

Accretion disk Equations

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1 Advective Disk with Outflows

Equations in brackets indicate Equations from Lipunova 1999. Equations without brackets refer to Equations from this pdf. From Equation (14) of Lipunova 1999

$$\frac{d}{dr} (\dot{M} \omega r^2) = -2\pi \frac{d}{dr} (W_{r\phi} r^2) + \frac{d\dot{M}}{dr} \omega r^2 \quad (1)$$

$$\cancel{\frac{d\dot{M}}{dr} \omega r^2} + \dot{M} \frac{d}{dr} (\omega r^2) = -2\pi \left(\frac{dW_{r\phi}}{dr} r^2 + 2rW_{r\phi} \right) + \cancel{\frac{d\dot{M}}{dr} \omega r^2} \quad (2)$$

$$-\frac{\dot{M} \frac{d}{dr} (\omega r^2)}{2\pi} - 2rW_{r\phi} = \frac{dW_{r\phi}}{dr} r^2 \quad (3)$$

$$-\frac{\frac{1}{2}\dot{M}\omega r^2}{2\pi} - 2rW_{r\phi} = \frac{dW_{r\phi}}{dr} r^2 \quad (4)$$

$$-\frac{\dot{M}\omega}{4\pi} - 2W_{r\phi} = \frac{dW_{r\phi}}{dr} r \quad (5)$$

$$-\left(\frac{\frac{M\omega}{4\pi} + 2W_{r\phi}}{r} \right) = \frac{dW_{r\phi}}{dr} \quad (6)$$

From Equation (9) (Hydrostatic equilibrium) we can eliminate any dependency on the pressure P

$$\frac{P}{\rho H} = \frac{GMH}{r^3} \quad (7)$$

$$\frac{P}{\rho H} = \omega^2 H \quad (8)$$

$$P = \omega^2 H^2 \rho \quad (9)$$

$$\frac{dP}{dr} = \omega^2 H \left(2 \frac{dH}{dr} \rho + H \frac{d\rho}{dr} - 3 \frac{H\rho}{r} \right) \quad (10)$$

Where we have used

$$\frac{d\omega}{dr} = \frac{-3}{2} \frac{\omega}{r} \quad (11)$$

From Equation (10) we can further eliminate dependencies on ϵ_{rad} and therefore T , assuming radiation pressure is the dominant pressure (i.e. neglecting gas pressure):

$$P = \frac{\epsilon_{\text{rad}}}{3} \quad (12)$$

From Equation (8) and using the found Equations for the pressure (Equation 7) and the radiative energy (Equation 12) we can find Q_{rad} as a function of H only:

$$Q_{\text{rad}} = \frac{1}{3} \frac{c}{k_T \rho H} \epsilon_{\text{rad}} = \frac{P c}{k_T \rho H} = \frac{w^2 H^2 \rho c}{k_T \rho H} \quad (13)$$

$$Q_{\text{rad}} = \frac{\omega^2 H c}{k_T} \quad (14)$$

This implies that the scale height is determined by the radiation pressure of the disk. There is sometimes a factor 4 added, which we keep there to keep track of it in the Equations. Using Equation (12) from Lipunova 1999 and the expression for

Q_{rad} (Equation 13)

$$\frac{\omega^2 r}{8\pi} \frac{d\dot{M}}{dr} = Q_{\text{rad}} = \frac{\omega^2 H c}{k_T} \quad (15)$$

$$\frac{\omega^2 r}{8\pi} \frac{d\dot{M}}{dr} = \frac{\omega^2 H c}{k_T} \quad (16)$$

$$\frac{d\dot{M}}{dr} = \frac{8\pi H c}{r k_T} \quad (17)$$

Note that the above equation is only valid for disks with outflows (i.e. when $\frac{d\dot{M}}{dr} \neq 0$.)

For **non-conservative** disks, we can also express $\frac{d\dot{M}}{dr}$ as a function of $W_{r\phi}$, which is useful when solving the differential equations ($W_{r\phi}(R), \dot{M}(R)$):

$$\frac{d\dot{M}}{dr} = 8\pi \frac{Q_{\text{rad}}}{\omega^2 r} \quad (18)$$

$$Q^+ = -\frac{3}{4}\omega W_{r\phi} = Q_{\text{rad}} \quad (19)$$

$$\frac{d\dot{M}}{dr} = -8\pi \frac{3}{4} \omega \frac{W_{r\phi}}{\omega^2 r} \quad (20)$$

$$\frac{d\dot{M}}{dr} = -6\pi \frac{W_{r\phi}}{\omega} \quad (21)$$

Using Equation (13), the mass-flux conservation, we can write

$$\frac{\dot{M}}{4\pi r} = \rho v_r H \quad (22)$$

and Q_{adv} (Equation (22)) can be reduced to $Q_{\text{adv}}(H, \rho, \dot{M})$ using Equations 22, 7 and 12:

