Nonlinear models for new product growth

New product life cycle: phases

1. Introduction
2. Growth
3. Maturity
4. Decline

What are the variables influencing a product's life cycle?
Marketing strategies play an essential role ...
but the success of a new product ultimately depends on consumers accepting them.

Diffusion of innovations

Diffusion is the process by which an innovation is communicated through certain channels over time among the members of a social system (Rogers, 2003).

Four key elements for describing an innovation diffusion process:

- innovation
- communication channels
- time
- social system

Innovation

An innovation is:

- New product, new service, new technology, new production process, new way of doing things (Schumpeter, 1947).
- Typical distinction: radical vs incremental innovations.
- Radical innovations could be hindered from barriers and social inertia.

New product growth models

General aim: depict the successive increases in the number of adopters and predict the continued development of a diffusion process already in progress (Mahajan and Muller, 1979).

- Fourt and Woodlock model (1960)
- Mansfield model (1961)
- Bass model (1969)
- Generalized Bass model (1994)

Bass Model

The Bass Model is defined by a first order differential equation

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))$$

Bass Model

innovation

$$z'(t) = \left(p + q\frac{z(t)}{m} \right) (m - z(t))$$

Bass Model

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))$$

imitation

Bass Model

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))$$

word-of-mouth

Bass Model

If we pose $\frac{z(t)}{m} = y(t)$ the model becomes

$$y'(t) = (p + qy(t))(1 - y(t))$$

Bass Model: solution

The Bass Model has a closed-form solution

$$y(t) = F(t; p, q) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} \qquad t > 0.$$

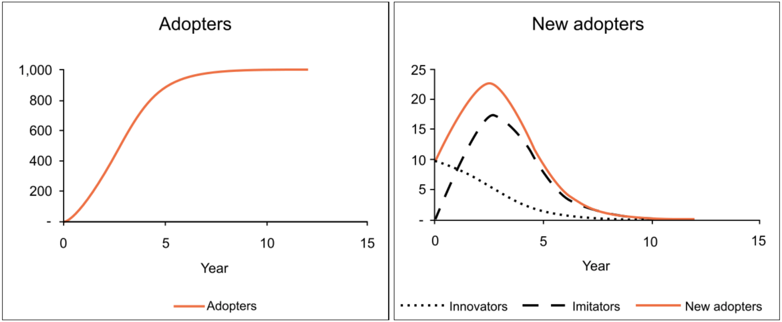
or, by posing $z = ym$

$$z(t) = m\, F(t; p, q) = m\, \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} \qquad t > 0.$$

Cumulative sales $z(t)$ 'depend' on parameters p and q.
The market potential m is a scale parameter and is assumed constant.

Bass Model

BM: cumulative and rate data



Bass Model: estimation

The Bass Model is a nonlinear model

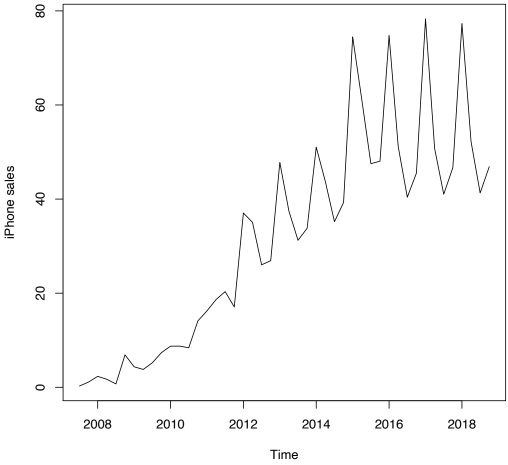
$$Z(t) = f(\beta, t) + \varepsilon(t)$$

where $Z(t)$ is the dependent variable, $f(\beta, t)$ is the deterministic term, function of $\beta \in R^k$ and of time t.
The second term, $\varepsilon(t)$, is the error term, for which usual assumptions hold, namely $M(\varepsilon(t)) = 0$,$Var(\varepsilon(t)) = \sigma^2$, $Cov(\varepsilon(t), \varepsilon(t')) = 0, t \neq t'$.

Bass Model: estimation

- Typical starting values for p and q are 0.01 and 0.1.
- Estimating m is the most difficult task.
- Parameter estimates are very sensitive to the number of available data.
- Reliable estimates are obtained after the maximum peak, but ... *"By the time sufficient observations have been developed for realiable estimation, it is too late to use the estimates for forecasting purposes"* (Mahajan, Muller, Bass, 1990).

