# LISTA PAZ E AMOR

Coletânea de Problemas Olímpicos de Astronomia

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CHAPTER

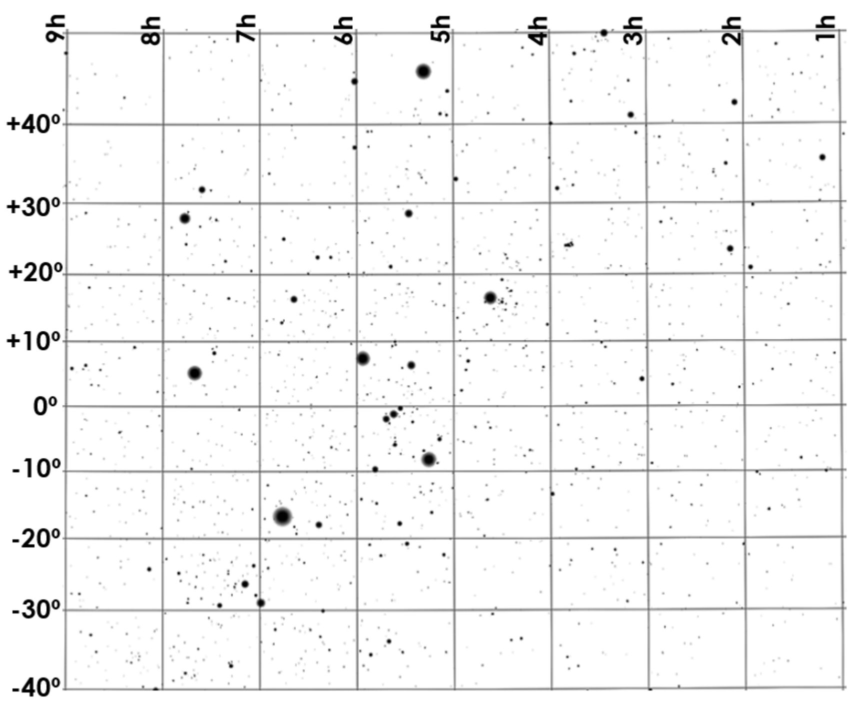
# 1

# PROBLEMS

## 1.1 Multiplayer Astronomy

Johannes Kleber travels to Asunción (25° 16' 55" S; 57° 38' 06" W; GMT: -4h) in Paraguay on February 21st to buy products for his astronomy store. In the past, he even bought two altazimuth telescopes, one of which he sold to Nill, his friend, and the other he kept for himself. Upon arrival, with nothing to do on a night, at 21h1min37s, he observes the sky and finds an interesting object at an altitude of 40°. At the same moment, he contacts Nill (who was in Imperatriz - Brazil (5° 31' 33" S; 47° 28' 33" W; GMT: -3h)) who coincidentally was also observing the object (both inaugurating their new telescopes). Nill tells Kleber that he observed the hour angle of the object was 1h15min. Excited by the discovery, they return to looking through their respective telescopes to find more information, but they realize that their telescopes were broken. Sad but curious, they think for a while, with the information exchanged earlier, and both conclude that the object is:

- Consider that the sidereal time in Greenwich on January 1st, 2021 at 00:00 is 6h43min28.5s.
- Consider the celestial chart below:



## 1.2 M666

The great astronomer José Lagranja discovered the star cluster M666.

### Part A: Photometry

**A.1)** The apparent magnitude of the i-th star is $m_i$. Show that the apparent magnitude of the cluster is determined as:

$$M = -\frac{5}{2}\log\left(\sum 10^{-\frac{2}{5}m_i}\right)$$

**A.2)** Lagranja's assistant, Frederico no Grauss, noticed that the apparent stellar magnitudes are in an arithmetic progression, starting from star 1 ($m_1 = m$), with a common difference $\Delta m > 0$. Knowing that the number of stars is very large (practically infinite), show that:

$$M = m + \frac{5}{2}\log\left(1 - 10^{-\frac{2}{5}\Delta m}\right)$$

**A.3)** Show that for $\Delta m \ll 1$:

$$M \approx m + \frac{5}{2}\log\left(\frac{2}{5}\ln(10)\Delta m\right)$$

**A.4)** Show that for $\Delta m \gg 1$:

$$M \approx m - \frac{5}{2\ln(10)}10^{-\frac{2}{5}\Delta m}$$

Hint: Use Taylor approximations for the appropriate ranges:

$$10^x \approx 1 + \ln(10)x$$

$$\ln(1+x) \approx x$$

### Part B: Physical Analysis

Sapphire, a young astronomer, was greatly excited by the discovery of the new cluster and set out to find more information about the object. With her detection tools, she managed to find the frequency $H_\alpha$ of each component of the cluster, denoted as $f_1, f_2, ..., f_i, ..., f_n$. Knowing that the velocities of the stellar components were non-relativistic, answer the following:

**B.1)** Find the recession velocity of the cluster $v_{cl}$ knowing that the rest frequency of the $H_\alpha$ line is $f_0$.

It is known that the cluster is spherical and isotropic in terms of velocity distribution and homogeneous in matter distribution.

**B.2)** Considering that all component stars have equal masses, find the total kinetic energy of the cluster with respect to its center of mass.

**B.3)** Using the Virial theorem, find the mass of the cluster, knowing that its radius is $R$.

## 1.3 Chaotic Observation

Nill das Graças was obsessed with observing the Solar System. Once, he decided to observe the movement of a specific object that orbited the sun periodically without orbital interferences. He made several measurements, all done while the Earth was at its perihelion, in an attempt to describe this orbit. For this, he used his favorite unit of measurement: the Guanômetro (Gm). He organized all his measurements (which were very precise, by the way) into coordinate pairs centered on the Earth, i.e., Earth = (0,0). It is worth noting that he was lucky that the Earth's and the comet's orbits were coplanar and that the line crossing the Perihelion and Aphelion of the comet passed through Earth in such a way that the Sun was at the focus closest to the planet. The coordinates are specified in the following table:

Table 1.1: Coordinates of points in the orbit of the misterious object.

| $x$ | $y$ |
|---|---|
| -10.89 | 10.30 |
| 29.84 | 5.01 |
| 42.18 | 29.61 |
| 0.68 | 44.25 |
| 25.39 | 56.51 |

Unfortunately, Nill was captured by Interpol for hacking the Hubble telescope to make these measurements and was executed by the government. You are the only one he trusted to deliver the measurements. However, unfortunately, he didn't have time to explain what a Guanômetro is. But you promised that you would continue his work (in your words: "Never gonna give you up, never gonna let you down!"). So, find out:

**a)** What is the value of a Guanômetro?

**b)** The semi-major axis of the orbit.

**c)** The eccentricity of the orbit.

**d)** The orbital period.

Hint: In GeoGebra, there is a function where selecting 5 points will provide the conic that contains these points.

## 1.4 Ballistics 2.0

Tired of conventional kinematics problems? Well, today might be your lucky day... or maybe not! Frustrated after losing a game of Donkey Kong, Ramanu Jan throws his PS5 controller into the air with an initial speed $v_0$ at an angle $\alpha$ with the horizontal. He needs to find the exact location where the controller will land so he can catch it. Fortunately, on his planet, with mass $M$ and radius $R$, there is no atmosphere.

Hint: ignore the planet's rotation and remember Kepler's laws (Ramanu fears he might have thrown the controller at quite a high speed).

**a)** Under these conditions, what is the maximum height reached by the controller? Use the necessary approximations to verify that for low speeds, the result matches the classical problem (when considering constant $\vec{g}$).

**b)** What is the displacement of the controller (along the spherical surface) when it lands? Use the necessary approximations to verify that for low speeds, the result matches the classical problem (when considering constant $\vec{g}$).

## 1.5 Stelar Photometry

Nill das Graças was an enthusiast of astronomical observation. One night, he was enthusiastically observing the binary system composed of Mizar and Alcor. After a few minutes of observation, something unexpected happened: the total magnitude of the system increased by $\Delta m_{sis} = 0.1$. After some time studying the event to determine what had occurred, Nill discovered the reason for this change: a dust cloud passed near the system, and Alcor absorbed part of its matter, forming an additional layer over the star's photosphere

Dados:

- Alcor's apparent magnitude: 3.9
- Mizar's apparent magnitude: 2.2
- Alcor's photosphere temperature: $T_0 = 8200K$
- Alcor's Mass: $1.84M_\odot$
- Alcor's radius: $1.85R_\odot$
- $\mu = 1.67 \cdot 10^{-27}kg$

**a)** Determine Alcor's apparent magnitude variation, $\Delta m_A$, exclusively.

**b)** Given that $\rho(r)$ is the matter density at a distance $r$ from the center of the star and that $m(r)$ is the mass contained in a sphere of radius $r$ concentric with the star, prove the hydrostatic equilibrium equation:

$$\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

Consider that the most important process that mantains the equilibrium at the surface of the star is the radiation pressure, represented by the following equation:

$$P = \frac{4\sigma}{3c}T^4$$

**c)** Assumindo a estrela composta de um gás ideal com partículas de massa $\mu$, encontre uma relação para a variação de temperatura $\Delta T$ entre a fotosfera e camada exterior de matéria que garante o equilíbrio. Não assuma nenhuma relação entre a pressão de radiação e de gás ideal. Assuma que a espessura $\Delta r << R$, sendo $R$ o raio de Alcor. Se necessário utilize que $\frac{dy^n}{dx} = ny^{n-1}\frac{dy}{dx}$.

**d)** Assumindo que $|\Delta T| << T_0$, encontre a espessura dessa camada de matéria. Se necessário utilize que, para $|x| << 1$: $\ln(1+x) \approx x$.

**e)** Algum tempo após a acreção de matéria a camada adicional de gás começou a se comportar como uma atmosfera estelar bem definida. Sabendo que a fotosfera de uma estrela é definida como a região com profundidade óptica $\tau = \frac{2}{3}$ vista de fora da estrela, encontre a opacidade $\kappa$ da atmosfera. Considere que a densidade na superfície da estrela é da ordem de 10 vezes menos densa que a densidade média da estrela.

**c)** Assuming the star is composed of an ideal gas with particles of mass $\mu$, find a relationship for the temperature variation $\Delta T$ between the photosphere and the outer layer of matter that ensures equilibrium. Do not assume any relation between radiation pressure and ideal gas pressure. Assume that the thickness $\Delta r$ is much smaller than $R$, where $R$ is Alcor's radius. If necessary, use the derivative property:

$$\frac{dy^n}{dx} = ny^{n-1}\frac{dy}{dx}$$

**d)** Assuming that $|\Delta T| \ll T_0$, determine the thickness of this layer of matter. If necessary, use the approximation:

$$\ln(1+x) \approx x, \quad \text{for } |x| \ll 1.$$

**e)** Some time after the accretion of matter, the additional gas layer started behaving like a well-defined stellar atmosphere. Knowing that the photosphere of a star is defined as the region with an optical depth $\tau = \frac{2}{3}$ when viewed from outside the star, determine the opacity $\kappa$ of the atmosphere. Consider that the density at the surface of the star is approximately ten times less than the average density of the star.

## 1.6 Orbit Control

A space probe with mass $m_0$ is in a circular orbit with velocity $v_0$ around a homogeneous spherical planet without an atmosphere, with mass $M$ ($M \gg m_0$) and radius $R$, at a distance $\alpha R$ ($\alpha > 1$) from the center of the planet. The team of astronauts, called *Vinhedeiros*, wants to perform an elliptical orbit as close as possible to the planet (without causing collisions, of course). To achieve this, the engines will be activated, releasing propellant at a velocity $u$ relative to the engines. It is known that the engine releases propellant at a constant rate equal to $K$.

**a)** How long should the engine be programmed to remain active? Hint: Although the propellant release is not instantaneous, consider that this time is much shorter than the orbital period, so intermediate changes can be disregarded.

**b)** What will this time be for a soft landing on the planet? Use the same considerations.

# 1.7 Black Holes

After many years, the parent star of the Kafshian stellar system enters a supernova state and becomes a black hole. Fortunately, the civilization managed to escape centuries before this event, and you, an inhabitant of the Kafsh II system, wish to study more about the history of your ancestors and, consequently, learn more about black holes and general relativity!

## Part A: Metrics

First, we need a way to describe the space-time around us. For this, we can define metrics capable of describing the curvature of space-time due to the presence of matter. To do this, we will define the concept of an interval $ds$ between two events $A$ and $B$ for flat space-time (infinitely distant from any mass):

$$(\text{interval})^2 = (\text{distance traveled by light in the time interval})^2 - (\text{distance between events } A \text{ and } B)^2$$

Or:

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

However, in the following problems, the spatial description will be easier if we use spherical coordinates instead of Cartesian coordinates.

**A.1)** Demonstrate the definition of the interval using spherical coordinates $(r, \theta, \phi)$.

For $(ds)^2 > 0$, the value of $(cdt)^2$ dominates, meaning that light has more than enough time to travel between events. For $(ds)^2 = 0$, only light can travel between events $A$ and $B$. For $(ds)^2 < 0$, as expected, not even light can access events $A$ and $B$.

By definition, the time between two events occurring at the same spatial coordinate is called proper time:

$$\delta\tau = \frac{\delta s}{c}$$

On the other hand, the distance between two events occurring at the same temporal coordinate is called proper distance:

$$\delta L = \sqrt{-(\delta s)^2}$$

When dealing with space-time that may be curved due to the presence of matter, the trajectory of a body—or even light—can be described using the concept of the interval. The trajectory of a freely falling body is defined as a geodesic of maximum interval, and the trajectory of light is defined as a geodesic of zero interval!

In 1916, the German physicist Schwarzschild solved Einstein's field equations and obtained a metric that describes space-time around[1] a single body with mass $M$ and radius $R$:

$$(ds)^2 = \left(cdt\sqrt{1 - \frac{2GM}{rc^2}}\right)^2 - \left(\frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}}\right)^2 - (rd\theta)^2 - (r\sin(\theta)d\phi)^2$$

[1]The equations are valid only outside the body.

However, be careful! A common mistake is to consider that $r$ is the distance between the center of the body and the point being analyzed. The value of $r$ simply represents the spatial coordinate of the point. To calculate the actual distance, relativistic effects that alter the proper distance must be considered (unlike in special relativity, the proper distance varies with respect to the coordinate $r$). This also means that the variables in the metric are those observed from an observer at an infinite distance from the body (essentially outside the curvature).

**A.2)** For this metric, find the values of $\Delta L(r_1 \to r_2)$ (measured radially) and $\Delta\tau(t_1 \to t_2)$. If necessary, use:

$$\int \frac{dx}{\sqrt{1-\frac{a}{x}}} = x\sqrt{1-\frac{a}{x}} + a\tanh^{-1}\left(\sqrt{1-\frac{a}{x}}\right) + C$$

**A.3)** Consider a radiation jet with frequency $\nu_0$ emitted radially from a spatial coordinate $r$. Find the frequency $\nu_\infty$ received by an external observer, considering only gravitational redshift. Hint: remember that frequency is inversely proportional to the period between two oscillations.

**A.4)** Now, we would like to analyze a circular orbit around the object. Resist the urge to use Newtonian physics and follow Schwarzschild's metric to find the angular velocity $\omega$ of the orbit.

## Part B: Black Holes

**B.1)** For some objects in the universe, the curvature in space-time caused by their existence is such that not even light can "escape" their attraction. Conveniently, we call these objects Black Holes! Find the relation for the radius $R$ of this limit, called the Event Horizon.

**B.2)** Find the value of $r$ for which a photon would describe a circular orbit around the black hole. The set of points with such a coordinate is called the Photon Sphere.

**B.3)** In some cases in general relativity, it is still possible to use Newtonian physics with slight corrections. The equation that determines the attractive "force"[2] of a test body $m$ at a distance $r$ from the central body $M$ with angular momentum $L$ is:

$$|\vec{F}| = \frac{GMm}{r^2} + \frac{3GML^2}{mc^2r^4}$$

Find the expression for the radius of the innermost stable circular orbit (*Innermost Stable Circular Orbit*, $r_{ISCO}$) with angular momentum $L$. Hint: Initially treat $L$ as a fixed value, then make the necessary considerations.

Another very useful metric (with broader applicability) is the Kerr metric, which deals with black holes with non-zero *spin*. Unlike in quantum mechanics, here *spin* literally means the rotation of the black hole. It makes sense to think that black holes have *spin* because stars initially have angular momentum that must be preserved. In these cases, we can describe the rotation of the black hole using a spin parameter represented by $a_*$, which can vary from $-1$ to 1, with 1 being the maximum rotation allowed by the *Cosmic Censorship Hypothesis*, which rules out the existence

[2]In this case, this "force" is more related to the effective potential energy than to an actual force acting on the body.

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of naked singularities. In this metric, one can develop the equation that relates $r_{ISCO}$ to the spin parameter $a_*$:

$$r_{ISCO} = \frac{GM}{c^2}\left(1 + \sqrt{8.354\left((2-a_*)^2 - 1\right)}\right)$$

**B.4)** One technique to determine $a_*$ is through the analysis of the observed frequency of the iron $K_\alpha$ emission lines near $r_{ISCO}$. You, a scientist from the Kafsh II system, observe a sample of the emission spectrum from the accretion disk of the black hole $M33\,X-7$, with a mass of $M_{M33X-7} = (15.65 \pm 1.45)M_\odot$, which has a companion star in a binary system. You notice that the minimum received $K_\alpha$ frequency was $\nu = 7.83 \cdot 10^{17}\,Hz$. For simplicity, we will use the approximation (which is invalid in this case, as it is a rotating black hole) obtained in item **A.3)**. Determine the spin parameter of this black hole. Consider that the iron $K_\alpha$ transition energy is $E_{FeK\alpha} = 6.38\,KeV$.

## Part C: "Black" Thermodynamics

As proposed by Stephen Hawking, the temperature of a black hole can be expressed as:

$$T_b = \frac{\hbar c^3}{8\pi G M k_b}$$

**C.1)** Thus, find the emission power of a black hole acting as a blackbody.

**C.2)** Consider that for a body at rest, $E = Mc^2$. What is the estimated lifetime of this black hole?

It is believed that a black hole has an associated entropy because, if $S = 0$ in any case, it would be terrifyingly possible for the universe's entropy to decrease as matter is absorbed by the black hole.

**C.3)** Knowing that $dS = \frac{dQ}{T}$ (the classical thermodynamic interpretation of entropy), what should be the entropy of a black hole?

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## 1.8 The Mass of the Curve

João Kleper, a world champion bowler known for his perfectly precise throws, was in his match competing against the Russians. On his last throw, he noticed that the ball was deflected to the right. This was not enough to cost him the victory in the competition, but it left him very curious as to why he had failed. In his quest for the truth, he analyzed the recording of the match and noticed that his ball had been deflected by approximately 1 degree. He also realized that the cause of the deflection was an irregularity in the lane. After a deep analysis, he realized that the curvature of the lane caused an effect similar to that of a properly positioned gravitational field (at least under the conditions of his throw). Thus, Kleper wants to find the mass that would cause this same effect, that is, the "mass of the curve."

The lane is a rectangle with a width of 4 meters. The ball was launched perpendicular to the width, initially at 10 m/s, from the left edge of the lane. Kleper noticed that the irregularity was radially symmetric and had its center at the geometric center of the lane.

Assume that initially the ball does not interact with this fictitious field (as if it came from infinity with the mentioned velocity). Use Newtonian theory (unless you are confident in general relativity), but remember that the actual gravitational deflection angle is twice that predicted by Newton.

Assume everything is ideal: no air resistance, point masses, no sabotage by the Russians, among others.

## 1.9 Nill's Spaceship

As is well known, Nill das Graças was responsible for the illegal use of the Hubble Telescope in order to study the solar system with precision. To be able to use the telescope, Nill had to launch a spacecraft that would be positioned close to the target. Nill's spacecraft is a perfect spherical shell (where he is inside, obviously) with an inner radius $r$ and an outer radius $R$. The satellite's orbit is circular with a radius $a$, and at a given moment, Nill is aligned with the center of the Earth and the center of the satellite at a distance $x$ $(x < r)$ from the center of the satellite, in the same direction and sense as the position vector of the satellite relative to the center of the Earth. Nill is in his own circular orbit. Find an algebraic expression for the time it takes for Nill to hit the wall of the satellite. Use any approximations you deem necessary, such as $(1 + x)^n \approx 1 + nx$ and the approximation for the arccosine function: $\arccos(x) \approx \sqrt{2 - 2x}$ for $x \to 1^-$.

## 1.10 Orbital Sniper

Ramanu Jan discovers that someone has attempted to enter the airspace of his planet and is now trying to escape. At a certain moment, the invader stops at a distance $d$ from the planet to refuel their ship. Ramanu, being very merciful, only wishes to destroy the invader's ship with his super-powerful laser; however, he is prevented from taking a direct shot (in a straight line). Fortunately, he has an ace up his sleeve: a satellite in the shape of a 100% mirrored plate that orbits the planet in a stationary orbit. The operation will consist of sending the satellite into a specific elliptical orbit, after which Ramanu will shoot at the satellite, which must reflect the radiation towards the ship.

Note: The mirror's axis will always be tangent to the orbit.

Disregarding radiation pressure effects and relativistic effects, answer the following:

**a)** What is the minimum $\Delta v$ required for the operation to succeed?

**b)** Considering that the laser can emit radiation with intensity $I_0$ and that the medium permeating this region of space has density $\rho$ and opacity as a function of light frequency given by $\kappa = \kappa_0 f^3$, what is the necessary frequency range for the ship to successfully explode, given that light with intensity $\frac{I_0}{2}$ is required for the engines to detonate?

# 1.11 Let There Be Light!

The Kamoto Universe is a universe where the laws of physics are nearly identical to ours, with one exception: gravitational attraction follows the relation:

$$\vec{F}_{ij} = -Gm_i m_j(\vec{r}_j - \vec{r}_i)$$

where $\vec{F}_{ij}$ is the force exerted on $j$ by $i$. This purely classical universe (which lacks quantum mechanics, relativity, etc.) has a total mass $M$ and an intrinsic gravitational constant $G$.

## Part A: Age of the Universe

**A.1)**Resultant Force in the Center of Mass Reference Frame

Write the resultant force in the center of mass frame due to the gravitational action of all particles in this universe on a particle $j$.

**A.2)**Equation of Motion

Derive the equation of motion for this particle.

**A.3)**Estimating the Universe's Lifetime

Estimate the lifetime of this universe considering it has the same parameters as our universe ($M = 10^{54}kg$ and $G = 6.67 \cdot 10^{-11}N/(kg^2 \cdot m)$).

## Part B: Kepler's Laws

In the following items, some orbital properties of this atypical universe will be analyzed.

**B.1)**Center of Mass and Center of Gravity

In our universe, the center of gravity of a body does not necessarily coincide with its center of mass (for most shapes, it does not). Discuss the relationship between the center of mass and the center of gravity in the Kamoto universe.

Consider a particle of mass $m$ orbiting $M$ (with $m \ll M$):

**B.2)**Second Law of Kepler

Prove that Kepler's second law remains valid in this scenario and find an expression for the areal velocity (area swept by the position vector per unit time) as a function of the angular momentum of the particle $m$, denoted as $L$.

**B.3)**Orbital Shape

Determine the shape of $m$'s orbit (analogous to Kepler's first law).

**B.4)**Third Law of Kepler Equivalent

Find the analog to Kepler's third law for this universe, proving that:

$$T = a^n k(M)$$

where $a$ is some measure of the orbital shape, $T$ is the period, and $k(M)$ is a constant depending only on the central mass $M$.

For example, in our universe, $n = \frac{3}{2}$ and $k(M) = \sqrt{\frac{4\pi^2}{GM}}$. Determine $n$ and $k(M)$ for the Kamoto universe.

## Part C: Orbital Transfers

In the following items, an orbital transfer situation will be analyzed.

**C.1)** Orbital Velocity at Extremes

A team of astronauts is orbiting a central mass $M$ in an elliptical orbit with semi-major axis $a$ and semi-minor axis $b$. Derive an equation for the orbital velocity when the spacecraft is at its maximum distance from the planet ($v_a$) and at its minimum distance ($v_b$).

**C.2)** Multi-Step Orbital Transfer

The team aims to reach a second circular orbit of radius $\alpha R$ (with $\alpha > 1$). The method used involves $n$ elliptical transfers such that the semi-major axis of ellipse $i$ corresponds to the semi-minor axis of ellipse $i+1$, as shown in Figure 1.1.

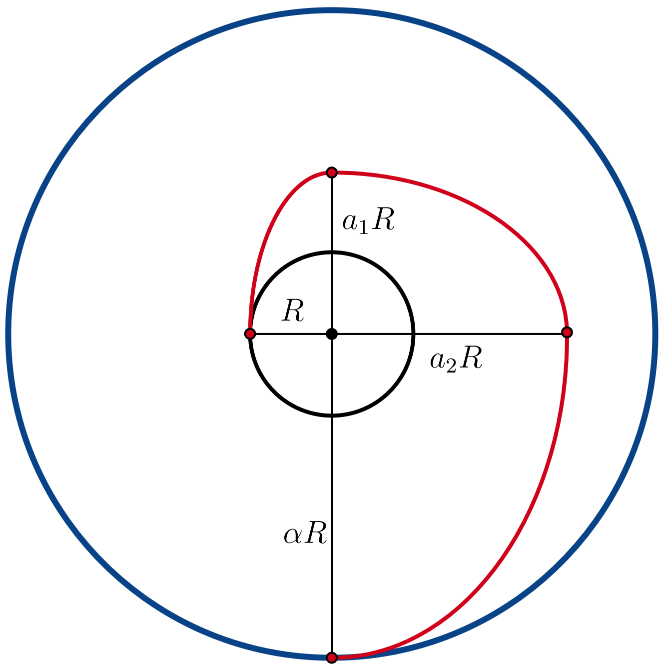


Figure 1.1: Scheme showing a transfer with $n = 3$.

Determine the total $\Delta v_T$, that is, the sum of all $\Delta v_i$ needed to complete the transfer.

**C.3)** Determine the time taken for complete orbital transfer.

## 1.12 The Conics of Narnia

Aslan planned to create a new universe in such a way that this time there would be no interference from evil, as there was with Narnia. At a certain moment during its creation, while designing gravitation, he defined that the attraction between two bodies would be given by the relation: $\vec{F} = -\frac{GMm}{d^2}\hat{d}$. Thus, help Aslan and prove some properties of the orbits of a mass $m$ around a mass $M$ with ($m \ll M$):

Define: $m(x,y)$ as the position of $m$ in the orbital plane, and analogously, $M(0,0)$ as the position of $M$ in the orbital plane. In this case, we consider $M$ as the origin.

**a)** Given that the velocity of $m$, at a distance $d$ from $M$, follows the relation:

$$v^2 < \frac{2GM}{d}$$

Consider a circumference $\Omega(x_j,y_j)$ centered at $m(x_j,y_j)$ and passing through $M(0,0)$. Define (at first without physical meaning) the orbit space as all points belonging to a given circumference $\Omega(x_i,y_i)$. Prove that the orbit space is a filled circle of radius $R = \frac{2GMd}{2GM-v^2d}$ centered at the other focus of the orbit (the focus different from $M(0,0)$).

**b)** Given that the velocity of $m$, at a distance $d$ from $M$, follows the relation:

$$v^2 = \frac{2GM}{d}$$

Show that there exists a line $r$ in the orbital plane such that the distance between $m(x_i,y_i)$ and $M(0,0)$ is equal to the distance between $m(x_i,y_i)$ and its orthogonal projection onto $r$.

**c)** What must be the velocity relation so that the region that does not belong to the orbit space (as defined in item **a)**) is a circumference centered at the other focus of the orbit (the focus different from $M(0,0)$)?

## 1.13 AstroMagnetism

### Part A: Relation Between Dipoles

**A.1)** Consider the planet Pluto II as a sphere with density $\rho(r)$ at a distance $r$ from the core and radius $R$. Recalling that the moment of inertia of a spherical shell is $\frac{2}{3}MR^2$, show that the moment of inertia $I$ of Pluto II around its symmetry axis is:

$$I = \frac{8\pi}{3}\int_0^R r^4\rho(r)dr$$

**A.2)** Consider a body (with unspecified geometry) of mass $M$ and charge $Q$ with a given distribution of charge and matter relative to a common axis. This distribution follows the criterion: for an element located at $\vec{r}$, we have $dq = \kappa dm$, where $\kappa$ is a constant. Prove the well-known *gyromagnetic ratio*:

$$\vec{\mu} = \frac{Q}{2M}\vec{L}$$

**A.3)** José Lagranja, a great scientist from Pluto III, a neighboring planet of Pluto II, in his book *Principia Plutonis*, shows that Pluto II is a poor conductor (fixed charges) with total charge $Q$ and satisfies the relation:

$$\int_0^R r^4\rho(r)dr = \mathcal{L}\int_0^R r^2\rho(r)dr$$

where $\mathcal{L}$ is the Plutonian constant.

Thus, show that:

$$\vec{\mu} = \frac{1}{3}Q\mathcal{L}\vec{\omega}$$

Now, consider the following scenario: the planet Pluto II is immersed in an environment with a magnetic field $\vec{B}$, such that $|\vec{B}| = 4.87$ nT, generated by its parent star, Scorp. The planet has a constant density and satisfies the conditions from the previous items, meaning it has mass $M = 6\cdot 10^{24}$ kg, a period $T = 25$ hours, and a total fixed charge $Q = 5.2\cdot 10^{21}$ C. The inclination of the planet's rotation axis relative to the magnetic field is $\theta = 18°24'$, as shown in Figure 1.2.

**A.4)** Find the precession period of the equinoxes of this planet. Disregard any other sources of precessional motion.

### Part B: Magnetic GPS

José Lagranja, in his quest to complete his pioneering research on the magnetic activity of his planet, sought to locate the magnetic north pole[3] of his planet, which possesses an intrinsic magnetic dipole. The magnetic effect of the planet resembles that of a perfect dipole, which produces an equivalent magnetic field of:

$$\vec{B}(r,\theta) = \frac{\mu_0|\vec{\mu}|}{4\pi|\vec{r}|^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}) \tag{1.1}$$

[3]At the magnetic north pole, the magnetic field lines enter the surface perpendicularly.

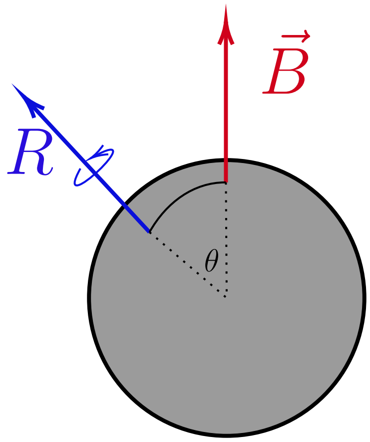


Figure 1.2: Representation of the angular separation between the magnetic field and the planet's axis of rotation.

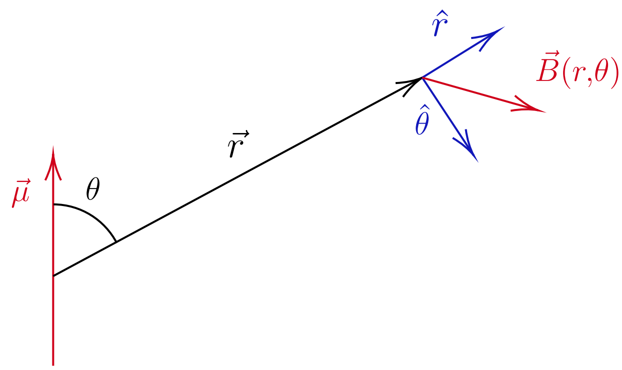


Figure 1.3: Scheme representing the magnetic field $\vec{B}$ generated by a dipole $\vec{\mu}$ at a position $\vec{r}$.

Given that $\mu_0$ is the magnetic permeability of vacuum, $\vec{\mu}$ is the planet's magnetic moment, and $\vec{r}$ is the position vector of the point where the field is being calculated, with its respective spherical coordinates ($\hat{r}$ and $\hat{\theta}$).

To find the pole, he devised an experiment: the scientist prepared a short and extremely thin ferromagnetic metal wire and held it by one end. The planet's magnetism oriented the wire both in the horizontal and vertical planes.

**B.1)** Upon conducting the experiment, the wire tilted about $39°20'$. Find the angular distance between the magnetic north pole and the location of Lagranja's experiment.

The scientist noticed that the angle between the meridional plane and the vertical plane containing the wire was $8°12'$ westward. It is known that Lagranja's coordinates are $17°22'$N, $130°52'$E.

**B.2)** Determine the coordinates of the magnetic north pole.

## Part C: Rotation of DSCOVR

Some time after publishing his book, Lagranja launched a spherical satellite with a radius of $r = 1.62$ m and mass $M = 570$ kg, named DSCOVR, and developed software to study his star: JStarViewer. However, a system failure occurred on the satellite, causing it to spin uncontrollably with an angular velocity $\vec{\omega}(t)$. Just before the failure, the satellite collected data on the magnetic field and the charge flux hitting it, and all this data was transmitted to JStarViewer.

Consider that initially, the satellite was electrically neutral, that the observed stellar wind

Table 1.2: Quantities obtained by satellite and tabulated by JStarViewer.

| Constant | Value |
|---|---|
| Density of the Stelar Wind | $3.608 \cdot 10^{-23}$ g cm$^{-3}$ |
| Speed of the Stelar Wind | 313 km s$^{-1}$ |
| Magnetic Field | 4.87 nT |

consists only of protons, and that the values of the magnetic field, solar wind density, and particle velocity remain constant.

**C.1)** Find the charge accretion rate $R$ onto the satellite, as well as the total magnetic field in the region.

**C.2)** Determine $\vec{\omega}(t)$.

**C.3)** Find the time $t(n)$ elapsed until the satellite completes its $n$th precession cycle.

## 1.14 Nill in the Telescope Wonderland

Nill das Graças, as a child, was fascinated by telescopes! He lived in the countryside of Maranhão and had no one to teach him the techniques and tricks for handling the instruments, so he was forced to learn everything on his own. Follow Nill on his optical journey until he becomes the great astronomer he would eventually be.

### Part A: Refracting Telescopes

Nill began his path with refracting telescopes. The boy was very fond of geometric optics, so he soon thought of creating his first telescope based on his knowledge of how lenses work. Consider the setup designed by him:

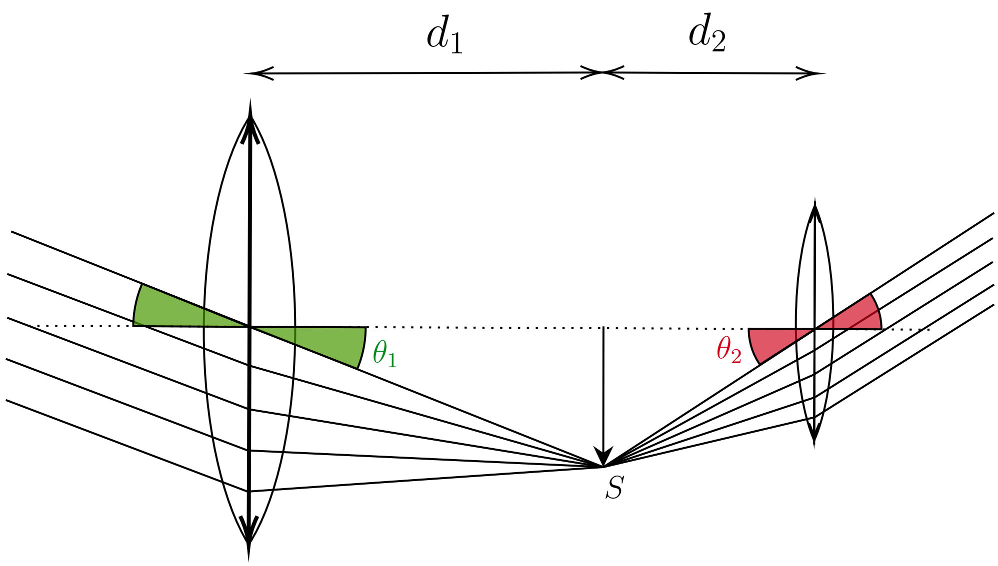


Figure 1.4: The interior of Nill's telescope and the functioning of the lenses in the deviation of light.

A recurring approximation is that the rays from the observed objects (stars, planets, among others) strike the objective lens—the larger one in the previous figure—in a parallel manner, since due to the great distances their angular size is close to zero. The telescope is also adjusted so that the rays exiting through the eyepiece—the smaller one in the image—also emerge parallel, in order to preserve the proportions of the original image.

Nill knows and masters the lensmaker's equation:

$$\frac{1}{p} + \frac{1}{p'} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{1.2}$$

**A.1)** Knowing that Nill used glass to manufacture the lenses ($n = 1.5$), that the radius of curvature of the faces of the objective lens is 70 cm and that of the faces of the eyepiece is 10 cm, determine the focal lengths of each lens.

**A.2)** Find the distances $d_1$ and $d_2$ and the length of the telescope.

**A.3)** Find a relation between the angles $\theta_1$ and $\theta_2$. From this, argue how to find the angular magnification of the telescope for an object close to the lens axis. Show this value.

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**A.4)** Nill also measured the diameter of each lens: $D_{ob} = 7.6$ cm and $D_{oc} = 1$ cm. Knowing that each lens has an absorbance of $\epsilon = 0.09$, what is the limiting magnitude of Nill's improvised telescope? The interior of the telescope is coated with black pigment that absorbs the incident radiation.

- Limiting magnitude of observation of the human eye = 6.
- Diameter of the human pupil = 6 mm.
- Consider that the radiation comes parallel to the lens axis (represented by the horizontal line).
- Hint: make the smallest possible number of assumptions!

Note that in this arrangement the final image is inverted along the vertical axis. To obtain an image with the correct orientation, a diverging eyepiece would be necessary.

## Part B: Reflecting Telescopes

Continuing his studies, Nill moved on to a new type of telescope: reflectors. In these, instead of lenses, mirrors shaped into specific forms are used to correctly direct and focus the light to the observer's eyes.

To this end, Nill decided to learn more about the reflective properties of conic sections.

**B.1)** Consider an ellipsoid mirrored on the inside. Prove that any ray of light that originates from one of the foci of the conic reflects and strikes its other focus. Hint: solving the problem backwards, that is, proving that the path that connects one focus to the other and touches the conic once can be realized by a ray of light, may be much more practical.

**B.2)** Consider a paraboloid mirrored on the inside (concave part). Prove that any ray of light that originates from the focus of the parabola reflects and emerges parallel to its axis. Hint: since you already know the result for ellipses from the previous item, could that help you here?

**B.3)** Consider a hyperboloid—half of a hyperboloid, that is, consisting of only one of the two branches of the curve—mirrored on the outside (convex part). Prove that any ray of light that is directed toward the focus of this half reflects off the curve and strikes the focus of the other half (which was not considered).

Nill built his first telescope with a Cassegrain focus, which uses a parabolic primary mirror and a hyperbolic secondary mirror, as shown in Figure 1.5.

The young man knows all the parameters of the telescope and wishes to find other important relations for the operation of the instrument, as tabulated in Table 1.3.

Table 1.3: Telescope construction parameters.

| Quantity | Value (mm) |
|---|---|
| $h$ | 233 mm |
| $p$ | 467 mm |
| $t$ | 534 mm |
| $D$ | 195 mm |

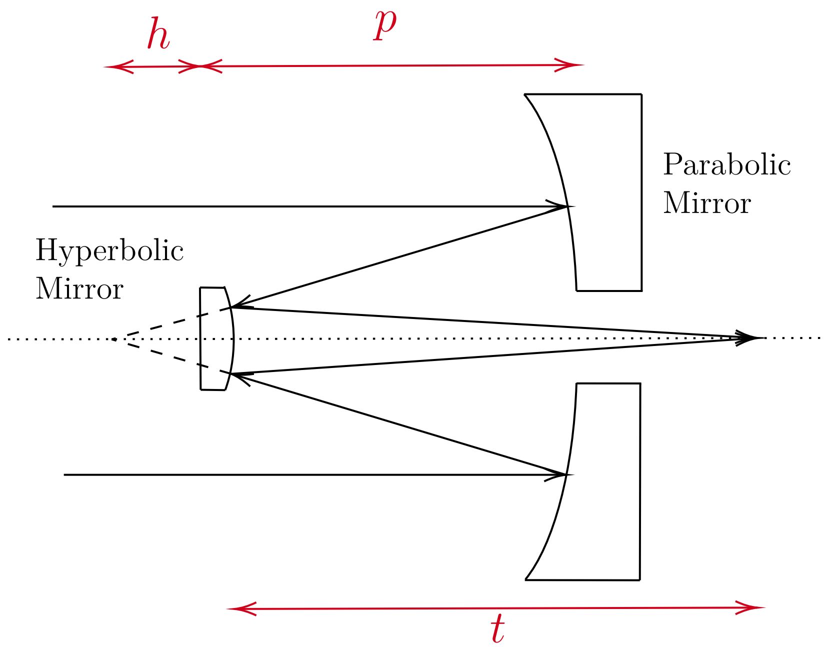


Figure 1.5: Image focusing apparatus with a Cassegrain focus, based on a parabolic mirror with a hole and a secondary hyperbolic mirror.

**B.4)** Starting from these data, find the focus of the primary mirror.

**B.5)** Find a relation between the effective focus of the system and the focus of the primary mirror. Define effective focus as the focus of a converging lens that would generate the same final effect as the system, that is, the same final angular deviation of the rays.

**B.6)** Find the focal ratio of the telescope.

**B.7)** Each mirror has an absorbance of $\epsilon = 0.05$, and the dimensions of the mirrors are adjusted so that parallel rays with the greatest possible separation that enter the tube are reflected at the edges of the mirrors. The hole in the primary mirror is adjusted to be aligned and smaller than the secondary mirror and large enough to allow all the radiation to pass through without obstruction so as to be focused into the observer's eye. In this way, find the limiting magnitude of the telescope.

After that, Nill continued his studies with image acquisition and processing devices, seeking to develop his own CCD! But that is already a story for another question.

# 1.15 Unfocused Photometry

Eugene, a Paraguayan telescope salesman, once bought a certain refracting telescope at a low price. Upon arriving home, he noticed that the telescope lens was terrible: the chromatic aberration was very large!

He noticed this because he was observing different stars and saw that the lens focus was extremely variable. He collected the data in the following table:

Table 1.4: Data of the temperature and focal distance of the rays from each star.

| Star | Temperature (K) | Focus (cm) |
|---|---|---|
| Spica | 25 000 | 3.11 |
| Betelgeuse | 3 600 | 149.93 |
| Aldebaran | 4 055 | 118.17 |
| Sirius | 9 845 | 20.05 |

Even frustrated with the purchase, he had an idea: to try to find information about a star from the undesired phenomenon. For this, he found a highly sensitive photoreceptor plate and placed it on the lens focal axis, observing the flux curve in relation to the distance (given in centimeters) to the center of the lens. The obtained curve was as follows:

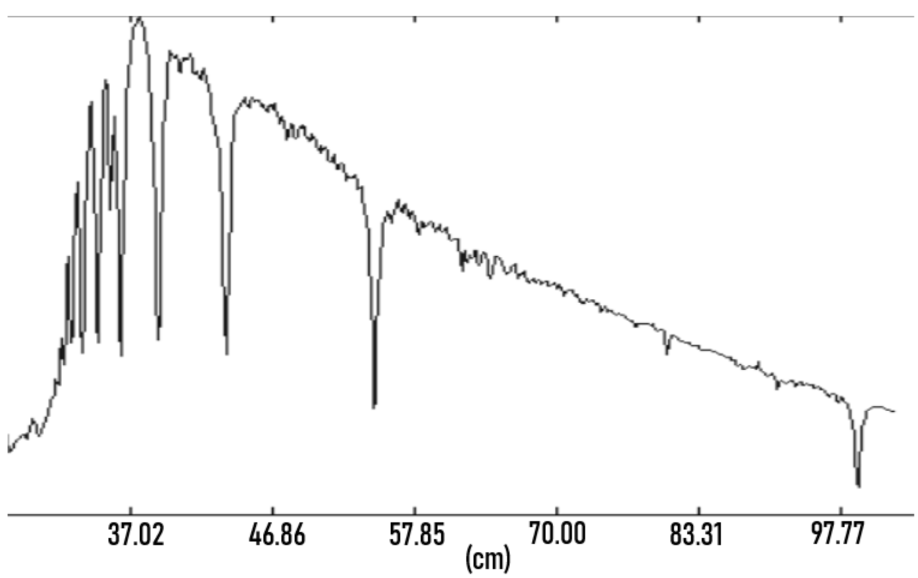


Figure 1.6: Radiation intensity received by the photoreceptors as a function of the distance to the lens.

Consider a known approximation for the refractive index of an object as a function of the wavelength of incident light:

$$n(\lambda) = n_0 + \frac{k}{\lambda^2}$$

**a)** Determine the temperature of the star in question.

To adjust the telescope, Eugene acquired two other lenses: A and B. Lens A has properties similar to the previous one, such that

$$n_A(\lambda) = n_A + \frac{k_A}{\lambda^2}$$

and lens B has a fixed focal distance $f_B$ (without chromatic aberration). In the following items, express your answers in terms of the presented parameters.

**b)** Consider that the two curvature radii of a lens are equal ($R_1 = R_2 = R$). Find the relation between the curvature radii of the initial lens and lens A so that the optical system is stigmatic, i.e., has a well-defined focus.

**c)** What is the maximum condition for $f_B$ so that the system functions as an objective lens?
Tips:
Can we approximate the value of $n_0$ from graphical analysis?
Lens manufacturer equation:

$$\frac{1}{f} = \left(\frac{n_{lens}}{n_{medium}} - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Combination of lenses placed very close together:

$$\frac{1}{f_T} = \sum_i \frac{1}{f_i}$$

# 1.16 Variable Solar "Constant"

Safira's house is a cube with 5 windows: one on each vertical wall and one on the roof, each with a cross-sectional area $A$. Observing the house from above, one notes that the angle of inclination of the house's axis with the projection of the local meridian is $\theta$, as shown in Figure 1.7.

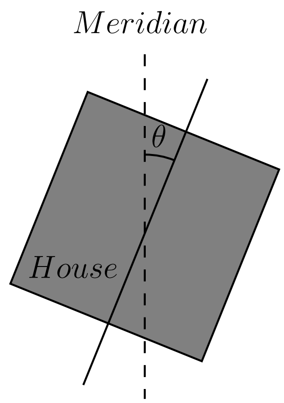


Figure 1.7: Geometric positioning of Safira's house.

**a)** Find the Sun's altazimuthal coordinates as a function of its declination $\delta$, the time elapsed since true solar noon $t$ ($t < 0$ for before noon), and the local latitude $\phi$.

**b)** Find an expression for the radiation luminosity entering the house as a function of the previous parameters, the window area, and the solar constant.

**c)** Sapphire, upset with her excessive energy costs, decided to install a solar panel system to collect energy. However, she did not plan properly and ended up installing the panels inside her house, and also renovated her house and covered all the side windows. Thus, find the total energy obtained by the panel over a full day.

**Data:**

- Total area of the panels: $A_p$
- Side length of the house: $L$
- Window area: $A_j$
- The top window has a film that allows radiation to enter but not escape. Also consider that the walls are reflective and the brightness becomes homogeneous inside the house.

# 1.17 Non-Homogeneous Cosmology

## Part A: Spherical Cavity

One of the universes in the multiverse is non-homogeneous. It is a sphere of matter density equal to $\rho$ that has a spherical cavity at a distance $a$ from the center. Inside this cavity, there is a total absence of matter, as shown in Figure 1.8.

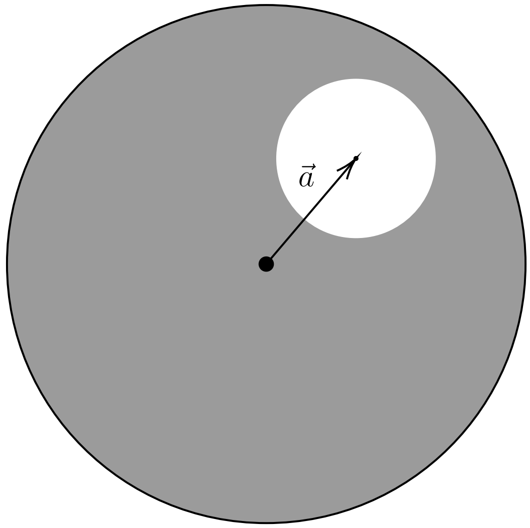


Figure 1.8: Composition and distribution of matter in the universe under consideration.

**A.1)** Find a relation for the gravitational field inside the cavity.
Hint: Think about the superposition of matter and remember the vectorial analysis of gravitation.

## Part B: Discrete Distribution

A similar situation occurs in universe M190i, a non-homogeneous universe consisting of an inner sphere with energy density $\epsilon_b$ and a spherical shell with energy density $\epsilon_a$, as shown in the figure:

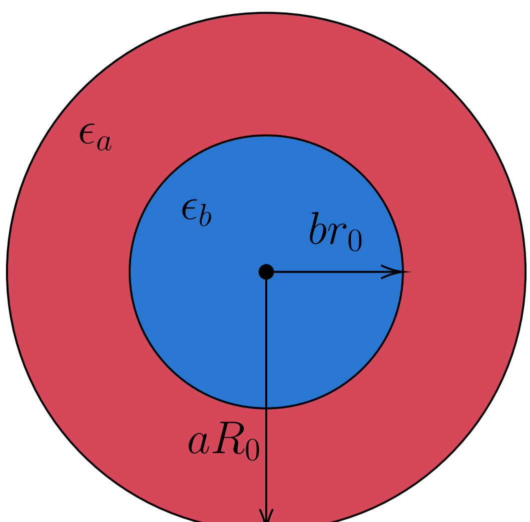


Figure 1.9: Density map of the M190i universe.

Let $b$ and $a$ be the scale factors for the inner and outer parts, respectively, and $r_0$ and $R_0$ the initial radii of the inner and outer parts, respectively.

In this problem, use $H_x = \frac{\dot{x}}{x}$ and $\Omega_x = \frac{\epsilon_x}{\epsilon_{x,c}}$, where $\epsilon_{x,c}$ is the critical energy density of parameter $x$. Also, take that parameters with subscript 0 refer to current values.

**B.1)** Show that the evolution relation of the inner part is given by:

$$H_b^2 = \frac{8\pi G}{3c^2}\epsilon_b$$

**B.2)** Now analyze the evolution law of the shell. Show that:

$$H_a^2 = \frac{8\pi G}{3c^2}\epsilon_a \left(1 + \left(\frac{br_0}{aR_0}\right)^3 \left(\frac{\epsilon_b}{\epsilon_a} - 1\right)\right)$$

A cosmologist from this universe noticed that, coincidentally, the gravitational field inside the inner region (with energy density $\epsilon_b$) was constant.

**B.3)** An interesting factor about the relation for energy density as a function of the scale factor comes from dark energy: it has been observed that such dependence is $\frac{d\epsilon_a}{dt} = 0$, so find the evolution function of the universe's shell.

Hint: When the integral becomes painful, a substitution like $u = \sqrt{1 - za^{-m}}$ might help a lot. Also use:

$$\int \frac{dx}{1 - x^2} = \tanh^{-1} x + C$$

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## 1.18 Miraculous Escape

You, one of the analysts of the ABFire empire, were studying a curious dust cloud in a strange planetary system. In your studies, you wished to enter the cloud and study it more closely; however, you only realized that there was a black hole inside the cloud. Too late! Your SPS (Space Positioning System) failed, and you lost control of the spacecraft. Only one resource was working: the emergency button. This button, when activated, can send the spacecraft on a parabolic escape trajectory in a desired direction. You, however, do not know which direction to take or how far you are from the black hole. Trying to calm yourself, you decide to find the probability of successfully escaping by choosing a random direction.

Consider that the radius of the black hole's event horizon is $R_b$ and the radius of the cloud is $R_n$.

**a)** Find the relation between the angle of the position vector and the velocity vector for the spacecraft to **enter** the event horizon, for a given position $r$.

**b)** Find the probability of the spacecraft escaping from a given position $r$.

**c)** Find the probability that the spacecraft is located at a distance $r$.

**d)** Determine the probability of the spacecraft escaping, considering that the radius of the black hole is $R_b = 69 \cdot 10^6$ m and the radius of the cloud is $R_n = 420 \cdot 10^7$ m.

Given:

$$\frac{1}{R_n^3 - R_b^3} \int_{R_b}^{R_n} \sqrt{1 - \frac{R_b}{r}}\, r^2 \, dr \approx 0.2895485$$

## 1.19 Quantum Star

Consider a star made of degenerate matter in which the electrons of the material are in stationary states within confined regions of space. These stationary states give the electrons a certain energy associated with their quantum states.

### Part A: Fermi Energy

In this model, the electron has a wave function associated with each Cartesian axis (so it may have different wavelengths along the $x$, $y$, and $z$ axes, since it may have different momenta along each axis). From the time-independent Schrödinger equation (since we are dealing with a stationary state) for a single axis $u$, we can find the corresponding stationary wave under the confinement conditions:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(u)}{du^2} + V(u)\psi = E\psi \tag{1.3}$$

Along this axis, $u = 0$ corresponds to one edge of the box, while $u = L$ corresponds to the other edge.

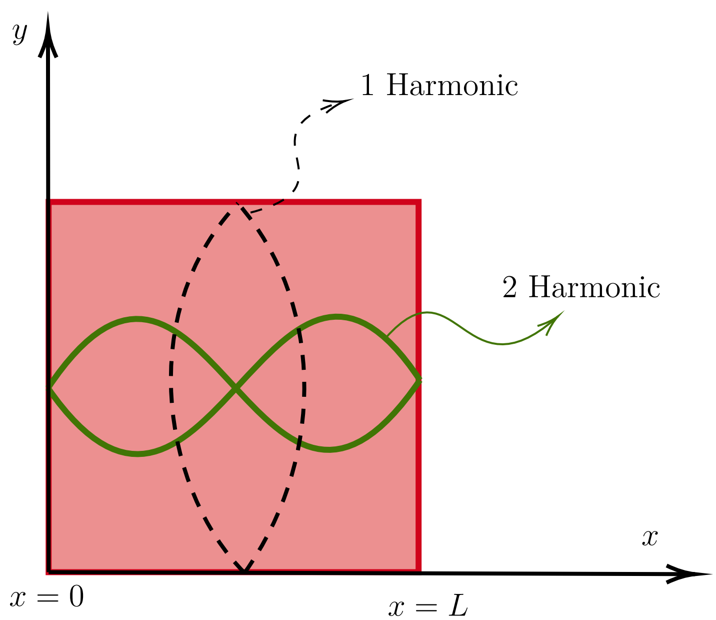


Figure 1.10: Example of two-dimensional arrangement.

