CD Dissertation

Geometric quantisation and Hamiltonian cobordism

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0 Introduction

It has long been understood that nature is described at the molecular scale by the laws of quantum mechanics. At large scales, the effects of quantum phenomena are negligible and mechanics is well-described classically. Classical mechanics can thus be thought of as approximating quantum mechanics, so it seems optimistic to expect to be able to go the other way: to canonically 'quantise' a classical system.

One way of carrying out this task is *geometric quantisation*, which turns classical observables into (twisted versions of) quantum observables. Geometric quantisation actually refers to a range of constructions with the same philosophy, with the $\mathcal{Q}^0$-quantisation of §II.3 being the simplest, and various quantisations via Spin^c Dirac operators and K-theory [GGK00; Met99] being the most technologically advanced. Our quantisation will be a sort of intermediary, defined for Kähler manifolds using twisted Dolbeault cohomology, which coincides with these more technical ones in the Kähler setting.

Symmetry is fundamental in physics. If a classical system has symmetries, we should expect that its quantisation is symmetric under the same group. Moreover, it should make no difference whether we 'quotient' by these symmetries before or after quantising; the reduced quantum system should be the same, so that "quantisation commutes with reduction", sometimes informally written [Q,R]=0. Proving this in certain settings is the goal of this dissertation.

0.1 Summary

In §I, we review some key definitions and concepts in the theory of symplectic and complex geometry. We introduce Hamiltonian vector fields, structures on complex vector bundles, and we describe the structure of coisotropic submanifolds in Theorem I.9.

In §II, we give two inequivalent constructions of geometric quantisation of Kähler manifolds. The first is a complex Hilbert space denoted $\mathcal{Q}^0$, defined as (the completion of) the space of polarised sections of a 'prequantum' line bundle. The second is a *virtual* Hilbert space denoted $\mathcal{Q}$, which can be seen as a refinement of $\mathcal{Q}^0$.

In §III, we introduce Hamiltonian group actions on symplectic manifolds, and develop the notion of *symplectic reduction*. We describe the special features of the abelian setting and give various examples.

In §IV, we state and prove the first result of its kind, both in this dissertation and in the literature [GS82b]: that $\mathcal{Q}^0$ commutes with reduction. More precisely, a Hamiltonian G-action on M defines a linear G-action on $\mathcal{Q}^0(M)$, and Theorem IV.4 states that (under some favourable conditions), the G-invariant part of $\mathcal{Q}^0(M)$ is isomorphic to $\mathcal{Q}^0(M//G)$.

The rest of the dissertation is dedicated to proving that $\mathcal{Q}$ also commutes with reduction in a similar sense as above, only that $\mathcal{Q}(M)$ is now a *virtual* G-representation and we restrict to the case where G is a torus. This is Theorem VI.6, which states that for (certain) integral weights $\alpha \in \mathfrak{g}^*$, the part of $\mathcal{Q}(M)$ of weight α is isomorphic to $\mathcal{Q}(M/\!/_\alpha G)$.

In §V, we describe a cobordism theory for manifolds equipped with Hamiltonian torus actions. The highlight of this section is a proof of the 'Linearisation Theorem' which states that M is Hamiltonian-cobordant to the normal bundles of its fixed point set.

In §VI, we prove the big theorem mentioned above by using the Linearisation Theorem to reduce the problem to the case of symplectic toric manifolds, which makes the problem relatively combinatorial.

0.2 Non-obvious conventions and notation

There are two coexisting definitions of the Fubini–Study symplectic form ω_{FS} on complex projective space $\mathbb{P}^n$, which differ by a factor of 2. We take the one such that $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 2\pi$ where $\mathbb{P}^1 \subseteq \mathbb{P}^n$ is a complex line. This is related to the factor of 2 in Example I.1.

The Hamiltonian equation for us is $\iota_{X_H}\omega = -\mathrm{d}H$. (There is an alternative sign convention in which $\iota_{X_H}\omega = \mathrm{d}H$ and $[X,Y] \coloneqq YX - XY$. This is not common, but it is used in the introductory text [MS17]; an explanation of this choice is given there in Remark 3.1.6.)

Suppose a group G acting on a set X. Write $\mathrm{Fix}_G(X)$ for the set of G-fixed points of X, and $GS \coloneqq \{gs \mid g \in G, s \in S\}$ for the union of orbits of a subset $S \subseteq X$.

Let V be a real vector space. For a smooth map $f\colon M \to V^*$ and vector $v \in V$, write $\langle f, v\rangle$ for the function $M \to \mathbb{R}\colon p \mapsto f(p)(v)$. Some books shorten this to f_v or f^v. It will be clear from context whether $\langle -, - \rangle$ refers to this or a Riemannian/Hermitian metric.

Some texts on geometric quantisation are strewn with factors of $\hbar$. We take $\hbar = 1$.

I Preliminary notions

The following is a sporadic introduction to concepts which are essential for subsequent sections. We only assume knowledge of the undergraduate mathematics curriculum at Oxford.

I.1 Symplectic manifolds

A *symplectic manifold* (M, ω) is a smooth manifold M equipped with a closed non-degenerate 2-form ω on it. The non-degeneracy of ω implies that M must have even dimension, say $2n$, and that its top exterior power ω^n is a volume form.

Example I.1 (Local model). The basic example, which is the local model for all symplectic manifolds (see Theorem I.3), is $M = \mathbb{R}^{2n}$ with symplectic form given by

$$\omega_{\mathrm{std}} = \sum_{k=1}^{n} \mathrm{d}x_k \wedge \mathrm{d}y_k.$$

Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $z_k = x_k + iy_k$, this form coincides (up to a factor of 2) with

$$\omega_0 = i \sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k.$$

These 2-forms are exact; we have $\omega_{\mathrm{std}} = -\mathrm{d}\lambda_{\mathrm{std}}$ and $\omega_0 = -\mathrm{d}\lambda_0$ where $\lambda_{\mathrm{std}} = \sum_{k=1}^{n} y_k \mathrm{d}x_k$ and $\lambda_0 = i \sum_{k=1}^{n} \bar{z}_k \mathrm{d}z_k$. Note that λ_{std} and λ_0 are *not* simply related by a factor of 2; they differ by an exact 2-form.

Example I.2 (Cotangent bundles). Let Q be a manifold. Let $\pi\colon T^*Q \to Q$ be its cotangent bundle and $\pi'\colon TT^*Q \to T^*Q$ be *its* tangent bundle. Consider the diagram

$$\begin{tikzcd}
 & TT^*Q \arrow[dl, "\pi'"'] \arrow[dr, "\mathrm{d}\pi"] & \\
T^*Q \arrow[r, hook] & T^*Q \otimes TQ \arrow[d, "\mathrm{ev}"] & TQ \arrow[l, hook'] \\
 & \mathbb{R} &
\end{tikzcd}$$

All this is happening over a given point in Q, so the evaluation map makes sense. The map $TT^*Q \to \mathbb{R}\colon X \mapsto \pi'(X)(\mathrm{d}\pi(X))$ defines a 1-form $\tau \in \Omega^1(T^*Q)$. Imprecisely speaking, the value of τ at $p \in T^*Q$ is p itself, which gives τ the title of the *tautological 1-form*. The closed 2-form $\omega = -\mathrm{d}\tau$ is non-degenerate, so this defines a canonical symplectic form on T^*Q. In local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on T^*Q, the tautological 1-form and symplectic 2-form are

$$\tau = \sum_{k=1}^{n} p_k \mathrm{d}q_k, \qquad \omega = \sum_{k=1}^{n} \mathrm{d}q_k \wedge \mathrm{d}p_k.$$

When $Q = \mathbb{R}^n$ then $T^*Q \cong \mathbb{R}^{2n}$ and this coincides with Example I.1.

Non-degeneracy of ω implies that $X \mapsto \iota_X\omega$ defines an isomorphism $\Gamma(TM) \cong \Omega^1(M)$, which is of fundamental importance. The closure condition ($\mathrm{d}\omega = 0$) has the following consequence:

Theorem I.3 (Darboux)**.** *Every symplectic form is locally diffeomorphic to* ω_{std}.

Proof. This is a direct corollary of a more general result [Bry+91, Theorem 3.3] which is also called Darboux's Theorem. The proof found in most symplectic geometry textbooks uses a trick called *Moser isotopy*; see [MS17, Theorem 3.2.2]. □

Basic linear algebra implies that for any $p \in M$ one can pick a basis of T_pM in which $\omega_p = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$. The content of Darboux's Theorem is that this can always be done consistently on open patches $U \ni p$, using appropriate local coordinates called *Darboux coordinates*.

I.2 Kähler manifolds

An *almost-complex structure* on a manifold M is a section $J \in \Gamma(\mathrm{End}(TM))$ which squares to $-\mathrm{id}_{TM}$. We say J is *compatible* with a symplectic form $\omega \in \Omega^2(M)$ if $\langle X, Y\rangle := \omega(X, JY)$ defines a Riemannian metric on M. It is a theorem going back to at least Gromov [Gro85] that the space of ω-compatible almost-complex structures is contractible (in particular nonempty).

Any almost-complex structure extends to an endomorphism of the *complexified tangent bundle* $TM \otimes_{\mathbb{R}} \mathbb{C}$, and defines a splitting

$$TM \otimes_{\mathbb{R}} \mathbb{C} = \underbrace{T^{1,0}M}_{+i \text{ eigenspace}} \oplus \underbrace{T^{0,1}M}_{-i \text{ eigenspace}} .$$

There are multiple equivalent notions of *integrability* of almost-complex structures [Huy05]. We say J is integrable if $T^{0,1}M$ is an involutive sub-bundle of $TM \otimes_{\mathbb{R}} \mathbb{C}$, means it is closed under the Lie bracket. Integrability is characterised by the vanishing of the *Nijenhuis tensor*, which is a global section of $\mathrm{Hom}(TM \otimes TM, TM)$ defined by

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

(This is the Frölicher–Nijenhuis bracket of J with itself; see [KSM93].) By the Newlander–Nirenberg Theorem [Huy05, Theorem 2.6.19], this is equivalent to J coming from an honest complex structure (i.e. a holomorphic atlas). A *Kähler manifold* (M, ω, J) is a symplectic manifold (M, ω) with an ω-compatible integrable almost-complex structure J.

Example I.4. Extending Example I.1, let $r \in \{0, \dots, n\}$ and define

$$\omega_r = i\left(-\sum_{k \leqslant r} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k + \sum_{r<k} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k\right).$$

This is exact; we have $\omega_r = -\mathrm{d}\lambda_r$ where $\lambda_r = i(-\sum_{k\leqslant r} \bar{z}k\mathrm{d}z_k + \sum_{r<k} \bar{z}k\mathrm{d}z_k)$. We see that ω_r is compatible with the standard complex structure on $\mathbb{C}^n$, and therefore $(\mathbb{C}^n, \omega_r)$ is Kähler, if and only if $r = 0$.

Example I.5. A vast supply of Kähler manifolds are the smooth quasi-projective varieties. The symplectic form is given by the pullback (under inclusion) of the *Fubini–Study* form ω_{FS} on complex projective space $\mathbb{P}^n$, which is defined as the unique form such that $\pi^*\omega_{\mathrm{FS}} = \iota^*\omega_0$, where ω_0 is the standard form on $\mathbb{C}^{n+1}$ from Example I.1 and ι, π are as in the following diagram.