Example: Apple iPhone life cycle



Quarterly sales data from 2007 to 2019
(source: Apple Inc.)

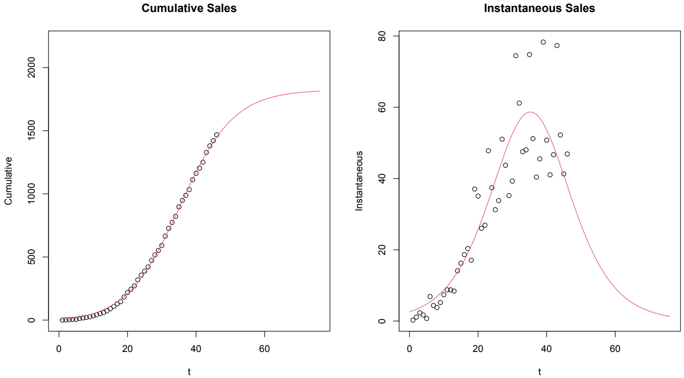
Example: Apple iPhone life cycle

Bass Model for iPhone life cycle: parameter estimates and 95% CIs

	Estimate	Std.Error	Lower	Upper	p-value
m	1823.7466	34.1251	1756.8627	1890.6306	< 0.0001
p	0.0014	0.0001	0.0013	0.0015	< 0.0001
q	0.1259	0.0027	0.1206	0.1311	< 0.0001

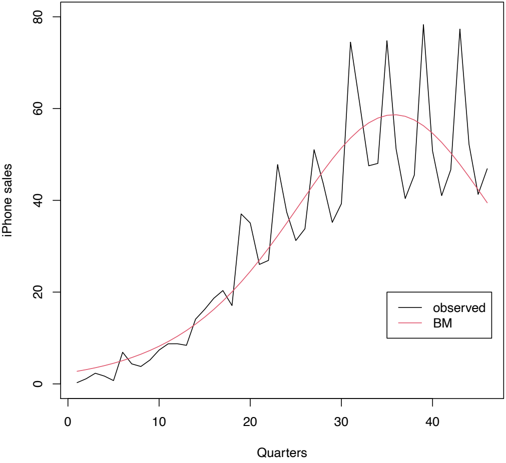
$R^2 = 0.9995$

Example: Apple iPhone life cycle



Cumulative and instantaneous sales data and forecasts with BM

Example: Apple iPhone life cycle



Instantaneous sales data and forecasts with BM

Bass Model: interesting properties

- Parsimonious model with just three parameters m, p, q.
- Only needs aggregate sales data.
- Easy to interpret.

Bass Model: limitations

- The market potential m is constant along the whole life cycle.
- The Bass Model does not account for marketing mix strategies.
- It is a model for products with a limited life cycle: needs a hypothesis.

Generalized Bass Model

The Bass Model does not account for the effect of exogenous variables, such as marketing mix, public incentives, environmental shocks.
Besides, in some cases the diffusion process does not have a bell shape curve, but a more complex structure.

Generalized Bass Model

The Generalized Bass Model (Bass et al., 1994) adds an intervention function $x(t)$

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))x(t).$$

where $x(t)$ is an integrable, non negative function.

- The Bass Model is a special case where $x(t) = 1$.
- if $0 < x(t) < 1$ the process slows down,
- if $x(t) > 1$ the process accelerates.

Generalized Bass Model: closed-form solution

The closed-form solution of the model is

$$z(t) = m\,\frac{1 - e^{-(p+q)\int_0^t x(\tau)d\tau}}{1 + \frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}}, \qquad t > 0.$$

Interesting: function $x(t)$ does not modify the market potential m!
Function $x(t)$ modifies the speed of the process.

Modelling $x(t)$: exponential shock

Function $x(t)$ may take several forms in order to describe various types of shock.
A strong and fast shock may take an exponential form

$$x(t) = 1 + c_1 e^{b_1(t-a_1)} I_{t \geq a_1},$$

where parameter c_1 is intensity and sign of the shock, b_1 is the 'memory' of the effect and is typically negative, and a_1 is the starting time of the shock.

Modelling $x(t)$: exponential shock

The use of exponential shock is suitable for identifying the positive effect of marketing strategies or incentive measures, in order to speed up the diffusion process.
Also, a negative shock may represent a fast slowdown in sales due to the entrance of a competitor.