$$Q_{\text{adv}} = \frac{\dot{M}}{4\pi r} \left(3 \frac{dP}{dr} + P \frac{d\frac{1}{\rho}}{dr} \right) \quad (23)$$

$$Q_{\text{adv}} = \frac{\dot{M}}{4\pi r} \left(3 \frac{dP}{dr} \frac{1}{\rho} + 3P \frac{d\frac{1}{\rho}}{dr} + P \frac{d\frac{1}{\rho}}{dr} \right) \quad (24)$$

$$Q_{\text{adv}} = \frac{\dot{M}}{4\pi r} \left(3 \frac{dP}{dr} \frac{1}{\rho} + 4P \frac{d\frac{1}{\rho}}{dr} \right) \quad (25)$$

$$= \frac{\dot{M}}{4\pi r} \left[3 \frac{\omega^2 H}{\rho} \left(2 \frac{dH}{dr} \rho + H \frac{d\rho}{dr} - 3 \frac{H\rho}{r} \right) - 4 \frac{d\rho}{dr} \frac{\omega^2 H^2 \rho}{\rho^2} \right] \quad (26)$$

$$\frac{\dot{M}\omega^2 H}{4\pi r \rho} \left[3 \left(2 \frac{dH}{dr} \rho + H \frac{d\rho}{dr} - 3 \frac{H\rho}{r} \right) - 4H \frac{d\rho}{dr} \right] \quad (27)$$

$$Q_{\text{adv}} = \frac{\dot{M}\omega^2 H}{4\pi r \rho} \left(6\rho \frac{dH}{dr} - H \frac{d\rho}{dr} - 9 \frac{H\rho}{r} \right) \quad (28)$$

the density can be replaced using the α -prescription

$$W_{r\phi} = -2\alpha H P = -2\alpha H \omega^2 H^2 \rho = -2\alpha H^3 \omega^2 \rho \quad (29)$$

$$\rho = -\frac{1}{2} \frac{W_{r\phi}}{\alpha \omega^2 H^3} \quad (30)$$

$$\frac{d\rho}{dr} = -\frac{1}{2\alpha H^3 \omega^2} \left(\frac{dW_{r\phi}}{dr} - \frac{3}{H} W_{r\phi} \frac{dH}{dr} + 3 \frac{W_{r\phi}}{r} \right) \quad (31)$$

Finally from the energy Equation ($Q^+ = Q_{\text{rad}} + Q_{\text{adv}}$) and Equation (8) we have that:

$$Q^+ = -\frac{3}{4} \mathcal{W} W_{r\phi} = \frac{w \mathcal{H} c}{k_T} + \frac{\dot{M} \omega \mathcal{H}}{4\pi r \rho} \left(6\rho \frac{dH}{dr} - H \frac{d\rho}{dr} - 9 \frac{H\rho}{r} \right) \quad (32)$$

$$-\frac{3}{4} W_{r\phi} - \frac{w H c}{k_T} = \frac{\dot{M} \omega H}{4\pi r \rho} \left(6\rho \frac{dH}{dr} - H \frac{d\rho}{dr} - 9 \frac{H\rho}{r} \right) \quad (33)$$

$$-3 \frac{W_{r\phi} \pi r \rho}{\dot{M} \omega H} - \frac{4\pi c r \rho}{\dot{M} k_T} = 6\rho \frac{dH}{dr} - H \frac{d\rho}{dr} - 9 \frac{H\rho}{r} \quad (34)$$

$$\frac{1}{6\rho} \left(-3 \frac{W_{r\phi} \pi r \rho}{\dot{M} \omega H} - \frac{4\pi c r \rho}{\dot{M} k_T} + H \frac{d\rho}{dr} + 9 \frac{H\rho}{r} \right) = \frac{dH}{dr} \quad (35)$$

we can now replace $\frac{d\rho}{dr}$ from Equation 32 to obtain $\frac{dH}{dr}$:

$$\frac{dH}{dr} = \frac{1}{6\rho} \left[-3 \frac{W_{r\phi} \pi r \rho}{\dot{M} \omega H} - \frac{4\pi c r \rho}{\dot{M} k_T} - \frac{H}{2\alpha H^3 \omega^2} \left(\frac{dW_{r\phi}}{dr} - \frac{3}{H} W_{r\phi} \frac{dH}{dr} + 3 \frac{W_{r\phi}}{r} \right) + 9 \frac{H\rho}{r} \right] \quad (36)$$

$$= -\frac{W_{r\phi} \pi r}{2\dot{M} \omega H} - \frac{2\pi c r}{3\dot{M} k_T} - \frac{1}{12\rho \alpha H^2 \omega^2} \left(\frac{dW_{r\phi}}{dr} - \frac{3}{H} W_{r\phi} \frac{dH}{dr} + 3 \frac{W_{r\phi}}{r} \right) + \frac{3}{2} \frac{H}{r} \quad (37)$$

