Accretion disk Equations

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1 Advective Disk with Outflows

From Equation (14) of Lipunova+99

$$\begin{align}
\frac{d}{dr}\left(\dot{M}\omega r^2\right) &= -2\pi\frac{d}{dr}\left(W_{r\phi}r^2\right) + \frac{d\dot{M}}{dr}\omega r^2 \tag{1}\\
\cancel{\frac{d\dot{M}}{dr}}\omega r^2 + \dot{M}\frac{d}{dr}\left(\omega r^2\right) &= -2\pi\left(\frac{dW_{r\phi}}{dr}r^2 + 2rW_{r\phi}\right) + \cancel{\frac{d\dot{M}}{dr}}\omega r^2 \tag{2}\\
-\frac{\dot{M}\frac{d}{dr}\left(\omega r^2\right)}{2\pi} - 2rW_{r\phi} &= \frac{dW_{r\phi}}{dr}r^2 \tag{3}\\
-\frac{\frac{1}{2}\dot{M}\omega\cancel{r}}{2\pi} - 2\cancel{r}W_{r\phi} &= \frac{dW_{r\phi}}{dr}r^{\cancel{2}} \tag{4}\\
-\frac{\dot{M}\omega}{4\pi} - 2W_{r\phi} &= \frac{dW_{r\phi}}{dr}r \tag{5}\\
-\left(\frac{\frac{\dot{M}\omega}{4\pi} + 2W_{r\phi}}{r}\right) &= \frac{dW_{r\phi}}{dr} \tag{6}
\end{align}$$

From Equation (9) (Hydrostatic equilibrium) we can eliminate any dependency on the pressure P

$$\begin{align}
\frac{P}{\rho H} &= \frac{GMH}{r^3} \tag{7}\\
\frac{P}{\rho H} &= \omega^2 H \tag{8}\\
P &= \omega^2 H^2 \rho \tag{9}\\
\frac{dP}{dr} &= \omega^2 H\left(2\frac{dH}{dr}\rho + H\frac{d\rho}{dr} - 3\frac{H\rho}{r}\right) \tag{10}
\end{align}$$

From Equation (10) we can further eliminate dependencies on $\epsilon_{\rm rad}$ and therefore T, assuming radiation pressure is the dominant pressure (i.e. neglecting gas pressure):

$$P = \frac{\epsilon_{\rm rad}}{3} \tag{11}$$

From Equation (8) and using the found Equations for the pressure (Equation 7) and the radiative energy (Equation 11) we can find $Q_{\rm rad}$ as a function of H only:

$$\begin{align}
Q_{\rm rad} = \frac{1}{3}\frac{c}{k_{\rm T}\rho H}\epsilon_{\rm rad} &= \frac{Pc}{k_{\rm T}\rho H} = \frac{w^2 H^{\cancel{2}}\cancel{\rho}c}{k_{\rm T}\cancel{\rho H}} \tag{12}\\
Q_{\rm rad} &= \frac{\omega^2 Hc}{k_{\rm T}} \tag{13}
\end{align}$$

This implies that the scale height is determined by the radiation pressure of the disk. There is sometimes a factor 4 added, which we keep there to keep track of it in the Equations. Using Equation (12) from Lipunova 1999 and the expression for

Q_{rad} (Equation 12)

$$\frac{\omega^2 r}{8\pi}\frac{d\dot{M}}{dr} = Q_{\mathrm{rad}} = \frac{\omega^2 Hc}{k_{\mathrm{T}}} \tag{14}$$

$$\frac{\cancel{\omega^2} r}{8\pi}\frac{d\dot{M}}{dr} = \frac{\cancel{\omega^2} Hc}{k_{\mathrm{T}}} \tag{15}$$

$$\frac{d\dot{M}}{dr} = \frac{8\pi Hc}{r k_{\mathrm{T}}} \tag{16}$$

Note that the above equation is only valid for disks with outflows (i.e. when $\frac{d\dot{M}}{dr} \neq 0$.)
For **non-advective** disks, we can also express $\frac{d\dot{M}}{dr}$ as a function of $W_{\mathrm{r}\phi}$, which is useful when solving the differential equations $(W_{r\phi}(R), \dot{M}(R))$:

$$\frac{d\dot{M}}{dr} = 8\pi\frac{Q_{\mathrm{rad}}}{\omega^2 r} \tag{17}$$

$$Q^+ = -\frac{3}{4}\omega W_{r\phi} = Q_{\mathrm{rad}} \tag{18}$$

$$\frac{d\dot{M}}{dr} = -8\pi\frac{3}{4}\cancel{\omega}\frac{W_{r\phi}}{\omega^{\cancel{2}} r} \tag{19}$$

$$\frac{d\dot{M}}{dr} = -6\pi\frac{W_{r\phi}}{\omega} \tag{20}$$

Q_{adv} (Equation (22)) can be reduced to $Q_{\mathrm{adv}}(H, \rho, \dot{M})$ using Equations (13), 7 and 11:

