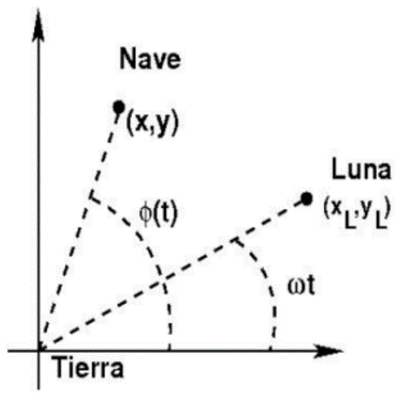
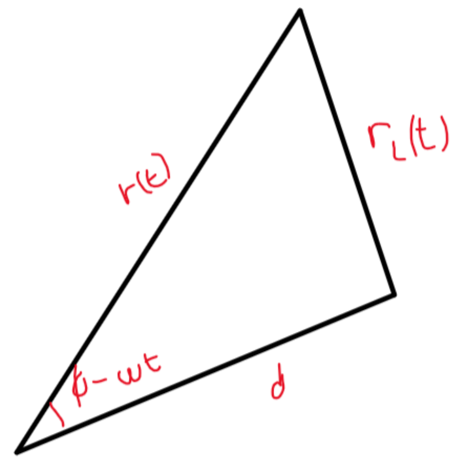
c) Se tiene que





Donde la distancia r_L se puede encontrar mediante la ley de cosenos

$$r_L(r,\phi,t\,)^2 = r(t)^2 + d^2 - 2r(t)d\cos(\phi - \omega t)$$

$$r_L(r,\phi,t) = \sqrt{r(t)^2 + d^2 - 2r(t)d\cos(\phi - \omega t)}$$

d)

Se tiene que la posición de la nave es

$$\vec{r} = (x,y) = (r\cos\phi\,, r\sin\phi)$$

$$\dot{\vec{r}} = \left(\dot{r}\cos\phi - r\dot{\phi}\ \sin\phi\,, \dot{r}\sin\phi + r\dot{\phi}\ \cos\phi\right)$$

$$|\dot{\boldsymbol{r}}|^2 = \left(\dot{r}\cos\phi - r\dot{\phi}\sin\phi\right)^2 + \left(\dot{r}\sin\phi + r\dot{\phi}\cos\phi\right)^2$$

$$|\dot{\boldsymbol{r}}|^2 = \dot{r}^2\cos^2\phi - 2\dot{r}r\dot{\phi}\cos\phi\ \sin\phi + r^2\dot{\phi}^2\ \sin\phi^2 + \dot{r}^2\sin^2\phi + 2\dot{r}r\dot{\phi}\sin\phi\cos\phi + r^2\dot{\phi}^2\cos^2\phi$$

$$|\dot{\boldsymbol{r}}|^2 = \dot{r}^2(\cos^2\phi + \sin^2\phi) + r^2\dot{\phi}^2(\cos^2\phi + \sin^2\phi)$$

$$|\dot{\boldsymbol{r}}|^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

Entonces la energía cinética es de la forma

$$T = \frac{1}{2}m|\dot{\boldsymbol{r}}|^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

Mientras la energía potencial es

$$U = -G\left(\frac{mm_T}{r} + \frac{mm_L}{r_L}\right)$$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + G\left(\frac{mm_T}{r} + \frac{mm_L}{r_L}\right)$$

De igual forma los momentos generalizados serán

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} = p_r$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\dot{\phi}r^2 = p_{\phi}$$

Además, el Hamiltoniano es definido como $H = \sum p_{q_i}\dot{q}_i - \mathcal{L}$

$$H = m\dot{r}^2 + m\dot{\phi}^2r^2 - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - G\left(\frac{mm_T}{r} + \frac{mm_L}{r_L}\right)$$

$$H = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{\phi}^2r^2 - G\left(\frac{mm_T}{r} + \frac{mm_L}{r_L}\right)$$

Veamos que con las definiciones de los momentos generalizados de $m\dot{r} = p_r$ y $m\dot{\phi}r^2 = p_{\phi}$ se puede reescribir como

