PRACTICE 4 - SUPPORT VECTOR MACHINE

Andrés Herencia and Antonio Fernández

MUTECI 2023-2024

Table of Contents

Exercise 1

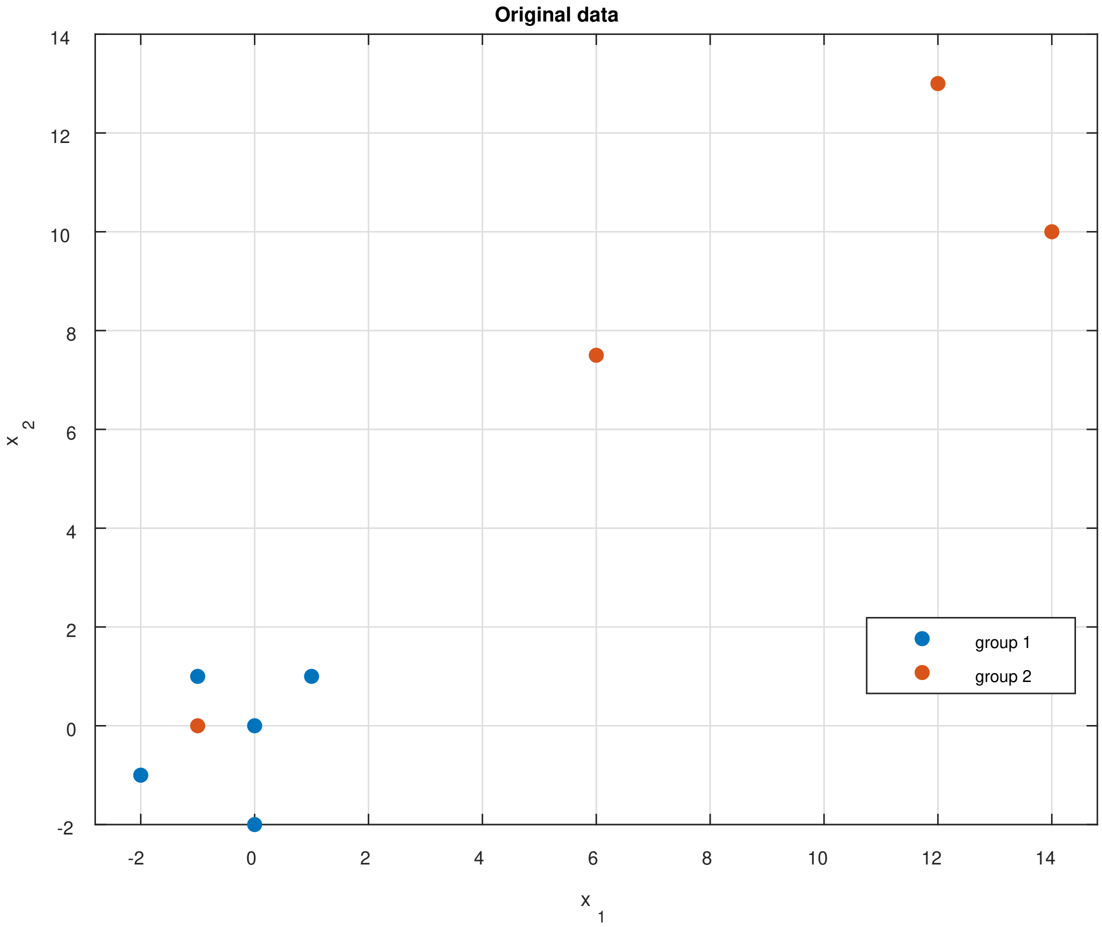
From the data given in the file `svm_nolineal.txt`, the following is requested:

- a) Apply the SVM model to separate lineally data into two categories.
- b) Apply the soft model with different values of the parameter `C > 0` to solve the classification problem.
- c) Use a polynomial kernel to solve the binary classification problem. If this is not possible, use the soft approach with the polynomial kernel by varying the parameter `C`.

NOTE*: use quadratic programming software and, if applicable, the appropriate kernel function.*

Firstly, we will represent our data.

```
clc, clear, clf
cd 'C:\Users\andre\Documents\UNIVERSIDAD\MUTECI\BLOQUE 2\TOPT\lab\P4'
% cd 'C:\Users\anton\OneDrive\TOPT\Práctica 4'

data = importdata('svm_nolineal.txt', '\t');
X1 = data.data(:, 1); X2 = data.data(:, 2); Y = data.data(:,3);

figure(1)
gscatter(X1, X2, Y)
xlabel("x_1")
ylabel("x_2")
grid on
legend('group 1', 'group 2')
title('Original data')
```



a) Apply the SVM model to separate lineally data into two categories.

The primal support vector machine problem will be:

$$P : \begin{cases} \min_{w,b} & \frac{1}{2}\|w\| \\ \text{such to} & -y^i(w^t x^i - b) \leq -1, \quad \forall i \in \{1, 2, \ldots, m\} \end{cases}$$

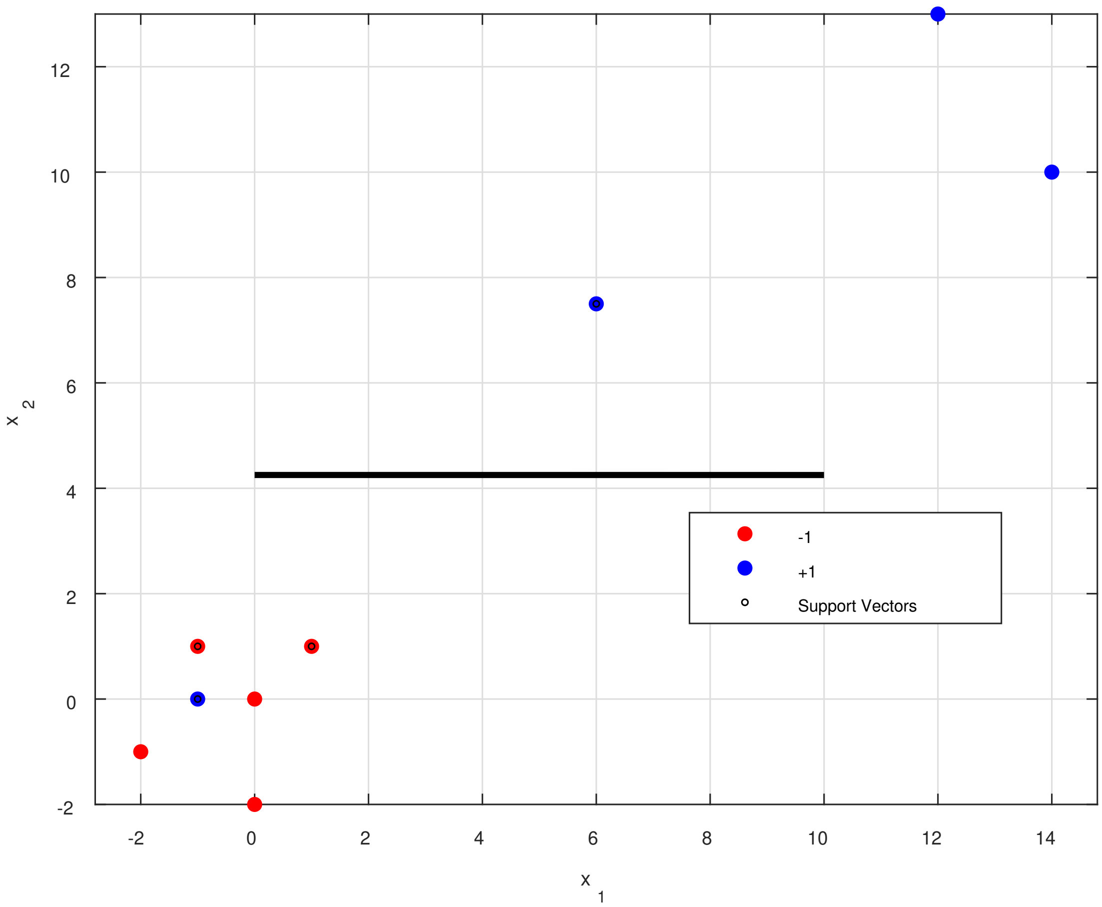
```
svm=fitcsvm([X1,X2],Y);
svm.Beta
```

```
ans = 2x1
    0.0003
    0.3075
```

```
b = svm.Bias
```

```
b = -1.3076
```

```
d = 0.02;
[x1Grid,x2Grid] = meshgrid(0:d:10,0:d:10);

xGrid = [x1Grid(:),x2Grid(:)];
[~,scores] = predict(svm,xGrid);

figure;
h(1:2) = gscatter(X1,X2,Y,'rb','.');
hold on
X=[X1,X2];
h(3) = plot(X(svm.IsSupportVector,1),X(svm.IsSupportVector,2),'ko', 'MarkerSize',2);
contour(x1Grid,x2Grid,reshape(scores(:,2),size(x1Grid)),[0 0],'k', 'LineWidth',2);
legend(h,{'-1','+1','Support Vectors'});
grid on
xlabel("x_1")
ylabel("x_2")
hold off
```



