BE Mathematical Extended Essay

Pseudo-holomorphic lines in the complex projective plane

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0 Introduction

Consider the following basic fact from projective geometry.

Proposition 0.1. *Let $\Bbbk$ be a field. Given $p, p' \in \Bbbk\mathbb{P}^2$ distinct, there exists a unique projective line which meets p and p'.* □

In the case of $\Bbbk = \mathbb{C}$ this can be reinterpreted in terms of holomorphic maps. From now on, we simply write $\mathbb{P}^n$ for $\mathbb{C}\mathbb{P}^n$.

Theorem 0.2. *Given $p, p' \in \mathbb{P}^2$ distinct, there exists a degree one holomorphic map $\mathbb{P}^1 \to \mathbb{P}^2$, unique up to reparametrisation, which meets p and p'.*

Note that the standard definition of the degree of an algebraic curve (via polynomials) does not make sense a priori, so by 'degree one' we mean that it represents the positive generator $[\mathbb{P}^1]$ of $H_2(\mathbb{P}^2; \mathbb{Z}) \cong \mathbb{Z}$, which is the class of a line.

Proof. The implication from Proposition 0.1 is simple; the image of a degree one holomorphic map $\mathbb{P}^1 \to \mathbb{P}^2$ is a projective line. This follows from Chow's theorem [Cho49] that every complex analytic variety in $\mathbb{P}^n$ is algebraic; an algebraic variety of degree one is a projective line. □

For an alternative proof more in keeping with the flavour of this essay, note that existence is immediate because the inclusions of complex projective lines are holomorphic. The uniqueness part follows from the 'positivity of intersections' phenomenon for holomorphic curves in 4-manifolds: see Corollary 1.15.

This essay concerns an extension of Theorem 0.2 from the complex-geometric setting to the almost-complex setting. A few definitions are in order before we can precisely state the result.

0.1 Pseudo-holomorphic curves

Definitions 0.3. An *almost-complex structure* on a smooth manifold M is a linear complex structure on its tangent bundle, which is a section $J \in \Gamma(\mathrm{End}(TM))$ satisfying $J^2 = -\mathrm{id}_{TM}$. The pair (M, J) is an *almost-complex manifold*. Write $\mathcal{J}(M)$ for the space of almost-complex structures on a smooth manifold M, with the C^∞-subspace topology from $\Gamma(\mathrm{End}(TM))$.

We will specifically be interested in a subclass of almost-complex structures called *tame* almost-complex structures. In practice one normally works with *compatible* almost-complex structures, but we will stick to tame ones here in the interest of greater generality.

Definitions 0.4. An almost complex structure J on a symplectic manifold (M, ω) is *ω-tame* if $\omega(v, Jv) > 0$ for all nonzero tangent vectors $v \in TM$. An ω-tame almost-complex structure J

is *ω-compatible* if $\omega(-,-) = \omega(J-,J-)$. Write $\mathcal{J}(\omega)$ for the space of ω-tame almost-complex structures on M.

Note immediately that if J is ω-tame then $-J$ is not, so the subset $\mathcal{J}(\omega) \subset \mathcal{J}(M)$ is always proper.

Example 0.5. The complex structure of a complex manifold gives rise to a natural almost-complex structure I given by $Iv = \sqrt{-1}v$ in coordinates, and we say I is *integrable* if it arises in this way. For example, consider the Fubini–Study form $\omega_{\mathbb{P}^n}$, the unique (up to scaling[1]) symplectic form on $\mathbb{P}^n$ which is invariant under the natural $U(n+1)$ action. It is easily checked that $\omega_{\mathbb{P}^n}(-,I-)$ defines a Riemannian metric on $\mathbb{P}^n$, so in particular I is $\omega_{\mathbb{P}^n}$-tame.

We come to a few central definitions.

Definitions 0.6. A *pseudo-holomorphic curve* is a smooth map $u : \Sigma \to M$ from a Riemann surface (Σ, j) to an almost-complex manifold (M,J) whose differential is complex-linear: the diagram

$$\begin{tikzcd} T\Sigma \arrow[r, "\mathrm{d}u"] \arrow[d, "j"] & u^*TM \arrow[d, "J"] \\ T\Sigma \arrow[r, "\mathrm{d}u"] & u^*TM \end{tikzcd}$$

commutes. We occasionally call this a *J-holomorphic curve* when J needs to be specified. A *J-holomorphic line (in $\mathbb{P}^2$)* is a J-holomorphic curve $(\mathbb{P}^1, I) \to (\mathbb{P}^2, J)$ of degree one (this is nonstandard).

Remark 0.7. In more general contexts than this essay, where j must also be specified, one might call them (j,J)-holomorphic curves. However, we will almost exclusively consider the case of the Riemann sphere $(\Sigma, j, \mathrm{dvol}_\Sigma) = (\mathbb{P}^1, I, \omega_{\mathbb{P}^1})$ here; in particular j is fixed.

A decomposition of $\mathrm{d}u$ into its complex-linear and -antilinear components

$$\mathrm{d}u = \tfrac{1}{2}(\mathrm{d}u - J \circ \mathrm{d}u \circ j) + \tfrac{1}{2}(\mathrm{d}u + J \circ \mathrm{d}u \circ j) =: \partial_J u + \bar{\partial}_J u$$

allows us to restate this condition simply as $\bar{\partial}_J u = 0$. It is clear that a smooth map between complex manifolds is holomorphic in the chart-transition sense if and only if it is pseudo-holomorphic for the induced almost-complex structures.