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & \mathbb{C}^{n+1} \\ \mathbb{P}^n \end{tikzcd}$$

This construction is a special case of symplectic reduction discussed in §III.2

Remark I.6. The Newlander–Nirenberg Theorem cited above says that $N_J = 0$ is an *integrability condition* for almost-complex structures. In light of Darboux's Theorem, the condition $\mathrm{d}\omega = 0$ for symplectic forms is another example of an integrability condition. (A non-degenerate 2-form is sometimes called an *almost symplectic form*.) Morally, integrablility means that a given structure is locally—rather than just pointwise—isomorphic to some "standard model". This unifying perspective can alternatively be understood via the intrinsic torsion of G-structures, as explained in [Sch10].

I.3 Complex vector bundles

Let M be a manifold with sheaf of smooth functions C^∞. Let $E \to M$ be a complex vector bundle over M. A *connection* on E is a C^∞-linear map

$$\nabla : TM \to \mathrm{End}_{\mathbb{C}}(E) : X \mapsto \nabla_X$$

satisfying the Leibniz rule $\nabla_X(fs) = (Xf)s + f\nabla_X s$, where X, f, s are respectively (local) vector fields, smooth functions, and sections of E. The difference of two connections on E is clearly C^∞-linear, which means the space of global connections is an affine space isomorphic to $\Omega^1(M, \mathrm{End}_{\mathbb{C}}(E))$. A connection ∇ fails to be a morphism of (sheaves of) Lie algebras by its *curvature*, which is the C^∞-bilinear map

$$\mathrm{curv}\,\nabla : TM \times TM \to \mathrm{End}_{\mathbb{C}}(E) : (X, Y) \mapsto [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

A *Hermitian structure* (or Hermitian metric) on E is a section $h \in \Gamma(E^* \otimes_{\mathbb{C}} E^*)$ such that each $h_p : E_p \otimes E_p \to \mathbb{C}$ is a Hermitian inner product.

Example I.7. Given a symplectic manifold (M, ω) with compatible almost-complex structure J, it is easy to check that the h defined by $h(X, Y) := \omega(X, JY) + i\omega(X, Y)$ is a Hermitian structure on the complex vector bundle TM.

We say h and ∇ are *compatible* if $Xh(s, t) = h(\nabla_X s, t) + h(s, \nabla_X t)$. This is equivalent to the statement that the parallel transport maps induced by ∇ preserve h.

Suppose now that $E = \mathbb{L}$ is a line bundle, which means the fibre has complex dimension 1. The dual bundle $\mathbb{L}^*$ is the inverse of $\mathbb{L}$ with respect to the tensor product, so $\mathrm{End}_{\mathbb{C}}(\mathbb{L}) \cong \mathbb{L} \otimes_{\mathbb{C}} \mathbb{L}^*$ is the trivial line bundle. A section of $\mathrm{End}_{\mathbb{C}}(\mathbb{L})$ is therefore just a $\mathbb{C}$-valued smooth function, and $\operatorname{curv}\nabla$ is an element of $\Omega^2(M,\mathbb{C})$. We will state some facts about $\operatorname{curv}\nabla$.

Proposition I.8. *Suppose ∇ is compatible with some Hermitian structure on $\mathbb{L}$. Then* $\operatorname{curv}\nabla$ *is a purely imaginary closed 2-form, and the cohomology class*

$$c_1(\mathbb{L}) := \frac{i}{2\pi}[\operatorname{curv}\nabla]$$

depends only on $\mathbb{L}$.

Proof. The fact that $\operatorname{curv}\nabla$ is imaginary is [Wei19, Lemma 2.69]. We want to show that $\operatorname{curv}\nabla$ is locally exact. Suppose that $\mathbb{L}$ is trivial; its global sections can be viewed as elements of $C^\infty(M,\mathbb{C}) = C^\infty(M)\otimes\mathbb{C}$, so that ∇ is given by

$$\nabla = \mathrm{id} + \beta$$

for some $\beta \in \Omega^1(M,\mathbb{C})$ called the *connection 1-form* of this trivialisation. A vector field X naturally acts on the first factor, and $\lambda(X)$ acts by multiplication. Its curvature is

$$\operatorname{curv}\nabla(X,Y) = [X+\beta(X), Y+\beta(Y)] - [X,Y] - \beta([X,Y]).$$

The general formula $\mathrm{d}\beta(X,Y) = X\beta(Y) - Y\beta(X) - \beta([X,Y])$ reduces this to $\mathrm{d}\beta(X,Y)$. Since $\operatorname{curv}\nabla$ is locally exact, it is closed. The last statement follows, because the same calculation shows that $\operatorname{curv}(\nabla+\eta) = \operatorname{curv}\nabla + \mathrm{d}\eta$ for all $\eta \in \Omega^1(M,\mathbb{C})$. □

In fact $\frac{i}{2\pi}[\operatorname{curv}\nabla]$ is an integral cohomology class known as the *first Chern class* of $\mathbb{L}$. This Proposition represents the beginning of a general approach to characteristic classes known as Chern–Weil theory; see [Zha01, Chapter 1] for a straightforward exposition.

I.4 Submanifolds of symplectic manifolds

Let (M,ω) be a symplectic manifold and $S \subseteq M$ a submanifold. Write TS^ω for the vector bundle over S which is *complementary* to TS (with respect to ω);

$$TS^\omega := \{(p,X) \in TM \mid p \in S, X \in T_pM, \iota_X\omega|_{TS} = 0\}.$$

We say that S is *symplectic* if $\omega|_{TS}$ is non-degenerate (it is already closed because $\mathrm{d}(\omega|_{TS}) = \mathrm{d}\omega|_{TS} = 0$). It is *isotropic* if $\omega|_{TS} = 0$ and *coisotropic* if $\omega|_{TS^\omega} = 0$. Finally, S is *Lagrangian* if it is both isotropic and coisotropic. If M has dimension $2n$, then isotropic submanifolds have dimension at most n, and coisotropic submanifolds have dimension at most n. Thus Lagrangian submanifolds have dimension n, and are equivalently characterised by being isotropic (resp. coisotropic) of maximal (resp. minimal) dimension. The properties of isotropy and coisotropy can be rephrased as saying that for all $p \in S$ and $X \in T_pS$ we have

$$
\begin{tikzcd}
Y \in T_pS \arrow[r, Rightarrow, bend left, "\text{isotropic}"] & \omega_p(X,Y) = 0 \arrow[l, Rightarrow, bend left, "\text{coisotropic}"]
\end{tikzcd}
$$

A *foliation* of a manifold S is an equivalence relation $\sim$ on S whose equivalence classes—the *leaves* of the foliation—are submanifolds of constant dimension. Important examples of foliations are *principal G-bundles* where G is a Lie group, which are spaces with free G-actions such that the quotient map is a locally trivial fibration [Aud04].

Theorem I.9. *Every coisotropic submanifold $S \subset M$ is foliated by isotropic leaves of dimension equal to the codimension of S. The quotient space $S/\sim$ is naturally a symplectic manifold.*

Proof. The first statement holds because TS^ω is an integrable isotropic distribution of rank given by the codimension of S [MS17]; one uses the Frobenius Theorem to integrate this to a foliation. The fact that the quotient is symplectic can be understood as follows. At each point $p \in S$, consider the isotropic tangent space $T_p\ell \cong (TS^\omega)_p$ of the leaf $\ell \subseteq S$ containing p. By definition of TS^ω, the degenerate form $\omega|_{TS}$ descends to a nondegenerate one on $TS/TS^\omega \cong T(S/\sim)$. The induced form on the quotient is closed because d commutes with pullback. □

Example I.10. If $S = M$ then the leaves of the foliation are just the points of M. If S is Lagrangian then the leaves are the connected components of S.

I.5 Hamiltonian vector fields

In this section, we explain how a symplectic form ω on M provides a way to associate a vector field to each smooth function on M, satisfying properties inspired by classical mechanics. Let $H \in C^\infty(M)$ be a smooth function on M. Because ω is non-degenerate, there exists a unique vector field $X_H \in \Gamma(TM)$ satisfying

$$\iota_{X_H}\omega = -\mathrm{d}H.$$

The flow of X_H preserves the function H, as can be seen by the following calculation:

$$X_H H = \mathrm{d}H(X_H) = -\omega(X_H, X_H) = 0.$$

When H is the classical-mechanical Hamiltonian of some phase space, the flow of X_H dictates the time-evolution of the system, and the identity $X_H H = 0$ amounts to the assertion that 'total energy is preserved'. Inspired by this example, we often call H a Hamiltonian function.

Importantly, the flow of X_H also preserves the symplectic form ω, as can be seen by applying Cartan's magic formula

$$\mathcal{L}_{X_H}\omega = \mathrm{d}(\iota_{X_H}\omega) + \iota_{X_H}\mathrm{d}\omega = \mathrm{d}(-\mathrm{d}H) + \iota_{X_H}0 = 0.$$

In fact, the same formula shows that the flow of an arbitrary vector field $X \in \Gamma(TM)$ preserves ω if and only if $\iota_X\omega$ is closed. Thus, we say a vector field X is *symplectic* if $\iota_X\omega$ is closed, and *Hamiltonian* if $\iota_X\omega$ is exact. The spaces of such vector fields are easily seen to be vector subspaces of $\Gamma(TM)$, which are denoted here by $\mathfrak{symp}(M,\omega)$ and $\mathfrak{ham}(M,\omega)$. Note that $\mathfrak{symp}(M,\omega)$ is the Lie algebra of the symplectomorphism group $\mathrm{Symp}(M,\omega) := \{\phi \in \mathrm{Diff}(M) \mid \phi^*\omega = \omega\}$, and $\mathfrak{ham}(\omega)$ is the Lie algebra of the Hamiltonian diffeomorphism group $\mathrm{Ham}(M,\omega) := \{\exp(X) \mid X \in \mathfrak{ham}(M,\omega)\}$.

If X is Hamiltonian then $X = X_H$ for some $H \in C^\infty(M)$ which we say *generates* X. Clearly such a H is only unique up to a choice of constant in $H^0(M;\mathbb{R})$.

The symplectic form ω makes $C^\infty(M)$ a Lie algebra with the bracket $\{F,H\} := \omega(X_F, X_H)$. (The fact that $\{-,-\}$ satisfies the Jacobi identity follows from the proof of Proposition I.11 below.) In fact, this is a *Poisson algebra*, which means the bracket satisfies a Leibniz rule:

$$\{FH,-\} = \{F,-\}H + F\{H,-\}.$$

Note also that $X_F H = \mathrm{d}H(X_F) = -\omega(X_H, X_F) = \omega(X_F, X_H)$. We say that F and H *commute* if $\{F,H\} = 0$; that is, if the flows of X_F and X_H are symplectically orthogonal. In mechanical terms, F is a conserved quantity for the system with Hamiltonian H.

Proposition I.11. *The subspaces $\mathfrak{symp}(M,\omega)$ and $\mathfrak{ham}(M,\omega)$ are Lie subalgebras of $\Gamma(TM)$. Moreover, the map*

$$X_\bullet \colon C^\infty(M) \to \mathfrak{ham}(M,\omega) \colon F \mapsto X_F$$

is a Lie algebra morphism.