How could we predict just by looking at the data, we cannot separate the two categories linearly. We will see that the quadratic problem is not feasible.

```
n = 2;
m = size(X1, 1);
H=zeros(n+1);
```

```
%indpos=Y>0;
%indneg=Y<0;
H(1:n,1:n)= eye(n);
f=zeros(n+1,1);
A=zeros(m,n+1);
A(:,1)=-1*Y.* X1;
A(:,2)=-1*Y.* X2;
A(:,3)=Y;
b= -1*ones(m,1);

quadprog(H,f,A,b,[],[],[],[]);
```

```
No feasible solution found.

quadprog stopped because it was unable to find a point that satisfies
the constraints within the value of the constraint tolerance.

<stopping criteria details>
```

b) Apply the soft model with different values of the parameter `C > 0` to solve the classification problem.

NOTE: *In this section, we will make use of a pre-built function,* `fitcsvm`*. The soft model primal-dual problem is solved using quadratic programming in Exercise 2.*

The soft model has the following form:

$$P : \begin{cases} \min_{w,b} & \frac{1}{2}\|w\|^2 + C\left(\sum_{i=1}^{m}\xi_i\right) & \\ \text{such to} & y^i(w^t x^i - b) \geq 1 - \xi_i, & \forall i \in \{1,2,\dots,m\} \\ & \xi_i \geq 0, & \forall i \in \{1,2,\dots,m\} \end{cases}$$

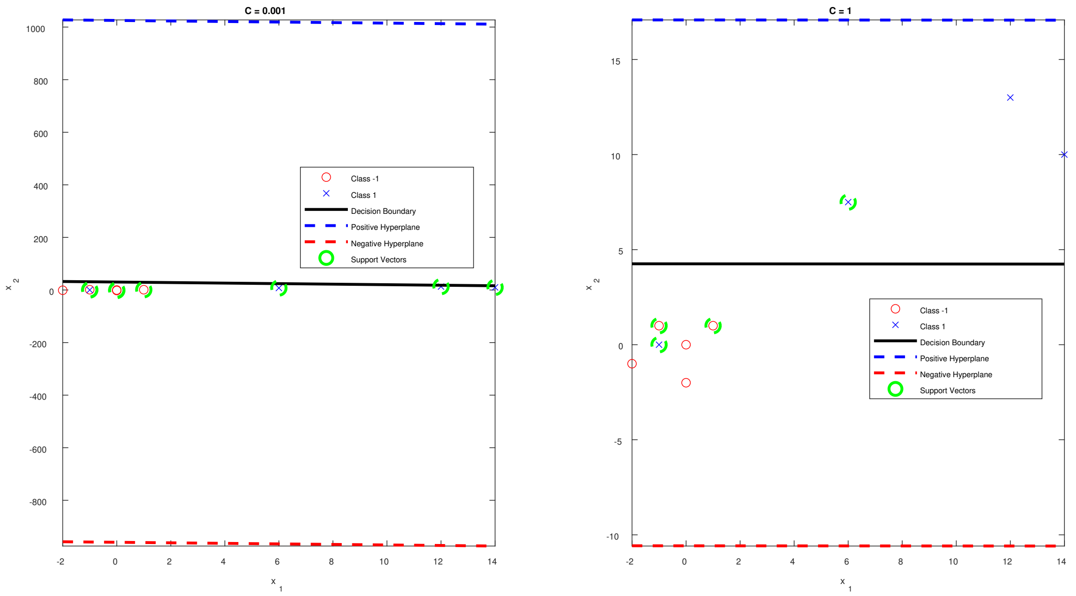
The dual problem is:

$$D_C : \begin{cases} \min_{a\in\Re^m} & \left\{\frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{m}\alpha_i\alpha_j y^i y^j (x^j)^t x^i - \sum_{i=1}^{m}\alpha_i\right\} \\ \text{such to} & \sum_{i=1}^{m}\alpha_i y^i = 0 \quad \forall i \in \{1,2,\dots,m\} \\ & 0 \leq \alpha_i \leq C, \quad \forall i \in \{1,2,\dots,m\} \end{cases}$$

We will try different C values : `[0.01, 1]`. We built a function that plot the decission boundary, which can be found on annexes.

```
C_values = [0.001, 1]; % Adjust this list as needed

% Initialize a cell array to store the SVM models
models = cell(length(C_values), 1);

% Train SVM models with different values of C
for i = 1:length(C_values)
    C = C_values(i);
    SVMModel = fitcsvm(X, Y, 'KernelFunction', 'linear', 'BoxConstraint', C);
    models{i} = SVMModel;
end
```

```
cols = 2;
rows = round(length(C_values)/cols);

% Plot the data and decision boundaries for each model
figure('Position', [100, 100, 1200, 600]);

for i = 1:length(C_values)
    subplot(rows, cols, i); % Adjust subplot parameters based on the number of C values
    SVMModel = models{i};
    h = plotDecisionBoundary(X, Y, SVMModel);
    title(['C = ' num2str(C_values(i))]);
    legend('Location', 'Best');
end
```