$$Q_{\mathrm{adv}} = \frac{\dot{M}}{4\pi r}\left(3\frac{dP}{dr}\frac{1}{\rho} + 4P\frac{d\frac{1}{\rho}}{dr}\right) \tag{21}$$

$$= \frac{\dot{M}}{4\pi r}\left[3\frac{\omega^2 H}{\rho}\left(2\frac{dH}{dr}\rho + H\frac{d\rho}{dr} - 3\frac{H\rho}{r}\right) - 4\frac{d\rho}{dr}\frac{\omega^2 H^2 \cancel{\rho}}{\rho^{\cancel{2}}}\right] \tag{22}$$

$$\frac{\dot{M}\omega^2 H}{4\pi r\rho}\left[3\left(2\frac{dH}{dr}\rho + H\frac{d\rho}{dr} - 3\frac{H\rho}{r}\right) - 4H\frac{d\rho}{dr}\right] \tag{23}$$

$$Q_{\mathrm{adv}} = \frac{\dot{M}\omega^2 H}{4\pi r\rho}\left(6\rho\frac{dH}{dr} - H\frac{d\rho}{dr} - 9\frac{H\rho}{r}\right) \tag{24}$$

the density can be replaced using the α-prescription

$$W_{r\phi} = -2\alpha HP = -2\alpha H\omega^2 H^2\rho = -2\alpha H^3\omega^2\rho \tag{25}$$

$$\rho = -\frac{1}{2}\frac{W_{r\phi}}{\alpha\omega^2 H^3} \tag{26}$$

$$\frac{d\rho}{dr} = -\frac{1}{2\alpha H^3\omega^2}\left(\frac{dW_{r\phi}}{dr} - \frac{3}{H}W_{r\phi}\frac{dH}{dr} + 3\frac{W_{r\phi}}{r}\right) \tag{27}$$

Finally from the energy Equation ($Q^+ = Q_{\mathrm{rad}} + Q_{\mathrm{adv}}$) and Equation (8) we have that:

$$Q^+ = -\frac{3}{4}\cancel{\omega}W_{r\phi} = \frac{w^{\cancel{2}}Hc}{k_{\mathrm{T}}} + \frac{\dot{M}\omega^{\cancel{2}}H}{4\pi r\rho}\left(6\rho\frac{dH}{dr} - H\frac{d\rho}{dr} - 9\frac{H\rho}{r}\right) \tag{28}$$

$$-\frac{3}{4}W_{r\phi} - \frac{wHc}{k_{\mathrm{T}}} = \frac{\dot{M}\omega H}{4\pi r\rho}\left(6\rho\frac{dH}{dr} - H\frac{d\rho}{dr} - 9\frac{H\rho}{r}\right) \tag{29}$$

$$-3\frac{W_{r\phi}\pi r\rho}{\dot{M}\omega H} - \frac{4\pi cr\rho}{\dot{M}k_{\mathrm{T}}} = 6\rho\frac{dH}{dr} - H\frac{d\rho}{dr} - 9\frac{H\rho}{r} \tag{30}$$

$$\frac{1}{6\rho}\left(-3\frac{W_{r\phi}\pi r\rho}{\dot{M}\omega H} - \frac{4\pi cr\rho}{\dot{M}k_{\mathrm{T}}} + H\frac{d\rho}{dr} + 9\frac{H\rho}{r}\right) = \frac{dH}{dr} \tag{31}$$

we can now replace $\frac{d\rho}{dr}$ from Equation 28 to obtain $\frac{dH}{dr}$:

$$\frac{dH}{dr} = \frac{1}{6\rho}\left[-3\frac{W_{r\phi}\pi r\rho}{\dot{M}\omega H} - \frac{4\pi c r\rho}{\dot{M}k_{\mathrm{T}}} - \frac{H}{2\alpha H^3\omega^2}\left(\frac{dW_{r\phi}}{dr} - \frac{3}{H}W_{r\phi}\frac{dH}{dr} + 3\frac{W_{r\phi}}{r}\right) + 9\frac{H\rho}{r}\right] \tag{32}$$

$$= -\frac{W_{r\phi}\pi r}{2\dot{M}\omega H} - \frac{2\pi c r}{3\dot{M}k_{\mathrm{T}}} - \frac{1}{12\rho\alpha H^2\omega^2}\left(\frac{dW_{r\phi}}{dr} - \frac{3}{H}W_{r\phi}\frac{dH}{dr} + 3\frac{W_{r\phi}}{r}\right) + \frac{3}{2}\frac{H}{r} \tag{33}$$