**A.1)** Considering that the potential inside the box of side $L$ is $V(u) = 0$, prove that:

$$\psi(u) = A\sin\left(\sqrt{\frac{2mE}{\hbar^2}}u\right) + B\cos\left(\sqrt{\frac{2mE}{\hbar^2}}u\right)$$

**A.2)** Argue why $B = 0$ is necessary.

**A.3)** From the wave equation: $\sqrt{\frac{2mE}{\hbar^2}} = \frac{2\pi}{\lambda_u}$, where $\lambda_u$ is the wavelength associated with the particle along axis $u$. Argue that:

$$E = \frac{\hbar^2 \pi^2}{2m^2 L^2} n_u^2$$

where $n_u$ is a positive integer characterizing the quantum state of the particle.

**A.4)** Show that the total energy of the system is of the form:

$$E(r) = \frac{\hbar^2 \pi^2}{2mL^2} r^2$$

where $r = \sqrt{n_x^2 + n_y^2 + n_z^2}$.

**A.5)** Define $N$ as the number of electrons with energy less than $E(r)$. Show that:

$$N = \frac{\pi r^3}{3}$$

**A.6)** Find the function $E(N)$ and integrate it from 0 to $N_t$, the total number of particles in the star, with respect to $N$. Show that:

$$E = \frac{3\hbar^2}{10m} \left(3\pi^2 \eta\right)^{\frac{2}{3}} N_t$$

where $\eta$ is the particle number density in the region.

## Part B: Gravitational Self-Potential Energy

**B.1)** Consider a sphere of mass and initial radius $m$ and $r$. Calculate the work required to add an incremental mass $dm$ to this sphere. Show that the increase in the system's potential energy is:

$$dE = -\frac{Gm}{r} dm$$

**B.2)** Relating $m$ and $r$ for a sphere of constant density $\rho$ and integrating, show that the total potential energy of the final system is:

$$E = -\frac{3}{5} \frac{GM^2}{R}$$

## Part C: Degenerate Stellar Remnants

**C.1)** Find the total energy of a white dwarf of mass $M$ and radius $R$, composed of an element $X$ with atomic number $Z$ and atomic mass $m_X$ (mass of one atom). Consider that other energy contributions, apart from gravitational, electron energy, and rest energy, are negligible.

**C.2)** Given a mass $M$, find the radius of the white dwarf associated with this mass. Show that this value is:

$$R = \frac{\hbar^2}{Gm_e} \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \left(\frac{Z}{m_X}\right)^{\frac{5}{3}} M^{-\frac{1}{3}}$$

where $m_e$ is the electron mass.

**C.3)** Find the radius of a white dwarf composed of hydrogen with the mass of the Sun.

## 1.20 Bulk Star

In this problem, we will develop a stellar model based on the formation of a star from a spherical nebula composed of a certain fluid (not necessarily a gas).

Consider a homogeneous and isotropic spherical nebula with initial radius $R_0$ and initial density $\rho_0$. Imagine that at this moment there are practically no interactions between the fluid particles. This fluid is compressible and has a Bulk modulus equal to $B$. From this scenario, we want to determine the parameters of the "star" formed after the particles reach hydrostatic equilibrium.

Recall the definition of the Bulk modulus:

$$B = -\frac{\Delta P}{\frac{\Delta V}{V}}$$

Here, $\Delta P$ is the change in pressure in the fluid caused by the relative change in its volume $\dfrac{\Delta V}{V}$.

**a)** If the final radius of the star is much smaller than the initial radius of the cloud and considering that gravitational interaction between particles is dominant throughout the motion (neglecting opposing pressure), find the formation time of the star as a function of the given parameters.

**b)** Consider a range of radii $r_0$ to $r_0 + dr_0$ in the initial cloud. The mass contained in this interval will move to the corresponding interval $r$ to $r + dr$ in the final star. Show that the pressure in the final star is:

$$P(r) = B\left(1 - \frac{\rho_0}{\rho(r)}\right)$$

**c)** Find the pressure gradient required to "support" the star in hydrostatic equilibrium. Show that:

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

where $m(r)$ is the mass contained within radius $r$.

**d)** Substitute the pressure using the relation found in (b) and demonstrate:

$$-\frac{B\rho_0}{4\pi G}\frac{d\rho(r)}{dr} = \frac{\rho(r)^3}{r^2}\int_0^r x^2\rho(x)dx$$

Solving this differential equation is your challenge! For this problem, we will make a simplifying assumption to make the calculations easier. Suppose the density as a function of distance is:

$$\rho(r) = ar^n$$

**e)** Find $a$ and $n$.

**f)** Find the pressure function as a function of distance from the center of the star.

**g)** Conclude: what should be the maximum radius of a star under these conditions?

**h)** Assuming the fluid that makes up the star has particle mass $\mu$ and behaves like an ideal gas, find its temperature profile.

**i)** What is the Mass-Radius relation for stars in this model?

**j)** Assuming stars in this model have radii much smaller than the limiting radius, find their Mass-Luminosity relation.

## 1.21 Celestial Chart

In this exercise, we will discuss a method of cartographic projection for constructing celestial charts, as well as its uses and distortions. For reference, see the figure below, taken from the celestial chart exam of the Vinhedo selection in 2021:

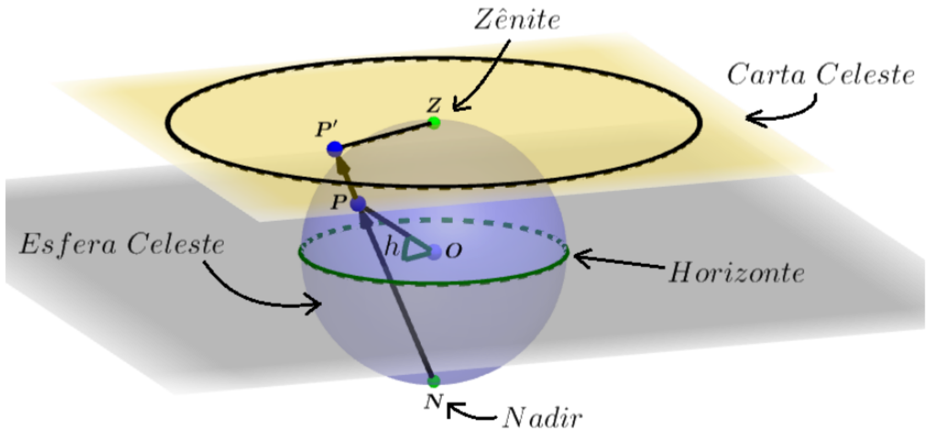


Figure 1.11: Representation of the construction of the projection in question. Source: OBA Selection Committee.

First, we will find a way to relate the distance of an object to the center of the chart $r$ in relation to the chart radius $R$ and the zenith distance $z$ of the object on the celestial sphere (recall that $h + z = 90°$).

**a)** Find the proper relation between $z$ and $r$ as a function of $R$.

**b)** Given that the distortion relation $\zeta(z)$ for a zenith distance $z$ is the ratio between an infinitesimal area element on the chart and the corresponding element on the celestial sphere, show that:

$$\zeta(z) = \sec^4\left(\frac{z}{2}\right)$$

Kaian, a young astronomer and Marvel fan, was studying the orbits of various satellites around the Earth. For this, he used his faithful "Celestial Chart." As expected, the orbits of the satellites are great circles projected from the center of the Earth onto the celestial sphere.

**c)** Kaian wants to know the shape of these orbits on his celestial chart. Therefore, find the shape and the polar and Cartesian equations governing this curve, as a function of the inclination $i$ of the orbit relative to the horizon.

Kaian's disciple, Menegron, was fascinated by geometry and loved calculating areas and angular distances of constellations. To this end, he developed a formula capable of giving the area of a spherical triangle in terms of its internal angles. See Figure 1.12:

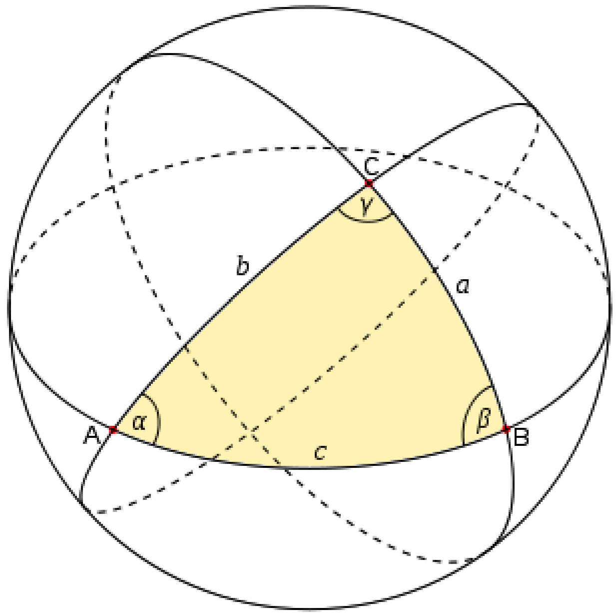


Figure 1.12: Representation of a generic spherical triangle and its constituent lunes.

**d)** Argue that the area of a lune[4] formed by two great circles is equal to $2\alpha R^2$, where $R$ is the radius of the sphere and $\alpha$ is the angle between them.

**e)** Using the principle of inclusion and exclusion, show that the area of the spherical triangle $ABC$ is:

$$A_{ABC} = (\alpha + \beta + \gamma - \pi)R^2.$$

On a given night, Menegron noticed that there were 3 satellites on the celestial sphere, with alt-azimuth coordinates (altitude, azimuth): $(90°, 0°)$, $(0°, 97°)$, and $(60°, 187°)$.

**f)** Calculate the area of this spherical triangle.

**g)** Calculate the projected area of this triangle on the chart.

**h)** Show that the ratio between the areas on the chart and on the sphere is:

$$\zeta(\Delta ABC) = 16 - \frac{12\sqrt{3}}{\pi}$$

[4] A lune is one of the two regions between two of these great circles.

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## 1.22 Plutonian Atmosphere II

The scientist José Lagranja, a resident of the planet Pluto II, in his planetary studies of his home planet, aimed to develop a model for the planet's atmosphere. He observed that the gases composing the atmosphere behave as ideal gases.

The planet Pluto II has radius $R$, mass $M$, and an atmosphere composed of particles of mass $m$.

### Part A: Physical Modeling

**A.1)** Given that the atmosphere is in thermal equilibrium, that the temperature at its surface is $T(R)$, and that the radial derivative of temperature at its surface is $\dot{T}(R)$, find an expression for the temperature of the atmosphere at a distance $r$ from the planet's center.

**A.2)** Using the equations of hydrodynamic equilibrium and the ideal gas law, show an expression for the density of the gases at a distance $r$ from the planet's center. Express your answer in terms of $\rho(R)$, the density of the atmosphere at the planet's surface.

**A.3)** Finally, find an expression for the pressure of the atmosphere at a distance $r$ from the planet's center.

**A.4)** Consider that the atmosphere of Pluto II consists of a very thin layer (compared to the planet's radius $R$) of thickness $e$, and that, coincidentally, the temperature at the planet's surface is given by: $T(R) = \dfrac{GMmE}{kR^2}$. Determine a simplified expression with the detailed approximations for $T(r)$ and for $\rho(r)$.

### Part B: Atmospheric Distortion

**B.1)** Consider a light ray arriving at the atmosphere with an angle $\theta$ relative to the normal of the outermost layer. Due to the variable refractive index of the different layers, the trajectory of the ray is distorted into a curve, as shown in Figure 1.13:

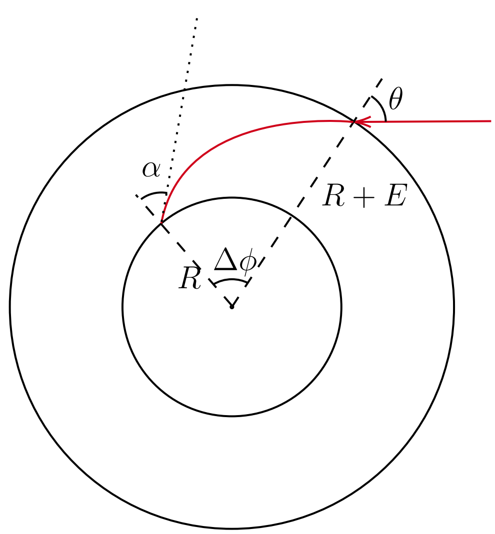



Figure 1.13: Demonstration of the refraction caused by the planet's atmosphere for a ray with an incidence angle $\theta$.

Note that, in the end, the celestial object in question is observed at a zenith distance $\alpha$. Find $\alpha$

knowing that the refractive index at a distance $r$ from the planet's center is given by: $n(r) = n_0 r^{-1}$ for $R \leq r \leq R + E$, such that at the outer edge of the atmosphere this index equals that of vacuum.

**B.2)** Using the specified approximations, find $\Delta\phi$.

**B.3)** Knowing that the opacity of the atmospheric gases is constant and equal to $\kappa$, find the change in magnitude $\Delta m$ caused by atmospheric absorption for an object at zenith angle $z$.

**B.4)** Lagranja also studied two stars: AMX-13 and TUC-33. He knows that the parallax of these stars, disregarding atmospheric effects, is $p_a = 0.13''$ and $p_t = 0.33''$, respectively. Knowing that, at a given moment of his observation, Lagranja notes that the zenith distance of AMX-13 is $25°$ and that of TUC-33 is $46°$, that the azimuthal separation between them is $36°$, and that the ratio $\dfrac{E}{R} = 3.3 \cdot 10^{-2}$. Find the distance between the stars in parsecs.

## 1.23 Binary Systems

In this problem, we aim to find the shape of the orbits of binary systems. After all, Kepler's basic laws referred only to a system where one body was considerably more massive and therefore remained practically stationary, so why should the orbits of binaries be ellipses? Obviously, we already know this fact, but understanding this reason will be very useful in the future.

### Part A: Orbital Parameters

Consider two bodies in the plane, as shown in the figure below:

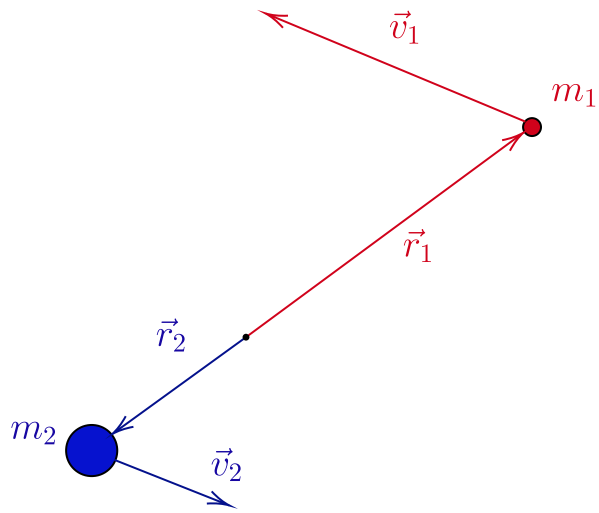



Figure 1.14: Binary System.

**A.1)** Find an expression for the gravitational force experienced between the bodies.

**A.2)** Remember that the center of mass in an isolated system moves with uniform velocity (use $v = 0$ without loss of generality). In this case, use the following trick: Suppose there were a body of mass $m_x$ at the center of mass and only this body acted on $m_i$ ($i \in \{1{,}2\}$), what should the value of $m_x$ be so that the orbit of $m_i$ is the same as in the previous case? Denote $m_j$ as the other component of the binary system.

**A.3)** Prove that the orbit is a conic section and find the values of $a_1$ and $e_1$ in terms of the velocity $v_{p,1}$ (velocity of body 1 at periapsis) and $L$, the minimum separation between the components. Then, show that $m_1 a_1 = m_2 a_2$ and that $e_1 = e_2 = e$.

### Part B: Relative Orbits

**B.1)** Write the total mechanical energy equation of the system in terms of the relative velocity $V$ of one star as seen from the other, $M$, the sum of the component masses, $\mu = \frac{m_1 m_2}{m_1 + m_2}$, the reduced mass of the system, and $L$, the distance between the components.

**B.2)** Reinterpret the result from the concept of relative orbit. What does each parameter represent?

### Part C: Lagrangian Points

Lagrangian points are points where, if a small test mass $m \ll m_1, m_2$ were placed, it would orbit with the same angular velocity as the system, that is, it would have the same orbital period and therefore would "follow" the revolutions of the binary. Get ready for the calculations!

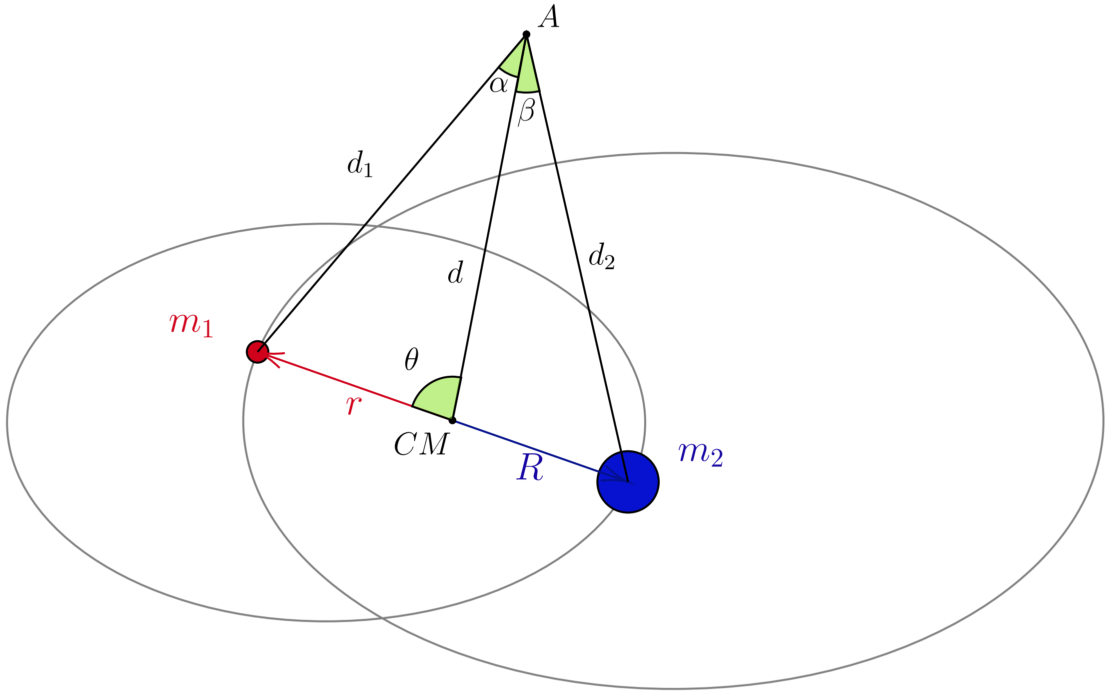



Figure 1.15: Components of the binary system ($m_1$ and $m_2$) and the Lagrangian point $A$.

Consider the binary system such that, at a given moment, body $m_1$ is located at a distance $r$ from the center of mass (CM) and $m_2$ at a distance $R$. Consider point A in the figure as a small test mass such that it has an orbit around the center of mass with the same period as the binary, equal to $T$. We wish to find the parameters that allow the existence of this orbit.

Thus, we have 2 cases: the case where $\sin(\theta) = 0$ ($\theta \in \{0{,}180°\}$), i.e., all three are aligned, and the case where $\sin(\theta) \neq 0$, where they are not aligned. In the following problems, the non-alignment cases of the Lagrangian point will be discussed. The other points are left as an exercise to the reader (checkmate!).

There are two points that satisfy this condition and they are symmetric with respect to the line connecting the components, so we only need to find one of them.

**C.1)** Given the gravitational forces of $m_1$ and $m_2$ on $A$, find a relation for the components parallel to $d$ and perpendicular to $d$. Again, use the trick that the orbit of $A$ will be dictated by a body of mass $m_x$, which will be determined during the calculations.

**C.2)** Prove that $d_1 = d_2 = D$ and that $d = kD$. Show the value of $k$ in terms of $m_x$, $m_1$, and $m_2$. Hint: Recall the law of sines and the law of cosines.

**C.3)** Relate the values of $a_x$ and $m_x$ with $a_1$, $a_2$, $m_1$, and $m_2$ using Kepler's third law, where $a_1$ and $a_2$ are the semimajor axes of the orbits of $m_1$ and $m_2$. Finally, show that $a_x = k(a_1 + a_2)$.

**C.4)** Find the eccentricity of $A$'s orbit from the calculation of the periapsis and apoapsis positions of $A$'s orbit. Hint: Recall the relation of $d$ with $L$.

**C.5)** Finally, relate the different equations for calculating $a_x$ and determine the value of $k$. Also show the values of $\cos(\theta)$, $m_x$, and $d$. Notice that since the value of angle $\theta$ is constant, the object maintains a constant angular position in the binary reference frame, which is precisely the condition for a Lagrangian point!

**C.6)** Show that $D = L$ and, therefore, the triangle $\Delta A, m_1, m_2$ is an equilateral triangle!
**Bonus:** it is possible to extend the previous result to any conic!

Paz e Amor

# 1.24 "Relative" Luminosity

In this problem, we aim to find an expression for the distribution of luminous power of a spherical wavefront emitted from a source in a reference frame $S'$, which emits light isotropically with total power $L$ and moves with velocity $v = \beta c$.

Before that, however, we will review some of the principles governing special relativity, so that we can proceed with the calculations.

## Part A: Lorentz Transformations

A transformation is nothing more than the relation between the coordinates of two reference frames, $S$ and $S'$. In the following derivations, consider $S$ as the stationary frame of an external observer and $S'$ as the frame moving with velocity $v$ along the $x$-axis relative to $S$.

Imagine that at the moment the origins of the two frames coincide, a light ray is emitted from the origin.

**A.1)** Show that:

$$x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2$$

where $x', y', z'$ and $t'$ are the coordinates in $S'$.

**A.2)** Argue, by contradiction, the necessity that $y = y'$ and $z = z'$ (note that this is not required for $x$ or $t$).

**A.3)** Note that in this way, it follows that $x^2 - (ct)^2 = x'^2 - (ct')^2$. Observe that this description is very similar to the squared modulus of a two-dimensional vector ($|\vec{v}|^2 = v_x^2 + v_y^2$), which remains constant after a given transformation. A linear transformation with this property is a rotation. Considering that in the transformation $S \to S'$, along the appropriate axes, there is a rotation by an angle $\theta$, find the relation $x \to x'$ and $t \to t'$.

**A.4)** Finally, find the Lorentz transformations, that is, the relation between coordinates in the two reference frames.

## Part B: Headlight Effect

**B.1)** Find the power $dL'$ passing through the area on the sphere (see the figure below) delimited by an angular interval $d\theta'$ from the $x'$-axis, as shown in Figure 1.16. Express the result in terms of the total power $L$, the angle $\theta'$, and $d\theta'$.

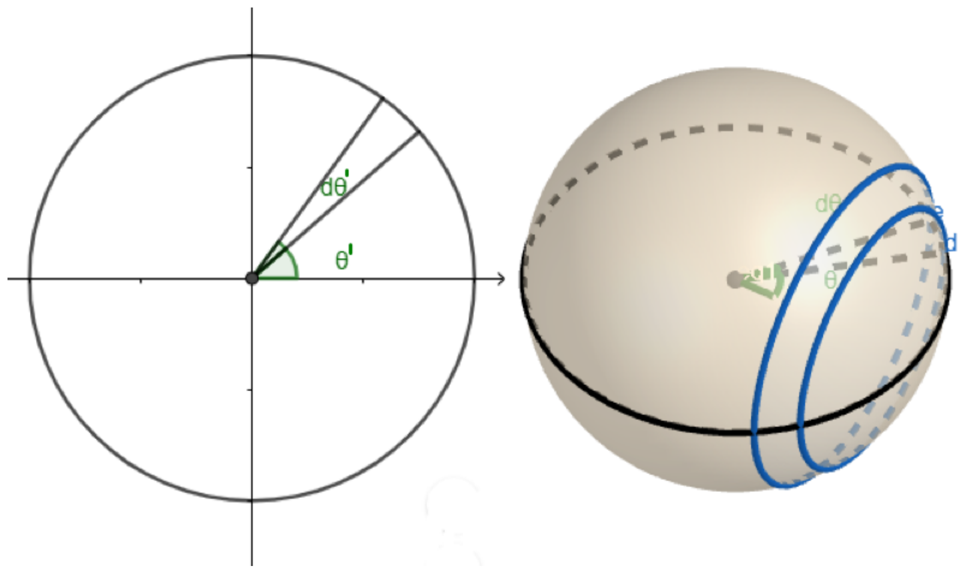



Figure 1.16: Representation of the considered angular interval.

**B.2)** Consider a light beam propagating at an angle $\theta'$ with respect to the $x'$-axis (figure below). In the reference frame $S$, this angle is $\theta$. Find an expression relating $\theta$ and $\theta'$.

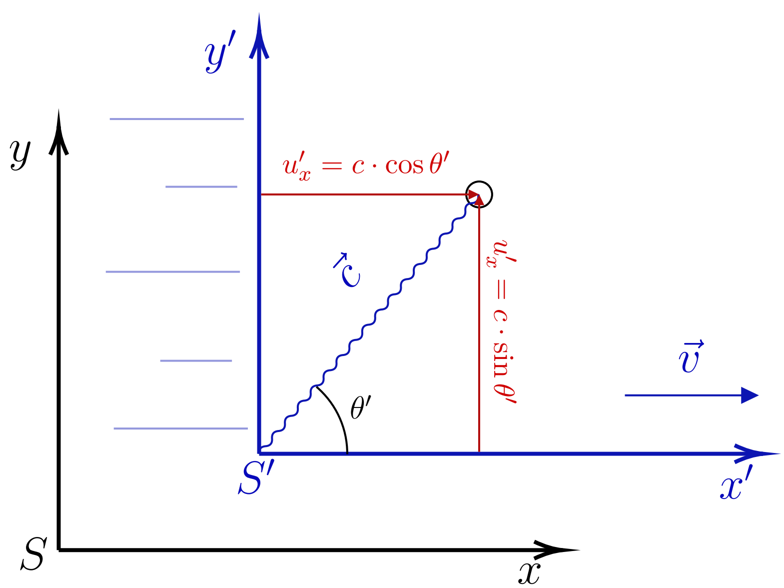



Figure 1.17: Headlight effect applied to the change in light propagation angle with the change of reference frame.

**B.3)** From the previous result, find the value of $d\theta'$ as a function of $d\theta$ and $\theta$.

## Part C: Redshift and Blueshift

Now we will consider the effects of changes in the frequency of light emitted due to relativistic effects of the source. Note that this change is also related to the energy of photons: $E \propto f$.

**C.1)** Considering the received light as a two-dimensional sinusoidal wave (we will analyze only the electric field) with phase $\phi = \vec{k} \cdot \vec{r} - \omega t$, rewrite the phase $\phi$ in terms of the angle $\theta$ and substituting the parameters $\vec{r}$ and $t$ with the parameters of $S'$: ($x'$,$y'$,$t'$).

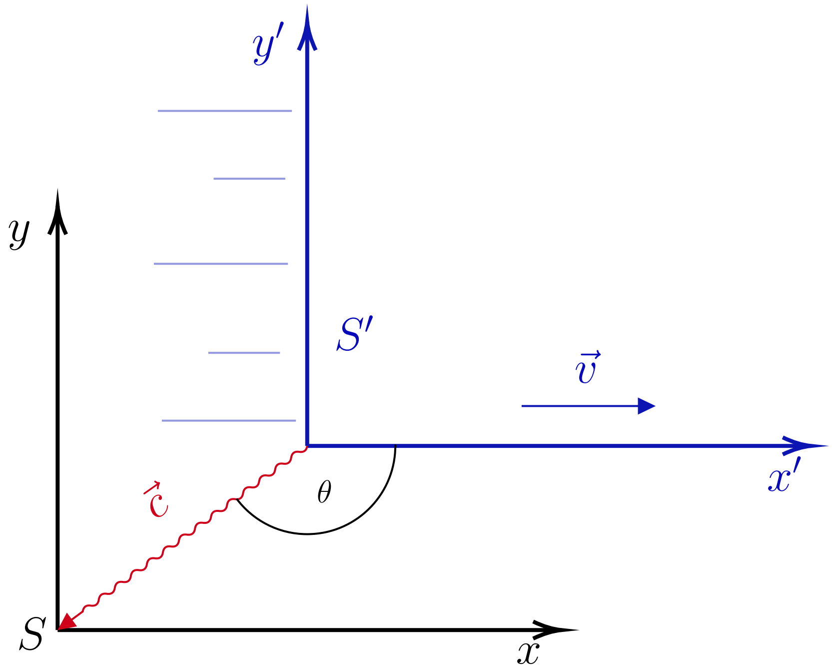


Figure 1.18: Incidence of light from the moving reference frame $S'$ relative to $S$.

**Reminder:** $\vec{r} = \vec{x} + \vec{y}$ and $\vec{k}$ is the angular wave number.

**C.2)** Compare the expression from the previous item with the phase $\phi'$ of light emission in the reference frame $S'$ and find the ratio between the received and emitted frequencies.

## Part D: The Grand Unification

Finally, we aim to find the distribution $dL(\theta)$ of power within the angular interval $d\theta$.

**D.1)** Find a relation between $\dfrac{dE}{dt}$ and $\dfrac{dE'}{dt'}$ ($E$ and $E'$ are the energies passing through the angular intervals).

**D.2)** Considering that $\dfrac{dE}{dt} = dL(\theta)$, find the distribution function $dL(\theta)$.

**D.3)** Integrate over the entire spherical surface to obtain the total luminosity observed in the reference frame $S$.

## 1.25 Intergalactic Spectroscopy

The proportion of baryonic matter determined the composition of matter in the universe! In the early universe, the concentration of baryonic matter was responsible for governing the formation of deuterium and helium in the short time interval when the universe was cold enough for stable nuclei to form and hot enough for nuclear fusion to occur. This matter composition has remained unchanged since the universe cooled, 20 minutes after its beginning, and we can currently use it to infer the proportion of baryonic matter relative to other cosmological components, such as dark matter and dark energy.

Until recently, computational simulations suggested that there should be a proportion of 5% of baryonic matter in the universe; however, only 2.5% was known, part of it found in galaxies, stars, intergalactic gases, among others. Where were the remaining 2.5%?! Fortunately, a recent phenomenon in the distant universe emitted radiation under conditions such that "invisible" particles could absorb this radiation in the $WHIM$ (Warm-Hot Intergalactic Medium). Using a technique known as the Lyman-Alpha Forest, high-frequency radiation underwent a redshift capable of allowing it to interact with particles in the $WHIM$, leaving absorption features in the spectrum. From these features, we were able to determine the remaining matter composition in these invisible regions of the universe.

Consider the mysterious light source emitting radiation at a temperature $T$ as an ideal blackbody and at an initial distance (when the radiation was emitted) of $D_e = 2469.03\ Mpc$. Denote the current time as $t_0 = 14.571$ billion years and the universal scale factor at the time of emission as $a_e$ (denote $a_0 = 1$ for the current scale factor). It is known that for absorption to occur in the $WHIM$, the wavelength[5] must lie between $\lambda_H$ and $\lambda_H + \Delta\lambda_H$. The particle density in the $WHIM$ was $\rho_e$ at the time of emission, and the medium's opacity is constant and equal to $\kappa$.

Cosmological studies show that the expansion of the universe is practically exponential, following the Hubble relation: $a = a_0 e^{H_0(t-t_0)}$, with $H_0$ being the Hubble constant.

**a)** Find the emission time $t_e$ of the radiation.

**b)** Find a relation between the redshift of the radiation and the scale factor of the universe at the moment.

**c)** Find a relation between the matter density of the particles in the $WHIM$ as a function of the scale factor.

**d)** Consider the portion of radiation with initial wavelength[6] equal to $\lambda$. Find the scale factor at which this portion starts to undergo absorption, and also the scale factor at which this portion finishes being absorbed.

**e)** Considering that the flux of this portion before absorption was $F_0$, find the flux after the full absorption in the $WHIM$. Neglect other factors that could alter the flux. It is known that $\Delta\lambda_H \ll \lambda_H$.

An astronomer on Earth, upon receiving the spectrum of the body, notices that an interval of wavelengths does not obey Planck's law in the flux distribution. The astronomer then related the obtained flux to the expected flux by correcting the problematic interval with a correction factor

[5] Also known as the Lyman-Alpha wavelength, which is the wavelength predominantly absorbed in the Lyman-Alpha Forest.

[6] Consider $\lambda < \lambda_H$.

$$\alpha(\lambda) = \frac{F_{obtained}}{F_{expected}}.$$

**f)** Find $\alpha(\lambda)$.

**g)** What is the observed wavelength interval in which this effect can be observed? Are there subintervals where the effect occurs partially? Determine them.

**h)** The astronomer collected the following data:

Table 1.5: Collected parameter data.

| Quantity | Value |
|---|---|
| $\kappa$ | $5.3 \cdot 10^{-2}\ \mathrm{m^2\,kg^{-1}}$ |
| $\Delta\lambda_H$ | 1.7 nm |
| $\lambda_H$ | 121.6 nm |
| $H_0$ | $67.15\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ |

He also created the following table relating the correction factor to the observed wavelength:

Table 1.6: Results of $\alpha$ as a function of wavelength $\lambda$.

| $\lambda$ ($\pm$1.4 nm) | $\alpha(\lambda)$ |
|---|---|
| 130.0 | 0.99964 |
| 160.0 | 0.99933 |
| 190.0 | 0.99887 |
| 220.0 | 0.99825 |
| 250.0 | 0.99744 |

Rewrite the previous table by adding the proper uncertainties in the correction factor.

**i)** Use the linear regression method, performing the necessary substitutions and algebraic manipulations, to determine the density of the $WHIM$ at the emission time, as well as its corresponding measurement uncertainty.

# 1.26 Apollo XXVI

## Part A: "Optical" Rocket

Consider a converging lens with radius $R$ and focal ratio $\beta$ in the vacuum of space. A celestial object illuminates the lens, which is perpendicular to the direction of light propagation. The radiation flux in that region of space is $F_{\odot}$. The lens has mass $m$. The object is far enough that the flux remains practically constant with distance.

**A.1)** Neglecting relativistic effects, determine the thrust force of the lens.

**A.2)** The Hubble telescope, with mass $m = 11\,110$ kg, has a primary lens with $\beta = 24$ and diameter $D = 2.4$ m. Determine how long it would take for it to travel a distance equal to the Earth's diameter (neglecting the gravitational attraction of the Earth and Sun).

**A.3)** What should be the mass of a star with the same luminosity as the Sun for the telescope to remain stationary in space?

## Part B: Matter Cannon

Consider a rocket with initial mass $m_0$ that ejects propellant at speed $u$ relative to its engines and starts from rest at $t = 0$. The rocket's mass can be described by the relation $m(t) = m_0 f(t)$.

**B.1)** Considering relativistic effects, find the relation for the rocket's velocity as a function of time.

**B.2)** Approximate the expression for non-relativistic cases and show that the equation becomes:

$$V(t) = -u \ln(f(t))$$

## Part C: Radiation Cannon

Consider a rocket with initial mass $m_0$ that ejects radiation as propulsion, starting from rest at $t = 0$. The rocket's mass can be described by $m(t) = m_0 f(t)$. Considering relativistic effects, find the relation for the rocket's velocity as a function of time.

## Part D: Ion Propulsion

Sapphire, a researcher from the planet Pluto II, contributed greatly to the development and improvement of rocket propulsion techniques and science on her world. In the year 3170, she developed a new liquid fuel for combustion engines, and later, in 3173, she created a design for engines powered by ions accelerated by potentials.

Regarding this last propulsion method, consider the scheme in Figure 1.19.

The engine works by injecting both elemental xenon[7] and free electrons in order to collide with the xenon and ionize it, generating a cation that will proceed toward the base of the nozzle and be accelerated by a set of parallel plates, separated by a distance $E$ and having a potential difference of $\Delta V = 5\ kV$ (do not worry about the sign of $\Delta V$, it is understood that the engine would only function if the outermost plate had the lowest potential).

[7] Element chosen for being inert.

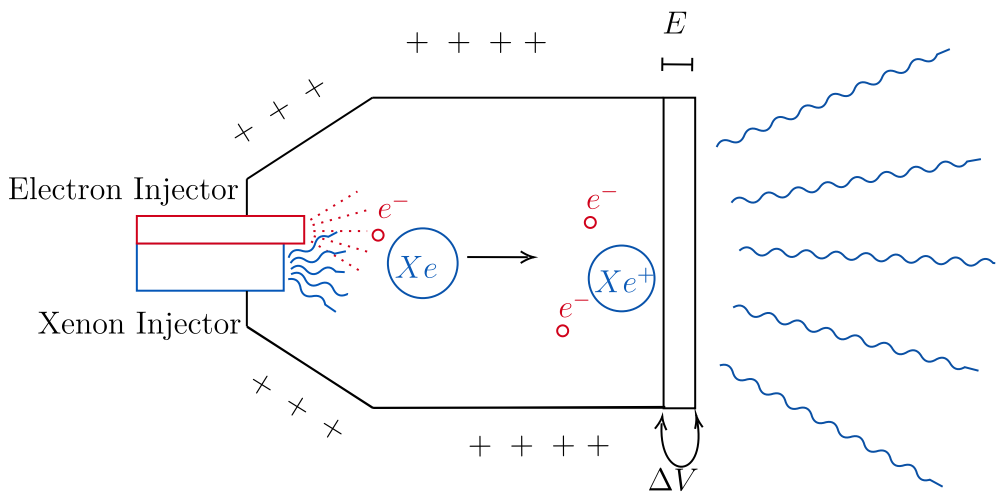



Figure 1.19: Theoretical model of the operation of a xenon ion engine.

**D.1)** Note that the walls of the nozzle are charged positively. Considering the charge dynamics and the tendency to avoid wear on the rocket, explain the purpose of this electrical protection.

**D.2)** Neglecting the initial velocity of the ions, find an expression for $\Delta V$ as a function of the surface charge densities of each plate.

**D.3)** Find the exit velocity of the cations from the rocket's nozzle. Consider the acceleration plates as infinite planes (while the ions are passing between them). Take the elementary charge as $1e^- = 1.602 \cdot 10^{-19}\ C$ and the mass of a xenon atom as $m = 219.66 \cdot 10^{-27}\ kg$.

**D.4)** Determine the velocity variation equation for a rocket with ion propulsion in non-relativistic cases, whose mass is $m(t) = m_0 f(t)$.

However, although the ion engine is capable of expelling ions at extremely high velocities, it does not have a high mass ejection rate, which significantly reduces the $\Delta v$ of the ion engine, making chemical engines much more powerful[8].

Therefore, both on Pluto II and on our beloved Earth, ion engines are not used for the initial planetary launch since extremely high thrust is required. However, ion engines, although slower, are extremely efficient, making them better in vacuum and in situations allowing gradual velocity changes, as they require much less propellant for propulsion.

## Part E: Space Exploration

Sapphire, to test the peculiarities of her engines, decided to send an analysis capsule to the neighboring planet Genibals using a rocket of her own design, also to learn more about her own solar system. The rocket will have multiple stages of complete ejection; that is, once activated, the fuel container will eject all the fuel contained and be detached at the end, using either an ion or chemical propulsion engine (chosen according to Sapphire's preference).

Preparing for her round-trip mission, she collected the following data:

- Both planets revolve and rotate in the same direction.
- Both planets have their equators in the orbital plane, which is the same for both.

[8] Currently, ion engines can take up to 4 days to achieve a $\Delta v = 90$ km/s

- The rotation period of Pluto II is 25 hours, and of Genibals is 20 hours.
- Pluto II and Genibals have circular orbits around Scorp, the system's star, with radii of 1.5 AU and 2 AU, respectively.
- The rocket will be launched from latitude 15° on Pluto II.
- The radius of Pluto II is $R_p = 6 \cdot 10^6$ m, and of Genibals, $R_g = 10^7$ m.
- The masses of Scorp, Genibals, and Pluto II are, respectively, $M_s = 3 \cdot 10^{30}$ kg, $M_g = 7 \cdot 10^{25}$ kg, and $M_p = 6 \cdot 10^{24}$ kg.

**E.1)** How many propulsion stages will need to be built in the rocket, and which engine should be used for each stage to achieve maximum efficiency?

**E.2)** At what latitude will the rocket arrive on Genibals?

**E.3)** Determine how many Plutonian years[9] the Sapphire capsule will have to stay on Genibals before it is possible to return to its home planet.

The fuel fluid of the chemical engine has a density $\rho = 1.30 \cdot 10^3$ kg/m$^3$, and the fuel tanks are hollow cylinders with an outer radius of 2 m and thickness $e = 10$ cm, made of material with density $\rho' = 4.01 \cdot 10^3$ kg/m$^3$, with the specifications shown in the figure below:

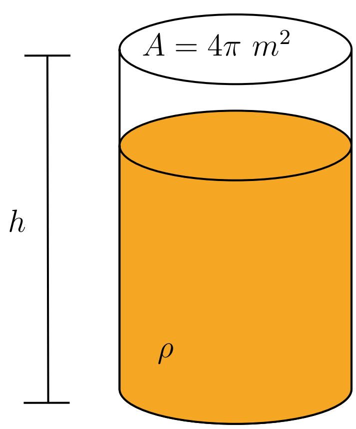



Figure 1.20: Example of a rocket fuel tank.

Consider that both the fluid and the container are ejected at a velocity of $u = 4$ km/s, and that the mass of the circular caps can be neglected, and that the mass of the capsule is $m = 10$ kg (the only part that will land on the ground). Also consider that even though all the fuel, when activated, is ejected, the spacecraft pilot has full control over the direction of the thrust, and can even adjust the direction during the impulse.

In the case of the ion engine, the ionization chamber stores the atoms and ions at a pressure of 10 atm and a temperature of 73.88 K. It is known that the material constituting the chamber is the same as that of the combustion containers, and the chamber also has the same specifications. The gas constant is $R = 8.314$ S.I.[10] and the molar mass of Xe is $M = 131.3$ g/mol. The containers are ejected at the same velocity as the gas.

[9] 1 Plutonian year is the orbital period of Pluto II

[10] Out of laziness, the author did not explicitly write the units

**E.4)** Determine the total mass, in tons, of the rocket at the moment of launch. Use the velocity equation for non-relativistic cases.

## 1.27 Hyperuniverses

Tired of doing Astronomy and Physics problems in three-dimensional universes?! In this problem, we will study the cosmological expansion of an n-dimensional universe!

Suppose that, for an n-dimensional universe, the gravitational attraction law between two bodies is given by:

$$\vec{F} = -\frac{G(n)m_1m_2}{r^{n-1}}\hat{r}$$

where $r$ is the distance between them and $G(n)$ is the gravitational constant for each universe. You can imagine the $n-1$ dependence as arising from the fact that the gravitational field lines of a mass would spread over the "area" of that universe, similarly to what happens in our three-dimensional universe.

### Part A: Hypersolid Angles

Consider the locus of all points at a distance $r$ from the origin in $\mathbb{R}^n$, in other words: an n-dimensional ball. One can define a hypersolid angle (terminology invented for this problem) as:

$$\Omega(n) = \frac{A(n)}{r^{n-1}}$$

where $A(n)$ is the surface area of the n-dimensional ball.

To calculate this area, we will use a recurrence principle and thus relate the terms $\Omega(i)$ and $\Omega(i+1)$.

However, instead of using areas, we will use volumes! It is known that:

$$A(n) = \frac{dV(n)}{dr}$$

**A.1)** Prove that:

$$V(n) = \frac{\Omega(n)}{n}r^n$$

We will now make an analogy with our universe in order to expand this idea: Normally, when we want to demonstrate the volume of a sphere, we use the fact that the volume of the sphere can be associated with the volume of several cylinders with circular bases that, when summed (integrated in the infinitesimal case), give us the volume of the sphere.

To do this, consider an $(n-1)$-dimensional sphere. Now project this sphere into the n-th dimension (just as a cylinder is the projection of a circle in the third dimension). This will form a kind of "cylinder" for an n-dimensional being. The volume of this cylinder will be the volume of the $(n-1)$-dimensional sphere times the $dh$ of the projection. You can imagine that you are one of these beings, but in 3 dimensions, and see a 2-dimensional sphere (a circle) projected into a cylinder. The "volume" of this circle would be its area.

From the next figure, you can see that the radius of the $(n-1)$-sphere (define sphere $i$ as the i-dimensional sphere) is $r = R\cos(\theta)$, where $R$ is the radius of the n-sphere. And $dh$ is $d(R\sin(\theta)) = R\cos(\theta)d\theta$.

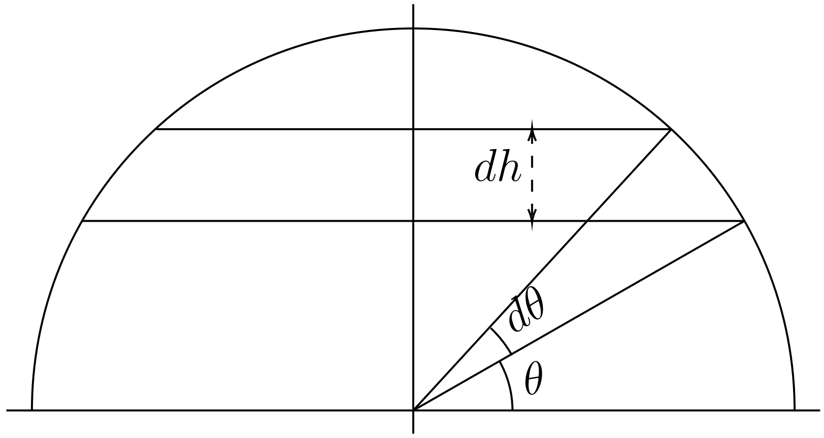



Figure 1.21: Representation of an n-dimensional sphere in a 2D space. For each angular position $\theta$, with an infinitesimal deviation $d\theta$, there is a cylinder with a base given by the $(n-1)$-dimensional sphere and thickness $dh$.

The volume of the projected cylinder will then be: $dV(n) = V(n-1)dh$.

**A.2)** Show that:

$$dV(n) = \frac{\Omega(n-1)}{n-1} R^n (\cos(\theta))^n d\theta$$

You can integrate the expression to find:

$$V(n) = \frac{\Omega(n)}{n} R^n = \frac{\Omega(n-1)}{n-1} R^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^n d\theta$$

Which implies:

$$\frac{\Omega(n)}{n} = \frac{\Omega(n-1)}{n-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^n d\theta$$

Since $d\sin(\theta) = \cos(\theta)d\theta$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^n d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{n-1} d\sin(\theta)$$

Also, remember the integration by parts formula:

$$\int_b^a f dg = [f(a)g(a) - f(b)g(b)] - \int_b^a g df$$

**A.3)** Using integration by parts, show that:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^n d\theta = \frac{n-1}{n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{n-2} d\theta$$

**A.4)** Prove that:

$$\Omega(n) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{n-2} d\theta \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{n-3} d\theta \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) d\theta \cdot 2\pi$$

**A.5)** Using the result from (A.3), show that:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{2k} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{2k-1} d\theta = \frac{\pi}{k}$$

where $k$ is a positive integer.

Now we will take an important leap: it doesn't matter whether $k$ is an integer or not! This step will not be explained here, but it works! Just as Newton expanded the binomial for non-integer exponents, we will use the same trick[11].

**A.6)** Finally, show that the hypersolid angle of dimension $n$ is given by[12]:

$$\Omega(n) = \frac{2\pi^{n/2}}{\left(\frac{n}{2} - 1\right)!}$$

**BONUS:** Another way to represent this result is using the following fact:

$$\Gamma\left(m + \frac{1}{2}\right) = \left(m - \frac{1}{2}\right)! = \frac{(2m)!}{2^{2m} m!}\sqrt{\pi}$$

From this, the result becomes:

$$\Omega(n) = \frac{2^n \pi^{\frac{n-1}{2}}}{(n-1)!}\left(\frac{n-1}{2}\right)!$$

Each of the previous formulas represent the same results, but it is easier to use one or the other depending on the parity of $n$.

## Part B: Gauss's Law

We will now analyze Gauss's Law for a Gaussian surface in $n$ dimensions surrounding a mass $m$.

Remember that Gauss's Law is used to find the flux of a field through a region of space. This flux is given by the relation:

$$d\phi = \vec{g} \cdot d\vec{A}$$

where the area vector has a magnitude equal to the area of the region and a direction and sense equal to the normal vector pointing outward from the region. Notice that this dot product can be written as the product of the magnitude of the field $\vec{g}$ and the effective area perpendicular to this field, as represented in the low-quality figure below.

Now imagine the Gaussian surface and analyze an area element $d\vec{A}$. This element has an effective area perpendicular to the field equal to $dA_t$.

**B.1)** Find the flux element passing through this area element.

[11] We can define this "trick" more formally using the gamma function (associated with factorials), such that:

$$\cos(x) = \frac{\pi}{\Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right)\Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right)}$$

[12] This result may seem strange, since for odd $n$, such as our 3-dimensional case, the formula has to handle factorials of fractional numbers, and $\pi$ would have a fractional exponent. However, the gamma function allows the existence of such "crazy" factorials. For example, for $n = 3$, $\frac{1}{2}! = \frac{\sqrt{\pi}}{2}$, which, when substituted into the formula, yields exactly $4\pi$, the solid angle of a 3-dimensional sphere!

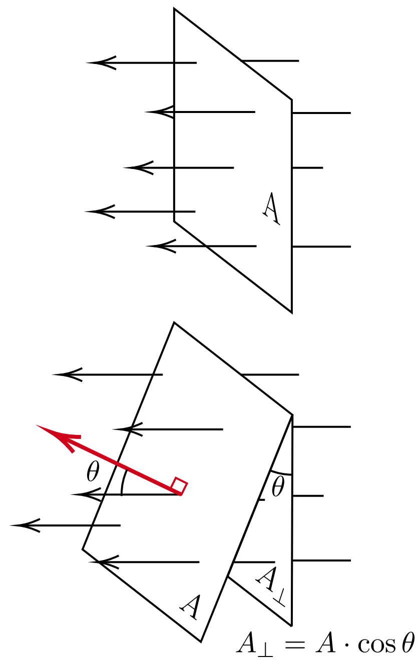



Figure 1.22: Gravitational flux with respect to area.

**B.2)** Show that, integrating over the entire closed surface, the result is:

$$\phi = \oint_S \vec{g} \cdot d\vec{A} = -\Omega(n)G(n)m$$

## Part C: Hyper Cosmology

Consider a flat $n$-dimensional universe, isotropic and homogeneous, with density $\rho$. At a distance $a(t)R$ from the center of the universe there is a test particle whose motion will be our object of study, where $a(t)$ is the scale factor governing the expansion of this universe. The mass enclosed within this radius is $m$.

**C.1)** Using Gauss's Law from the previous part, show that:

$$a^{n-1}\frac{d^2a(t)}{dt^2} = -\frac{G(n)m}{R^n}$$

**C.2)** Develop the previous relation and show the first Friedmann equation for such a universe:

$$H^2 = \frac{2G(n)}{n(n-2)}\Omega(n)\rho$$

where $H = \frac{\dot{a}}{a}$ and $\dot{k} = \frac{dk}{dt}$ for any quantity $k$.

**C.3)** Now we derive the second Friedmann equation for this universe, considering an adiabatic transformation. We can then associate simultaneously pressure ($P$), density ($\rho$), and scale factor ($a$), as follows:

$$\dot{\rho}c^2 + \gamma(P + \rho c^2)\frac{\dot{a}}{a} = 0$$

Show that $\gamma = n$.

**C.4)** Consider that the composition of the universe is given in two scenarios:

- Non-relativistic matter
- Radiation

Find the dependence of $\rho$ on $a$ for each scenario, and solving the first Friedmann equation determine how each model evolves over time.

## 1.28 Charge-Coupled Device

### Part A: A Bit of Quantum Mechanics

As mentioned in question 14, Nill continued his optical studies with the analysis of CCD operation. The CCD, which stands for *Charge-Coupled Device*, is a photosensitive device capable of converting the received radiation into electric charge. A CCD is composed of cells (pixels) that absorb radiation and use the energy of incident photons to release electrons from a semiconductor and move them to a transfer zone that confines the particles in a potential well. The charge is then transferred to an adjacent cell, recursively, until it reaches the reader at the edge of the device, as shown in the diagram below:

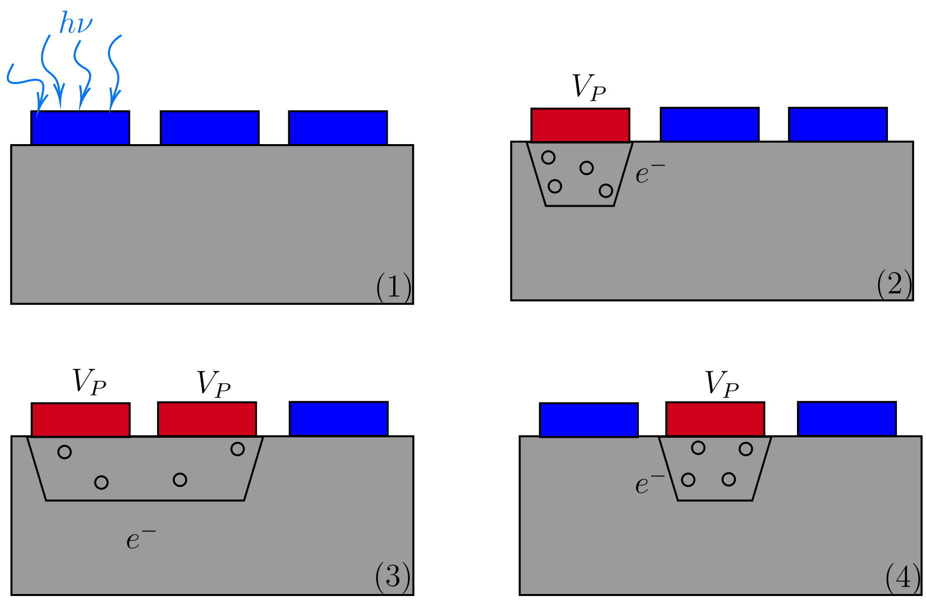



Figure 1.23: Operating principle of energy storage in the CCD and the movement of photon-electrons through the CCD *array*.

In this problem, we will analyze the effect caused by the storage of electrons in the potential well generated by the CCD's capacitors and how, due to quantum effects, the charge can escape and be lost.

For this, we will use a simplified model where each incident photon with energy greater than $E_\phi$, the energy between the conduction and valence bands of the semiconductor[13], excites an electron with energy equivalent to $E_\phi$. Consider that any excess energy is lost as thermal energy in the material.

**A.1)** Silicon is a semiconductor material commonly used in this type of device. It has a *band gap* of approximately 1.1 eV. Calculate the largest wavelength that can be detected by a CCD made of silicon.

