

Accretion disk Equations

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1 Advective Disk with Outflows

From Equation (14) of Lipunova+99

$$\frac{d}{dr} (\dot{M} \omega r^2) = -2\pi \frac{d}{dr} (W_{r\phi} r^2) + \frac{d\dot{M}}{dr} \omega r^2 \quad (1)$$

$$\cancel{\frac{d\dot{M}}{dr} \omega r^2} + \dot{M} \frac{d}{dr} (\omega r^2) = -2\pi \left(\frac{dW_{r\phi}}{dr} r^2 + 2r W_{r\phi} \right) + \cancel{\frac{d\dot{M}}{dr} \omega r^2} \quad (2)$$

$$-\frac{\dot{M}}{2\pi} \frac{d}{dr} (\omega r^2) - 2r W_{r\phi} = \frac{dW_{r\phi}}{dr} r^2 \quad (3)$$

$$-\frac{\frac{1}{2}\dot{M}\omega r'}{2\pi} - 2r' W_{r\phi} = \frac{dW_{r\phi}}{dr} r'^2 \quad (4)$$

$$-\frac{\dot{M}\omega}{4\pi} - 2W_{r\phi} = \frac{dW_{r\phi}}{dr} r \quad (5)$$

$$-\left(\frac{\frac{\dot{M}\omega}{4\pi} + 2W_{r\phi}}{r} \right) = \frac{dW_{r\phi}}{dr} \quad (6)$$

From Equation (9) (Hydrostatic equilibrium) we can eliminate any dependency on the pressure P

$$\frac{P}{\rho H} = \frac{GMH}{r^3} \quad (7)$$

$$\frac{P}{\rho H} = \omega^2 H \quad (8)$$

$$P = \omega^2 H^2 \rho \quad (9)$$

$$\frac{dP}{dr} = \omega^2 H \left(2 \frac{dH}{dr} \rho + H \frac{d\rho}{dr} - 3 \frac{H\rho}{r} \right) \quad (10)$$

From Equation (10) we can further eliminate dependencies on ϵ_{rad} and therefore T , assuming radiation pressure is the dominant pressure (i.e. neglecting gas pressure):

$$P = \frac{\epsilon_{\text{rad}}}{3} \quad (11)$$

From Equation (8) and using the found Equations for the pressure (Equation 7) and the radiative energy (Equation 11) we can find Q_{rad} as a function of H only:

$$Q_{\text{rad}} = \frac{1}{3} \frac{c}{k_{\text{T}} \rho H} \epsilon_{\text{rad}} = \frac{Pc}{k_{\text{T}} \rho H} = \frac{w^2 H^{\frac{1}{2}} \rho c}{k_{\text{T}} \rho H} \quad (12)$$

$$Q_{\text{rad}} = \frac{w^2 H c}{k_{\text{T}}} \quad (13)$$

This implies that the scale height is determined by the radiation pressure of the disk. Using Equation (12) and using the expression of Q_{rad} (Equation 12)

$$\frac{\omega^2 r}{8\pi} \frac{d\dot{M}}{dr} = Q_{\text{rad}} = \frac{\omega^2 H c}{k_{\text{T}}} \quad (14)$$

$$\frac{\omega^2 r}{8\pi} \frac{d\dot{M}}{dr} = \frac{\omega^2 H c}{k_{\text{T}}} \quad (15)$$

$$\frac{d\dot{M}}{dr} = \frac{8\pi H c}{r k_{\text{T}}} \quad (16)$$

Note that the above equation is only valid for disks with outflows. For conservative disks $\frac{d\dot{M}}{dr} = 0$.

Q_{adv} (Equation (22)) can be reduced to $Q_{\text{adv}}(H, \rho, \dot{M})$ using Equations 13, 7 and 11:

$$Q_{\text{adv}} = \frac{\dot{M} \omega^2 H}{4\pi r \rho} \left(6\rho \frac{dH}{dr} - H \frac{d\rho}{dr} - 9 \frac{H\rho}{r} \right) \quad (17)$$

the density can be replaced using the α -prescription

$$W_{r\phi} = -2\alpha H P = -2\alpha H \omega^2 H^2 \rho \quad (18)$$

$$\rho = -\frac{1}{2} \frac{W_{r\phi}}{\alpha \omega^2 H^3} \quad (19)$$

$$\frac{d\rho}{dr} = -\frac{1}{2\alpha H^3 \omega^2} \left(\frac{dW_{r\phi}}{dr} - \frac{3}{H} W_{r\phi} \frac{dH}{dr} + 3 \frac{W_{r\phi}}{r} \right) \quad (20)$$

Finally from the energy Equation ($Q^+ = Q_{\text{rad}} + Q_{\text{adv}}$) and Equation (8) we have that:

$$Q^+ = -\frac{3}{4} \omega W_{r\phi} = \frac{w^2 H c}{k_{\text{T}}} + \frac{\dot{M} \omega^2 H}{4\rho} \left(6\rho \frac{dH}{dr} - H \frac{d\rho}{dr} - 9 \frac{H\rho}{r} \right) \quad (21)$$

we can now replace $\frac{d\rho}{dr}$ from Equation 21 to obtain $\frac{dH}{dr}$:

$$\frac{dH}{dr} = \frac{9 \frac{H\rho}{r} - \left(\frac{3\pi r \rho W_{r\phi}}{\dot{M} \omega H} + \frac{\frac{dW_{r\phi}}{dr}}{2\alpha H^2 \omega^2} + \frac{3}{2} \frac{W_{r\phi}}{\alpha r H^2 \omega^2} + \frac{4\pi r \rho c}{\dot{M} k_{\text{T}}} \right)}{6\rho - \frac{3}{2} \frac{W_{r\phi}}{\alpha H^3 \omega^2}} \quad (22)$$

we can further eliminate ρ using Equation 18

$$\frac{dH}{dr} = \frac{1}{9} \left(\frac{12H}{r} - \frac{3\pi r W_{r\phi}}{\omega H \dot{M}} + \frac{dW_{r\phi}}{dr} \frac{H}{W_{r\phi}} - \frac{4\pi r c}{\dot{M} k_T} \right) \quad (23)$$

after that we have $\frac{dW_{r\phi}}{dr}(W_{r\phi}, \dot{M})$, $\frac{d\dot{M}}{dr}(H)$ and $\frac{dH}{dr}(W_{r\phi}, \dot{M}, H)$ (Equations 1, 14 and 23) that depend only on $W_{r\phi}$, H and \dot{M} and need to be solved to find these three quantities. The boundary conditions are:

- $Q_{\text{rad}}(r_{in}) = 0 \rightarrow H(r_{in}) = 0$ i.e all the energy (Q^+) is advected into the black hole
- $W_{r\phi}(r_{in}) = 0$ i.e. zero-torque boundary condition
- $\dot{M}(r_{sp}) = \dot{M}_0$ i.e. the mass-transfer rate at the spherization radius (a parameter of the problem) is equal to the initial mass-transfer rate