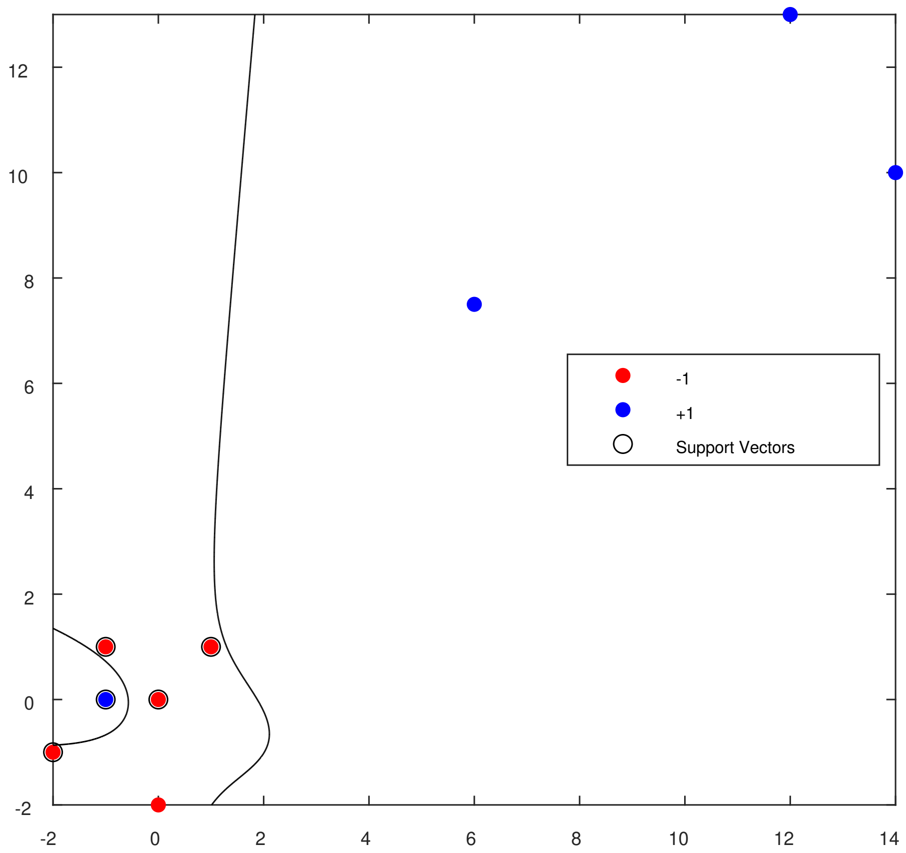
We can observe how the separation line changes its slope. For C values greater than 1, there are no differences in position or slope. Additionally, it can be observed that from $C = 0.001$ onwards, the accuracy does not improve.

c) Use a polynomial kernel to solve the binary classification problem. If this is not possible, use the soft approach with the polynomial kernel by varying the parameter C.

Firstly, we will try to solve the problem with $C = \infty$.

```
svm=fitcsvm(X,Y,'KernelFunction','polynomial');

sv = svm.SupportVectors;
var_dual = zeros(m,1);
var_dual(svm.IsSupportVector)=svm.Alpha;

[x1Grid,x2Grid] = meshgrid(linspace(min(X(:, 1)), max(X(:, 1)), 1000), ...
                           linspace(min(X(:, 2)), max(X(:, 2)), 1000));

xGrid = [x1Grid(:),x2Grid(:)];
```

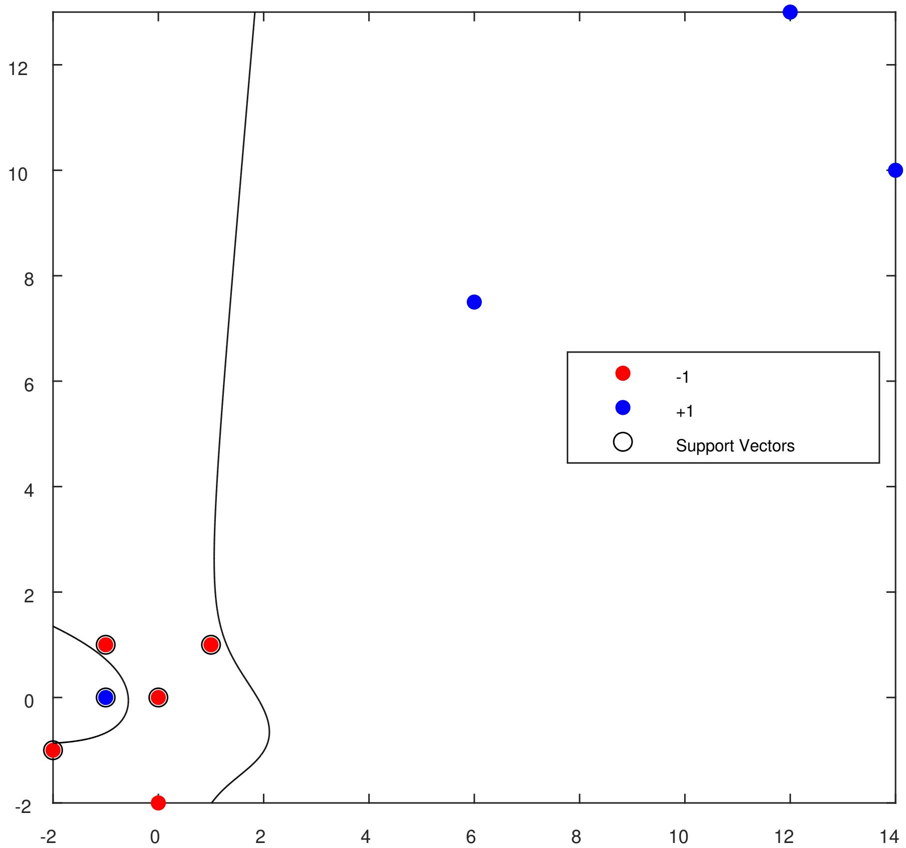
```
[~,scores] = predict(svm,xGrid);

figure;
h(1:2) = gscatter(X(:,1),X(:,2),Y,'rb','.');
hold on

h(3) = plot(X(svm.IsSupportVector,1),X(svm.IsSupportVector,2),'ko');
contour(x1Grid,x2Grid,reshape(scores(:,2),size(x1Grid)),[0 0],'k');
legend(h,{'-1','+1','Support Vectors'});
axis equal
hold off
```



It can be observed that the polynomial kernel perfectly solves the problem, and therefore, a relaxed approach is not necessary. Anyway, let's see that with $C = 1$, the result doesn't change.

```
svm = fitcsvm(X,Y,'KernelFunction','polynomial','BoxConstraint', 1);

[~,scores] = predict(svm,xGrid);

figure;
h(1:2) = gscatter(X(:,1),X(:,2),Y,'rb','.');
hold on

h(3) = plot(X(svm.IsSupportVector,1),X(svm.IsSupportVector,2),'ko');
```

```
contour(x1Grid,x2Grid,reshape(scores(:,2),size(x1Grid)),[0 0],'k');
legend(h,{'-1','+1','Support Vectors'});
axis equal
hold off
```