In this model, the potential well has energy $V_P < E_\phi$, while the surrounding region has energy $V_0 > E_\phi$, as shown in the figure below, for the $x$ axis:

To analyze the probability of the electron escaping the well, we will use the time-independent

[13] Commonly known as the *Band Gap*

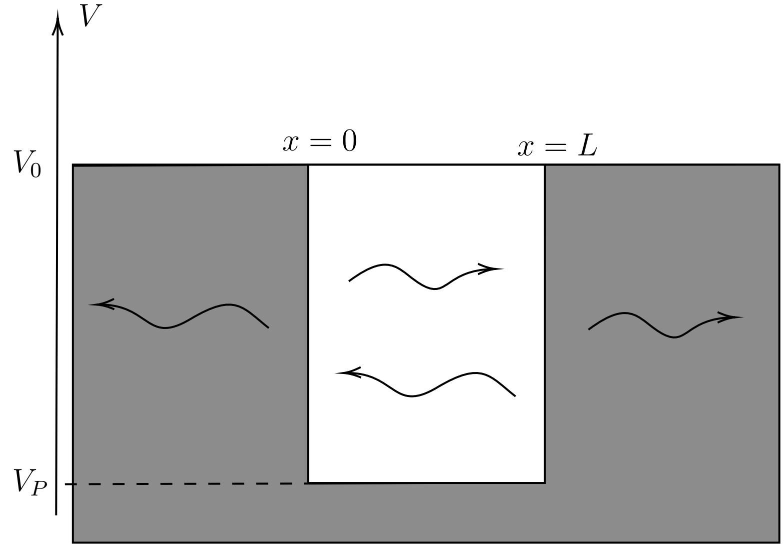



Figure 1.24: Potential well generated by the CCD for capturing photon-electrons.

Schrödinger equation[14], considering that changes occur on a timescale much longer than that required to reach equilibrium:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) + V(x,y,z)\psi = E\psi$$

Assume that the potential $V_P$ is equally generated in all three spatial dimensions, so that the wave function behaves independently in each direction.

**A.2)** Isolating the $x$-axis direction, and paying attention to the conditions for the existence of the function, show that the wave function can be described as:

$$\psi(x) = \begin{cases} Ae^{k_0 x} & \text{for } x < 0 \\ B_1 \sin(k_1 x) + B_2 \cos(k_2 x) & \text{for } 0 < x < L \\ Ce^{-k_0 x} & \text{for } x > L \end{cases}$$

Where $A, B_1, B_2$, and $C$ are generic constants, $k_0 = \sqrt{\frac{2m(V_0 - E_\phi)}{\hbar^2}}$ and $k_1 = \sqrt{\frac{2m(E_\phi - V_P)}{\hbar^2}}$.

**A.3)** Considering the boundary conditions at the interfaces between regions of different potential (where the values of the wave functions and their first derivatives must coincide), express all other coefficients in terms of $A$.

**A.4)** Prove the following relation for the well thickness $L$:

$$L = \frac{1}{k_1}\left(n\pi + \arctan\left(\frac{2k_1k_0}{k_1^2 - k_0^2}\right)\right)$$

With $n \in \mathbb{Z}$.

**A.5)** Considering the following data and assuming that $L$ is the first positive value that satisfies the previous relation, find the length of the potential well:

[14]In the equation, the operator $\nabla^2$ means: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

**DATA:**

- $V_0 = 3.43$ eV
- $V_1 = 0.85$ eV
- $E_\phi = 1.1$ eV
- $m = 9.11 \cdot 10^{-31}$ kg

Finally, discuss the applicability and accuracy of the current model.

**A.6)** According to the Copenhagen interpretation of the wave function found, the probability of finding the particle between positions $x_0$ and $x_1$, with $x_1 > x_0$, is:

$$P(x_0, x_1) = \int_{x_0}^{x_1} \psi^2(x)dx$$

Thus, calculate the probability that the generated electron is inside the potential well along the $x$-axis.

**A.7)** Finally, considering the three spatial dimensions, find the efficiency $\epsilon$ of the CCD's charge storage.

## Part B: Spectral Analysis

Consider that the edge flux (power emitted per unit area on the surface of the material) generated by the radiation of a spherical blackbody of temperature $T$ and radius $R$ has the following frequency distribution:

$$dF(\nu) = \frac{2\pi h}{c^2}\nu^3 \frac{1}{e^{\frac{h\nu}{k_b T}} - 1}d\nu$$

This distribution is known as Planck's law, responsible for solving the ultraviolet catastrophe problem through the implementation of energy quantization of radiation!

**NOTE:** Do not hesitate to use graphical or computational tools in this part!

**B.1)** Find the relation of power $dP(\nu)$ per frequency that reaches a CCD of area $A$, perpendicular to the direction of radiation propagation, at a distance $d$ from the center of the blackbody. Assume that $d >> R$ so that the rays arrive practically parallel in the analysis region.

**B.2)** Find the total current ($I$) recorded by the CCD array generated by the body's radiation, considering that the CCD's total efficiency is $\beta$.

Note that this CCD model is only capable of quantifying the total intensity of incident light but cannot specify the intensities of subintervals of wavelengths, that is, it cannot generate colors! One way to circumvent this problem is to filter the light before it reaches the device, in order to select the desired wavelength range.

**B.3)** Determine a relation for the current detected by a CCD with a filter that allows the passage of radiation with $\lambda \in [\lambda_1, \lambda_2]$ from the blackbody in question.

Nill had two different filters: U (ultraviolet) and V (visible). These filters have a radiation passband of $[332\text{nm}, 398\text{ nm}]$ and $[507\text{nm}, 595\text{ nm}]$, respectively.

**B.4)** Show that $M(U,V) = m_U - m_V$ (the difference of magnitudes of the blackbody for each frequency interval) is a function only of $T$. Also determine this function and plot its qualitative graph.

**B.5)** Find the value of $T$ for which $M(U,V) = 0$. What implications can be drawn from analyzing the color of an object's radiation with its temperature? Find the color index $M(U,V)$ for the Sun ($T \approx 5670$ K).

## 1.29 Pulsars

Pulsars are magnetized neutron stars that appear to emit short periodic pulses of radio radiation with periods between 1.4 ms and 8.5 s. Such objects, as they expel excess matter and shrink from millions to a few tens of kilometers in diameter, conserve their initial angular momentum while accelerating to extreme rotational speeds. This process also drastically boosts the object's magnetic field, which can increase by more than $10^{10}$ times.

However, in such objects, it is common for the magnetic dipole moment not to be exactly aligned with the star's rotation axis, generating a rapid variation in the surrounding magnetic field. As predicted by electromagnetism, radiation—and thus energy—can escape the star, slightly decreasing its rotation period.

In this problem, we aim to calculate the energy loss rate of a pulsar given its initial conditions and what this implies for its physical evolution.

### Part A: Let there be light!

In this section, we will study how the laws of electromagnetism govern the existence of light and how it is possible to find the relationships between the electric and magnetic field intensities of an electromagnetic wave, as well as its propagation speed.

Consider the following Maxwell equations, along with the fundamental wave equation, where $v$ is the wave propagation speed:

**Maxwell's Equations**

1. $\nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$
2. $\nabla \cdot \vec{B} = 0$
3. $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
4. $\nabla \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

**Fundamental Wave Equation**

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

**A.1)** Consider an electromagnetic wave moving in the positive $x$-direction, with its associated electric and magnetic fields oscillating in the $y$ and $z$ directions, respectively, as shown in Figure 1.25. Using relations 3 and 4, find the propagation speed $c$ as a function of the constants $\mu_0$ and $\epsilon_0$.

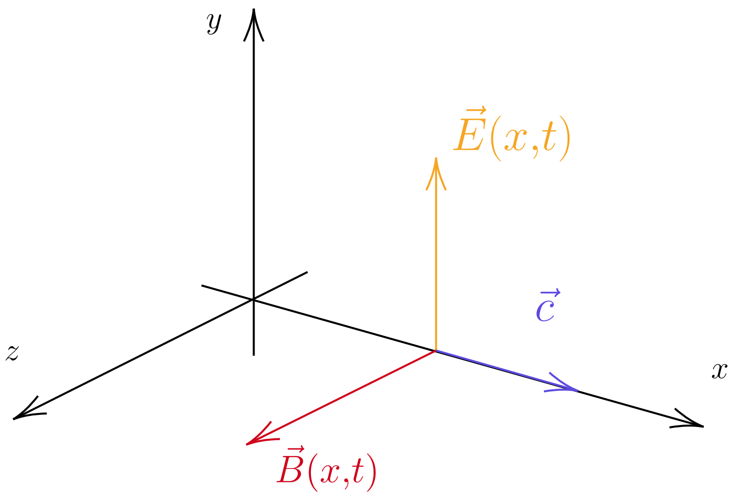



Figure 1.25: Schematic representation of the propagation of an electromagnetic wave.

A simple solution to the wave equation found in the previous item is a sinusoidal function, such that:

$$\begin{cases} E(x,t) = E_0 \sin(k_1 x - \omega_1 t + \phi_1) \\ B(x,t) = B_0 \sin(k_2 x - \omega_2 t + \phi_2) \end{cases}$$

**A.2)** Show that, for an electromagnetic wave, $\phi_1 = \phi_2$, $k_1 = k_2$, and $\omega_1 = \omega_2$, i.e., the excitations of the electric and magnetic fields are always coherent.

**A.3)** Show that $|\vec{E}(x,t)| = c|\vec{B}(x,t)|$, $\forall x, t \in \mathbb{R}$.

## Part B: Time-Varying Dipole

In this part of the problem, we will analyze the energy dissipation effect of an electric dipole rotating in uniform circular motion. Consider a pair of particles with charges $q$ and $-q$ such that one is fixed and the other orbits the fixed charge in a circle of radius $R$ with angular velocity $\Omega << \dfrac{c}{R}$, as shown in Figure 1.26.

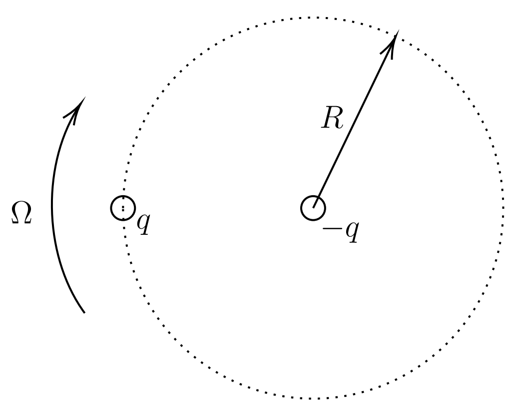



Figure 1.26: Diagram of an electric dipole rotating in uniform circular motion.

Consider that at a given instant $t_0$ the particle $q$ has velocity $\vec{v}$. In this case, adopt the inertial frame $S$ moving with $\vec{v}$, so that in this frame the particle $q$ is initially at rest at $t_0$, although it has an acceleration (perpendicular to $\vec{v}$).

This careful choice of reference frame makes the problem now that of a particle at rest which is accelerating with acceleration $|\vec{a}| = \Omega R^2$:

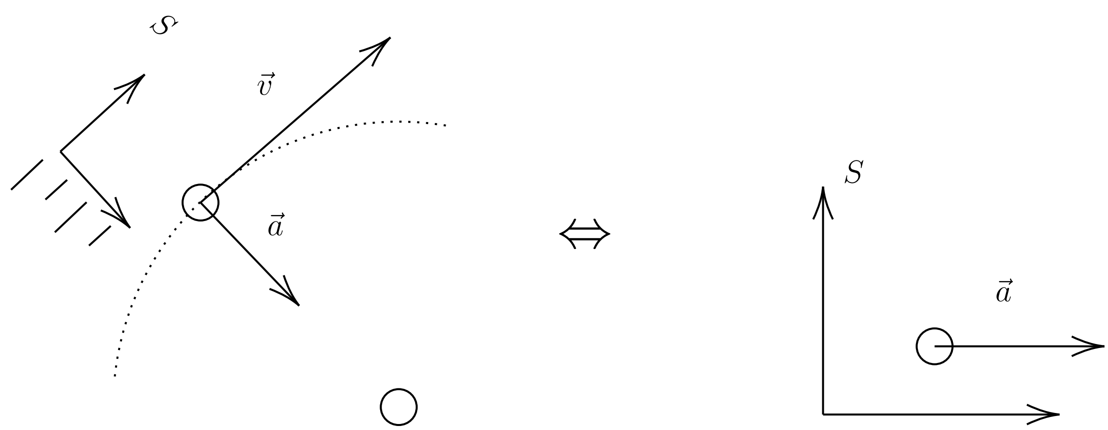



Figure 1.27: Reference frame simplification.

In this situation, the particle "remains" for a time $dt$ in this reference frame, until its velocity changes due to the circular influence of the acceleration. However, in this time interval from $t_0$ to $t_0 + dt$, the particle increases its velocity by $dv = a\,dt$, so that at time $t_0 + T$, the disturbance in the electric field lines caused by the sudden motion of the particle has propagated a distance $R = cT$, as shown in Figure 1.28.

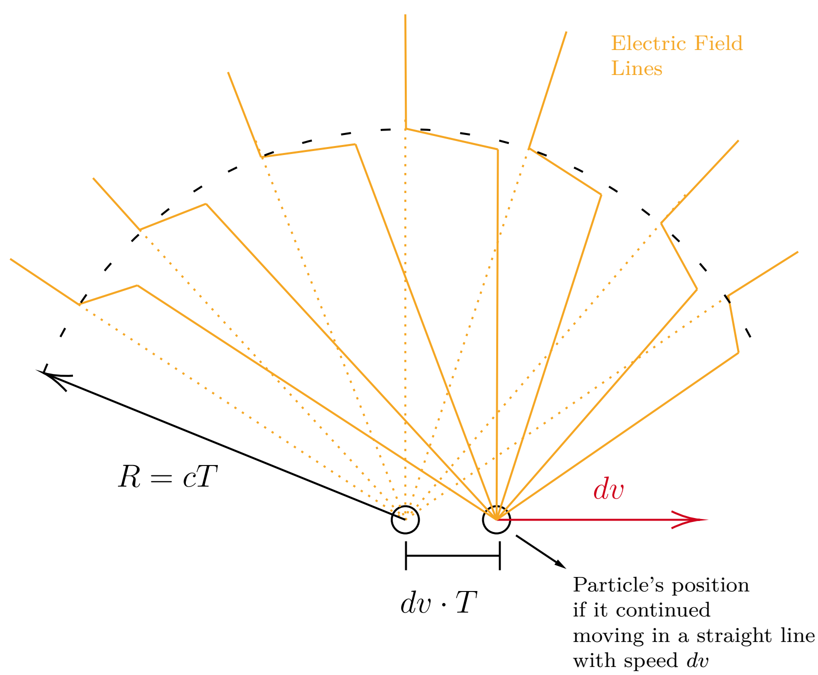



Figure 1.28: Variation of electric field lines due to the particle's motion. As the particle moves, the field lines adjust to its new position, generating a disturbance that propagates at the speed of light $c$.

**B.1)** Note that in the disturbance zone the electric field has a component perpendicular $\vec{E}_\perp$ to the particle (not only the radial component: $\vec{E}_r$). Thus, find the value of $\dfrac{|\vec{E}_\perp|}{|\vec{E}_r|}$ for the region at an angle $\theta$ with $dv$, as a function of $a$, $T$, $\theta$, and $c$.

**B.2)** Knowing that $E_r$ is normally calculated by Coulomb's law, find the value of $E_\perp$ as a function of $q$, $a$, $\theta$, $R$, and $c$.

**B.3)** Argue that, in a region sufficiently far from the dipole, the electric field can be approximated as $E_\perp$.

**B.4)** Considering that the radiation flux (power emitted per unit area) in a region of space is defined as:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

Thus, find the flux at a region at a distance $R$ from the particle's initial position and at an angle $\theta$ from the direction of $dv$.

**B.5)** Show that the total power emitted due to the dipole's rotation can be expressed as:

$$P_{ele} = \frac{p^2\Omega^4}{6\pi\epsilon_0 c^3}$$

where $p = qR$ is the dipole moment of the particle system.

## Part C: Pulsars

Consider for this part a neutron star that has a magnetic field at its north magnetic pole equal to $B$, radius $R$, mass $M$, and rotation period $P$. This celestial body has its magnetic axis tilted by an angle $\theta$ relative to the rotation axis, as shown in Figure 1.29.

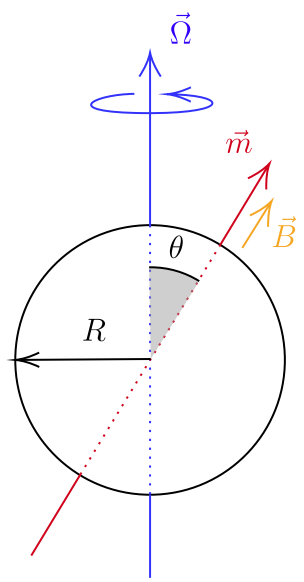



Figure 1.29: Graphical scheme of the pulsar in question, the rotation axis is represented by the angular velocity vector with magnitude $\frac{2\pi}{P}$.

Note that in this case, the variable dipole moment is not an electric dipole, but rather a magnetic dipole. Nevertheless, a similar intuition is used to calculate the power dissipated by a magnetic dipole $m$ rotating in uniform circular motion with angular velocity $\Omega$, whose equation is presented as follows:

$$P_{mag} = \frac{\mu_0}{6\pi c^3} m^2 \Omega^4$$

**C.1)** Considering that the magnetic field in a region of space with $R \leq |\vec{r}|$ is described by equation 1.1, find the power emitted by the pulsar as a function of $B$, $R$, $\theta$, $P$ and other natural constants.

Measuring the magnetic field of a distant celestial body is a complex task; however, from the relation in the previous item, it is possible to relate this physical quantity to other parameters that are easier to detect. Therefore:

**C.2)** Considering that the pulsar has a homogeneous matter distribution, that the dissipated power comes from the rotational kinetic energy of the pulsar, and that it has a period variation rate $\dot{P}$, calculate the magnetic field $B$ at the magnetic north pole of the star.

CHAPTER

# 2

# SOLUTIONS

Let $\phi_K$ be Kleber's latitude and $h$ the altitude of the object, using the known relation between hour angle ($H$), declination ($\delta$), latitude ($\phi$), and altitude ($h$):

$$\cos H = \sin h \sec \delta \sec \phi - \tan \delta \tan \phi$$

Given the latitude and altitude of the object, the hour angle is needed to find the declination. For this, we will use Nill's observation.

The hour angle for Nill is $1h15min$. Considering the difference in longitude: $\Delta\lambda = \lambda_K - \lambda_L = -57°38'06'' - (-47°28'33'') = -10°9'33'' = -40min38.2s$

Thus: $H_K = H_L + \Delta\lambda = 1h15min - 40min38.2s = 34min21.8s = 8°35'27''$. Substituting the values:

$$\cos 8°35'27'' = \sin 40° \sec \delta \sec -25°16'55'' - \tan \delta \tan -25°16'55''$$

Define: $a = \tan 25°16'55'' = 0.4723$; $b = \sin 40° \sec 25°16'55'' = 0.7109$ and $c = \cos 8°35'27'' = 0.9888$. Remembering that $(\sec \theta)^2 = (\tan \theta)^2 + 1$, and letting $\tan \delta$ be $x$:

$$ax + b\sqrt{x^2+1} = c$$

$$x^2 + 1 = \left(\frac{c}{b} - \frac{a}{b}x\right)^2$$

Define $\frac{c}{b} = C = 1.3909$ and $\frac{a}{b} = A = 0.6644$:

$$(A^2-1)x^2 - 2ACx + C^2 - 1 = 0$$

$$-0.5586x^2 - 1.8481x + 0.9346 = 0$$

Solving for $x$:

$$x = \tan \delta = \frac{1.8481 \pm 2.346}{-1.1172} = -1.6542 \pm 2.1$$

Finally, we find two values:

$$\delta_1 \approx 24^o$$

$$\delta_2 \approx -75^o$$

Now think: since the hour angle is approximately 8°, the object is "almost" on the local meridian. Thus, from the previous diagram, we conclude that the value we should adopt is $\delta_1 = 24°$:

Knowing the declination of the object, we must now find its right ascension $\alpha$. For this, we will find the sidereal time $T_L$ of Leibniz at the time of observation, knowing that $T_L = H + \alpha$. Since at the beginning of the year the sidereal time at Greenwich was $\alpha_G$, the sidereal time at the longitude $\lambda_L$ of Imperatriz-MA was $T_L = \alpha_G + \lambda_L$: $T_L = 6h43min28.5s + (-3h9min54.2s) = 3h33min34.3s$.

To determine the sidereal time at the time of observation, remember that there are two effects to be considered: the Earth's translation and the Earth's rotation, therefore:

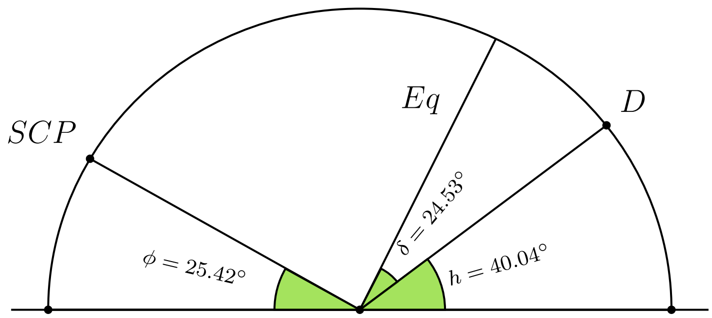
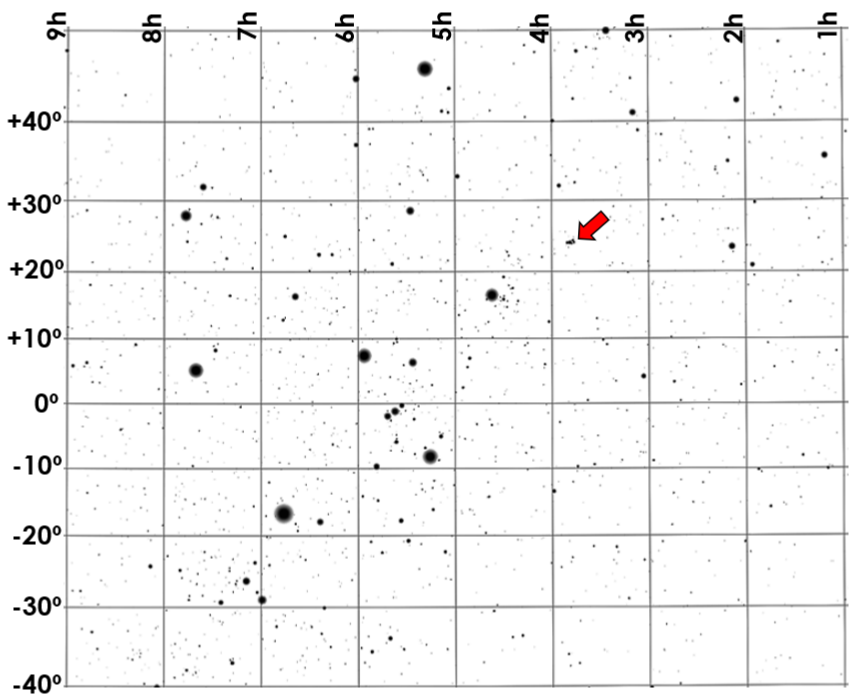



Figure 2.1: Representation of the geometry and the position of the points on the celestial sphere.

$$\Delta T_L = T_L - T_{L,0} = \frac{24h}{365.24}\Delta T_{days} + \frac{24h}{23.9447}\Delta T_{hours}$$

For the time of the analysis: $\Delta T_{days} = 51.920 days$ and $\Delta T_{hours} = 22.027 hours$ (remember to add one hour when converting from Kleber's timezone to Leibniz's). Thus, we finally find that $T_L = 29h2min56.6s = 5h2min56.6s$. From there, we determine that $\alpha = 3h47min56.6s$. In the presented image, this corresponds to the following point:




This means that the observed object was the **Pleiades (M45)**.

# 2.2 M666

## Part A: Photometry

**A.1)** Writing Pogson's equation for an individual star:

$$m_i = -2.5 \log \frac{F_i}{F_v}$$

Rearranging to obtain the value of $F_i$ (flux):

$$F_i = F_v 10^{-\frac{m_i}{2.5}}$$

Note that the flux of the cluster is simply the sum of the flux of all stars (since the distance to Earth is approximately the same, the flux is proportional to the intrinsic luminosity), thus the total flux $F_t = \sum F_i$, and the equivalent magnitude is:

$$M = -2.5 \log \left( \frac{\sum F_i}{F_v} \right)$$

$$M = -2.5 \log \left( \frac{F_v \sum 10^{-\frac{m_i}{2.5}}}{F_v} \right)$$

$$\therefore M = -\frac{5}{2} \log \left( \sum 10^{-\frac{2m_i}{5}} \right)$$

**A.2)** If the apparent magnitudes are in an arithmetic progression, it means: $m_{i+1} = m_i + \Delta m$, where $\Delta m$ is the common difference of this progression. Thus, the sum $\sum 10^{-\frac{2m_i}{5}}$ can be written as:

$$10^{-\frac{2m_1}{5}} + 10^{-\frac{2m_2}{5}} + 10^{-\frac{2m_3}{5}} + ... = 10^{-\frac{2m_1}{5}} + 10^{-\frac{2(m_1+\Delta m)}{5}} + 10^{-\frac{2(m_1+2\Delta m)}{5}} + ...$$

Assuming a large number of stars, we can approximate as follows:

$$\sum 10^{-\frac{2m_i}{5}} = 10^{-\frac{2m_1}{5}} + 10^{-\frac{2m_1}{5} - \frac{2\Delta m}{5}} + 10^{-\frac{2m_1}{5} - \frac{4\Delta m}{5}} + ...$$

Notice that this is the sum of an infinite geometric series (since $\Delta m > 0$, $10^{-\frac{2\Delta m}{5}} < 1$, so the sum converges). Therefore:

$$\sum 10^{-\frac{2m_i}{5}} = \frac{10^{-\frac{2m_1}{5}}}{1 - 10^{-\frac{2\Delta m}{5}}} = 10^{-\frac{2m_1}{5}} \cdot \left(1 - 10^{-\frac{2\Delta m}{5}}\right)^{-1}$$

Applying it in Pogson's equation:

$$M = -\frac{5}{2} \log \left( 10^{-\frac{2m_1}{5}} \cdot \left(1 - 10^{-\frac{2\Delta m}{5}}\right)^{-1} \right)$$

$$M = -\frac{5}{2} \log \left( 10^{-\frac{2m_1}{5}} \right) - \frac{5}{2} \log \left(1 - 10^{-\frac{2\Delta m}{5}}\right)^{-1}$$

Finalizing:

$$M = m_1 + \frac{5}{2} \log \left(1 - 10^{-\frac{2\Delta m}{5}}\right)$$

**A.3)**
If $\Delta m \ll 1$: $10^{\frac{-2\Delta m}{5}} \approx 1$ (with $10^{\frac{-2\Delta m}{5}} < 1$), so $1 - 10^{\frac{-2\Delta m}{5}} \approx 0$.
Using the approximation $\ln(1+\alpha x) \approx \alpha x$, for $\alpha x \ll 1$ (in this case $10^{\frac{-2\Delta m}{5}} - 1 = \alpha x$:

$$\therefore \ln\left(1 + 10^{\frac{-2\Delta m}{5}} - 1\right) \approx 10^{-\frac{2\Delta m}{5}} - 1 \approx \ln\left(10^{\frac{-2\Delta m}{5}}\right)$$

Changing the base: $10 = e^{\ln 10}$, we get:

$$10^{-\frac{2\Delta m}{5}} - 1 \approx \ln\left(10^{\frac{-2\Delta m}{5}}\right) = -\frac{2\Delta m}{5}\ln 10$$

To find $1 - 10^{-\frac{2\Delta m}{5}}$:

$$1 - 10^{-\frac{2\Delta m}{5}} = \frac{2\Delta m}{5}\ln 10$$

Finalizing:

$$M = m_1 + \frac{5}{2}\log\left(\frac{2\Delta m}{5}\ln 10\right)$$

**A.4)** For $\Delta m \gg 1$: $10^{\frac{-2\Delta m}{5}} \approx 0$, so:

$$1 - 10^{\frac{-2\Delta m}{5}} = 1 + \left(-10^{\frac{-2\Delta m}{5}}\ln 10\right) \cdot \frac{1}{\ln 10}$$

As $-10^{\frac{-2\Delta m}{5}} \cdot \ln 10 \approx 0$, we can approximate: $1 + \left(-10^{\frac{-2\Delta m}{5}}\ln 10\right) \cdot \frac{1}{\ln 10} = 10^{\frac{-10^{\frac{-2\Delta m}{5}}}{\ln 10}}$.

$$\therefore \log 10^{\frac{-10^{\frac{-2\Delta m}{5}}}{\ln 10}} = \frac{-10^{\frac{-2\Delta m}{5}}}{\ln 10}$$

Finalizing:

$$M = m_1 - \frac{5}{2}\frac{10^{\frac{-2\Delta m}{5}}}{\ln 10}$$

## Part B: Physical Analysis

**B.1)**
For non-relativistic velocities, we can use the Doppler effect to find the redshift of the H$\alpha$ frequency:

$$f = f_0\left(1 - \frac{v}{c}\right)$$

Solving for velocity $v$, in relation to the cluster:

$$v = c\left(1 - \frac{f}{f_0}\right)$$

For a large number of stars, we assume the average of the stars is:

$$v_{cl} = c\left(1 - \frac{\sum f_i}{f_0 n}\right)$$

In which $n$ is the total number of stars.

**B.2)**

As the object is in recession and it is homogeneous (with stars equally distributed) and isotropic (velocity vectors equally distributed among stars), the system's internal kinetic energy is given by:

$$E = \frac{1}{2}\sum m_i v_i^2$$

Given that each $m_i$ and $v_i$ is equal:

$$E = \frac{1}{2}nmv^2$$

**B.3)**

The Virial theorem states that for a bound system in equilibrium:

$$2U + E_p = 0$$

Where $U$ is the internal kinetic energy, and $E_p$ is the internal potential energy. Since $U = \frac{1}{2}E_p$:

$$U = \frac{1}{2}E_p = \frac{1}{2}\frac{-GM^2}{R}$$

Where $G$ is the gravitational constant and $R$ is the radius of the cluster. Rewriting:

$$U = \frac{1}{2}\frac{-GM^2}{R}$$

$$\therefore M = \frac{UR}{G}$$

Rewriting to account for internal kinetic energy $U$:

$$M = \frac{3nmv^2 R}{G}$$

## 2.3 Chaotic Observation

First, we must construct the orbit. From there, we can find all the other important elements of the orbit following these steps:

1. Draw the semi-major axis of the ellipse, finding point $G$, which is the farthest from the origin (this happens because Earth is located on the ellipse's axis).
2. Draw the perpendicular bisector of the major axis, find the center $K$ of the ellipse, and identify the minor axis of the ellipse.
3. Find the focal distance ($c$) knowing that $a^2 = b^2 + c^2$ and mark the position of the Sun (the Sun is at one of the foci, so it is a distance $c$ from the ellipse's center).
4. Calculate the distance between Earth and the Sun, knowing that this equals $1.471 \cdot 10^{11}$ m (the Earth's perihelion distance), calculate the semi-major axis of the orbit, and other relevant parameters.

After this, we will have something like:

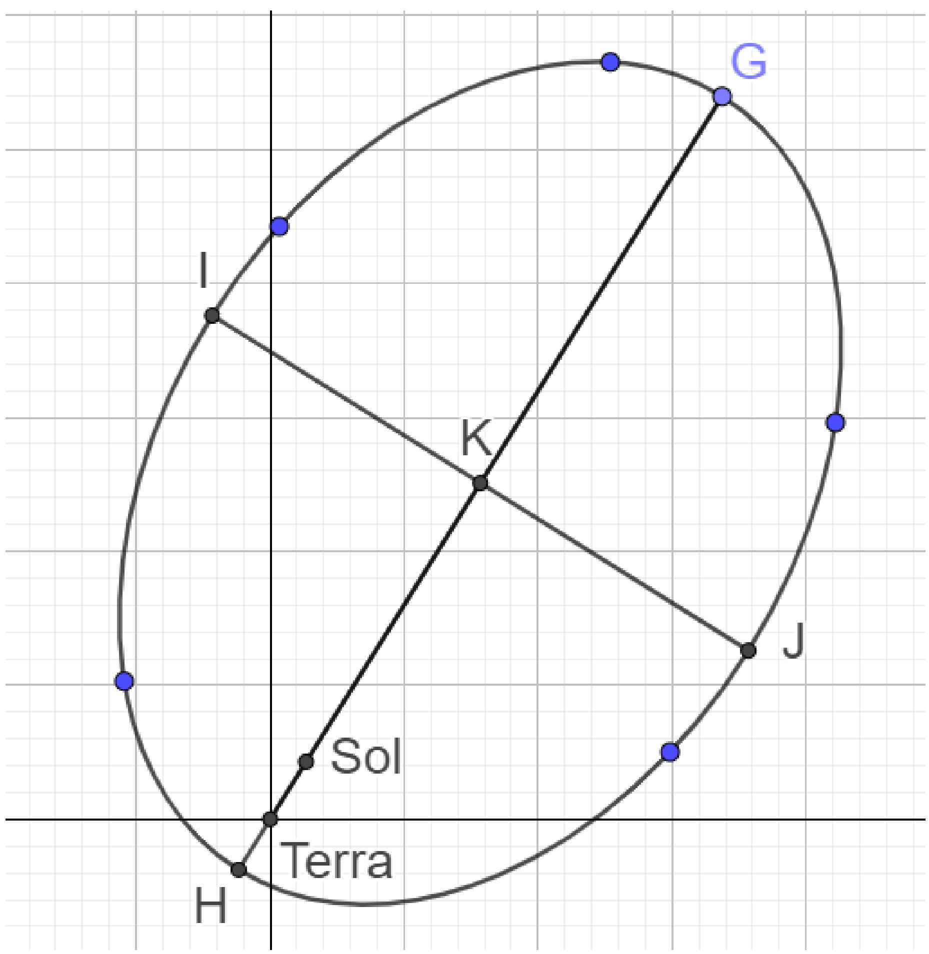



Figure 2.2: Conic associated with the 5 points provided by the statement. Conic generated using the free software GeoGebra.

Given that $HG$ is the major axis ($2a = 68.06721$ Gm) and $IJ$ is the minor axis ($2b = 47.17954$ Gm). The distance from $K$ to the Sun is the parameter $c$ of the ellipse, which can be calculated as $c = \sqrt{a^2 - b^2} = 24.53180$ Gm.

**a)** In the diagram, the distance from Earth to the Sun is 5.05501 Gm, which corresponds to $1.471 \cdot 10^{11}$ m. Therefore: 1 Gm $= 2.91 \cdot 10^{10}$ m $= 0.1945$ AU.

**b)** Just convert from Guanômetro to meters: $a = 9.9 \cdot 10^{11}$ m $= 6.619$ AU.

**c)** The eccentricity of the orbit is simply: $e = \frac{c}{a}$, which is independent of the unit of measure:

$$e = \frac{24.53180}{68.06721/2} = 0.72$$

**d)** By Kepler's third law for the solar system:

$$P_{\text{years}} = (a_{\text{AU}})^{\frac{3}{2}}$$

$$P = 17 \text{ years}$$

# 2.4 Ballistics 2.0

Consider the following orbital scheme:

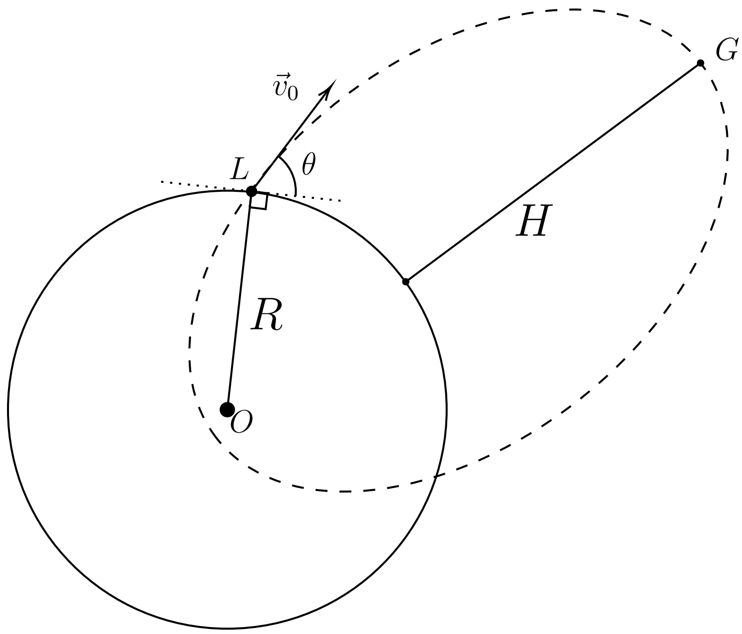



Figure 2.3: Trajectory of Ramanu Jan's controller.

**a)** In order to find $H$, notice that we only need to determine the apogee: $a(1+e)$ and subtract the planet's radius $R$ from it. We can find the semi-major axis of this orbit using the velocity relation:

$$v_0^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right)$$
$$\therefore a = \frac{GMR}{2GM - v_0^2 R}$$

To find the eccentricity $e$, we will use the conservation of angular momentum:

$$v_0 R \sin(\theta + 90^\circ) = v_p r_p$$

Where $v_p$ and $r_p$ are the velocity and distance at perihelion, respectively. We know that $v_p = \sqrt{GM\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)}$ and $r_p = a(1-e)$. Note that $\sin(\theta + 90^\circ) = \cos(\theta)$, therefore:

$$v_0 R \cos(\theta) = \sqrt{GMa(1-e^2)}$$

Thus:

$$e = \sqrt{1 - \frac{v_0^2 R^2 \cos^2(\theta)}{GMa}}$$
$$\therefore e = \sqrt{1 - \frac{v_0^2 R \cos^2(\theta)}{G^2 M^2}(2GM - v_0^2 R)}$$

Finally:

$$H = a(1+e) - R = \frac{GMR}{2GM - v_0^2 R}\left(\frac{v_0^2 R}{GM} + \sqrt{1 - \frac{v_0^2 R\cos^2(\theta)}{G^2M^2}(2GM - v_0^2 R)} - 1\right)$$

In the case where $v_0^2 \ll \dfrac{2GM}{R}$, we use the approximation $\sqrt{1+x} \approx 1 + \dfrac{x}{2}$:

$$H = \frac{GMR}{2GM}\left(\frac{v_0^2 R}{GM} + 1 - \frac{v_0^2 R\cos^2(\theta)}{2G^2M^2}(2GM) - 1\right)$$

$$H = \frac{R}{2}\left(\frac{v_0^2 R}{GM} - \frac{v_0^2 R}{GM}\cos^2(\theta)\right)$$

It is known that $1 - \cos^2 x = \sin^2 x$, therefore:

$$H = \frac{R^2 v_0^2}{2GM}\sin^2(\theta)$$

Knowing that the gravitational acceleration on Earth's surface is $g = \dfrac{GM}{R^2}$, we conclude:

$$H = \frac{v_0^2 \sin^2(\theta)}{2g}$$

Which is the classical result!

**b)** We can calculate the angular displacement by the angle swept by the position vector during the trajectory. Thus, the satellite reaches the ground when $r = R$:

$$R = \frac{a(1-e^2)}{1 + e\cos(\phi)}$$

$$\phi = \arccos\left(\frac{a(1-e^2) - R}{Re}\right)$$

The angular displacement is $2\pi - 2\phi$ (remember that $\phi$ is the angle of the position vector at perihelion). Therefore, the displacement over the planet is:

$$\Delta L = 2R\left(\pi - \arccos\left(\frac{a(1-e^2) - R}{Re}\right)\right)$$

In the case where $v_0^2 \ll \dfrac{2GM}{R}$:

$$a = \frac{GMR}{2GM - v_0^2 R} \approx \frac{R}{2}$$

and

$$e = \sqrt{1 - \frac{v_0^2 R^2 \cos^2(\theta)}{GMa}}$$

In this case, we will use another approximation: the $\Delta L$ is practically equal to the linear displacement between the two points (launch point and landing point), as shown in Figure 2.4.

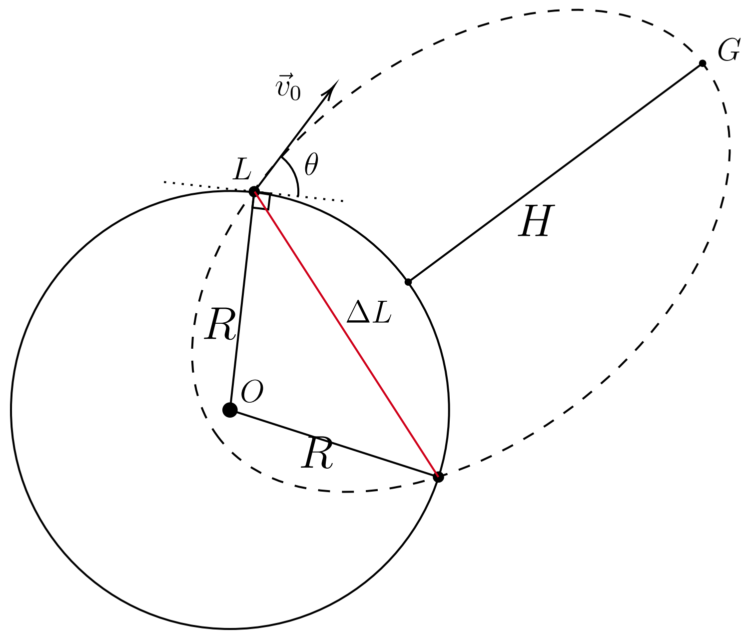



Figure 2.4: Final displacement of the controller trajectory.

$$\Delta L = 2R\sin(\pi - \phi) = 2R\sqrt{1-\cos^2\phi}$$

$$\cos\phi = \frac{a(1-e^2)-R}{Re}$$

Substituting the expressions:

$$\cos\phi = \frac{v^2R\cos^2\theta}{GMe} - \frac{1}{e}$$

$$1-\cos^2\phi = 1 - \frac{v^4R^2\cos^4\theta}{G^2M^2e^2} + \frac{2v^2R\cos^2\theta}{GMe^2} - \frac{1}{e^2}$$

Rearranging a bit, we arrive at the following expression:

$$1-\cos^2\phi = \frac{R^2v^2\cos^2\theta}{G^2M^2e^2}\left(\frac{2GM}{R} - \frac{GM}{a} - v^2\cos^2\theta\right)$$

We see that the previous expression is the mechanical energy of the body (remember again that $\cos^2\theta + \sin^2\theta = 1$):

$$\frac{m}{2}\left(v^2\cos^2\theta + v^2\sin^2\theta\right) - \frac{GMm}{R} = -\frac{GMm}{2a}$$

Multiplying everything by $\dfrac{2}{m}$:

$$v^2\cos^2\theta + v^2\sin^2\theta - \frac{2GM}{R} = -\frac{GM}{a}$$

Thus:

$$\frac{2GM}{R} - \frac{GM}{a} - v^2\cos^2\theta = v^2\sin^2\theta$$

Substituting:

$$1 - \cos^2\phi = \frac{R^2 v^4 \cos^2\theta \sin^2\theta}{G^2 M^2 e^2}$$

$$\Delta L = 2R^2 \frac{v^2 \cos\theta \sin\theta}{GMe}$$

Approximating $e$ to 1 (just observe the expression and notice that $e$ tends to 1), knowing that the surface gravity $g = \frac{GM}{R^2}$ and that $\sin(2\theta) = 2\cos\theta\sin\theta$:

$$\Delta L = \frac{v^2 \sin(2\theta)}{g}$$

Which is the classical result!

Paz e Amor

**a)** Writing Poyson's equation for a individual star:

$$m_i = -2.5 \log \frac{F_i}{F_v}$$

$$F_i = F_v 10^{-\frac{m_i}{2.5}}$$

The system's radiation flux is the sum of the radiation of the components:

$$M = -2.5 \log \left( \frac{F_v \sum 10^{-\frac{m_i}{2.5}}}{F_v} \right)$$

$$\therefore M = -2.5 \log \left( 10^{\frac{(2.2)}{-2.5}} + 10^{\frac{(3.9)}{-2.5}} \right) \approxeq 2.0$$

But it is known that:

$$M + \Delta m_{sis} = 2.1 = -2.5 \log \left( 10^{\frac{(2.2)}{-2.5}} + 10^{\frac{(3.9+\Delta m_A)}{-2.5}} \right)$$

Which leads to:

$$\boxed{\Delta m_A = 0.84}$$

**b)** Consider a cylindrical element of mass $dm$ at a distance $r$ from the center of the star, with cross section area $dA$ and length $dr$. The pressure at the base of the element is $P(r)$ and at the top is $P(r+dr)$. The pressure difference generates a net outward force that compensates for the gravitational attraction: $\frac{Gm(r)dm}{r^2}$.

Which means that:

$$(P(r) - P(r+dr))dA = -\frac{Gm(r)dm}{r^2}$$

Notice that $\rho(r) = \frac{dm}{dAdr}$ and that $P(r+dr) - P(r) = dP(r)$, which leads to:

$$\boxed{\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r)} \tag{2.1}$$

**c)** Now considering the general case in which the relation between the radiation and ideal gas pressure is unknown, we have that:

$$P_{rad}(r) = \frac{4\sigma}{3c} T^4(r)$$

$$P_{gas}(r) = \frac{\rho(r)}{\mu} k_b T(r)$$

$$\frac{dP_{tot}(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

Since $P_{tot} = P_{rad} + P_{gas}$, then:

$$\frac{16\sigma}{3c}T^3(r)\frac{dT(r)}{dr} + \frac{k_b}{\mu}T(r)\frac{d\rho(r)}{dr} + \frac{k_b}{\mu}\rho(r)\frac{dT(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

Applying the relation above to the photosphere, where $r = R$, $m(R) = M$ and, by approximation, $dr = \Delta r$ where $\Delta r$ is the thickness of the atmosphere, $dT(R) = \Delta T$ and $d\rho(r) = -\rho(R)$ (since the density drops from $\rho(R)$, the density inside the photosphere, to 0, outside the atmosphere):

$$\frac{16\sigma}{3c}T_0^3\frac{\Delta T}{\Delta r} - \frac{k_b}{\mu}T_0\frac{\rho(R)}{\Delta r} + \frac{k_b}{\mu}\rho(R)\frac{\Delta T}{\Delta r} = -\frac{GM}{R^2}\rho(R)$$

Notice that since $|\Delta T| \ll T_0$:

$$\left|\frac{k_b}{\mu}\Delta T\frac{\rho(R)}{\Delta r}\right| \ll \frac{k_b}{\mu}T_0\frac{\rho(R)}{\Delta r}$$

Therefore it is possible to ignore the first term in relation to the second:

$$\frac{16\sigma}{3c}T_0^3\frac{\Delta T}{\Delta r} - \frac{k_b}{\mu}T_0\frac{\rho(R)}{\Delta r} = -\frac{GM}{R^2}\rho(R)$$

Isolating $\Delta T$:

$$\Delta T = \frac{3\rho(R)c}{16\sigma T_0^3}\left(\frac{k_b}{\mu}T_0 - \frac{GM\Delta r}{R^2}\right)$$

**d)** It is known that the radiation flux if dependent of the fourth power of the outside temperature: $F \propto T^4$. Now using the Pogson relation for the magnitudes:

$$m - m_0 = -2.5\log\left(\frac{F}{F_0}\right) = -2.5\log\left(\frac{T^4}{T_0^4}\right)$$

$$\Delta m_A = -10\log\left(\frac{T_0 + \Delta T}{T_0}\right) = -\frac{10}{\ln(10)}\ln\left(1 + \frac{\Delta T}{T_0}\right)$$

$$\Delta r = \frac{R^2 T_0}{GM}\left(\frac{k_b}{\mu} + \frac{8\sigma T_0^3\ln(10)}{15\rho(R)c}\right)$$

Now pay attention to the order of magnitude of the terms:

$$\frac{k_b}{\mu} \approx 8.3 \cdot 10^3 [S.I.]$$

Now, in order to determine the order of magnitude of the second term, we have to find an approximation to the density. Consider the density being the average density of the star:

$$\rho(R) \approx \bar{\rho} = \frac{3M}{4\pi R^3}$$

So the second term would be equal to:

$$\frac{32\pi\sigma T_0^3\ln(10)R^3}{45Mc}$$

$$\frac{32\pi\sigma T_0^3 \ln(10) R^3}{45Mc} \approx 3.12 \cdot 10^{-7}[S.I.]$$

Notice that this approach overestimates the value of the density of the atmosphere, which would make this term bigger. But even if the photosphere was a billion times less dense then the average density of the star, which is unlikely, the term would be less then 4% of the first term, so it is safe to ignore it:

$$\frac{k_b}{\mu} + \frac{8\sigma T_0^3 \ln(10)}{15\rho(R)c} \approx \frac{k_b}{\mu}$$

Which leads to:

$$\boxed{\Delta r = \frac{R^2 T_0 k_b}{GM\mu}}$$

Substituting the values we get that:

$$\boxed{\Delta r \approx 460\,\text{km}}$$

**e)** Utilizing the hydrostatic equilibrium equation in the photosphere, we have that:

$$\frac{dP(R)}{dr} \approx \frac{\Delta P}{\Delta r} = -\frac{GM}{R^2}\rho$$

By the definition of optical depth:

$$\tau = \frac{2}{3} = \kappa\rho\Delta r$$

$$\therefore \Delta P = -\frac{GM}{R^2}\rho\Delta r = \Delta P = -\frac{2GM}{3\kappa R^2}$$

Since $\Delta P = P_{ext} - P_{fot}$ and the outside pressure is $P_{ext} = 0$: $P_{fot} = \frac{2GM}{3\kappa R^2}$. Comparing this with the expression of the total pressure at the photosphere (gas + radiation) we find that:

$$\frac{2GM}{3\kappa R^2} = \frac{4\sigma}{3c}T_0^4 + \frac{\rho(R)}{\mu}k_b T_0$$

Comparing the orders of magnitude of the term in the right:

$$\frac{4\sigma}{3c}T_0^4 + \frac{\rho(R)}{\mu}k_b T_0 \approx \frac{\rho(R)}{\mu}k_b T_0$$

Now, considering as stated by the problem statement[1]:

$$\rho(R) = \alpha\frac{3M}{4\pi R^3}$$

$$\therefore \kappa = \frac{8\pi GR\mu}{9\alpha k_b T_0}$$

[1] $\alpha = 10$.

Substituting the values we have that:

$$\kappa = 86\,\mathrm{m}^2\,\mathrm{kg}^{-1}$$

**a)** The first step to solve the problem is to better visualize the orbits and to identify the transfers that need to occur, like shown in the Figure 2.5:

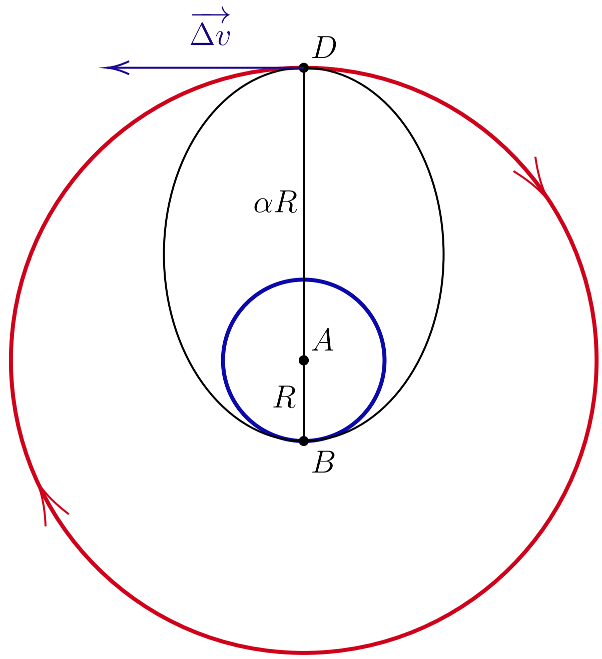



Figure 2.5: Transfer orbits for the given situation.

Consider the initial orbit as the largest circle, with the planet represented by the colored circumference, and the intermediate ellipse as the transfer orbit.

Note that since we want the spacecraft to pass as close as possible to the planet's surface without colliding, we aim for the periapsis of the elliptical orbit to approach the planet's radius as closely as possible. Thus, in the limiting case:

$$2a = R + \alpha R = (1+\alpha)R$$

Therefore, we have:

$$a = \frac{1+\alpha}{2}R$$

Calculating the $\Delta v$ between the two orbits (from circular to elliptical):

$$\Delta v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} - \sqrt{GM\frac{1}{r}}$$

$$\Delta v = \sqrt{GM\left(\frac{2}{\alpha R} - \frac{2}{(1+\alpha)R}\right)} - \sqrt{\frac{GM}{\alpha R}}$$

$$\therefore \Delta v = \sqrt{\frac{GM}{(1+\alpha)\alpha R}}(\sqrt{2} - \sqrt{1+\alpha})$$

Note that since $\alpha > 1$, we have $1 + \alpha > 2$, thus $\Delta v < 0$, as expected.

Now we can relate the time the engines remain active with the $\Delta v$ required for the transfer. Using the rocket equation:

$$|\Delta v| = u \ln \frac{m_0}{m_f}$$

Since mass is ejected at a rate $K$, we can express the final mass of the rocket as $m_f = m_0 - K\Delta t$. Applying and rearranging:

$$\frac{m_0}{K}\left(1 - e^{-\frac{|\Delta v|}{u}}\right) = \Delta t$$

Finalizing:

$$\Delta t = \frac{m_0}{K}\left(1 - e^{\frac{1}{u}\sqrt{\frac{GM}{(1+\alpha)\alpha R}}(\sqrt{2}-\sqrt{1+\alpha})}\right)$$

**b)** In this case, it is enough to stop the spacecraft to achieve a soft landing, meaning $\Delta v = v$. For this situation:

$$v = \sqrt{GM\left(\frac{2}{R} + \frac{2}{(\alpha+1)R}\right)}$$

$$v = \sqrt{\frac{2GM(\alpha+2)}{(\alpha+1)R}}$$

Thus, using the formula found in the previous item:

$$\Delta t = \frac{m_0}{K}\left(1 - e^{-\sqrt{\frac{2GM(\alpha+2)}{u^2 R(\alpha+1)}}}\right)$$

## 2.7 Black Holes

### Part A: Metrics

**A.1)** Consider what it is shown on the Figure 2.6.

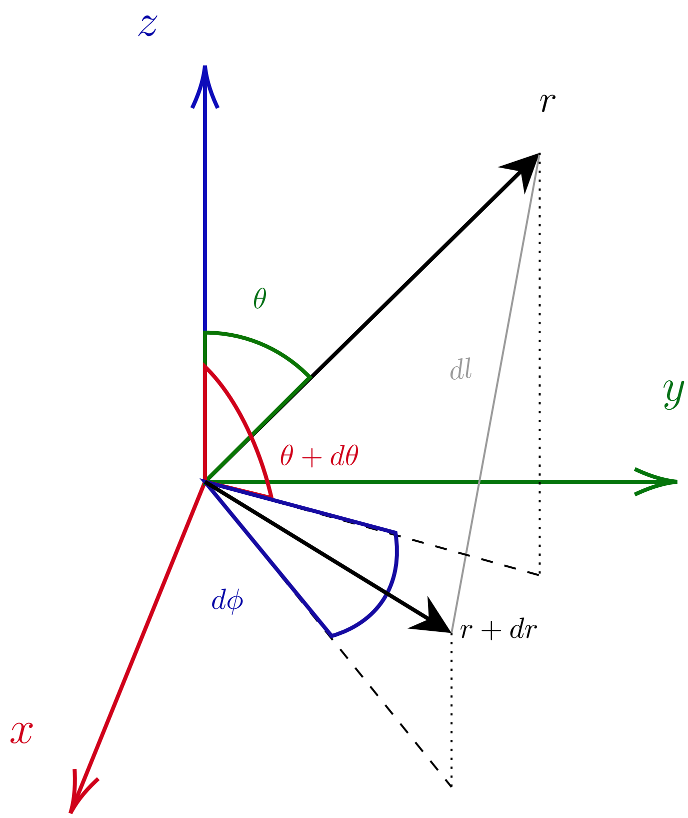



Figure 2.6: Applied spherical coordinates.