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - G\left(\frac{mm_T}{r} + \frac{mm_L}{r_L}\right)$$

e) Las ecuaciones de movimiento entonces son

$$\frac{\partial H}{\partial P_r} = \dot{r} = \frac{\partial}{\partial p_r}\left(\frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - G\left(\frac{mm_T}{r} + \frac{mm_L}{r_L}\right)\right) = \frac{p_r}{m}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \frac{\partial}{\partial p_\phi}\left(\frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - G\left(\frac{mm_T}{r} + \frac{mm_L}{r_L}\right)\right) = \frac{p_\phi}{mr^2}$$

$$\dot{p_r} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - G\frac{mm_T}{r^2} + G\frac{\partial}{\partial r}\left(\frac{mm_L}{\sqrt{r^2 + d^2 - 2rd\cos(\phi - \omega t)}}\right)$$

$$\frac{\partial}{\partial r}\left(\frac{mm_L}{\sqrt{r^2 + d^2 - 2rd\cos(\phi - \omega t)}}\right) = -\frac{1}{2}\frac{mm_L[2r - 2d\cos(\phi - \omega t)]}{[r^2 + d^2 - 2rd\cos(\phi - \omega t)]^{\frac{3}{2}}}$$
$$= -\frac{mm_L}{r_L^3}[r - d\cos(\phi - \omega t)]$$

$$\dot{p_r} = \frac{p_\phi^2}{mr^3} - G\frac{mm_T}{r^2} - G\frac{mm_L}{r_L^3}[r - d\cos(\phi - \omega t)]$$

$$\dot{p_\phi} = -\frac{\partial H}{\partial \phi} = G\frac{\partial}{\partial \phi}\left(\frac{mm_L}{\sqrt{r^2+d^2-2rd\cos(\phi-\omega t)}}\right)$$

$$= G\left[-\frac{1}{2}\frac{mm_L 2rd\sin(\phi-\omega t)}{[r^2+d^2-2rd\cos(\phi-\omega t)]^3}\right]$$

$$\dot{p_\phi} = -G\frac{mm_L}{r_L^3}rd\sin(\phi-\omega t)$$

f) definiendo $\tilde{r} = \frac{r}{d}$, $\widetilde{p_r} = \frac{p_r}{md}$, $\widetilde{p_\phi} = \frac{p_\phi}{md^2}$

Dividiendo la primera ecuación de movimiento por d se obtiene

$$\frac{\dot{r}}{d} = \frac{p_r}{md} \to \dot{\tilde{r}} = \tilde{p}_r$$

$$\dot{\phi} = \frac{p_\phi}{mr^2} = \tilde{p}_\phi\frac{d^2}{r^2} = \frac{\tilde{p}_\phi}{\tilde{r}^2}$$

$$\frac{\dot{p_r}}{md} = \frac{p_\phi^2}{mr^3}\frac{1}{md} - \frac{1}{md}G\frac{mm_T}{r^2} - \frac{1}{md}G\frac{mm_L}{r_L^3}[r-d\cos(\phi-\omega t)]$$

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_\phi^2}{r^3}d^3 - \frac{d^2}{d^2}\frac{1}{d}G\frac{m_T}{r^2} - \frac{1}{md}G\frac{mm_L}{[r^2+d^2-2rd\cos(\phi-\omega t)]^{\frac{3}{2}}}[r-d\cos(\phi-\omega t)]$$

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - G\frac{m_T}{\tilde{r}^2 d^3} - G\frac{m_L}{[r^2+d^2-2rd\cos(\phi-\omega t)]^{\frac{3}{2}}}[\tilde{r}-\cos(\phi-\omega t)]$$

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - G\frac{m_T}{\tilde{r}^2 d^3} - G\frac{\frac{m_L}{d^3}}{\frac{[r^2+d^2-2rd\cos(\phi-\omega t)]^{\frac{3}{2}}}{d^3}}[\tilde{r}-\cos(\phi-\omega t)]$$

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - G\frac{m_T}{\tilde{r}^2 d^3} - G\frac{\frac{m_L}{d^3}m_T[\tilde{r}-\cos(\phi-\omega t)]}{\left(\left(\frac{r^2+d^2-2rd\cos(\phi-\omega t)}{d^2}\right)^{\frac{1}{2}}\right)^3 m_T}$$

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - G\frac{m_T}{\tilde{r}^2 d^3} - G\frac{\frac{m_L}{d^3}m_T[\tilde{r}-\cos(\phi-\omega t)]}{\left((\tilde{r}^2+1-2\tilde{r}\cos(\phi-\omega t))^{\frac{1}{2}}\right)^3 m_T}$$

Con

$$\Delta \equiv \frac{Gm_T}{d^3};\;\; \mu \equiv \frac{m_L}{m_T};\;\;\; \tilde{r}' = \sqrt{1+\tilde{r}^2-2\tilde{r}\cos(\phi-\omega t)}$$