It turns out that pseudo-holomorphic curves behave much like ordinary (integrable) holomorphic curves. The moral conclusion is that the rigidity of complex geometry comes from the

[1]The Fubini–Study form is often normalised so that $\int_{\mathbb{P}^1} \omega_{\mathbb{P}^n} = \pi$ (e.g. [Mor07]) but sometimes so that $\int_{\mathbb{P}^1} \omega_{\mathbb{P}^n} = 1$ (e.g. [Eli98]). The choice is entirely inconsequential for our purposes, so we simply define the constant

$$\hbar := \int_{\mathbb{P}^1} \omega_{\mathbb{P}^n},$$

which will reappear in the bubbling discussion.

complex-linearity of the differential—as opposed to the existence of a holomorphic atlas—and we will later see results which demonstrate this. For example, the following structure theorem shows that every pseudo-holomorphic curve is a multiple cover of a somewhere injective one.

Proposition 0.8 ([McD87, Lemma 4.4(i)]). *Let Σ be a connected Riemann surface and M an almost-complex manifold. Every pseudo-holomorphic curve $u : \Sigma \to M$ factors as*

$$\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[d, "\phi"'] & M \\ \Sigma' \arrow[ur, "u'"'] & \end{tikzcd}$$

for a connected Riemann surface Σ', a holomorphic map $\phi : \Sigma \to \Sigma'$, and a somewhere injective pseudo-holomorphic curve $u' : \Sigma' \to M$. □

Accordingly, we call a somewhere injective pseudo-holomorphic curve is *simple*.

0.2 Statement of Theorem 0.9

A more specific case of the properties of pseudo-holomorphic curves comes from the seminal paper of Gromov [Gro85], where he introduced moduli spaces of pseudo-holomorphic curves into symplectic geometry. The ideas presented therein have since completely transformed the field, and the aim of this essay is to exposit these ideas by describing the proof of the following relatively simple result.

Theorem 0.9 (Improved from [Gro85, Example 2.4.A]). *Let $J \in \mathscr{J}(\omega_{\mathbb{P}^2})$. Given $p, p' \in \mathbb{P}^2$ distinct, there exists an embedded J-holomorphic line, unique up to reparametrisation, which meets p and p'.*

Remark 0.10. If $\mathscr{J}(\omega)$ were just a point $\{I\}$, the proof of Theorem 0.9 would be very short indeed! Note that for all $J \in \mathscr{J}(\omega)$, the image $u(\Sigma) \subseteq M$ of a J-holomorphic embedding u is a symplectic submanifold of M, because ω is non-degenerate on the J-invariant bundle $\operatorname{im} \mathrm{d}u \to u(\Sigma)$ by definition. In fact, the converse is true: every symplectically embedded surface $S \subseteq M$ is the image of a J-holomorphic curve for some $J \in \mathscr{J}(\omega)$. To see this, take some $J \in \mathscr{J}(\omega|_S)$ on TS, then extend it: first to $TM|_S$, then to TM. (Full details of this construction can be found in the proof of [Wen18, Proposition 2.2].) Since $(\mathbb{P}^2, \omega_{\mathbb{P}^2})$ contains many symplectically embedded spheres which are not algebraic curves, we can be assured that the problem is nontrivial.

In the original version of the result, Gromov could only guarantee that the curve is immersed. The improvement is due to McDuff's adjunction formula (Corollary 1.13), which provides a purely homological criterion for a pseudo-holomorphic curve in a four dimensional manifold to be embedded. It is worth mentioning that we found this application of the adjunction formula before we discovered it already in the literature. Consequently, we present no novel results.

Let us take this opportunity to describe the relative character of this essay. The content of Gromov's paper, while revolutionary, is presented in a very sketchy way. Since then, many of the foundations have been set for the subject, notably in the monograph [MS12]. We differ from the existing literature by taking advantage of special cases and developing only what is necessary for the proof of Theorem 0.9. For example, the full statement of Gromov's compactness theorem is unnecessary because bubbling is impossible for minimal spherical homology classes: see Subsection 4.3.

Unfortunately, this area of mathematics rests on a huge amount of background in the form of incredibly deep theorems (e.g. the Atiyah–Singer Index Theorem) and heavy functional analysis (e.g. elliptic regularity theory). It is therefore impossible to give a completely self-contained proof of Theorem 0.9, so we are forced to draw a veil over many details, each of which could be the topic of entire Extended Essays in their own right. We instead focus on the geometric content of the ideas developed by Gromov, in the spirit of his remark

> "... the only meaningful objects are holomorphic curves which are unsensible to a choice of the infinite dimensional phraseology".

0.3 Strategy and organisation

Here we outline the structure of this essay by describing our strategy for the proof of Theorem 0.9. For $J \in \mathscr{J}(\mathbb{P}^2)$, write $\widetilde{\mathscr{M}}(J)$ for the moduli space of (parametrised) J-holomorphic lines. Since $[\mathbb{P}^1] \in H_2(\mathbb{P}^2;\mathbb{Z})$ is not a multiple of any other positive class, every J-holomorphic line is somewhere injective (see Proposition 0.8) so the Möbius group $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1) = \mathrm{PSL}_2(\mathbb{C})$ acts freely on $\widetilde{\mathscr{M}}(J)$ by reparametrisation via

$$\phi.u = u \circ \phi^{-1}.$$

Write $\mathscr{M}(J) := \widetilde{\mathscr{M}}(J)/\mathrm{PSL}_2(\mathbb{C})$ for the quotient space of unparametrised lines, which can be thought of as the space of images in $\mathbb{P}^2$. Clearly $\widetilde{\mathscr{M}}(J)$ has no hope of being compact unless it is empty, but we will later prove the nontrivial fact that $\mathscr{M}(J)$ is compact when J is $\omega_{\mathbb{P}^2}$-tame.