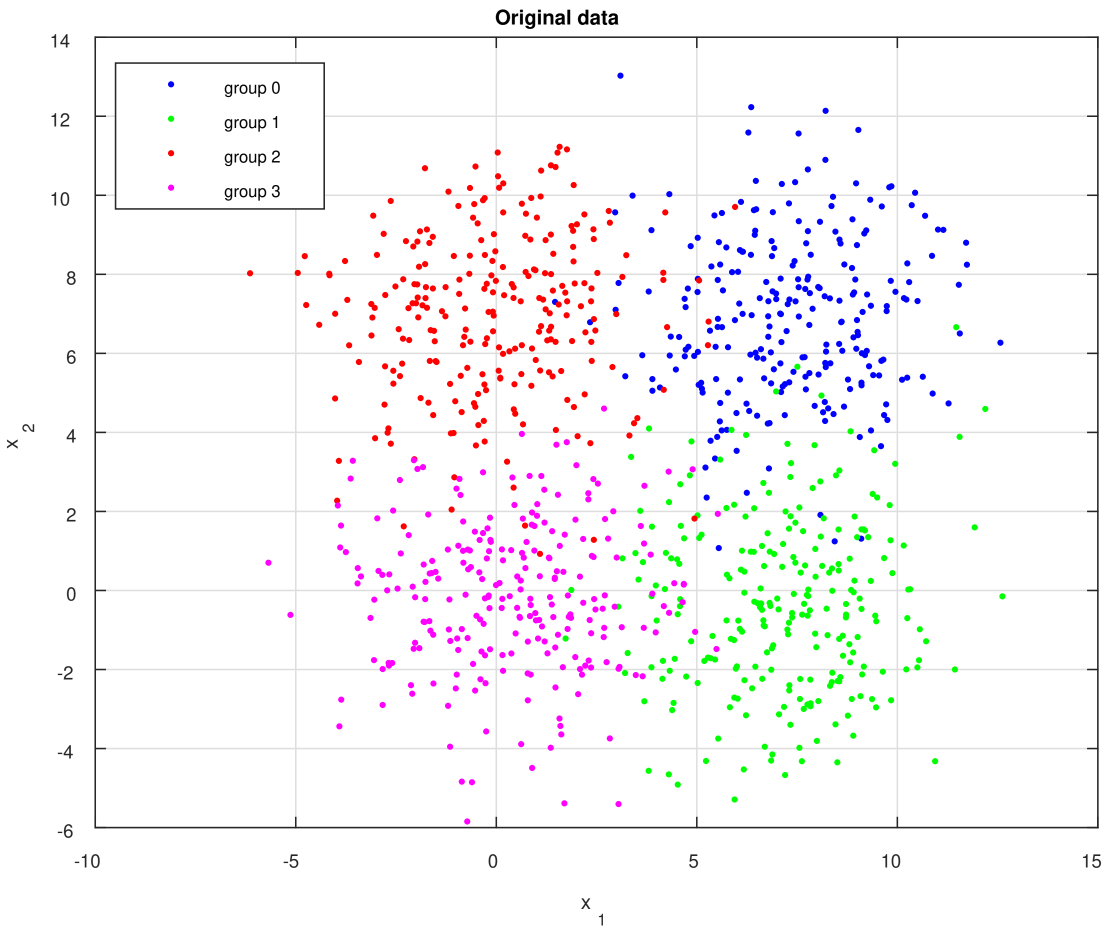
Exercise 2

From the data given in the file square4.txt or square4.xlsx, the following is requested:

- a) Apply a Support Vector Machine for separating lineally the groups 0 and 2.
- b) Apply the soft model with different values for the parameter `C>0` to solve the classification problem.
- c) Determine which value of `C` is the most suited for this problem. Justify the answer.

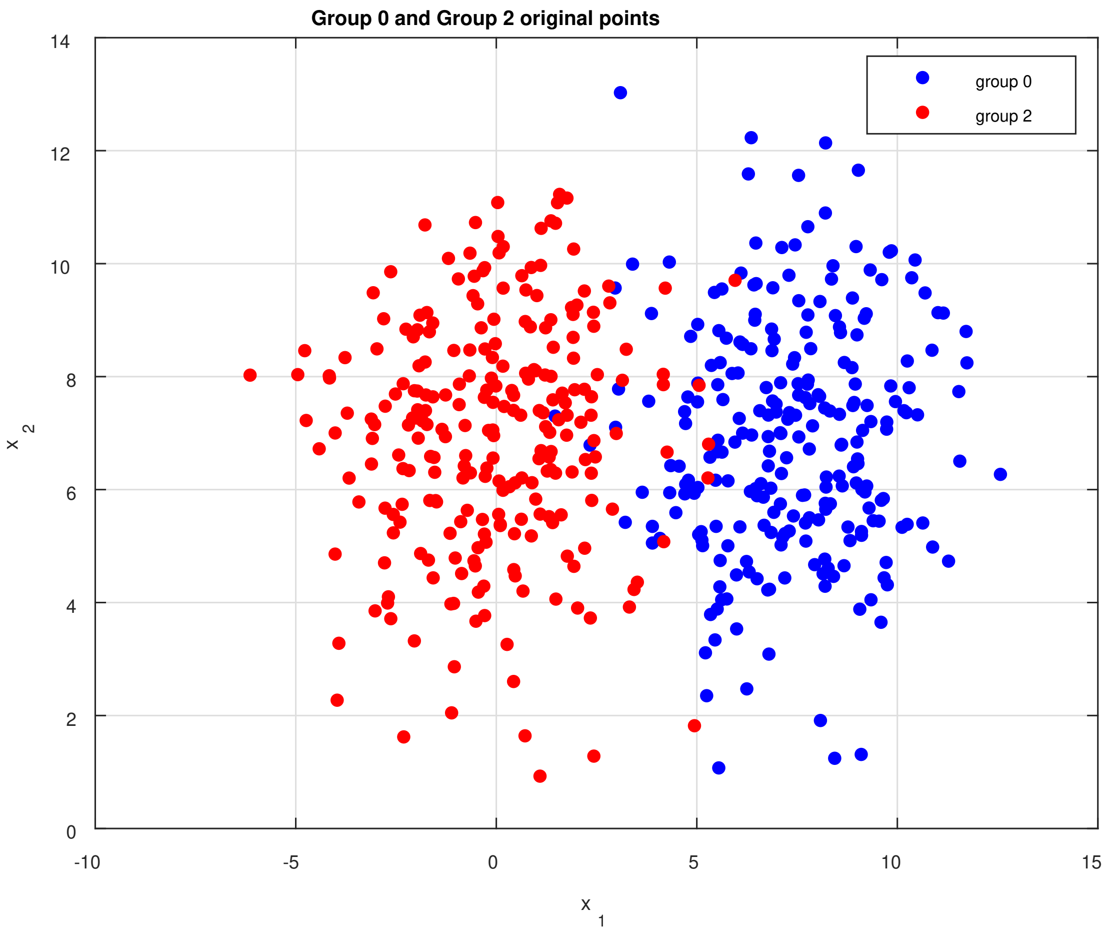
```
clear; clc; clf

data = readmatrix('square4.txt');
X1 = data(:, 1); X2 = data(:, 2); Y = data(:, 3);

figure(1)
gscatter(X1, X2, Y, 'bgrm')
grid on
xlabel("x_1")
ylabel("x_2")
```

```
legend('group 0', 'group 1','group 2', 'group 3')
title('Original data')
```



a) Apply a Support Vector Machine for separating lineally the groups 0 and 2.

```
ind1 = Y==0;
ind2 = Y==2;
x1 = X1(ind1 | ind2);
x2 = X2(ind1 | ind2);
y = Y(ind1 | ind2);
y(y==0) = -1; y(y==2) = +1;
figure(2)
gscatter(x1, x2, y, 'br')
grid on
xlabel("x_1")
ylabel("x_2")
legend('group 0', 'group 2')
title('Group 0 and Group 2 original points')
```



We can observe graphically that there is **no linear separator for this data groups**. If we try to optimize the primal problem for the SVM...

```
% 1. Quadratic term of the objective function
n = 2; % number of groups
m = size(x1,1); % size of each group
H = zeros(n+1); % hessian
H(1:n,1:n) = eye(n)
```

```
H = 3x3
     1     0     0
     0     1     0
     0     0     0
```

```
% 2. Linear term of the objective function (0)
f = zeros(n+1,1);
% 3. A matrix with linear conditions
A = zeros(m,n+1);
A(:,1) = -1*y.* x1;
A(:,2) = -1*y.* x2;
A(:,3) = y;
```

```
A
```

```
A = 500x3
    6.2853   11.5879   -1.0000
    9.3978    5.4489   -1.0000
    5.7362    8.6803   -1.0000
    6.1020    9.8327   -1.0000
    9.2948    5.6737   -1.0000
    8.0270    7.6795   -1.0000
    5.0270    6.0370   -1.0000
    3.0486    7.7805   -1.0000
    2.9635    9.5684   -1.0000
    6.8647    8.8412   -1.0000
```

```
% b term
b = -1*ones(m,1);
[wb,ff,exitflag] = quadprog(H,f,A,b,[],[],[],[])
```

```
No feasible solution found.

quadprog stopped because it was unable to find a point that satisfies
the constraints within the value of the constraint tolerance.

<stopping criteria details>
wb = 3x1
   -0.2891
    0.0096
   -0.9591

ff = 0.0418
exitflag = -2
```

We can see that, indeed, there is no feasible solution to this problem.