Notice that $(dx)^2 + (dy)^2 + (dz)^2 = (dl)^2$, by the Pythagorean Theorem. Therefore we want to determine $(dl)^2$ in terms of the spherical coordinates. In order to do so, we can use the approximation for infinitesimal segments, as shown in the figure 2.7:

$$(dl)^2 = r^2(d\delta)^2 + (dr)^2$$

In order to calculate the angle $d\delta$, between the two vectors, podemos utilizar a lei dos cossenos na trigonometria esférica, de tal forma que:

$$\cos(d\delta) = \cos(\theta)\cos(\theta + d\theta) + \sin(\theta)\sin(\theta + d\theta)\cos(d\phi)$$

You might be tempted to use approximations like $\cos(dx) = 1$ or even $\cos(\theta + d\theta) = \cos(\theta) - \sin(\theta)d\theta$. However, all these approximations ignore terms of order $d\delta^2$, $d\theta^2$, and $d\phi^2$, which should not be neglected in this case, as these are precisely the terms we need to describe $dl^2$. Therefore, we will have to make approximations that preserve such terms, using $\cos(dx) = 1 - \frac{dx^2}{2}$, and we will need to apply the sum formulas for sine and cosine.

$$1 - \frac{d\delta^2}{2} = \cos(\theta)\left(\cos(\theta)\left(1 - \frac{d\theta^2}{2}\right) - \sin(\theta)d\theta\right)$$

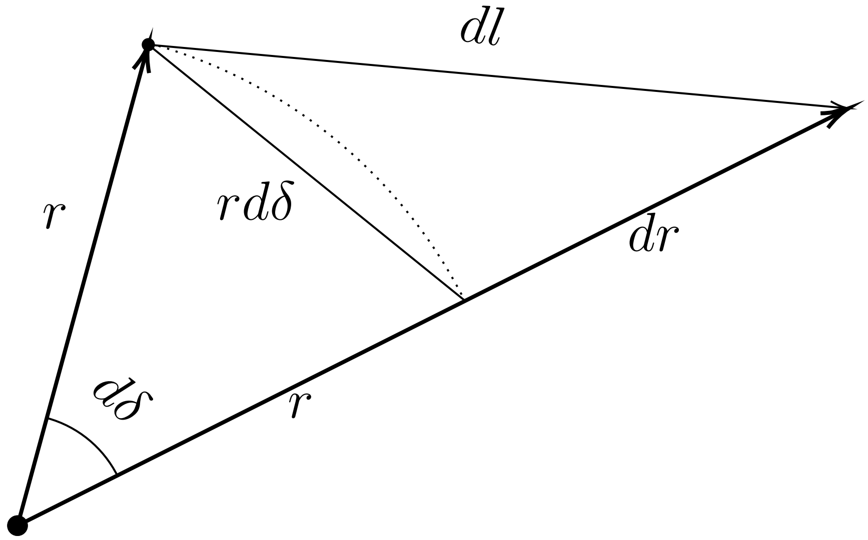


Figure 2.7: Infinitesimal variation on the position vector with the angle.

$$+\sin(\theta)\left(\sin(\theta)\left(1-\frac{d\theta^2}{2}\right)+\cos(\theta)d\theta\right)\left(1-\frac{d\phi^2}{2}\right)$$

After all the necessary simplifications and the exclusion of third-order terms, we find that:

$$d\delta^2 = d\theta^2 + \sin^2(\theta)d\phi^2$$

Thus, substituting into the expression for distance, we obtain:

$$(dl)^2 = (rd\theta)^2 + (r\sin(\theta)d\phi)^2 + (dr)^2$$

Hence, we have proven that the expression for the metric in flat spacetime, in its polar form, is:

$$(ds)^2 = (cdt)^2 - (rd\theta)^2 - (r\sin(\theta)d\phi)^2 - (dr)^2$$

**A.2)** By the given definition, the calculation of $\Delta L$ implies an instantaneous measurement, meaning $dt = 0$ and, since it is radial, $d\theta = 0$ and $d\phi = 0$:

$$(ds)^2 = -\left(\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}\right)^2$$

Since $dL = \sqrt{-(ds)^2}$, we find that:

$$dL = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$\rightarrow \int dL = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}$$

Using the integral provided in the problem statement:

$$\Delta L = r_2\sqrt{1-\frac{2GM}{r_2c^2}}+\frac{2GM}{c^2}\tanh^{-1}\left(\sqrt{1-\frac{2GM}{r_2c^2}}\right)-r_1\sqrt{1-\frac{2GM}{r_1c^2}}+\frac{2GM}{c^2}\tanh^{-1}\left(\sqrt{1-\frac{2GM}{r_1c^2}}\right)$$

For the proper time, by definition, we know that in this analysis the spatial coordinates remain constant, implying:

$$(ds)^2 = \left(cdt\sqrt{1-\frac{2GM}{rc^2}}\right)^2$$

$$\therefore \Delta\tau = \sqrt{1-\frac{2GM}{rc^2}}\,\Delta t$$

**A.3)** It is known that $\nu \propto T^{-1}$, where $T$ is the oscillation period of the electromagnetic wave. Thus:

$$\frac{T_0}{T_\infty} = \frac{\nu_\infty}{\nu_0}$$

Using the relation from the previous item[2], we can relate:

$$T_0 = \sqrt{1-\frac{2GM}{rc^2}}\,T_\infty$$

Hence:

$$\nu_\infty = \nu_0\sqrt{1-\frac{2GM}{rc^2}}$$

**A.4)** Assume a circular orbit located in the equatorial plane of the coordinate system, so that $\theta = \frac{\pi}{2}$ and $d\phi = \omega dt$. Thus, we know that $dr = d\theta = 0$:

$$(ds)^2 = \left(cdt\sqrt{1-\frac{2GM}{rc^2}}\right)^2 - (r\omega dt)^2$$

Simplifying the expression, we obtain the interval for an orbit as:

$$\Delta s = \int_0^{\frac{2\pi}{\omega}} \sqrt{c^2-\frac{2GM}{r}-r^2\omega^2}\,dt$$

In the case of a free orbit without external forces, the body will follow the path with the smallest possible interval. Thus, we determine the value of $r$ that provides this minimum:

$$\frac{d}{dr}\Delta s = \int_0^{\frac{2\pi}{\omega}} \frac{d}{dr}\sqrt{c^2-\frac{2GM}{r}-r^2\omega^2}\,dt = 0$$

For this, we require:

[2]Remember that the definition of $dt$ refers to the time interval observed by an observer outside the spacetime curvature, as it is the temporal coordinate.

$$\frac{d}{dr}\left(\frac{2GM}{r}+r^2\omega^2\right)=0$$

Solving, we find that, “coincidentally,” the result matches the prediction from Newtonian physics:

$$\omega=\sqrt{\frac{GM}{r^3}}$$

## Part B: Black Holes

**B.1)** In this situation, light does not make progress (does not move) in the spatial coordinates, as it is “trapped,” so $dr=d\theta=d\phi=0$, and furthermore, since we know that it must always move along a null geodesic: $ds=0$. Therefore:

$$cdt\sqrt{1-\frac{2GM}{rc^2}}=0$$

Since there is no restriction on its temporal coordinate, it is understood that $dt\neq 0$, hence:

$$r=\frac{2GM}{c^2}$$

Which is the same result obtained by the Newtonian prediction!

**B.2)** In this case, using the same convention as in **A.4)**, it is known that $dr=d\theta=ds=0$, $\sin(\theta)=1$, and that $d\phi=\omega dt$. Note also that the result of item **A.4)** is not restricted to massive particles, since light also moves along a geodesic such that $\frac{d}{dr}\Delta s=0$ (which is obvious, since $\Delta s$ is essentially 0), so we can directly substitute the value of $\omega$ found:

$$cdt\sqrt{1-\frac{2GM}{rc^2}}=r\sqrt{\frac{GM}{r^3}}dt$$

Thus, we find that:

$$r=\frac{3GM}{c^2}$$

**B.3)** Equating the gravitational force to the centripetal force ($v=\frac{L}{mr}$):

$$\frac{GMm}{r^2}+\frac{3GML^2}{mc^2r^4}=\frac{L^2}{mr^3}$$

Rearranging, we find that:

$$r^2-\frac{L^2}{GMm^2}r+3\frac{L^2}{m^2c^2}=0$$

Solving for $r$:

$$r=\frac{L^2}{2GMm^2}\left(1\pm\sqrt{1-\frac{12G^2M^2m^2}{L^2c^2}}\right)$$

Paz e Amor

Note that in the case where we return to classical physics by making $c \to \infty$, it is expected that we find the classical result. When we do this, the solution with the minus sign shows that $r \to 0$ when $c \to \infty$, so it is understood that the stable orbit is:

$$r = \frac{L^2}{2GMm^2}\left(1 + \sqrt{1 - \frac{12G^2M^2m^2}{L^2c^2}}\right)$$

Note that to have a minimum $r$, we need $L$ to also be minimum. Therefore, respecting the existence condition for the square root, we will set $L^2 = \frac{12G^2M^2m^2}{c^2}$:

$$r = \frac{12G^2M^2m^2}{2GMm^2c^2}$$

$$r_{ISCO} = \frac{6GM}{c^2}$$

**B.4)** Note that the minimum frequency is the one that has undergone the maximum possible redshift, so it is the one that departed from $r_{ISCO}$. It is known that the rest frequency of the emission line can be found as: $E_{FeK\alpha} = h\nu_0$, which gives us the data: $\nu_0 = 1.539 \cdot 10^{18} Hz$. From the gravitational redshift equation:

$$7.83 \cdot 10^{17} Hz = 1.539 \cdot 10^{18}\sqrt{1 - \frac{2GM}{r_{ISCO}c^2}}$$

Substituting the data, we find that $r_{ISCO} = 2.7\frac{GM}{c^2}$. Note that there is no need to substitute the mass of the black hole, because we can now conclude that:

$$1 + \sqrt{8.354\left((2 - a_*)^2 - 1\right)} = 2.7$$

Finally, we find that:

$$a_* = 0.84$$

## Part C: "Black" Thermodynamics

**C.1)** Starting from the proposed temperature relation, we can find a relation for the "luminosity" (it does not mean light emission, since the hole is black) using the same Stefan-Boltzmann relation:

$$P_b = 4\pi R_b^2 \sigma T_b^4$$

Where $R_b$ is the radius of the event horizon of the star, given by the Schwarzschild relation:

$$R_b = \frac{2GM}{c^2}$$

Therefore, we have:

$$P_b = 4\pi\left(\frac{2GM}{c^2}\right)^2 \sigma \left(\frac{\hbar c^3}{8\pi GMk_b}\right)^4$$

Simplifying:

$$P_b = \frac{\hbar^4 c^8 \sigma}{256\pi^3 G^2 M^2 k_b^4}$$

**C.2)** Where does this energy being released come from? From the black hole's own mass (as defined by the model in the statement), so we have that $E_b = Mc^2$ and, for small energy variations: $\Delta E_b = \Delta M c^2$, or, analyzing infinitesimal variations: $dE_b = dMc^2$. Note that the energy released in an interval $dt$ will be $P_b dt$, so we will have:

$$P_b dt = -dE_b = -dMc^2$$

The negative sign is used because there is energy release (decrease in the internal energy of the black hole)

$$\frac{\hbar^4 c^8 \sigma}{256\pi^3 G^2 M^2 k_b^4} dt = -dMc^2$$

Rearranging:

$$\frac{\hbar^4 c^6 \sigma}{256\pi^3 G^2 k_b^4} dt = -M^2 dM$$

Integrating the expression, we have:

$$\frac{\hbar^4 c^6 \sigma}{256\pi^3 G^2 k_b^4}(t - t_0) = -\left(\frac{M^3}{3} - \frac{M_0^3}{3}\right)$$

Where $t_0$ is the initial time (t=0) and $M_0$ is the initial mass, so the black hole "dies" when it is completely evaporated, that is, $M = 0$. Thus, we have:

$$\frac{\hbar^4 c^6 \sigma}{256\pi^3 G^2 k_b^4}(\Delta t) = -\left(\frac{0^3}{3} - \frac{M_0^3}{3}\right)$$

$$\therefore \Delta t = \left(\frac{256\pi^3 G^2 k_b^4 M_0^3}{3\hbar^4 c^6 \sigma}\right)$$

Bonus: Lifetime of a black hole with the mass of the Sun: $2.1 \cdot 10^{67}$ years.

**C.3)** Note that we already have an expression for the emitted power and we know that $dMc^2 = dQ$, so:

$$dS = dMc^2 \frac{1}{T} = dMc^2 \frac{8\pi G M k_b}{\hbar c^3}$$

Integrating, we will have:

$$S - S_0 = \frac{4\pi G M^2 k_b}{\hbar c} - \frac{4\pi G M_0^2 k_b}{\hbar c}$$

Thus, we associate:

$$S = \frac{4\pi G M^2 k_b}{\hbar c}$$

## 2.8 The Mass of the Curve

Consider the following image as a representation of the situation:

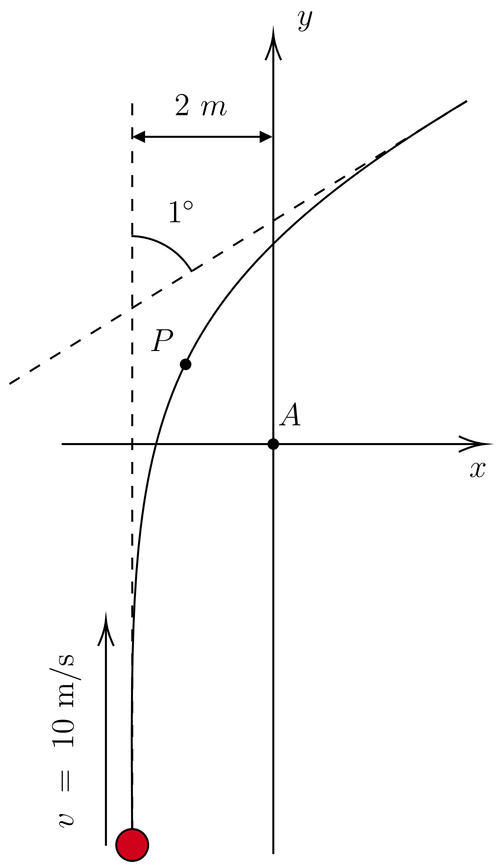



Figure 2.8: Bowling Ball trajectory near the deformation.

Assume that the initial velocity occurs without "gravitational" interference from the hole, so the mechanical energy of the object is $\frac{1}{2}mv_0^2$, which is the same as the total energy of the hyperbolic orbit $\frac{GMm}{2a}$. Therefore, we find that $a = \frac{GM}{v_0^2}$. From the velocity equation for a hyperbolic orbit, at the periapsis, we have:

$$v_p = \sqrt{GM\left(\frac{2}{a(e-1)} + \frac{1}{a}\right)}$$

By conservation of angular momentum:

$$v_0 r_0 \sin\alpha = v_p r_p$$

Note that $r_0 \sin\alpha$ is the distance AB multiplied by the sine of the angle between the velocity and AB, which is the opposite leg to that angle, that is: $r_0 \sin\alpha = 2$ m. Substituting the expression for $v_p$ and knowing that $r_p = a(e-1)$:

$$v_0 r_0 \sin\alpha = \sqrt{GMa(e^2-1)} = \sqrt{GM\frac{GM}{v_0^2}(e^2-1)}$$

From this, we derive:

$$M = \frac{v_0^2 r_0 \sin\alpha}{G\sqrt{e^2-1}}$$

To determine the value of "$e$," we will study the deflection angle. This angle represents the angle between the asymptotes of the hyperbola. A known property of the hyperbola is that the tangent of the angle between the asymptote and the axis is equal to $\frac{b}{a}$ (where $b$ and $a$ are the semi-minor and semi-major axes, respectively):

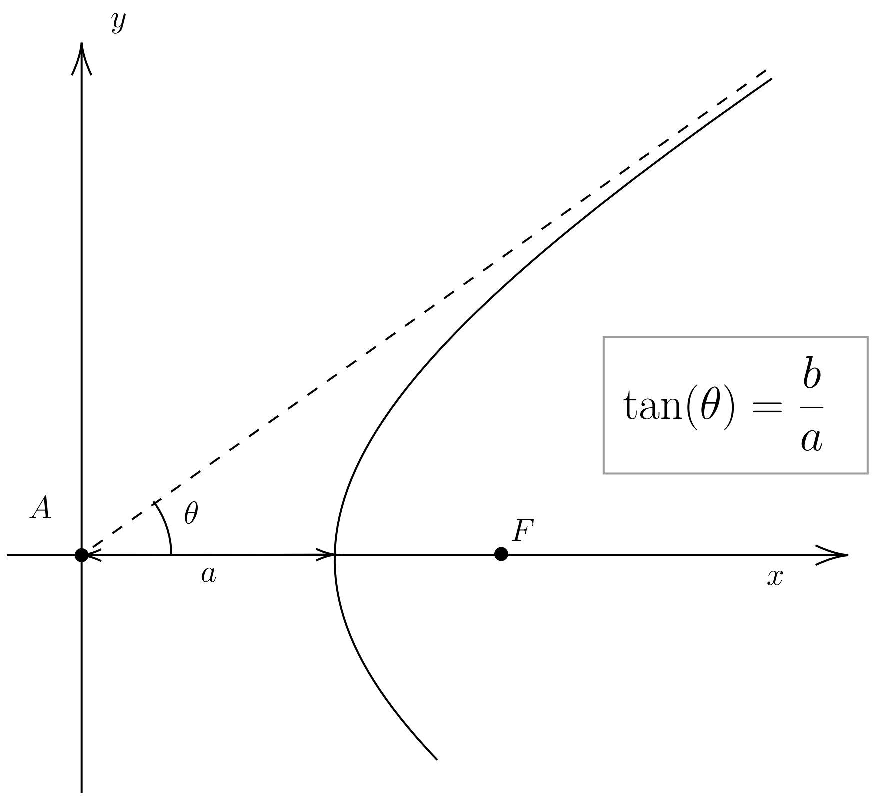



Figure 2.9: Calculation of the angle of inclination of the upper asymptote of the hyperbola.

In this case, the deflection is $\delta = 180^o - 2\theta$, so: $\theta = 89.5^o$. Recalling that $a^2 + b^2 = c^2$ (for the hyperbola) and that $e = \frac{c}{a}$, we have:

$$e^2 - 1 = \frac{c^2 - a^2}{a^2} = \left(\frac{b}{a}\right)^2 = \tan^2 89.5^o$$

Therefore:

$$M = \frac{v_0^2 r_0 \sin\alpha}{G} \cot 89.5^o$$

Substituting the values:

$$M = 2.617 \cdot 10^{10}\text{kg}$$

## 2.9 Nill's Spaceship

Consider the following image as the representation of the described situation:

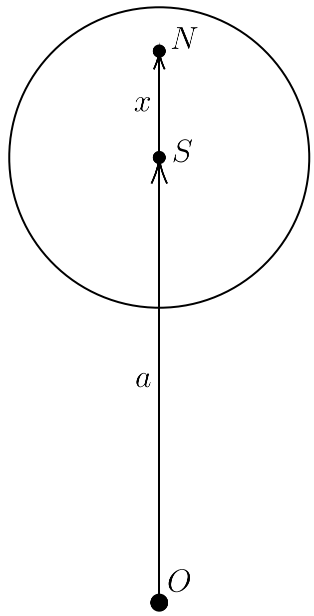



Figure 2.10: Representation of the radial position of Nill in regard with his spaceship.

Let "O" be the center of the Earth, "S" the center of the satellite, and "N" the position of Nill. Note that, by the shell theorem, there is no gravitational force between Nill and the satellite (since he is inside it), so the only force acting on Nill is the one from the Earth, which causes him to orbit in a circle of radius $(a + x)$. Since the satellite orbits in a circle of radius $a$, there is a difference between the orbital periods of Nill and the satellite. Equating the orbital periods:

$$\frac{T_S^2}{a^3} = \frac{T_N^2}{(a+x)^3} = \frac{4\pi^2}{GM}$$

$$\therefore T_N = \sqrt{\frac{4\pi^2(a+x)^3}{GM}}$$

$$\therefore T_S = \sqrt{\frac{4\pi^2 a^3}{GM}}$$

After a time $\Delta t$, the position vectors of S and N would have separated by an angle $\alpha$, as shown in Figure 2.11:

This angle can be calculated by the difference in the angular velocities of the objects (assuming they orbit in the same direction):

$$\omega_{\text{rel}} = \omega_s - \omega_n = \frac{2\pi}{T_s} - \frac{2\pi}{T_n}$$

$$\omega_{\text{rel}} = \frac{2\pi\sqrt{GM}}{\sqrt{4\pi^2 a^3}} - \frac{2\pi\sqrt{GM}}{\sqrt{4\pi^2(a+x)^3}} = \sqrt{\frac{GM}{a^3}} - \sqrt{\frac{GM}{(a+x)^3}}$$

Assuming $x \ll a$ and using the approximation $(1+y)^n \approx 1 + ny$, we have:

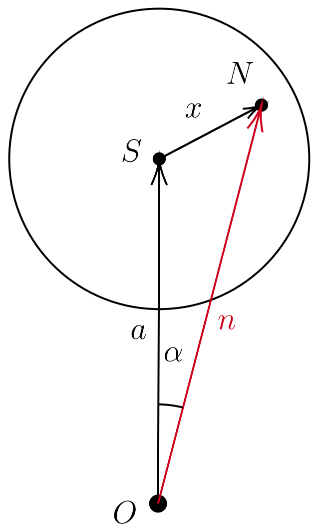



Figure 2.11: Variation of Nill's position in regards to time.

$$\omega_{\text{rel}} = \sqrt{\frac{GM}{a^3}} - \sqrt{\frac{GM}{a^3(1+x/a)^3}} = \sqrt{\frac{GM}{a^3}} - \sqrt{\frac{GM}{a^3}}(1+x/a)^{-3/2}$$

$$\omega_{\text{rel}} \approx \sqrt{\frac{GM}{a^3}} - \sqrt{\frac{GM}{a^3}}\left(1 - \frac{3x}{2a}\right)$$

$$\therefore \omega_{\text{rel}} \approx \frac{3x}{2a}\sqrt{\frac{GM}{a^3}}$$

Therefore, the angle $\alpha$:

$$\alpha = \omega_{\text{rel}}\Delta t = \sqrt{\frac{9x^2GM}{4a^5}}\Delta t$$

Note that Nill will hit the edge of the satellite when the distance between N and S is equal to the inner radius of the spacecraft. Applying the law of cosines to triangle OSN to find SN:

$$r^2 = a^2 + (a+x)^2 - 2a(a+x)\cos(\alpha)$$

$$\cos(\alpha) = \frac{a^2 + (a+x)^2 - r^2}{2a(a+x)}$$

$$\sqrt{\frac{9x^2GM}{4a^5}}\Delta t = \arccos\left(\frac{a^2 + (a+x)^2 - r^2}{2a(a+x)}\right)$$

$$\therefore \Delta t = \sqrt{\frac{4a^5}{9x^2GM}}\arccos\left(\frac{a^2 + (a+x)^2 - r^2}{2a(a+x)}\right)$$

$$\Delta t = \sqrt{\frac{4a^5}{9x^2GM}}\arccos\left(1 + \frac{x^2 - r^2}{2a(a+x)}\right)$$

Since $a \gg x$, we can approximate $a + x \approx a$:

$$\Delta t = \sqrt{\frac{4a^5}{9x^2GM}} \arccos\left(1 - \frac{r^2 - x^2}{2a^2}\right)$$

Since $r^2 - x^2 \ll a^2$, we use the Taylor expansion of $\arccos x$: $\arccos x \approx \sqrt{2 - 2x}$ for $x \to 1^-$:

$$\Delta t = \sqrt{\frac{4a^5}{9x^2GM}} \frac{\sqrt{r^2 - x^2}}{a}$$

$$\therefore \Delta t = \frac{2a}{3x}\sqrt{\frac{a(r^2 - x^2)}{GM}}$$

## 2.10 Orbital Sniper

**a)** The mirror must always be aligned with its orbital velocity, so it is as if Ramanu were shooting at the ellipse of the orbit itself, and it reflects. By Poncelet's theorem for an ellipse, it is known that if a light ray originates from one focus of the mirrored conic from within, this ray will converge at the other focus. Thus, we want the enemy ship to be located at this secondary focus so that the rays originating from the planet reach it. Therefore, $d = 2ae$, where "d" is the focal distance, "a" is the semi-major axis of the orbit, and "e" is the eccentricity of the orbit.

To find the minimum $\Delta v$, we must determine the vector difference between the velocities at the transfer point of the elliptical and circular orbits. For this, consider the diagram in Figure 2.12:

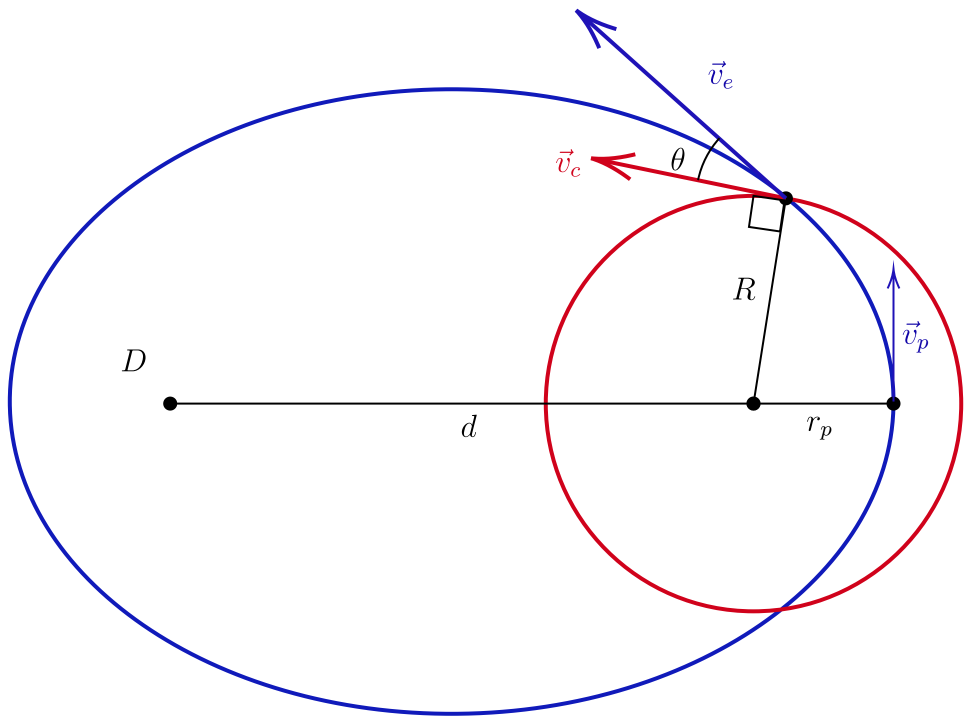



Figure 2.12: Representation of orbital transfer to elliptical orbit.

By the velocity equation, we have:

$$|\vec{v_e}| = \sqrt{GM\left(\frac{2}{R} - \frac{1}{a}\right)}$$

$$|\vec{v_c}| = \sqrt{\frac{GM}{R}}$$

And by the law of cosines:

$$\Delta\vec{v}^2 = v_e^2 + v_c^2 - 2v_e v_c \cos\theta$$

For the calculation of $\cos(\theta)$, we use the law of angular momentum conservation for the elliptical orbit, as shown in the figure:

$$v_e R \sin(\alpha) = v_p r_p$$

Substituting the velocity at perihelion ($r = a(1-e)$) and the fact that $\alpha = 90^o + \theta$, which implies $\sin(\alpha) = \cos(\theta)$:

$$\cos(\theta) = \frac{\sqrt{GMa(1-e^2)}}{v_e R}$$

$$\therefore \Delta\vec{v}^2 = v_e^2 + v_c^2 - 2v_c\frac{\sqrt{GMa(1-e^2)}}{R}$$

Substituting the velocities:

$$\Delta\vec{v}^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right) + \frac{GM}{R} - 2\sqrt{\frac{GM}{R}}\frac{\sqrt{GMa(1-e^2)}}{R}$$

Substituting $e = \frac{d}{2a}$, we obtain:

$$\Delta\vec{v}^2 = GM\left(\frac{3}{R} - \frac{1}{a} - \sqrt{\frac{(4a^2-d^2)}{aR^3}}\right)$$

Existence conditions are:

$$4a^2 > d^2 \Rightarrow a > \frac{d}{2}$$

And for the intersection between the ellipse and the circle (transfer point):

$$a(1-e) \leq R \Rightarrow a\left(1 - \frac{d}{2a}\right) \leq R \Rightarrow a \leq R + \frac{d}{2}$$

Finally:

$$\frac{d}{2} < a \leq R + \frac{d}{2}$$

Now, analyze the following relation:

$$GM\left(\frac{3}{R} - \frac{1}{a}\right) - \Delta\vec{v}^2 = GM\sqrt{\frac{4a^2-d^2}{aR^3}}$$

Make the following substitution: $x = \dfrac{a}{d}$. Thus, the function inside the square root becomes: $\dfrac{d}{R^3}\left(4x - \frac{1}{x}\right)$. Note that $4x - \dfrac{1}{x}$ is increasing $\forall x > 0$, therefore:

$$\forall a_2 > a_1 \implies x_2 > x_1 \implies GM\sqrt{\frac{4a_2^2-d^2}{a_2R^3}} > GM\sqrt{\frac{4a_1^2-d^2}{a_1R^3}}$$

$$\implies GM\left(\frac{3}{R} - \frac{1}{a_2}\right) - \Delta\vec{v}_2^2 > GM\left(\frac{3}{R} - \frac{1}{a_1}\right) - \Delta\vec{v}_1^2$$

$$\therefore \Delta\vec{v}_1^2 - \Delta\vec{v}_2^2 > GM\left(\frac{1}{a_2} - \frac{1}{a_1}\right) > 0$$

Thus, we find that $\forall a_2 > a_1 \implies |\Delta\vec{v}_2| < |\Delta\vec{v}_1|$. Therefore, to choose the smallest possible velocity variation, we must select the largest possible value of $a$. That is, we will choose $a$ in the extreme case of $a = R + \frac{d}{2}$, when the perihelion of the elliptical orbit meets the circumference.

In this case:

$$\Delta v^2 = GM\left(\frac{3}{R} - \frac{2}{2R+d} - \sqrt{\frac{2((2R+d)^2 - d^2)}{(2R+d)R^3}}\right)$$

$$\Delta v = \sqrt{GM\frac{4R + 3d - 2\sqrt{4R^2 + 6Rd + 2d^2}}{(2R+d)R}}$$

**b)** The effect of the medium's opacity will cause an exponential decrease in the intensity of the radiation as a function of the distance traveled, as follows:

$$I(r) = I_0 e^{-\kappa\rho r}$$

where $r$ is the distance traveled by the light ray, which will be equal to $2a$ since, by definition, in an ellipse, the sum of the distances from a point to the foci is equal to the major axis. Since we want $I(r) \geq I_0/2$:

$$I(r) = I_0 e^{-2\kappa\rho a} \geq \frac{I_0}{2}$$

$$I_0 e^{-2\kappa\rho a} \geq I_0 e^{-\ln(2)}$$

$$\therefore -2\kappa\rho a \geq -\ln(2) \Rightarrow 2\rho\kappa a \leq \ln(2)$$

Finally, since $\kappa = \kappa_0 f^3$:

$$0 < f \leq \sqrt[3]{\frac{\ln(2)}{\kappa_0\rho(2R+d)\rho}}$$

# 2.11 Let There Be Light!

## Part A: Age of the Universe

**A.1)** To find the resultant force on particle $j$, we simply sum vectorially all the forces acting on it:

$$\vec{F}_{res} = \sum_{i \neq j}(-Gm_i m_j(\vec{r}_j - \vec{r}_i))$$

Note: We sum over all possible indices $i$, except for $j$, because the particle does not exert a force on itself.

Recalling that $m_j$ and $r_j$ are constants (the mass and position of a chosen particle) and expanding, we obtain:

$$\vec{F}_{res} = -Gm_j \sum_{i \neq j}(m_i(\vec{r}_j - \vec{r}_i))$$

$$\vec{F}_{res} = -Gm_j \left(\sum_{i \neq j}(m_i \vec{r}_j) - \sum_{i \neq j}(m_i \vec{r}_i)\right)$$

$$\vec{F}_{res} = -Gm_j \left(\vec{r}_j \sum_{i \neq j}(m_i) - \sum_{i \neq j}(m_i \vec{r}_i)\right)$$

Now note that $\sum_{i \neq j}(m_i)$ is the sum of the masses of all the particles in the universe except for $j$, so $\sum_{i \neq j}(m_i) = M - m_j$. Thus:

$$\vec{F}_{res} = -Gm_j \left(\vec{r}_j(M - m_j) - \sum_{i \neq j}(m_i \vec{r}_i)\right)$$

$$\vec{F}_{res} = -Gm_j \left(M\vec{r}_j - m_j\vec{r}_j - \sum_{i \neq j}(m_i \vec{r}_i)\right)$$

Grouping $m_j\vec{r}_j$ into the summation, we get:

$$\vec{F}_{res} = -Gm_j \left(M\vec{r}_j - \sum_{i}(m_i \vec{r}_i)\right)$$

Since these positions refer to the center of mass, we know from the definition of the center of mass that $\sum_i(m_i\vec{r}_i) = 0$, so:

$$\vec{F}_{res} = -GMm_j\vec{r}_j$$

**A.2)** Applying Newton's second law to $m_j$: $\vec{F}_{res} = m_j\vec{a}_{res}$, where $\vec{a}_{res}$ is the total acceleration of $j$ in the center of mass reference frame. Finally:

$$\vec{a}_{res} = -GM\vec{r}_j$$

**A.3)** Note that the acceleration varies linearly with the position of the particle, so we have a force of the type $F = -kx$, meaning the particle undergoes simple harmonic motion (SHM). Also, observe that the result found in item (b) does not depend on mass, so ALL particles will follow this SHM (since they all start from the same point at the beginning of the universe).

Thus, we need to find half of the period of SHM (only half because the particles start at position 0, reach the maximum, and return to 0).

$$a_{res} = -\omega^2 r_j$$

$$T = \frac{1}{2}\left(\frac{2\pi}{\omega}\right)$$

$$\omega = \sqrt{GM}$$

$$\therefore T = \sqrt{\frac{\pi^2}{GM}} \approx 3.85 \cdot 10^{-22} s$$

## Part B: Kepler's Laws

**B.1)** As demonstrated in item **A.2)**, the total acceleration generated by a rigid body on a test mass is given by:

$$\vec{a} = GM\vec{r}$$

Where $\vec{r}$ is the position vector originating from the center of mass of the rigid body. From this relation, we see that the center of application of gravitational force coincides with the center of mass of the body! Thus, unlike in our universe, we can consider the effect of a body of any shape or size as that of a point mass located at the system's center of mass.

**B.2)** Consider a moment when the body $m$ is at a distance $r$ from $M$ and the angle between the velocity $v$ of $m$ and $r$ is $\theta$. After an infinitesimal interval $dt$, the body will travel a distance $vdt$ in the direction and sense of the velocity vector. Thus, the radial vector will have swept an area in the shape of a triangle, and we can calculate this area:

$$dA = \frac{1}{2} rvdt \sin\theta \rightarrow \frac{dA}{dt} = v_A = \frac{1}{2m} mvr \sin\theta = \frac{L}{2m}$$

Since $L$ is a conserved quantity (because the gravitational force is a central force, meaning it points in the same direction as the position vector relative to $M$), we know that $v_A$ is constant and equal to $\frac{L}{2m}$.

**B.3)** Write the acceleration and position vectors of the body:

$$\begin{bmatrix} a_x \\ a_y \end{bmatrix} = -GM \begin{bmatrix} x \\ y \end{bmatrix}$$

From this, we see that, by associating each component:

$$\begin{cases} a_x = -GMx \\ a_y = -GMy \end{cases}$$

This implies that the body undergoes SHM with the same period in each axis:

$$\begin{cases} x = A\cos\left(\sqrt{GM}t + \phi_x\right) \\ y = B\cos\left(\sqrt{GM}t + \phi_y\right) \end{cases}$$

For simplification, let $\sqrt{GM} = \omega$. Now we will use an algebraic trick to find the equation of the curve:

$$\cos\left(\omega t + \phi_x - \omega t + \phi_y\right) = \cos\left(\omega t + \phi_x\right)\cos\left(\omega t + \phi_y\right) + \sin\left(\omega t + \phi_x\right)\sin\left(\omega t + \phi_y\right)$$

$$\therefore \cos\left(\omega t + \phi_x - \omega t + \phi_y\right) = \cos\left(\phi_x - \phi_y\right) = \frac{x}{A}\frac{y}{B} + \sqrt{1 - \frac{x^2}{A^2}}\sqrt{1 - \frac{y^2}{B^2}}$$

Let $\cos\left(\phi_x - \phi_y\right) = k$:

$$\sqrt{1 - \frac{x^2}{A^2} - \frac{y^2}{B^2} + \frac{x^2y^2}{A^2B^2}} = k - \frac{x}{A}\frac{y}{B}$$

Simplifying, we find:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2kxy}{AB} + k^2 - 1 = 0$$

The previous equation is nothing more than the equation of a rotated conic! Since the motion of $m$ is periodic, it cannot be parabolic or hyperbolic, so we infer that it must be an ellipse!

**B.4)** Note that the period can be found in the same way as did previously, that is:

$$T = 2\pi\sqrt{\frac{1}{GM}}$$

Thus:

$$n = 0 \text{ and } k(M) = \sqrt{\frac{4\pi^2}{GM}}$$

## Part C: Orbital Transfers

**C.1)**

First, define the coordinate axes: $y$ represents the Cartesian axis containing the minor axis of the ellipse, and $x$ represents the Cartesian axis containing the major axis of the ellipse. Notice the following: the velocity of the body in a Cartesian axis is maximal when it is at the maximum amplitude in the complementary elliptical axis. For example, the velocity when the body passes through the major axis equals the maximum velocity of the simple harmonic motion (SHM) in the $y$ axis, as illustrated in the following scheme:

From the equations obtained in item **B.2)**, we know that the velocity equation in the $y$ axis can be written as: $v_y = -B\sqrt{GM}\sin\left(\sqrt{GM}t + \phi_b\right)$. In the case where the spacecraft is at the semi-major axis, the velocity in the $y$ axis is maximal, so $v_a = B\sqrt{GM}$. Notice that the value of $B$ is nothing more than the amplitude of the simple harmonic motion (SHM) in the $y$ axis, so we see that $B = b$, thus:

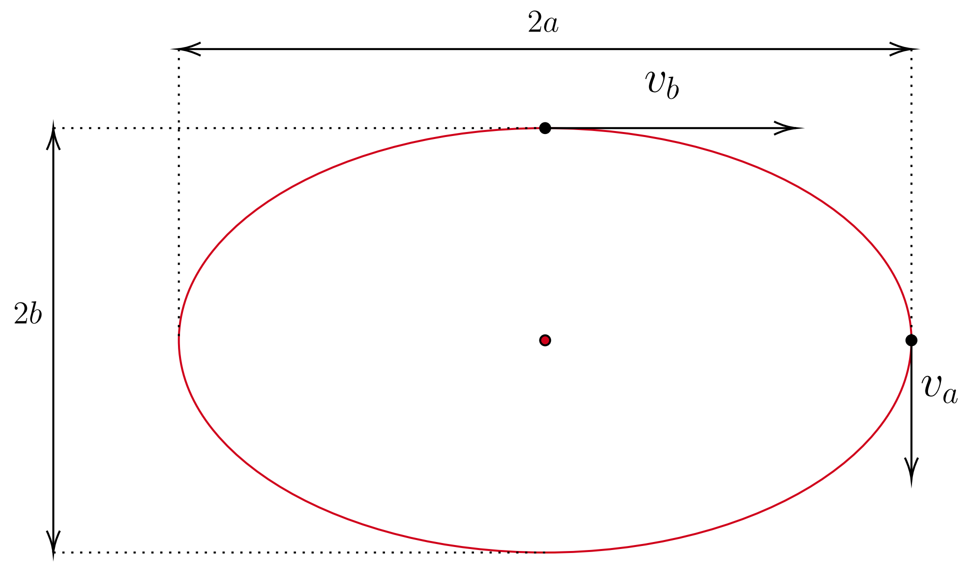


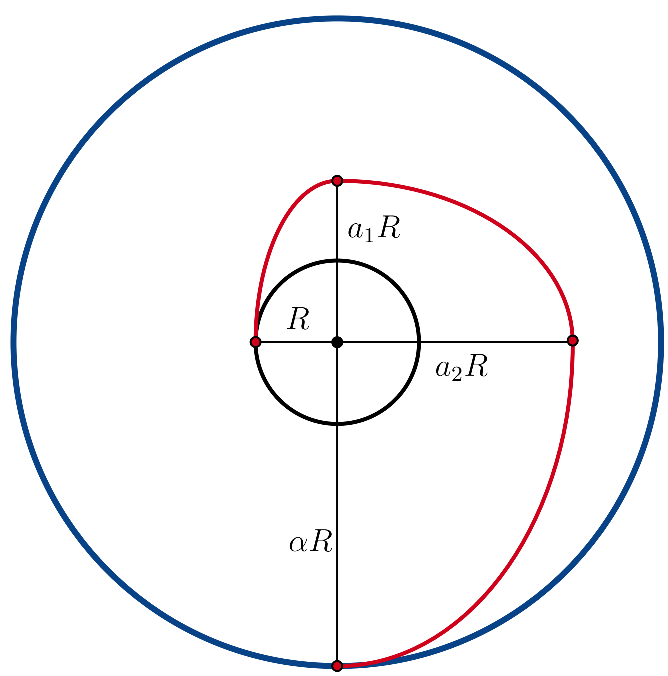

Figure 2.13: Representation of the orbit and apoastron and periastron velocities.

$$v_a = b\sqrt{GM}$$

Similarly:

$$v_b = a\sqrt{GM}$$

**C.2)** By convention, we will call $R\sqrt{GM}$ as $v$ in this solution. Consider the diagram proposed by the statement (Figure 1.1):




In the first orbital transfer, we calculate the velocity difference $v_{b,1}$ (velocity at the point of minimum separation of the first transfer orbit) minus the initial velocity in the circular orbit $v_{circ,1}$:

$$\Delta v_1 = a_1 v - v$$

Next, we continue finding the velocity variations between transfer orbits as: $\Delta v_i = v_{b,i+1} - v_{a,i}$ and the final variation for the desired circular orbit as: $\Delta v_{circ,2} = v_{circ,2} - v_{a,n-1}$:

$$\Delta v_1 = a_1 v - v$$
$$\Delta v_2 = a_2 v - v$$
$$\Delta v_3 = a_3 v - a_1 v$$
$$\Delta v_4 = a_4 v - a_2 v$$
$$\vdots$$
$$\Delta v_n = a_n v - a_{n-2} v$$
$$\Delta v_{circ,2} = a_n v - a_{n-1} v$$

Summing all the values to find $\Delta v_T$, we conclude that, regardless of the number or size of the transfer orbits:

$$\Delta v_T = 2(a_n - 1)v$$

Since the problem statement provides that $a_n = \alpha$ and knowing that $v = R\sqrt{GM}$:

$$\Delta v_T = 2(\alpha - 1)R\sqrt{GM}$$

**C.3)** Notice that in each transfer orbit, the radius vector starts at the semi-minor axis and reaches the semi-major axis, covering $\frac{1}{4}$ of the total ellipse area. Applying Kepler's Second Law, we know that covering one-quarter of the total area also corresponds to staying in orbit for one-quarter of the total period. Therefore, the total time interval for all $n$ transfers is:

$$\Delta t = n\frac{1}{4}T = \frac{n\pi}{2\sqrt{GM}}$$

## Bonus

Below are some bonus details about orbits in Kamoto.

- Orbital velocity as a function of the distance $r$ to $M$:

$$v = \sqrt{GM(a^2 + b^2 - r^2)}$$

- Polar equation of an ellipse centered at its geometric center, with semi-major axis $a$ and eccentricity $e$:

$$r = \frac{a\sqrt{1-e^2}}{\sqrt{1 - e^2\cos^2(\theta)}}$$

- Total orbital energy:

$$E_T = \frac{1}{2}GMm(a^2 + b^2)$$

- Angular momentum of the test mass:

$$L = mab\sqrt{GM}$$

## 2.12 The Conics of Narnia

**a)** Note that the gravitational law is the same as in our universe, so all physical principles remain the same. Since $v^2 < \frac{2GM}{d}$, we know that this is a closed orbit (velocity lower than the escape velocity), meaning it is an ellipse, as shown in Figure 2.14 ("B" represents the other focus of the orbit).

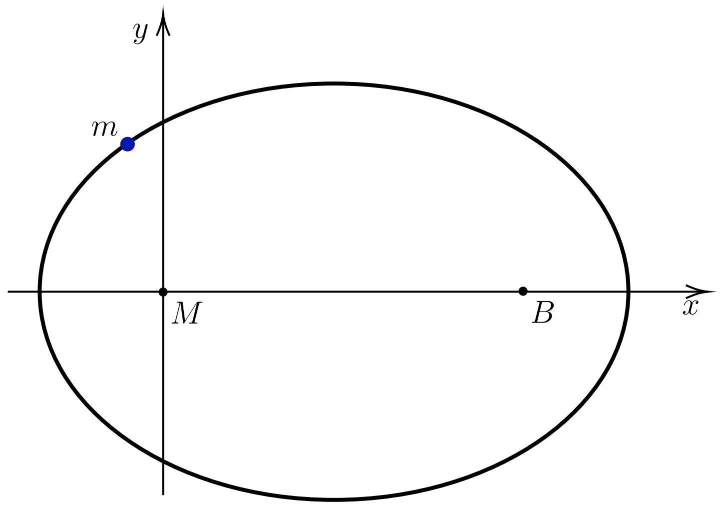



Figure 2.14: Scheme of an elliptical orbit, from point $m$, from foci $M$ and $B$.

Now, tracing the circle $\Omega$ with center at "$m$" and passing through "$M$" we will have the representation of the figure 2.15:

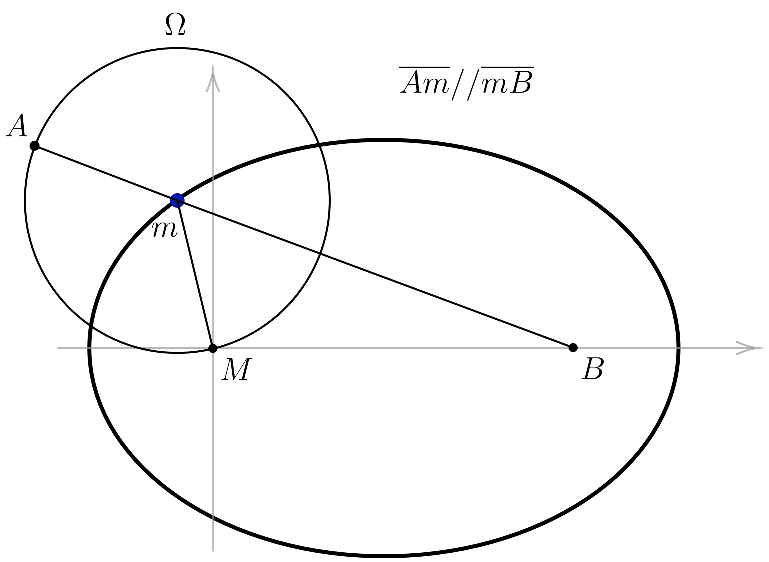



Figure 2.15: Geometric construction of points and the circle centered at $m$ that passes through $M$.

Note that, by definition, any point A belonging to $\Omega$ is at the same distance from $m$ as from $M$. Thus, we can say that the sum of the distances from A to $m$ and from $m$ to B is the same as the sum of the focal distances of the ellipse, which equals the length of its major axis. Also, note that the maximum distance A can be from B occurs when the lines connecting these points to $m$

are collinear (which is easy to see by the triangle inequality). Therefore, the points of maximum separation will always lie on a circle of radius $2a$ centered at B (where $a$ is the semi-major axis of the ellipse).

From the equation for velocity in an elliptical orbit, we have:

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$\therefore a = \frac{GMd}{2GM - v^2 d}$$

$$R = 2a = \frac{2GMd}{2GM - v^2 d}$$

Now consider a point "D" located at a distance $r < 2a$ (i.e., within this reachability circle). The perpendicular bisector of the segment connecting M and D is drawn. It is evident that this bisector contains all the centers of circles passing through M and D. Notice that this bisector obviously intersects the ellipse at least at one point, meaning that for any specified D, there is a point $m$ on the orbit that serves as the center of a possible circle passing through D.

Thus, it is proven that the orbital space is the internal region of a circle of radius

$$R = \frac{2GMd}{2GM - v^2 d}$$

centered at B.

**b)** For $v^2 = \frac{2GM}{d}$, the body is at its escape velocity, meaning its trajectory will be parabolic. Therefore, there will always exist a line $r$ for which the distance from $m$ to $r$ is equal to the distance from $m$ to $M$. This follows from the very definition of a parabola: the geometric locus of all points equidistant from a point (focus) and a line (directrix).

**c)** Note that for the region outside the orbital space to be the internal region of a finite circle, the body $m$ must be able to move arbitrarily far from $M$ (so that it can "reach" points infinitely distant from $M$). This means that the orbit must be hyperbolic. For this to occur:

$$v^2 > \frac{2GM}{d}$$

## BONUS

Consider the same orbital space diagram as in Figure 2.16.

Given the hyperbola with foci $A$ and $B$, let $D$ be a point on it. By the definition of a hyperbola, we know that $DA - DB = k$, where $k$ is a constant. Since we also know that $DB = DE = R$, we observe that

$$DA - DB = AE + DE - DB = k \Rightarrow AE = k = \text{constant}.$$

Since $E$ is the point on the circle closest to $A$, the geometric locus of these points forms a circle of radius $k$ (which is the parameter of the hyperbola).

Now, considering any point $P$ outside this circle, when we draw the perpendicular bisector of segment $PB$, we obtain the geometric locus of the centers of circles passing through both $P$ and $B$. This bisector necessarily intersects the hyperbola. Thus, we conclude that all—and only—points at a distance greater than or equal to $k$ from focus $A$, that is, outside the circle of radius $k$ centered at $A$, belong to the orbital space.

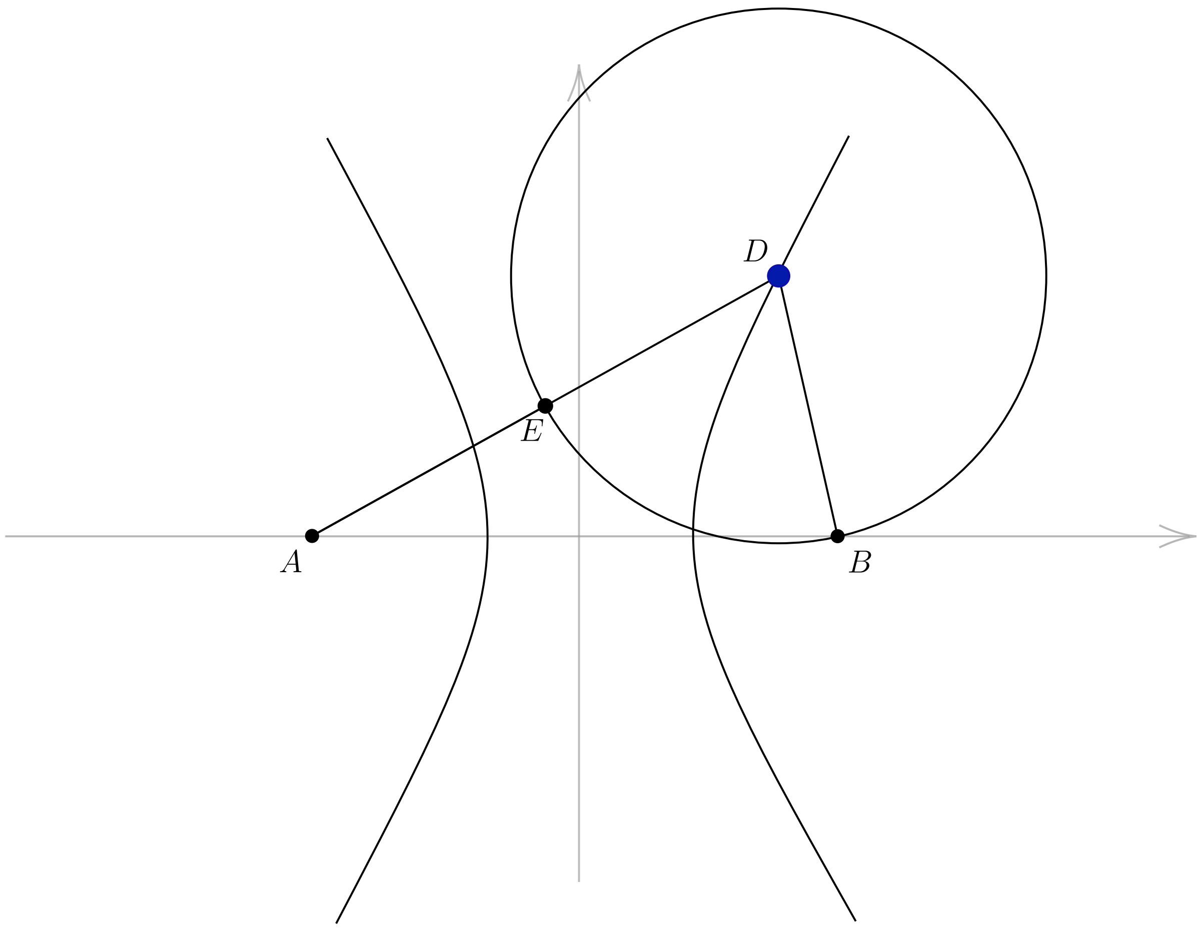


Figure 2.16: Schematization of the circumference of the point $D$ in the hyperbolic orbit.