Example 0.11. By Theorem 0.2, the moduli space $\mathscr{M}(I)$ is the space of complex projective lines in $\mathbb{P}^2$. By projective duality, this is isomorphic to $\mathbb{P}^2$ itself, which is compact.

Proving the uniqueness aspect of Theorem 0.9 is the focus of Section 1. Along the way, we show that the space of $\omega_{\mathbb{P}^2}$-tame almost-complex structures is contractible. Section 2 introduces the perspective on $\widetilde{\mathscr{M}}(J)$ as the zero set of a section of an infinite-dimensional vector bundle. This shows that $\widetilde{\mathscr{M}}(J)$ is a smooth finite-dimensional manifold if the pertinent section is transverse to the zero section, which always holds in our setting; this is the topic of Section 3. Section 4 proves that any two moduli spaces $\mathscr{M}(J)$ of images are compact cobordant, thus nonempty. The proof of Theorem 0.9 is finished in Section 5, where we conclude with an interpretation of the

Theorem as a Gromov–Witten computation, and an application to Hamiltonian dynamics.

1 Intersections in 4-manifolds

Note that Theorem 0.9 is essentially an existence and uniqueness result for a system of non-linear partial differential equations ($\bar{\partial}_J u = 0$). In general for PDE problems, there is no principle for whether existence or uniqueness is harder to prove. However, one of the features of four dimensions, which makes this theory particularly powerful, is the fact that they two pseudo-holomorphic curves can intersect transversely. This gives rise to purely homological bounds for the number of intersections of pseudo-holomorphic curves, and ultimately to uniqueness statements for these curves.

We begin this Section by introducing the first Chern class, an important invariant of complex vector bundles which will appear in multiple contexts in this essay.

1.1 The first Chern class

Recall that an almost-complex structure on a manifold M endows TM with the structure of a complex vector bundle.

Definition 1.1. The *first Chern class* is an assignment of a cohomology class $c_1(E) \in H^2(B;\mathbb{Z})$ to each complex complex vector bundle $E \to B$, satisfying

(NATURALITY) $c_1(f^*E) = f^*c_1(E)$ for all maps $f : A \to B$, where f^*E is the pullback bundle and $f^* : H^2(B;\mathbb{Z}) \to H^2(A;\mathbb{Z})$ is the induced map;

(HOMOMORPHISM) $c_1(E \oplus F) = c_1(E) + c_1(F)$, where $F \to B$ is another complex vector bundle;

(NORMALISATION) $\langle c_1(\mathcal{O}(-1)), [\mathbb{P}^1]\rangle = -1$, where $\mathcal{O}(-1) \to \mathbb{P}^1$ is the tautological line bundle and $[\mathbb{P}^1] \in H_2(\mathbb{P}^1;\mathbb{Z})$ is the fundamental class.

The existence and uniqueness of c_1 is a theorem in and of itself, which we will not bother to prove, but can be found in most textbooks on characteristic classes (e.g. [Hat17]).

Remark 1.2. The (HOMOMORPHISM) property is only a H^2-slice of the formula

$$c(E \oplus F) = c(E) \smile c(F),$$

where $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in H^*(B)$ is the *total Chern class* of a bundle $E \to B$.

In the context of complex vector bundles $E \to \Sigma$ over oriented surfaces, it is common to abuse notation by writing $c_1(E)$ for the *first Chern number* $\langle c_1(E), [\Sigma]\rangle$. This is simply an identification under the isomorphism $H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$, which does not cause much confusion in practice.

Example 1.3. Holomorphic line bundles over $\mathbb{P}^n$ are (isomorphic to) the twisting bundles $\mathcal{O}(k)$ for $k \in \mathbb{Z}$; for example, see [OSS80]. We have $c_1(\mathcal{O}(k)) = k$ and $\mathcal{O}(k) \otimes \mathcal{O}(m) = \mathcal{O}(k+m)$, so the first Chern number is precisely the degree $c_1 = \deg : \mathrm{Pic}(\mathbb{P}^n) \to \mathbb{Z}$ of the line bundle.

Example 1.4. If $E \to \Sigma$ is a complex line bundle over an oriented surface then the first Chern class is the Euler class of the underlying real bundle. Thus, given a global section $s : \Sigma \to E$ which is transverse to the zero section, $c_1(E)$ is Poincaré dual to the homology class $[s^{-1}(0)]$. In particular, the first Chern number $c_1(T\Sigma)$ is the Euler characteristic $\chi(\Sigma)$. Clearly $c_1(\mathcal{O}(k))$ cannot be defined in this way for $k < 0$ since these bundles admit no global sections, but in this case we just take $c_1(\mathcal{O}(k)) = -c_1(\mathcal{O}(-k))$.