A lot of points have been misclassified. Then, we need to "relax" this model for penalizing those points that would have been misclassified using a soft model.

b) Apply the soft model with different values for the parameter `C>0` to solve the classification problem.

The soft model has the following form:

$$P : \begin{cases} \min_{w,b} & \frac{1}{2}\|w\|^2| + C\left(\sum_{i=1}^{m}\xi_i\right) & \\ \text{such to} & y^i(w^t x^i - b) \geq 1 - \xi_i, & \forall i \in \{1,2,\ldots,m\} \\ & \xi_i \geq 0, & \forall i \in \{1,2,\ldots,m\} \end{cases}$$

From the previous model, they have been included m variables $\xi_i \geq 0$ to identify those points x^i that have been misclassified. These variables act on those points that are on the wrong side of the plane and penalize them based on their value.

On the other hand, the value of C penalizes failure to comply with prior constraints. This is, the higher the value, the greater the penalization. In the limit case, when $C \to \infty$ tends to be infinite, the previous model converges to the first model with the proposed separator hyperplane (H), narrowing the band where the hyperplane H^+ and H^- are located. Meanwhile, a low value ($C \to 0$) will widen the strip, allowing it to contain more points that should be misclassified.

The dual problem is:

$$D_C : \begin{cases} \min_{a \in \Re^m} & \left\{ \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^m \alpha_i \alpha_i y^i y^j (x^j)^t x^i - \sum_{i=1}^m \alpha_i \right\} \\ \text{such to} & \sum_{i=1}^m \alpha_i y^i = 0 \quad \forall i \in \{1, 2, \ldots, m\} \\ & 0 \leq \alpha_i \leq C, \quad \forall i \in \{1, 2, \ldots, m\} \end{cases}$$

We can determine the vector $\bar{w}$ in the same way as the strict model does, as a function of $\bar{\alpha}$: $\bar{w} = \sum_{i=1}^m \bar{\alpha}_i y^i x^i$.

Thus, to solve this problem we have to:

- 1.- Characterize the quadratic term of the objective function.
- 2.- Compute the linear term of the objective function.
- 3.- Obtain the equation matrix and the independent term of the objective function.
- 4.- Apply the model with some values of C and represent graphically their hyperplane bounds. We will take the following values of C in a logarithmic scale, between `1e-4` and `1e+4`.
- 5.- Compare the models (in section c).

```
% 1.
H=(([x1,x2]).*y)*([x1,x2].*y)';
```

```
% 2.
f=-1*ones(m,1);
```

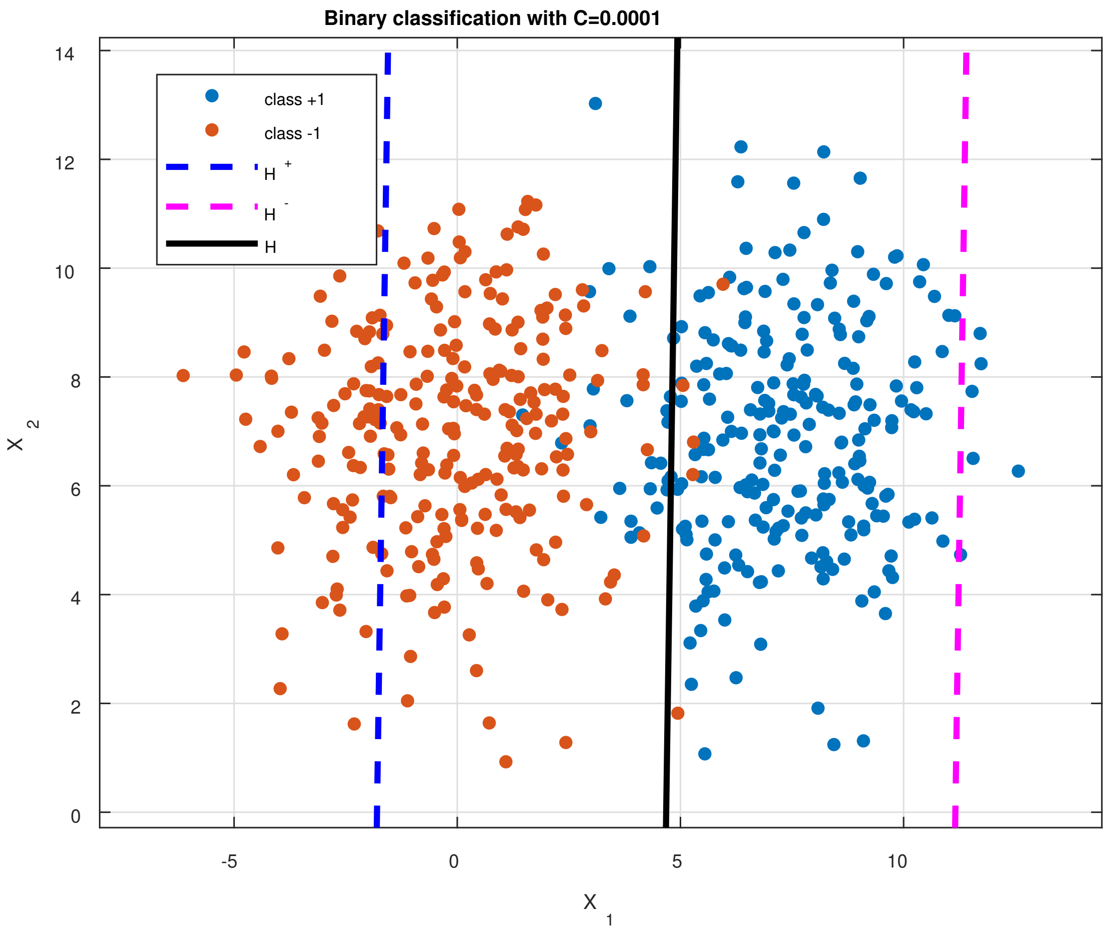
```
% 3.
Aeq = zeros(1,m);
Aeq(1,:) = y';
```

```
beq = 0;
```