Thus, we can express the time derivative of the angular momentum direction as:

$$\frac{dL}{dt} = \mu B \sin\theta$$

Since precession is defined as the slow rotation of the angular momentum vector around the external field $\vec{B}$, the angular velocity of precession $\Omega_p$ is given by:

$$\Omega_p = \frac{\tau}{L} = \frac{\mu B \sin\theta}{I\omega}$$

Using the previously derived result:

$$\mu = \frac{1}{3} Q \mathcal{L} \omega$$

we substitute into the expression for $\Omega_p$:

$$\Omega_p = \frac{\frac{1}{3} Q \mathcal{L} \omega B \sin\theta}{I\omega}$$

Canceling $\omega$:

$$\Omega_p = \frac{1}{3} \frac{Q \mathcal{L} B \sin\theta}{I}$$

Since the period of precession $T_p$ is given by:

$$T_p = \frac{2\pi}{\Omega_p}$$

we obtain:

$$T_p = \frac{6\pi I}{Q \mathcal{L} B \sin\theta}$$

This is the period of precession of Pluto II's equinoxes.

The angle of inclination of the ferromagnetic wire in José Lagranja's experiment is given as $39°20'$. This inclination corresponds to the magnetic dip angle, which relates to the latitude $\lambda$ of the experiment's location via the following equation for a dipole field:

$$\tan I = 2 \tan\lambda$$

where $I$ is the inclination angle. Substituting the given value:

$$\tan(39°20') = 2 \tan\lambda$$

Solving for $\lambda$:

$$\lambda = \tan^{-1}\left(\frac{\tan(39°20')}{2}\right)$$

This latitude difference $\lambda$ represents the angular distance between the experiment's location and the magnetic pole. Therefore, we compute:

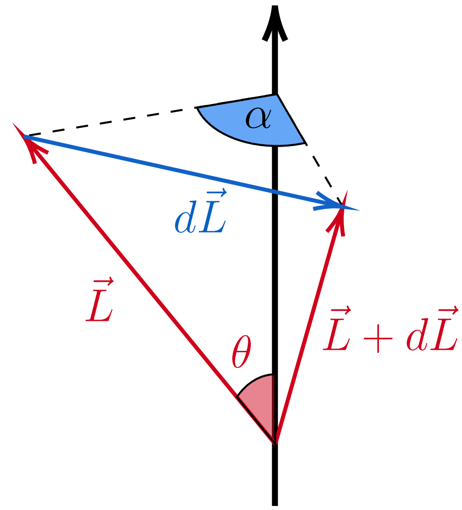



Figure 2.17: Angular variation of angular momentum due to precession.

$$\Delta\lambda = 90^\circ - \lambda$$

which provides the latitude of the magnetic north pole.

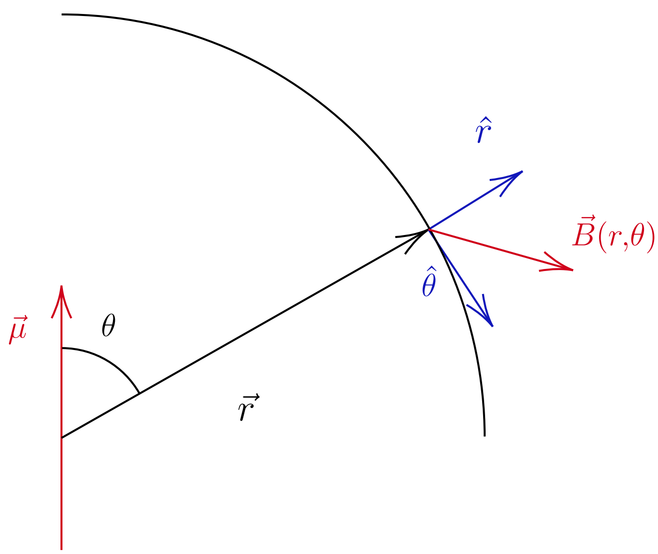



Figure 2.18: Magnetic Field on the planet's surface as a function of its dipole.

Notice that the direction $\hat{\theta}$ aligns with the ground, while $\hat{r}$ is perpendicular to it. Thus:

$$\vec{B}(r,\theta) = \frac{\mu_0|\vec{\mu}|}{4\pi|\vec{r}|^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta})$$

To calculate the angle of elevation of the wire, observe from the diagram that the tangent of this angle is the ratio between the vertical and horizontal components of the field:

$$\tan(\kappa) = \frac{2\cos(\theta)}{\sin(\theta)} = 2\cot(\theta)$$

Since $\kappa = 39^\circ 20'$, as the wire tilted, it follows that $\theta = 67^\circ 43'$.

**B.2)** Analyzing the positioning diagram on the planet, we can construct the following spherical triangle:

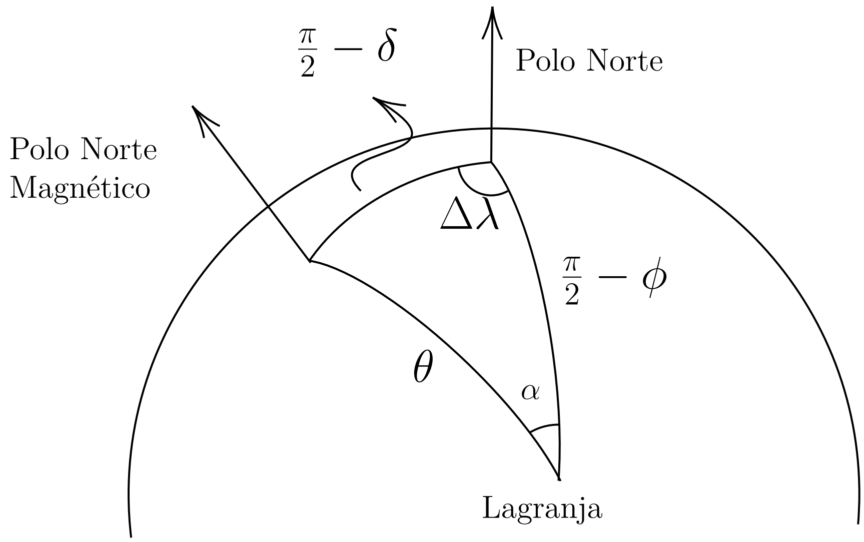



Figure 2.19: Spherical positions of the magnetic north pole, north pole and Lagranja.

Notice that the wire will align with the plane that passes through Lagranja and contains the Magnetic North Pole. Thus, the angle between the wire and the meridian plane (which passes through the Planetary North Pole) is precisely the angle $\alpha$ as represented in figure 2.19. By the spherical law of cosines, it is known that:

$$\cos\left(\frac{\pi}{2} - \delta\right) = \cos(\theta)\cos\left(\frac{\pi}{2} - \phi\right) + \sin(\theta)\sin\left(\frac{\pi}{2} - \phi\right)\cos(\alpha)$$

$$\therefore \sin(\delta) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)\cos(\alpha)$$

Thus, we can conclude that the latitude of the magnetic pole is $\delta = 80°51'$. Now, by the spherical law of sines, it is possible to find the relation for $\Delta\lambda$:

$$\frac{\sin(\Delta\lambda)}{\sin(\theta)} = \frac{\sin(\alpha)}{\sin\left(\frac{\pi}{2} - \delta\right)}$$

$$\therefore \Delta\lambda = 56°5'$$

Assume that the coordinate system of Pluto III is similar to Earth's (longitude is counted from left to right from an observer in space). Thus, it can be seen that the magnetic pole is $\Delta\lambda$ degrees west of Lagranja's position. Therefore, simply subtract $\Delta\lambda$ from the researcher's longitude: $\lambda_{PMN} = 130°52' - 56°5' = 74°47'$.

Thus, the coordinates of the magnetic north pole are: $80°51'N$ $74°47'E$.

## Part C: Rotation of the DSCOVR

**C.1)** Given the particle density $\rho$, the mass of the particles $\mu$ (the particle number density is therefore $\dfrac{\rho}{\mu}$), and the velocity of the particles $v$, consider the number of particles that will hit the

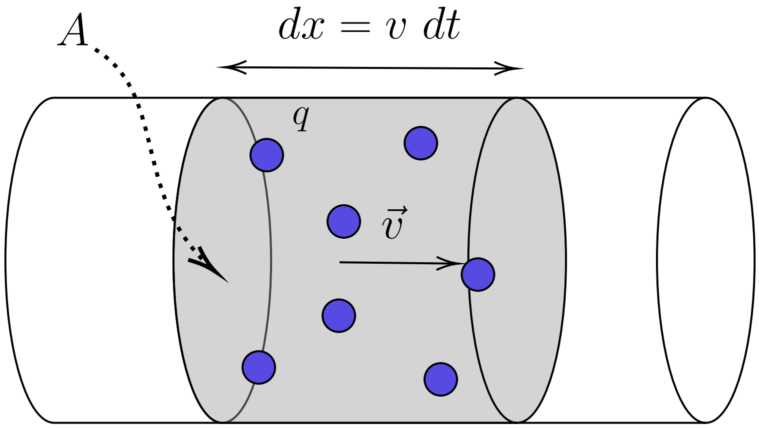


Figure 2.20: Representation of charge flow in space.

satellite within an interval $dt$. These particles are within a cylinder with the area of the satellite and are at most a distance $vdt$ from the satellite, as shown in figure 2.20.

Therefore, the charge that reaches the satellite is: $\frac{\rho}{\mu}qAvdt$, where $q$ is the charge of each particle. Therefore:

$$R = \frac{\rho\pi r^2 vq}{\mu}$$

Since the effective cross-sectional area of the satellite is $\pi r^2$.

Substituting the values, we find:

$$R = 8.916 \cdot 10^{-6}\,\mathrm{C} \cdot \mathrm{s}^{-1}$$

**C.2)** Define:

$$\vec{\omega}(t) = \omega_x(t)\hat{\imath} + \omega_y(t)\hat{\jmath} + \omega_z(t)\hat{k}$$

Let us define the coordinate space based on the position of the magnetic field (since this is the invariant factor in the problem):

$$\vec{B} = B\hat{k}$$

$$\vec{L} = I\vec{\omega}$$

The charge of the body grows linearly with the relation $Q(t) = Rt$, therefore:

$$\vec{\mu}(t) = \frac{Rt}{2M} I\omega\vec{(t)}$$

Considering the torque exerted by the field:

$$\vec{\tau} = I\vec{\alpha} = \vec{\mu} \times \vec{B}$$

$$\vec{\alpha} = \frac{Rt}{2M}\omega\vec{(t)} \times \vec{B}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

By the well-known relation for finding the cross product between two vectors:

$$\vec{\omega}(t) \times \vec{B} = \begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x(t) & \omega_y(t) & \omega_z(t) \\ 0 & 0 & B \end{Vmatrix}$$

$$\vec{\omega}(t) \times \vec{B} = \omega_y(t)B\hat{i} - \omega_x(t)B\hat{j}$$

$$\frac{d\omega_x(t)}{dt}\hat{i} + \frac{d\omega_y(t)}{dt}\hat{j} + \frac{d\omega_z(t)}{dt}\hat{k} = \frac{Rt}{2M}\omega_y(t)B\hat{i} - \frac{Rt}{2M}\omega_x(t)B\hat{j}$$

Since the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are linearly independent, for the vectors to be equal, their components along each axis must be equal:

$$\begin{cases} \frac{d\omega_z(t)}{dt} = 0 \\ \frac{d\omega_x(t)}{dt} = \frac{Rt}{2M}\omega_y(t)B \\ \frac{d\omega_y(t)}{dt} = -\frac{Rt}{2M}\omega_x(t)B \end{cases}$$

Dividing the last two equations:

$$\omega_x(t)d\omega_x(t) = -\omega_y(t)d\omega_y(t)$$

Integrating and rearranging:

$$\omega_x^2(t) + \omega_y^2(t) = \omega_0^2$$

Where $\omega_0$ is an integration constant. Having found the relationship between $\omega_x$ and $\omega_y$, let us proceed:

$$d\omega_y(t) = d\sqrt{\omega_0^2 - \omega_x^2(t)} = \frac{-2\omega_x(t)d\omega_x(t)}{2\sqrt{\omega_0^2 - \omega_x^2(t)}}$$

Substituting this result into the last equation of the system:

$$\frac{d\omega_x(t)}{\sqrt{\omega_0^2 - \omega_x^2(t)}} = \frac{Rt}{2M}Bdt$$

$$\int_{\omega_{x,0}}^{\omega_x(t)} \frac{d\omega_x(t)}{\sqrt{\omega_0^2 - \omega_x^2(t)}} = \frac{Rt^2}{4M}B$$

Substituting $\omega_x(t) = \omega_0 \sin(y)$, and knowing that: $d\omega_x(t) = \omega_0 \cos(y)dy$:

$$\int_{y_0}^{y} \frac{\cos(y)dy}{\sqrt{1 - \sin^2(y)}} = \frac{Rt^2}{4M}B$$

$$y - y_0 = \frac{Rt^2}{4M}B = \arcsin\left(\frac{\omega_x(t)}{\omega_0}\right) - \arcsin\left(\frac{\omega_{x,0}}{\omega_0}\right)$$

$$\omega_x(t) = \omega_0 \sin\left(\frac{Rt^2}{4M}B + \arcsin\left(\frac{\omega_{x,0}}{\omega_0}\right)\right)$$

$$\omega_y(t) = \omega_0 \cos\left(\frac{Rt^2}{4M}B + \arccos\left(\frac{\omega_{y,0}}{\omega_0}\right)\right)$$

$$\omega_z(t) = \omega_{z,0}$$

Finally:

$$\vec{\omega}(t) = \omega_0 \sin\left(\frac{Rt^2}{4M}B + \arcsin\left(\frac{\omega_{x,0}}{\omega_0}\right)\right)\hat{\imath} + \omega_0 \cos\left(\frac{Rt^2}{4M}B + \arccos\left(\frac{\omega_{y,0}}{\omega_0}\right)\right)\hat{\jmath} + \omega_{z,0}\hat{k}$$

**C.3)** Note that for the satellite to complete a full precessional rotation, the angular velocity configuration must return to the same value, which occurs when the argument of the trigonometric functions increases by $2\pi$ (a full rotation). Therefore, for the $n$-th rotation: $\Delta\theta = 2n\pi$, thus:

$$\frac{Rt^2(n)}{4M}B = 2n\pi$$

Therefore:

$$t(n) = \sqrt{\frac{8M\pi}{RB}}\sqrt{n}$$

Substituting the values:

$$t(n) = 18.214\sqrt{n}\,\text{years}$$

**BONUS:** The period for the $n$-th rotation is given by:

$$P(n) = \Delta t(n) = t(n+1) - t(n) = 18.214(\sqrt{n+1} - \sqrt{n})$$

For large values of $n$, Bernoulli's approximation can be used: $\sqrt{n+1} = \sqrt{n}\sqrt{1+\frac{1}{n}} \approx \sqrt{n}\left(1+\frac{1}{2n}\right)$:

$$P(n) = 18.214\frac{\sqrt{n}}{2n} = \frac{9.107}{\sqrt{n}}$$

This reveals that the satellite rotates faster and faster, as the precessional period decreases with the factor $\sqrt{n}$.

# 2.14 Nill in the Telescope Wonderland

## Part A: Refracting Telescopes

**A.1)** As shown by the famous lens equation:

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{f}$$

We then know that, since in both cases $R_1 = R_2 = R$:

$$f = \frac{R}{2(n-1)}$$

As $n = 1.5$, we find that $f = R$! Therefore: $f_{ob} = 70cm$ and $f_{oc} = 10cm$.

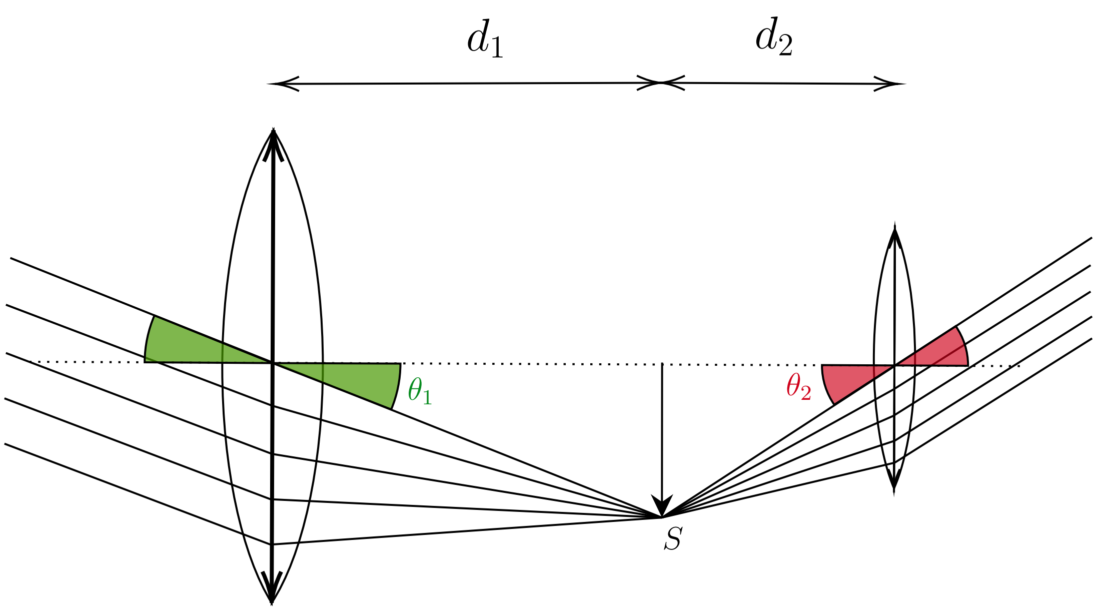
**A.2)** In order for the conversion of parallel rays into the intermediate image and vice versa to occur, it is necessary that each distance be equal to the focal length of its respective lens; thus $d_1 = 70cm$ and $d_2 = 10cm$.

**A.3)** From the image of the intermediate image, we can conclude that the length $l$ of the intermediate image can be obtained in two equivalent ways:

$$\begin{cases} l = d_1 \tan(\theta_1) \\ l = d_2 \tan(\theta_2) \end{cases}$$

Thus we have $f_{ob} \tan(\theta_1) = f_{oc} \tan(\theta_2)$.

Note that when the angle of the light rays is changed, the angular size of the object also changes. In this case, consider the scheme below, which presents the analysis of an object located on the lens axis and having an angular diameter of $2\theta_1$. One wishes to know the variation of this parameter after passing through the tube and reaching the observer's eye, as shown in Figure 1.4.



In this way, it is observed that the size of the image is proportional to the angle $\theta$, so we find that the magnification factor is $A = \dfrac{\theta_2}{\theta_1}$:

$$A = \frac{2\arctan\left(\frac{f_{ob}}{f_{oc}}\tan\left(\frac{\delta}{2}\right)\right)}{\delta}$$

With $\delta = 2\theta_1$ being the initial angular diameter. In common cases, such angles are infinitesimal, so we can approximate $\tan(\theta) \approx \theta$, thus we find the famous relation:

$$A = \frac{f_{ob}}{f_{oc}}$$

**A.4)** Check the following scheme:

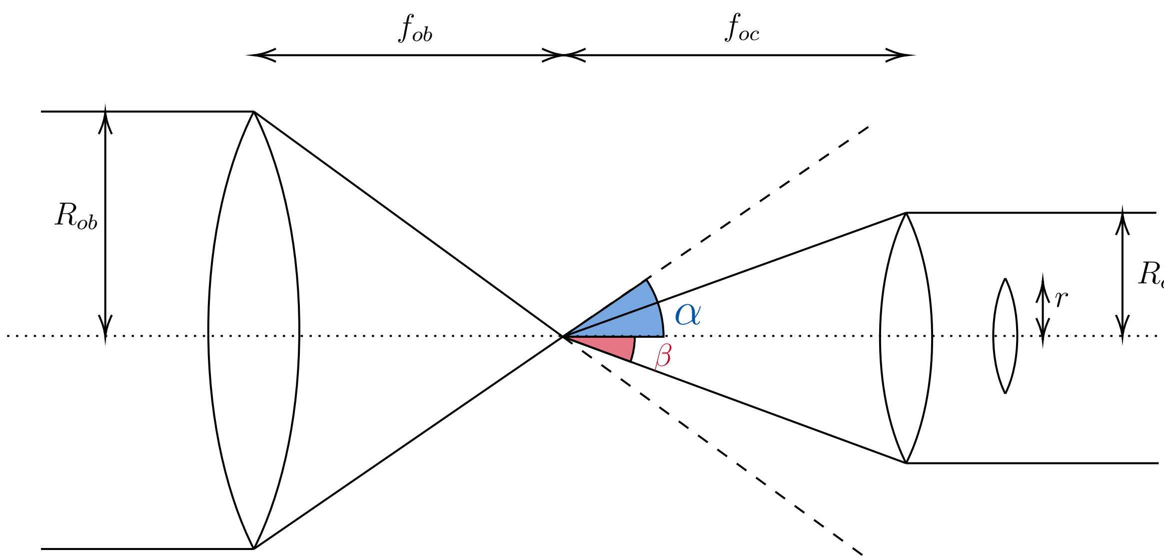



Figure 2.21: Representation of the fraction of light that effectively passes through the two lenses of Nill's refracting telescope.

We will perform the analysis by comparing the flux that reaches the observer's eye with the actual flux emitted by the celestial object. We will call the original flux $F$ and the limiting flux for human vision $F_0$ (the smallest flux at which something can be observed). Note that the power that effectively enters the telescope is $P = F\pi R_{ob}^2(1-\epsilon)$ ($1-\epsilon$ because this corresponds to the transmitted part—the complement of the absorbed part). This power is concentrated at the focal point of the lenses and then spreads out toward the eyepiece. Note that since $\dfrac{R_{oc}}{f_{oc}} < \dfrac{R_{ob}}{f_{ob}}$, we have $\tan(\beta) < \tan(\alpha)$, which implies $\beta < \alpha$, and therefore not all the power passes through the eyepiece.

The fraction that passes depends on the solid angle corresponding to this aperture. Thus:

$$\frac{P_{oc}}{P_{total}} = \frac{\Omega(\beta)}{\Omega(\alpha)}$$

As discussed in other problems, we know that $\Omega(\theta) = 2\pi(1-\cos(\theta))$, so the power that passes through the eyepiece is:

$$P_{oc} = F\pi R_{ob}^2(1-\epsilon)^2\frac{1-\cos(\beta)}{1-\cos(\alpha)}$$

Notice that we multiply again by $1-\epsilon$ to take into account only the transmitted part. Note that even so, not all the radiation (which exits parallel to the axis) reaches the observer's eye, since the diameter of the pupil is smaller than the diameter of the eyepiece. Thus, the part that effectively enters is proportional to the area:

$$\frac{P_{olho}}{P_{oc}} = \frac{\pi r^2}{\pi R_{oc}^2}$$

Thus:

$$P_{olho} = F\pi\left(\frac{R_{ob}}{R_{oc}}\right)^2 (1-\epsilon)^2 \frac{1-\cos(\beta)}{1-\cos(\alpha)} r^2$$

The flux is the power divided by the area ($\pi r^2$):

$$F_{olho} = F\left(\frac{R_{ob}}{R_{oc}}\right)^2 (1-\epsilon)^2 \frac{1-\cos(\beta)}{1-\cos(\alpha)}$$

It is interesting that in this case the dimensions of the pupil are irrelevant, as long as $r < R_{oc}$! In the limiting case, we set $F_{olho} = F_0$, thus:

$$F = F_0\left(\frac{R_{oc}}{R_{ob}}\right)^2 (1-\epsilon)^{-2} \frac{1-\cos(\alpha)}{1-\cos(\beta)}$$

Since $m_0 = 6 = -2.5\log(F_0/F_r)$ (with $F_r$ being the reference flux), we can find that:

$$m_{lim} = -2.5\log\left[\frac{F_0}{F_r}\left(\frac{R_{oc}}{R_{ob}}\right)^2 (1-\epsilon)^{-2} \frac{1-\cos(\alpha)}{1-\cos(\beta)}\right]$$

Simplifying and rearranging, we obtain:

$$m_{lim} = m_0 + 5\log\left[\frac{R_{ob}}{R_{oc}}(1-\epsilon)\sqrt{\frac{1-\cos(\beta)}{1-\cos(\alpha)}}\right]$$

Substituting the values:

$$m_{lim} = 10.02$$

## Part B: Reflecting Telescopes

**B.1)** Choose a point on the ellipse with foci $A$ and $B$ and draw the tangent that passes through this point. A ray of light, upon striking the mirror, would behave as if it were incident on a plane mirror coinciding with the tangent line, so we want to prove that the angles $\alpha$ and $\beta$ in Figure 2.22.

Draw also a segment perpendicular to the tangent that starts from each focus (segments $K$ and $H$). Note that the sum of the distances from the point that defines the tangent to the two foci is equal, by definition, to the major axis of the ellipse $2a$; however, note another important factor: this is the point on the tangent line that minimizes this sum (just look at the figure).

In this way, we want to prove that for this point of minimum $\alpha = \beta$. This distance from the point to the foci is given by:

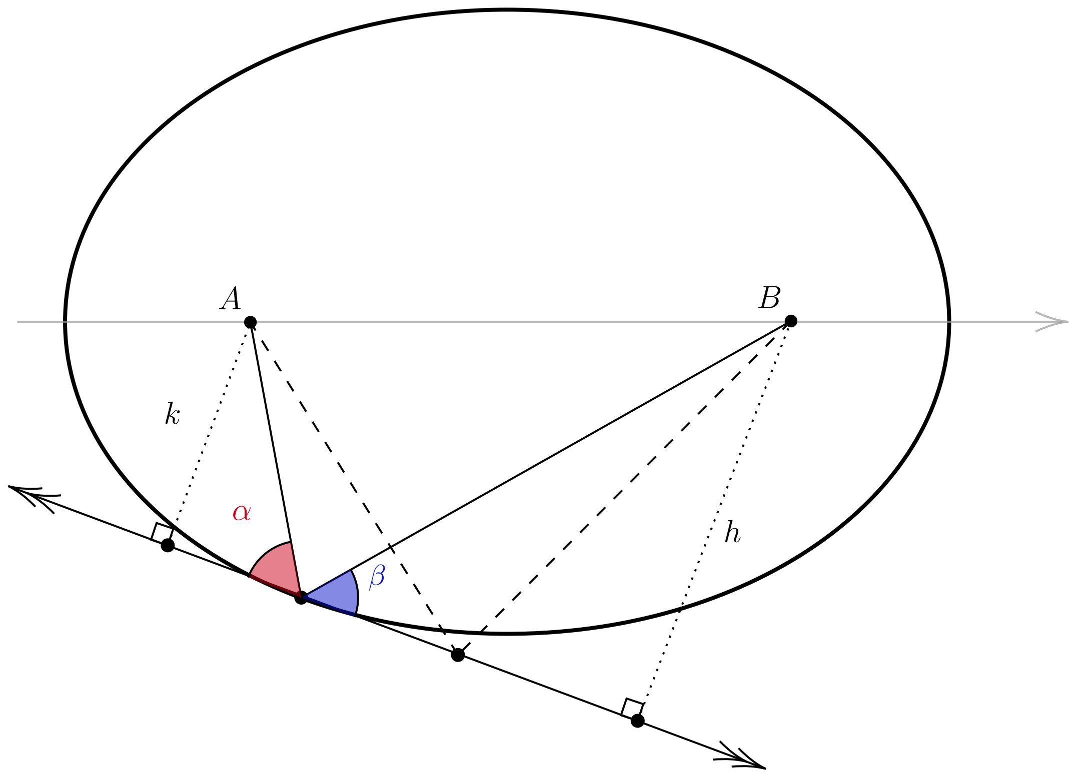



Figure 2.22: Geometric relation of the points of the ellipse with the tangent line at a generic point.

$$d = \frac{K}{\sin(\alpha)} + \frac{H}{\sin(\beta)}$$

For $d$ to be minimum, it suffices to differentiate and set it equal to zero:

$$\frac{K}{\sin^2(\alpha)} \cos(\alpha) d\alpha = -\frac{H}{\sin^2(\beta)} \cos(\beta) d\beta$$

Note that the sum of the distances from a point to the orthogonal projections of $A$ and $B$ onto the tangent line is constant for points within this interval:

$$\frac{K}{\tan(\alpha)} + \frac{H}{\tan(\beta)} = \text{constant}$$

Thus:

$$\frac{K}{\sin^2(\alpha)} d\alpha = -\frac{H}{\sin^2(\beta)} d\beta$$

Which implies that $\cos(\alpha) = \cos(\beta)$, which, under the conditions of the analysis, means that $\alpha = \beta$. Thus, the relation that the angle of incidence is equal to the angle of reflection is satisfied, and therefore each ray that originates from one of the foci strikes the ellipse and goes to the other focus.

**B.2)** We can prove this item in isolation using analytic geometry and going through the calculations (writing the equation of the tangent line, finding the relations between the angles from trigonometric relations and derivatives, etc.), or even by plane geometry, starting from the definition of a parabola and studying the points that belong to the tangent line and noting that the only point that is at equal distance from the directrix and the focus (similar to what we did in the proof for the ellipse). However, we will do something different!

Recall that ellipses have eccentricity $0 \leq e < 1$. Note that $e \neq 1$, but we can fix one focus at the origin, for example, and let $e \to 1$ (that is, $e$ tends to 1), which would make the ellipse in question approach arbitrarily close to a parabola. Also note that as $e \to 1$, the unfixed focus tends to infinity, so a ray of light coming from it would tend to come almost parallel to the axis of the ellipse and converge at the fixed focus, which proves[3] the stated property!

**B.3)** We will follow a line of reasoning similar to that of the proof for the ellipse[4], for this, consider the following scheme, where a ray of light "aims" at focus $A$ and is reflected at point $D$, as shown in Figure 2.23:

We wish to prove that the angle $\alpha$—which is vertically opposite to the angle of incidence of the ray (which means they are equal)—is equal to the angle $\beta$ (the angle of reflection), since this would guarantee that this is the path followed by the reflected ray.

In the diagram, the tangent line to the conic at $D$ was drawn and the angles shown were defined, with $A$ and $B$ being the foci of the hyperbola. Note that among all the points on the tangent line, $D$ is the one for which $L = \overline{BD} - \overline{DA}$ is maximum. Proceeding to the relations to be developed, note that:

$$\begin{cases} \overline{BD} = \frac{K}{\sin(\beta)} \\ \overline{DA} = \frac{h}{\sin(\alpha)} \\ \overline{IB} = K + h \\ p = \frac{K}{\tan(\beta)} - \frac{h}{\tan(\alpha)} \end{cases}$$

Note that when the point $D$ varies along the tangent, the only quantities that change are the angles and the distances $\overline{DA}$ and $\overline{BD}$. Thus, fortunately, $p$, $K$, $h$, and $c$ remain constant:

$$dp = 0 = \frac{-K}{\sin^2(\beta)} d\beta - \frac{-h}{\sin^2(\alpha)} d\alpha$$

$$\therefore \frac{K}{\sin^2(\beta)} = \frac{h}{\sin^2(\alpha)}$$

For $L$ to be maximum:

$$dL = 0 = \frac{-K\cos(\beta)}{\sin^2(\beta)} d\beta - \frac{-h\cos(\alpha)}{\sin^2(\alpha)} d\alpha$$

Comparing the two previous equations, we finally find that $\cos(\alpha) = \cos(\beta)$, which, under the conditions of the problem, implies that $\alpha = \beta$, which guarantees the path of the light ray.

**B.4)** We know that the primary mirror (parabolic) focuses parallel rays at a focal distance of $h + p$, as shown in the figure; thus $f_p = 700mm$.

**NOTE: Consider Figure 2.24 below for the solution of the next items**

[3]Our mathematical colleagues were probably quite dissatisfied with this non-rigorous "proof", but in this case practicality spoke louder than rigor!

[4]If you stop to pay attention, the difference between the definition of a hyperbola and an ellipse is just a sign: while one involves the sum of the distances, the other involves the difference of the distances.

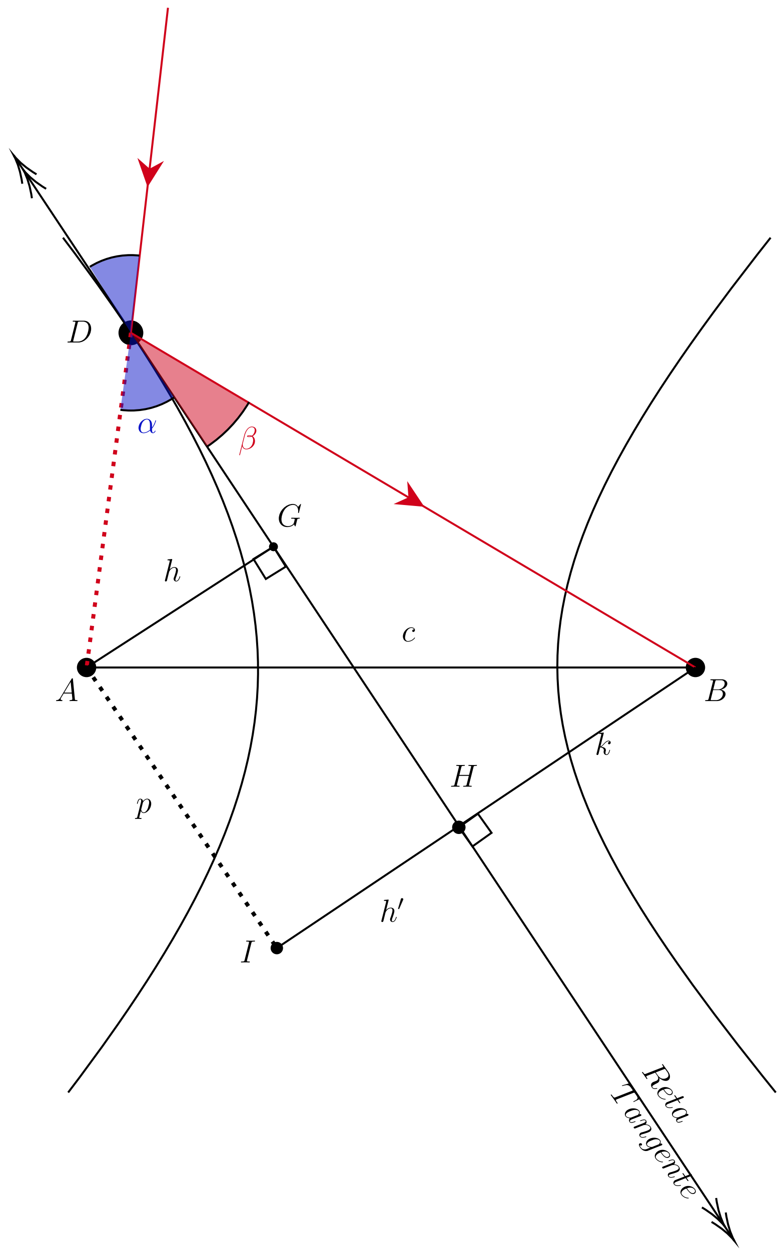



Figure 2.23: Reflection scheme on the hyperbola.

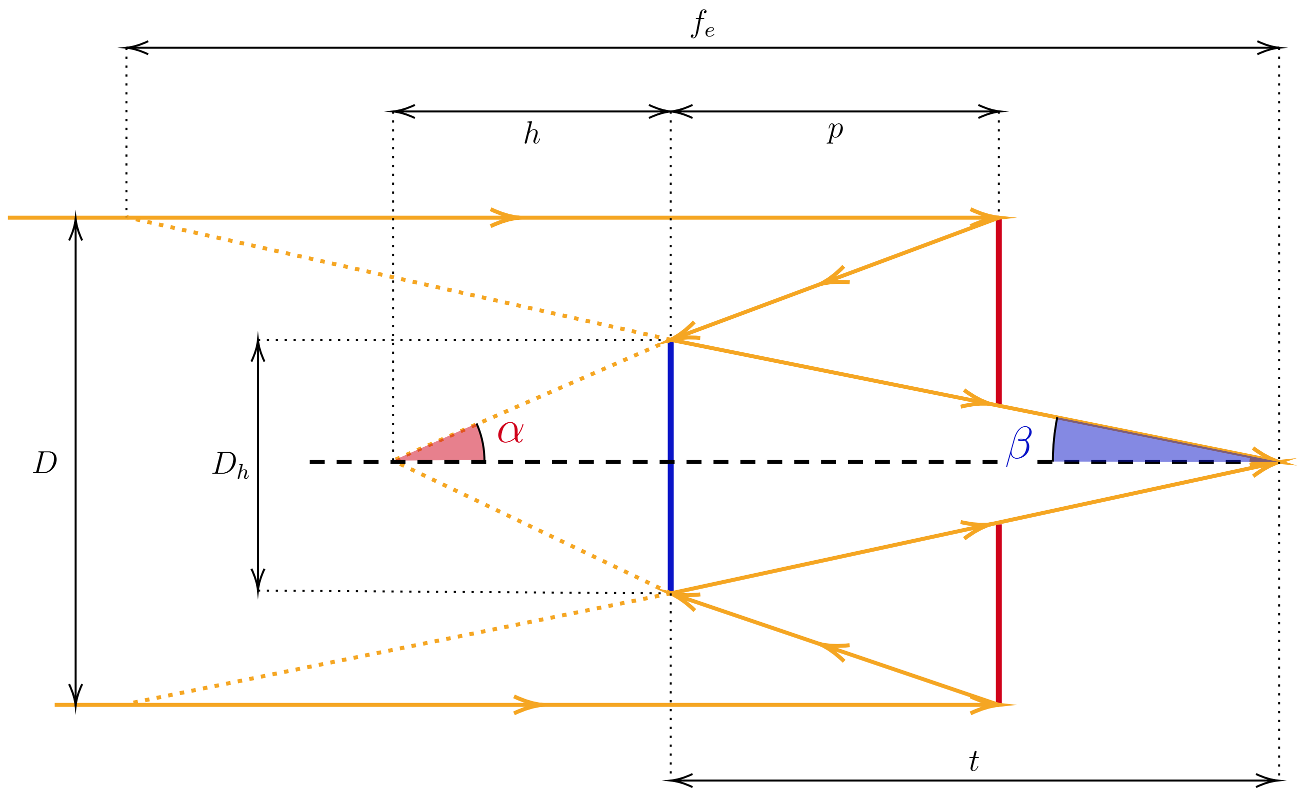



Figure 2.24: Scheme of convergence of light rays inside the tube of the reflecting telescope. Consider the red trace as the parabolic mirror and the blue as the hyperbolic mirror (all curvatures were neglected).

The parabolic mirror is represented by the red line and the hyperbolic one by the blue; the rays under study enter at the edge of the telescope with diameter $D$.

**B.5)** From the image, we observe that the effective focus of the telescope is $f_e$. Note that we can find $f_e$ from $\beta$: $f_e = \frac{D}{2\tan(\beta)}$; we therefore need to find the value of $\tan(\beta)$.

Thus, we will first find a relation between $\alpha$ and $\beta$, starting from the triangles formed with the hyperbolic mirror and with the parabolic one. From this we can derive the following relations:

$$\begin{cases} \tan(\alpha) = \frac{D_h}{2h} \\ \tan(\beta) = \frac{D_h}{2t} \\ \tan(\alpha) = \frac{D}{2(p+h)} = \frac{D}{2f_p} \end{cases}$$

From the previous relations, we find that $\tan(\beta) = \dfrac{Dh}{2tf_p}$. Thus:

$$f_e = \frac{2Dtf_p}{2Dh} = f_p\frac{t}{h}$$

Therefore $f_e = 1604.3mm$.

**B.6)** The focal ratio is defined as: $R = \dfrac{f}{D}$; thus $R = 8.23$. Hence this telescope is an $f/8.23$.

**B.7)** We will once again analyze the power that reaches the observer's eye, similarly to what we did in item **A.4)**. Let $F$ denote the limiting flux of observation of the telescope. From the previous image, it is known that the power that enters the tube is $P = F\pi\frac{D^2}{4}$; however, only $F\frac{\pi}{4}(D^2 - D_h^2)$

reaches the primary mirror, due to the blockage by the back of the secondary mirror. Note that the light that reflects off the primary mirror also reflects off the secondary mirror and is focused at the system's focus, so we must consider the effect of two reflections with regard to the absorbance of the mirrors:

$$P_{obs} = F\frac{\pi}{4}\left(D^2 - D_h^2\right)(1-\epsilon)^2$$

Note that, by the similarity of triangles, $\dfrac{D}{f_e} = \dfrac{D_h}{t}$; thus:

$$P_{obs} = F\frac{\pi D^2}{4}\left(1 - \frac{t^2}{f_e^2}\right)(1-\epsilon)^2$$

Thus, the flux observed by the observer, whose pupil diameter is $d$, will be:

$$F_0 = F\frac{D^2}{d^2}\left(1 - \frac{t^2}{f_e^2}\right)(1-\epsilon)^2$$

In the limiting case $m_0 = -2.5\log\left(F_0/F_r\right) = 6$; thus:

$$m_0 = -2.5\log\left[\frac{F}{F_r}\frac{D^2}{d^2}\left(1 - \frac{t^2}{f_e^2}\right)(1-\epsilon)^2\right]$$

Knowing that $m_{lim} = -2.5\log\left(F/F_r\right)$ and rearranging the terms, we find that:

$$m_{lim} = m_0 + 5\log\left[\frac{D}{d}(1-\epsilon)\sqrt{1 - \frac{t^2}{f_e^2}}\right]$$

Applying the values:

$$m_{lim} = 13.32$$

## 2.15 Unfocused Photometry

**a)** From the temperature of each star, we can find its respective wavelength of maximum emission (this is what we will use to relate to the focus) through Wien's law: $\lambda = \frac{0.0028976}{T}$. From this, we find the relations for each star:

Table 2.1: Data of wavelength adjusted from temperature and focal distance of the rays from each star.

| Star | Wavelength (nm) | Focus (cm) |
|---|---|---|
| Spica | 115.90 | 3.11 |
| Betelgeuse | 804.89 | 149.93 |
| Aldebaran | 714.57 | 118.17 |
| Sirius | 294.32 | 20.05 |

Assuming the refractive index of air is equal to 1 (a very good approximation), we want to find the refractive index of the lens for some cases. For this, we calculate the ratio of focal distances between any focus and a reference focus. Notice that the ratio between the reference focus and that of any star $i$ is of the form:

$$\frac{f_r}{f_i} = \frac{n_0 - 1 + \frac{k}{\lambda_i^2}}{n_0 - 1 + \frac{k}{\lambda_r^2}}$$

Let us define $\dfrac{f_r}{f_i} = y_i$, $\dfrac{1}{\lambda_i^2} = x_i$, $\dfrac{k}{n_0 - 1 + \dfrac{k}{\lambda_r^2}} = a$ and $\dfrac{n_0 - 1}{n_0 - 1 + \dfrac{k}{\lambda_r^2}} = b$

From this we get:

$$y_i = b + a x_i$$

Performing a linear regression with all possible combinations[5] we obtain the graphs in Figure 2.25.

From this we find the following lines for each reference star, in SI units:

- **Spica:** $y = 1.34331 \cdot 10^{-14} x - 1.91033 \cdot 10^{-5}$
- **Betelgeuse:** $y = 64.75980 \cdot 10^{-14} x - 92.09508 \cdot 10^{-5}$
- **Aldebaran:** $y = 51.04159 \cdot 10^{-14} x - 72.58637 \cdot 10^{-5}$
- **Sirius:** $y = 8.660267 \cdot 10^{-14} x - 12.315789 \cdot 10^{-5}$

Now we need to find the distance where the rays of highest intensity were focused. From the figure we can estimate this distance to be approximately $f = 37.60$ cm (even though the graph is not perfectly linear, the local variation is small, so we can reasonably approximate it as linear locally). From this, we apply this value to the functions above and find the value of $x$ ($\lambda^{-2} = x$):

[5] This is not really necessary at the moment, it is enough to find a single law, but it is interesting to see how all the lines behave.

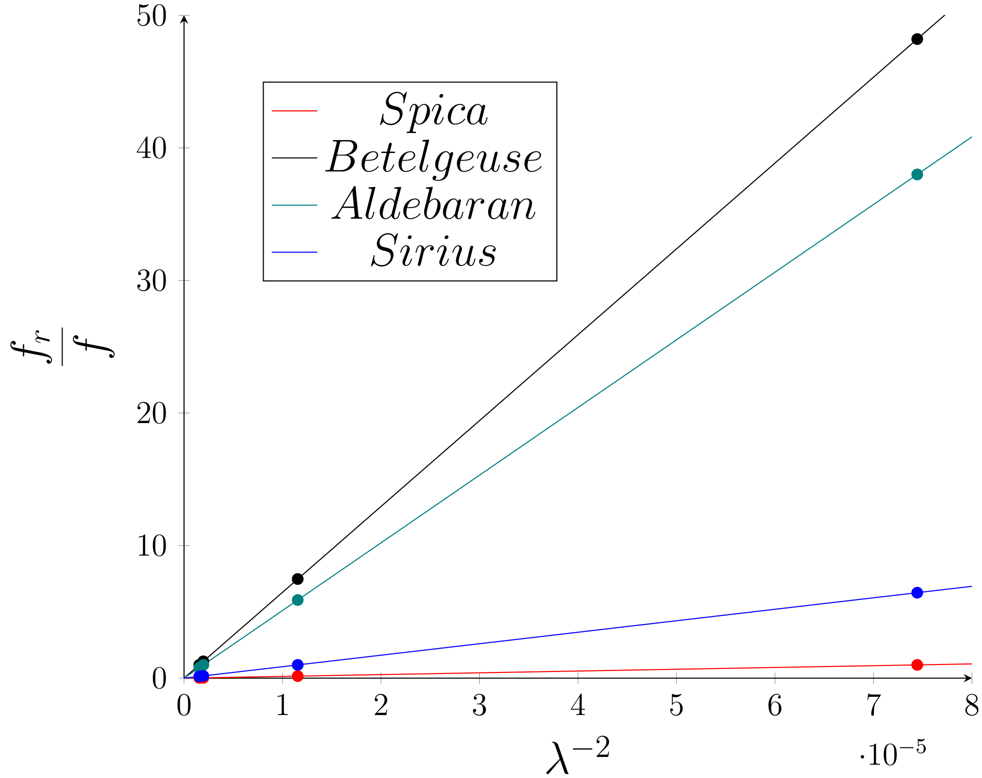



Figure 2.25: Regression lines found for each pair of stars.

- **Spica:** $\lambda = 402.950nm$
- **Betelgeuse:** $\lambda = 402.951nm$
- **Aldebaran:** $\lambda = 402.950nm$
- **Sirius:** $\lambda = 402.950nm$

As the Chinese sage would say: "it's as if it was meant to work." Finally, we can calculate the average temperature of the star from the mean wavelength of maximum emission ($\lambda = 402.95nm$):

$$T = 7190K$$

**b)** We notice that from the equations found previously, the linear coefficient of the found lines is very small, so we can approximate it to zero, so that $n_0 \approx 1$[6].

From the lens combination relation we have:

$$\frac{1}{f_t} = \frac{1}{f_A} + \frac{1}{f_B} + \frac{1}{f_0}$$

$$\frac{1}{f_t} = \left(n_A - 1 + \frac{k_A}{\lambda^2}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{f_B} + \frac{k}{\lambda^2}\frac{2}{R}$$

[6] Even though the slope is also very small, remember that the value of $x$ will be extremely large in most cases, so the product approximates a number with magnitude much larger than the linear coefficient, which can alter it

$$\frac{1}{f_t} = (n_A - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{f_B} + \frac{k_A}{\lambda^2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{2k}{\lambda^2 R}$$

To make $f_t$ independent of $\lambda$, we want:

$$\frac{k_A}{\lambda^2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{2k}{\lambda^2 R} = 0$$

From which we find:

$$\frac{1}{R_1} + \frac{1}{R_2} = -\frac{2k}{Rk_A}$$

**c)** The condition that needs to be satisfied for the lens to act as a converging lens is that $f_t > 0$:

$$\frac{1}{f_t} = (n_A - 1)\left(-\frac{2k}{Rk_A}\right) + \frac{1}{f_B} > 0$$

Finally:

$$f_B < \frac{Rk_A}{2k(n_A - 1)}$$

## 2.16 Variable Solar "Constant"

**a)** Using the well-known relationships of altazimuthal coordinates:

$$\cos(H) = \frac{\sin(h) - \sin(\delta)\sin(\phi)}{\cos(\delta)\cos(\phi)}$$

Such that:

$$\sin(h) = \cos(H)\cos(\phi)\cos(\delta) + \sin(\phi)\sin(\delta)$$

And from spherical triangle relations, we can show that:

$$\frac{\cos(h)}{\sin(H)} = -\frac{\cos(\delta)}{\sin(A)}$$

Therefore:

$$\sin(A) = -\frac{\sin(H)\cos(\delta)}{\cos(h)}$$

Note that $H = \omega t$, where $\omega = \dfrac{2\pi}{T}$ and $T = 24h$.
From this, we obtain the desired functions:

$$\sin(h) = \cos(\omega t)\cos(\phi)\cos(\delta) + \sin(\phi)\sin(\delta)$$

$$\sin(A) = -\frac{\sin(\omega t)\cos(\delta)}{\sqrt{1 - (\cos(\omega t)\cos(\phi)\cos(\delta) + \sin(\phi)\sin(\delta))^2}}$$

**b)** In this case, we need to consider the angle that sunlight makes when entering each window, since it determines how much light actually enters the house. To do this, we project the normal vector of each window onto the celestial sphere (essentially projecting the window itself onto the celestial sphere). These windows have the following projections ($h$,$A$):

$$(0, \theta)$$
$$\left(0, \frac{\pi}{2} + \theta\right)$$
$$(0, \pi + \theta)$$
$$\left(0, \frac{3\pi}{2} + \theta\right)$$
$$\left(\frac{\pi}{2}, 0\right)$$

The amount of radiation entering a given window is the solar flux (solar constant $F$) multiplied by the area perpendicular to the radiation: $A_t = A\cos(\Delta)$.

The angle $\Delta$ can be calculated using spherical trigonometry:

$$\cos(\Delta) = \sin(h_1)\sin(h_2) + \cos(h_1)\cos(h_2)\cos(A_1 - A_2)$$

For each window we have:

$$\cos(\Delta) = \cos(h)\cos(A-\theta)$$
$$\cos(\Delta) = \cos(h)\sin(A-\theta)$$
$$\cos(\Delta) = -\cos(h)\cos(A-\theta)$$
$$\cos(\Delta) = -\cos(h)\sin(A-\theta)$$
$$\cos(\Delta) = \sin(h)$$

Notice that if $\cos(\Delta) < 0$, the angle is larger than 90º, so the ray will not even enter the window. Therefore, we only consider positive values for the above expressions. Hence, the total luminosity entering through all windows is:

$$L = FA\left(|\cos(h)\cos(A-\theta)| + |\cos(h)\sin(A-\theta)| + |\sin(h)|\right)$$

**c)** In the case where all windows are blocked except the top one, the fraction of flux entering the house is $F_\odot \cos(\Delta) = F_\odot \sin(h)$ or:

$$F = F_\odot\left(\cos(\omega t)\cos(\phi)\cos(\delta) + \sin(\phi)\sin(\delta)\right)$$

The power reaching a panel of area $A_p$ is the total power distributed over the area of radiation incidence. After reflections on the walls, the whole room is uniformly illuminated. The total power is $P_t = FA_j$, while the power reaching each wall is $P_i = F\frac{A_j A_p}{6L^2}$. Since $P = \frac{dE}{dt}$:

$$dE = \frac{A_p A_j}{6L^2} F_\odot \left(\cos(\omega t)\cos(\phi)\cos(\delta) + \sin(\phi)\sin(\delta)\right) dt$$

Integrating this result, we have:

$$\Delta E = \frac{A_p A_j}{6L^2} F_\odot \left(\cos(\phi)\cos(\delta)\frac{1}{\omega}\left(\sin(\omega t_f) - \sin(\omega t_i)\right) + \sin(\phi)\sin(\delta)(t_f - t_i)\right)$$

From the solutions of $\sin(h) = 0$, we find:

$$\begin{cases} t_i = -|\arccos(-\tan(\phi)\tan(\delta))|\frac{1}{\omega} \\ t_f = |\arccos(-\tan(\phi)\tan(\delta))|\frac{1}{\omega} \end{cases}$$

Therefore:

$$\Delta E = \frac{A_p A_j}{3\omega L^2} F_\odot \left(\sqrt{\cos(\phi-\delta)\cos(\phi+\delta)} + \sin(\phi)\sin(\delta)|\arccos(-\tan(\phi)\tan(\delta))|\right)$$

The value of $\omega$ is given relative to the period of a solar day: $\omega = \frac{2\pi}{T_\odot}$.

$$\Delta E = \frac{A_p A_j}{6\pi L^2} F_\odot T_\odot \left(\sqrt{\cos(\phi-\delta)\cos(\phi+\delta)} + \sin(\phi)\sin(\delta)|\arccos(-\tan(\phi)\tan(\delta))|\right)$$

# 2.17 Non-Homogeneous Cosmology

## Part A: Spherical Cavity

**A.1)** Imagine that the cavity inside the sphere is the superposition of a sphere of mass $m$ with another sphere of mass $-m$ (because in the end they cancel each other out). Thus, we can calculate the field at a point in space as the sum of the field generated by the full sphere and by a sphere of the same density magnitude but negative.

For a person inside a sphere, the field at a point is only due to the sphere inside them. So, the field generated by the larger sphere at a point with position vector $\vec{r}$ relative to the sphere's center is:

$$\vec{g}_1 = -\frac{GM_i}{r^2}\hat{r}$$

where $M_i = \frac{4}{3}\pi r^3 \rho$ (internal mass), hence:

$$\vec{g}_1 = -4\pi G\rho\vec{r}$$

For the sphere of density $\rho$, we have $\vec{r}_2 = \vec{r} - \vec{a}$ (relative position to the center). Analogously:

$$\vec{g}_2 = 4\pi G\rho\vec{r}_2 = 4\pi G\rho\vec{r} - 4\pi G\rho\vec{a}$$

Adding the two fields vectorially, we get:

$$\vec{g}_t = -4\pi G\rho\vec{a}$$

Notice that the field is constant and points in a constant direction.

