Accretion disk Equations

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1 Advective Disk with Outflows

From Equation (14) of Lipunova+99

$$
\begin{aligned}
\frac{d}{dr}\left(\dot{M}\omega r^2\right) &= -2\pi\frac{d}{dr}\left(W_{r\phi}r^2\right) + \frac{d\dot{M}}{dr}\omega r^2 && (1)\\
\cancel{\frac{d\dot{M}}{dr}\omega r^2} + \dot{M}\frac{d}{dr}\left(\omega r^2\right) &= -2\pi\left(\frac{dW_{r\phi}}{dr}r^2 + 2rW_{r\phi}\right) + \cancel{\frac{d\dot{M}}{dr}\omega r^2} && (2)\\
-\frac{\dot{M}\frac{d}{dr}\left(\omega r^2\right)}{2\pi} - 2rW_{r\phi} &= \frac{dW_{r\phi}}{dr}r^2 && (3)\\
-\frac{\frac{1}{2}\dot{M}\omega \cancel{r}}{2\pi} - 2\cancel{r}W_{r\phi} &= \frac{dW_{r\phi}}{dr}r^{\cancel{2}} && (4)\\
-\frac{\dot{M}\omega}{4\pi} - 2W_{r\phi} &= \frac{dW_{r\phi}}{dr}r && (5)\\
-\left(\frac{\frac{\dot{M}\omega}{4\pi} + 2W_{r\phi}}{r}\right) &= \frac{dW_{r\phi}}{dr} && (6)
\end{aligned}
$$

From Equation (9) (Hydrostatic equilibrium) we can eliminate any dependency on the pressure P

$$
\begin{aligned}
\frac{P}{\rho H} &= \frac{GMH}{r^3} && (7)\\
\frac{P}{\rho H} &= \omega^2 H && (8)\\
P &= \omega^2 H^2 \rho && (9)\\
\frac{dP}{dr} &= \omega^2 H\left(2\frac{dH}{dr}\rho + H\frac{d\rho}{dr} - 3\frac{H\rho}{r}\right) && (10)
\end{aligned}
$$

From Equation (10) we can further eliminate dependencies on ϵ_{rad} and therefore T, assuming radiation pressure is the dominant pressure (i.e. neglecting gas pressure):

$$P = \frac{\epsilon_{\mathrm{rad}}}{3} \quad (11)$$

From Equation (8) and using the found Equations for the pressure (Equation 7) and the radiative energy (Equation 11) we can find Q_{rad} as a function of H only:

$$Q_{\mathrm{rad}} = \frac{1}{3}\frac{c}{k_{\mathrm{T}}\rho H}\epsilon_{\mathrm{rad}} = \frac{Pc}{k_{\mathrm{T}}\rho H} = \frac{w^2 H^{\cancel{2}}\cancel{\rho} c}{k_{\mathrm{T}}\cancel{\rho H}} \tag{12}$$

$$Q_{\mathrm{rad}} = \frac{w^2 H c}{k_{\mathrm{T}}} \tag{13}$$

This implies that the scale height is determined by the radiation pressure of the disk. Using Equation (12) and using the expression of Q_{rad} (Equation 12)

$$\frac{\omega^2 r}{8\pi}\frac{d\dot{M}}{dr} = Q_{\mathrm{rad}} = \frac{\omega^2 H c}{k_{\mathrm{T}}} \tag{14}$$

$$\frac{\cancel{\omega^2} r}{8\pi}\frac{d\dot{M}}{dr} = \frac{\cancel{w^2} H c}{k_{\mathrm{T}}} \tag{15}$$

$$\frac{d\dot{M}}{dr} = \frac{8\pi H c}{r k_{\mathrm{T}}} \tag{16}$$

Note that the above equation is only valid for disks with outflows. For conservative disks $\frac{d\dot{M}}{dr} = 0$.

Q_{adv} (Equation (22)) can be reduced to $Q_{\mathrm{adv}}(H, \rho, \dot{M})$ using Equations 13, 7 and 11:

$$Q_{\mathrm{adv}} = \frac{\dot{M}\omega^2 H}{4\pi\rho}\left(6\frac{dH}{dr} - H\frac{d\rho}{dr} - 9\frac{H\rho}{r}\right) \tag{17}$$

the density can be replaced using the α-prescription

$$W_{r\phi} = -2\alpha H P = -2\alpha H \omega^2 H^2 \rho \tag{18}$$

$$\rho = -\frac{1}{2}\frac{W_{r\phi}}{\alpha\omega^2 H^3} \tag{19}$$

$$\frac{d\rho}{dr} = -\frac{1}{2\alpha H^3\omega^2}\left(\frac{dW_{r\phi}}{dr} - \frac{3}{H}W_{r\phi}\frac{dH}{dr} + 3\frac{W_{r\phi}}{r}\right) \tag{20}$$

Finally from the energy Equation ($Q^+ = Q_{\mathrm{rad}} + Q_{\mathrm{adv}}$) and Equation (8) we have that:

$$Q^+ = -\frac{3}{4}\omega W_{r\phi} = \frac{w^2 H c}{k_{\mathrm{T}}} + \frac{\dot{M}\omega^2 H}{4\pi\rho}\left(6\frac{dH}{dr} - H\frac{d\rho}{dr} - 9\frac{H\rho}{r}\right) \tag{21}$$

we can now replace ρ and $\frac{d\rho}{dr}$ from Equation 21 to obtain $\frac{dH}{dr}$:

$$\frac{dH}{dr} = \frac{9\frac{H}{\rho} - \left(\frac{3\pi r\rho W_{r\phi}}{\dot{M}\omega H} + \frac{\frac{dW_{r\phi}}{dr}}{2\alpha H^2\omega^2} + \frac{3}{2}\frac{W_{r\phi}}{\alpha r H^2\omega^2} + \frac{4\pi r\rho c}{\dot{M}k_T}\right)}{6\rho - \frac{3}{2}\frac{W_{r\phi}}{H^3\omega^2}} \tag{22}$$

after that we have $\frac{dW_{r\phi}}{dr}(W_{r\phi}, \dot{M})$, $\frac{d\dot{M}}{dr}(H)$ and $\frac{dH}{dr}(W_{r\phi}, \dot{M}, H)$ (Equations 1, 14 and 22) that depend only on $W_{r\phi}, H$ and $\dot{M}$ and need to be solved to find these three quantities. The boundary conditions are:

- $Q_{\rm rad}(r_{in}) = 0 \rightarrow H(r_{in}) = 0$ i.e all the energy (Q^{+}) is advected into the black hole
- $W_{r\phi}(r_{in}) = 0$ i.e. zero-torque boundary condition
- $\dot{M}(r_{sp}) = \dot{M}_0$ i.e. the mass-transfer rate at the spherization radius (a parameter of the problem) is equal to the initial mass-transfer rate