Parameter C $= 10^{-4}$ The `svm_soft` function (own-made) can be found on annexes.

```
C0 = 1e-4;
[pr0, ph0] = svm_soft(x1,x2,y,H,f,Aeq,beq,C0,true)
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

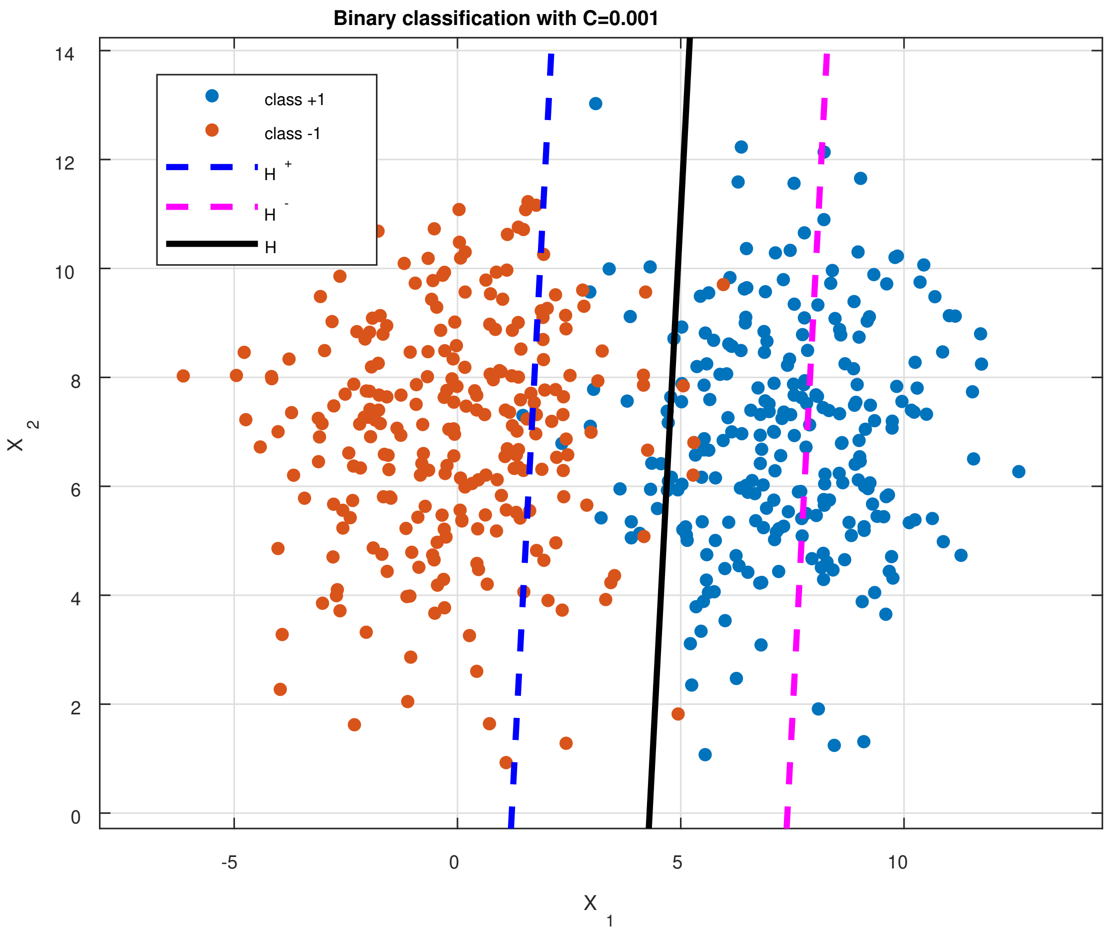



```
pr0 = 0.9420
ph0 = 0.1360
```

Parameter C $= 10^{-3}$

```
C1 = 0.001;
[pr1, ph1] = svm_soft(x1,x2,y,H,f,Aeq,beq,C1,true)
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

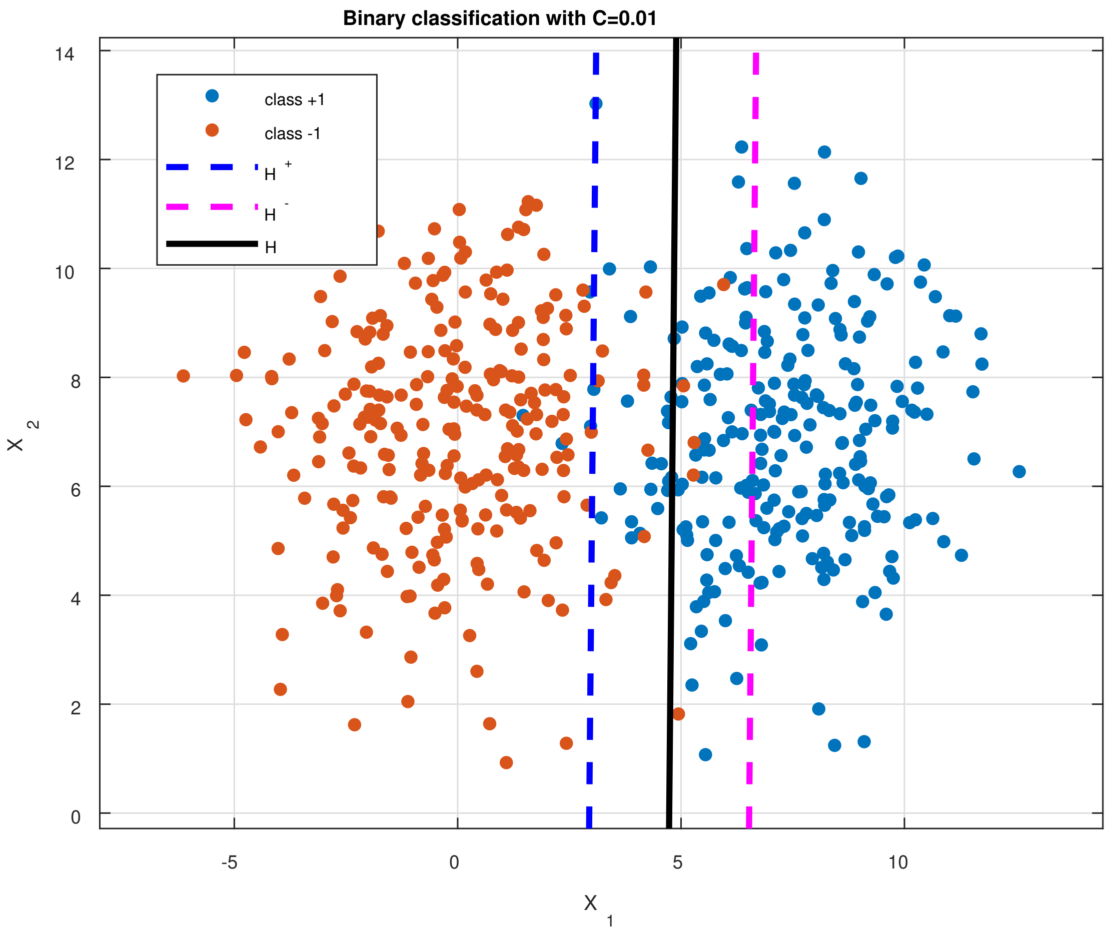



```
pr1 = 0.9440
ph1 = 0.6040
```

Parameter C = 0.01

```
C2 = 1e-2;
[pr2, ph2] = svm_soft(x1,x2,y,H,f,Aeq,beq,C2,true)
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

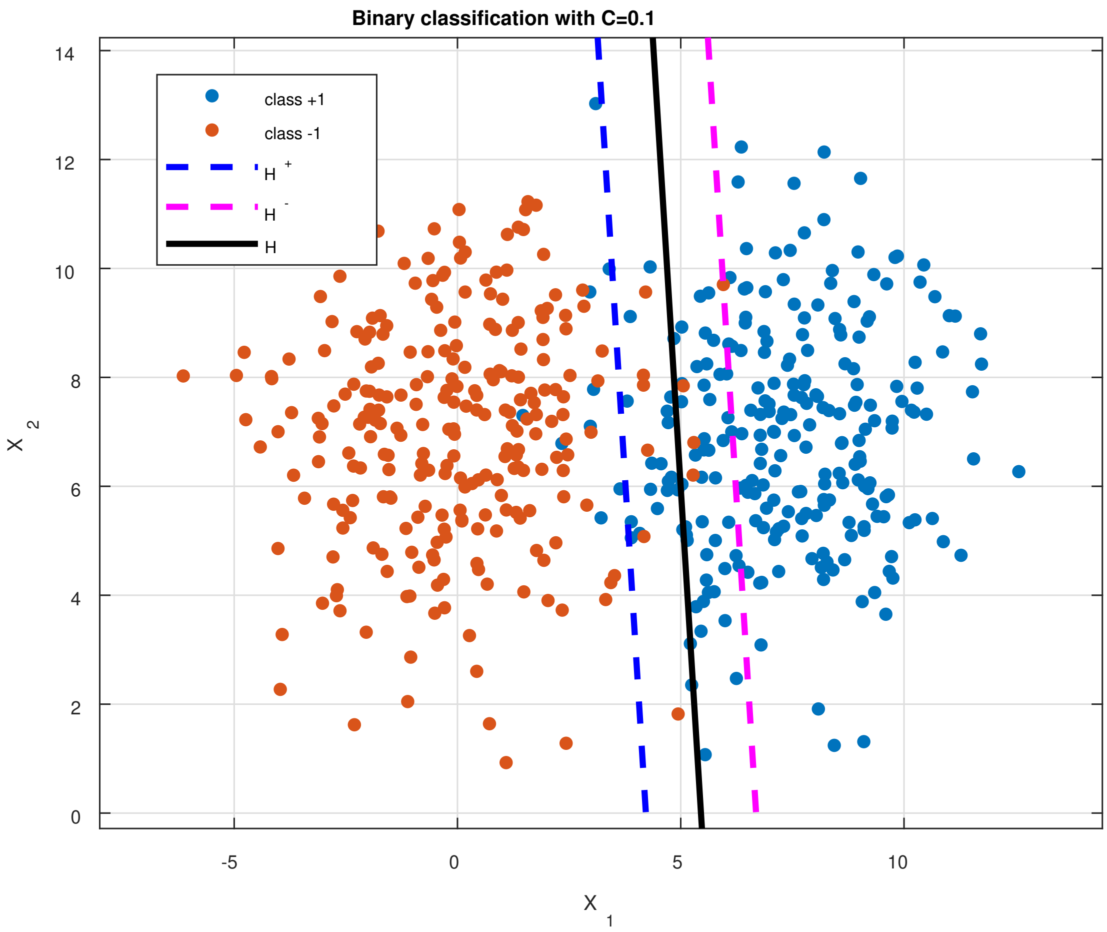