Proof. Let $X, Y \in \mathfrak{symp}(M,\omega)$. From the relation $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$ we get

$$\iota_{[X,Y]}\omega = \mathcal{L}_X\iota_Y\omega - \iota_Y\mathcal{L}_X\omega = \mathrm{d}(\iota_X\iota_Y\omega) + \iota_X\mathrm{d}(\iota_Y\omega) - \iota_Y 0 = -\mathrm{d}(\omega(X,Y)).$$

Therefore $\omega(X,Y)$ generates the (a posteriori Hamiltonian) vector field $[X,Y]$, which means that $\mathfrak{symp}(M,\omega)$ and $\mathfrak{ham}(M,\omega)$ are Lie algebras. Specialising this to $X = X_F$ and $Y = X_H$ for Hamiltonians $F, H \in C^\infty(M)$, we see that $X_\bullet$ is indeed a morphism of Lie algebras. □

II Geometric quantisation

Fix a symplectic manifold (M, ω), which we can heuristically think of as a 'curved phase space' by analogy with the case of cotangent bundles. Classical observables are real valued smooth functions on M.

II.1 Prequantising exact symplectic manifolds

Thinking now of a quantum system, wave functions are elements of some complex Hilbert space $\mathscr{H}$—or rather rays in $\mathscr{H}$, but we will not make this distinction—and quantum observables are self-adjoint operators from $\mathscr{H}$ to itself. Write $\mathfrak{O}(\mathscr{H})$ for the space of self-adjoint operators from $\mathscr{H}$ to itself ($\mathfrak{O}$ stands for 'observables'). Physically, the self-adjoint property comes from the fact that their eigenvalues must be real, as these are what is observed in experiments.

Recall from the previous section that $C^\infty(M)$ has a Lie algebra structure given by the Poisson bracket $\{F, H\} = \omega(X_F, X_H)$. Similarly, given self-adjoint operators $A, B \in \mathfrak{O}(\mathscr{H})$ it is readily seen that the commutator $[A, B]$ is skew-adjoint, and the operator

$$[\![A, B]\!] := i[A, B]$$

is self-adjoint. The bracket $[\![-, -]\!]$ also gives $\mathfrak{O}(\mathscr{H})$ a Lie algebra structure.

There are several axioms posed by Dirac [Woo92], which describe what a transformation passing from classical to quantum should look like. Together, they say that *prequantisation* of the classical system M is a Lie algebra morphism

$$C^\infty(M) \to \mathfrak{O}(\mathscr{H})\colon F \mapsto \widehat{F}$$

which sends the constant function $\mathbf{1}\colon M \to \mathbb{R}\colon p \mapsto 1$ to the identity operator $\mathrm{id}_{\mathscr{H}}$.

Let us attempt to find a prequantisation of M. Fix a volume form $\mathrm{d}V$ on M; together with a Hermitian structure from Example I.7 and the standard inner product on $\mathbb{C}$, this defines the Hilbert space $\mathscr{H} := L^2(M, \mathbb{C})$. Our first guess for the prequantisation morphism might be

$$\widehat{F} = -iX_F,$$

since then for all $F, H \in C^\infty(M)$ we have

$$[\![\widehat{F}, \widehat{H}]\!] = i[-iX_F, -iX_H] = -i[X_F, X_H] = -iX_{\{F,H\}} = \widehat{\{F, H\}}$$

by Proposition I.11. However, this does not work because then $\widehat{\mathbf{1}} = 0$. We remedy this with a second guess

$$\widehat{F} = -iX_F + F$$

where, by a slight abuse of notation, F stands for multiplication by F (as a linear operator it is $F\,\mathrm{id}_{\mathscr{H}}$). This does not work either, because now $F \mapsto \widehat{F}$ fails to be a Lie algebra morphism:

$$\begin{aligned}
[\![\widehat{F}, \widehat{H}]\!] &= i\widehat{F}\widehat{H} - i\widehat{H}\widehat{F} \\
&= \Big(-iX_F X_H + \underbrace{X_F H}_{\{F,H\}+HX_F} + FX_H + iFH \Big) - \Big(-iX_H X_F + \underbrace{X_H F}_{\{H,F\}+FX_H} + HX_F + iHF \Big) \\
&= -i[X_F, X_H] + \{F,H\} - \{H,F\} \\
&= -iX_{\{F,H\}} + 2\{F,H\} \\
&= \widehat{\{F,H\}} + \{F,H\}.
\end{aligned}$$

This is the only place where our abuse of notation could be confusing: the expression $X_F H$ is short for the composite operator $X_F \circ (H\,\mathrm{id}_{\mathscr{H}})$, not $(X_F H)\,\mathrm{id}_{\mathscr{H}}$. The latter is $\{F,H\}$, and the Leibniz rule for derivations gives the calculation in the underbraces.

There is nothing obvious we can do to rectify the failure of this second guess without introducing more issues. However, if we make the additional assumption that the symplectic form ω is exact, then we can make a third guess:

$$\widehat{F} = -iX_F + \lambda(X_F) + F. \tag{†}$$

Here λ is the symplectic potential satisfying the equation $\omega = -\mathrm{d}\lambda$. This can be shown to be satisfy the desired conditions, via an unseemly computation which we omit. Thus, we have prequantised exact symplectic manifolds! However, this ignores many cases of interest; it is an easy exercise to check that compact (positive-dimensional) symplectic manifolds cannot be exact. In the next subsection, we will see how to extend this formula to the non-exact setting.

II.2 Prequantising symplectic manifolds

Since closed forms are locally exact, the above construction can be done locally. The difficulty is in patching together these local trivialisations to get a globally defined $\widehat{F}$. In fact, this is impossible if we insist that the operators act on the Hilbert space $L^2(M, \mathbb{C})$.

View $L^2(M,\mathbb{C})$ as the L^2-completion of the space of smooth sections of a trivial Hermitian line bundle over M. The key idea is to replace this with some other Hermitian line bundle $\mathbb{L}$ equipped with a connection ∇ whose curvature coincides with $-i\omega$. In this case, setting

$$\widehat{F} = -i\nabla_{X_F} + F$$

and repeating the computation for the second guess of the previous section, we get

$$[\![\widehat{F}, \widehat{H}]\!] = -i[\nabla_{X_F}, \nabla_{X_H}] + 2\{F,H\}$$

$$
\begin{aligned}
&= -i\big(\nabla_{[X_F,X_H]} - i\omega(X_F,X_H)\big) + 2\{F,H\} \\
&= -i\nabla_{X_{\{F,H\}}} + \{F,H\} \\
&= \widehat{\{F,H\}}.
\end{aligned}
$$

Remark II.1. Suppose again that $\omega = -\mathrm{d}\lambda$ is exact, so $\mathbb{L}$ is trivial. Substituting the connection $\nabla = \mathrm{id} + i\lambda$ (which has curvature $-i\omega$ by the proof of Proposition I.8) recovers the formula (†).

We have therefore arrived at a formula which works for all symplectic forms which arise as the curvature of a connection on a complex line bundle. By Chern–Weil theory (see §I.3) this occurs precisely when $[\omega] \in 2\pi H^2(M;\mathbb{Z})$; in this case (M,ω) is said to be *prequantisable*. This is still quite a restrictive condition in the sense that $\mathbb{Z}$ is a small subset of $\mathbb{R}$, but it does not limit the topology of M in the same way that exactness does.

The Hermitian line bundle $(\mathbb{L}, \langle -,-\rangle_{\mathbb{L}})$ is called the *prequantum line bundle* and ∇ is the *prequantum connection*. The *prequantisation* of (M,ω) is the complex Hilbert space $L^2(\mathbb{L})$ of square-integrable sections, together with the map $F \mapsto \widehat{F}$ above.

Remark II.2. With our sign conventions we have $c_1(\mathbb{L}) = [\omega]/2\pi$. Working backwards, we see that this is because $\operatorname{curv}\nabla = -i\omega$, which is necessary because $\widehat{F} = -i\nabla_{X_F} + F$. The effect of instead taking $\widehat{F} = i\nabla_{X_F} + F$ would be replacing the prequantum line bundle with its inverse.

We have demonstrated existence; the question of uniqueness is answered by the following.

Proposition II.3. *Prequantisations of (M,ω) are classified up to isomorphism by the torus $H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$. In particular, if M is simply connected then uniqueness holds.*

Proof. Omitted. This is [GGK00, Proposition 6.5]. □

Remark II.4. The failure of uniqueness in the non-simply connected case is not a defect of the construction. Mathematically, this comes from the fact that a flat connection can have nontrivial holonomy along topologically nontrivial loops. In physics, this is the *Aharonov–Bohm effect*.

II.3 Polarisations and $\mathcal{Q}^0$-quantisation

Prequantising $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ produces the Hilbert space $L^2(\mathbb{R}^{2n}, \mathbb{C})$. The problem is that this is too large because it includes functions dependent on all coordinates and momenta, which contradicts the uncertainty principle. We expect the smaller space $L^2(\mathbb{R}^n, \mathbb{C})$ instead.

Indeed, the additional condition imposed by Dirac which upgrades a prequantisation to a bonafide quantisation is the *irreducibility axiom* which states that a complete set of (classical) observables $\{F_k\}$ must map to a complete set of quantum observables $\{\widehat{F_k}\}$. (A subset is *complete* if the only elements which commute with it are the constants.) We therefore need a systematic way of halving the number of dimensions used to define the Hilbert space of states from M.

If M were foliated by Lagrangian submanifolds, we could consider sections of the prequantum line bundle which are constant on each leaf. Unfortunately, such foliations are hard to find, so this will not do in general. The solution is to instead consider a *complex polarisation* of M, which is an integrable Lagrangian sub-bundle $D \subseteq TM \otimes_{\mathbb{R}} \mathbb{C}$. A smooth section $s \in \Gamma(\mathbb{L})$ is *polarised* if it covariantly constant along D with respect to the prequantum connection: $\nabla_X s = 0$ for all $X \in \Gamma(D)$.

The advantage of using complex polarisations is that if M is Kähler with almost-complex structure J, then the sub-bundle $T^{0,1}M$ is integrable because J is integrable, and Lagrangian because J is ω-compatible. Unless otherwise mentioned, we henceforth assume that M is is Kähler and equipped with the polarisation $T^{0,1}M$.

Remark II.5. In general, a *Kähler polarisation* is an integrable Lagrangian sub-bundle $D \subseteq TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $D \cap \overline{D} = 0$. The existence of a Kähler polarisation is an alternative characterisation of Kähler manifolds, as proved in [Bay19, §6.3].

We can hence view $\mathbb{L}$ as a holomorphic line bundle over M; polarised sections of $\mathbb{L}$ are then precisely holomorphic sections. Define the $\mathcal{Q}^0$*-quantisation* of M to be

$$\mathcal{Q}^0(M) \coloneqq L^2_{\text{hol}}(\mathbb{L}),$$

the L^2-completion of the space of holomorphic sections of $\mathbb{L}$. Note that $\mathcal{Q}^0(M)$ is finite-dimensional if M is compact, which looks very different from an infinite-dimensional Hilbert space of states which one might be used to from quantum mechanics.

Remark II.6. Polarisation also necessarily restricts the *domain* of the quantisation map, since we can only quantise observables F such that $\widehat{F}$ sends polarised sections to polarised sections. This means we can only quantises small Lie subalgebras of $C^\infty(M)$; depending on the polarisation, one can even get subalgebras which do not contain the Hamiltonian. This is quite disappointing from a physics perspective. It is true that quantisation has found applications in modern theoretical physics (perhaps most notably through Witten's work on the Jones polynomial [Wit89]) but a large part of the physical interest lies in its formalisation of loose analogies between the mathematical frameworks of classical and quantum mechanics.

Henceforth we ignore the observables, and focus solely on the Hilbert space produced from M.