## Part B: Discrete Distribution

**B.1)** For the inner part, we can simply use the classical demonstration of the Friedmann law since the shell does not influence the development of the inner part (easily seen via Gauss's law for gravitation, also known as the shell theorem):

Consider a reference point at a distance $br_0$ from the universe's center, using the gravitational Gauss law:

$$\oint \vec{g} \cdot d\vec{A} = -4\pi Gm_i$$

Since $|\vec{g}|$ is constant and always perpendicular to the area elements:

$$\vec{g}\oint \cdot d\vec{A} = \vec{g}4\pi r^2 = -4\pi Gm_i$$

$$\vec{g} = -\frac{Gm_i}{r^2}\hat{r}$$

Considering mass conservation in the expansion and $g = \frac{d^2r}{dt^2} = \frac{dv}{dt}$, integrate the expression in terms of $dr$:

$$\int \frac{dv}{dt}dx = -Gm_i \int r^{-2}dr$$

$$\int vdv = \frac{Gm_i}{r} + C$$

$$v^2 = \frac{2Gm_i}{r} + C'$$

Now note that $m_i = \frac{4}{3}\pi r^3 \rho$, and we can write $r$ and $v$ as $br_0$ and $\dot{b}r_0$, respectively.

$$\dot{b}^2 = \frac{8\pi G}{3}\rho b^2 + C''$$

Dividing both sides by $b^2$ and noting that $C'' = 0$ for a flat universe:

$$\left(\frac{\dot{b}}{b}\right)^2 = \frac{8\pi G}{3}\rho$$

Since $H_b = \dfrac{\dot{b}}{b}$ and relativistic energy includes the factor $c^2$ from $E = mc^2$:

$$H_b^2 = \frac{8\pi G}{3c^2}\epsilon_b$$

**B.2)** For the outer sphere, we can follow the same procedure, but now the internal mass is dictated by another relation. Still:

$$v^2 = \frac{2Gm_i}{r} + C''$$

Here we consider the superposition of a larger sphere with density $\rho_a$ and a smaller sphere with density $\rho_b - \rho_a$, so that the total density in the intersection region (smaller sphere) is $\rho_b$.

Now $r = aR_0$, $v = R_0\dot{a}$, and

$$m_i = \frac{4\pi b^3 r_0^3}{3}(\rho_b - \rho_a) + \frac{4\pi a^3 R_0^3}{3}\rho_a$$

Rewriting:

$$\dot{a}^2 R_0^2 = \frac{8\pi G}{3aR_0}\left(b^3 r_0^3(\rho_b - \rho_a) + a^3 R_0^3 \rho_a\right)$$

Simplifying:

$$\dot{a}^2 = \frac{8\pi G}{3}\left(\left(\frac{br_0}{aR_0}\right)^3 \frac{\epsilon_b - \epsilon_a}{c^2} + \frac{\epsilon_a}{c^2}\right)$$

Finally:

$$H_a^2 = \frac{8\pi G}{3c^2}\epsilon_a\left(1 + \left(\frac{br_0}{aR_0}\right)^3\left(\frac{\epsilon_b}{\epsilon_a} - 1\right)\right)$$

**B.3)** For the gravitational field inside the inner region to be constant, it is necessary that $\epsilon_b = 0$ (any other value would create a central attractive force). Substituting $\epsilon_b = 0$:

$$H_b^2 = 0 \therefore \dot{b} = 0$$

So $b$ is constant: $b = b_0 = 1$. Then:

$$H_a^2 = \frac{8\pi G}{3c^2}\epsilon_a \left(1 - 1\left(\frac{r_0}{aR_0}\right)^3\right)$$

Isolating $\dot{a}$:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_a a^2 \left(1 - 1\left(\frac{r_0}{aR_0}\right)^3\right)$$

Assuming $\epsilon_a$ is constant over time:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_{a,0} a^2 \left(1 - 1\left(\frac{r_0}{aR_0}\right)^3\right)$$

Let $n = \dfrac{8\pi G}{3c^2}\epsilon_{a,0}$ and $m = \dfrac{r_0^3}{R_0^3}$:

$$\dot{a}^2 = na^2\left(1 - ma^{-3}\right)$$

$$\dot{a} = \sqrt{n}a\sqrt{1 - ma^{-3}}$$

Separating variables to integrate:

$$\int_1^a a^{-1}\frac{da}{\sqrt{1 - ma^{-3}}} = \int_{t_0}^t \sqrt{n}dt$$

Using the suggested substitution $u = \sqrt{1 - ma^{-3}}$:

$$du = \frac{3ma^{-4}da}{2\sqrt{1 - ma^{-3}}}$$

Substituting:

$$\frac{2a^4du}{3m} = \frac{da}{\sqrt{1 - ma^{-3}}}$$

$$\int_{\sqrt{1-m}}^{\sqrt{1-ma^{-3}}} a^3\frac{2du}{3m} = \int_{t_0}^t \sqrt{n}dt$$

Since $a = \left(\frac{1-u^2}{m}\right)^{-\frac{1}{3}}$, the integral becomes:

$$\int_{\sqrt{1-m}}^{\sqrt{1-ma^{-3}}} \frac{m}{1 - u^2}\frac{2du}{3m} = \int_{t_0}^t \sqrt{n}dt$$

Simplifying:

$$\int_{\sqrt{1-m}}^{\sqrt{1-ma^{-3}}} \frac{du}{1 - u^2} = \frac{3}{2}\Delta t\sqrt{n}$$

Paz e Amor

Knowing that:

$$\int \frac{du}{1-u^2} = \tanh^{-1} u + C$$

Thus we find:

$$\tanh^{-1}\sqrt{1-ma^{-3}} - \tanh^{-1}\sqrt{1-m} = \frac{3}{2}\Delta t\sqrt{n}$$

Assuming $t_0 = 0$ (present):

$$a(t) = \left(\frac{m}{1-\tanh^2\left(\tanh^{-1}\sqrt{1-m}+\frac{3}{2}t\sqrt{n}\right)}\right)^{\frac{1}{3}}$$

$$a(t) = \frac{r_0}{R_0}\left(1-\tanh^2\left(\tanh^{-1}\sqrt{1-\frac{r_0^3}{R_0^3}}+\frac{3}{2}t\sqrt{\frac{8\pi G}{3c^2}\epsilon_{a,0}}\right)\right)^{-\frac{1}{3}}$$

## 2.18 Miraculous Escape

**a)** Consider the following diagram:

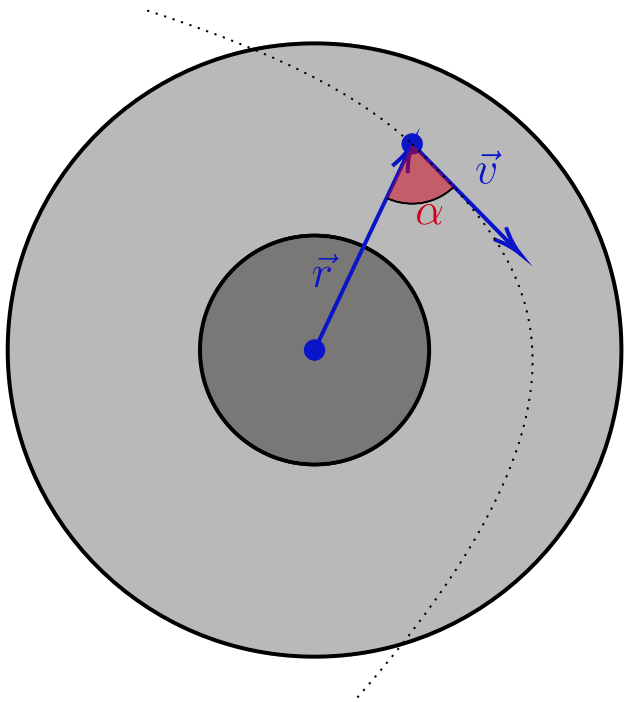



Figure 2.26: Representation of the spacecraft's parabolic trajectory at a given distance from the black hole's center.

We will use the law of conservation of angular momentum between the current position and the periapsis:

$$vr\sin(\alpha) = v_p r_p$$

Notice that $0 \leq \alpha \leq \pi$. Knowing that the orbit is parabolic, which implies that the velocity at each point is the escape velocity:

$$\sqrt{\frac{2GM}{r}}\, r\sin(\alpha) = \sqrt{\frac{2GM}{r_p}}\, r_p$$

$$\sin(\alpha) = \sqrt{\frac{r_p}{r}}$$

Since we want to know the angles for which the periapsis radius is inside the black hole (the spacecraft enters), we set $r_p \leq R_b$, which leads to:

$$\sin(\alpha) \leq \sqrt{\frac{R_b}{r}}$$

**b)** By interpreting this result (and observing the unit circle), we find that the angle $\alpha$ is of the form:

$$\alpha \in \left[0, \arcsin\sqrt{\frac{R_b}{r}}\right] \cup \left[\pi - \arcsin\sqrt{\frac{R_b}{r}}, \pi\right]$$

Illustrating the situation for the spacecraft, we have Figure 2.27.

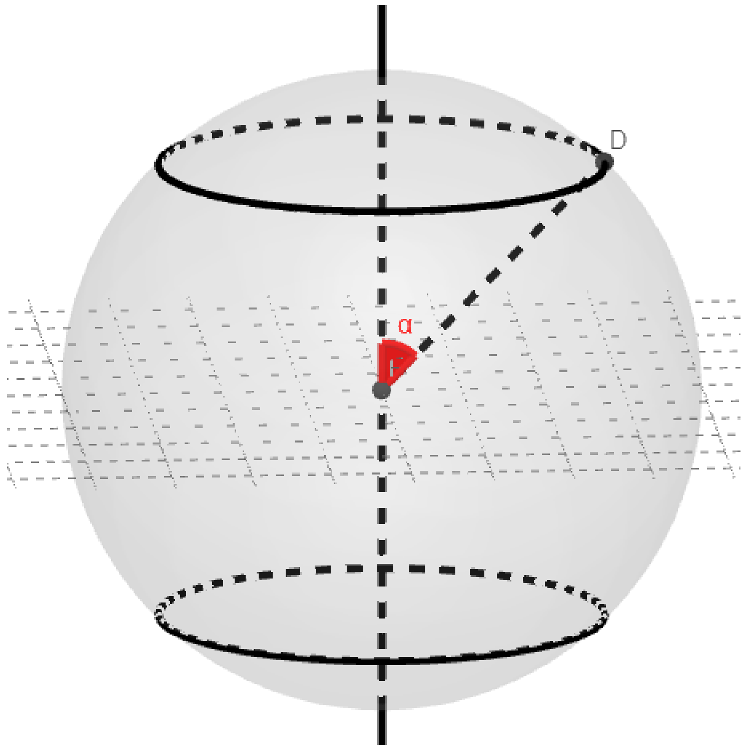



Figure 2.27: Representation of the solid angle and section of the sphere for which escape is possible.

The spacecraft will have a periapsis inside the black hole if its velocity vector is directed within one of the spherical caps (top or bottom). The total area of the two caps is given by:

$$A_t = 2(A_{\text{cap}}) = 4\pi R^2(1 - \cos(\alpha))$$

Thus, the probability that the velocity lies within one of these areas is the ratio between this area and the total area ($4\pi R^2$):

$$P = 1 - \cos(\alpha)$$

To calculate the probability that the spacecraft escapes, we first compute the probability of **not escaping** and then subtract this from 1 to obtain the desired probability.

One might expect the probability of not escaping to be $P$, but it is possible that the spacecraft is moving in the opposite direction; that is, even if the periapsis is inside the black hole, it could be moving away or toward it (in one case the angle is $\alpha$, in the other it is $\pi - \alpha$). For one of these cases, the spacecraft will still pass through periapsis (and never return), and for the other, it is as if it has already passed. Therefore, the probability of not escaping should be:

$$P_{\text{NS}} = \frac{P}{2} = \frac{1 - \cos(\alpha)}{2}$$

Substituting the values and knowing that $(\sin(\alpha))^2 + (\cos(\alpha))^2 = 1$, we can find the probability of escaping:

$$P_S = \frac{1 + \sqrt{1 - \frac{R_b}{r}}}{2}$$

**c)** The probability that the spacecraft is located at a distance $r$ from the black hole is the ratio between the infinitesimal volume element of a sphere of radius $r$ and the total volume of a sphere of radius $R_n$ minus a sphere of radius $R_b$ (since the spacecraft is not inside the black hole):

$$P_r = \frac{d\left(\frac{4}{3}\pi r^3\right)}{\frac{4}{3}\pi(R_n^3 - R_b^3)} = \frac{3r^2 dr}{R_n^3 - R_b^3}$$

**d)** By the multiplicative principle, the probability that the spacecraft is at a distance $r$ and escapes is:

$$dP(r) = P_S P_r$$

Thus, the total probability of escaping is the integral of this function from $R_b$ to $R_n$:

$$P = \int_{R_b}^{R_n} \frac{1 + \sqrt{1 - \frac{R_b}{r}}}{2} \frac{3r^2 dr}{R_n^3 - R_b^3}$$

$$P = \frac{3}{2(R_n^3 - R_b^3)} \int_{R_b}^{R_n} \left(1 + \sqrt{1 - \frac{R_b}{r}}\right) r^2 dr$$

We can separate the integral into two parts:

$$\int_{R_b}^{R_n} \left(1 + \sqrt{1 - \frac{R_b}{r}}\right) r^2 dr = \int_{R_b}^{R_n} r^2 dr + \int_{R_b}^{R_n} \sqrt{1 - \frac{R_b}{r}} r^2 dr$$

The first integral is trivial:

$$\int_{R_b}^{R_n} r^2 dr = \frac{R_n^3 - R_b^3}{3}$$

The second integral can use the provided value from the problem statement, giving:

$$P = \frac{3}{2(R_n^3 - R_b^3)} \left(\frac{R_n^3 - R_b^3}{3} + 0.2895485(R_n^3 - R_b^3)\right)$$

Finally, we find that:

$$P = 93.43\%$$

# 2.19 Quantum Star

## Part A: Fermi Energy

**A.1)** Notice that $0 \leq u \leq L$ since the particle is confined within this region. Therefore, $\psi(u) = 0$ for $u \leq 0$ and $u \geq L$. Inside the imposed interval, we have:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(u)}{du^2} = E\psi$$

$$\frac{d^2\psi(u)}{du^2} = -\frac{2mE}{\hbar^2}\psi$$

Solving this differential equation (which resembles a simple harmonic motion equation), we find:

$$\psi(u) = A\sin\left(\sqrt{\frac{2mE}{\hbar^2}}u\right) + B\cos\left(\sqrt{\frac{2mE}{\hbar^2}}u\right)$$

**A.2)** As mentioned before, at $u = 0$ there is a "node" of the particle's wavefunction, similar to the boundary condition of a stationary wave on a string. Therefore, $\psi(0) = 0$. This condition is only satisfied if $B = 0$, since $\cos(0) = 1$.

**A.3)** Similarly: $\psi(L) = 0$ represents the other "node." To make the sine function zero at this point, the argument must be a multiple of $\pi$:

$$\sqrt{\frac{2mE}{\hbar^2}}L = n_u\pi$$

From which we obtain:

$$E = \frac{\hbar^2\pi^2}{2m^2L^2}n_u^2$$

**A.4)** To find the total energy of the particle, we must consider the energy associated with its oscillatory motion along each axis. Therefore, we sum the energies for each axis: $\sum n_u^2 = n_x^2 + n_y^2 + n_z^2$.

**A.5)** To count all particles with energy less than $E(R)$, we count how many have $r < R$. This can be visualized as counting how many particles have coordinates ($n_x$,$n_y$,$n_z$) inside a sphere of radius $R$ in this space, ensuring $\sqrt{n_x^2 + n_y^2 + n_z^2} < R$.

However, we are only concerned with one eighth of the sphere: the part containing all positive values (positive and negative $n$ correspond to the same state). Each electron also has a fourth quantum number (spin), so at most two electrons can occupy the same position without violating the Pauli exclusion principle:

$$N = 2\frac{1}{8}\frac{4\pi R^3}{3} = \frac{\pi R^3}{3}$$

**A.6)** Substituting this into (A.4):

$$E(N) = \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{3N}{\pi}\right)^{\frac{2}{3}}$$

This expression gives the energy of a single particle. To find the total energy of the system, multiply by the number of particles with that energy and integrate from 0 to $N_t$:

$$dE(N) = \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{3N}{\pi}\right)^{\frac{2}{3}} dN$$

Integrating gives:

$$E = \frac{3\hbar^2}{10m}\left(3\pi^2\eta\right)^{\frac{2}{3}} N_t$$

with $\eta = \dfrac{N_t}{L^3}$.

## Part B: Gravitational Self-Potential Energy

**B.1)** The work required to add this incremental mass is the change in potential energy of $dm$:

$$W = \frac{Gmdm}{r} - 0$$

Here, 0 is the potential at infinite distance from the source (where the mass $dm$ comes from). Therefore, the increase in potential energy of the system is:

$$dE = \frac{Gmdm}{r}$$

**B.2)** We know that:

$$m = \frac{4\pi}{3}r^3\rho$$

and

$$dm = 4\pi r^2 dr\rho$$

Thus:

$$dE = -G\frac{4\pi r^2}{3}\rho 4\pi r^2\rho dr$$

Integrating:

$$E = -\frac{16\pi^2\rho^2 G}{3}\frac{R^5}{5}$$

Finally, simplifying and substituting back the total mass:

$$E = -\frac{3GM^2}{5R}$$

## Part C: Degenerate Stellar Remnants

**C.1)** We can consider the contributions from rest energy ($E = Mc^2$), gravitational potential energy $\left(-\frac{3GM^2}{5R}\right)$, and the Fermi energy of the electrons.

To calculate the Fermi contribution, we need the number of electrons and the star's volume:

$$N_t = \frac{M}{m_X} Z$$

This represents the product of the number of atoms of element $X$ and the number of electrons per atom. The volume is:

$$V = \frac{4\pi R^3}{3}$$

Substituting into the Fermi energy formula:

$$E = \frac{3\hbar^2}{10m_e}(3\pi^2)^{\frac{2}{3}} \frac{N_t^{\frac{5}{3}}}{V^{\frac{2}{3}}}$$

$$E = \frac{3\hbar^2}{10m_e}(3\pi^2)^{\frac{2}{3}} \frac{\left(\frac{M}{m_X} Z\right)^{\frac{5}{3}}}{\left(\frac{4\pi R^3}{3}\right)^{\frac{2}{3}}}$$

$$E = \frac{3\hbar^2}{10m_e}\left(\frac{9}{4}\pi\right)^{\frac{2}{3}} \left(\frac{M}{m_X} Z\right)^{\frac{5}{3}} \frac{1}{R^2} = k\frac{M^{\frac{5}{3}}}{R^2}$$

Thus, the total energy of the star is:

$$E_t = Mc^2 - \frac{3GM^2}{5R} + k\frac{M^{\frac{5}{3}}}{R^2}$$

with

$$k = \frac{3\hbar^2}{10m_e}\left(\frac{9}{4}\pi\right)^{\frac{2}{3}} \left(\frac{Z}{m_X}\right)^{\frac{5}{3}}$$

**C.2)** The equilibrium situation occurs when the total energy is minimized, i.e., when the derivative of $E_t$ is zero and the second derivative is negative (ensuring a minimum). Fixing the mass $M$, we find the radius $R$ by setting $\frac{dE}{dR} = 0$:

$$\frac{dE}{dR} = 0 = \frac{3GM^2}{5R^2} - 2k\frac{M^{\frac{5}{3}}}{R^3}$$

$$\frac{d^2E}{dR^2} = -2\frac{3GM^2}{5R^3} + 6k\frac{M^{\frac{5}{3}}}{R^4}$$

Solving for $R$:

$$R = \frac{10k}{3GM^{\frac{1}{3}}}$$

Substituting $k$ gives:

$$R = \frac{\hbar^2}{Gm_e}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left(\frac{Z}{m_X}\right)^{\frac{5}{3}} M^{-\frac{1}{3}}$$

**C.3)** Substituting the appropriate values, we find the radius of a hydrogen white dwarf with solar mass.

$$R(M_{Sun}) = 22830km \approx 3.57R_{Earth}$$

## 2.20 Bulk Star

**a)** We can consider the outermost part of the star (at a distance $R_0$ from the center) which will practically reach the center of the star (since the final radius can be considered negligible). Using the degenerate ellipse technique and Kepler's law for orbital period:

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

The free-fall time is half a period (since it does not return to $R_0$) and $2a = R_0$:

$$(2\Delta t)^2 = \frac{\pi^2 R_0^3}{2GM}$$

$$\Delta t^2 = \frac{\pi^2 R_0^3}{8G\left(\frac{4\pi}{3}R_0^3\rho_0\right)}$$

$$\Delta t = \sqrt{\frac{3\pi}{32G\rho_0}}$$

**b)** Consider a fixed mass element $dm$:

$$dm = 4\pi r^2 dr\rho(r) = 4\pi r_0^2 dr_0\rho_0$$

$$B = \frac{\Delta P}{\Delta V}V$$

$$\Delta V = 4\pi r_0^2 dr_0\left(1 - \frac{\rho_0}{\rho(r)}\right)$$

Since $V = 4\pi r_0^2 dr_0$:

$$P(r) = B\left(1 - \frac{\rho_0}{\rho(r)}\right)$$

**c)** Consider a cylindrical element of mass $dm$ at a distance $r$ from the center with base area $dA$ and height $dr$. The pressure at the lower base is $P(r)$ and at the upper base is $P(r+dr)$. The pressure difference generates a force balancing the gravitational attraction on this element, $\frac{Gm(r)dm}{r^2}$:

$$(P(r) - P(r+dr))dA = -\frac{Gm(r)dm}{r^2}$$

Note that $\rho(r) = \frac{dm}{dAdr}$ and $P(r+dr) - P(r) = dP(r)$, so rearranging:

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

**d)** By definition:

$$m(r) = 4\pi \int_0^r x^2 \rho(x) dx$$

Substitute:

$$\frac{B\rho_0}{\rho(r)^2} \frac{d\rho(r)}{dr} = -4\pi G \frac{\rho(r)}{r^2} \int_0^r x^2 \rho(x) dx$$

Thus:

$$-\frac{B\rho_0}{4\pi G} \frac{d\rho(r)}{dr} = \frac{\rho(r)^3}{r^2} \int_0^r x^2 \rho(x) dx$$

**e)** Assume:

$$\rho(r) = ar^n$$

Substituting into the previous equation:

$$\begin{aligned}
-\frac{B\rho_0}{4\pi G} anr^{n-1} &= \frac{a^3 r^{3n}}{r^2} \int_0^r x^2 (ax^n) dx \\
-\frac{B\rho_0}{4\pi G} nr^{n-1} &= \frac{a^3 r^{3n-2}}{n+3} r^{n+3} \\
-\frac{B\rho_0}{4\pi G} n &= \frac{a^3 r^{3n+2}}{n+3}
\end{aligned}$$

Hence: $n = -\frac{2}{3}$
Then:

$$a = \sqrt[3]{\frac{B\rho_0}{2\pi G} \frac{7}{9}}$$

Thus:

$$\rho(r) = \sqrt[3]{\frac{7B\rho_0}{18\pi G r^2}}$$

**f)** Substituting into (b):

$$P(r) = B\left(1 - \sqrt[3]{\frac{18\pi G r^2 \rho_0^2}{7B}}\right)$$

**g)** For the star to exist, pressure must remain positive:

$$r < \sqrt{\frac{7B}{18\pi G \rho_0^2}}$$

**h)** Applying the Ideal Gas Law:

$$P(r)V = NkT(r)$$

For particle mass $\mu$:

$$P(r) = \frac{N}{V} kT(r)$$

$$P(r) = \frac{\rho(r)}{\mu} kT(r)$$

Hence:

$$T(r) = \left(1 - \sqrt[3]{\frac{18\pi G r^2 \rho_0^2}{7B}}\right) \sqrt[3]{\frac{18\pi G r^2 B^2}{7\rho_0}} \frac{\mu}{k}$$

**i)** Note that:

$$M = 4\pi \int_0^R r^2 \rho(r) dr$$

Then:

$$M = 4\pi \int_0^R \sqrt[3]{\frac{7B\rho_0}{18\pi G}} r^{4/3} dr$$

$$M = \sqrt[3]{\frac{96\pi^2 B\rho_0}{49G}} R^{7/3}$$

The mass-radius relation is therefore $M \propto R^{7/3}$.

**j)** For stars with radius much smaller than the limiting radius, we can approximate the temperature:

$$T(r) = \sqrt[3]{\frac{18\pi G r^2 B^2}{7\rho_0}} \frac{\mu}{k}$$

Using:

$$L = 4\pi R^2 \sigma T(R)^4$$

We get:

$$L = 4\pi R^2 \left(\frac{18\pi G B^2}{7\rho_0}\right)^{4/3} \frac{\mu^4}{k^4} R^{8/3}$$

Hence:

$$L \propto R^{14/3}$$

Relating this to the Mass-Radius relation, we find:

$$L \propto M^2$$

Remarkably, this result is similar to the actual Mass-Luminosity relation observed for some stars in the universe.

## 2.21 Celestial Chart

**a)** Consider the system proposed in the figure 2.28.

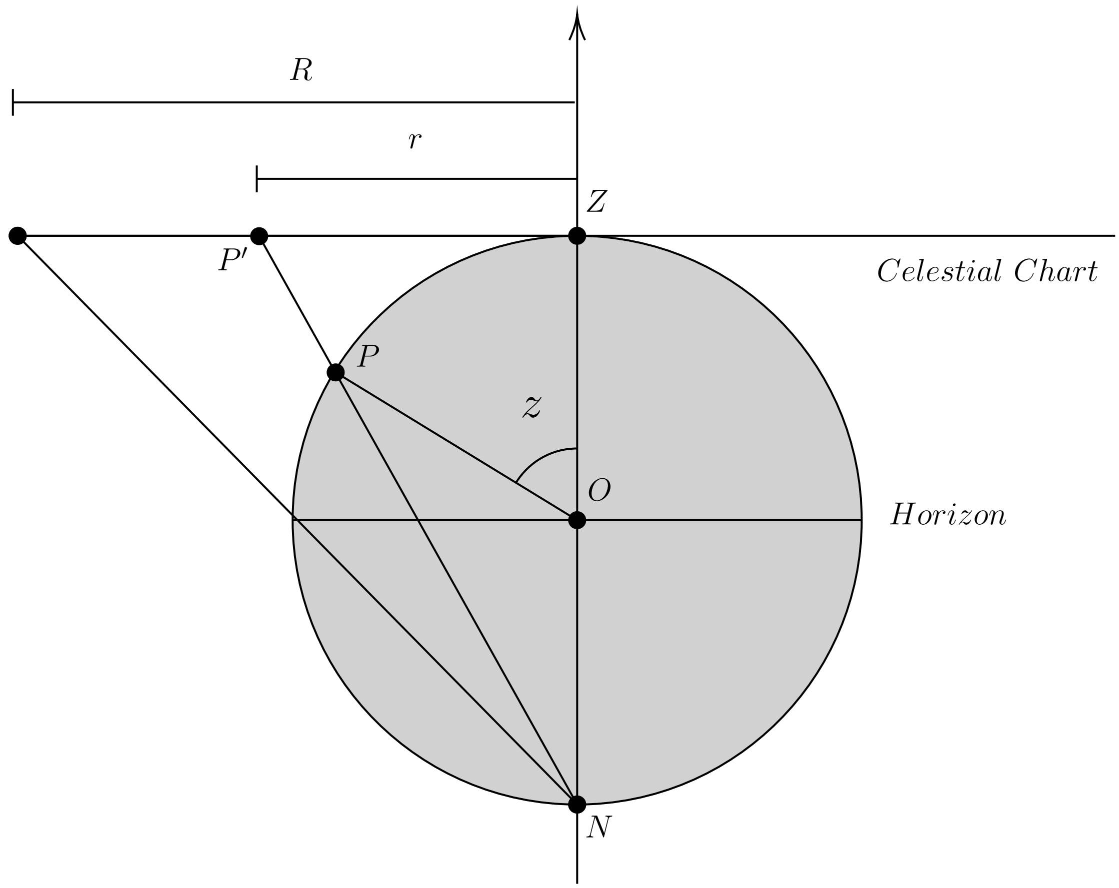



Figure 2.28: Two-dimensional representation of the projection method. $O$ is the origin of the celestial sphere, $N$ is the nadir, $Z$ is the zenith, and $P'$ is the point $P$ projected onto the chart.

Notice that $\angle POZ = 2\angle PNO$. From this, we find that:

$$\tan\left(\frac{z}{2}\right) = \frac{r}{R}$$

**b)** Consider the infinitesimal area element of all points with $z_0 < z < z_0 + dz$. This area will be projected onto the chart such that $r_0 < r < r_0 + dr$.

This area on the celestial sphere is:

$$dA_E = 2\pi\frac{R}{2}\sin(z)\frac{R}{2}dz$$

$$dA_E = \frac{\pi R^2}{2}\sin(z)dz$$

The projected area on the chart will be:

$$dA_C = 2\pi r dr$$

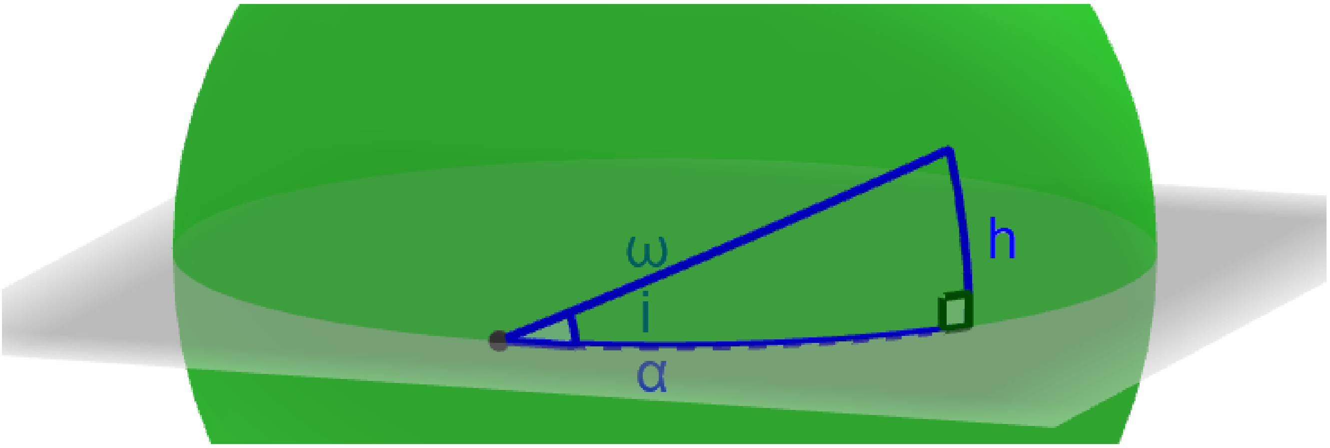



Figure 2.29: Spherical triangle of an orbit with inclination $i$.

So that:

$$\frac{dr}{dz} = R\sec^2\left(\frac{z}{2}\right)\frac{1}{2}$$

Thus:

$$dA_C = \pi R^2 \tan\left(\frac{z}{2}\right)\sec^2\left(\frac{z}{2}\right)dz$$

Finally:

$$\zeta(z) = \frac{dA_C}{dA_E} = \frac{2\tan\left(\frac{z}{2}\right)\sec^2\left(\frac{z}{2}\right)}{\sin(z)} = \frac{2\tan\left(\frac{z}{2}\right)\sec^2\left(\frac{z}{2}\right)}{2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)}$$

$$\zeta(z) = \sec^4\left(\frac{z}{2}\right)$$

**c)** Given the following orbit with inclination $i$:

Using spherical trigonometry equations, we have the following relations, using the law of cosines and the law of sines, respectively:

$$\cos(\omega) = \cos(h)\cos(\alpha) + \sin(\alpha)\sin(h)\cos(90^\circ)$$

$$\sin(h) = \sin(i)\sin(\omega)$$

Note that $\sin(h) = \cos(z)$, $\cos(h) = \sin(z)$, and $\cos(90^\circ) = 0$:

$$\cos(\omega) = \sin(z)\cos(\alpha)$$

$$\cos(z) = \sin(i)\sin(\omega)$$

Squaring both equations and using $\sin^2 x + \cos^2 x = 1$:

$$\cos^2(\omega) = \sin^2(z)\cos^2(\alpha) = 1 - \frac{\cos^2(z)}{\sin^2(i)}$$

$$\sin^2(z) - \sin^2(z)\sin^2(\alpha) = 1 - \frac{\cos^2(z)}{\sin^2(i)}$$

$$1 - \cos^2(z) - \sin^2(z)\sin^2(\alpha) = 1 - \frac{\cos^2(z)}{\sin^2(i)}$$

From this, we find:

$$\sin^2(\alpha) = \cos^2(z)\left(\frac{1}{\sin^2(i)} - 1\right)$$

Finally:

$$\sin^2(\alpha) = \frac{\cot^2(z)}{\tan^2(i)}$$

However, we have:

$$\tan(z) = \frac{2\tan\left(\frac{z}{2}\right)}{1 - \tan^2\left(\frac{z}{2}\right)}$$

$$\tan(z) = \frac{2\frac{r}{R}}{1 - \frac{r^2}{R^2}} = \frac{2Rr}{R^2 - r^2}$$

Thus:

$$\sin(\alpha) = \frac{R^2 - r^2}{2Rr\tan(i)}$$

Converting to Cartesian coordinates:

$$x = r\cos(\alpha)$$

$$y = r\sin(\alpha)$$

We then notice:

$$y = \frac{R^2 - r^2}{2R\tan(i)}$$

$$r^2 = R^2 - 2Ry\tan(i)$$

Since $x^2 = r^2 - y^2$:

$$x^2 = R^2 - 2Ry\tan(i) - y^2$$

Completing the equation, we find:

$$x^2 + (y + R\tan(i))^2 = (R\sec(i))^2$$

Which characterizes a circle.

**d)** We can think of a lune as a union of infinitesimal lunes with angle $d\alpha$. Since the areas of these lunes are constant for a constant $d\alpha$, it is expected that the area of the lune is proportional to its opening angle. Hence:

$$A_L = 4\pi R^2 \frac{\alpha}{2\pi} = 2\alpha R^2$$

**e)** We can think that the area of the whole sphere is the sum of the areas of the lunes minus 4 times the area of the spherical triangle, since when summing the areas of the lunes we are counting the area of the triangle twice on the front (triangle $ABC$) and twice on the back (the triangle diametrically opposite $ABC$). This implies:

$$4\pi R^2 = 2(2\alpha R^2 + 2\beta R^2 + 2\gamma R^2) - 4A_{ABC}$$

Therefore:

$$A_{ABC} = (\alpha + \beta + \gamma - \pi)R^2$$

**f)** It is easy to see that the angles of this triangle are 90°, 90°, and 30°:

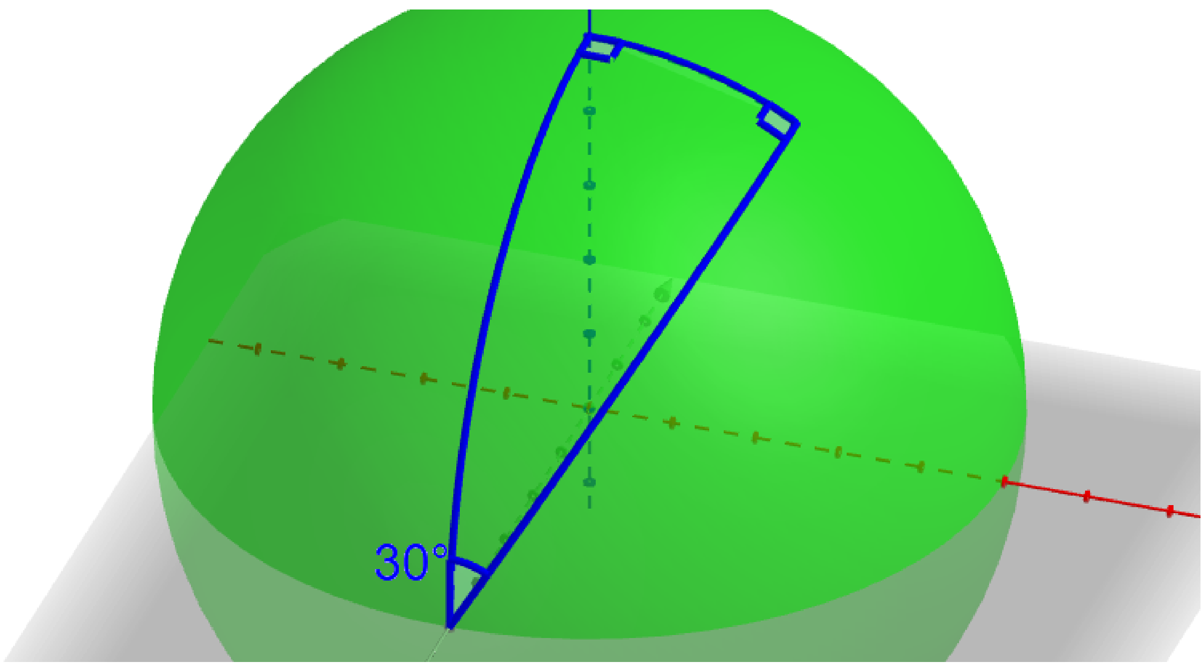



Figure 2.30: Representation of the spherical triangle proposed in the problem statement.

Thus:

$$A_{ABC} = \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{6} - \pi\right)\frac{R^2}{4}$$

$$A_{ABC} = \frac{\pi R^2}{24}$$

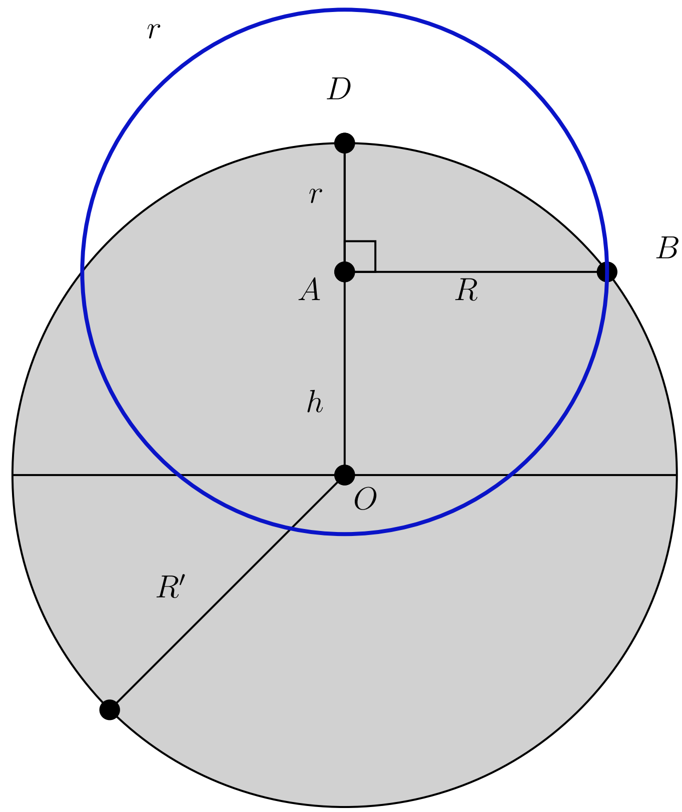



Figure 2.31: Projection of the orbital circle onto the celestial chart.

**g)** For this item, we need to project each point onto the chart and find the area between them, knowing that the sides will be circles (or straight lines, which correspond to circles of infinite radius). For this, observe the desired projection:

Denote point $A$ as the vertex of the triangle at the zenith, point $B$ as the point on the horizon, and point $D$ as the point with altitude 60°. As explained earlier, the side $DB$ of the spherical triangle is projected onto a circle passing through $D$ and $B$.

The area of this element $ADB$ is given by the area of the circular sector $ODB$ minus the area of the triangle $OAB$, where $O$ is the center of the larger circle. Notice that $h = R\tan(i)$ from the equation of the larger circle, so the angle $AB$ is:

$$\tan(AB) = \cot(i)$$

Which implies: $AB = 90° - i$. The area of the sector is:

$$A_S = \frac{1}{2}\left(\frac{\pi}{2} - i\right) R^2 \sec^2(i)$$

The area of the triangle is:

$$A_T = \frac{1}{2} R^2 \tan(i)$$

So the lune area is:

$$A_l = \frac{1}{2}\left(\frac{\pi}{2} - i\right) R^2 \sec^2(i) - \frac{1}{2} R^2 \tan(i) = \frac{R^2}{2}\left(\frac{\pi}{2}\sec^2(i) - i\sec^2(i) - \tan(i)\right)$$

Since $i = 60°$:

$$A_l = \frac{R^2}{2}\left(\frac{4\pi}{3} - \sqrt{3}\right)$$

**h)** Hence:

$$\zeta(\Delta ABC) = \frac{12\left(\frac{4\pi}{3} - \sqrt{3}\right)}{\pi}$$

$$\zeta(\Delta ABC) = 16 - \frac{12\sqrt{3}}{\pi} \approx 9.38$$

# 2.22 Plutonian Atmosphere II

## Part A: Physical Modeling

**A.1)** First, separate the atmosphere into infinitesimal layers of thickness $dr$. Each of these layers has a slightly different temperature from its neighbors, which causes heat to be transferred primarily through conduction.

Consider the following layer $dr$ of the atmosphere, located at a distance $r$ from the center:

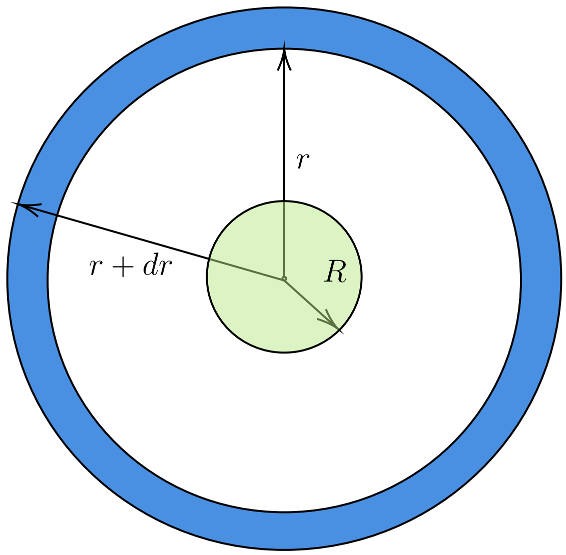



Figure 2.32: Two-dimensional representation of the Plutonian II atmosphere distribution.

For thermal equilibrium to occur, it is necessary that the heat flux entering the layer at $r$ from the layer at $r - dr$ is equal to the heat flux leaving $r$ towards $r + dr$, as shown in the scheme below:

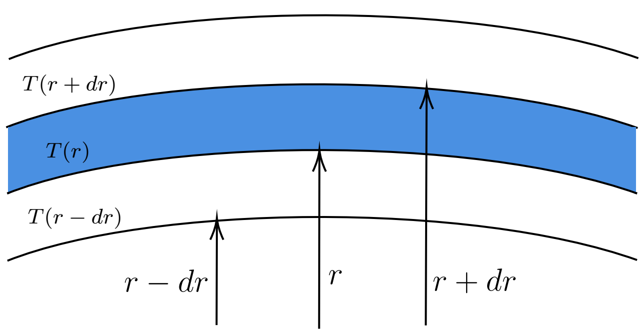



Figure 2.33: Radial variation of temperature in the planet's atmosphere.

The energy flux (power) transferred by conduction can be found using Fourier's law:

$$\phi = \frac{kA\Delta T}{L}$$

Adapting this to the conditions of the problem and setting the net flux in layer $r$ to zero:

$$\frac{k(4\pi r^2)(T(r - dr) - T(r))}{dr} - \frac{k(4\pi(r + dr)^2)(T(r) - T(r + dr))}{dr} = 0$$

Note that we can divide both sides of the equation by $4\pi k$, and simplify $\frac{T(y+dy)-T(y)}{dy} = \frac{dT(y)}{dy}$:

$$r^2\frac{dT(r-dr)}{dr} = (r^2+2rdr)\frac{dT(r)}{dr}$$

Combining like terms (multiples of $r^2$), we have:

$$-2rdr\frac{dT(r)}{dr} = r^2\left(\frac{dT(r)}{dr} - \frac{dT(r-dr)}{dr}\right)$$

$$-2r\frac{dT(r)}{dr} = r^2\frac{\left(\frac{dT(r)}{dr} - \frac{dT(r-dr)}{dr}\right)}{dr} = r^2\frac{d^2T(r-dr)}{dr^2}$$

However, in the limit $dr \to 0$: $\frac{d^2T(r-dr)}{dr^2} \to \frac{d^2T(r)}{dr^2}$. Using the notation provided in the problem statement (which indicates the derivative from points above the function):

$$-2r\dot{T}(r) = r^2\ddot{T}(r) \to \frac{d\dot{T}(r)}{\dot{T}(r)} = -2\frac{dr}{r}$$

$$\int_{\dot{T}(R)}^{\dot{T}(r)}\frac{d\dot{T}(r)}{\dot{T}(r)} = -2\int_R^r\frac{dr}{r}$$

$$\therefore \dot{T}(r) = \frac{dT(r)}{dr} = \dot{T}(R)\left(\frac{R}{r}\right)^2$$

$$\int_{T(R)}^{T(r)} dT(r) = \dot{T}(R)R^2\int_R^r r^{-2}dr$$

$$\therefore T(r) = T(R) + \dot{T}(R)\left(R - \frac{R^2}{r}\right)$$

**A.2)** Reviewing the hydrodynamic equilibrium equation, shown in previous chapters:

$$\dot{P}(r) = -\frac{GM_i}{r^2}\rho(r)$$

Where $M_i$ is the mass enclosed within radius $r$, $P(r)$ is the gas pressure, and $\rho(r)$ is the gas density. Neglecting the mass of the atmosphere relative to that of the planet, we have for any $r \geq R$: $M_i = M$. Using the ideal gas law:

$$P(r)dV = dNkT(r)$$

Where $k$ is Boltzmann's constant. Note that $mdN$ is the total gas mass in the interval $r \to r+dr$, so:

$$P(r) = \frac{k}{m}T(r)\frac{dm}{dV} = \frac{k}{m}T(r)\rho(r)$$

Differentiating with respect to $r$:

$$\frac{m}{k}\dot{P}(r) = \dot{T}(r)\rho(r) + \dot{\rho}(r)T(r)$$

$$-\frac{GMm}{kr^2}\rho(r) - \dot{T}(r)\rho(r) = T(r)\dot{\rho}(r)$$

Simplifying:

$$-\left(\frac{GMm}{k} + \dot{T}(R)R^2\right)\frac{dr}{\left(T(R) + \dot{T}(R)\left(R - \frac{R^2}{r}\right)\right)r^2} = \frac{d\rho(r)}{\rho(r)}$$

For convenience, define $a = T(R) + \dot{T}(R)R$, $b = \dot{T}(R)R^2$, and $c = \dfrac{GMm}{k} + \dot{T}(R)R^2$:

$$-c\int_R^r \frac{dr}{ar^2 - br} = \ln\left(\frac{\rho(r)}{\rho(R)}\right)$$

$$-\frac{c}{a}\int_R^r \frac{dr}{r^2 - \frac{b}{a}r} = \ln\left(\frac{\rho(r)}{\rho(R)}\right)$$

Note that $r^2 - \dfrac{b}{a}r = \left(r - \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2}$. Substituting $u = \dfrac{2a}{b}r - 1$: $\left(r - \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2} = \dfrac{b^2}{4a^2}(u^2 - 1)$:

$$\frac{2c}{b}\int_{\frac{2a}{b}R-1}^{\frac{2a}{b}r-1} \frac{du}{1-u^2} = \ln\left(\frac{\rho(r)}{\rho(R)}\right)$$

As known:

$$\int \frac{du}{1-u^2} = \tanh^{-1}(u) + C = \ln\sqrt{\frac{1+u}{1-u}} + C$$

Thus:

$$\frac{2c}{b}\left(\ln\sqrt{\frac{\frac{2a}{b}r}{2 - \frac{2a}{b}r}} - \ln\sqrt{\frac{\frac{2a}{b}R}{2 - \frac{2a}{b}R}}\right) = \ln\frac{\rho(r)}{\rho(R)}$$

Simplifying further:

$$\frac{2c}{b}\left(\ln\sqrt{\frac{(b-aR)ar}{(b-ar)aR}}\right) = \ln\frac{\rho(r)}{\rho(R)}$$

Finally we find:

$$\rho(r) = \rho(R)\left(\frac{r(b-aR)}{R(b-ar)}\right)^{\frac{c}{b}}$$

**A.3)** To find the desired expression for pressure, simply use the ideal gas law with the relation from the previous item:

$$P(r) = \frac{k}{m}\rho(R)\left(\frac{T(R)r}{ar-b}\right)^{\frac{c}{b}}\left(T(R) + \dot{T}(R)\left(R - \frac{R^2}{r}\right)\right)$$

With:

$$\begin{cases} a = T(R) + \dot{T}(R)R \\ b = \dot{T}(R)R^2 \\ c = \frac{GMm}{k} + \dot{T}(R)R^2 \end{cases}$$

**A.4)** In this case, the temperature variation at the planet's surface ($T(R)$) and at the top of the atmosphere ($T(R+E) = 0$) is almost infinitesimal. We can approximate $\dfrac{\Delta T(R)}{\Delta r} = -\dfrac{T(R)}{E} \approx \dfrac{dT(R)}{dr} = T(\dot{R})$. Therefore, the temperature can be written as:

$$T(r) = T(R) - \frac{T(R)}{E}\left(R - \frac{R^2}{r}\right)$$

For the density $\rho(r)$, we calculate the variable $c$:

$$c = \frac{GMm}{k} + \dot{T}(R)R^2 = \frac{GMm}{k} - \frac{T(R)}{E}R^2$$

$$c = \frac{GMm}{k} - \frac{GMmE}{kR^2}\frac{R^2}{E} = 0$$

Since $c = 0$, we see that $\rho(r) = \rho(R)(...)^0 = \rho(R)$, meaning the density is constant for all $R \leq r \leq R + E$.

## Part B: Atmospheric Distortion

**B.1)** First, we will analyze the case where light rays refract at spherical interfaces, as shown in Figure 2.34:

Note that $n_1 \sin(\alpha) = n_2 \sin(\gamma)$. Applying the law of sines in the triangle formed by the radial segments and the path in medium $n_2$:

$$\frac{r_2}{\sin(\gamma)} = \frac{r_1}{\sin(\pi - \beta)} \rightarrow \sin(\gamma) = \frac{r_2}{r_1}\sin(\beta)$$

Thus, we have $n_1 r_1 \sin(\alpha) = n_2 r_2 \sin(\beta) = \text{constant}$.

Note that the ray starts its path in the vacuum of space, outside the atmosphere, so $n_1 = 1$:

$$n(r) r \sin(\theta(r)) = 1 \cdot (R+E)\sin(\theta) \rightarrow n_0 \sin(\theta(r)) = (R+E)\sin(\theta)$$

$$\therefore \sin(\theta(r)) = \frac{R+E}{n_0}\sin(\theta)$$

However, by the condition of the problem, so that $n(R+E) = 1$, we have $n_0 = R + E$:

$$\sin(\theta(r)) = \sin(\theta)$$

This result holds for any $r$! Therefore, $\alpha = \theta$.

**B.2)** Consider the schematic of Figure 2.35 (note that $dr < 0$):

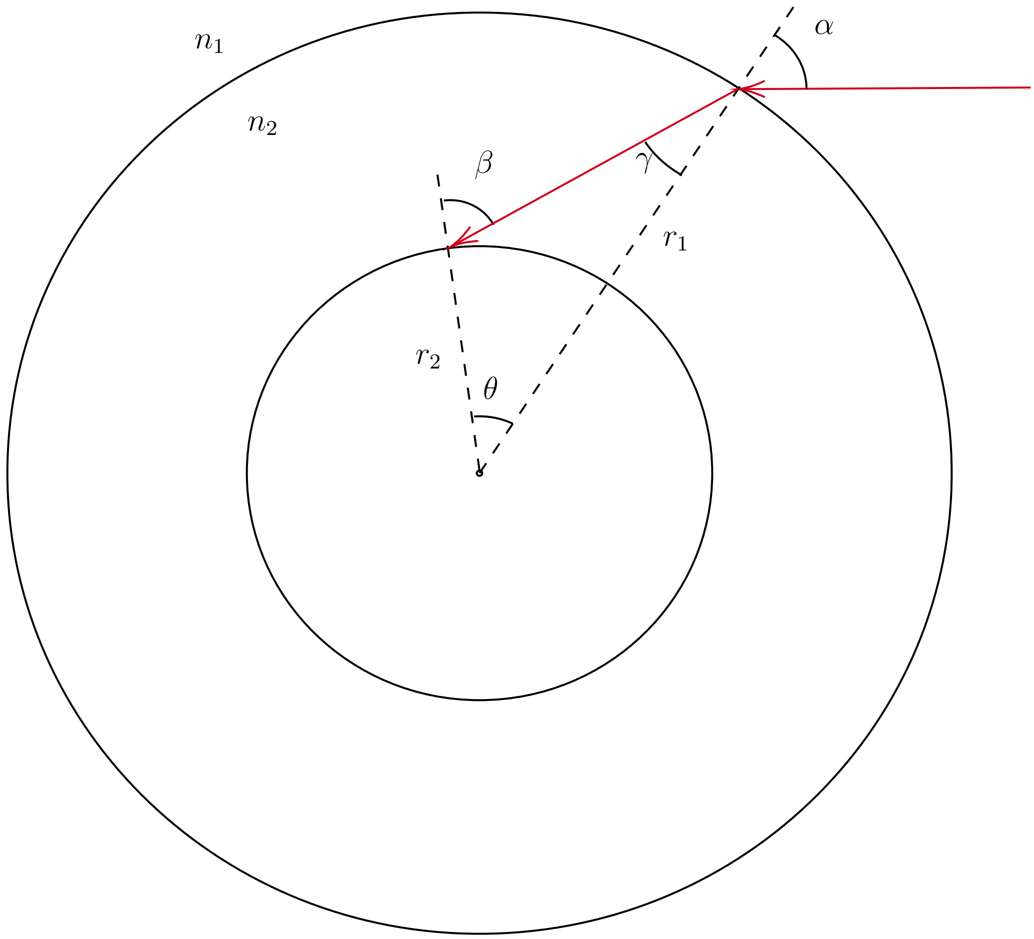



Figure 2.34: Refraction at interfaces with curved rays and different refractive indices.

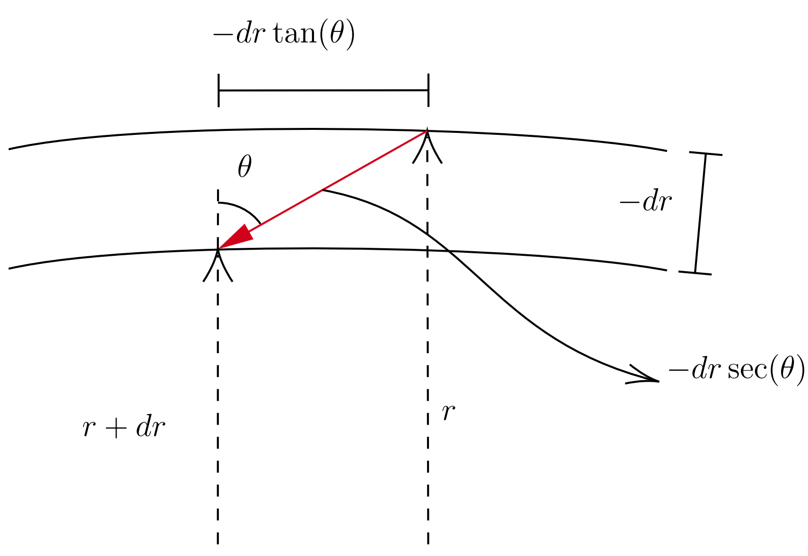



Figure 2.35: Infinitesimal displacement in a radial section of the atmosphere.