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - \frac{\Delta}{\tilde{r}^2} - \frac{\Delta\mu[\tilde{r} - \cos(\phi - \omega t)]}{\tilde{r}'^3}$$

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - \Delta\left\{\frac{1}{\tilde{r}^2} + \frac{\mu}{\tilde{r}'^3}[\tilde{r} - \cos(\phi - \omega t)]\right\}$$

$$\frac{\dot{p_\phi}}{md^2} = -G\frac{m_L}{r_L^3 d^2} rd\sin(\phi - \omega t)$$

$$\dot{\tilde{p}}_\phi = -\left(\frac{m_T}{m_T}\right)G\frac{m_L}{r_L^3}\tilde{r}\sin(\phi - \omega t) = -G\frac{\mu m_T}{r_L^3}\tilde{r}\sin(\phi - \omega t)$$

$$\dot{\tilde{p}}_\phi = -G\frac{\frac{\mu m_T}{d^3}}{\left(\left(\frac{r^2 + d^2 - 2rd\cos(\phi - \omega t)}{d^2}\right)^{\frac{1}{2}}\right)^3}\tilde{r}\sin(\phi - \omega t)$$

$$\dot{\tilde{p}}_\phi = -\frac{\Delta\mu\tilde{r}}{\left((\tilde{r}^2 + 1 - 2\tilde{r}\cos(\phi - \omega t))^{\frac{1}{2}}\right)^3}\sin(\phi - \omega t)$$

$$\dot{\tilde{p}}_\phi = -\frac{\Delta\mu\tilde{r}}{\tilde{r}'^3}\sin(\phi - \omega t)$$

g) Finalmente para asignar los momentos canónicos originales se demuestra que

Como se sabe p_r no es mas que el momento lineal, entonces

$$r_0 = (r_0\cos\phi, r_0\sin\phi);\ \dot{r}_0 = (v_0\cos\theta, v_0\sin\theta)$$

$$p_r = m\frac{dr}{dt} = m\frac{d}{dt}\sqrt{x^2 + y^2}$$

$$\tilde{p}_r^0 = \frac{p_r^0}{md} = \frac{m}{md}\frac{dr_0}{dt} = \frac{1}{d}\frac{d}{dt}\left(\sqrt{x_0^2 + y_0^2}\right) = \frac{2x_0\dot{x}_0 + 2y_0\dot{y}_0}{2d\sqrt{x_0^2 + y_0^2}} = \frac{x_0\dot{x}_0 + y_0\dot{y}_0}{dr_0}$$

$$\tilde{p}_r^0 = \frac{r_0\cos\phi\, v_0\cos\theta + r_0\sin\phi\, v_0\sin\theta}{dr_0} = \frac{\cos\phi\, v_0\cos\theta + v_0\sin\phi\sin\theta}{d}$$

$$= \tilde{v}_0(\cos\phi\, v_0\cos\theta + \sin\phi\sin\theta)$$

$$\tilde{p}_r^0 = \tilde{v}_0\cos(\theta - \phi)$$

Como $\arctan\left(\frac{y}{x}\right) = \phi$ y p_ϕ no es más que el momento angular, que sigue entonces la forma $p_\phi = \frac{d\phi}{dt}I$. Para una partícula $I = mr^2$

$$\tilde{p}_\phi^0 = \frac{p_\phi^0}{md^2} = \frac{1}{md^2} mr^2 \frac{d\phi}{dt} = \tilde{r}^2 \frac{d}{dt}\left(\arctan\left(\frac{y}{x}\right)\right) = \tilde{r}^2 \frac{1}{\left(\frac{y}{x}\right)^2 + 1} \frac{d}{dt}\left(\frac{y}{x}\right)$$

$$= \tilde{r}^2 \frac{1}{\left(\frac{y}{x}\right)^2 + 1} \frac{\dot{y}x - \dot{x}y}{x^2} = \tilde{r}^2 \frac{1}{y^2 + x^2}(\dot{y}x - \dot{x}y) = \frac{\tilde{r}^2}{r^2}(\dot{y}x - \dot{x}y)$$

$$\frac{1}{d^2}(v_0 r_0 \sin\theta \cos\phi - v_0 r_0 \cos\theta \sin\phi)$$

$$\tilde{p}_\phi^0 = \tilde{v}_0 \tilde{r}_0 (\sin\theta \cos\phi - \cos\theta \sin\phi) = \tilde{v}_0 \tilde{r}_0 \sin(\theta - \phi)$$