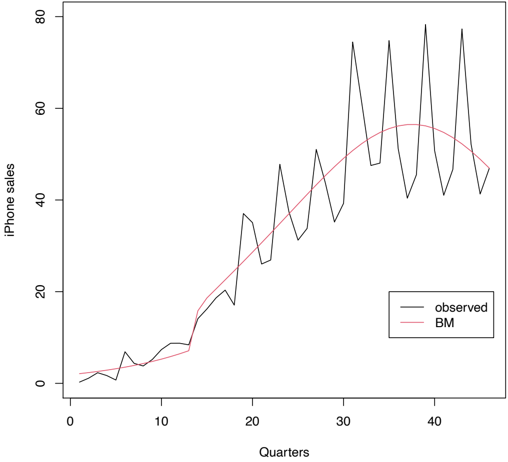
Example: Apple iPhone life cycle

GBM for iPhone life cycle: parameter estimates and 95% CIs

	Estimate	Std.Error	Lower	Upper	p-value
m	2080.9397	105.6182	1873.9319	2287.9475	< 0.001
p	0.0010	0.0001	0.0008	0.0011	< 0.001
q	0.1042	0.0092	0.0861	0.1222	< 0.001
a_1	13.1034	0.9609	11.2201	14.9867	< 0.001
b_1	-0.1587	0.0632	-0.2827	-0.0348	0.016
c_1	1.1086	0.1808	0.7542	1.4629	< 0.001

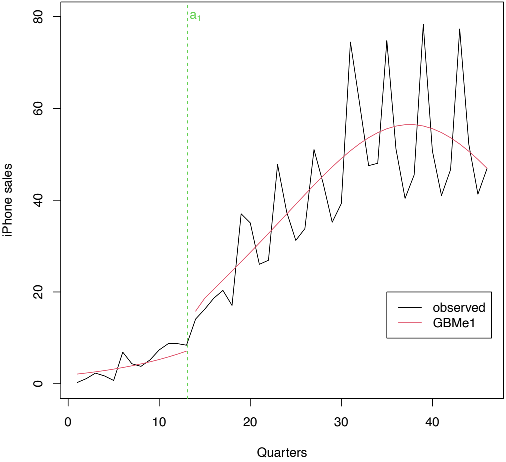
$R^2 = 0.9998$

Example: Apple iPhone life cycle



Instantaneous sales data and forecasts with GBMe1

Example: Apple iPhone life cycle



We may appreciate the starting time of the shock . . .

Modelling $x(t)$: rectangular shock

A more stable shock, acting on a longer period of time, may be modeled through a rectangular shock

$$x(t) = 1 + c_1 I_{t \geq a_1} I_{t \leq b_1},$$

where parameter c_1 describes intensity of the shock, either positive or negative, parameters a_1 and b_1 define beginning and end of the shock (con $a_1 < b_1$).
The rectangular shock is useful to identify the effect of policies and measures within a limited time interval.

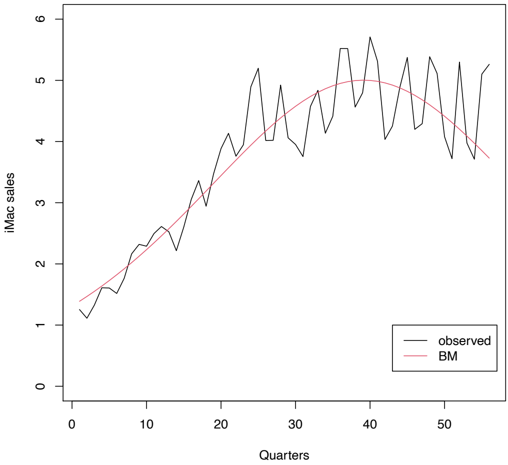
Example: Apple iMac life cycle

BM for iMac life cycle: parameter estimates and 95% CIs

	Estimate	Std.Error	Lower	Upper	p-value
m	281.66	3.58	274.65	288.68	< 0.0001
p	0.0047	0.0042	0.0047	0.0048	< 0.0001
q	0.061	0.001	0.059	0.063	< 0.0001