One might expect the first Chern class $c_1(TM, J)$ to depend on the almost-complex structure J. This is the case in general, but the following useful result shows that $c_1(TM, J)$ is independent of J when we restrict to the subset of tame almost-complex structures.

Lemma 1.5. *Given a symplectic manifold (M, ω), the space $\mathcal{J}(\omega)$ is contractible.*

In particular, since $\mathcal{J}(\omega)$ is connected we have the following.

Corollary 1.6. *The map $\mathcal{J}(\omega) \to H^2(M; \mathbb{Z}) : J \mapsto c_1(TM, J)$ is constant.* □

The argument originally given by Gromov in [Gro85, Corollary 2.3.C$'_2$] to prove this statement is purely homotopy-theoretic. In a more general setting, the non-emptiness of $\mathcal{J}(\omega)$ is part of this result, and consequently part of the proof. However, we will apply these results where $(M, \omega) = (\mathbb{P}^2, \omega_{\mathbb{P}^2})$, and we already know that $\mathcal{J}(\omega_{\mathbb{P}^2})$ is non-empty; it contains the integrable $\omega_{\mathbb{P}^2}$-compatible structure I.

The existence of a compatible almost-complex structure affords us the liberty of a more direct proof as in [MS17, Proposition 2.6.4(i)].

Proof of Lemma 1.5 when M is Kähler. Let $\langle v, w\rangle_I = \omega(v, Iw)$ be the Kähler metric and $\|v\| = \sqrt{\langle v, v\rangle_I}$ the associated norm. Write

$$\|E_p\| = \sup_{0 \neq v \in T_pM} \frac{\|E_p v\|}{\|v\|}$$

for the operator norm of a fibre endomorphism $E_p : T_pM \to T_pM$. Define

$$\mathscr{E} = \{E \in \Gamma(\mathrm{End}(TM)) : EI + IE = 0,\ \|E_p\| < 1\ \forall p \in M\}.$$

Then $J \mapsto (\mathrm{id}_{TM} + IJ)(\mathrm{id}_{TM} - IJ)$ defines a homeomorphism $\mathcal{J}(\omega) \to \mathscr{E}$, with inverse

$$E \mapsto I(\mathrm{id}_{TM} + E)^{-1}(\mathrm{id}_{TM} - E).$$

Since $\mathscr{E}$ is convex, this completes the proof. □

Remark 1.7. The Kähler property is only used to assert the existence of a compatible almost-complex structure; integrability is not used at all.

In light of this Lemma, we can think of $\omega_{\mathbb{P}^2}$-tame almost-complex structures on $\mathbb{P}^2$ as deformations of the integrable almost-complex structure on $\mathbb{P}^2$ (the definite article is justified here by the content of Remark 1.9).

We end this Subsection with a comment on the literature. It is common to see $c_1(u_*[\Sigma])$ for the Kronecker pairing of $c_1(TM) \in H^2(M;\mathbb{Z})$ with the pushforward integral class $u_*[\Sigma] \in H_2(M;\mathbb{Z})$. However, for computational purposes it is often better to consider bundles over Σ, so much of the literature uses $c_1(u^*TM)$. They turn out to be equal because the naturality of c_1 implies $u^*c_1(TM) = c_1(u^*TM)$, and the equality

$$\langle u^*c_1(TM), [\Sigma]\rangle = \langle c_1(TM), u_*[\Sigma]\rangle$$

follows from the definition of the Kronecker pairing. The advantage of $c_1(u_*[\Sigma])$ is that it clearly depends only on the homology of u, while the advantage of $c_1(u^*TM)$ is that it is more easily computed; see Examples 1.10 and 1.11.

1.2 Intersection forms

On a smooth oriented 4-manifold M there is a bilinear form $H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$ called the *intersection form*, which is Poincaré dual to the cup product on cohomology. As the name suggests, it can be interpreted as counting intersections; if integral classes A and B are represented by transverse submanifolds then

$$A \cdot B = \sum_{p \in A \pitchfork B} k_p$$

where $k_p \in \{-1, 1\}$ depends on the orientation of the intersection at p. This breaks down if A and B are not transverse (for example if they are equal), but can be recovered by perturbing A and B to be transverse. When A and B are not transverse but still have finite intersection, this effectively corresponds to counting intersection points 'with signed multiplicity'.

Example 1.8. By the isomorphism $H_2(\mathbb{P}^2;\mathbb{Z}) \cong \mathbb{Z}$, the intersection form of $\mathbb{P}^2$ is determined by the integer $[\mathbb{P}^1] \cdot [\mathbb{P}^1]$, which is equal to 1 by Proposition 0.1.

It is easy to see that the homology groups of any simply connected closed oriented M are

$$H_0(M;\mathbb{Z}) = H_4(M;\mathbb{Z}) = \mathbb{Z}, \quad H_1(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = 0, \quad H_2(M;\mathbb{Z}) = \mathbb{Z}^r$$

for some $r \geqslant 0$. Thus, in a sense the intersection form of $\mathbb{P}^2$ is as simple as it could be without being trivial; we have $r = 1$ and the form itself is, under the identification $H_2(\mathbb{P}^2;\mathbb{Z}) \cong \mathbb{Z}$, given by multiplication. In general, the intersection form is a very powerful algebro-topological invariant of 4-manifolds. For example, we have the following fact.