```
pr2 = 0.9400
ph2 = 0.7980
```

Parameter C = 0.1

```
C3 = 1e-1;
[pr3, ph3] = svm_soft(x1,x2,y,H,f,Aeq,beq,C3,true)
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

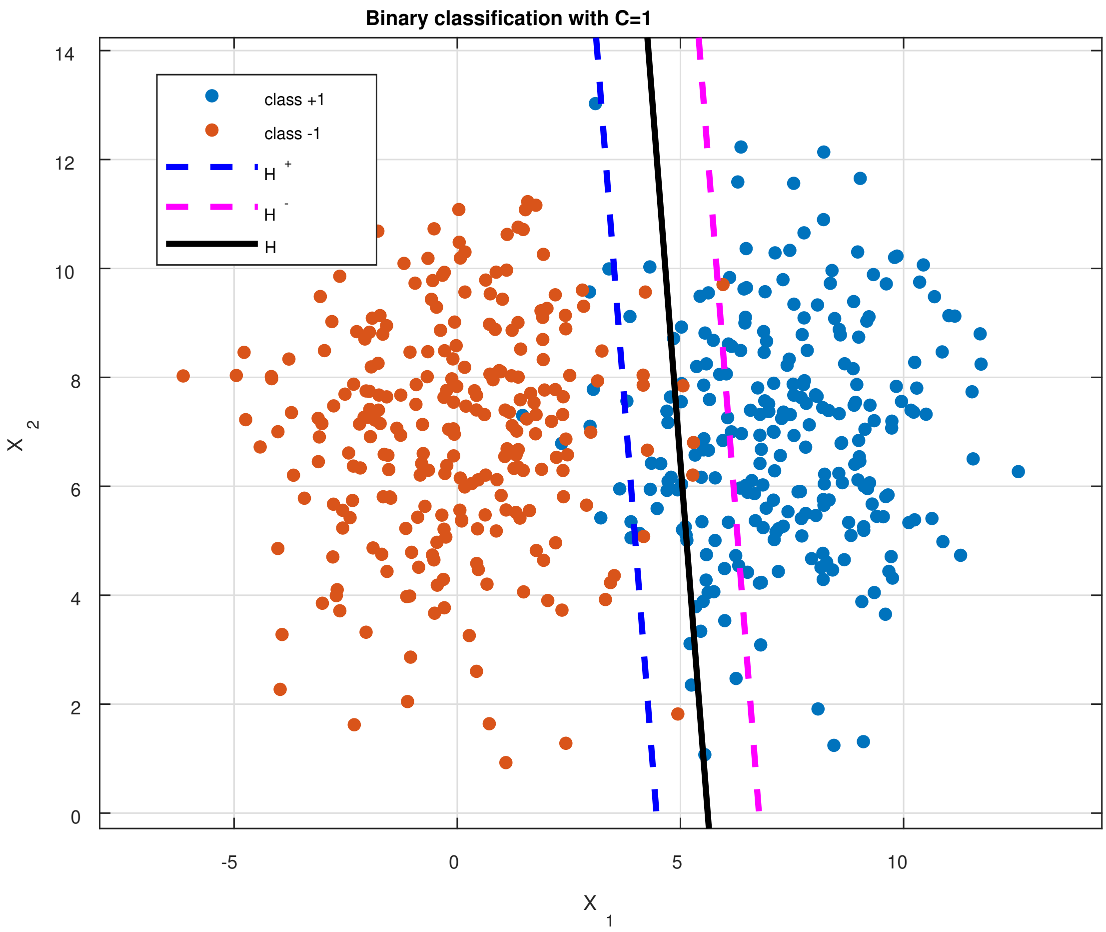



```
pr3 = 0.9340
ph3 = 0.8560
```

Parameter C = 1

```
C4 = 1;
[pr4, ph4] = svm_soft(x1,x2,y,H,f,Aeq,beq,C4,true)
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```




```
pr4 = 0.9300
ph4 = 0.8640
```

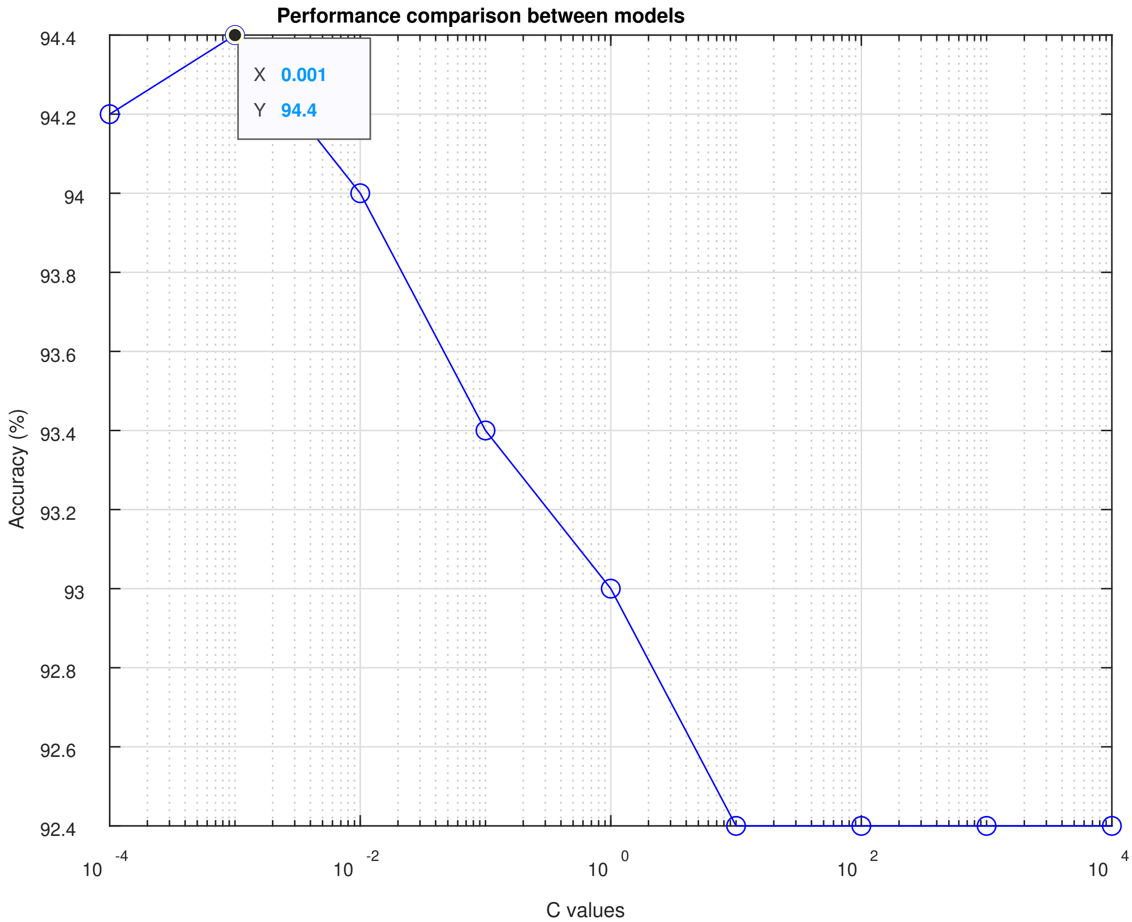
From now on forward, the quadratic optimization problem associated with computing the optimum hyperplane distance returns the same result. Then, we will not show those plots.

c) Determine which value of C is the most suited for this problem. Justify the answer.

Now we must determine which value of C is the most suited for this binary classification.