$R^2 = 0.9999088$

Example: Apple iMac life cycle



Instantaneous sales data and forecasts with BM

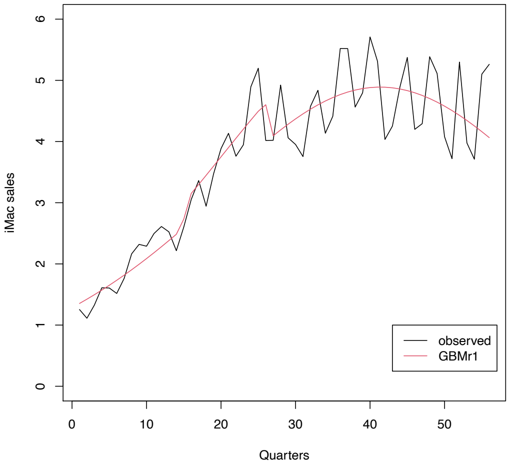
Example: Apple iMac life cycle

GBM for iMac life cycle: parameter estimates and 95% CIs

	Estimate	Std.Error	Lower	Upper	P-value
m	304.16	3.67	296.96	311.36	< 0.0001
p	0.0043	0.00001	0.0042	0.0044	< 0.0001
q	0.055	0.00	0.053	0.056	< 0.0001
a_1	14.67	0.96	12.79	16.54	< 0.0001
b_1	25.95	0.71	24.55	27.35	< 0.0001
c_1	0.16	0.02	0.13	0.20	< 0.0001

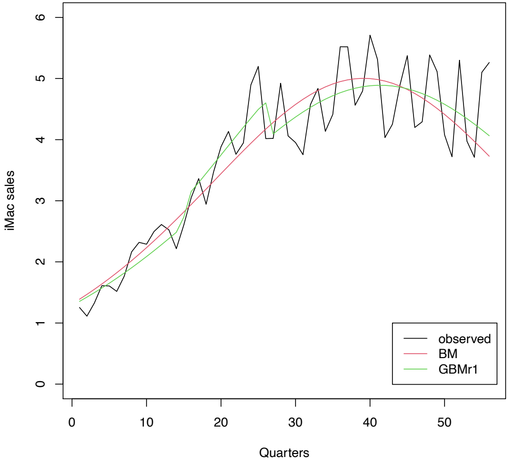
$R^2 = 0.9999733$

Example: Apple iMac life cycle



Instantaneous sales data and forecasts with GBMr1

Example: Apple iMac life cycle



Comparison between models . . . how can we evaluate the difference between these?

Modelling $x(t)$: mixed shock

It may be useful to have more than one shock of different nature. A simple case is made of a couple of shocks, rectangular and exponential,

$$x(t) = 1 + c_1 I_{t \geq a_1} I_{t \leq b_1} + c_2 e^{b_2 (t - a_2)} I_{t \geq a_2}$$

Other combinations are possible.

Model performance and selection

The usual performance indicator is the R^2

$$R^2 = \frac{\mathsf{SST} - \mathsf{SSE}}{\mathsf{SST}} = \frac{\sum(y_i - \bar{y})^2 - \sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2}$$

where y_i, $i = 1, 2, ..., n$ are calculated with the selected model. Further evaluations are performed through analysis of residuals (e.g. residual plots, Durbin-Watson statistic).

Model selection: $\tilde{R}^2$

In order to select between two 'nested' models, a suitable tool is the $\tilde{R}^2$

$$\tilde{R}^2 = \frac{\mathsf{SSE}_{m_1} - \mathsf{SSE}_{m_2}}{\mathsf{SSE}_{m_1}} = (R^2_{m_2} - R^2_{m_1})/(1 - R^2_{m_1}),$$

where $R^2_{m_i},\ i = 1, 2$ is the R^2 of model m_i.
If $\tilde{R}^2 > 0.3$ then the more complex model is significant.

Dynamic market potential, $m(t)$

A generalization of the Bass Model considers a dynamic market potential, $m(t)$

$$z'(t) = m(t)\left\{ \left(p + q\frac{z(t)}{m}\right)\left(1 - \frac{z(t)}{m(t)}\right) \right\} + z(t)\frac{m'(t)}{m(t)}$$

$$\frac{z'(t)m(t) - z(t)m'(t)}{m^2(t)} = \left(\frac{z(t)}{m(t)}\right)' = \left(p + q\frac{z(t)}{m(t)}\right)\left(1 - \frac{z(t)}{m(t)}\right)$$

and, by setting $y(t) = z(t)/m(t)$, we have

$$y'(t) = p + qy(t)(1 - y(t))$$

which is a standard Bass Model.

