i(0)] = \overline{Y}(1) - \overline{Y}(0),$$

where:

$$\overline{Y}(1) = \frac{1}{N}\sum_{i=1}^{N} Y_i(1), \quad \overline{Y}(0) = \frac{1}{N}\sum_{i=1}^{N} Y_i(0).$$

Definition of $\hat{\tau}_{\text{diff}}$

Define the estimator $\hat{\tau}_{\text{diff}}$ as the difference of the sample means:

$$\hat{\tau}_{\text{diff}} = \overline{Y}_{\text{obs},t} - \overline{Y}_{\text{obs},c},$$

where:

$$\overline{Y}_{\text{obs},t} = \frac{1}{N_t} \sum_{i:W_i=1} Y_i^{\text{obs}}, \quad \overline{Y}_{\text{obs},c} = \frac{1}{N_c} \sum_{i:W_i=0} Y_i^{\text{obs}}.$$

Theorem: Unbiasedness of $\hat{\tau}_{\text{diff}}$

Theorem: The estimator $\hat{\tau}_{\text{diff}}$ is an unbiased estimator of the average treatment effect τ_{fs}.

Proof:

Definition of Sampling Variance of the Neyman Estimator

The sampling variance of $\hat{\tau}_{\text{diff}}$ is:

$$V_W\left(\hat{\tau}_{\text{diff}}\right) = \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} - \frac{S_{tc}^2}{N},$$

where:

$$S_c^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i(0) - \overline{Y}(0)\right)^2,$$

$$S_t^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i(1) - \overline{Y}(1)\right)^2,$$

$$S_{tc}^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left([Y_i(1) - Y_i(0)] - \tau_{fs}\right)^2.$$

Theorem 6.2

Theorem 6.2: The sampling variance of $\hat{\tau}_{\text{diff}}$ is given by:

$$V_W\left(\hat{\tau}_{\text{diff}}\right) = \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} - \frac{S_{tc}^2}{N}.$$

Intuition:

Theorem 6.3

Theorem 6.3: If the treatment effect $Y_i(1) - Y_i(0)$ is constant across units, then an unbiased estimator for the sampling variance is:

$$\hat{V}_{\text{Neyman}} = \frac{\hat{s}_c^2}{N_c} + \frac{\hat{s}_t^2}{N_t},$$

where s_c^2 and s_t^2 are sample variances calculated from the observed data.

Estimation:

Confidence Intervals

To construct a $(1-\alpha)\times 100\%$ confidence interval for τ_{fs}, we use:

$$\mathsf{CI}_{1-\alpha}(\tau_{fs}) = \left[\hat{\tau}_{\mathsf{diff}} + z_{\alpha/2}\sqrt{\hat{V}},\quad \hat{\tau}_{\mathsf{diff}} + z_{1-\alpha/2}\sqrt{\hat{V}}\right],$$

where $z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution.

Testing

To test the null hypothesis $H_0 : \tau_{fs} = 0$ against the alternative $H_a : \tau_{fs} \neq 0$, we compute the test statistic:

$$t = \frac{\hat{\tau}_{\text{diff}}}{\sqrt{\hat{V}}}.$$

We compare t to the critical values from the standard normal distribution.

Content

Classical Additive Approach: Improving Precision Under Linearity

We start with the assumption that the conditional expectation function is exactly linear:

$$\mathbb{E}[Y \mid D, W] = D\alpha + \beta^\top X,$$

where:

- D is the treatment indicator.
- W represents pre-treatment covariates.
- $X = (1, W^\top)^\top$ includes an intercept and W.

