Computation of KKT Points

There seems to be confusion on how one computes KKT points. In general this is a hard problem. The problems I give you to do by hand are not necessarily easy, but they are doable. The basic is idea is to make some reasonable guesses and then to use elimination techniques. I will illustrate this with the following homework problem.

Problem: Locate all of the KKT points for the following problem. Are these points local solutions? Are they global solutions?

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 - 4x_1 - 4x_2 \\ \text{subject to} & x_1^2 \leq x_2 \\ & x_1 + x_2 \leq 2 \ . \end{array}$$

Solution: First write the problem in the standard form required for the application of the KKT theory:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, 2, \ldots, s \\ & f_i(x) = 0, \quad i = s+1, s+2, \ldots, m. \end{array}$$

In our example there are no equality constraints, so $s = m = 2$ and we have

$$\begin{aligned} f_0(x_1, x_2) &= x_1^2 + x_2^2 - 4x_1 - 4x_2 = (x_1 - 2)^2 + (x_2 - 2)^2 - 8 \\ f_1(x_1, x_2) &= x_1^2 - x_2 \\ f_2(x_1, x_2) &= x_1 + x_2 - 2 \ . \end{aligned}$$

Note that we can ignore the constant term in the objective function since it does not effect the optimal solution, so henceforth $f_0(x_1, x_2) = x_1 - 2)^2 + (x_2 - 2)^2$. At this point it is often helpful to graph the solution set if possible, as it is in this case. It is a slice of a parabola.

Since all of these functions are convex, this is an example of a convex programming problem and so the KKT conditions are both necessary and sufficient for global optimality. Hence, if we locate a KKT point we know that it is necessarily a globally optimal solution.

The Lagrangian for this problem is

$$L((x_1, x_2), (u_1, u_2)) = (x_1 - 2)^2 + (x_2 - 2)^2 + u_1(x_1^2 - x_2) + u_2(x_1 + x_2 - 2) \ .$$

Let us now write the KKT conditions for this problem.

1. (Primal Feasibility) $x_1^2 \leq x_2$ and $x_1 + x_2 \leq 2$
2. (Dual Feasibility) $0 \leq u_1$ and $0 \leq u_2$
3. (Complementarity) $u_1(x_1^2 - x_2) = 0$ and $u_2(x_1 + x_2 - 2) = 0$
4. (Stationarity of the Lagrangian)

$$0 = \nabla_x L((x_1, x_2), (u_1, u_2)) = \begin{pmatrix} 2(x_1 - 2) + 2u_1 x_1 + u_2 \\ 2(x_2 - 2) - u_1 + u_2 \end{pmatrix},$$

or equivalently

$$\begin{aligned} 4 &= 2x_1 + 2u_1 x_1 + u_2 \\ 4 &= 2x_2 - u_1 + u_2. \end{aligned}$$

Next observe that the global minimizer for the objective function is $(x_1, x_2) = (2, 2)$. Thus, if this point are feasible, it would be the global solution and the multipliers would both be zero. But it is not feasible. Indeed, both constraints are violated by this point. Hence, we conjecture that both constraints are active at the solution. In this case, the KKT pair $((x_1, x_2), (u_1, u_2))$ must satisfy the following 4 key equations

$$\begin{aligned} x_2 &= x_2^2 \\ 2 &= x_1 + x_2 \\ 4 &= 2x_1 + 2u_1x_1 + u_2 \\ 4 &= 2x_2 - u_1 + u_2. \end{aligned}$$

This is 4 equations in 4 unknowns that we can try to solve by elimination. Using the first equation to eliminate x_2 from the second equation, we see that x_1 must satisfy

$$0 = x_1^2 + x_1 - 2 = (x_1 + 2)(x_1 - 1),$$

so $x_1 = -2$ or $x_1 = 1$. Thus, either $(x_1, x_2) = (-2, 4)$ or $(x_1, x_2) = (1, 1)$. Since $(1, 1)$ is closer the global minimizer of the objective f_0, let us first investigate $(x_1, x_2) = (1, 1)$ to see if it is a KKT point. For this we must find the KKT multipliers (u_1, u_2).

By plugging $(x_1, x_2) = (1, 1)$ into the second of the key equations given above, we get

$$2 = 2u_1 + u_2 \quad \text{and} \quad 2 = -u_1 + u_2 \ .$$

By subtracting these two equations, we get $0 = 3u_1$ so $u_1 = 0$ and $u_2 = 2$. Since both of these values are non-negative, we have found a KKT pair for the original problem. Hence, by convexity we know that $(x_1, x_2) = (1, 1)$ is the global solution to the problem.