My presentation

A cool subtitle

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Why are we interested in the Ising model?

The Ising model is a mathematical model of ferromagnetism in statistical mechanics.
Probability of a configuration σ (*Boltzmann distribution*):

$$P_\beta(\sigma) = e^{-\beta H(\sigma)}/Z(G),$$

$$Z(G) = \sum_{\sigma\colon V \to \{+,-\}} e^{-\beta H(\sigma)}.$$

- $Z(G)$ is *the partition function of the Ising model*.
- $Z(G)$ encodes information about the physical system.

Why are we interested in the Ising model?

$$P_\beta(\sigma) = e^{-\beta H(\sigma)}/Z(G),$$
$$Z(G) = \sum_{\sigma:\, V \to \{+,-\}} e^{-\beta H(\sigma)}.$$

1. **Problem:** What is the complexity of evaluating $Z(G)$?
   $\#P$-hard in some cases.
2. For some of those cases, $Z(G)$ admits a *fully polynomial-time randomised approximation scheme*
   (Jerrum and Sinclair, 1993, Gödel prize in 1996).
3. **Problem:** What is the complexity of approximating $Z(G)$?
4. **Application:** Strongly simulating *instantaneous quantum polynomial-time circuits* is equivalent to approximating $|Z(G)|$ for some parameters.

Outline

1. Preliminaries
2. The Ising Model
3. Conclusion

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Complexity of counting

Definition (*The complexity class* #P)

A function $f\colon \{0,1\}^* \to \mathbb{N}$ is in #P if there is a nondeterministic polynomial-time Turing machine M_f such that, for each input w, M_f has exactly $f(w)$ accepting paths.

Example (#*SAT*)

Instance: A boolean formula F in conjunctive normal form.
Example: $F \equiv (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_1 \vee x_3) \wedge (\neg x_2 \vee x_4)$.
Output: The number of satisfying assignments of F.

Definition (#*P-hardness*)

A computational problem C is #P-hard if, for any $f \in$ #P, the problem "evaluating f" is Turing reducible to C.

Complexity of counting

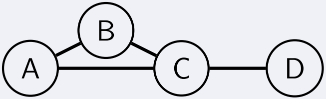
Definition (*#P-completeness*)

A function $f\colon \{0,1\}^* \to \mathbb{N}$ is #P-complete if $f \in$ #P and evaluating f is #P-hard.

1. **Cook-Levin theorem:** $\#SAT$ is #P-complete.
2. There are #P-complete problems whose decision version is trivial.

Example (*Counting independent sets is #P-complete*)

Instance: A graph G.
Example:



Output: The number of independent sets of G.

Partition functions

Definition (*Partition function*)

Given a family $\mathcal{F}_n$ of subsets of the set $\{1, \ldots, n\}$, we define the partition function of $\mathcal{F}_n$ as the polynomial

$$P_{\mathcal{F}_n}(x_1, \ldots, x_n) = \sum_{S \in \mathcal{F}_n} \prod_{j \in S} x_j.$$

- Hard to compute: enumerating $\mathcal{F}_n$ is usually not feasible.
- Many partition functions arise in statistical mechanics.

Example (*The independent sets polynomial*)

Let G be a graph. The independent sets polynomial of G is

$$Z(G; y) = \sum_{I \text{ independent set of } G} y^{|I|}.$$

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The partition function of the Ising model

Definition (*The Ising model*)

Let $G = (V, E)$ be a graph. The partition function of the Ising model on G is

$$Z_{\text{Ising}}(G; y, z) = \sum_{\sigma\colon V \to \{0,1\}} y^{m(\sigma)} z^{n_1(\sigma)}.$$

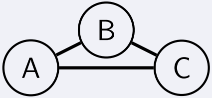
- $m(\sigma)$ is the number of monochromatic edges of σ.
- $n_1(\sigma)$ is the number of vertices v with $\sigma(v) = 1$.
- y is *the edge interaction*.
- z is *the external field*.

1. Setting 1: there is no external field ($z = 1$).
   We write $Z_{\text{Ising}}(G; y) = Z_{\text{Ising}}(G; y, 1)$.
2. Setting 2: there is an external field ($z \neq 1$).

Example

$$Z_{\mathrm{Ising}}(G; y, z) = \sum_{\sigma:\ V \to \{0,1\}} y^{m(\sigma)} z^{n_1(\sigma)}.$$

Example (*The Ising model*)



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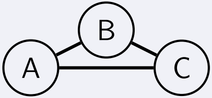

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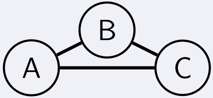

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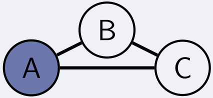

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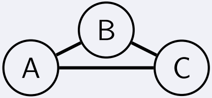

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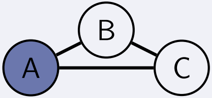

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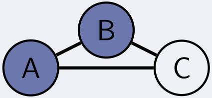

$$Z_{\text{Ising}}(G; y, z) = y^3 + 3yz + 3yz^2 + \cdots$$

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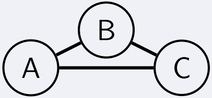

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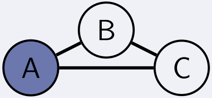

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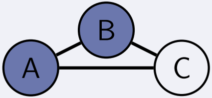

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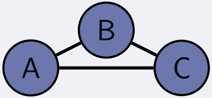
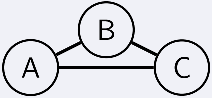


$$Z_{\text{Ising}}(G; y, z) = y^3 + 3yz + 3yz^2 + y^3z^3$$

Example

$$Z_{\text{Ising}}(G; y, z) = \sum_{\sigma:\ V \to \{0,1\}} y^{m(\sigma)} z^{n_1(\sigma)}.$$

Example (*The Ising model*)



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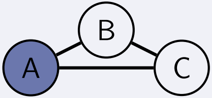

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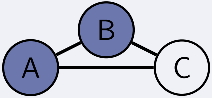
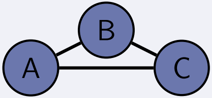


$$Z_{\text{Ising}}(G; y, z) = y^3 + 3yz + 3yz^2 + \cdots$$



$$Z_{\text{Ising}}(G; y, z) = y^3 + 3yz + 3yz^2 + y^3z^3$$

$$Z_{\text{Ising}}(G; y) = 2y^3 + 6y$$

Computational problems on the Ising model

Proposition

Computing the polynomial $Z_{\text{Ising}}(G; y)$ is #P-hard.

Computational problem: Ising(y, z)

Instance: *A (multi)graph G.*
Output: $Z_{Ising}(G; y, z)$.

Computational problem: Factor-K-NormIsing(y, z)

Instance: *A (multi)graph G.*
Output: *A rational number $\hat{N}$ such that*

$$\frac{1}{K}\hat{N} \leq |Z_{Ising}(G; y, z)| \leq K\hat{N}.$$

Factor-K-NormIsing$(y, z) \leq_T$ Ising(y, z).

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1. The Ising model arises in physics, and it is also studied in computer science because of its interesting complexity properties.
2. **Main problem:** complexity of approximating $Z_{\text{Ising}}(G; y, z)$.
3. When there is no external field ($z = 1$):
   - Complexity of exact evaluation is fully classified.
   - Complexity of approximation: #P-hardness is not fully characterised.
4. When there is an external field ($z \neq 1$):
   - Only one result was known: case when y and z are roots of unity.
   - We have made progress when $|y| < 1$ or y is a root of unity.
   - For $|y| \geq 1$ and non root of unity, the complexity is unknown.

Thank you for your time and attention

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