Let N be a Riemannian manifold, and let $\iota\colon M \hookrightarrow N$ be a submanifold. Write νM for the normal bundle of M. The Levi-Civita connection of N induces a pullback connection

$$\nabla\colon TM \to \mathrm{End}(\iota^* TN)$$

on ι^*TN. Under the splitting $\iota^*TN \cong TM \oplus \nu M$, and in view of the equality $S_\xi(X) = -(\nabla_X \xi)^T$, we write this in block matrix form as

$$\nabla = \begin{pmatrix} \nabla^M & -\Sigma \\ B & \nabla^\perp \end{pmatrix}$$

where for each vector field $X \in \Gamma(TM)$,

$$\begin{aligned} \nabla^M_X &\in \mathrm{Hom}(TM, TM) && \text{is the Levi-Civita connection on } M \\ B_X &\in \mathrm{Hom}(TM, \nu M) && \text{is the second fundmental form } B_X Y = B(X,Y) \\ \Sigma_X &\in \mathrm{Hom}(\nu M, TM) && \text{is the curried shape operator } \Sigma_X \xi = S_\xi(X) \\ \nabla^\perp_X &\in \mathrm{Hom}(\nu M, \nu M) && \text{is the normal connection on } M. \end{aligned}$$

Given $X, Y \in \Gamma(TM)$, the matrix for $\nabla_X \nabla_Y$ is given as follows:

$$\begin{pmatrix} \nabla^M_X & -\Sigma_X \\ B_X & \nabla^\perp_X \end{pmatrix} \begin{pmatrix} \nabla^M_Y & -\Sigma_Y \\ B_Y & \nabla^\perp_Y \end{pmatrix} = \begin{pmatrix} \nabla^M_X \nabla^M_Y - \Sigma_X B_Y & -\nabla^M_X \Sigma_Y - \Sigma_X \nabla^\perp_Y \\ B_X \nabla^M_Y + \nabla^\perp_X B_Y & -B_X \Sigma_Y + \nabla^\perp_X \nabla^\perp_Y \end{pmatrix}$$

The matrix for the Riemann curvature operator $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is thus

$$\begin{pmatrix} \nabla^M_X \nabla^M_Y - \nabla^M_Y \nabla^M_X - \nabla^M_{[X,Y]} + \Sigma_Y B_X - \Sigma_X B_Y & -\nabla^M_X \Sigma_Y + \nabla^M_Y \Sigma_X - \Sigma_X \nabla^\perp_Y + \Sigma_Y \nabla^\perp_X + \Sigma_{[X,Y]} \\ B_X \nabla^M_Y - B_Y \nabla^M_X + \nabla^\perp_X B_Y - \nabla^\perp_Y B_X - B_{[X,Y]} & \nabla^\perp_X \nabla^\perp_Y - \nabla^\perp_Y \nabla^\perp_X - \nabla^\perp_{[X,Y]} - B_X \Sigma_Y + B_Y \Sigma_X \end{pmatrix}$$

This matrix describes the four components of $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(TM \oplus \nu M)$. We claim that this fact recovers the Fundamental Equations of Gauss, Codazzi and Ricci.

We derive the Codazzi equation; the others are similar. Given $Z \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$,

$$\begin{aligned} R(X,Y,Z,\xi) &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \xi \rangle \\ &= \langle B_X \nabla^M_Y Z - B_Y \nabla^M_X Z + \nabla^\perp_X B_Y Z - \nabla^\perp_Y B_X Z - B_{[X,Y]} Z, \xi \rangle. \end{aligned}$$

Because ∇^M is torsion-free we have $B_{[X,Y]} = B_{\nabla^M_X Y} - B_{\nabla^M_Y X}$. Then

$$\begin{aligned} &= \langle \nabla^\perp_X B_Y Z - B_{\nabla^M_X Y} Z - B_Y \nabla^M_X Z, \xi \rangle - \langle \nabla^\perp_Y B_X Z - B_{\nabla^M_Y X} Z - B_X \nabla^M_Y Z, \xi \rangle \\ &= \langle (\nabla_X B)(Y,Z), \xi \rangle - \langle (\nabla_Y B)(X,Z), \xi \rangle \end{aligned}$$

which is the Codazzi equation.

This formulation hints at a naturally occurring 'fourth Fundamental equation' describing the $\mathrm{Hom}(\nu M, TM)$ part, but this is equivalent to the Codazzi equation by the symmetry of the Riemann curvature tensor.