Remark 1.9. By a theorem of Milnor and Whitehead [Mil58], simply connected closed 4-manifolds are classified up to homotopy by their intersection form. Moreover, a corollary [Yau77, Theorem 5] of Yau's solution to Calabi's conjecture is that that a compact complex surface (a complex manifold with two complex dimensions) is biholomorphic to $\mathbb{P}^2$ if it and only if it is homotopy equivalent to it. Therefore $\mathbb{P}^2$ is the unique simply connected compact complex surface with the above intersection form.

1.3 The adjunction inequality

First we make a couple of observations. Until the Section ends, all Riemann surfaces are assumed to be closed, but not necessarily connected.

Example 1.10. The pullback bundle of an embedding $u : \Sigma \to M$ from a surface to a 4-manifold splits as $u^*TM = \operatorname{im} \mathrm{d}u \oplus \nu_u$, where ν_u is the normal bundle to $u(\Sigma)$. Note that $\operatorname{im} \mathrm{d}u \cong T\Sigma$ so $c_1(\operatorname{im} \mathrm{d}u) = \chi(\Sigma)$; see Example 1.4. Furthermore, the Tubular Neighbourhood Theorem identifies sections of ν_u with unparametrised curves near $u(\Sigma)$, so we have $c_1(\nu_\nu) = u_*[\Sigma] \cdot u_*[\Sigma]$. Thus

$$c_1(u^*TM) = \chi(\Sigma) + u_*[\Sigma] \cdot u_*[\Sigma],$$

by the (HOMOMORPHISM) property.

Example 1.11. If $u : \Sigma \to M$ is immersed with k double points, the same argument goes through, except now each double point contributes -2 to the number $c_1(\nu_u)$, so again we have

$$c_1(u^*TM) = \chi(\Sigma) + u_*[\Sigma] \cdot u_*[\Sigma] - 2k,$$

again by the (HOMOMORPHISM) property.

We now aim to describe a converse to the above calculations. Given two simple J-holomorphic curves $u : \Sigma \to M$ and $u' : \Sigma' \to M$ with simple union, define the number

$$\mathfrak{x}(u, u') := \#\{(z, z') \in \Sigma \times \Sigma' : u(z) = u'(z'), z \neq z'\};$$

the condition $z \neq z'$ is superfluous when $\Sigma \neq \Sigma'$, but important otherwise. For example, $\mathfrak{x}(u, u)$ is twice the number of self-intersection points of u. We have the following result, which was proved by McDuff in [McD91] (modulo some minor technical corrections in her chapter of [AL94]).

Theorem 1.12 (Adjunction inequality)**.** *Let Σ be a closed Riemann surface.*

$$c_1(u_*[\Sigma]) + \mathfrak{x}(u, u) \leqslant \chi(\Sigma) + u_*[\Sigma] \cdot u_*[\Sigma]$$

with equality if and only if u is a self-transverse immersion. □

As we have seen, one direction is relatively easy, at least for self-transverse immersions. The converse depends crucially on the singularity theory of pseudo-holomorphic curves. However, the Theorem is not directly based on an idea suggested in [Gro85], which is our excuse for skipping the proof entirely.

That being said, it does imply the following very useful corollary.

Corollary 1.13. *Every J-holomorphic line is an embedding for all $J \in \mathcal{J}(\omega_{\mathbb{P}^2})$.*

Proof. Every I-holomorphic line is embedded by the proof of Theorem 0.2, so by Example 1.10 we have $\langle c_1(T\mathbb{P}^2, I), [\mathbb{P}^1]\rangle = \chi(\mathbb{P}^1) + [\mathbb{P}^1] \cdot [\mathbb{P}^1]$. By Corollary 1.6, the equality

$$\langle c_1(T\mathbb{P}^2, J), [\mathbb{P}^1]\rangle = \chi(\mathbb{P}^1) + [\mathbb{P}^1] \cdot [\mathbb{P}^1]$$

holds for all $J \in \mathcal{J}(\omega_{\mathbb{P}^2})$. It is here that we use the full power of the adjunction inequality: by Theorem 1.12, we have $\mathfrak{x}(u, u) \leqslant 0$ for every J-holomorphic line u. Equality must hold because $\mathfrak{x}$ is non-negative, so u is an injective immersion, thus an embedding because $\mathbb{P}^1$ is compact. □

Recall that this is the improvement upon Gromov's original statement noted in the introduction. An important consequence of this, which we will use many times in the following section, is that the pullback bundle always splits as in Example 1.10.

1.4 Positivity of intersections

Transverse complex subvarieties always intersect positively, so the number of times they intersect is bounded by the homological intersection number. We can also describe what happens when intersections are non-transverse, or occur on singular points of the varieties. In particular, we have the following fact from classical algebraic geometry.

Example 1.14 (Bézout's Theorem)**.** If C and D are algebraic curves represented by integral classes $[C], [D] \in H_2(\mathbb{P}^2; \mathbb{Z})$, then each point $p \in C \cap D$ contributes a positive integer to the intersection number $[C] \cdot [D]$. In fact, we have

$$|C \cap D| \leqslant \deg(C) \deg(D) = [C] \cdot [D]$$

with equality if and only if C and D are transverse.

It turns out that this phenomenon persists for J-holomorphic curves. Note that the almost-complex structure J should be the same for both curves. Otherwise this would imply the result for intersections of arbitrary symplectic surfaces!