II.4 L^2 Dolbeault cohomology

We momentarily take a brief interlude to develop L^2 Dolbeault cohomology, with the aim of defining a more refined version of geometric quantisation for Kähler manifolds. In the compact setting everything is finite-dimensional so there is no difficulty with restrictions and completions; one can just take ordinary Dolbeault cohomology, which is a purely algebraic construction. However, our eventual goal of quantising (non-compact) linear space means we must take more care in this respect. Our treatment largely follows [Dui11].

In the Kähler setting, the splitting $TM \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$ induces the decomposition $\mathrm{d} = \partial + \bar{\partial}$. The complexified de Rham chain complex splits into the *Dolbeault bicomplex*:

$$\begin{tikzcd}
\vdots & \vdots & \\
\Omega^{1,0}(M) \arrow[u, "\partial"] \arrow[r, "\bar{\partial}"] & \Omega^{1,1}(M) \arrow[u, "\partial"] \arrow[r, "\bar{\partial}"] & \cdots \\
C^{\infty}(M, \mathbb{C}) \arrow[u, "\partial"'] \arrow[r, "\bar{\partial}"] & \Omega^{0,1}(M) \arrow[u, "\partial"'] \arrow[r, "\bar{\partial}"] & \cdots
\end{tikzcd}$$

where $\Omega^{p,q}(\bigwedge^{p,q} T^*M)$ where $\bigwedge^{p,q} T^*M = \left(\bigwedge^p (T^{1,0}M)^*\right) \wedge \left(\bigwedge^q (T^{0,1})^*M\right)$. This is modified to differential forms with coefficients in the prequantum line bundle $\mathbb{L}$, in the following way. Focusing on the bottom row of this bicomplex, we get a chain complex

$$\Gamma(\mathbb{L}) \xrightarrow{\bar{\partial}_{\mathbb{L}}} \Omega^{0,1}(M, \mathbb{L}) \xrightarrow{\bar{\partial}_{\mathbb{L}}} \cdots$$

where $\Omega^{0,q}(M, \mathbb{L}) = \Gamma(\bigwedge^{0,q} T^*M \otimes_{\mathbb{C}} \mathbb{L})$ and $\bar{\partial}_{\mathbb{L}}$ is defined on $\Omega^{0,q}(M, \mathbb{L})$ by linearly extending

$$\bar{\partial}_{\mathbb{L}}(\beta \otimes s) := \bar{\partial}_{\mathbb{L}}\beta \otimes s + (-1)^q \beta \wedge \bar{\partial}_{\mathbb{L}} s.$$

Since M is Kähler, there is a natural Hermitian structure h_{TM} on its tangent bundle as in Example I.7. Transporting this under the $\mathbb{C}$-linear isomorphism

$$TM \xrightarrow{\sim} \textstyle\bigwedge^{0,1} T^*M \colon X \mapsto \iota_X h_{TM}$$

produces a Hermitian structure on $\bigwedge^{0,1}T^*M$, and similarly on $\bigwedge^{0,q}T^*M$ for each q. Combining this in the natural way with the Hermitian structure $\langle -, - \rangle_{\mathbb{L}}$ on $\mathbb{L}$ produces Hermitian structures h_k on each $\bigwedge^{0,q}T^*M \otimes_{\mathbb{C}} \mathbb{L}$. Given a volume form $\mathrm{d}V$ on M, we can define an inner product $(-, -)_q$ on each $\Omega^{0,q}(M, \mathbb{L})$ by

$$(s, t)_q = \int_M h_q(s, t) \mathrm{d}V.$$

These assemble into an inner product on the whole Dolbeault complex $\Omega^{0,\bullet}(M, \mathbb{L})$; write $\|-\|$ for the associated norm. Let

$$W^{\bullet} := \{ s \in \Omega^{0,\bullet}(M, \mathbb{L}) : \|s\|, \left\|\bar{\partial}_{\mathbb{L}} s\right\| < \infty \}$$

be the subcomplex of all s such that both s and $\bar{\partial}_{\mathbb{L}} s$ have finite L^2-norm. Then write

$$H^{0,q}_{L^2}(M,\mathbb{L}) := \frac{\ker(\bar{\partial}_{\mathbb{L}} : W^q \to W^{q+1})}{\overline{\operatorname{im}}(\bar{\partial}_{\mathbb{L}} : W^{q-1} \to W^q)}$$

for the L^2*-Dolbeault cohomology* of M, where $\overline{\operatorname{im}}(\bar{\partial}_{\mathbb{L}})$ denotes the closure with respect to $\|-\|$.

Suppose M is compact. Then we can safely take $\mathrm{d}V$ to be a constant multiple of ω^n, and we get $W^\bullet = \Omega^{0,\bullet}(M,\mathbb{L})$. As a consequence of Dolbeault's Theorem [Voi02, Theorem 4.2]—which links Dolbeault cohomology to sheaf cohomology—we have $H^{0,k}_{L^2}(M,\mathbb{L}) \cong H^k(M,\mathbb{L})$.

II.5 Quantising Kähler manifolds

Because the global sections functor on holomorphic vector bundles is not exact, the quantisation $\mathcal{Q}^0$ (which only incorporates the global sections of the prequantum line bundle) may not be very well behaved. Earlier, we defined $\mathcal{Q}^0(M) = L^2_{\mathrm{hol}}(\mathbb{L})$, which coincides with the zeroth Dolbeault cohomology $H^{0,0}_{L^2}(M,\mathbb{L})$. Our final definition of geometric quantisation is a refinement of this definition, adding correction terms in the form of higher cohomologies:

$$\mathcal{Q}(M) := \sum_{k=0}^{\infty} (-1)^k H^{0,k}_{L^2}(M,\mathbb{L}).$$

Note that, as opposed to $\mathcal{Q}^0$, this is a *virtual vector space*; its dimension is $\dim_{\mathbb{C}} \mathcal{Q}(M) = (-1)^k \dim_{\mathbb{C}} H^{0,k}_{L^2}(M,\mathbb{L})$, which could be negative.

Example II.7. Let $\alpha \in \{1,2,\dots\}$. The prequantum line bundle for the Kähler manifold $(\mathbb{P}^n, \alpha\omega_{\mathrm{FS}})$ is the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^n}(\alpha)$. It is a standard result (see [Vak, 18.1.3. Theorem]) that

$$H^{0,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\alpha)) = \begin{cases} \mathbb{C}[z_0,\dots,z_n]_\alpha & q = 0 \\ 0 & q \neq 0 \end{cases}$$

where $\mathbb{C}[z_0,\dots,z_n]_\alpha$ denotes the space of homogeneous complex polynomials of degree α in $n+1$ variables. Therefore $\mathcal{Q}(\mathbb{P}^n) \cong \mathbb{C}[z_0,\dots,z_n]_\alpha$, which has dimension $\binom{n+\alpha}{\alpha}$. Note that $\mathcal{Q}^0$ and $\mathcal{Q}$ agree in this example. Reiterating the point of Remark II.2, with a different sign convention we would have $\mathbb{L} = \mathcal{O}_{\mathbb{P}^n}(-\alpha)$ instead, which has no global sections.

We now turn to the quantisation of the nonstandard linear space $(\mathbb{C}^n, \omega_r)$ defined in Example I.4. Note that this is not Kähler unless $r = 0$, but the integrable sub-bundle $T^{0,1}\mathbb{C}^n \subseteq T\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}$ is nevertheless still Lagrangian because ω_r is obtained from ω_0 the signs of some summands. The volume form $\mathrm{d}V$ we use is the standard Lebesgue measure on $\mathbb{C}^n$, and the Hermitian structure on $T\mathbb{C}^n$ is the standard one positive-definite one, which is *not* the one defined by substituting $\omega = \omega_r$ into Example I.7.

Theorem II.8 ([GGK00, Theorem 7.3])**.** *The L^2 Dolbeault cohomology of is*

$$H^{0,k}_{L^2}(\mathbb{C}^n, \mathbb{L}) = \begin{cases} \mathcal{B}(\overline{\mathbb{C}}^r \times \mathbb{C}^{n-r}) \otimes \operatorname{span}\{s \mathrm{d}\bar{z}_1 \wedge \cdots \mathrm{d}\bar{z}_r\} & k = r, \\ 0 & k \neq r. \end{cases}$$

Here $\mathcal{B}(\overline{\mathbb{C}}^r \times \mathbb{C}^{n-r})$ is the Bargmann space*, defined as the completion of*

$$\{\bar{z}_1^{\ell_1} \cdots \bar{z}_r^{\ell_r} \cdot z_{r+1}^{\ell_{r+1}} \cdots z_n^{\ell_n} \mid (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n\}$$

with respect to the inner product

$$(f, g) \mapsto \int_{\mathbb{C}^n} f \bar{g} e^{-|\mathbf{z}|^2} \mathrm{d}\mathbf{z} < \infty,$$

and $s \in \Gamma(\mathbb{L})$ is a trivialising section satisfying $\nabla_X s = \lambda_r(X) s$ for all vector fields X. □

This is isomorphic as a Hilbert space to the Bargmann space $\mathcal{B}(\overline{\mathbb{C}}^r \times \mathbb{C}^{n-r})$. However, tensoring with

$$\operatorname{span}\{s \otimes \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_r\}$$

changes the complex behaviour, and will change the associated weights of the virtual representation (see §VI.1 for this phenomenon).

Example II.9. Suppose $r = 0$. Then $\mathcal{Q}(\mathbb{C}^n) = \mathcal{B}(\mathbb{C}^n)$, which coincides with $L^2_{\mathrm{hol}}(\mathbb{C}^n)$. In this instance $\mathcal{Q}^0$ and $\mathcal{Q}$ agree. However, when r is odd then $\mathcal{Q}$ has non-positive dimension so $\mathcal{Q}(\mathbb{C}^n)$ and $\mathcal{Q}(\mathbb{C}^n)$ only coincide if they both vanish.

III Hamiltonian actions and reduction

III.1 Symplectic and Hamiltonian actions

Let G be a connected Lie group. A *symplectic action* on a symplectic manifold (M, ω) by G is a Lie group homomorphism $G \to \operatorname{Symp}(M, \omega)$. The derivative of this map at the identity of G is the Lie algebra morphism

$$\sharp \colon \mathfrak{g} \to \mathfrak{symp}(M, \omega) \colon \xi \mapsto \xi^\sharp$$

defined by

$$\xi^\sharp_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi) p.$$

One might expect that Hamiltonian actions be defined analogously, as Lie group morphisms $G \to \operatorname{Ham}(M, \omega)$ (or equivalently, those for which the image of $\sharp$ is contained in $\mathfrak{ham}(M, \omega)$). In fact, this is only the definition of a *weakly Hamiltonian* action. We now give the necessary background in order to define genuine Hamiltonian actions.

Suppose we have a weakly Hamiltonian action $G \to \mathrm{Ham}(M, \omega)$. For all $\xi \in \mathfrak{g}$, there exists a smooth function on M which generates $\xi^\sharp$. One can always arrange that this assignment is linear, so there exists a linear map $\mu^* \colon \mathfrak{g} \to C^\infty(M)$ making this big diagram commute:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & H^0(M;\mathbb{R}) \arrow[d] & 0 \arrow[d] & & \\
 & C^\infty(M) \arrow[d, "X_\bullet"] & \mathfrak{g} \arrow[l, "\mu^*"'] \arrow[d, "\sharp"] & 0 \arrow[l] & \\
0 \arrow[r] & \mathfrak{ham}(M,\omega) \arrow[r] \arrow[d] & \mathfrak{symp}(M,\omega) \arrow[r] & H^1(M;\mathbb{R}) \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

Explicitly, we have $\xi^\sharp = X_{\mu^*(\xi)}$ for all $\xi \in \mathfrak{g}$. If μ^* is a Lie algebra morphism (which amounts to $\mu^*([\xi,\zeta]) = \omega(\xi^\sharp, \zeta^\sharp)$ for all $\xi, \zeta \in \mathfrak{g}$), then it is called the *comoment map* of the G-action.

