3. Demuestre la Ecuacion 4.50

$$U_{i,j}^{\ell+1} = v^2 \Big[U_{i+1,j}^{\ell} - 2\, U_{i,j}^{\ell} + U_{i-1,j}^{\ell} + \frac{\Delta p}{p[i]} (U_{i,j}^{\ell} - U_{i-1,j}^{\ell}) + \left(\frac{\lambda}{p[i]}\right)^2 (U_{i,j+1}^{\ell} - 2\, U_{i,j}^{\ell} + U_{i,j-1}^{\ell}) \Big] + 2\, U_{i,j}^{\ell} - U_{i,j}^{\ell-1}$$

Ahora, la ecuación de onda en coordenadas cilíndricas:

$$\frac{\partial^2 U}{\partial t^2} = c^2 \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} \right)$$

Entonces podemos discretizar la ecuación con lo siguiente:

✓ Temporalmente

$$\frac{\partial^2 U}{\partial t^2} = \frac{U_{i,j}^{\ell+1} - 2\, U_{i,j}^{\ell} + U_{i,j}^{\ell-1}}{(\Delta t)^2}$$

✓ Espacialmente

$$\frac{\partial^2 U}{\partial r^2} = \frac{U_{i+1,j}^{\ell} - 2\, U_{i,j}^{\ell} + U_{i-1,j}^{\ell}}{(\Delta p)^2}$$

$$\frac{\partial U}{\partial r} = \frac{1}{p[i]\Delta p} (U_{i+1,j}^{\ell} - U_{i-1,j}^{\ell})$$

$$\frac{\partial^2 U}{\partial \phi^2} = \frac{1}{p[i]^2 (\Delta \phi)^2} (U_{i,j+1}^{\ell} - 2\, U_{i,j}^{\ell} + U_{i,j-1}^{\ell})$$

DD MM AA

Entonces

$$\frac{\overbrace{U_{i,j}^{l+1} - 2U_{i,j}^{l} + U_{i,j}^{l-1}}^{1}}{(\Delta t)^2} = c^2 \left[\overbrace{\frac{U_{i+1,j}^{l} - 2U_{i,j}^{l} + U_{i-1,j}^{l}}{(\Delta \rho)^2}}^{2} + \frac{1}{\rho[i]\Delta\rho} \overbrace{(U_{i+1,j}^{l} - U_{i-1,j}^{l})}^{3} \right.$$

$$\left. + \frac{1}{\rho[i]^2(\Delta\phi)^2} \underbrace{(U_{i,j+1}^{l} - 2U_{i,j}^{l} + U_{i,j-1}^{l})}_{4} \right]$$

$V = c\,\frac{\Delta t}{\Delta \rho}$ $\lambda = \frac{\Delta \rho}{\Delta \phi}$ y multiplicando por $(\Delta t)^2$

$$1 = V^2 \left[2 + \frac{\Delta\rho}{\rho[i]}\, 3 + \frac{\lambda^2}{\rho[i]^2}\, 4 \right]$$

Entonces podemos expresarlo como

$$U_{i,j}^{l+1} = V^2 \left[U_{i+1,j}^{l} - 2U_{i,j}^{l} + U_{i-1,j}^{l} + \frac{\Delta\rho}{\rho[i]} (U_{i,j}^{l} - U_{i-1,j}^{l}) \right.$$

$$\left. + \frac{\lambda^2}{\rho[i]^2} (U_{i,j+1}^{l} - 2U_{i,j}^{l} + U_{i,j-1}^{l}) \right] + 2U_{i,j}^{l} - U_{i,j}^{l-1}$$

Lo que coincide con la ecuación 4,50 ✓