Almost Real Trajectories — Linear Drag

For very low velocities (before turbulence sets in) the drag force $\vec{\boldsymbol{D}}$ on an object moving through a fluid [*e.g.*, air] is proportional to the velocity $\vec{\boldsymbol{v}}$ of the object and (as always, for friction) directed opposite to the motion:

$$\vec{\boldsymbol{D}} \;=\; -\kappa\,\vec{\boldsymbol{v}}. \tag{1}$$

This vector equation separates into two scalar equations for the horizontal (x) components and vertical (y) components:

$$D_x \;=\; -\kappa\,v_x \qquad \text{and} \qquad D_y \;=\; -\kappa\,v_y. \tag{2}$$

Combined with a constant gravitational force mg in the downward ($-y$) direction, this allows us to separate Newton's Second Law ($\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) into independent horizontal and vertical equations:

$$ma_x \;=\; -\kappa\,v_x \qquad \text{and} \qquad ma_y \;=\; -mg\;-\kappa\,v_y. \tag{3}$$

Dividing through by m and noting that $a \equiv \frac{dv}{dt}$, we get

$$\frac{dv_x}{dt} \;=\; -k\,v_x \qquad \text{and} \qquad \frac{dv_y}{dt} \;=\; -g\;-k\,v_y, \qquad \text{where} \qquad k \equiv \frac{\kappa}{m}. \tag{4}$$

The solutions to these equations are

$$v_x(t) \;=\; v_{x_0}\,e^{-kt} \qquad\qquad \text{and} \tag{5}$$

$$v_y(t) \;=\; -v_f \;+\; (v_{y_0} + v_f)e^{-kt} \tag{6}$$

$$\text{where} \qquad v_f \equiv \frac{g}{k} \qquad \text{is the } \textbf{terminal velocity}. \tag{7}$$

These equations can also be "solved" for the horizontal and vertical *positions* as functions of time,

$$x(t) \;=\; x_0 \;+\; \frac{v_{x_0}}{k}\left(1 - e^{-kt}\right) \qquad \text{and} \tag{8}$$

$$y(t) \;=\; y_0 \;-\; v_f t \;+\; \left(\frac{v_{y_0} + v_f}{k}\right)\left[1 - e^{-kt}\right]. \tag{9}$$

You can check these results yourself by taking the time derivatives.

Note: the *truth* ($D \propto v^2$) is even worse! Such problems in the real world are usually treated numerically by a computer, stepping through many small increments of position.

Linear Drag: $v(x)$

$$\text{We have} \quad a \;=\; -k\,v \quad \text{so} \quad adx \;=\; -k\,v\,dx$$

$$\text{But in general} \quad a \equiv \frac{dv}{dt} \quad \text{and} \quad v \equiv \frac{dx}{dt} \quad \Longrightarrow \quad dt = \frac{dv}{a} \quad \text{and} \quad dt = \frac{dx}{v} \quad \text{so}$$

$$\frac{dv}{a} = \frac{dx}{v} \quad \text{or} \quad \boxed{v\,dv \;=\; a\,dx} \tag{10}$$

$$\text{Therefore} \quad adx \;=\; vdv \;=\; -k\,v\,dx \quad \text{so} \quad dv \;=\; -k\,dx$$

$$\text{which integrates easily to} \quad \boxed{v \;-\; v_0 \;=\; -k\,(x \;-\; x_0)}$$

— *i.e.*, the velocity drops off *linearly* with distance travelled when a linear drag force acts.

This can also be shown by noting that

$$v(t) \;=\; v_0\,e^{-kt} \quad \Longrightarrow \quad e^{-kt} \;=\frac{v}{v_0}$$

and substituting for e^{-kt} in the equation for $x(t)$.