Recall that G acts linearly on $\mathfrak{g}^*$ by the coadjoint representation, and on $C^\infty(M)$ by precomposition. It is a general fact (for example, see [MS17, Lemma 5.2.1]) that μ^* being a Lie algebra morphism is equivalent to μ^* being G-equivariant. We only prove this in the abelian case, since this will be all that we need for applications.

Proof when G is abelian. The triviality of the coadjoint representation gives the following easy equivalences:

$$\begin{tikzcd}
\mu^* \text{ is } G\text{-equivariant} \arrow[d, Leftrightarrow] & \{\mu^*(\xi), \mu^*(\zeta)\} = 0 \text{ for all } \xi, \zeta \in \mathfrak{g} \arrow[d, Leftrightarrow] \\
\mu^*(\xi) \text{ is } G\text{-invariant for all } \xi \in \mathfrak{g} \arrow[r, Leftrightarrow] & \zeta^\sharp \mu^*(\xi) = 0 \text{ for all } \xi, \zeta \in \mathfrak{g}
\end{tikzcd}$$

The Lie bracket on $\mathfrak{g}$ vanishes identically, so the top-right proposition is equivalent to μ^* being a Lie algebra morphism. □

By definition, the *transpose* of μ^* is a smooth map $\mu \colon M \to \mathfrak{g}^*$ satisfying $\langle \mu, \xi \rangle = \mu^*(\xi)$. Clearly equivariance of μ^* and of μ are equivalent, which justifies the following definition of a Hamiltonian action: A *moment map* is a G-equivariant smooth map $\mu \colon M \to \mathfrak{g}^*$ such that, for each $\xi \in \mathfrak{g}$, the vector field $\xi^\sharp$ is generated by $\langle \mu, \xi \rangle$. A G-action on a symplectic manifold is *Hamiltonian* if it admits a moment map.

As noted in [Dui11, p. 193], the moment map μ can be shifted by any constant element $\alpha \in \mathfrak{g}^*$

which satisfies $\alpha([\mathfrak{g},\mathfrak{g}]) = 0$. If G is abelian then $[\mathfrak{g},\mathfrak{g}] = 0$, so the condition holds for all α. On the other hand, if G is semisimple then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, which forces $\alpha = 0$.

Also important is the notion of a coherent moment map. Given $\zeta \in \mathfrak{g}$, a moment map $\mu\colon M \to \mathfrak{g}^*$ is said to be *ζ-coherent* if $\langle \mu, \zeta\rangle$ is bounded below; μ is said to be *coherent* if it is ζ-coherent for some nonzero ζ. If M is compact all moment maps are coherent, but some care has to be taken otherwise. Note that what we call *coherent* maps are called *polarised* maps in [GGK00]. Our terminology was chosen so it would not clash with the notion of polarisation defined in §II.

Moment maps interact nicely with group homomorphisms in the following way.

Proposition III.1. *Let $\tilde{G}, G$ be Lie groups with respective Lie algebras $\tilde{\mathfrak{g}}, \mathfrak{g}$. A morphism $\phi\colon \tilde{G} \to G$ induces a map $(\mathrm{d}\phi)^*\colon \mathfrak{g}^* \to \tilde{\mathfrak{g}}^*$. A Hamiltonian G-action on M (with moment map μ) induces a Hamiltonian $\tilde{G}$-action on M (with moment map $\tilde{\mu} = (\mathrm{d}\phi)^* \circ \mu$).*

Proof. Omitted. This can be found, for example, in [Ren, Lemma 4.4]. □

Various examples of (coherent) moment maps, and a specialisation to the case where G is a torus, are presented below in §III.4. Before that, we need to develop the notion of reduction.

III.2 Symplectic reduction

Given a symplectic action of G on M, we may ask how to quotient M by G to get a symplectic manifold. The naive quotient M/G is not going to work, because the resulting space—if it is even a manifold at all—could be odd-dimensional. As we will see, the correct notion is of *symplectic reduction*, which requires the action to be Hamiltonian.

Consider, therefore, a Hamiltonian G-action on M with moment map μ. Given $p \in M$, denote by G_p the stabiliser subgroup $\{g \in G\colon g.p = p\}$ of group elements fixing p. Its Lie algebra is denoted by $\mathfrak{g}_p$.

Proposition III.2. *Let $p \in M$. Then $\mathrm{d}\mu_p\colon T_pM \to \mathfrak{g}^*$ is surjective if and only if G_p is discrete.*

Proof. Unravelling the definitions, we see that the dual map to $\mathrm{d}\mu_p$ is the linear map

$$(\mathrm{d}\mu_p)^*\colon \mathfrak{g} \to T_p^*M\colon \xi \mapsto (\iota_{\xi^\sharp}\omega)_p.$$

Now we see that $\mathfrak{g}_p = \ker(\mathrm{d}\mu_p)^*$. By standard linear algebra, this means the annihilator of $\mathfrak{g}_p$ is $\mathrm{im}(\mathrm{d}\mu_p)$. It follows that $\mathfrak{g}_p = 0$ if and only if $\mathrm{d}\mu_p$ is surjective. □

In particular, if $\alpha \in \mathfrak{g}^*$ is a regular value of μ then G acts locally freely on $\mu^{-1}(\alpha)$. This observation brings us to the notion of the *reduction* of M by G.

Theorem III.3. *Let $\alpha \in \mathfrak{g}^*$ be a coadjoint fixed point; the* symplectic reduction of M *at α is $M/\!/_\alpha G := \mu^{-1}(\alpha)/G$. Suppose that α is a regular value of μ and that G acts freely on $\mu^{-1}(\alpha)$. Then $M/\!/_\alpha G$ is a symplectic manifold when equipped with the 2-form $\omega^{/\!/}$ satisfying $\pi^*\omega^{/\!/} = \iota^*\omega$, where π is as in the following diagram.*

$$\begin{tikzcd}
(\mu^{-1}(\alpha), \iota^*\omega) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & (M, \omega) \\
(M/\!/_\alpha G, \omega^{/\!/}) &
\end{tikzcd}$$

Proof. From the freeness of the action, we see that $\mu^{-1}(\alpha)$ is a principal G-bundle. Denote by $[p]$ the orbit of $p \in \mu^{-1}(\alpha)$. We have $T_p\mu^{-1}(\alpha) = \ker \mathrm{d}\mu_p$ and $T_p[p] = \{\xi^\sharp_p \mid \xi \in \mathfrak{g}\}$. Also

$$(\ker \mathrm{d}\mu_p)^\omega = \left(\bigcap_{\xi\in\mathfrak{g}} \ker \mathrm{d}\langle \mu, \xi\rangle_p\right)^\omega = \sum_{\xi\in\mathfrak{g}} (\ker \mathrm{d}\langle\mu,\xi\rangle_p)^\omega = \sum_{\xi\in\mathfrak{g}} \mathrm{span}_\mathbb{R}\{\xi^\sharp_p\} = \{\xi^\sharp_p \mid \xi \in \mathfrak{g}\},$$

so $\left(T_p\mu^{-1}(\alpha)\right)^\omega = T_p[p] \subseteq T_p\mu^{-1}(\alpha)$. This implies that $\mu^{-1}(\alpha)$ is coisotropic, with isotropic leaves given by the G-orbits. Apply Theorem I.9 to get the result. ◻

Remark III.4. If G acts only locally freely on $\mu^{-1}(\alpha)$, then the reduction $M/\!/_\alpha G$ will be an orbifold in general. This occurs in many cases of interest (such as the moduli space of flat connections modulo gauge, as described in [MS17, §5.3]) but we will focus on the manifold case here for simplicity.

An important issue is the quantisability of reduced spaces. As we have already seen, for $M/\!/_\alpha G$ to be quantisable, we must have $[\omega^{/\!/}] \in 2\pi H^2(M/\!/_\alpha G; \mathbb{Z})$. This clearly depends on the choice of $\alpha \in \mathfrak{g}^*$, but not in a way that exclusively depends on the group G; rescaling ω will rescale μ and therefore the set of such α.

III.3 Reducing quantisation data

Suppose that the Hamiltonian G-action on M lifts to an equivariant action on the prequantum line bundle $\mathbb{L}$. The infinitesimal action is given by

$$\mathfrak{g} \to \mathrm{End}_\mathbb{C}(\mathbb{L}) \colon \xi \mapsto \nabla_{\xi^\sharp} + 2\pi i \langle \mu, \xi\rangle.$$

Assume that $\alpha \in \mathfrak{g}^*$ satisfies the hypotheses of Theorem III.3—that is, α is a regular value of μ and G acts freely on the fibre $\mu^{-1}(\alpha)$—and that $M/\!/_\alpha G$ is quantisable. We want to define the Hermitian line bundle over $M/\!/_\alpha G$ induced from reduction of M. This is completely described by its local sections, which are precisely the G-invariant sections of $\iota^*\mathbb{L}$. This relationship is illustrated in the following pullback diagram.

$$\begin{tikzcd} \iota^*\mathbb{L} \arrow[r] \arrow[d] & \mathbb{L}^{/\!/} \arrow[d] \\ \mu^{-1}(\alpha) \arrow[r, "\pi"] & M/\!/_\alpha G \end{tikzcd}$$

The identity $\pi^*\mathbb{L}^{/\!/} = \iota^*\mathbb{L}$ actually specifies the reduced line bundle uniquely. Reduction of the prequantum connection is analogous, but it is a more difficult result:

Proposition III.5. *There is a unique connection $\nabla^{/\!/}$ on $\mathbb{L}^{/\!/}$ such that $\pi^*\nabla^{/\!/} = \iota^*\nabla$. Moreover, we have* $\operatorname{curv}\nabla^{/\!/} = -i\omega^{/\!/}$*, so $(\mathbb{L}^{/\!/}, \nabla^{/\!/})$ is a prequantum line bundle for $M/\!/_\alpha G$.*

Proof. This combines Theorem 3.2 and Corollary 3.4 of [GS82b]. □

The final part of the quantisation data is the polarisation of M. Regarding this, we assume that the polarisation is G-invariant. Since we will always use the holomorphic polarisation $T^{0,1}M$ of a Kähler manifold, this amounts to G acting by biholomorphisms. The polarisation on M therefore descends in a straightforward way to a polarisation on the reduction $M/\!/_\alpha G$.

III.4 Examples and torus actions

We come to some examples. Given an abelian Lie group G, we identify $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathbb{R}^{\dim G}$.