Dynamic market potential, $m(t)$

1. Market of new products is unstable and uncertain in the first phase of diffusion: incubation
2. Advertising and promotional efforts play a central role to overcome this phase
3. These efforts influence the structure of the market potential, which depends on information on the product
4. Communication and adoption are two separate phases, needing a distinct modelling

Dynamic market potential, $m(t)$

We may notice that di $m(t)$ is 'free'

$$z(t) = m(t)F(t) = m(t)\,\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}$$

Dynamic market potential, $m(t)$: GGM

The GGM (Guseo and Guidolin, 2009) is a generalization of the Bass Model, where $m(t)$ is time-dependent

$$z(t) = m(t)F(t) = m(t)\ \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}$$

and function of a communication process

$$z(t) = KG(t)F(t) = K\ \sqrt{\frac{1 - e^{-(p_c+q_c)t}}{1 + \frac{q_c}{p_c}e^{-(p_c+q_c)t}}}\frac{1 - e^{-(p_s+q_s)t}}{1 + \frac{q_s}{p_s}e^{-(p_s+q_s)t}}$$

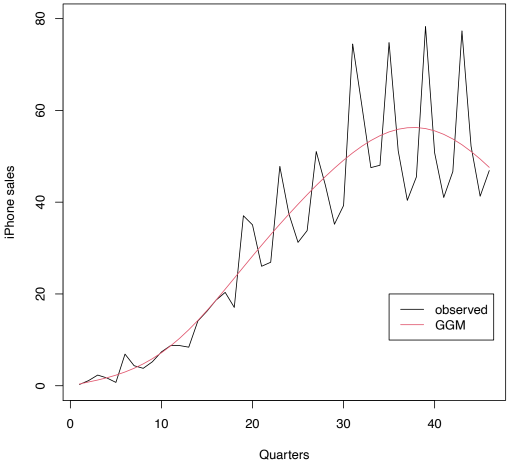
Example: Apple iPhone

GGM for iPhone: estimates and 95% CIs

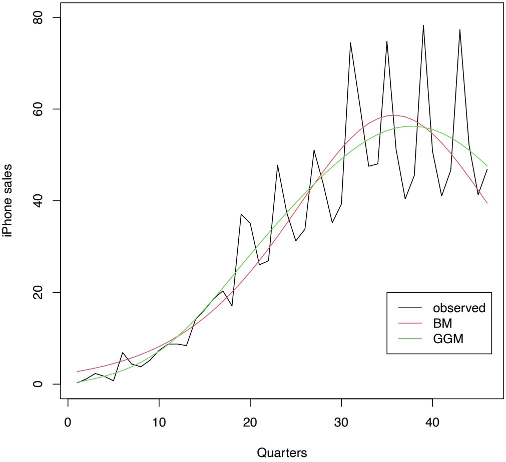
	Estimate	Std.Error	Lower	Upper	P-value
K	2116.78	97.50	1925.68	2307.88	< 0.0001
p_c	0.00059	0.00	0.0028	0.009	< 0.0001
q_c	0.21	0.04	0.13	0.28	< 0.0001
p_s	0.0021	0.00	0.0015	0.0026	< 0.0001
q_s	0.10	0.01	0.09	0.11	< 0.0001

$R^2 = 0.99986$

Example: Apple iPhone



Example: Apple iPhone



Model comparison . . . what is the difference between the two models?

Competition between two products

Unbalanced competition and regime change diachronic model

$$
\begin{aligned}
z_1'(t) &= m\left\{ \left[p_{1a} + q_{1a}\frac{z(t)}{m}\right](1 - I_{t>c_2}) \right. \\
&\quad \left. + \left[p_{1c} + (q_{1c} + \delta)\frac{z_1(t)}{m} + q_{1c}\frac{z_2(t)}{m}\right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m}\right], \\
z_2'(t) &= m\left[p_2 + (q_2 - \gamma)\frac{z_1(t)}{m} + q_2\frac{z_2(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right] I_{t>c_2},
\end{aligned}
$$

Competition between two products

Unbalanced competition and regime change diachronic model

$$
\begin{aligned}
z_1'(t) &= m\Bigg\{\left[p_{1a} + q_{1a}\frac{z(t)}{m}\right](1 - I_{t>c_2}) \\
&\quad + \left[p_{1c} + (q_{1c} + \delta)\frac{z_1(t)}{m} + q_{1c}\frac{z_2(t)}{m}\right] I_{t>c_2}\Bigg\}\left[1 - \frac{z(t)}{m}\right], \\
z_2'(t) &= m\left[p_2 + (q_2 - \gamma)\frac{z_1(t)}{m} + q_2\frac{z_2(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right] I_{t>c_2},
\end{aligned}
$$

