$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

$x = r\cos\theta$ $\qquad$ $\sqrt{x^2+y^2} = r$

$y = r\sin\theta$ $\qquad$ $\tan\theta = \frac{y}{x}$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{2x}{2\sqrt{x^2+y^2}} + \frac{\partial U}{\partial \theta}\left(\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}\right)$$

$$= \frac{\partial U}{\partial r}\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial U}{\partial \theta}\frac{y}{x^2+y^2}$$

$$= \frac{\partial U}{\partial r}\frac{r\cos\theta}{r} - \frac{\partial U}{\partial \theta}\frac{r\sin\theta}{r^2}$$

$$= \frac{\partial U}{\partial r}\underbrace{\cos\theta}_{\frac{\partial r}{\partial x}} - \frac{\partial U}{\partial \theta}\underbrace{\frac{\sin\theta}{r}}_{\frac{\partial \theta}{\partial x}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial x}\right)\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial}{\partial r}\left[\frac{\partial U}{\partial r}\cos\theta - \frac{\partial U}{\partial \theta}\frac{\sin\theta}{r}\right]\left[\cos\theta\right] - \dots$$

$$\dots - \frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r}\cos\theta - \frac{\partial U}{\partial \theta}\frac{\sin\theta}{r}\right)\left[\frac{1}{r}\sin\theta\right]$$

$$= \left(\frac{\partial^2 U}{\partial U^2}\cos\theta - \frac{\partial^2 U}{\partial r\partial\theta}\frac{1}{r}\sin\theta + \frac{\partial U}{\partial \theta}\frac{1}{r^2}\sin\theta\right)\cos\theta - \dots$$

$$\dots - \left(\frac{\partial^2 U}{\partial\theta\partial r}\cos\theta - \frac{\partial U}{\partial r}\sin\theta - \frac{\partial^2 U}{\partial \theta^2}\frac{1}{r}\sin\theta - \frac{\partial U}{\partial \theta}\frac{1}{r}\cos\theta\right)\frac{1}{r}\sin\theta$$

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Reagrupando

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial r^2}\cos^2\theta - \frac{\partial^2 U}{\partial r\partial\theta}\frac{2\sin\theta\cos\theta}{r} + \frac{\partial U}{\partial \theta}\frac{2}{r^2}\sin\theta\cos\theta + \frac{\partial U}{\partial r}\frac{1}{r}\sin^2\theta + \frac{\partial^2 U}{\partial \theta^2}\frac{1}{r^2}\sin^2\theta$$

DD MM AA

Haciendo el mismo procedimiento para $\frac{\partial^2 U}{\partial y^2}$ ...

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2}\operatorname{sen}^2\theta + \frac{\partial^2 U}{\partial r\partial\theta}\frac{2}{r}\operatorname{sen}\theta\cos\theta - \frac{\partial U}{\partial\theta}\frac{2}{r^2}\operatorname{sen}\theta\cos\theta + \frac{\partial U}{\partial r}\frac{1}{r}\cos^2\theta + \frac{\partial^2 U}{\partial\theta^2}\frac{1}{r^2}\cos^2\theta$$

Haciendo $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \nabla^2 U$

$$\to \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} = \nabla^2 U$$

Con diferencias finitas:

$$\frac{\partial U}{\partial r} = \frac{U^l_{i,j} - U^l_{i-1,j}}{\Delta r}$$

$$\frac{\partial^2 U}{\partial\theta^2} = \frac{U^l_{i,j+1} - 2U^l_{i,j} + U^l_{i,j-1}}{(\Delta\theta)^2}$$

$$\frac{\partial^2 U}{\partial r^2} = \frac{U^l_{i+1,j} - 2U^l_{i,j} + U^l_{i-1,j}}{(\Delta r)^2}$$

$$\frac{\partial^2 U}{\partial t^2} = \frac{U^{l+1}_{i,j} - 2U^l_{i,j} + U^{l-1}_{i,j}}{(\Delta t)^2}$$

Ecuación de onda con $\alpha = \beta$ (desplazamiento homogéneo)

$$\frac{U^{l+1}_{i,j} - 2U^l_{i,j} + U^{l-1}_{i,j}}{(\Delta t)^2} = \alpha^2\left[\frac{U^l_{i+1,j} - 2U^l_{i,j} + U^l_{i-1,j}}{(\Delta r)^2} + \cdots\right.$$

$$\left.\cdots + \frac{1}{r_i^2}\frac{U^l_{i,j+1} - 2U^l_{i,j} + U^l_{i,j-1}}{(\Delta\theta)^2} + \frac{1}{r_i}\frac{U^l_{i,j} - U^l_{i-1,j}}{\Delta r}\right]$$

$$\to U^{l+1}_{i,j} = (\Delta t)^2\alpha^2\left[\frac{U^l_{i+1,j} - 2U^l_{i,j} + U^l_{i-1,j}}{(\Delta r)^2} + \cdots\right.$$

$$\cdots + \frac{1}{r_i^2}\frac{U^l_{i,j+1} - 2U^l_{i,j} + U^l_{i,j-1}}{(\Delta\theta)^2} + \cdots$$

$$\left.\cdots + \frac{1}{r_i}\frac{U^l_{i,j} - U^l_{i-1,j}}{(\Delta r)}\right] + 2U^l_{i,j} - U^{l-1}_{i,j}$$

Factorizando $\frac{1}{(\Delta r)^2}$

$$U^{l+1}_{i,j} = \left(\frac{\alpha\Delta t}{\Delta r}\right)^2\left[U^l_{i+1,j} - 2U^l_{i,j} + U^l_{i-1,j} + \cdots\right.$$

$$\cdots + \left(\frac{\Delta r}{r_i\Delta\theta}\right)^2\left(U^l_{i,j+1} - 2U^l_{i,j} + U^l_{i,j-1}\right) + \cdots$$

$$\left.\cdots + \frac{\Delta r}{r_i}\left(U^l_{i,j} - U^l_{i-1,j}\right)\right] + 2U^l_{i,j} - U^{l-1}_{i,j}$$

$$\lambda = \frac{\Delta r}{\Delta\theta} \qquad \gamma = \frac{\alpha\Delta t}{\Delta r}$$

$$U^{l+1}_{i,j} = \gamma^2\left[U^l_{i+1,j} - 2U^l_{i,j} + U^l_{i-1,j} + \cdots\right.$$

$$\cdots + \frac{\Delta r}{r_i}\left(U^l_{i,j} - U^l_{i-1,j}\right) + \cdots$$

$$\left.\cdots + \left(\frac{\lambda}{r_i}\right)^2\left(U^l_{i,j+1} - 2U^l_{i,j} + U^l_{i,j-1}\right)\right] + \cdots$$

$$\cdots + 2U^l_{i,j} - U^{l-1}_{i,j}$$