Centered Covariates and Covariate Balance

We assume that the covariates are centered:

$$\mathbb{E}[W] = 0.$$

By the assumption of covariate balance in randomized experiments:

$$\mathbb{E}[W \mid D = 1] = \mathbb{E}[W \mid D = 0].$$

Average Treatment Effect under Linearity

Using centered covariates, we have:

$$\mathbb{E}[Y(0)] = \mathbb{E}\left[\mathbb{E}[Y \mid D = 0, X]\right] = \beta_1,$$
$$\mathbb{E}[Y(1)] = \mathbb{E}\left[\mathbb{E}[Y \mid D = 1, X]\right] = \beta_1 + \alpha.$$

Therefore, the Average Treatment Effect (ATE) is:

$$\delta = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)] = \alpha.$$

Statistical Inference on the ATE

Even without assuming linearity, the projection coefficient α recovers the ATE δ.
Under regularity conditions, the Ordinary Least Squares (OLS) estimators satisfy:

$$\sqrt{n}\begin{pmatrix}\hat{\alpha}-\alpha\\ \hat{\beta}_1-\beta_1\end{pmatrix} \overset{approx}{\sim} \mathcal{N}(0, V),$$

where the covariance matrix V has components:

$$V_{11} = \frac{\mathbb{E}[\epsilon^2\tilde{D}^2]}{\left(\mathbb{E}[\tilde{D}^2]\right)^2},$$

$$V_{22} = \frac{\mathbb{E}[\epsilon^2\tilde{1}^2]}{\left(\mathbb{E}[\tilde{1}^2]\right)^2},$$

$$V_{12} = V_{21} = \frac{\mathbb{E}[\epsilon^2\tilde{D}\tilde{1}]}{\mathbb{E}[\tilde{D}^2]\mathbb{E}[\tilde{1}^2]},$$

with:

Relative Average Treatment Effect

We can also perform inference on the Relative ATE α/β_1.
Using the Delta Method, we have:

$$\sqrt{n}\left(\frac{\hat{\alpha}}{\hat{\beta}_1} - \frac{\alpha}{\beta_1}\right) \overset{approx}{\sim} \mathcal{N}\left(0, G^\top V G\right),$$

where:

$$G = \begin{pmatrix} 1/\beta_1 \\ -\alpha/\beta_1^2 \end{pmatrix}.$$

Improvement in Precision Under Linearity

When we do not include covariates, the OLS estimator $\bar{\alpha}$ estimates α in:

$$Y = \alpha D + \beta_1 + U, \quad \mathbb{E}[U] = \mathbb{E}[UD] = 0,$$

where:

$$U = \beta^\top (X - \mathbb{E}[X]) + \epsilon.$$

Under the linear model, including covariates improves precision:

$$V_{11} \leq \bar{V}_{11},$$

where $\bar{V}_{11}$ is the variance when covariates are omitted.

Limitations Without Linearity

Without the linearity assumption, the improvement in precision is not guaranteed.

- The variance V_{11} and $\bar{V}_{11}$ are not generally comparable.
- Including covariates may increase the standard errors if the linear model is misspecified.

The Interactive Approach: Capturing Heterogeneity

Consider the interactive linear model:

$$\mathbb{E}[Y \mid D, W] = \alpha^\top XD + \beta^\top X.$$

Here, the Conditional Average Treatment Effect (CATE) is:

$$\delta(W) = \mathbb{E}[Y(1) \mid W] - \mathbb{E}[Y(0) \mid W] = \alpha^\top X.$$

The Average Treatment Effect (ATE) is:

$$\delta = \mathbb{E}[\delta(W)] = \alpha_1,$$

where α_1 is the first component of α.

Estimation and Inference with Interactions

The coefficient α is estimated from the linear projection:

$$Y = \alpha^\top (DX) + \beta^\top X + \epsilon, \quad \epsilon \perp (X, DX).$$

We can treat DX as a vector of technical treatments and apply standard inference methods.

Advantages of the Interactive Approach

- Always delivers improvements in precision for estimating the ATE δ, even if the linearity assumption does not hold.
- Allows for the discovery of treatment effect heterogeneity.

This result was demonstrated by Lin (2013).

Conclusion

- Including pre-treatment covariates can improve the precision of ATE estimates in RCTs.
- The classical additive approach relies on linearity and may not always improve precision.
- The interactive approach, which includes interactions between treatment and covariates, always improves precision and uncovers heterogeneity.

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