Corollary 1.15 (Positivity of intersections for embeddings)**.** *Let $J \in \mathcal{J}(M)$ and consider two J-holomorphic embeddings $u : \Sigma \to M$ and $u' : \Sigma' \to M$. Suppose that the union $u \amalg u' : \Sigma \amalg \Sigma' \to M$ is simple. Then*

$$\mathfrak{x}(u, u') \leqslant u_*[\Sigma] \cdot u'_*[\Sigma']$$

with equality if and only if $u(\Sigma)$ and $u'(\Sigma')$ are transverse.

This follows (as suggested in [MS12, Exercise 2.6.7]) from applying the adjunction formula to the union $\Sigma \amalg \Sigma' \to M$.

Proof of Corollary 1.15. Since u, u' are embeddings and consequently $\mathfrak{x}(u, u) = \mathfrak{x}(u', u') = 0$, we have $2\mathfrak{x}(u, u') = \mathfrak{x}(u \amalg u', u \amalg u')$, which is bounded above by

$$\chi(\Sigma \amalg \Sigma') + (u \amalg u')_*[\Sigma \amalg \Sigma'] \cdot (u \amalg u')_*[\Sigma \amalg \Sigma'] - c_1((u \amalg u')_*[\Sigma \amalg \Sigma'])$$

As the Euler characteristic is additive under disjoint union and the evaluation map $H_2(M; \mathbb{Z}) \to \mathbb{Z} : A \mapsto c_1(A)$ is linear, we have

$$\chi(\Sigma \amalg \Sigma') = \chi(\Sigma) + \chi(\Sigma'), \qquad c_1((u \amalg u')_*[\Sigma \amalg \Sigma']) = c_1(u_*[\Sigma]) + c_1(u'_*[\Sigma']).$$

The only 'non-linear' part comes from the intersection number:

$$(u \amalg u')_*[\Sigma \amalg \Sigma'] \cdot (u \amalg u')_*[\Sigma \amalg \Sigma'] = u_*[\Sigma] \cdot u_*[\Sigma] + 2u_*[\Sigma] \cdot u'_*[\Sigma'] + u'_*[\Sigma'] \cdot u'_*[\Sigma'].$$

Since equality holds in the adjunction inequalities for u and u', substituting Theorem 1.12 and simplifying using the above identities yields

$$2\mathfrak{x}(u, u') \leqslant 2u_*[\Sigma] \cdot u'_*[\Sigma'],$$

where equality holds if and only if $u(\Sigma)$ and $u'(\Sigma')$ are transverse. □

This treatment, while efficient for our exposition, does not represent how this result originally came about. Gromov described the phenomenon in its full generality (i.e. for nonregular curves) in his original paper [Gro85], and the proof given by McDuff does not follow the argument above. For the culture, we give an outline of the original proof strategy for immersions. (If $p \in u(\Sigma) \cap u'(\Sigma')$ is a critical point of both u and u' then one needs to do more work still.)

Proof sketch. Essentially, one shows that intersections are isolated and that we can identify a small neighbourhood of each $p \in u(\Sigma) \cap u'(\Sigma')$ with a neighbourhood of 0 in $\mathbb{C}^2$ such that the almost-complex structure of $\mathbb{C}^2$ coincides with I at the point $0 \in \mathbb{C}^2$, and u, u' are respectively given by

$$z \mapsto (z, 0), \quad z \mapsto (f(z), az^k + O(z^{k+1}))$$

for some holomorphic function f. Perturbing u' slightly, it becomes clear that u and u' have algebraic intersection number $k_p = k$. □

The conclusion of this result is an extremely powerful tool for uniqueness results: for a fixed J, two J-holomorphic lines with distinct images intersect transversely exactly once.

Corollary 1.16. *Given $J \in \mathcal{J}(\omega_{\mathbb{P}^2})$ and $p, p' \in \mathbb{P}^2$ distinct, there exists* ***at most one*** *J-holomorphic line, unique up to reparametrisation, which meets p and p'.*

Proof. If two J-holomorphic lines $u, u' : \mathbb{P}^1 \to \mathbb{P}^2$ intersect twice, then their union cannot be simple by Corollary 1.15. Thus the images of u and u' coincide, so they differ by a reparametrisation. □

It is worth noting that nothing so strong holds in higher dimensions. Indeed, if we assume the result of Theorem 0.9 then we can see explicitly that uniqueness completely fails when one replaces $\mathbb{P}^2$ by $\mathbb{P}^3$ in the statement of the theorem. Given $p, p' \in \mathbb{P}^3$ distinct, there is a pencil of projective hyperplanes $\mathbb{P}^2 \subset \mathbb{P}^3$ which intersect at the unique straight line through p and p'. Theorem 0.9 then guarantees the existence of a different[2] J-holomorphic line through each copy of $\mathbb{P}^2$.

As we will see, the fact that one generically has a whole family of pseudo-holomorphic lines through two points in $\mathbb{P}^3$ can be explained analytically by the positive *virtual dimension* of a corresponding moduli space.