$$= -\frac{W_{r\phi} \pi r}{2\dot{M} \omega H} - \frac{2\pi c r}{3\dot{M} k_T} - \frac{1}{12\rho \alpha H^2 \omega^2} \frac{dW_{r\phi}}{dr} + \frac{W_{r\phi}}{4\rho \alpha H^3 \omega^2} \frac{dH}{dr} - \frac{W_{r\phi}}{4\rho \alpha H^2 \omega^2 r} + \frac{3}{2} \frac{H}{r} \quad (38)$$

$$\frac{dH}{dr} - \frac{W_{r\phi}}{4\rho \alpha H^3 \omega^2} \frac{dH}{dr} = -\frac{W_{r\phi} \pi r}{2\dot{M} \omega H} - \frac{2\pi c r}{3\dot{M} k_T} - \frac{1}{12\rho \alpha H^2 \omega^2} \frac{dW_{r\phi}}{dr} - \frac{W_{r\phi}}{4\rho \alpha H^2 \omega^2 r} + \frac{3}{2} \frac{H}{r} \quad (39)$$

$$\frac{dH}{dr} \left(1 - \frac{W_{r\phi}}{4\rho \alpha H^3 \omega^2} \right) = -\frac{W_{r\phi} \pi r}{2\dot{M} \omega H} - \frac{2\pi c r}{3\dot{M} k_T} - \frac{1}{12\rho \alpha H^2 \omega^2} \frac{dW_{r\phi}}{dr} - \frac{W_{r\phi}}{4\rho \alpha H^2 \omega^2 r} + \frac{3}{2} \frac{H}{r} \quad (40)$$

$$\frac{dH}{dr} = \frac{9 \frac{H\rho}{r} - \left(\frac{3\pi r \rho W_{r\phi}}{\dot{M} \omega H} + \frac{\frac{dW_{r\phi}}{dr}}{2\alpha H^2 \omega^2} + \frac{3}{2} \frac{W_{r\phi}}{\alpha r H^2 \omega^2} + \frac{4\pi r \rho c}{\dot{M} k_T} \right)}{6\rho - \frac{3}{2} \frac{W_{r\phi}}{\alpha H^3 \omega^2}} \quad (41)$$

$$= \frac{9 \frac{H\rho}{r} - \left(\frac{3\pi r \rho W_{r\phi}}{\dot{M} \omega H} + \frac{\frac{dW_{r\phi}}{dr}}{2\alpha H^2 \omega^2} + \frac{3}{2} \frac{W_{r\phi}}{\alpha r H^2 \omega^2} + \frac{4\pi r \rho c}{\dot{M} k_T} \right)}{9\rho} \quad (42)$$

we can further eliminate ρ using Equation 29

$$\frac{dH}{dr} = \frac{1}{9} \left(\frac{12H}{r} - \frac{3\pi r W_{r\phi}}{\omega H \dot{M}} + \frac{dW_{r\phi}}{dr} \frac{H}{W_{r\phi}} - \frac{4\pi r c}{\dot{M} k_T} \right) \quad (43)$$

we can further eliminate $\frac{dW_{r\phi}}{dr}$ using Equation 1:

$$\frac{dH}{dr} = \frac{1}{9} \left(\frac{12H}{r} - \frac{3\pi r W_{r\phi}}{\omega H \dot{M}} - \frac{\dot{M} \omega}{4\pi r} \frac{H}{W_{r\phi}} - \frac{2\cancel{W_{r\phi}}}{r} \frac{H}{\cancel{W_{r\phi}}} - \frac{4\pi r c}{\dot{M} k_T} \right) \quad (44)$$

$$= \frac{4H}{3r} - \frac{\pi r W_{r\phi}}{3\omega H \dot{M}} - \frac{\dot{M} \omega}{36\pi r} \frac{H}{W_{r\phi}} - \frac{2}{9r} H - \frac{4\pi r c}{9\dot{M} k_T} \quad (45)$$

$$= \frac{10H}{9r} - \frac{\pi r W_{r\phi}}{3\omega H \dot{M}} - \frac{\dot{M} \omega}{36\pi r} \frac{H}{W_{r\phi}} - \frac{4\pi r c}{9\dot{M} k_T} \quad (46)$$

after that we have $\frac{dW_{r\phi}}{dr}(W_{r\phi}, \dot{M})$, $\frac{d\dot{M}}{dr}(H)$ and $\frac{dH}{dr}(W_{r\phi}, \dot{M}, H)$ (Equations 1, 15 and 44) that depend only on $W_{r\phi}, H$ and \dot{M} and need to be solved to find these three quantities. The boundary conditions are:

- $Q_{\text{rad}}(r_{in}) = 0 \rightarrow H(r_{in}) = 0$ i.e all the energy (Q^+) is advected into the black hole
- $W_{r\phi}(r_{in}) = 0$ i.e. zero-torque boundary condition
- $\dot{M}(r_{sp}) = \dot{M}_0$ i.e. the mass-transfer rate at the spherization radius (a parameter of the problem) is equal to the initial mass-transfer rate