Example III.6. Let $r \geqslant 1$, let $\mathbf{w}_1, \dots, \mathbf{w}_n \in \mathbb{Z}^r$ and consider an effective T^r action on $\mathbb{C}^n$:

$$(e^{i\theta_1}, \dots, e^{i\theta_r}).(z_0, \dots, z_n) = (e^{i\mathbf{w}_0\cdot\boldsymbol{\theta}}z_0, \dots, e^{i\mathbf{w}_n\cdot\boldsymbol{\theta}}z_n)$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$. This is a complex T^r-representation with weights $\{\mathbf{w}_k\}$. A moment map for this action is given by $\mu\colon \mathbb{C}^n \to \mathbb{R}^r\colon (z_1, \dots, z_n) \mapsto \sum_{k=0}^n \mathbf{w}_k|z_k|^2$. The prequantum line bundle is trivial, with prequantum connection given by

$$\nabla = \mathrm{id} + i\lambda_0.$$

Now choose $\zeta \in \mathbb{R}^r$ such that $\{\mathbf{w}_k \cdot \zeta\}$ are all nonzero; clearly μ would be ζ-coherent if we replaced the $\mathbf{w}_k$ by $\mathbf{w}_k^+ := \operatorname{sgn}(\mathbf{w}_k \cdot \zeta)\mathbf{w}_k$. However, we want to preserve the action and complex structure of $\mathbb{C}^n$, so this is achieved by replacing the standard symplectic form ω_0 with

$$\omega_0^+ := i\sum_{k=1}^n \operatorname{sgn}(\mathbf{w}_k \cdot \zeta)\mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k.$$

Note that $(\mathbb{C}^n, \omega_0^+)$ is *not* Kähler unless $\omega_0 = \omega_0^+$.

If $r = n$ then this is an example of a symplectic toric manifold. In general, a *symplectic toric manifold* is a symplectic manifold with an effective action of $T^{\frac{1}{2}\dim M}$. Complex projective spaces are also symplectic toric manifolds, as described in the following example adapted from [Aud04, Example III.2.18].

Example III.7. Let S^1 act diagonally by multiplication on $\mathbb{C}^{n+1}$ (on which we use coordinates $(z_0, \dots, z_n)$). A moment map is given by $\mu(z_0, \dots, z_n) = \sum_{k=0}^n |z_k|^2$. For each $\alpha \in [0, \infty)$, the moment fibre $\mu^{-1}(\alpha)$ is the sphere S^{2n+1} of radius $\sqrt{\alpha}$, and the reduction at α is $\mathbb{C}^{n+1}/\!/_\alpha S^1 = (\mathbb{P}^n, \alpha\omega_{\mathrm{FS}})$, where ω_{FS} is the Fubini–Study form defined in Example I.5. Note that the quotient map $S^{2n+1} \xrightarrow{\pi} \mathbb{P}^n$ is (a rescaling of) the Hopf fibration. The T^r-action of Example III.6 preserves the unit sphere S^{2n+1}, so it naturally induces a (possibly non-effective) T^r-action on $\mathbb{P}^n$ with moment map

$$\mu\colon \mathbb{P}^n \to \mathbb{R}^r\colon [z_0, \dots, z_n] \mapsto \alpha \frac{\sum_{k=0}^n \mathbf{w}_k |z_k|^2}{\sum_{k=0}^n |z_k|^2}.$$

If $r = n+1$ then $\mathbb{P}^n$ becomes a symplectic toric variety with big torus T^{n+1}/S^1.

In the above example, the moment image $\mu(\mathbb{P}^n)$ is a simplex defined by the convex hull of the points $\{\alpha\mathbf{w}_k\}$. (The 'traditional' picture of the polytope associated to $\mathbb{P}^n$—which is the hull of $\{0, \mathbf{e}_1, \dots, \mathbf{e}_n\} \subset \mathbb{R}^n$—is recovered by picking the appropriate weights $\mathbf{w}_k$ and projecting the resulting standard simplex onto a coordinate hyperplane. See Example VI.5.) The moment image being a polytope is not an accident, as the following result shows.

Theorem III.8. *Let M be a compact connected symplectic manifold. Suppose a torus T acts on M in a Hamiltonian fashion, with moment map μ. Then $\mu(M) \subseteq \mathfrak{t}^*$ is a convex polytope which coincides with the convex hull of $\mu(\mathrm{Fix}_T(M))$.*

Proof. This is attributed to independent papers of Atiyah [Ati82] and Guillemin and Sternberg [GS82a]. □

Not all polytopes can be obtained in this way. Those that can are called *Delzant*, and we have the following correspondence [Del88] between these polytopes and symplectic toric manifolds.

Theorem III.9 (Delzant)**.** *There is a correspondence*

$$\frac{\textit{symplectic toric manifolds}}{\textit{equivariant symplectomorphism}} \longleftrightarrow \frac{\textit{Delzant polytopes}}{\textit{translation}}$$

given by $(M, \mu) \leftrightarrow \mu(M)$. □

IV The Guillemin–Sternberg Theorem

In this short section, we show that the $\mathcal{Q}^0$-quantisation of Kähler manifolds commutes with reduction, under quite favourable hypotheses. The argument is identical to the one presented in

[GS82b], and all proofs sketched here can be found in that paper.

IV.1 The complexified action

Let G be a compact Lie group. Its *complexification* $G_{\mathbb{C}}$ is a complex Lie group equipped with a morphism $G \to G_{\mathbb{C}}$, satisfying the obvious universal property:

$$\begin{tikzcd} G_{\mathbb{C}} & \\ G \arrow[u, hook] \arrow[r] & H \arrow[ul, dashed, "\exists!"'] \end{tikzcd}$$

for all complex Lie groups H. For example, the complexification of S^1 is $\mathbb{C}^*$.

Proposition IV.1. *Suppose G acts holomorphically on a compact complex manifold (M, J). Then it extends to a holomorphic $G_{\mathbb{C}}$-action.*

This is [GS82b, Theorem 4.4]. The idea is to extend $\sharp$ to $\mathfrak{g}_{\mathbb{C}}$ by $(i\xi)^\sharp := J\xi^\sharp$, then check that this is still a Lie algebra morphism. The key ingredient is that the Lie derivative $\mathcal{L}_{\xi^\sharp}$ commutes with J for all $\xi \in \mathfrak{g}$. One then shows that this integrates to a $G_{\mathbb{C}}$ action.

Proposition IV.2. *The G-action on $\mathbb{L}$ extends canonically to a $G_{\mathbb{C}}$-action. Moreover, G-invariant holomorphic sections are automatically $G_{\mathbb{C}}$-invariant.*

Proof. We need to extend the infinitesimal action $\xi \mapsto \nabla_{\xi^\sharp} + 2\pi i\langle \mu, \xi\rangle$ to the complexification $\mathfrak{g}_{\mathbb{C}}$. Suppose $s \in \Gamma(\mathbb{L})$ is holomorphic. Define $(i\xi).s := i(\xi.s)$.

$$\begin{aligned} (i\xi).s &= i(\xi.s) && \text{(by definition)} \\ &= i(\nabla_{\xi^\sharp} + 2\pi i\langle \mu, \xi\rangle)s && \\ &= i(i\nabla_{J\xi^\sharp} + 2\pi i\langle \mu, \xi\rangle)s && \text{(s is holomorphic)} \\ &= -\nabla_{(i\xi)^\sharp}s - 2\pi\langle \mu, \xi\rangle s. && \text{(by definition)} \end{aligned}$$

The action of $i\xi$ for all sections is obtained by setting

$$(i\xi).(fs) = ((i\xi)^\sharp f)s + f(i\xi).s$$

for all smooth $f \in C^\infty(M)$. One then integrates this to a $G_{\mathbb{C}}$ action. The second assertion is obvious in view of the formulae above. □

IV.2 $\mathcal{Q}^0$ commutes with reduction

For the remainder of this section we consider reduction at $0 \in \mathfrak{g}^*$. Note that $G\mu^{-1}(0) = \mu^{-1}(0)$; by [GS82b, Theorem 4.5] the set

$$\mathring{M} := G_{\mathbb{C}}\mu^{-1}(0)$$

is an open subset of M, and $G_{\mathbb{C}}$ acts freely on it.

Example IV.3. The Hamiltonian T^n-action of a symplectic toric n-manifold M extends to a $(\mathbb{C}^*)^n$-action, where $\mathring{M} \cong (\mathbb{C}^*)^n$. This is what allows us to view symplectic toric manifolds as toric varieties in the sense of complex algebraic geometry. A good comparison of these perspectives is [Ren].

In general we see that

$$M//G = \mu^{-1}(0)/G = (G\mu^{-1}(0))/G = \mathring{M}/G_{\mathbb{C}}.$$

This perspective allows us to view symplectic reduction as a genuine group-theoretic quotient of M, by enlarging the pertinent group and—informally speaking—deleting a badly behaved divisor from M.

We come to the following result, known as the Guillemin–Sternberg Theorem.

Theorem IV.4. *Suppose that (M, ω) is a compact prequantisable Kähler manifold. Let G be a compact Lie group which acts on M in a holomorphic and Hamiltonian fashion, with moment map μ. Suppose that $0 \in \mathfrak{g}^*$ is a regular value of μ, that G acts freely on $\mu^{-1}(0)$ and that $M//G$ is prequantisable. Then there is a canonical isomorphism*

$$\mathcal{Q}^0(M)^G \cong \mathcal{Q}^0(M//G)$$

of complex vector spaces.

Proof. By the content of §III.3 there is a canonical map

$$\mathcal{Q}^0(M)^G \to \mathcal{Q}^0(M//G) \colon s \mapsto [s|_{\mu^{-1}(0)}]. \qquad (\star)$$

We show in Lemmas IV.5 and IV.6 that this decomposes into two maps which are isomorphisms:

$$\begin{tikzcd}
\mathcal{Q}^0(\mathring{M})^G \arrow[dr, "\text{Lemma IV.6}"] & \\
\mathcal{Q}^0(M)^G \arrow[u, "\text{Lemma IV.5}"] \arrow[r, "(\star)"] & \mathcal{Q}^0(M//G)
\end{tikzcd}$$

Essentially, Lemma IV.5 says that every G-invariant holomorphic section of $\mathbb{L}$ on $\mathring{M}$ comes from restricting a section on M. Lemma IV.6 says that a G-equivariant section of $\mathbb{L}$ over $\mathring{M}$ is determined uniquely by its values on the moment fibre $\mu^{-1}(0) \subset \mathring{M}$. □

It remains only to prove the Lemmas.