The angular variation that occurs along this infinitesimal path is $d\phi = -\dfrac{dr\tan(\theta)}{r}$. Integrating this result, we have:

$$\Delta\phi = \tan(\theta)\ln\left(\frac{R+E}{R}\right)$$

Since $E \ll R$, we have:

$$\ln\left(\frac{R+E}{R}\right) = \ln\left(1+\frac{E}{R}\right) \approx \frac{E}{R}$$

Thus, we know that:

$$\Delta\phi = \frac{E}{R}\tan(\theta)$$

**B.3)** From the scheme in the previous item, note that the distance the ray travels within the radial interval $r$ to $r+dr$ is $-dr\sec(\theta)$. The expression for the optical depth $\tau$ is given by $d\tau = \kappa\rho(r)dl$, where $dl$ is the distance traveled:

$$\tau = \kappa\rho(R)\sec(\theta)E$$

Hence, the observed flux is reduced by the factor:

$$F = F_0 e^{-\tau}$$

$$\therefore m = -2.5\log\left(\frac{F_0}{F_v}e^{-\tau}\right) = -2.5\log\left(\frac{F_0}{F_v}\right) + 2.5\kappa\rho(R)\sec(\theta)E\log(e)$$

Knowing that $m_0 = -2.5\log\left(\dfrac{F_0}{F_v}\right)$:

$$\Delta m = 2.5\kappa\rho(R)\sec(\theta)E\log(e)$$

**B.4)** First, calculate the distances of each star relative to the Earth, in parsecs:

$$d_a = \frac{1}{0.13}\ \text{pc} \quad \text{and} \quad d_t = \frac{1}{0.33}\ \text{pc}$$

$$\therefore d_a = 7.69\ \text{pc} \quad \text{and} \quad d_t = 3.03\ \text{pc}$$

For a star observed at a zenith angle $z$, we know that $z = \theta$, which is the same as the angle of entry into the atmosphere. However, the actual zenith angle at which the star is located is given by $z_r = z + \Delta\phi$:

$$z_r = z + \frac{E}{R}\tan(z)$$

Thus, we have $z_{r,a} = 25.88°$ and $z_{r,t} = 47.96°$. Note that $\Delta\phi$ is being calculated in radians, so a unit conversion is necessary. Finally, using the law of cosines:

$$\cos(\Delta) = \cos(z_{r,a})\cos(z_{r,t}) + \sin(z_{r,a})\sin(z_{r,t})\cos(\Delta A)$$

Paz e Amor

Using the values, we find:

$$\Delta = 30.14^\circ$$

By the law of cosines for plane geometry (considering the triangle formed by the stars and the Earth), we can finally find the distance between the stars:

$$d_{a\to t} = \sqrt{d_a^2 + d_t^2 - 2d_a d_t \cos\Delta}$$

$$d_{a\to t} = 5.29pc$$

# 2.23 Binary Systems

## Part A: Orbital Parameters

**A.1)** Just use the universal law of gravitation:

$$|\vec{F}| = \frac{Gm_1m_2}{(r_1+r_2)^2}$$

**A.2)** Consider that the point between the masses is the center of mass. Recall the definition of center of mass:

$$\vec{x}_{CM} = \frac{\sum_i m_i\vec{x}_i}{M} \tag{2.2}$$

If we define the center of mass as the origin of the Cartesian plane, in this example we notice that:

$$\vec{x}_{CM} = 0 = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

Thus we find:

$$m_1|\vec{r}_1| = m_2|\vec{r}_2| \tag{2.3}$$

If we take the time derivative of both sides (remember that the time derivative of position is velocity):

$$\frac{d}{dt}m_1\vec{r}_1 = \frac{d}{dt}m_2\vec{r}_2$$

$$\frac{dm_1}{dt}\vec{r}_1 + m_1\vec{v}_1 = \frac{dm_2}{dt}\vec{r}_2 + m_2\vec{v}_2$$

Since the masses of the components do not vary, we can set the time derivatives of the masses to 0:

$$m_1\vec{v}_1 = m_2\vec{v}_2 \tag{2.4}$$

Equations (2.4) and (2.3) reveal important facts: the ratios of velocities and positions relative to the center of mass depend on the component masses, and equally important, the velocities are parallel and the position vectors are collinear (you can see this since the velocity and position vectors are linearly dependent, i.e., one is a multiple of the other).

Now considering the force acting on body $m_i$:

$$\vec{F}_{ji} = -\frac{Gm_im_j}{(r_i+r_j)^2}\hat{r}_i$$

Notice that, by equation (2.3), we can rewrite $r_j$ as $\frac{m_ir_i}{m_j}$, then we have:

$$\vec{F}_{ji} = -\frac{Gm_im_j}{(r_i+\frac{m_ir_i}{m_j})^2}\hat{r}_i = -\frac{Gm_im_j^3}{(m_i+m_j)^2r_i^2}\hat{r}_i$$

Paz e Amor

In the case of a fixed central mass $m_x$:

$$\vec{F}_{ji} = -\frac{Gm_i m_x}{r_i^2}\hat{r}_i$$

Thus, we see that:

$$m_x = \frac{m_j^3}{(m_i + m_j)^2}$$

**A.3)** In this analysis, the orbit of body $i$ is "identical" to this fictitious orbit where the central body is fixed at the center of mass. This result is great because it ensures that each component of the binary orbits in an ellipse, allowing us to apply the well-known Kepler laws.

Using this technique, we can find some orbital parameters of each component. First, we can find the semimajor axis of the orbit from the total mechanical energy equation:

$$\frac{1}{2}m_1 v_{p,1}^2 - \frac{Gm_1 m_2^3}{(m_1 + m_2)^2 r_{p,1}} = -\frac{Gm_1 m_2^3}{(m_1 + m_2)^2 a_1}$$

Since $m_1 r_{p,1} = m_2 r_{p,2}$ and $r_{p,1} + r_{p,2} = L$, we find:

$$r_{p,1} = \frac{m_2 L}{m_1 + m_2}$$

Thus we find:

$$a_1 = \frac{Gm_2^3 L}{4Gm_2^2(m_1 + m_2) - 2v_{p,1}^2 L(m_1 + m_2)^2}$$

By the vis-viva equation, we have:

$$v_p = \sqrt{G\frac{m_2^3}{(m_1 + m_2)^2}\left(\frac{2}{a_1(1 - e_1)} - \frac{1}{a_1}\right)}$$

$$v_{p,1} r_{p,1} = \sqrt{G\frac{m_2^3}{(m_1 + m_2)^2} a_1(1 - e_1^2)}$$

Thus:

$$e_1^2 = 1 - 2\frac{v_{p,1}^2 L(m_1 + m_2)}{Gm_2^2} + \frac{v_{p,1}^4 L^2(m_1 + m_2)^2}{G^2 m_2^4}$$

Simplifying:

$$e_1 = \frac{v_{p,1}^2 L(m_1 + m_2)}{Gm_2^2} - 1$$

Notice that $v_{p,1}^2 = \frac{(m_1 v_{p,1})^2}{m_1^2}$, so:

$$e_1 = \frac{(m_1 v_{p,1})^2 L(m_1 + m_2)}{Gm_2^2 m_1^2} - 1$$

Doing the same for body $m_2$ (swap indices):

$$e_1 = \frac{(m_2 v_{p,2})^2 L(m_2 + m_1)}{G m_1^2 m_2^2} - 1$$

Since $m_1 v_{p,1} = m_2 v_{p,2}$, all expressions are identical:

$$e_1 = e_2$$

At periapsis:

$$m_1 a_1 (1 - e_1) = m_2 a_2 (1 - e_2)$$

And since $e_1 = e_2$, we have $m_1 a_1 = m_2 a_2$.

## Part B: Relative Orbits

**B.1)** In this situation, the relative position of $m_2$ with respect to $m_1$ is $\vec{L} = \vec{r}_2 - \vec{r}_1$ and the relative velocity is (neglecting relativistic effects) $\vec{V} = \vec{v}_2 - \vec{v}_1$, but what really matters is the magnitude of these parameters ($L = r_1 + r_2$ and $V = v_1 + v_2$).

Thus, we write the energy equation of the system in terms of these relative parameters:

$$E_m = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_1 v_1^2 - \frac{G m_1 m_2}{r_1 + r_2}$$

Note that $V = v_2 + \frac{m_2 v_2}{m_1}$, therefore $v_2 = \frac{m_1 V}{m_1 + m_2}$, analogously $v_1 = \frac{m_2 V}{m_1 + m_2}$. Let $M = m_1 + m_2$:

$$E_m = \frac{1}{2} \frac{m_2 m_1^2}{M^2} V^2 + \frac{1}{2} \frac{m_1 m_2^2}{M^2} V^2 - \frac{G m_1 m_2}{R}$$

Simplifying:

$$E_m = \frac{1}{2} \frac{m_1 m_2}{M} V^2 - \frac{G m_1 m_2}{R}$$

Rewrite: $m_1 m_2 = \frac{m_1 m_2 M}{M}$:

$$E_m = \frac{1}{2} \mu V^2 - \frac{G \mu M}{R}$$

with $\mu = \frac{m_1 m_2}{m_1 + m_2}$, the reduced mass of the system.

**B.2)** Notice that this relation is identical to that of a body of mass $\mu$ orbiting a fixed central mass $M$, with relative parameters ($L$ and $V$). Therefore, the relative orbit of one star with respect to the other can be modeled as this auxiliary orbit.

## Part C: Lagrangian Points

**C.1)** First, note that the resultant gravitational force between $A$, $m_1$ and $A$, $m_2$ must point toward the center of mass (under our conditions), so the forces in directions perpendicular to the line connecting $A$ and CM must cancel:

$$F_{A,m_1} \sin(\alpha) = F_{A,m_2} \sin(\beta)$$

Paz e Amor

$$\frac{Gm_1m_A}{d_1^2}\sin(\alpha) = \frac{Gm_2m_A}{d_2^2}\sin(\beta)$$

$$\frac{m_1}{d_1^2}\sin(\alpha) = \frac{m_2}{d_2^2}\sin(\beta) \tag{2.5}$$

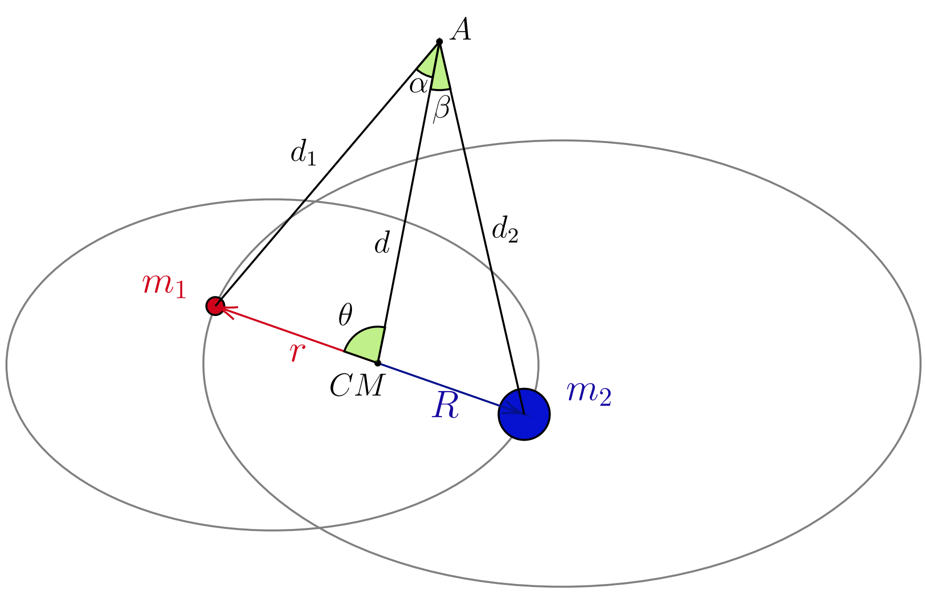
On the other hand, the sum of the forces in the direction of the center of mass must result in a gravitational force that puts body $A$ in the desired orbit of period $T$:

$$F_{A,m_1}\cos(\alpha) + F_{A,m_2}\cos(\beta) = F_{A,CM}$$

$$\frac{Gm_1m_A}{D^2}\cos(\alpha) + \frac{Gm_2m_A}{D^2}\cos(\beta) = \frac{Gm_xm_A}{d^2}$$

Notice that $m_x$ is used to represent the effective mass that would attract $m_A$. Thus:

$$\frac{m_1}{D^2}\cos(\alpha) + \frac{m_2}{D^2}\cos(\beta) = \frac{m_x}{d^2} \tag{2.6}$$




**C.2)** Using the law of sines in triangles $A$,$m_1$,$CM$ and $A$,$m_2$,$CM$, we find[7]:

$$\frac{\sin(\alpha)}{r} = \frac{\sin(\theta)}{d_1}$$

$$\frac{\sin(\beta)}{R} = \frac{\sin(\theta)}{d_2}$$

Thus, substituting the sine values into equation (2.5), we have:

$$\frac{m_1r}{d_1^3}\sin(\theta) = \frac{m_2R}{d_2^3}\sin(\theta)$$

[7]Remember that $\sin(180° - \theta) = \sin(\theta)$.

Notice, however, that $m_1 r = m_2 R$, which is precisely the definition of the center of mass. Therefore, we can simplify the equation and show that $d_1 = d_2$! From now on, we will denote $d_1$ and $d_2$ simply as $D$, since we know they are equal.

Applying the law of cosines to triangles $A$,$m_1$,$CM$ and $A$,$m_2$,$CM$, we find:

$$D^2 = d^2 + R^2 + 2dR\cos(\theta)$$
$$D^2 = d^2 + r^2 - 2dr\cos(\theta)$$

Thus, obviously, we have:

$$d^2 + R^2 + 2dR\cos(\theta) = d^2 + r^2 - 2dr\cos(\theta)$$

From which we get $2d\cos(\theta) = r - R$. Now we make a move: since $m_1 r = m_2 R$, we will express $r$ and $R$ in terms of the total distance between the stars, defining $L \equiv r + R$, so that $r = \frac{m_2 L}{m_1+m_2}$ and $R = \frac{m_1 L}{m_1+m_2}$. Thus, we have:

$$2d\cos(\theta) = \frac{m_2 - m_1}{m_2 + m_1}L \tag{2.7}$$

Using the law of cosines in the same triangles, but with angles $\alpha$ and $\beta$:

$$R^2 = D^2 + d^2 - 2Dd\cos(\beta)$$
$$r^2 = D^2 + d^2 - 2Dd\cos(\alpha)$$

Solving for $\cos(\alpha)$ and $\cos(\beta)$:

$$\cos(\alpha) = \frac{D^2 + d^2 - r^2}{2Dd}$$
$$\cos(\beta) = \frac{D^2 + d^2 - R^2}{2Dd}$$

Substituting these results into the previous equation:

$$\frac{m_1}{D^2}\frac{D^2 + d^2 - r^2}{2Dd} + \frac{m_2}{D^2}\frac{D^2 + d^2 - R^2}{2Dd} = \frac{m_x}{d^2}$$

Simplifying further:

$$m_1(D^2 + d^2 - r^2) + m_2(D^2 + d^2 - R^2) = \frac{2D^3 m_x}{d}$$

$$D^2(m_1 + m_2) + d^2(m_1 + m_2) - m_1 r^2 - m_2 R^2 = \frac{2D^3 m_x}{d}$$

Now, substitute $r$ and $R$ in terms of the masses and total distance $L$, as shown previously:

$$m_1 r^2 + m_2 R^2 = m_2\left(\frac{m_1 L}{m_1 + m_2}\right)^2 + m_1\left(\frac{m_2 L}{m_1 + m_2}\right)^2 = \frac{m_2 m_1^2 L^2}{(m_1 + m_2)^2} + \frac{m_1 m_2^2 L^2}{(m_1 + m_2)^2}$$

$$m_1 r^2 + m_2 R^2 = \frac{m_2 m_1 L^2}{m_1 + m_2}$$

Paz e Amor

Hence:

$$D^2(m_1+m_2)+d^2(m_1+m_2)-\frac{m_2m_1L^2}{m_1+m_2}=\frac{2D^3m_x}{d}$$

$$D^2+d^2-\frac{m_2m_1L^2}{(m_1+m_2)^2}=\frac{2D^3m_x}{d(m_1+m_2)} \tag{2.8}$$

Also, as shown before (law of cosines) $D^2=d^2+R^2+2dR\cos(\theta)$, substituting $R=\frac{m_1L}{m_1+m_2}$ and $2d\cos(\theta)=\frac{m_2-m_1}{m_2+m_1}L$:

$$D^2=d^2+\frac{m_1^2L^2}{(m_1+m_2)^2}+\frac{m_2-m_1}{m_2+m_1}L\frac{m_1L}{m_1+m_2}$$

Simplifying:

$$D^2=d^2+\frac{m_1m_2}{(m_1+m_2)^2}L^2 \tag{2.9}$$

Substituting this result into the previous expression:

$$d^2+\frac{m_1m_2}{(m_1+m_2)^2}L^2+d^2-\frac{m_2m_1L^2}{(m_1+m_2)^2}=\frac{2D^3m_x}{d(m_1+m_2)}$$

We arrive at:

$$d^3=\frac{m_x}{m_1+m_2}D^3 \tag{2.10}$$

Substituting the definition given in the problem statement: $k=\left(\frac{m_x}{m_1+m_2}\right)^{1/3}$, so that $d=kD$.

**C.3)** Substituting the previous result into relation (11):

$$\frac{d^2}{k^2}=d^2+\frac{m_1m_2}{(m_1+m_2)^2}L^2$$

$$d=\sqrt{\frac{m_1m_2k^2}{(m_1+m_2)^2(1-k^2)}}L$$

Using Kepler's third law, we can relate periods to the semimajor axes of the orbits:

$$\frac{GT^2}{4\pi^2}=\frac{(a_1+a_2)^3}{m_1+m_2}=\frac{a_x^3}{m_x}$$

From this we see that $a_x=k(a_1+a_2)$.

**C.4)** Applying some notable values for $d$ and $L$:
**Stars at Periapsis:**

$$a_x(1-e_x)=\sqrt{\frac{m_1m_2k^2}{(m_1+m_2)^2(1-k^2)}}(a_1+a_2)(1-e)$$

**Stars at Apoapsis:**

$$a_x(1+e_x) = \sqrt{\frac{m_1 m_2 k^2}{(m_1+m_2)^2(1-k^2)}}(a_1+a_2)(1+e)$$

Dividing one expression by the other:

$$\frac{1-e_x}{1+e_x} = \frac{1-e}{1+e}$$

Hence, we find that necessarily $e = e_x$.

**C.5)** Starting from the periapsis equation and substituting $e = e_x$:

$$a_x = \sqrt{\frac{m_1 m_2 k^2}{(m_1+m_2)^2(1-k^2)}}(a_1+a_2)$$

Notice we have two formulas for $a_x$, so we can relate them:

$$\sqrt{\frac{m_1 m_2 k^2}{(m_1+m_2)^2(1-k^2)}}(a_1+a_2) = a_x = k(a_1+a_2)$$

Proceeding with the calculations we find:

$$k = \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{m_1+m_2}$$

Therefore, the following relations hold:

$$a_x = \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{m_1+m_2}(a_1+a_2)$$

$$m_x = \frac{(m_1^2 + m_1 m_2 + m_2^2)^{3/2}}{(m_1+m_2)^2}$$

$$d = \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{m_1+m_2}L$$

From the relation $2d\cos(\theta) = \frac{m_2-m_1}{m_2+m_1}L$ and the relation between $d$ and $L$:

$$\cos(\theta) = \sqrt{\frac{(m_2-m_1)^2}{4(m_1^2+m_1m_2+m_2^2)}}$$

**C.6)** Since $k = \frac{\sqrt{m_1^2+m_1m_2+m_2^2}}{m_1+m_2}$ and $d = kD = kL$, we finally find that $D = L$. Notice then that the triangle $\Delta A, m_1, m_2$ **is an equilateral triangle!**, since all sides are equal. This result is general for any ellipse!

# 2.24 "Relative" Luminosity

## Part A: Lorentz Transformations

**A.1)** For a generic reference frame, the distance from a point $(x,y,z)$ to the origin is simply $\sqrt{x^2+y^2+z^2}$. In this case, as this is the distance traveled by light, which, by the principle of relativity, is constant in all reference frames, this distance is $ct$, assuming that $t=0$ when the ray is emitted. Thus, we find that, for any reference frame: $x^2+y^2+z^2=(ct)^2$, therefore:

$$x^2+y^2+z^2-(ct)^2=x'^2+y'^2+z'^2-(ct')^2=0$$

**A.2)** Imagine that there is a certain scaling distortion, without loss of generality, along the axes perpendicular to the direction of motion. Imagine two identical rulers $S$ and $S'$, each in its respective reference frame, both with a pen placed at the 10 cm mark.

For $S$, $S'$, which is moving, would have its length increased by a certain factor, so the pen of $S$ would mark a smaller than 10 cm position on $S'$, and $S'$ would mark above 10 cm on $S$. Conversely, in the reference frame of $S'$, it is $S$ that is moving, so the observed effect would be opposite. By Einstein's principle of relativity, both reference frames must agree on the occurrence of the same events, which does not happen in this case!

**A.3)** To avoid the inconvenience of the subtraction sign, we move to the use of complex numbers. Define the coordinate vector $(x,ict)$ and its rotation $(x',ict')$; to convert one into the other, we can use the rotation matrix $M(\theta)$, defined as:

$$M=\begin{Vmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{Vmatrix} \tag{2.11}$$

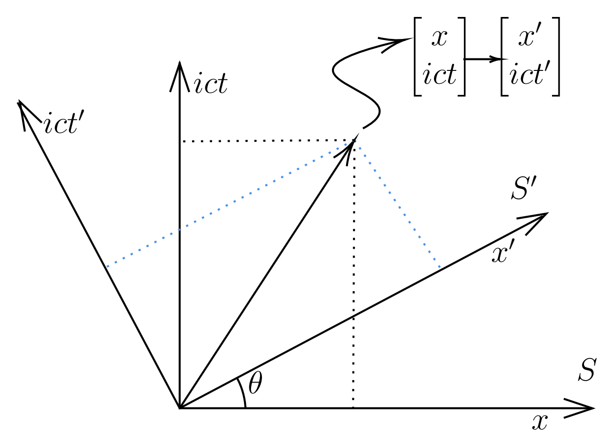



Figure 2.36: Change of reference frame observed as a rotation in a complex space.

Therefore, we understand that:

$$\begin{Vmatrix}x'\\ ict'\end{Vmatrix}=\begin{Vmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{Vmatrix}\begin{Vmatrix}x\\ ict\end{Vmatrix}$$

From this, we find that:

$$\begin{cases} x' = x\cos(\theta) + ict\sin(\theta) \\ ict' = -x\sin(\theta) + ict\cos(\theta) \end{cases}$$

**A.4)** Note that, if we fix $x' = 0$, i.e., the origin of $S'$, and knowing that in this case $x/t$ will be exactly the definition of the velocity of $S'$ as seen from $S$, we get:

$$x\cos(\theta) + ict\sin(\theta) = 0$$

$$\therefore \tan(\theta) = \frac{vi}{c}$$

We also conclude that $\sin(\theta) = \dfrac{vi}{c\sqrt{1-\dfrac{v^2}{c^2}}}$ and $\cos(\theta) = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$. Substituting these into the previous relations and defining:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

we obtain the Lorentz transformations:

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{vx}{c^2}\right) \end{cases}$$

## Part B: Headlight Effect

**B.1)** Since the radiation distribution is isotropic, the luminosity (power) is equally distributed over the wavefront:

$$dL' = \frac{L}{4\pi R^2}dA$$

where $dA$ is the area over which this luminosity is distributed (the area between the blue circles in the first figure). It is known that $dA = 2\pi R^2\sin(\theta')d\theta'$. Therefore:

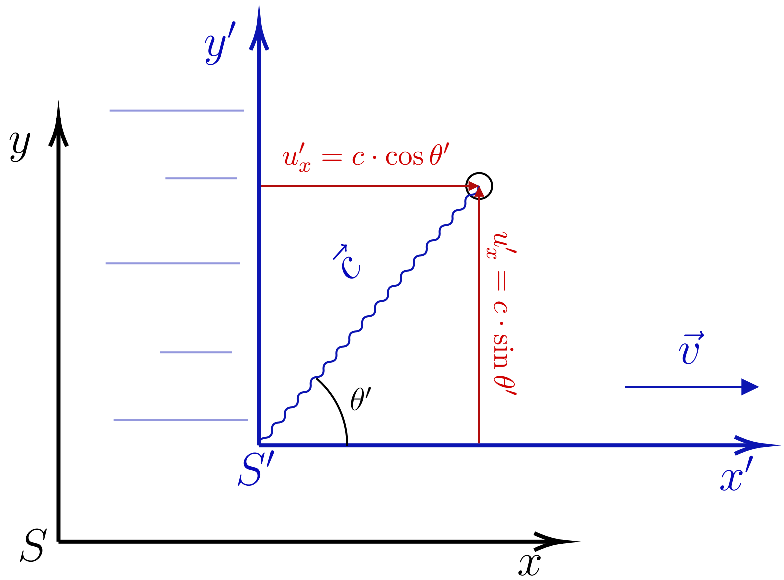
$$dL' = \frac{L}{2}\sin(\theta')d\theta'$$

**B.2)** Consider the scheme shown in Figure 1.17:

Notice that, in the $S'$ frame, the light has the following velocity components: $u'_y = c\sin(\theta')$ and $u'_x = c\cos(\theta')$. Applying the Lorentz velocity transformations:

$$dx' = \gamma(dx - vdt)$$

$$dy' = dy$$




$$dt' = \gamma\left(dt - \frac{vdx}{c^2}\right)$$

For the horizontal component:

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma\left(dt - \frac{vdx}{c^2}\right)} = \frac{dx - vdt}{dt - \frac{vdx}{c^2}}$$

For $S$, the velocity component is $u_x = \frac{dx}{dt}$:

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

Solving for $u_x$:

$$u_x = \frac{v + u'_x}{1 + \frac{vu'_x}{c^2}}$$

Since $u_x = c \cdot \cos(\theta)$ and $u'_x = c \cdot \cos(\theta')$, and recalling that $v = \beta c$:

$$\cos(\theta) = \frac{\beta + \cos(\theta')}{1 + \beta\cos(\theta')}$$

**Note:** This relation alone is sufficient to relate the angles, but working with the sine will be very useful.

For $u'_y$:

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{vdx}{c^2}\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

Since $u_x = c\cos(\theta)$, $u_y = c\sin(\theta)$, and $u'_y = c\sin(\theta')$:

$$\sin(\theta) = \frac{\sin(\theta')}{\gamma\,(1 + \beta\cos(\theta'))}$$

**B.3)** It is sufficient to differentiate the obtained expressions:

$$d \sin(\theta) = d\frac{\sin(\theta')}{\gamma(1+\beta\cos(\theta'))}$$

$$\cos(\theta) d\theta = \frac{1}{\gamma}\frac{\beta+\cos(\theta')}{(1+\beta\cos(\theta'))^2} d\theta'$$

Using the cosine relations used previously and rearranging:

$$\frac{d\theta}{\gamma(1-\beta\cos(\theta))} = d\theta'$$

## Part C: Redshift and Blueshift

**C.1)** Recall the dot product of two vectors: $\vec{v}\cdot\vec{u} = v_x u_x + v_y u_y$, thus:

$$\vec{k}\cdot\vec{r} = k_y y + k_x x$$

We can separate the components of $\vec{k}$ based on the propagation angle:

$$\vec{k}\cdot\vec{r} = k\sin(\theta) y + k\cos(\theta) x$$

Substituting the Lorentz transformations:

$$\vec{k}\cdot\vec{r} - \omega t = k\sin(\theta) y' + k\cos(\theta)\gamma(x' + vt') - \omega\gamma\left(t' + \frac{vx'}{c^2}\right)$$

**C.2)** For $S'$:

$$\phi' = \vec{k'}\cdot\vec{r'} - \omega' t'$$

Notice that both equations (the previous item and this one) describe the same wave function in the $S'$ frame, so we can associate the terms. All terms accompanying $t'$ correspond to the frequency $\omega'$:

$$k\cos(\theta)\gamma(vt') - \omega\gamma(t') = -\omega' t'$$

Since $k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$ and $\omega = 2\pi f$:

$$\left(\frac{v}{c}\cos(\theta) - 1\right) 2\pi\gamma f = -2\pi f'$$

Finally:

$$f = \frac{f'}{\gamma(1-\beta\cos(\theta))}$$

## Part D: The Grand Unification

**D.1)** Considering the angular interval, the energy $dE$ is the product of the number of photons passing through the area section and the energy of each photon. Therefore:

$$\frac{dE}{dt} = \frac{nE}{dt}$$

And for $S'$, the number of photons passing through the section is the same (even with angular changes, the number of photons remains unchanged):

$$\frac{dE'}{dt'} = \frac{nE'}{dt'}$$

Recalling that $E = hf = hf'\frac{1}{\gamma(1-\beta\cos(\theta))} = \frac{E'}{\gamma(1-\beta\cos(\theta))}$ and that $dt = \gamma dt'$ (time dilation):

$$\frac{dE}{dt} = \frac{1}{(1-\beta\cos(\theta))\gamma^2}\frac{dE'}{dt'}$$

**D.2)** From this, we have:

$$dL(\theta) = \frac{1}{(1-\beta\cos(\theta))\gamma^2}dL'$$

$$dL(\theta) = \frac{1}{(1-\beta\cos(\theta))\gamma^2}\frac{L}{2}\sin(\theta')d\theta'$$

Substituting the relations for $\sin(\theta')$ and $d\theta'$ found in item B.3:

$$dL(\theta) = \frac{L}{2\gamma^4}\frac{\sin(\theta)d\theta}{(1-\beta\cos(\theta))^3}$$

**D.3)** Integrating the luminosity from 0 to $L_t$ and the angle from $\pi$ to 0:

$$\int_0^{L_t} dL(\theta) = \int_\pi^0 \frac{L}{2\gamma^4}\frac{\sin(\theta)d\theta}{(1-\beta\cos(\theta))^3}$$

Since $\sin(\theta)d\theta = d\cos(\theta)$ and substituting $\cos(\theta) = u$:

$$L_t = \frac{L}{2\gamma^4}\int_\pi^0 \frac{d\cos(\theta)}{(1-\beta\cos(\theta))^3} = \frac{L}{2\gamma^4}\int_{-1}^1 \frac{du}{(1-\beta u)^3}$$

Substituting $z = 1-\beta u$, so $dz = -\beta du$:

$$L_t = -\frac{L}{2\beta\gamma^4}\int_{1+\beta}^{1-\beta}\frac{dz}{z^3}$$

Finally:

$$L_t = \frac{L}{4\beta\gamma^4}\left(\frac{1}{(1-\beta)^2} - \frac{1}{(1+\beta)^2}\right)$$

Notice that:

$$\left(\frac{1}{(1-\beta)^2} - \frac{1}{(1+\beta)^2}\right) = \frac{1+2\beta+\beta^2-1+2\beta-\beta^2}{(1-\beta^2)^2} = 4\beta\gamma^4$$

Therefore: $L_t = L$

Paz e Amor

**EXTRA:** Imagine a star whose energy is given solely by its rest mass. Its energy in $S'$ is $E' = m_0c^2$, while in $S$ it is $E = \gamma m_0 c^2$. Note that the rest mass is the same in both frames (by definition). Considering luminosity as the rate of energy change over time:

$$L = \frac{dE}{dt} = \frac{\gamma c^2 dm_0}{dt}$$

Substituting $dt = \gamma dt'$:

$$L = \frac{dE}{dt} = \frac{\gamma c^2 dm_0}{\gamma dt'} = L'$$

Which confirms the result obtained!

## 2.25 Intergalactic Spectroscopy

**a)** Consider the moment when the light is at a distance* $r$ from the source (*this distance corresponds to the comoving distance, i.e., the distance from the point where it is now to the source if measured at the time of emission). Over an interval $dt$, the distance traveled will be: $cdt = \frac{a(t)}{a_e}dr = e^{H_0(t-t_e)}dr$. Therefore, we find:

$$\frac{dt}{e^{H_0(t-t_e)}} = \frac{dr}{c} \rightarrow \int_{t_e}^{t_0} e^{-H_0(t-t_e)}dt = \frac{D_e}{c}$$

Thus:

$$e^{-H_0(t_0-t_e)} - 1 = -\frac{D_e H_0}{c}$$

$$t_e = t_0 + \frac{\ln\left(1 - \frac{D_e H_0}{c}\right)}{H_0}$$

Substituting the values, we find $t_e = 2.674$ billion years! Therefore, we deduce $a_e = 0.442$.

**b)** By the definition of redshift: $z = \frac{\lambda - \lambda_e}{\lambda_e}$, however, $\lambda$ is nothing but $\lambda_e$ scaled by a factor $\frac{a(t)}{a_e}$:

$$z = \frac{\lambda_e \frac{a(t)}{a_e}}{\lambda_e} - 1 = \frac{a(t)}{a_e} - 1$$

Thus:

$$a(t) = a_e(z+1)$$

**c)** Let the initial density be $\rho_e$. After the universe expands, the volume increases by a factor of $a(t)^3$, so, being inversely proportional to the volume, the density behaves as:

$$\rho(t) = \frac{\rho_e a_e^3}{a(t)^3}$$

**d)** For a wavelength to start being absorbed, it must redshift such that it equals $\lambda_H$. The end of absorption occurs when the wavelength after redshift equals $\lambda_H + \Delta\lambda_H$.

At the starting condition:

$$\lambda_H = \lambda \frac{a(t_1)}{a_e}$$

$$a(t_1) = a_e \frac{\lambda_H}{\lambda}$$

At the ending condition:

$$\lambda_H + \Delta\lambda_H = \lambda \frac{a(t_2)}{a_e}$$

$$a(t_2) = a_e \frac{\lambda_H + \Delta\lambda_H}{\lambda}$$

**e)** Knowing that the optical depth in this situation is given by $\tau = \kappa\rho(t)dR$, with $dR$ being the distance traveled, which can also be interpreted as $dR = cdt$. There is also the effect of wavelength change, since the flux is proportional to photon energy; when the photon wavelength changes, so does the energy, and therefore the flux, meaning the flux is multiplied by a factor $\frac{a(t_2)}{a(t_1)}$ over the interval $t_2 - t_1$.

Analyzing a small interval $dt$, consider first the absorption effect. This can be done because the effects are independent: one changes the number of photons (absorption) and the other changes the photon wavelength (expansion). Thus we can analyze them separately:

$$dF = -F(t)\tau = -F(t)\kappa c\rho_e a_e^3 a(t)^{-3}dt$$

Integrating:

$$\int_{F_0}^{F} \frac{dF}{F} = -\int_{t_1}^{t_2} \kappa c\rho_e a_e^3 e^{-3H_0(t-t_0)}dt$$

$$\ln\left(\frac{F}{F_0}\right) = \kappa c\rho_e a_e^3 \left(e^{-3H_0(t_2-t_0)} - e^{-3H_0(t_1-t_0)}\right)\frac{1}{3H_0}$$

$$\ln\left(\frac{F}{F_0}\right) = \kappa c\rho_e a_e^3 \left(a(t_2)^{-3} - a(t_1)^{-3}\right)\frac{1}{3H_0}$$

$$\ln\left(\frac{F}{F_0}\right) = \kappa c\rho_e \lambda^3 \left((\lambda_H + \Delta\lambda_H)^{-3} - \lambda_H^{-3}\right)\frac{1}{3H_0}$$

Using the approximation $(1+x)^n \approx 1 + nx$, since $\Delta\lambda_H \ll \lambda_H$:

$$\ln\left(\frac{F}{F_0}\right) = -\frac{\kappa c\rho_e \lambda^3 \Delta\lambda_H}{H_0\lambda_H^4}$$

$$F = F_0 e^{-\frac{\kappa c\rho_e \lambda^3 \Delta\lambda_H}{H_0\lambda_H^4}}$$

Finally, including the factor $\frac{a_e}{a_0}$:

$$F = 0.442 \cdot F_0 e^{-\frac{\kappa c\rho_e \lambda^3 \Delta\lambda_H}{H_0\lambda_H^4}}$$

**f)** The expected flux, ignoring $WHIM$ absorption, would only consider the universe expansion factor:

$$F_{expected} = 0.442F_0$$

Remember that all other factors affecting the flux (distance effect and redshift) are independent of the initial flux. Therefore, the correction factor is constant after the absorption interval, giving:

$$\alpha(\lambda) = e^{-\frac{\kappa c\rho_e \lambda^3 \Delta\lambda_H}{H_0\lambda_H^4}}$$

**g)** The expansion of the universe causes the redshift of radiation, i.e., an increase in wavelength. Absorption occurs only if $\lambda_H < \lambda < \lambda_H + \Delta\lambda_H$. If the initial $\lambda$ is greater than $\lambda_H + \Delta\lambda_H$, no effect occurs due to redshift. This initial wavelength corresponds to the observed wavelength:

$$\lambda_{observed} = \lambda_{emitted}\frac{a_0}{a_e} = (\lambda_H + \Delta\lambda_H)\frac{1}{a_e}$$

Substituting values, we find:

$$\lambda_{max} = 278.6 \text{ nm}$$

For smaller wavelengths, redshift has not had enough time to bring the initial wavelength to at least $\lambda_H$. Therefore, at the minimum, these wavelengths enter the necessary range just before reaching Earth:

$$\lambda_{min} = \lambda_H$$

So the interval where the anomaly occurs is:

$$[121.6 \text{ nm}, 278.6 \text{ nm}]$$

However, for initial wavelengths $\lambda_H < \lambda_0 < \lambda_H + \Delta\lambda_H$, absorption occurs partially in the $WHIM$. For observed wavelengths:

$$\frac{\lambda_H}{a_e} < \lambda < \frac{\lambda_H + \Delta\lambda_H}{a_e}$$

the effect is only partial:

$$274.8 \text{ nm} < \lambda < 278.6 \text{ nm}$$

Similarly, for radiation arriving at Earth with wavelength $\lambda_H < \lambda < \lambda_H + \Delta\lambda_H$, only part of the interval is traversed:

$$121.6 \text{ nm} < \lambda < 123.3 \text{ nm}$$

**h)** We use the standard error propagation equation for a function $f(x,y,z,...) : \mathbb{R} \to \mathbb{R}$:

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial f}{\partial y}\right)^2 (\Delta y)^2 + \left(\frac{\partial f}{\partial z}\right)^2 (\Delta z)^2 + ...}$$

Here, the function is $\alpha(\lambda)$:

$$\Delta\alpha(\lambda) = \left|\frac{d\alpha(\lambda)}{d\lambda}\Delta\lambda\right|$$

Be careful! The wavelengths in the table are observed wavelengths, different from the $\lambda$ used for uncertainty calculations. Convert each provided wavelength by multiplying by $a_e$:

| $\lambda_{obs.}(nm) \pm 1.4$ nm | $\lambda_{emit.}(nm) \pm 0.6$ nm |
|---|---|
| 130.0 | 57.5 |
| 160.0 | 70.8 |
| 190.0 | 84.1 |
| 220.0 | 97.3 |
| 250.0 | 110.6 |

Since $\alpha(\lambda) = e^{-k\lambda^3}$:

$$\Delta\alpha(\lambda) = 3k\lambda^2 e^{-k\lambda^3}\Delta\lambda$$

Given[8] $\Delta\lambda = 0.26$ nm $\forall\lambda$. Since $k$ is unknown (because $\rho_e$ is unknown), use $\ln(\alpha(\lambda)) = -k\lambda^3$, giving:

$$\Delta\alpha(\lambda) = -3\ln(\alpha(\lambda))\alpha(\lambda)\frac{\Delta\lambda}{\lambda}$$

Substitute values to construct the complete table:

| $\lambda(nm) \pm 1.4$ nm | $\alpha(\lambda)$ |
|---|---|
| 130.0 | $0.99964 \pm 0.00001$ |
| 160.0 | $0.99933 \pm 0.00002$ |
| 190.0 | $0.99887 \pm 0.00002$ |
| 220.0 | $0.99825 \pm 0.00003$ |
| 250.0 | $0.99744 \pm 0.00004$ |

**i)** To use linear regression, we need a linear function. How can we linearly relate $\alpha(\lambda)$ and $\lambda$? From the previous item: $\ln(\alpha(\lambda)) \propto \lambda^3$. Thus, a plot of $\ln(\alpha(\lambda))$ vs $\lambda^3$ yields a straight line.

Use emitted wavelengths:

| $\lambda(nm) \pm 0.6$ nm | $\alpha(\lambda)$ |
|---|---|
| 57.5 | $0.99964 \pm 0.00001$ |
| 70.8 | $0.99933 \pm 0.00002$ |
| 84.1 | $0.99887 \pm 0.00002$ |
| 97.3 | $0.99825 \pm 0.00003$ |
| 110.6 | $0.99744 \pm 0.00004$ |

Construct the table for linear regression (optional):

| $\lambda^3(nm^3)$ | $\ln(\alpha(\lambda))$ |
|---|---|
| 190109.4 | $-3.6006 \cdot 10^{-4}$ |
| 354894.9 | $-6.7022 \cdot 10^{-4}$ |
| 594823.3 | $-11.3064 \cdot 10^{-4}$ |
| 921167.3 | $-17.5153 \cdot 10^{-4}$ |
| 1352899.0 | $-25.6328 \cdot 10^{-4}$ |

After performing linear regression $y = A + Bx$, we find:

$$\begin{cases} A = 2.28314097 \cdot 10^{-8} \\ B = -1.896911019 \cdot 10^{-9} \text{ nm}^{-3} \\ r = -0.999993384 \end{cases}$$

Errors are calculated using (with $N = 5$ samples):

[8] After redshift correction

$$\Delta B = B\sqrt{\frac{r^{-2}-1}{N-2}}$$

$$\Delta A = \Delta B\sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N}}$$

Finally:

$$A = (0.02 \pm 3.18) \cdot 10^{-6}$$

$$B = (-1.897 \pm 0.004) \cdot 10^{-9} \text{ nm}^{-3}$$

The uncertainty in $A$ is larger than its value, implying $A$ cannot be determined and can be neglected (expected to be 0 in the ideal case).

Note:

$$B = -\frac{\kappa c \rho_e \Delta\lambda_H}{H_0 \lambda_H^4}$$

Using wavelengths in $nm$ as previously. Thus we find:

$$\rho_e = (3.341 \pm 0.007) \cdot 10^{-26}\ kg \cdot m^{-3}$$

## 2.26 Apollo XXVI

### Part A: "Optical" Rocket

**A.1)** Consider the schematic in Figure 2.37:

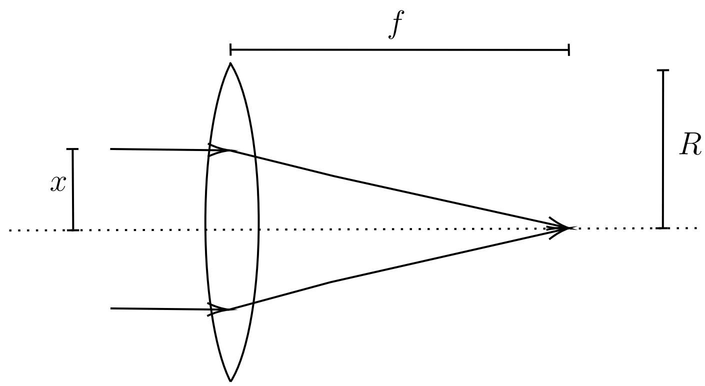



Figure 2.37: Lens and convergence of light rays.

Considering the radiation passing through the lens within the range of rays $x$ to $x + dx$, the power transmitted under these conditions is: $L = F_\odot 2\pi x dx$. The energy is then described over an exposure time $dt$: $E = 2\pi F_\odot x dx dt$. Using Einstein's relation for photons, we know that $E = pc$, where $p$ is the momentum of the radiation. Thus, we can find the initial momentum of the photons passing through the lens under these conditions: $p = \frac{2\pi F_\odot}{c} x dx dt$.

After passing through the lens, this same momentum is redirected toward the focus, so that the momentum in the original direction of motion is $p\cos(\theta)$, where $\theta$ is the deflection angle, such that $\tan(\theta) = \dfrac{x}{f}$, with $f$ being the focal length. Therefore, the change in momentum along the horizontal axis[9] is $dp = p(\cos(\theta) - 1)$. By conservation of momentum, the change in momentum of the lens is $dp = p(1 - \cos(\theta))$:

$$\cos(\theta) = \frac{1}{\sqrt{1 + \tan^2(\theta)}}$$

$$dp = \frac{2\pi F_\odot}{c} x dx dt \left(1 - \frac{f}{\sqrt{f^2 + x^2}}\right)$$

Integrating over all values of $x$:

$$\int_0^d p_x dp = \frac{2\pi F_\odot}{c} dt \int_0^R \left(1 - \frac{f}{\sqrt{f^2 + x^2}}\right) x dx$$

$$\frac{c dp_x}{2\pi F_\odot dt} = \int_0^R x dx - \int_0^R \frac{x dx}{\sqrt{1 + \frac{x^2}{f^2}}}$$

Substituting $u = 1 + x^2/f^2$ $(du = 2x dx/f^2)$:

[9]The change in momentum along axes perpendicular to the horizontal is zero, due to the cylindrical symmetry of the lens

$$\frac{cdp_x}{2\pi F_\odot dt} = \frac{R^2}{2} - \frac{f^2}{2}\int_1^{1+\frac{R^2}{f^2}} \frac{du}{\sqrt{u}}$$

Solving, we find:

$$\frac{cdp_x}{2\pi F_\odot dt} = \frac{R^2}{2} - f^2\left(\sqrt{1+\frac{R^2}{f^2}} - 1\right)$$

Using the known focal ratio, we have $\frac{f}{D} = \frac{f}{2R} = \beta$. Substituting this into the previous result and simplifying slightly:

$$dp_x = \frac{2\pi F_\odot R^2}{c}\left(\frac{1}{2} - 2\beta\left(\sqrt{4\beta^2+1} - 2\beta\right)\right)dt$$

By the definition of force, $F_x = \dfrac{dp_x}{dt}$, so:

$$F_x = \frac{2\pi F_\odot R^2}{c}\left(\frac{1}{2} - 2\beta\left(\sqrt{4\beta^2+1} - 2\beta\right)\right)$$

**A.2)** Substituting the values, we find the desired answer: $\Delta t \approx 357.4$ years.

**A.3)** Note that the force $F_x$ can be written as $F_x = kF_\odot = k\frac{L_\odot}{4\pi r^2}$, where $r$ is the distance from the satellite to the center of the star. To balance the gravitational force, we want:

$$\frac{L_\odot R^2}{2r^2 c}\left(\frac{1}{2} - 2\beta\left(\sqrt{4\beta^2+1} - 2\beta\right)\right) = \frac{GMm}{r^2}$$

$$M = \frac{L_\odot R^2}{2Gmc}\left(\frac{1}{2} - 2\beta\left(\sqrt{4\beta^2+1} - 2\beta\right)\right)$$

In this case, $M = 6.72 \cdot 10^{19}$ kg, approximately a thousand times lighter than the Moon!

## Part B: Mass Cannon

**B.1)** To solve this problem, it is useful to first move to the reference frame of the moving body, where the parameters are easier to understand, and then apply the necessary transformations. In this problem, we will denote the reference frame $S$ as that of the moving spacecraft[10], while $S'$ is for an inertial observer outside the spacecraft.

For this, consider the following scheme in reference frame $S$.

The spacecraft has a mass $m$ at a generic instant $t$, and at a time $dt$ later, its mass is $dm$. Note that $dm < 0$, so we must consider that the mass released is actually $-dm$. By conservation of linear momentum (initially $p = 0$):

$$0 = (m + dm)(dv)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - (-dm)u\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

Ignoring terms containing the product of two infinitesimals and approximating the Lorentz factor for $dv$ as 1, we have:

[10]Note the interesting fact that the reference frame $S$ is accelerated, since it follows the spacecraft; however, we will consider a $dt$ over which the spacecraft's velocity is practically constant and work in this regime.

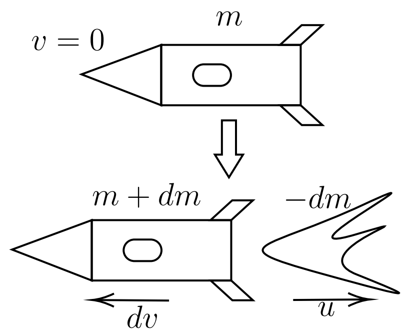



Figure 2.38: Operating principle of a mass-ejection rocket.

$$mdv = -\frac{udm}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$dv = -\frac{udm}{m\sqrt{1-\frac{u^2}{c^2}}}$$

Having done this analysis, we must transfer this result to reference frame $S'$ in order to find $dv'$, i.e., the differential velocity in $S'$.

By the Lorentz transformations, from the velocity $v$ of a point moving along the $x$-axis in both frames, we can find the velocity $v'$ of the same point in $S'$, with $V$ being the velocity of the spacecraft relative to $S'$:

$$\begin{cases} dx' = \gamma(x+Vt) \\ dt' = \gamma\left(t+\frac{Vx}{c^2}\right) \end{cases}$$

Dividing both equations:

$$v' = \frac{dx+Vdt}{dt+\frac{Vdx}{c^2}} = \frac{v+V}{1+\frac{Vv}{c^2}}$$

Differentiating both sides allows us to relate the changes in velocities:

$$dv' = \frac{\left(1+\frac{Vv}{c^2}\right)dv-(v+V)\frac{V}{c^2}dv}{\left(1+\frac{Vv}{c^2}\right)^2}$$

Now consider that this “random” point is the spacecraft after the addition of $dv$, our object of interest. In this case: $v'=V$ and $v=0$:

$$dV = dv - \frac{V^2}{c^2}dv = \left(1-\frac{V^2}{c^2}\right)dv$$

$$\frac{dV}{1-\frac{V^2}{c^2}} = -\frac{udm}{m\sqrt{1-\frac{u^2}{c^2}}}$$

$$\int_0^{V(t)} \frac{dV}{1-\frac{V^2}{c^2}} = -\int_{m_0}^{m(t)} \frac{udm}{m\sqrt{1-\frac{u^2}{c^2}}}$$

Substitute $y=V/c$:

$$c\int_0^{\frac{V}{c}} \frac{dy}{1-y^2} = -\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\int_{m_0}^{m(t)} \frac{dm}{m}$$

The following integrals are well known and result in $\tanh^{-1}(y)$ and $\ln(m)$, respectively:

$$\tanh^{-1}\left(\frac{V(t)}{c}\right) = -\frac{u}{c\sqrt{1-\frac{u^2}{c^2}}}\ln\left(\frac{m(t)}{m_0}\right)$$

Finally:

$$V(t) = \tanh\left(-\frac{u}{\sqrt{c^2-u^2}}\ln(f(t))\right)c$$

**B.2)** In the classical limit $c \to \infty$, the argument of the hyperbolic tangent tends to 0. The function $\tanh(x)$ can be written as:

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Taking $x \to 0$: $e^x \to 1 + x$ (Taylor expansion), so:

$$\tanh(x) \to \frac{1+x-1+x}{1+x+1-x} = x$$

Therefore:

$$V(t) = -\frac{uc}{\sqrt{c^2-u^2}}\ln(f(t))$$

Since $u \ll c$:

$$V(t) = -u\ln(f(t))$$

## Part C: Radiation Cannon

We will use the same technique described in part B. Consider the spacecraft emitting radiation as follows:

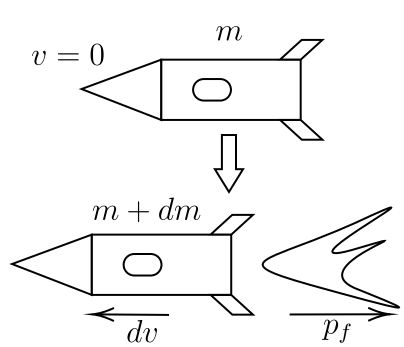



Figure 2.39: Operating principle of a photon-ejection rocket.

From the conservation of linear momentum we can conclude:

$$0 = (m + dm)dv - p_f$$

However, using Einstein's energy-momentum relation for photons, we have $E = pc$:

$$(m + dm)dv = \frac{E}{c}$$

Now, by conservation of the total energy of the system[11]:

$$mc^2 = (m + dm)c^2 + E \to E = -c^2 dm$$

$$\therefore mdv = dmc \to dv = -\frac{dm}{m}c$$

Using the reference frame transformation demonstrated in item B:

$$dV = \left(1 - \frac{V^2}{c^2}\right) dv$$

$$\frac{dV}{1 - \frac{V^2}{c^2}} = -\frac{dm}{m}c$$

Integrating the previous equation:

$$c \tanh^{-1}\left(\frac{V(t)}{c}\right) = -c \ln\left(f(t)\right)$$

$$\therefore V(t) = \tanh\left(\ln\left(f(t)\right)\right)c$$

However, as shown previously:

$$\tanh\left(-\ln(f(t))\right) = \frac{e^{-\ln(f(t))} - e^{\ln(f(t))}}{e^{-\ln(f(t))} + e^{\ln(f(t))}} = \frac{-f(t) + \frac{1}{f(t)}}{f(t) + \frac{1}{f(t)}}$$

Simplifying the expression we find:

$$V(t) = \frac{1 - (f(t))^2}{1 + (f(t))^2}c$$

## Part D: Ion Propulsion

**D.1)** The walls of the ionization chamber are electrified both to attract the remaining electrons from the reaction and to "guide" the $Xe^+$ cations towards the acceleration plates, as well as to prevent collisions with the walls which, due to their high velocity, could damage the chamber structure.

**D.2)** To calculate the electric field between the plates, we use Gauss's law:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

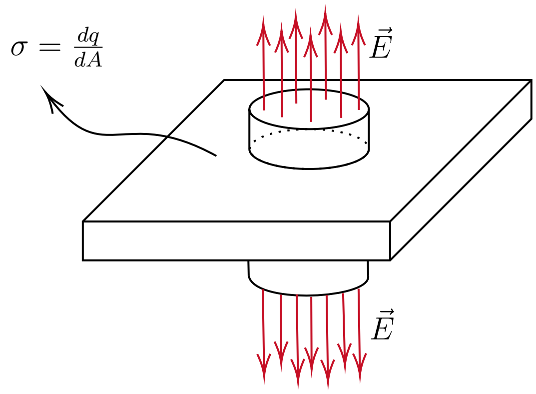



Figure 2.40: Electric field lines for an "infinite" flat surface. For real objects, an infinite dimension is impossible, however, such a phenomenon can be approximated with a plate much larger than the size of the chosen cylinder, so as to cancel edge effects..

