a) Teniendo la ecuación diferencial ordinaria

$$\frac{d^2y}{dx^2} - R(x)y = S(x)$$

Expandiendo en series de Taylor se tiene que

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y^{III}(x) + \frac{h^4}{24}y^{IV}(x) + \frac{h^5}{5!}y^{V}(x) + \mathcal{O}(h^6)$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2}y''(x) - \frac{h^3}{6}y^{III}(x) + \frac{h^4}{24}y^{IV}(x) - \frac{h^5}{5!}y^{V}(x) + \mathcal{O}(h^6)$$

$$y(x+h) + y(x-h) = 2y(x) + h^2y''(x) + \frac{h^4}{12}y^{IV}(x) + \mathcal{O}(h^6)$$

$$y(x+h) - 2y(x) + y(x-h) = \left[y''(x) + \frac{h^2}{12}y^{IV}(x)\right]h^2 + \mathcal{O}(h^6)$$

$$y_{n+1} - 2y_n + y_{n-1} = \left[y_n'' + \frac{h^2}{12}y_n^{IV}\right]h^2 + \mathcal{O}(h^6)$$

Fijémonos que la ecuación diferencial original se puede discretizar de modo que quede de la forma

$$y_n'' = \frac{y_{n+1} - 2y + y_{n-1}}{h^2} = R_n y_n + S_n$$

$$y_n^{IV} = \frac{y''_{n+1} - 2y'' + y''_{n-1}}{h^2} = \frac{R_{n+1}y_{n+1} + S_{n+1} - 2(R_n y_n + S_n) + R_{n-1}y_{n-1} + S_{n-1}}{h^2}$$

$$y_{n+1} - 2y_n + y_{n-1} = \left[R_n y_n + S_n + \frac{h^2}{12}y_n^{IV}\right]h^2 + \mathcal{O}(h^6)$$

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} \\ &= \Big[R_n y_n + S_n \\ &\quad + \frac{1}{12}(R_{n+1}y_{n+1} + S_{n+1} - 2(R_n y_n + S_n) + R_{n-1}y_{n-1} + S_{n-1})\Big]h^2 + \mathcal{O}(h^6) \end{aligned}$$

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} \\ &= \left[\left(\frac{R_{n+1}}{12}\right)y_{n+1} + \left(R_n - \frac{1}{6}R_n\right)y_n + \left(\frac{1}{12}R_{n-1}\right)y_{n-1} + \frac{S_{n+1}}{12} + S_n - \frac{1}{6}S_n \right. \\ &\quad \left. + \frac{1}{12}S_{n-1}\right]h^2 + \mathcal{O}(h^6) \end{aligned}$$

Despejando se llega finalmente a:

$$\left(1-\frac{h^2R_{n+1}}{12}\right)y_{n+1}+\left(-2-\frac{5h^2}{6}R_n\right)y_n+\left(1-\frac{h^2}{12}R_{n-1}\right)y_{n-1}$$
$$=\frac{h^2}{12}(S_{n+1}+10S_n+S_{n-1})+\ \mathcal{O}(h^6)$$
$$\left(1-\frac{h^2}{12}R_{n+1}\right)y_{n+1}-2\left(1+\frac{5h^2}{12}R_n\right)y_n+\left(1-\frac{h^2}{12}R_{n-1}\right)y_{n-1}$$
$$=\frac{h^2}{12}(S_{n+1}+10S_n+S_{n-1})+\ \mathcal{O}(h^6)$$

b) Teniendo la ecuación de Schrödinger para el oscilador armónico cuántico

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$

Comparando con $\frac{d^2y}{dx^2}-R(x)y=S(x)$

$$\frac{d^2\psi}{dx^2}-\frac{2m}{\hbar^2}\ V(x)\psi=-\frac{2m}{\hbar^2}E\psi$$
$$\frac{d^2\psi}{dx^2}=\frac{2m}{\hbar^2}(\ V(x)-E)\psi$$

Se puede afirmar que

$$-\frac{2m}{\hbar^2}(E-V(x))=R(x)$$
$$\frac{2m}{\hbar^2}\left(\frac{1}{2}m\omega^2x^2-E\right)=R_n$$
$$S_n=0$$