2 Moduli spaces as zero sets

Recall that a smooth map $u : \Sigma \to M$ from a Riemann surface Σ to an almost-complex manifold M is J-holomorphic if and only if $\bar{\partial}_J u = 0$. It is therefore reasonable to attempt to view the moduli spaces of J-holomorphic curves as the zero set of $\bar{\partial}_J$, which sends maps to anti-holomorphic one-forms. But the one-form $\bar{\partial}_J u$ is an element of $\Omega^{0,1}(u^*TM) = \Gamma(\Lambda^{0,1}\Sigma \otimes_{\mathbb{C}} u^*TM)$, which clearly depends on u, so the correct language to describe this setup is that of bundles. Define an (infinite-dimensional) vector bundle $\mathcal{E}$ over an appropriate base space $\mathcal{B} \subset C^\infty(\Sigma, M)$, where the fibre above $u \in \mathcal{B}$ is $\mathcal{E}_u := \Omega^{0,1}(u^*TM)$. Then $\bar{\partial}_J$ defines a section $\mathcal{B} \to \mathcal{E} : u \mapsto (u, \bar{\partial}_J u)$, and the set of J-holomorphic curves from Σ to M—satisfying certain conditions encoded in the choice of $\mathcal{B}$—is the preimage of the zero section in $\mathcal{E}$.

Example 2.1. If $\mathcal{B} = C^\infty(\mathbb{P}^1, \mathbb{P}^2)$ then $\bar{\partial}_J^{-1}(0)$ will consist of *all* J-holomorphic curves $\mathbb{P}^1 \to$

[2]One subtlety is that, a priori, the J-holomorphic line through p and p' might be exactly this straight line for all $J \in \mathcal{J}(\omega_{\mathbb{P}^3})$. However, we have seen that every symplectically embedded sphere is the image of some tame pseudo-holomorphic sphere, so this certainly does not always happen.

$\mathbb{P}^2$, including those of higher degree than one. Thus, our first guess for $\mathcal{B}$ should be

$$\mathcal{B} = \{u \in C^\infty(\mathbb{P}^1, \mathbb{P}^2) : u_*[\mathbb{P}^1] = [\mathbb{P}^1]\}$$

so that we have $\widetilde{\mathscr{M}}(J) = \bar{\partial}_J^{-1}(0)$.

It is not clear what the moduli space $\bar{\partial}_J^{-1}(0)$ is like in general; it could be horribly singular. However, in Section 3 we will see that if $\bar{\partial}_J$ is transverse to the zero section then $\bar{\partial}_J^{-1}(0)$ is about as nice as one could hope for: it is a finite-dimensional smooth manifold. Before this can be shown, though, we must set up the pertinent operators and describe their properties.

2.1 The vertical differential

At a pseudo-holomorphic curve u we have a canonical splitting of the tangent space

$$T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u = \Gamma(u^*TM) \oplus \Omega^{0,1}(u^*TM)$$

into horizontal and vertical subspaces. Define the operator

$$D_u \colon \Gamma(u^*TM) \to \Omega^{0,1}(u^*TM)$$

to be the *vertical differential* of $\bar{\partial}_J$, given by $\mathrm{d}_{(u,0)}\bar{\partial}_J(\xi) = (\xi, D_u\xi)$. Importantly, transversality of the section $\bar{\partial}_J$ is expressed as the condition that D_u is surjective for all $u \in \bar{\partial}_J^{-1}(0)$.

In local coordinates, a J-holomorphic curve $u : \mathbb{C} \to \mathbb{R}^{2n}$ satisfies $\partial_s u + J\partial_t u = 0$. Differentiating this in the direction of a vector field $\xi : \mathbb{C} \to \mathbb{R}^{2n}$ gives

$$D_u\xi = \bar{\partial}_J\xi - \frac{1}{2}(J\partial_\xi J)(u)\partial_J(u).$$

It follows that D_u fits into the diagram

$$\begin{tikzcd} \Gamma(T\Sigma) \arrow[r, "\bar{\partial}_j"] \arrow[d, "\mathrm{d}u"] & \Gamma(T\Sigma) \arrow[d, "\mathrm{d}u"] \\ \Gamma(u^*TM) \arrow[r, dashed, "D_u"] & \Omega^{0,1}(u^*TM) \end{tikzcd} \qquad (\star)$$

where $\bar{\partial}_j\xi = \frac{1}{2}(\xi + j \circ \xi \circ j)$ is the complex-antilinear part of a vector field ξ.

For $J = I$ the local formula reduces to $D_u\xi = \bar{\partial}_I\xi$, which means that the differential D_u is simply the Dolbeault operator $\bar{\partial}_{u^*TM}$ associated to the holomorphic vector bundle $u^*TM \to \Sigma$. The next Subsection gives a brief overview of Dolbeault operators, with an emphasis on the results which will (nontrivially) extend to D_u.

2.2 Dolbeault cohomology

Given a complex vector bundle $E \to X$ over a complex manifold X, we have a bisequence

$$\begin{tikzcd}
\vdots & \vdots & \\
\Omega^{1,0}(E) \arrow[u, "\partial_E"] \arrow[r, "\bar{\partial}_E"] & \Omega^{1,1}(E) \arrow[u, "\partial_E"] \arrow[r, "\bar{\partial}_E"] & \cdots \\
\Omega^{0}(E) \arrow[u, "\partial_E"] \arrow[r, "\bar{\partial}_E"] & \Omega^{0,1}(E) \arrow[u, "\partial_E"] \arrow[r, "\bar{\partial}_E"] & \cdots
\end{tikzcd}$$

In particular, $\bar{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfies the Leibniz rule

$$\bar{\partial}_E(f\xi) = f(\bar{\partial}_E\xi) + (\bar{\partial}f)\xi$$

for functions f and vector fields $\xi \in \Omega^{p,q}(E)$. If E is holomorphic[3] then this bisequence is actually a bicomplex, and one can define the *Dolbeault cohomology groups*

$$H^{p,q}_{\bar{\partial}}(X, E) = \frac{\ker(\Omega^{p,q}(E) \xrightarrow{\bar{\partial}_E} \Omega^{p,q+1}(E))}{\operatorname{im}(\Omega^{p,q-1}(E) \xrightarrow{\bar{\partial}_E} \Omega^{p,q}(E))}.$$

In fact, the converse also holds [Mor07, Proposition 9.2]: a $\bar{\partial}_E$ operator which squares to zero induces a holomorphic structure on the complex vector bundle E. Roughly, this is because one can use $\bar{\partial}_E$ to trivialise E around each $p \in X$ by holomorphic sections.