$$= -\frac{W_{r\phi}\pi r}{2\dot{M}\omega H} - \frac{2\pi c r}{3\dot{M}k_{\mathrm{T}}} - \frac{1}{12\rho\alpha H^2\omega^2}\frac{dW_{r\phi}}{dr} + \frac{W_{r\phi}}{4\rho\alpha H^3\omega^2}\frac{dH}{dr} - \frac{W_{r\phi}}{4\rho\alpha H^2\omega^2 r} + \frac{3}{2}\frac{H}{r} \tag{34}$$

$$\frac{dH}{dr} - \frac{W_{r\phi}}{4\rho\alpha H^3\omega^2}\frac{dH}{dr} = -\frac{W_{r\phi}\pi r}{2\dot{M}\omega H} - \frac{2\pi c r}{3\dot{M}k_{\mathrm{T}}} - \frac{1}{12\rho\alpha H^2\omega^2}\frac{dW_{r\phi}}{dr} - \frac{W_{r\phi}}{4\rho\alpha H^2\omega^2 r} + \frac{3}{2}\frac{H}{r} \tag{35}$$

$$\frac{dH}{dr}\left(1 - \frac{W_{r\phi}}{4\rho\alpha H^3\omega^2}\right) = -\frac{W_{r\phi}\pi r}{2\dot{M}\omega H} - \frac{2\pi c r}{3\dot{M}k_{\mathrm{T}}} - \frac{1}{12\rho\alpha H^2\omega^2}\frac{dW_{r\phi}}{dr} - \frac{W_{r\phi}}{4\rho\alpha H^2\omega^2 r} + \frac{3}{2}\frac{H}{r} \tag{36}$$

$$\frac{dH}{dr} = \frac{9\frac{H\rho}{r} - \left(\frac{3\pi r\rho W_{r\phi}}{\dot{M}\omega H} + \frac{\frac{dW_{r\phi}}{dr}}{2\alpha H^2\omega^2} + \frac{3}{2}\frac{W_{r\phi}}{\alpha r H^2\omega^2} + \frac{4\pi r\rho c}{\dot{M}k_T}\right)}{6\rho - \frac{3}{2}\frac{W_{r\phi}}{\alpha H^3\omega^2}} \tag{37}$$

$$= \frac{9\frac{H\rho}{r} - \left(\frac{3\pi r\rho W_{r\phi}}{\dot{M}\omega H} + \frac{\frac{dW_{r\phi}}{dr}}{2\alpha H^2\omega^2} + \frac{3}{2}\frac{W_{r\phi}}{\alpha r H^2\omega^2} + \frac{4\pi r\rho c}{\dot{M}k_T}\right)}{9\rho} \tag{38}$$

we can further eliminate ρ using Equation 25

$$\frac{dH}{dr} = \frac{1}{9}\left(\frac{12H}{r} - \frac{3\pi r W_{r\phi}}{\omega H\dot{M}} + \frac{dW_{r\phi}}{dr}\frac{H}{W_{r\phi}} - \frac{4\pi r c}{\dot{M}k_T}\right) \tag{39}$$

we can further eliminate $\frac{dW_{r\phi}}{dr}$ using Equation 1:

$$\frac{dH}{dr} = \frac{1}{9}\left(\frac{12H}{r} - \frac{3\pi r W_{r\phi}}{\omega H\dot{M}} - \frac{\dot{M}\omega}{4\pi r}\frac{H}{W_{r\phi}} - \frac{2\cancel{W_{r\phi}}}{r}\frac{H}{\cancel{W_{r\phi}}} - \frac{4\pi r c}{\dot{M}k_T}\right) \tag{40}$$

$$= \frac{4H}{3r} - \frac{\pi r W_{r\phi}}{3\omega H\dot{M}} - \frac{\dot{M}\omega}{36\pi r}\frac{H}{W_{r\phi}} - \frac{2}{9r}H - \frac{4\pi r c}{9\dot{M}k_T} \tag{41}$$

$$= \frac{10H}{9r} - \frac{\pi r W_{r\phi}}{3\omega H\dot{M}} - \frac{\dot{M}\omega}{36\pi r}\frac{H}{W_{r\phi}} - \frac{4\pi r c}{9\dot{M}k_T} \tag{42}$$

after that we have $\frac{dW_{r\phi}}{dr}(W_{r\phi}, \dot{M})$, $\frac{d\dot{M}}{dr}(H)$ and $\frac{dH}{dr}(W_{r\phi}, \dot{M}, H)$ (Equations 1, 14 and 40) that depend only on $W_{r\phi}, H$ and $\dot{M}$ and need to be solved to find these three quantities. The boundary conditions are:

- $Q_{\mathrm{rad}}(r_{in}) = 0 \rightarrow H(r_{in}) = 0$ i.e all the energy (Q^+) is advected into the black hole
- $W_{r\phi}(r_{in}) = 0$ i.e. zero-torque boundary condition
- $\dot{M}(r_{sp}) = \dot{M}_0$ i.e. the mass-transfer rate at the spherization radius (a parameter of the problem) is equal to the initial mass-transfer rate