Lemma IV.5. *There is a canonical bijection $\mathcal{Q}^0(M)^G \to \mathcal{Q}^0(\mathring{M})^G$.*

Proof. Fix a section $s \in \mathcal{Q}^0(\mathring{M})$. Take $p \in \mu^{-1}(0), \xi \in \mathfrak{g}$ and consider the curve $\gamma(t) = \exp(it\xi)p$. Now $(i\xi)^\sharp.s = 0$ by the $G_\mathbb{C}$-invariance of s, so the sketch proof of Proposition IV.2 gives $\nabla_{(i\xi)^\sharp} = -2\pi\langle\mu,\xi\rangle s$. From the compatibility of ∇ and $\langle -, -\rangle_\mathbb{L}$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle s,s\rangle_\mathbb{L} = (i\xi)^\sharp\langle s,s\rangle_\mathbb{L} = \langle\nabla_{(i\xi)^\sharp}s,s\rangle_\mathbb{L} + \langle s,\nabla_{(i\xi)^\sharp}s\rangle_\mathbb{L} = -4\pi\langle\mu,\xi\rangle\langle s,s\rangle_\mathbb{L}.$$

The compatibility of ω and J ensures that $(i\xi)^\sharp$ is the gradient vector field of $\langle\mu,\xi\rangle$, which implies that $\langle\mu,\xi\rangle$ is increasing along γ. Because $\langle\mu,\xi\rangle(0) = 0$, this in turn implies that $\langle s,s\rangle_\mathbb{L}^2$ is decreasing along γ. It follows that s is bounded, since it is bounded on $\mu^{-1}(0)$ by compactness. We claim that the restriction map $\mathcal{Q}^0(M)^G \to \mathcal{Q}^0(\mathring{M})^G$ is therefore bijective. This claim follows from a higher-dimensional analogue of the standard 'removal of singularities' theorem of complex analysis together with the fact that the complement $M \backslash \mathring{M}$ is contained in a subvariety of M of positive codimension [GS82b, Theorem 5.7]. □

Lemma IV.6. *There is a canonical bijection $\mathcal{Q}^0(\mathring{M})^G \to \mathcal{Q}^0(M/\!/G)$.*

Proof. The identity $M/\!/_0G = \mathring{M}/G_\mathbb{C}$ induces an identification $\mathcal{Q}^0(M/\!/G) \cong \mathcal{Q}^0(\mathring{M})^{G_\mathbb{C}}$. Under this identification the bijection in the statement of the Lemma is just the identity, which is well-defined by Proposition IV.2. □

V Hamiltonian cobordism

V.1 Hamiltonian G-spaces and cobordism

Let G be a Lie group. A *Hamiltonian G-space* is a smooth manifold M equipped with a smooth G-action, a closed 2-form ω which is preserved by G, and a G-equivariant map $\mu\colon M \to \mathfrak{g}^*$ such that $\langle\mu,\xi\rangle$ generates $\xi^\sharp$ for each $\xi \in \mathfrak{g}$. This definition is almost the same as the one for Hamiltonian actions in §III.1, except that we no longer require ω to be non-degenerate.

We begin this section by describing a notion of cobordism between Hamiltonian G-spaces. In general for cobordism theories, to avoid degeneration one needs to impose a compactness condition on the cobordisms. For example, every manifold M is cobordant to the empty set by setting $W = M \times [0,1)$. If we work with Hamiltonian G-spaces the issue may appear to persist because one can equip $M \times [0,1)$ with the G-action and 2-form induced from M. The caveat is that the associated moment map is not proper, because every preimage has a $[0,1)$ factor.

Suppose M and N are closed Hamiltonian G-spaces of the same dimension. A *Hamiltonian cobordism* between them is a Hamiltonian G-space W with boundary $\partial W = M \sqcup N$, whose Hamiltonian G-structures restrict to those on M and N, and whose moment map is proper.

V.2 Cobordism commutes with quantisation and reduction

Cobordism of Hamiltonian G-spaces interacts well with the two key constructions we have defined so far: quantisation and reduction. The former fact is too difficult to prove in complete generality. Here we suggest why such a thing might be reasonably expected by sketching the compact case.

Proposition V.1. *Let M and N be compact Hamiltonian G-spaces which are compactly Hamiltonian cobordant via some W. Then $\mathbb{Q}(M) \cong \mathbb{Q}(N)$.* □

Proof. The idea is as follows. Since M is compact, there are no issues with L^2 completeness, so its quantisation is just given by the alternating sums of sheaf cohomology groups: $\mathbb{Q}(M) \cong \sum_{k=0}^{\infty} H^k(M, \mathbb{L})$. Therefore its dimension is the sheaf Euler characteristic $\chi(M, \mathbb{L})$, which is given by the Hirzebruch–Riemann–Roch Theorem [Hir66] as the integral over M of a certain cohomology class which depends on the complex structure of M. Then $\mathbb{Q}(M) \cong \mathbb{Q}(N)$ follows from Stokes' theorem on compact manifolds with boundary, applied to W. To make the last line precise we would need to explain the notion of *stable complex structure* and much more besides. □

Commuting with reduction is more straightforward.

Proposition V.2. *Let M, N be Hamiltonian G-spaces with moment maps μ_M, μ_N. Suppose they are cobordant via some Hamiltonian G-space W with moment map μ_W and closed 2-form ω_W. Suppose that $\alpha \in \mathfrak{g}^*$ is a regular value of both μ_M and μ_N. Then $M//_\alpha G$ is cobordant to $N//_\alpha G$.*

Proof. If α is also a regular value of μ_W then we are immediately done; the reduction $W//_\alpha G$ provides the desired cobordism. If not, then [GGK96, Lemma 1] guarantees that one can perturb μ_W and ω_W on an open set away from ∂W, so that α is a regular value for μ_W. □

Remark V.3. As far as we know, this pertubation cannot always be performed such that G acts freely on $\mu_W^{-1}(\alpha)$, even if it does on $\mu_M^{-1}(\alpha)$ and $\mu_N^{-1}(\alpha)$. Therefore, in general the reductions of M and N are only cobordant through orbifolds; recall the caveat in Remark III.4.

V.3 The Linearisation Theorem

In this section is to prove that a Hamiltonian G-space is cobordant to the normal bundle of its fixed point set $\mathrm{Fix}_G(M)$. First, we show that $\mathrm{Fix}_G(M)$ is actually a symplectic manifold.

Lemma V.4. *Let a compact Lie group G act symplectically and holomorphically on a Kähler manifold (M, ω, J). Then $\mathrm{Fix}_G(M)$ is a symplectic submanifold of M.*

Proof. We want to construct the tangent space to each point $p \in \mathrm{Fix}_G(M)$. At a fixed point $p \in \mathrm{Fix}_G(M)$, the action of G induces a representation

$$G \to \mathrm{Aut}(T_p M)$$

which is unitary with respect to J. Note that G acts by isometries with respect to the Riemannian metric $\langle X, Y\rangle = \omega(X, JY)$, so its exponential map $\exp\colon TM \to M$ is equivariant. It follows that $\exp_p$ identifies a neighbourhood of $p \in \mathrm{Fix}_G(M)$ with the G-invariant subspace of T_pM. This subspace is precisely the tangent space $T_p\mathrm{Fix}_G(M)$, which is symplectic because it is a complex subspace and J is compatible with ω. □

At this point, we specialise to $G = T$ a torus. We will see that the Linearisation Theorem (Theorem V.6) arises naturally by fixing what goes wrong with trying to make M nullcobordant.

Consider a Hamiltonian T-action on a symplectic manifold (M, ω), with ζ-coherent moment map μ. Consider the product $W^{\mathrm{pre}} = M \times [0, 1)$ with the induced T-action from M, and let π be the natural projection $\pi\colon W^{\mathrm{pre}} \to M$. Then W^{pre} is a Hamiltonian T-space when equipped with the closed 2-form $\pi^*\omega$ and moment map $\pi^*\mu$. However, $\pi^*\mu$ is not proper since every preimage has a factor of $[0, 1)$, so this is not a valid Hamiltonian cobordism.

This non-properness is remedied by adding to $\pi^*\mu$ an increasing function $\rho\colon [0, 1) \to [0, \infty)$. More precisely, fix a positive parameter ε which will be taken as small as necessary, and let ρ be a smooth function which vanishes on $[0, 1-\varepsilon)$ before tending to ∞ on $[1-\varepsilon, 1)$. For example:

$$\rho(t) = \begin{cases} \dfrac{1}{1-t}\exp\left(-\dfrac{1}{(1-\varepsilon-t)^2}\right) & t \geqslant 1-\varepsilon, \\ 0 & t < 1-\varepsilon. \end{cases}$$

Write $\zeta^\sharp$ for the vector field on W^{pre} induced by the T-action as in §III.1, and write $g = \pi^*\langle -, -\rangle$. Define the 1-form $\beta^{\mathrm{pre}} \in \Omega^1(W^{\mathrm{pre}})$

$$\beta^{\mathrm{pre}}(X) := \begin{cases} \rho(t)\dfrac{g(\zeta^\sharp, X)}{g(\zeta^\sharp, \zeta^\sharp)} & \zeta^\sharp \neq 0 \\ 0 & \zeta^\sharp = 0 \end{cases}.$$

Now we equip W^{pre} with 2-form $\tilde{\omega}^{\mathrm{pre}} := \pi^*\omega - \mathrm{d}\beta$; a straightforward calculation shows that for each $\xi \in \mathfrak{t}$, the function

$$\langle \mu, \xi \rangle + \beta(\xi^\sharp) \colon W^{\mathrm{pre}} \to \mathbb{R}$$

generates $\xi^\sharp$. Moreover, it is proper because μ is ζ-coherent, and ζ-coherent because $\beta^{\mathrm{pre}}(\zeta^\sharp) = \rho$ by construction. It may appear as if W^{pre} is thus a Hamiltonian T-space with moment map $\tilde{\mu}^{\mathrm{pre}}$, which would constitute a complete degeneration of the cobordism theory. The saving grace is that $\tilde{\mu}^{\mathrm{pre}}$ is *not* a moment map, because it is not equivariant; it is evidently not constant on the subspace $\mathrm{Fix}_T(W) = \mathrm{Fix}_T(M) \times [0, 1)$.

By construction, $\tilde{\mu}^{\mathrm{pre}}$ is only non-constant on $\mathrm{Fix}_T(M) \times [1 - \varepsilon, 1)$. Let $W := W^{\mathrm{pre}} \backslash B$, where

$$B := \left\{ (p, t) \in W^{\mathrm{pre}} : \left(\inf_{q \in \mathrm{Fix}_T(M)} \right)^2 + |1 - t|^2 < (2\varepsilon)^2 \right\}$$

is a small neighbourhood of the submanifold $\mathrm{Fix}_T(M) \times \{0\}$; here d is the standard distance function derived from the metric on M. Here we make our choice of ε: small enough so that ∂B has one component for each $S \in \pi_0(\mathrm{Fix}_T(M))$. Use π to symplectically identify this component with a disc bundle on S, which itself is identified with the normal bundle νS of S.

We want to view these normal bundles as Hamiltonian T-spaces in their own right. The T-action on each νS is the induced one, and this requires no further discussion. On the other hand, the 2-form and moment map are not canonical. They are, however, unique up to cobordism, which we now prove in the case where S is a point. The statement in its full generality is [GGK00, Lemma 4.6].

Lemma V.5. *Let a torus T act linearly on $\mathbb{C}^n$. Let ω_0, ω_1 be closed 2-forms on $\mathbb{C}^n$, with moment maps μ_0, μ_1 which agree at the origin. Then there exists a Hamiltonian cobordism $(\mathbb{C}^n, \omega_0, \mu_0) \sim (\mathbb{C}^n, \omega_1, \mu_1)$.*

Proof. Let β be a T-invariant 1-form, which vanishes at 0, such that $\mathrm{d}\beta = \omega_0 - \omega_1$. Then

$$\mathfrak{t} \to C^\infty(\mathbb{C}^n) \colon \xi \mapsto \langle \mu_0, \xi \rangle + \beta(\xi^\sharp)$$

is a comoment map for ω_1. Then it is easily seen that the Hamiltonian T-space

$$\left(\mathbb{C}^n \times [0, 1], \mathrm{pr}^*\omega_0 - \mathrm{d}(t\mathrm{pr}^*\beta), (1 - t)\mathrm{pr}^*\mu_0 + t\mathrm{pr}^*\mu_1 \right)$$

provides the desired Hamiltonian cobordism, where $\mathrm{pr} \colon \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n$ is the projection. □

Write ω_S and μ_S for the symplectic form and moment map on νS. By flipping signs in a local coordinate expression of ω_S (as in Example III.6 for the case of a point) we can arrange that each μ_S is coherent. We have arrived at the following:

Theorem V.6. *Let (M, ω) be a Hamiltonian T-space, with coherent moment map μ. Then we have a Hamiltonian cobordism*

$$(M, \omega, \mu) \sim \bigsqcup_{S \in \pi_0 \mathrm{Fix}_T(M)} (\nu S, \omega_S, \mu_S)$$

where each μ_S is coherent. □

One can also incorporate the (pre)quantum data into the Linearisation Theorem, which leads to the following 'Quantum Linearisation Theorem' which is [GGK00, Theorem 7.31]. Among other things, its proof uses the cobordism-invariance of quantisation mentioned in §V.2.