Consider the following setup with a cylindrical analysis surface:

Such a cylinder is designed so that its length above and below the plate is equal and it has a cross-sectional area equal to $dA$. The charge inside the surface is simply $dq = \sigma dA$. By symmetry, $\vec{E}$ is constant over the surface, so that:

$$\oint_S \vec{E} \cdot d\vec{A} = \vec{E} \cdot (2dA) = \frac{\sigma dA}{\epsilon_0}$$

$$\therefore \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{r}$$

Where $\hat{r}$ is the unit vector pointing in the direction and sense from the point relative to the plate (so that $\vec{E}$ always "leaves" or "enters" the plate, depending on whether the plate has positive or negative charge, respectively).

Thus, between the plates we have:

$$|\vec{E}| = \frac{\sigma_+ - \sigma_-}{2\epsilon_0}$$

Where $\sigma_+$ and $\sigma_-$ are the surface charge densities of the positive and negative plates, respectively. Note that $\vec{E}$ "points" outward from the engine, perpendicular to the plates!

By the definition of potential:

$$\Delta V = -\int_{r_1}^{r_2} \vec{E} \cdot d\vec{r}$$

$$\therefore |\Delta V| = \frac{\sigma_+ - \sigma_-}{2\epsilon_0}E$$

**Note:** Do not confuse the $E$ in the previous equation with the electric field; in this case, it is the distance between the plates!

**D.3)** The amount of energy given to the ions passing through the acceleration plates is $\Delta U = q\Delta V$, where $q$ is the charge of an ion ($q = 1e^- = 1.602 \cdot 10^{-19}C$):

$$\frac{mv^2}{2} = q\Delta V$$

[11] Since the rocket's speed increased only infinitesimally, its total energy is basically contained in the rest energy

$$\therefore v = \sqrt{2\frac{q}{m}\Delta V}$$

$$v = 85.4 \text{ km/s}$$

**D.4)** Starting from the relation obtained in part B:

$$\Delta v(t) = -\sqrt{2\frac{q}{m}\Delta V}\ln\left(f(t)\right)$$

## Part E: Space Exploration

**E.1)** Each stage will represent a $\Delta v$ for orbital change. A $\Delta v_1$ is needed to enter the transfer orbit to Genibals, performed by the combustion engine (high thrust for vehicle ejection), and a $\Delta v_2$ to exit the transfer orbit and enter the same orbit as Genibals, which will be performed by the ion engine.

A $\Delta v_3$ is needed to land on Genibals, performed by the combustion engine (to counter the high descent speeds), a $\Delta v_4$ to leave Genibals again, also performed by the combustion engine, and a $\Delta v_5$ to enter the transfer orbit, which can be performed by the ion engine.

For the return to Plutão II, a $\Delta v_6$ is needed to enter the same orbit as Plutão II and exit the transfer orbit, performed by the ion engine, and finally a last $\Delta v_7$ to land on Plutão II, performed by the combustion engine.

Thus, the rocket requires **7 propulsion stages**[12].

**E.2)** Notice that, since the rocket is not launched directly from the equator, it will have a certain elevation relative to the orbital plane. Therefore, upon reaching Genibals, this elevation must be taken into account. Analyze the figure beside, which (not to scale) represents the situation:

By triangle similarity, we can relate the data:

$$\frac{R_p \sin(\delta)}{r_p} = \frac{R_g \sin(\delta_g)}{r_g}$$

Therefore:

$$\delta_g = \arcsin\left(\frac{R_p r_g}{R_g r_p}\sin(\delta)\right)$$

Substituting the values, we find $\delta_g = -11°57'$. Note that we changed the latitude value to negative, which becomes clear from the figure.

**E.3)** Consider that at the moment of launch, Plutão II was at angular position 0°. Upon arriving at Genibals, Plutão II will be in a different position and Genibals will be at position 180°. To know the position of Plutão II, we just need to relate the travel time of the outbound trip to the planet's period.

By Kepler's third law, $T \propto a^{3/2}$. The semi-major axis of the transfer orbit is $a = \frac{r_p + r_g}{2}$, so:

[12]A common question is why the arrival at Genibals and Plutão II, as well as departure from Genibals, is divided into two distinct impulses: one to enter/exit orbit and another to land/enter the transfer orbit. This is because these impulses occur in different directions, so if they were applied at once (as if summed), the spacecraft would have too high a velocity and would enter a different elliptical orbit.

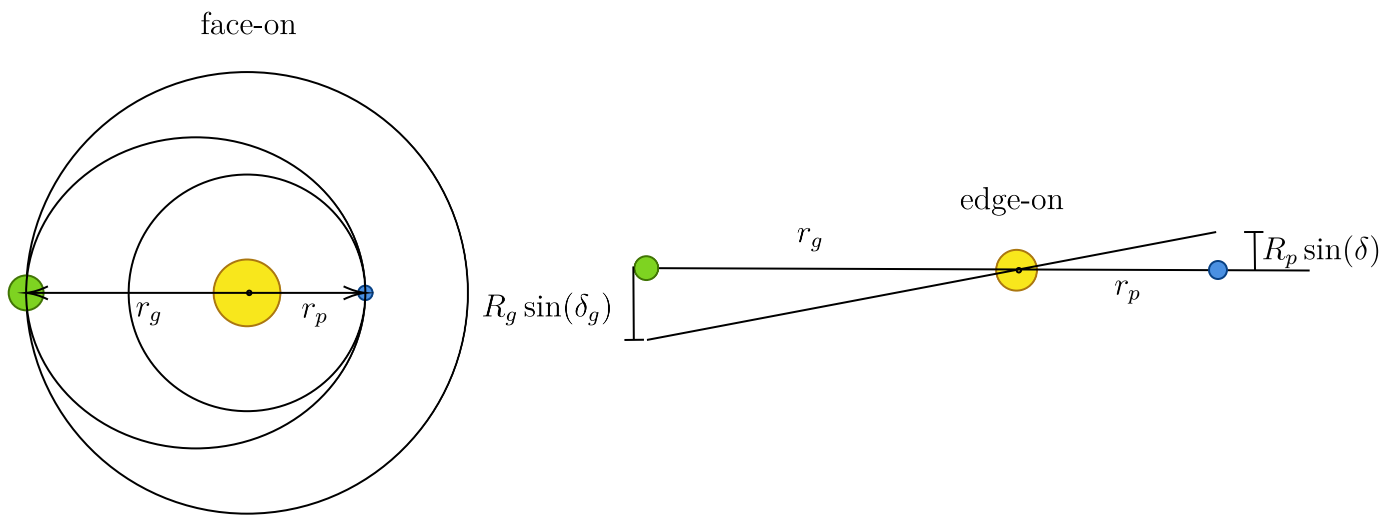



Figure 2.41: Representation of the inclination of the transfer orbit. In the *edge-on* figure, the planets were shown smaller than the departure and arrival lines intentionally to facilitate the visualization of distances..

$$T_{transfer} = T_p \left( \frac{r_p + r_g}{2r_p} \right)^{3/2}$$

However, the outbound travel time is only half of the total period of the transfer orbit, thus: $T_{outbound} = 0.63T_p$. This means that Plutão II traveled an angular distance of $\Delta\phi = 0.63 \cdot 360°$. Therefore, the angle $\theta$ in the figure below is $\theta_1 = 46.8°$.

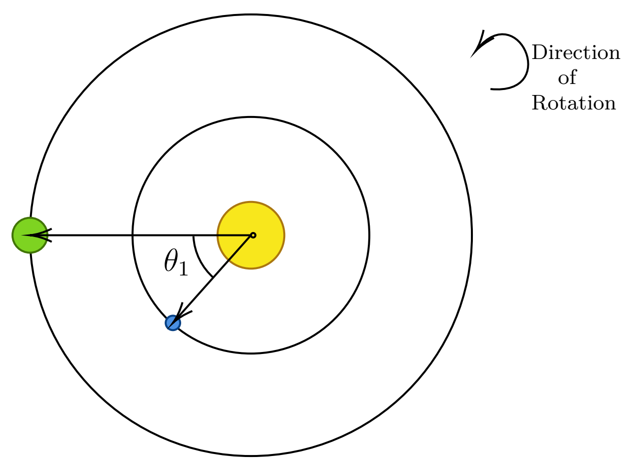



Figure 2.42: Representation of the orbital configuration and determination of the direction of rotation of the planets around the star.

Note that during the course of a one-way trip, Plutão II travels an angle of $\Delta\phi = 180° + \theta_1$. Therefore, for the spacecraft to reach the planet, it must be launched when Plutão II is half a revolution plus $\theta_1$ degrees before the landing position. The landing position is always half a revolution ahead of Genibals, so the orbital configuration at the time of launch for the return trip must be:

Note that the new angle, measured in the direction of rotation, between the planets is $\theta_2 = 360° - \theta_1$. So the next question is: how much time must pass for the angle between the planets to increase to $\theta_2$? From the angular velocity of each planet we have:

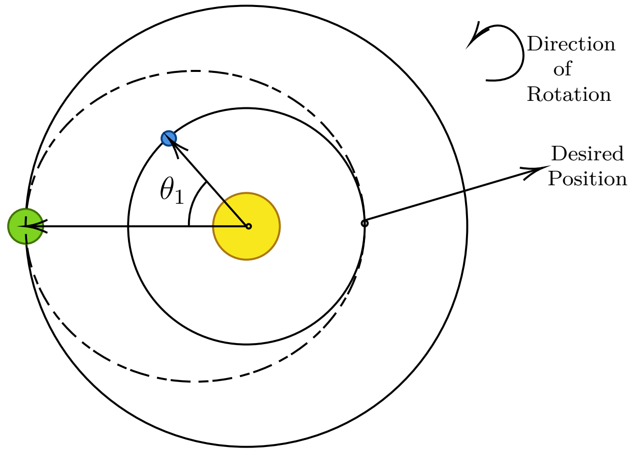



Figure 2.43: Geometry of the transfer curve and prediction of arrival position.

$$\frac{360^\circ}{T_p}\Delta t - \frac{360^\circ}{T_g}\Delta t = \Delta\theta = 360^\circ - 2\theta_1$$

Substituting the values we find:

$$\Delta t = 2.11 T_p = 2.11 \text{ Plutonian years}$$

**E.4)** For this, we must calculate the $\Delta v$ required for each maneuver.

1. $\Delta v_1$

   For the first maneuver, we must not only escape the gravitational field of Plutão II but also increase the spacecraft's velocity to enter the transfer orbit. From the launch location, the tangential velocity is $v_t = \dfrac{2\pi R_p \cos(\delta)}{t_p}$:

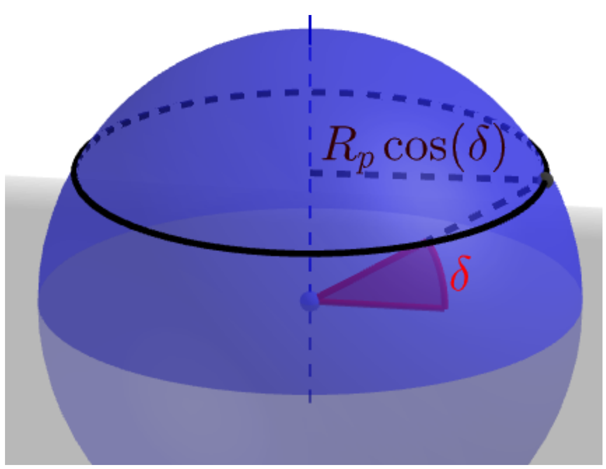



Figure 2.44: Launch latitude and radius of the circle of points with the same latitude.

Since the planet rotates in the same direction as its orbit, this velocity aids the launch. To escape the planet, the total spacecraft speed must be $v = \sqrt{\dfrac{2GM_p}{R_p}}$, therefore $\Delta v_1' =$

$\sqrt{\frac{2GM_p}{R_p}} - \frac{2\pi R_p \cos(\delta)}{t_p} = 11.145$ km/s. After this velocity change, the craft leaves Plutão II's gravitational influence and, relative to Scorp, moves with the same orbital velocity as Plutão II[13]. This means another $\Delta v_1''$ is required:

$$\Delta v_1'' = \sqrt{GM_s\left(\frac{2}{r_p} - \frac{1}{a}\right)} - \sqrt{\frac{GM_s}{r_p}}$$

$$\Delta v_1'' = \sqrt{\frac{2GM_s r_g}{r_p(r_p + r_g)}} - \sqrt{\frac{GM_s}{r_p}}$$

Substituting the values: $\Delta v_1'' = 2.061$ km/s. Therefore, $\Delta v_1 = 13.206$ km/s.

2. $\Delta v_2$

   To leave the transfer orbit and enter Genibals' orbit:

$$\Delta v_2' = \sqrt{\frac{GM_s}{r_g}} - \sqrt{\frac{2GM_s r_p}{r_g(r_p + r_g)}} = 1.918 \text{ km/s}$$

3. $\Delta v_3$

   When the spacecraft reaches the planet, after performing $\Delta v_2$, it would have a speed $v = \sqrt{\frac{2GM_g}{R_g}}$. However, it must arrive with a speed $v' = \frac{2\pi R_g \cos\delta_g}{t_g}$ for a perfect landing (due to the planet's rotation), so $|\Delta v_3| = \sqrt{\frac{2GM_g}{R_g}} - \frac{2\pi R_g \cos\delta_g}{t_g} = 29.704$ km/s.

4. $\Delta v_4$

   For $\Delta v_4$, we simply undo $\Delta v_3$ to leave Genibals: $\Delta v_4 = \Delta v_3 = 29.704$ km/s.

5. $\Delta v_5$

   For $\Delta v_5$, we undo $\Delta v_2$ to enter the transfer orbit: $\Delta v_5 = \Delta v_2 = 1.918$ km/s.

6. $\Delta v_6$

   To enter Plutão II's orbit, the rocket must reduce its velocity:

$$|\Delta v_6| = \sqrt{\frac{2GM_s r_g}{r_p(r_p + r_g)}} - \sqrt{\frac{GM_s}{r_p}} = 2.061 \text{ km/s}$$

7. $\Delta v_7$

   The rocket reaches the planet's surface at $v = \sqrt{\frac{2GM_p}{R_p}}$, but for a safe landing, it must have $v' = \frac{2\pi R_p \cos(\delta)}{t_p}$, so we apply $|\Delta v_7| = \sqrt{\frac{2GM_p}{R_p}} - \frac{2\pi R_p \cos(\delta)}{t_p} = 11.145$ km/s.

[13] This occurs because the rocket ends at rest relative to Plutão II

Now that we have all $\Delta v$ values, we need to find the corresponding mass changes to determine the total mass. To achieve a given $\Delta v$, a certain amount of fluid and its container must be ejected; thus, there are two propulsions, with the fluid propulsion obeying the relation $\Delta v' = u \ln\left(\frac{m_0}{m_F}\right)$.

First, we find a relation between the fluid mass and container mass. The fluid mass is $m_f = \pi(2-0.1)^2 h\rho$ (considering the container thickness), while the container mass is $m_c = \pi(2^2 - 1.9^2)h\rho'$. Note that $\frac{m_c}{m_f} = \frac{0.39(4.01)}{3.61(1.3)}$, meaning $m_c \approx \frac{1}{3}m_f$.

$$\Delta v = \Delta v_f + \Delta v_c$$

$$\Delta v_f = u \ln\left(\frac{m_0}{m_0 - m_f}\right)$$

For $\Delta v_c$, we must consider linear momentum conservation:

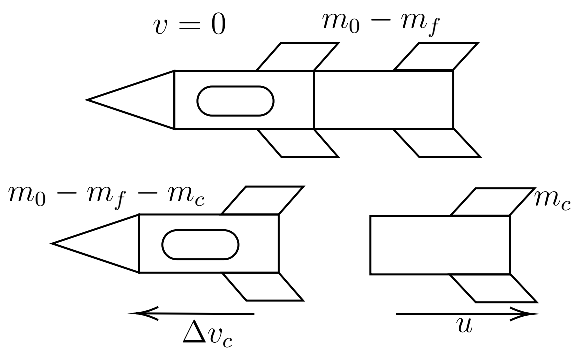



Figure 2.45: Representation of stage change and release of rocket modules to achieve thrust.

$$(m_0 - m_f - m_c)\Delta v_c = m_c u$$

However, we want to express the mass factors only in terms of the final mass $m_F$ and the initial mass $m_0$ for the maneuver: $m_F = m_0 - m_c - m_f = m_0 - \frac{4}{3}m_f$. Therefore, $m_c = \frac{m_0 - m_F}{4}$ and $m_f = \frac{3(m_0 - m_F)}{4}$.

$$\Delta v = u \ln\left(\frac{4m_0}{m_0 + 3m_F}\right) + u\frac{m_0 - m_F}{4m_F}$$

Let $x = \frac{m_0}{m_F}$:

$$4\frac{\Delta v}{u} = 4\ln\left(\frac{4x}{x+3}\right) + x - 1$$

$$\boxed{\therefore x = 4\frac{\Delta v}{u} + 1 - 4\ln\left(\frac{4x}{x+3}\right)} \tag{2.12}$$

For the ion engine, we have a similar scheme. Using the ideal gas equation for the gases inside the chamber:

$$V = \frac{nRT}{P} = \frac{m_{ion}RT}{MP}$$

This is the internal volume of the container, so:

$$V = (1.9)^2\pi \cdot h$$

$$\therefore h = \frac{V}{1.9^2\pi}$$

The volume of the container material is:

$$V' = \pi(2^2 - 1.9^2)\frac{V}{\pi 1.9^2} = \left(\left(\frac{2}{1.9}\right)^2 - 1\right)V$$

Therefore, the final expression for the mass of the container in terms of the gas mass is:

$$\therefore m_{chamber} = \rho'\left(\left(\frac{2}{1.9}\right)^2 - 1\right)\frac{m_{ion}RT}{MP}$$

Substituting the values, we find:

$$m_{chamber} \approx 2m_{ion}$$

Applying the velocity change equations:

$$\Delta v_1 = v_{ion}\ln\left(\frac{m_0}{m_0 - m_{ion}}\right)$$

And

$$(m_0 - m_{ion} - m_{chamber})\Delta v_2 = m_{chamber}v_{ion}$$

$$m_F = m_0 - 3m_{ion} \therefore m_{chamber} = \frac{2}{3}(m_0 - m_F)$$

$$\therefore \Delta v_2 = \frac{2(m_0 - m_F)}{3m_F}v_{ion}$$

Adding the velocity increments and letting $x = \dfrac{m_0}{m_F}$:

$$\Delta v = v_{ion}\left(\ln\left(\frac{3x}{2x+1}\right) + \frac{2}{3}x - \frac{2}{3}\right)$$

$$\boxed{\therefore x = \frac{3\Delta v}{2v_{ion}} + 1 - \frac{3}{2}\ln\left(\frac{3x}{2x+1}\right)} \tag{2.13}$$

Using equations 2.12 and 2.13 and iterating the results, we find the following values of $x$:

$$\begin{cases} x_1 = 9.735 \\ x_2 = 1.022 \\ x_3 = 25.602 \\ x_4 = 25.602 \\ x_5 = 1.022 \\ x_6 = 1.024 \\ x_7 = 7.889 \end{cases}$$

Note that $x_i = \dfrac{m_{i-1}}{m_i}$, where $i$ is the $i$-th stage and $m_i$ is the mass after the $i$-th stage. Considering that only the command module lands at the end, we have:

$$x_1 x_2 \dots x_7 = \frac{m_0}{m_1}\frac{m_1}{m_2} \dots \frac{m_6}{m_7} = \frac{m_0}{m_7}$$

Since $m_7$ is the final mass ($m_7 = 10$ kg):

$$m_0 = 10 \prod_{i=1}^{7} x_i \text{ kg}$$

Finally: $m_0 = 538.4$ tons.

### Part B: Gauss's Law

**B.1)** By definition of flux:

$$d\phi = g \cdot dA \cdot \cos(\theta)$$

Note that $dA \cdot \cos(\theta) = dA_t$, where $dA_t$ is the area perpendicular to the field $\vec{g}$. Where:

$$g = -\frac{G(n)m}{r^{n-1}}$$

Also, since $dA_t = d\Omega(n)r^{n-1}$:

$$d\phi = -\frac{G(n)m}{r^{n-1}} d\Omega(n) r^{n-1}$$

$$d\phi = -G(n)m \, d\Omega(n)$$

**B.2)** Integrating the previous expression yields:

$$\phi = -G(n)m \, \Omega(n)$$

### Part C: Hyper Cosmology

**C.1)** Using the relation found from Gauss's Law, at a distance $aR$ from the center of the universe the field $g$ has constant magnitude:

$$\oint g \, dA = -\Omega(n)G(n)m$$

$$g \oint dA = -\Omega(n)G(n)m$$

Thus:

$$g = -\frac{G(n)m}{(aR)^{n-1}}$$

Note that in this context $g$ also corresponds to the acceleration of a body at this distance:

$$g = \frac{d^2(aR)}{dt^2} = R\frac{d^2a}{dt^2}$$

Hence:

$$\frac{d^2a(t)}{dt^2} = -\frac{G(n)m}{R^n a^{n-1}}$$

**C.2)** An interesting property:

$$\frac{d^2a}{dt^2}da = \dot{a}d\dot{a}$$

which can be verified by substituting $\frac{d^2a}{dt^2} = \frac{d\dot{a}}{dt}$.
Thus:

$$\frac{d^2a(t)}{dt^2}da = \dot{a}d\dot{a} = -\frac{G(n)m}{R^n a^{n-1}}da$$

Integrating:

$$\frac{G(n)m}{R^n a^{n-2}(n-2)} = \frac{\dot{a}^2}{2} + C$$

For a flat universe, $C = 0$. Substituting also $m = \frac{\Omega(n)}{n}a^n R^n \rho$:

$$\frac{G(n)m}{R^n a^{n-2}(n-2)} = \frac{\dot{a}^2}{2}$$

Simplifying, we obtain:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2G(n)}{n(n-2)}\Omega(n)\rho$$

**C.3)** From the first law of thermodynamics:

$$dQ = dU + dW$$

where $dW$ is the work done by the universe, $dU$ the internal energy change (mainly from rest energy), and $dQ$ the heat added. In the adiabatic regime: $dQ = 0$.

$$dU + dW = 0$$

With $dU = c^2 dm$ and $dW = PdV$:

$$c^2 dm + PdV = 0$$

Since $m = \rho V$, $dm = V d\rho + \rho dV$:

$$V d\rho\, c^2 + \rho c^2 dV + PdV = 0$$

Dividing by $dV$:

$$Vc^2\frac{d\rho}{dV} + \rho c^2 + P = 0$$

With $\frac{d\rho}{dV} = \frac{\dot{\rho}}{\dot{V}}$:

$$\frac{\dot{\rho}}{\dot{V}}Vc^2 + \rho c^2 + P = 0$$

Multiplying by $\frac{\dot{V}}{V}$:

$$\dot{\rho}c^2 + (P + \rho c^2)\frac{\dot{V}}{V} = 0$$

Since $V \propto a^n$, $\dot{V}/V = n\dot{a}/a$:

$$\dot{\rho}c^2 + n(P + \rho c^2)\frac{\dot{a}}{a} = 0$$

Hence:

$$\therefore \gamma = n$$

**C.4)** For both cases, the energy density depends on $a^k$:

$$\rho_{\text{MNR}} = \rho_0 a^k$$

Solving the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2G(n)}{n(n-2)}\Omega(n)\rho_0 a^k$$

$$\dot{a}^2 = \frac{2G(n)}{n(n-2)}\Omega(n)\rho_0 a^{k+2}$$

$$a^{-\frac{k}{2}-1}da = \sqrt{\frac{2G(n)}{n(n-2)}\Omega(n)\rho_0}dt$$

Integrating gives:

$$-\frac{2}{k}a^{-\frac{k}{2}} = \sqrt{\frac{2G(n)}{n(n-2)}\Omega(n)\rho_0}t$$

For a universe dominated by non-relativistic matter, total energy remains constant but energy density decreases with $a^n$: $\rho \propto a^{-n}$, so $k = -n$:

$$a = \left(\frac{nG(n)}{2(n-2)}\Omega(n)\rho_0\right)^{1/n} t^{2/n}$$

For a universe dominated by radiation, total energy decreases with $a$ due to the redshift of radiation, and energy density also decreases with $a^n$: $\rho \propto a^{-n-1}$, so $k = -n-1$:

$$a = \sqrt[n+1]{\frac{(n+1)^2G(n)}{2n(n-2)}\Omega(n)\rho_0}t^{\frac{2}{n+1}}$$

# 2.28 Charge-Coupled Device

## Part A: A Bit of Quantum Mechanics

**A.1)** For photons to excite electrons, it is necessary that $h\nu \geq E_\phi$, thus: $h\dfrac{c}{\lambda} \geq 1.1 \cdot 1.602 \cdot 10^{-19}$ J. Finally, performing the calculations: $\lambda \leq 1.128\mu$m.

**A.2)** Isolating the problem along the $x$-axis, we have the following relation:

$$-\frac{\hbar}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi = E\psi$$

Since there are three different regions, we separate the equation and solve it in each one: $x < 0$ (I), $0 < x < L$ (II) and $x > L$ (III). For cases (I) and (III), note that $V(x) > E_\phi$, thus:

$$-\frac{\hbar}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = -(V_0 - E)\psi$$

$$\therefore \frac{\partial^2\psi(x)}{\partial x^2} = \frac{2m(V_0 - E)}{\hbar^2}\psi$$

Assigning $k_0 = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}}$ and solving for $\psi(x)$, we find a solution composed of exponentials for each region:

$$\begin{cases} \psi_{(I)}(x) = Ae^{k_0x} + A'e^{-k_0x} \\ \psi_{(III)}(x) = Ce^{-k_0x} + C'e^{k_0x} \end{cases}$$

However, in cases where $x \to -\infty$ for region (I) and $x \to \infty$ for region (III), the functions diverge to plus or minus infinity if the terms $A'$ and $C'$ are non-zero, which cannot occur under the problem conditions! Therefore, we conclude that these terms must necessarily be zero!

$$\therefore \begin{cases} \psi_{(I)}(x) = Ae^{k_0x} \\ \psi_{(III)}(x) = Ce^{-k_0x} \end{cases}$$

For region (II), since $E_\phi > V_P$, a slight change must be made:

$$-\frac{\hbar}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = (E_\phi - V_P)\psi$$

$$\therefore \frac{\partial^2\psi(x)}{\partial x^2} = -\frac{2m(E_\phi - V_P)}{\hbar^2}\psi$$

Notice that we now have an ODE of the type $y'' = -k_1^2 y$, assigning $k_1 = \sqrt{\dfrac{2m(E_\phi - V_P)}{\hbar^2}}$, which results in trigonometric solutions:

$$\psi_{(II)}(x) = B_1\sin(k_1x) + B_1\cos(k_1x)$$

**A.3)** Since it is necessary that $\psi_{(I)}(0) = \psi_{(II)}(0)$ and $\psi_{(II)}(L) = \psi_{(III)}(L)$, as well as $\frac{d\psi_{(I)}}{dx}(0) = \frac{d\psi_{(II)}}{dx}(0)$ and $\frac{d\psi_{(II)}}{dx}(L) = \frac{d\psi_{(III)}}{dx}(L)$, it is possible to relate the values in the following system of equations:

$$\begin{cases} A = B_2 \\ Ak_0 = B_1 k_1 \\ B_1 \sin(k_1 L) + B_2 \cos(k_1 L) = Ce^{-k_0 L} \\ B_1 k_1 \cos(k_1 L) - B_2 k_1 \sin(k_1 L) = -k_0 C e^{-k_0 L} \end{cases}$$

The first relations are trivial: $\boxed{B_2 = A}$ and $\boxed{B_1 = \frac{k_0}{k_1} A}$. To find the relation for $C$, just substitute these into the third or fourth equation:

$$\boxed{C = e^{k_0 L} \left( \frac{k_0}{k_1} \sin(k_1 L) + \cos(k_1 L) \right) A} \qquad \text{from the third}$$

$$\boxed{C = e^{k_0 L} \left( \frac{k_1}{k_0} \sin(k_1 L) - \cos(k_1 L) \right) A} \qquad \text{from the fourth}$$

**A.4)** It is left as an exercise for the reader to prove that $C = C$. Knowing this, we can relate:

$$\frac{k_0}{k_1} \sin(k_1 L) + \cos(k_1 L) = \frac{k_1}{k_0} \sin(k_1 L) - \cos(k_1 L)$$

Simplifying, we find:

$$\tan(k_1 L) = \frac{2k_1 k_0}{k_1^2 - k_0^2}$$

The period of the tangent function is $\pi$, which means $\tan(\theta + p\pi) = \tan(\theta)$, with $p$ an integer. Thus:

$$\tan(k_1 L + p\pi) = \frac{2k_1 k_0}{k_1^2 - k_0^2}$$

$$\therefore L = \frac{1}{k_1} \left( -p\pi + \arctan\left( \frac{2k_1 k_0}{k_1^2 - k_0^2} \right) \right)$$

Since $p$ is an integer, $-p$ is also an integer (call $n = -p$):

$$\therefore L = \frac{1}{k_1} \left( n\pi + \arctan\left( \frac{2k_1 k_0}{k_1^2 - k_0^2} \right) \right)$$

**A.5)** First, it is necessary to assign numerical values to the variables. Knowing $V_0$, $V_P$, and $E_\phi$, it is possible to find $k_0$ and $k_1$. Applying the values, we find $k_0 = 7.8153 \cdot 10^9 m^{-1}$ and $k_1 = 2.5600 \cdot 10^9 m^{-1}$.

Thus, we can find the value of $L$:

$$\tan(k_1 L) = -0.73387$$

Therefore, since $L > 0$, for the tangent to be negative, the first positive angle is $\pi + \tan^{-1}(-0.73387)$, using $n = 1$. Finally, we find $\boxed{L = 9.79879 \cdot 10^{-10} m}$. This size is approximately equal to the covalent atomic radius of beryllium!

This reveals that this model is quite far from reality, since the lengths of potential barriers have an order of magnitude of a few micrometers, not nanometers ($L \approx 1nm$).

**A.6)** The probability $P_{(II)}$ that the particle is within the interval $[0,L]$ can be found from the following relation:

$$P_{(II)} = \frac{P(0,L)}{P(-\infty,\infty)}$$

This is because the total probability $P(-\infty,\infty)$ must equal 1:

$$P_{(II)} = \frac{\int_0^L \psi^2_{(II)}(x)dx}{\int_{-\infty}^{\infty} \psi^2(x)dx}$$

Note that the integral in the denominator can be expressed as:

$$\int_{-\infty}^{\infty} \psi^2(x)dx = \int_{-\infty}^{0} \psi^2_{(I)}(x)dx + \int_0^L \psi^2_{(II)}(x)dx + \int_L^{\infty} \psi^2_{(III)}(x)dx$$

$$\int_{-\infty}^{\infty} \psi^2(x)dx = A^2 \int_{-\infty}^{0} e^{2k_0 x}dx + B_1^2 \int_0^L \sin^2(k_1 x)dx + 2B_1B_2 \int_0^L \sin(k_1 x)\cos(k_1 x)dx +$$

$$B_2^2 \int_0^L \cos^2(k_1 x)dx + C^2 \int_L^{\infty} e^{-2k_0 x}dx$$

Solving the integrals, we find:

$$\int_{-\infty}^{\infty} \psi^2(x)dx = A^2 \frac{1}{2k_0} + B_1^2\left(\frac{L}{2} - \frac{\sin(2k_1L)}{4k_1}\right) + B_1B_2\frac{\sin^2(k_1L)}{k_1} + B_2^2\left(\frac{L}{2} + \frac{\sin(2k_1L)}{4k_1}\right) + C^2\frac{e^{-2k_0L}}{2k_0}$$

While the numerator integral is

$$\int_{-\infty}^{\infty} \psi^2_{(II)}(x)dx = B_1^2\left(\frac{L}{2} - \frac{\sin(2k_1L)}{4k_1}\right) + B_1B_2\frac{\sin^2(k_1L)}{k_1} + B_2^2\left(\frac{L}{2} + \frac{\sin(2k_1L)}{4k_1}\right)$$

Notice that when taking the ratio, the value of $A$ is irrelevant since both numerator and denominator contain $A^2$ as a multiplicative factor. Substituting the given values, we conclude:

$$\boxed{P_{(II)} \approx 98\%}$$

**A.7)** Since the tunneling effect calculated previously is symmetric and independent in the three dimensions, we can assume that the probability of finding the particle confined in the well is 98% in any direction. Thus, the total storage efficiency is simply $0.98 \cdot 0.98 \cdot 0.98$:

$$\boxed{\epsilon \approx 94\%}$$

## Part B: Spectral Analysis

**B.1)** The first correction to be made in the problem's relation is the distance from the CCD to the blackbody. Since the radiation emission is homogeneous and isotropic, we can assume that intensity decreases with the square of the distance. Therefore, the correction factor $\frac{R^2}{d^2}$ should be multiplied.

Thus, we find the flux at a distance $d$ from the body; multiplying this by the CCD area gives the received power:

$$dP(\nu) = \frac{2\pi A h \nu^3 R^2}{c^2 d^2} \frac{1}{e^{\frac{h\nu}{k_b T}} - 1} d\nu$$

**B.2)** First, calculate the number of photons capable of generating electrons that hit the CCD and then multiply by the electron charge and the efficiency factor. Remember that a photon only generates electrons in the semiconductor if its energy is greater than the band gap $E_\phi$: $\nu_0 > \frac{E_\phi}{h}$.

In a time interval $\Delta t$, the energy hitting the CCD in the frequency interval $d\nu$ is:

$$dE(\nu) = \frac{2\pi A h \nu^3 R^2}{c^2 d^2} \frac{1}{e^{\frac{h\nu}{k_b T}} - 1} \Delta t d\nu$$

This energy consists of $n(\nu)$ photons of frequency $\nu$ hitting the CCD:

$$dE(\nu) = n(\nu) h\nu$$

$$\therefore n(\nu) = \frac{2\pi A \nu^2 R^2}{c^2 d^2} \frac{1}{e^{\frac{h\nu}{k_b T}} - 1} \Delta t d\nu$$

Integrating this result from $\nu_0$ and multiplying by $\beta$ gives the number of photons, hence electrons, generated:

$$N_{e^-} = \frac{2\pi A R^2}{c^2 d^2} \beta \Delta t \int_{\nu_0}^{\infty} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}} - 1} d\nu$$

The total charge generated is the number of electrons times the electron charge:

$$\Delta Q = \frac{2\pi A R^2 e}{c^2 d^2} \beta \Delta t \int_{\nu_0}^{\infty} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}} - 1} d\nu$$

$$\therefore I = \frac{2\pi A R^2 e}{c^2 d^2} \beta \int_{\nu_0}^{\infty} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}} - 1} d\nu$$

**B.3)** Using $c = \nu\lambda$, we find $\nu_1 = \frac{c}{\lambda_1}$ and $\nu_2 = \frac{c}{\lambda_2}$. Also note that if $\lambda_2 > \lambda_1 \implies \nu_2 < \nu_1$, so the integration limits must be inverted:

$$I = \frac{2\pi A R^2 e}{c^2 d^2} \beta \int_{\nu_2}^{\nu_1} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}} - 1} d\nu$$

**B.4)** Using Pogson's equation: $m_U - m_V = -2.5\log\left(\frac{F_U}{F_V}\right)$.

Note that $F_U \propto I_U$ (flux is directly proportional to the generated current). For the same body:

$$I \propto \int_{\nu_2}^{\nu_1} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}} - 1} d\nu$$

Thus:

$$M(U,V) = -2.5\log\left(\frac{\int_{\nu_{U,2}}^{\nu_{U,1}} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}}-1} d\nu}{\int_{\nu_{V,2}}^{\nu_{V,1}} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}}-1} d\nu}\right) = -2.5\log\left(\int_{\nu_{U,2}}^{\nu_{U,1}} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}} - 1} d\nu\right) + 2.5\log\left(\int_{\nu_{V,2}}^{\nu_{V,1}} \frac{\nu^2}{e^{\frac{h\nu}{k_b T}} - 1} d\nu\right)$$

The integration limits $\nu_{U,1}, \nu_{U,2}, \nu_{V,1}, \nu_{V,2}$ are constants, so the only variable in the equation is $T$ (with $\nu$ being just the integration variable, not an independent variable).

Substituting the values from the problem:

$$M(U,V) = -2.5\log\left(\int_{7.53\times10^{14}}^{9.03\times10^{14}} \frac{\nu^2}{e^{4.80\times10^{-11}\frac{\nu}{T}} - 1} d\nu\right) + 2.5\log\left(\int_{5.04\times10^{14}}^{5.91\times10^{14}} \frac{\nu^2}{e^{4.80\times10^{-11}\frac{\nu}{T}} - 1} d\nu\right) \tag{2.14}$$

Analyze this function qualitatively from the graph in Figure 2.46:

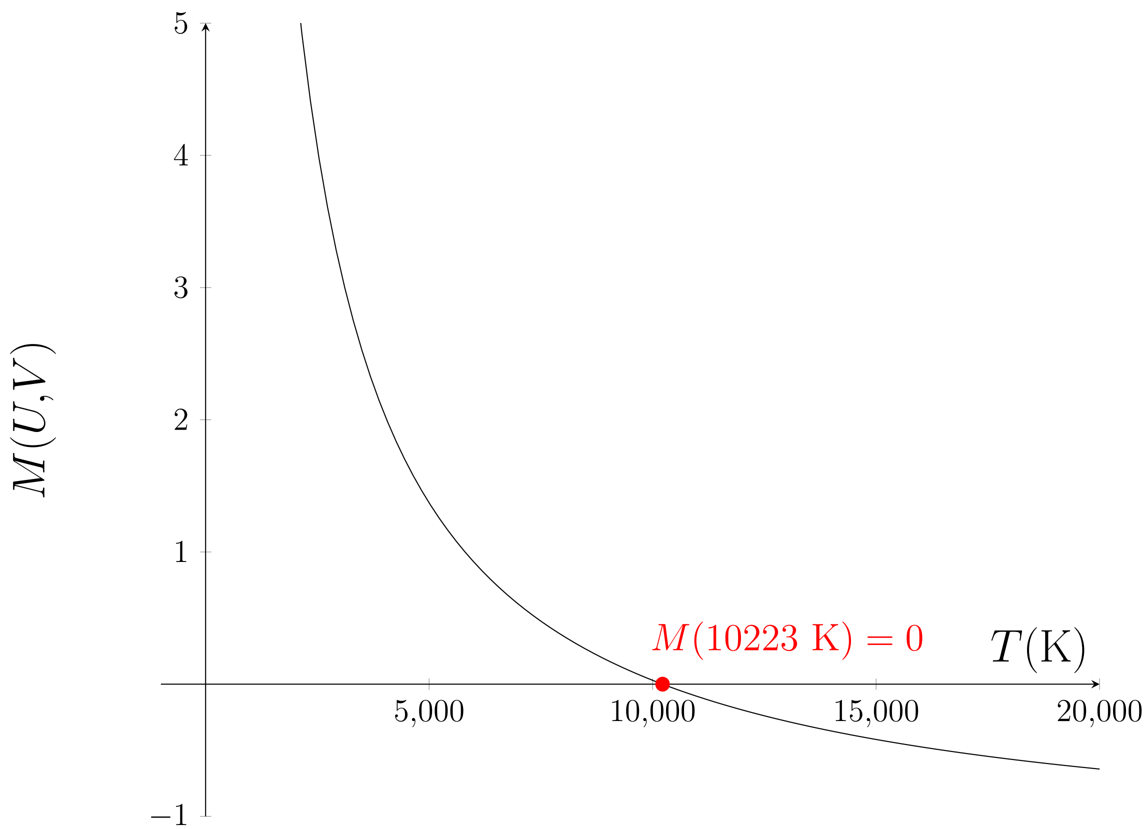



Figure 2.46: Graph of $M(U,V)$ *vs* $T$, for the relation 2.14.

**B.5)** From the graph in Figure 2.47, we find that $M(U,V) = 0$ for $T \approx 10\,220$ K. Notice that for any $T > 10\,220$ K, the value of $M(U,V) < 0$, which indicates that the magnitude of the body in the ultraviolet band is greater than in the visible band. This means that most of the emitted radiation intensity is concentrated near blue and violet, giving the body a bluish coloration.

On the other hand, for lower temperatures ($T < 10\,220$ K), the magnitude in the visible band is greater than in the ultraviolet, which means the body appears more reddish.

From the analysis of the graph, the Sun's UV color index can be determined as $M_\odot(U,V) \approx 1.07$.

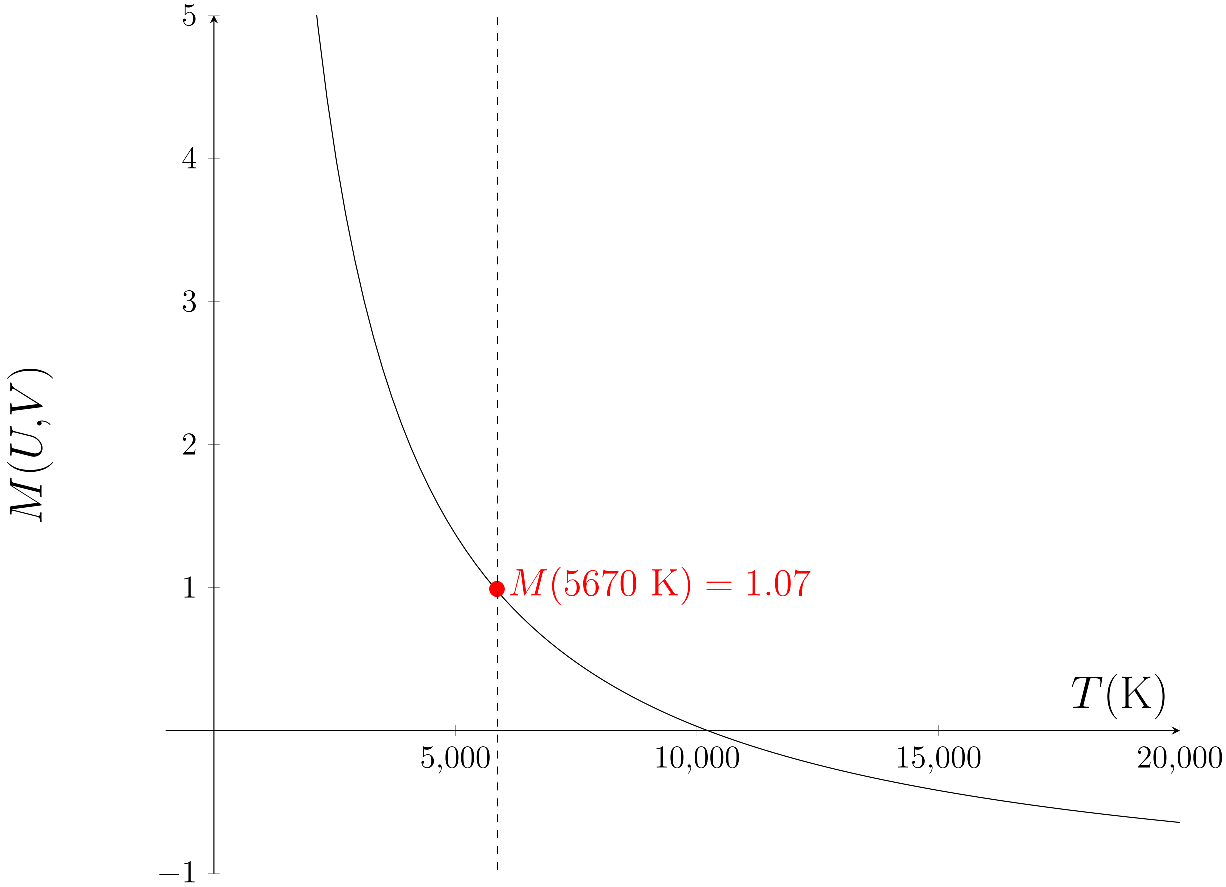



Figure 2.47: Graph of $M(U,V)$ *vs* $T$, for the relation 2.14 and the value of $M(5670\text{ K})$.

# 2.29 Pulsars

## Part A: Let There Be Light!

**A.1)** For the electromagnetic wave described in the statement, it is possible to describe its electric and magnetic fields as functions of position $x$ and time $t$ such that:

$$\begin{cases} \vec{E} = E(x,t)\hat{j} \\ \vec{B} = B(x,t)\hat{k} \end{cases}$$

So that:

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E(x,t) & 0 \end{vmatrix}$$

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial B(x,t)}{\partial t}\hat{k}$$

Therefore, it is possible to write Maxwell's relation 3 as:

$$\frac{\partial E(x,t)}{\partial x}\hat{k} - \frac{\partial E(x,t)}{\partial z}\hat{i} = -\frac{\partial B(x,t)}{\partial t}\hat{k}$$

Since $E(x,t)$ does not depend on $z$, we can state that $\dfrac{\partial E(x,t)}{\partial z} = 0$. Therefore, we have:

$$\frac{\partial E(x,t)}{\partial x} = -\frac{\partial B(x,t)}{\partial t} \tag{2.15}$$

Similarly, for Maxwell's relation 4:

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & B(x,t) \end{vmatrix} = \frac{\partial B(x,t)}{\partial y}\hat{i} - \frac{\partial B(x,t)}{\partial x}\hat{j}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{\partial E(x,t)}{\partial t}\hat{j}$$

Therefore, knowing that $\dfrac{\partial B(x,t)}{\partial y} = 0$, we can conclude:

$$\frac{\partial B(x,t)}{\partial x} = -\mu_0\epsilon_0\frac{\partial E(x,t)}{\partial t} \tag{2.16}$$

We will use the fact that $E(x,t)$ and $B(x,t)$ are smooth functions[14], so it is possible to apply Fubini's theorem, according to which, given $f(x,y) : D \to \mathbb{R}$ a $C^2$ function:

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

Thus, taking the partial derivative with respect to time of equation 2.15 and the partial derivative with respect to $x$ of equation 2.16, we get:

[14] That is, functions of class $C^2$.

$$\begin{cases} \dfrac{\partial^2 E(x,t)}{\partial t \partial x} = -\dfrac{\partial^2 B(x,t)}{\partial t^2} \\ \\ \dfrac{\partial^2 B(x,t)}{\partial x^2} = -\mu_0 \epsilon_0 \dfrac{\partial^2 E(x,t)}{\partial x \partial t} \end{cases}$$

Equating the partial derivatives of $E(x,t)$ by Fubini's theorem, we have:

$$\therefore \frac{\partial^2 B(x,t)}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B(x,t)}{\partial t^2}$$

Therefore, from the fundamental wave equation, we can conclude that the propagation speed of the electromagnetic wave is always $c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$.

**A.2)** Note that for this type of wave function (the same relation holds for $B(x,t)$):

$$\frac{\partial E}{\partial x} = -\frac{1}{c}\frac{\partial E}{\partial t}$$

Thus, from equations 2.15 and 2.16, the maxima and minima of the magnetic and electric field oscillations must occur simultaneously[15]. This implies that the functions must be coherent with each other; hence we conclude that $k_1 = k_2$, $\omega_1 = \omega_2$, and $\phi_1 = \phi_2$ (for simplicity, we assume $\phi_1 = 0$).

**A.3)** From equation 2.15:

$$E_0 k \cos(kx - \omega t) = B_0 \omega \cos(kx - \omega t)$$

$$\therefore E_0 = \frac{\omega}{k} B_0$$

Since $\omega = 2\pi f$ and $k = \dfrac{2\pi}{\lambda}$, it follows that $\dfrac{\omega}{k} = \lambda f = c$:

$$\therefore E_0 = cB_0$$

Therefore, we can see that $|\vec{E}(x,t)| = |\vec{B}(x,t)|$, $\forall x,t \in \mathbb{R}$.

## Part B: Variable Dipole

**B.1)** Consider figure 2.48, which addresses the situation presented, taking into account the problem conditions.

Note that, in the perturbation region, it is possible to compare the intensities of the radial and transverse components of the electric field using the distances in the formed triangle, where $E_r$ is proportional to $E_\perp$ just as $c \cdot dt$ is proportional to $dv \cdot T \cdot \sin(\theta)$. Therefore:

$$\frac{E_\perp}{E_r} = \frac{dv \cdot T \cdot \sin(\theta)}{c \cdot dt} = \frac{aT \sin(\theta)}{c}$$

[15]If one derivative is zero, the other must also be zero.

Paz e Amor

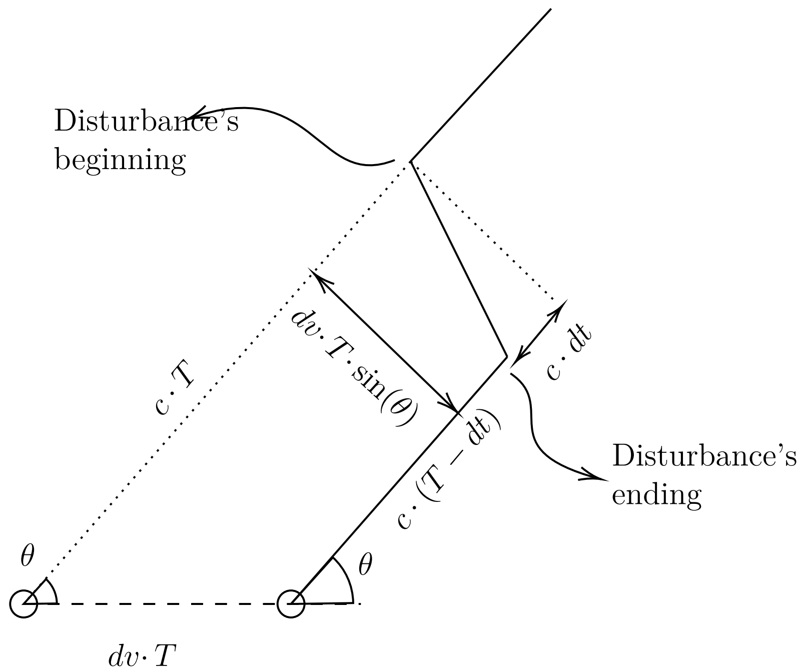



Figure 2.48: Variation of the electric field lines due to the acceleration of a charged particle.

**B.2)** Knowing that $E_r = \dfrac{kq}{r^2}$, we have:

$$E_\perp = \frac{kqaT\sin(\theta)}{cr^2}$$

Since the distance $r$ from the point of field application is $r = c\cdot T \implies T = \dfrac{r}{c}$, as stated in the problem, it is possible to write $E_\perp$ as:

$$E_\perp = \frac{kqa\sin(\theta)}{c^2 r}$$

**B.3)** Note the convenient fact that $E_\perp \propto \dfrac{1}{r}$ and not $\dfrac{1}{r^2}$ as $E_r$, so that, for a region sufficiently far from the dipole, we can assert that $E_\perp >> E_r$ and therefore it corresponds to the total electric field in the region (allowing us to neglect the radial component). Even considering the particle that is not accelerated (the one with charge $-q$), the electric field from it is also proportional to $r^{-2}$, so the only "relevant" component at large distances from the dipole is the transverse one.

**B.4)** Since this is an electromagnetic wave, the magnetic field is perpendicular to the electric field with magnitude such that $B = E/c$, therefore the radiation flux can be written as:

$$S = \frac{1}{\mu_0 c}E^2 = \frac{k^2 q^2 a^2 \sin^2(\theta)}{\mu_0 c^5 r^2}$$

**B.5)** In order to find the total power emitted throughout space, one just needs to integrate this result over all surface elements. For an interval $[\theta, \theta + d\theta]$, there exists a spherical region of area $2\pi r^2 \sin(\theta) d\theta$, so the integral becomes:

$$P = \oint SdA = \int_0^\pi S(2\pi r^2 \sin(\theta))d\theta = \frac{2\pi k^2 q^2 a^2}{\mu_0 c^5} \int_0^\pi \sin^3(\theta)d\theta$$

Note that $\sin^3(\theta)d\theta = \sin^2(\theta) \cdot (\sin(\theta)d\theta) = (1 - \cos^2(\theta))(-d\cos(\theta))$, therefore:

$$\int_0^\pi \sin^3(\theta)d\theta = \int_{-1}^1 1 - x^2 dx = \frac{4}{3}$$

$$\therefore P = \frac{8\pi k^2 q^2 a^2}{3\mu_0 c^5}$$

Substituting $k = \dfrac{1}{4\pi\epsilon_0}$, the particle's acceleration $a = \Omega^2 R$, and the particle's charge $q = \dfrac{p}{R}$:

$$P = \frac{1}{16\pi^2\epsilon_0^2} \frac{p^2}{R^2} \frac{8\pi\Omega^4 R^2}{3\mu_0 c^5} = \frac{p^2\Omega^4}{6\pi\epsilon_0^2\mu_0 c^5}$$

Since $\mu_0\epsilon_0 = c^{-2}$:

$$P_{ele} = \frac{p^2\Omega^4}{6\pi\epsilon_0 c^3}$$

## Part C: Pulsars

**C.1)** The magnetic north pole is the one where $\theta$ in equation 1.1 equals 0, so that:

$$B = \frac{\mu_0 m}{2\pi R^3}$$

Therefore, its magnetic dipole moment is $m = \dfrac{2\pi}{\mu_0} BR^3$. Note, however, that the pulsar's magnetic dipole moment is inclined at an angle $\theta$, so we can decompose it into two moment components: one parallel to the rotation axis and one perpendicular, which rotates in uniform circular motion with period $P$. Only this last component is responsible for dissipating energy, so the effective dissipating moment is $m\sin(\theta)$. Thus, the dissipated power can be calculated as:

$$P_{mag} = \frac{\mu_0}{6\pi c^3}\left(\frac{2\pi}{\mu_0} BR^3 \sin(\theta)\right)^2 \left(\frac{2\pi}{P}\right)^4$$

$$P_{mag} = \frac{\mu_0}{6\pi c^3}\frac{4\pi^2}{\mu_0^2} B^2 R^6 \sin^2(\theta) \frac{16\pi^4}{P^4} = \frac{32\pi^5 B^2 R^6 \sin^2(\theta)}{3\mu_0 c^3 P^4}$$

**C.2)** The rotational kinetic energy of the pulsar is $K = \dfrac{1}{2} I\Omega^2$, where $I = \dfrac{2}{5} MR^2$ is its moment of inertia, so that:

$$K = \frac{8\pi^2 MR^2}{5P^2}$$

$$\therefore \frac{dK}{dt} = -\frac{8\pi^2 MR^2}{5P^3}\dot{P}$$

Equating the values (and ignoring the sign, which only indicates energy loss), we obtain:

$$\frac{8\pi^2 M R^2}{5P^3}\dot{P} = \frac{32\pi^5 B^2 R^6 \sin^2(\theta)}{3\mu_0 c^3 P^4}$$

$$B = \frac{c}{2\pi R^2 \sin(\theta)}\sqrt{\frac{3\mu_0 c M P \dot{P}}{5\pi}}$$