```
% Definir tus datos
pr = [pr0, pr1, pr2, pr3, pr4, pr5, pr6, pr7, pr8].*100;
C = [C0, C1, C2, C3, C4, C5, C6, C7, C8];

figure;
semilogx(C, pr, 'bo-');
ylabel('Accuracy (%)');
xlabel('C values');

grid on;
title('Performance comparison between models');

ax = gca;
chart = ax.Children(1);
```

```
datatip(chart,0.001,94.4);
```



As we can see, the accuracy gets better when $C = 10^{-3}$. But in general, a low value of C allows us to get better results, because both groups are too mixed between them, and a wider margin penalizes less those points.

Annexes

Exercise 1

Plot decision boundary for SVM soft model

This function takes a SVMModel trained with the package `fitcsvm`.

```
function h = plotDecisionBoundary(X, Y, SVMModel)
%% Description:
%  Plots the decision boundary, positive hyperplane, negative hyperplane,
%  and support vectors for the given SVM model.
%
%  Inputs:
%  - X: Input data matrix with two features.
%  - Y: Vector of class labels (-1 or 1).
%  - SVMModel: SVM model trained using fitcsvm.
%
%  Outputs:
%  - h: Figure handle.

gscatter(X(:, 1), X(:, 2), Y, 'rb', 'ox');
hold on;

% Extract parameters
k = SVMModel.KernelParameters.Scale;
b = SVMModel.Bias;

% Calculate slope and intercept
slope = -SVMModel.Beta(1) / SVMModel.Beta(2);
intercept = b / SVMModel.Beta(2);

% Plot decision boundary
xline = linspace(min(X(:, 1)), max(X(:, 1)), 100);
yline = slope * xline - intercept;
plot(xline, yline, 'k-', 'LineWidth', 2);

% Plot positive hyperplane
yline_pos = slope * xline - (intercept - 1) / SVMModel.Beta(2);
plot(xline, yline_pos, 'b--', 'LineWidth', 2);

% Plot negative hyperplane
yline_neg = slope * xline + (intercept + 1) / SVMModel.Beta(2);
plot(xline, yline_neg, 'r--', 'LineWidth', 2);

% Highlight support vectors
idx = SVMModel.IsSupportVector;
plot(X(idx, 1), X(idx, 2), 'go', 'MarkerSize', 10, 'LineWidth', 2);
hold off;
xlabel('x_1');
ylabel('x_2');
legend('Class -1', 'Class 1', 'Decision Boundary', ...
    'Positive Hyperplane', 'Negative Hyperplane', 'Support Vectors');
```

```
axis tight;
h = gcf;

end
```

Exercise 2

SVM soft model using quadratic programming.

```
function [performance_rate, points_bet_hyp] = svm_soft(x1, x2, y, H, f, Aeq, beq, C, plt)
%% Description:
% Own-made MATLAB function implements a Support Vector Machine (SVM) with a
% soft margin for binary classification.
%
% Inputs:
% - x1: Feature values for the first dimension.
% - x2: Feature values for the second dimension.
% - y: Class labels (+1 or -1).
% - H: Hessian matrix in the quadratic programming problem.
% - f: Linear coefficient vector in the quadratic programming problem.
% - Aeq: Coefficients for linear equality constraints.
% - beq: Values for linear equality constraints.
% - C: Regularization parameter for the soft margin.
% - plt: Boolean indicating whether to plot the decision boundaries and support vectors (true
%
% Outputs:
% - performance_rate: The rate of correctly classified data points.
% - points_bet_hyp: The proportion of data points located between the decision boundaries.

% Lower and upper bound for the alpha value.
m= size(x1,1);
lb = zeros(m,1);
ub = C * ones(m,1);

[alpha, fun, exitflag] = quadprog(H, f, [], [], Aeq, beq, lb, ub);

if exitflag == -2
    return
else
    % normal vector to the hyperplanes
    w = [x1,x2]'*(alpha.*y);

    % identify support vectors for the class +1:
    for i=1:m
        if y(i) > 0
            indk = i;
        else
            indh = i;
        end
    end
```

```
b = ([x1(indk),x2(indk)]*w + [x1(indh),x2(indh)]*w)/2;

% Plotting
% indneg = y<0; indpos = y>0;
% H^+
xx1 = min(x1):0.1:max(x1);
xx2p = (b + 1 - w(1) * xx1)./ w(2);
% H^-
xx2n = (b - 1 - w(1) * xx1)./ w(2);
% H
xx2 = (b - w(1) * xx1)./ w(2);

if plt == true
    gscatter(x1, x2, y)
    hold on
    plot(xx1,xx2p,'b--',xx1,xx2n,'m--', xx1,xx2,'k-','LineWidth',2);
    title("Binary classification with C="+num2str(C));
    xlabel('X_1');
    ylabel('X_2');
    ylim([min(x2) - abs(max(x2) - min(x2))/10, ...
        max(x2) + abs(max(x2) - min(x2))/10]);
    xlim([min(x1) - abs(max(x1) - min(x1))/10, ...
        max(x1) + abs(max(x1) - min(x1))/10]);
    legend('class +1','class -1','H^+','H^-','H')
    grid on
    hold off
end

% parameters of each hyperplane
Xx1 = [ones(length(xx1),1), xx1'];
bn = Xx1\xx2n';
bp = Xx1\xx2p';
bh = Xx1\xx2';
hn = @(x1) bn(1) + bn(2).*x1;
hp = @(x1) bp(1) + bp(2).*x1;
h = @(x1) bh(1) + bh(2).*x1;

% points in a wrong class
missclassified = 0;
% points between hyperplanes
points_hyperplane = 0;
% validation criterion

% checking if wether the points fits in the hyperplane or are
% misclassified
for k = 1:length(x1)
    if bn(2) < 0
        if (x2(k) > hn(x1(k))) || (x2(k) < hp(x1(k)))
            points_hyperplane = points_hyperplane + 1;
        end
        if (x2(k) > h(x1(k))) && (y(k) ~= -1) || (x2(k) < h(x1(k))) && (y(k) ~= 1)
            missclassified = missclassified + 1;
```

```
            end
        else
            if (x2(k) < hn(x1(k))) || (x2(k) > hp(x1(k)))
                points_hyperplane = points_hyperplane + 1;
            end
            if (x2(k) < h(x1(k))) && (y(k) ~= -1) || (x2(k) > h(x1(k))) && (y(k) ~= 1)
                missclassified = missclassified + 1;
            end
        end
    end
    performance_rate = (length(x1)-missclassified)/length(x1);
    points_bet_hyp = points_hyperplane/length(x1);
end
end
```