within imitation

Competition between two products

Unbalanced competition and regime change diachronic model

$$
\begin{aligned}
z_1'(t) &= m\left\{ \left[p_{1a} + q_{1a}\frac{z(t)}{m}\right](1 - I_{t>c_2}) \right.\\
&\quad \left. + \left[p_{1c} + (q_{1c} + \delta)\frac{z_1(t)}{m} + q_{1c}\frac{z_2(t)}{m}\right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m}\right],\\
z_2'(t) &= m\left[p_2 + (q_2 - \gamma)\frac{z_1(t)}{m} + q_2\frac{z_2(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right] I_{t>c_2},
\end{aligned}
$$

within imitation

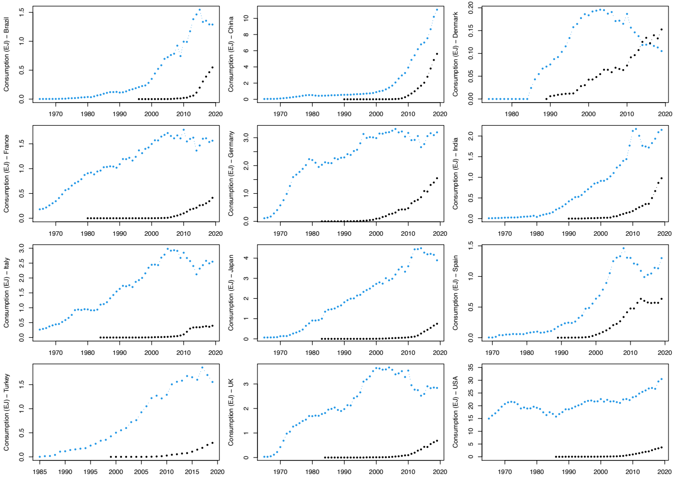
cross imitation

Model

Sign of cross-imitation coefficients: competition-collaboration

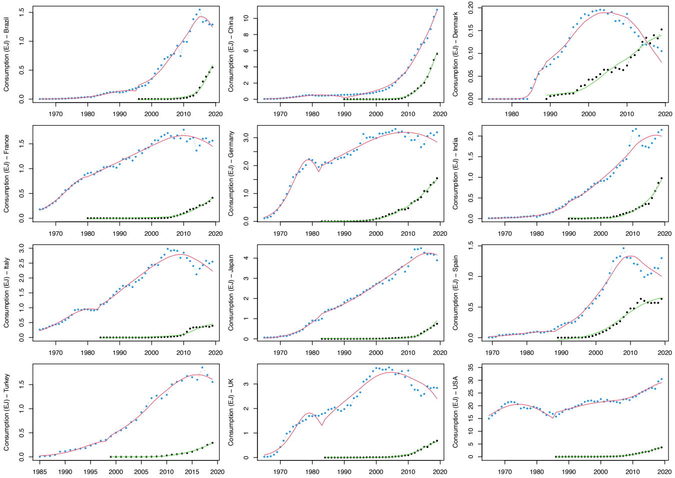
q_{1c}	$q_2 - \gamma$	interpretation
negative	negative	full competition
negative	positive	2 competes with 1, 1 collaborates with 2
positive	negative	2 collaborates with 1, 1 competes with 2
positive	positive	full collaboration

Example: energy technologies in competition



Is there a significant interplay?

Example: energy technologies in competition



UCRCD fit: there is a significant interplay...what kind?

Example: energy technologies in competition

Country	m_c	p_{1c}	$(q_{1c}+\delta)$	q_{1c}	q_2	$(q_2-\gamma)$	δ	γ
Brazil	61	0.003	0.12	-0.29	0.41	0.002	0.41	
China	2429	0.000	0.13	-0.05	0.2	0.010	0.19	
Denmark	10	0.007	0.11	-0.19	0.22	-0.010	0.30	0.23
France	139	0.006	0.04	-0.20	0.24	0.001	0.24	
Germany	409	0.004	0.03	-0.10	0.14	0.003	0.13	
India	158	0.002	0.08	-0.16	0.24	-0.001	0.24	
Italy	132	0.008	0.07	-0.18	0.33	0.001	0.25	0.33
Japan	532	0.002	0.04	-0.27	0.32	-0.001	0.32	
Spain	48	0.002	0.14	-0.09	0.24	0.004	0.23	
Turkey	52	0.007	0.13	-0.32	0.45	-0.0002	0.45	
UK	153	0.009	0.07	-0.33	0.40	0.001	0.40	
USA	1257	0.013	0.04	1.35	0.39	-0.0002	-1.3	0.40

Dynamic relationship between natural gas and renewables for the 12 countries selected