Theorem V.7. *We have $\mathcal{Q}(M) \cong \sum_S \mathcal{Q}(\nu S)$ as virtual unitary T-representations.* □

Example V.8. If the T-action has no fixed points, the right-hand sum is empty so $\mathcal{Q}(M) = 0$.

Henceforth we assume that the fixed point set is discrete, so that $\nu\{p\} = T_p M$ for each fixed point $p \in \mathrm{Fix}_T(M)$. By an equivariant unitary transformation we can identify $T_p M \cong \mathbb{C}^n$ in such a way that ω_S is mapped to the possibly non-Kähler linear space $(\mathbb{C}^n, \omega_r)$ from Example I.4, where $r = 0$ if $\mu_{\{p\}}$ was already ζ-coherent.

VI Quantisation commutes with reduction

The prototypical example to keep at the back of one's mind is the following. Let S^1 act on $(\mathbb{C}^{n+1}, \omega_0)$ diagonally by multiplication. We saw in Example II.9 that $\mathcal{Q}(\mathbb{C}^{n+1})$ is a certain L^2-completion of the space of all monomials $z_0^{\ell_0} \cdots z_n^{\ell_n}$. This is an S^1-representation, and the span of such a monomial is a sub-representation of weight $\ell_0 + \cdots + \ell_n$. In fact, the part of $\mathcal{Q}(\mathbb{C}^{n+1})$ of weight α, denoted $\mathcal{Q}(\mathbb{C}^{n+1})^\alpha$, is clearly given by $\mathbb{C}[z_0, \ldots, z_n]_\alpha$, the subspace of homogeneous polynomials of degree α. This coincides with the quantisation of $(\mathbb{P}^n, \alpha\omega_{\mathrm{FS}})$ as in Example II.7. This is not an coincidence, as we will see.

VI.1 Partition functions and Delzant polytopes

Let $(\mathbb{C}^n, \omega_r)$ be the modified linear space of Example I.4, with a linear T-action with coherent weights $\mathbf{w}_1, \ldots, \mathbf{w}_n \in \mathfrak{t}^*$, and moment map μ given by $\mu(z_1, \ldots, z_n) = \mu(0) + \sum_{k=1}^n \mathbf{w}_k |z_k|^2$. This T-action endows the quantisation

$$\mathcal{Q}(\mathbb{C}^n) \cong (-1)^r \mathcal{B}(\overline{\mathbb{C}}^r \times \mathbb{C}^{n-r}) \otimes (s\mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_r)$$

with the structure of a virtual T-representation. The elements $z_k, \bar{z}_k$, and $s\mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_r\}$ transform according to the weights $\mathbf{w}_k, -\mathbf{w}_k$ and $\mu(0) - \sum_{k \leqslant r} \mathbf{w}_k$. Thus if

$$X = \bar{z}_1^{\ell_1} \cdots \bar{z}_r^{\ell_r} z_{r+1}^{\ell_{r+1}} \cdots z_n^{\ell_n} \otimes s\mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_r$$

is a monomial element of $(-1)^r \mathcal{Q}(\mathbb{C}^n)$, the T-representation $\operatorname{span}_{\mathbb{C}}\{X\}$ has weight

$$\underbrace{-\sum_{k \leqslant r} \ell_k \mathbf{w}_k + \sum_{r<k} \ell_k \mathbf{w}_k}_{\text{from } \bar{z}_1^{\ell_1} \cdots \bar{z}_r^{\ell_r} z_{r+1}^{\ell_{r+1}} \cdots z_n^{\ell_n}} + \underbrace{\mu(0) - \sum_{k \leqslant r} \mathbf{w}_k}_{\text{from } s\mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_r} .$$

We can rewrite this as

$$\sum_{1 \leqslant k \leqslant n} \ell_k \mathbf{w}_k^+ + \mu(0) + \sum_{1 \leqslant k \leqslant r} \mathbf{w}_k^+,$$

where

$$\mathbf{w}_k^+ := \begin{cases} -\mathbf{w}_k & k \leqslant r \\ \mathbf{w}_k & r \leqslant k. \end{cases}$$

It follows immediately that $(-1)^r \dim \mathcal{Q}(\mathbb{C}^n)^\alpha$ is precisely the number of lattice points of the preimage $P^{-1}(\alpha)$, where the function $P\colon [0,\infty)^n \to \mathfrak{t}^*$ is given by

$$P(\ell_1, \dots, \ell_n) = \mu(0) + \sum_{1 \leqslant k \leqslant r} \mathbf{w}_k^+ + \sum_{1 \leqslant k \leqslant n} \ell_k \mathbf{w}_k^+.$$

We record this as a Proposition.

Proposition VI.1. *Consider the setup above. Then* $\dim \mathcal{Q}(\mathbb{C}^n)^\alpha = (-1)^r \#\big(\mathbb{N}^n \cap P^{-1}(\alpha)\big)$.

Note that $P^{-1}(\alpha)$ can only be finite and nonempty if the weights are coherent.

Example VI.2. Take $r = 0$ with a linear action of T^n with weights $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{R}^n$. Then each P-preimage is either empty or a single point: for $\alpha \in \mathbb{Z}^n$ we have

$$P^{-1}(\alpha) = \begin{cases} \{(\mathbf{e}_1 \cdot \alpha, \dots, \mathbf{e}_n \cdot \alpha)\} & \alpha \in [0,\infty)^n \\ \varnothing & \text{otherwise.} \end{cases}$$

This corresponds to the fact that if α is in the moment image then $\mathbb{C}^n /\!/_\alpha T^n$ is a point (in which case $\mathcal{Q}(\mathbb{C}^n /\!/_\alpha T^n) \cong \mathbb{C}$), and otherwise it is empty (in which case $\mathcal{Q}(\mathbb{C}^n /\!/_\alpha T^n) \cong 0$).

VI.2 Reduction of linear space

An effective Hamiltonian T^r-action on $\mathbb{C}^n$ (with coherent moment map $\mu\colon \mathbb{C}^n \to \mathbb{R}^r$) can be extended to an effective T^n action which makes $\mathbb{C}^n$ into a symplectic toric manifold; the reduction $\mathbb{C}^n /\!/ T^r$ inherits the structure of symplectic toric manifold with big torus $T^{n-r} \cong T^n/T^r$ (with moment map $\mu^{/\!/}\colon \mathbb{C}^n /\!/_\alpha T^r \to \mathbb{R}^{n-r}$).

We claim that the quantisation of this reduced space is given by the following formula.

Theorem VI.3. *Let α be a regular value of μ. Then* $\dim \mathcal{Q}(\mathbb{C}^n /\!/_\alpha T) = (-1)^r \#(\mathbb{N}^n \cap P^{-1}(\alpha))$.

Corollary VI.4. *Let α be a regular value of μ. Then $\mathcal{Q}(\mathbb{C}^n)^\alpha \cong \mathcal{Q}(\mathbb{C}^n /\!/_\alpha T)$.* □

Proof of Theorem VI.3. We sketch only the Kähler case ($r = 0$); the full statement is [GGK00, Proposition 8.5], whose proof occupies Chapter 7 therein. Note that $\mathbb{C}^n /\!/_\alpha T^r$ is the symplectic toric manifold corresponding under Theorem III.9 to the Delzant polytope

$$\left\{(s_1, \dots, s_n) \in [0, \infty)^n : \sum_k s_k \mathbf{w}_k = \alpha\right\}.$$

We turn to its quantisation. It turns out that the higher Dolbeault cohomology vanishes, and there is a folklore result [Dan78] which states that $\dim_{\mathbb{C}} H^{0,0}$ is given by the number of lattice points in the moment polytope $\mu^{/\!/}(\mathbb{C}^n /\!/_\alpha T^r)$. The crucial link comes when we see that the latter is precisely $P^{-1}(\alpha)$. In view of Proposition III.1, the moment image $\bar{\mu}(\mathbb{C}^n)$ surjects onto $\mu(\mathbb{C}^n)$ via P (where $\bar{\mu}\colon \mathbb{C}^n \to \mathbb{R}^n$ is the moment map for the T^n action) and the fibre over $\alpha \in \mathbb{Z}^r$ is the moment image $\mu^{/\!/}(\mathbb{C}^n /\!/_\alpha T^r)$. The result follows. □

Example VI.5. Suppose T^n acts on $(\mathbb{P}^n, \alpha\omega_{\mathrm{FS}})$ by

$$(e^{i\theta_1}, \dots, e^{i\theta_n}).[z_0, \dots, z_n] = [z_0, e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n].$$

As in Example III.7, a moment map is

$$\mu\colon \mathbb{P}^n \to \mathbb{R}^n\colon [z_0, \dots, z_n] \mapsto \alpha \frac{(|z_1|^2, \dots, |z_n|^2)}{\sum_{k=0}^n |z_k|^2}.$$

The moment polytope $\mu(\mathbb{P}^n)$ is the convex hull of $\{0, \alpha\mathbf{e}_1, \dots, \alpha\mathbf{e}_n\}$. For $\alpha = 1$ it is clear that the only intersections with $\mathbb{Z}^n$ are the vertices, of which there are $n+1$. For $\alpha = 2$ there are $n+1+\binom{n+1}{2}$ lattice points of $\mu(\mathbb{P}^n)$. In both cases this coincides with the calculation $\dim \mathcal{Q}(\mathbb{P}^n) = \binom{n+\alpha}{\alpha}$ carried out in Example II.7. As α grows, verifying Theorem VI.3 manually becomes more complicated.

VI.3 $\mathcal{Q}$ commutes with reduction

We now have all the pieces to prove the culmination of this dissertation. The reader should note that the general statement [GGK00, Theorem 8.3] greatly relaxes the regularity assumptions on α, which is one of the strengths of this proof strategy. However, accomodating the singular setting would be far too complicated to treat here; we have not even discussed singular reduction!

Theorem VI.6. *Suppose that (M, ω) is a compact prequantisable Kähler manifold. Let T be a torus which acts on M in a holomorphic and Hamiltonian fashion, with moment map μ, and with isolated fixed points. Suppose that $\alpha \in \mathfrak{g}^*$ is a regular value of μ and of that G acts freely* *on $\mu^{-1}(\alpha)$ and that $M//T$ is prequantisable. Then there is a canonical isomorphism*

$$\mathcal{Q}(M)^{\alpha} \cong \mathcal{Q}(M//_{\alpha}T)$$

of complex vector spaces.

Proof. Since M is compact, μ is coherent. Localise at the fixed points using the Linearisation Theorem, where the situation reduces to the case treated above:

$$\begin{aligned}
\mathcal{Q}(M)^{\alpha} &\cong \sum_{p \in \mathrm{Fix}_T(M)} \mathcal{Q}(T_pM)^{\alpha} && \text{(Theorem V.7)}\\
&\cong \sum_{p \in \mathrm{Fix}_T(M)} \mathcal{Q}(T_pM//_{\alpha}T) && \text{(Corollary VI.4)}\\
&\cong \mathcal{Q}(M//_{\alpha}T). && \text{(Proposition V.2)}
\end{aligned}$$

The result follows. □

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