Example 2.2. It can be seen that $\Omega^{p,q}(T\Sigma) = 0$ for $q \geqslant 2$. By the comment above, this implies that every almost-complex structure on a Riemann surface is integrable; there is simply not enough horizontal space in the sequence for successive compositions of the associated Dolbeault operator to be nonzero. In other words, every almost-complex manifold of real dimension 2 is a Riemann surface. This justifies the fact that the domains of pseudo-holomorphic curves are Riemann surfaces instead of two-dimensional almost-complex surfaces; no generality is lost, but this is not immediate from the definitions.

It is a theorem of Dolbeault that these are related to sheaf cohomology groups in the following way, which is essentially the complex analogue of de Rham's theorem.

Theorem 2.3 ([Voi02, Theorem 4.2])**.** *Let $E \to X$ be a holomorphic vector bundle. The Dolbeault cohomology $H^{0,q}_{\bar{\partial}}(X, E)$ is equal to the sheaf cohomology $H^q(X, E)$.* □

[3]It should be noted that recent progress has been made by Cirici and Wilson in defining Dolbeault cohomology for almost-complex manifolds (see [CW21]), but the operator $\bar{\mu}$ which squares to 0 has bidegree $(-1, 2)$ as opposed to $(0, 1)$.

Example 2.4. Consider the[4] holomorphic bundle $E = \bigoplus_{i=1}^{N} \mathcal{O}(k_i)$ over $\mathbb{P}^1$ and the following elliptic complex, which is the bottom row of the Dolbeault bicomplex (cf. Example 2.2):

$$0 \longrightarrow \Omega^0(E) \xrightarrow{\bar{\partial}_E} \Omega^{0,1}(E) \longrightarrow 0.$$

Note that $\bar{\partial}_E$ is just some operator between infinite-dimensional complex vector spaces. It is perhaps initially surprising, therefore, that the Dolbeault cohomology groups are finite-dimensional. By Theorem 2.3, the difference between these dimensions $\dim_{\mathbb{C}} \ker \bar{\partial}_E - \dim_{\mathbb{C}} \operatorname{coker} \bar{\partial}_E$ is equal to the sheaf cohomology Euler characteristic

$$\chi(\mathbb{P}^1, E) = \dim_{\mathbb{C}} H^0(\mathbb{P}^1, E) - \dim_{\mathbb{C}} H^1(\mathbb{P}^1, E).$$

Since $\mathbb{P}^1$ is compact, this Euler characteristic can be computed using the Hirzebruch–Riemann–Roch theorem [Hir66]:

$$\chi(\mathbb{P}^1, E) = \operatorname{rank}_{\mathbb{C}}(E) \cdot \frac{c_1(T\mathbb{P}^1)}{2} + c_1(E) = N + \sum_{i=1}^{N} k_i.$$

Alternatively, since $\bar{\partial}_E$ respects the splitting of E, one can arrive at this formula by considering each $\mathcal{O}(k_i)$ seperately. Using the additivity of χ, one can apply the ordinary Riemann–Roch formula for line bundles over Riemann surfaces to obtain

$$\chi(\mathbb{P}^1, E) = \sum_{i=1}^{N} (\deg(\mathcal{O}(k_i)) + 1 - 0) = N + \sum_{i=1}^{N} k_i.$$

The next Subsection is dedicated to an explanation of this phenomenon.

2.3 Elliptic operators on Sobolev spaces

Given two vector bundles E, F over a manifold X, an elliptic differential operator on a manifold X is a linear map $\Gamma(E) \to \Gamma(F)$ which looks like an elliptic partial differential operator in local coordinates and trivialisations. The most famous example of an elliptic differential operator is perhaps the Laplacian. The exterior derivative on the de Rham complex is also elliptic; see [Wel08, Example 2.5]. The following example is the most relevant here.

Example 2.5 ([Wel08, Example 2.7])**.** The Dolbeault operator $\bar{\partial}_E : \Gamma(E) \to \Omega^{0,1}(E)$ associated to a holomorphic vector bundle E is elliptic. (Here, $F = \Lambda^{0,1}\mathbb{P}^1 \otimes_{\mathbb{C}} E$.)

Together with the following theorem, this explains why $\dim \ker \bar{\partial}_E$ and $\dim \operatorname{coker} \bar{\partial}_E$ are finite-dimensional.

[4]By a theorem of Grothendieck [Gro57], every holomorphic vector bundle over $\mathbb{P}^1$ is isomorphic to such a sum.

Theorem 2.6 (See [Hör94]). *Elliptic operators on compact manifolds have finite-dimensional kernels and cokernels.* □

Definition 2.7. A bounded linear operator D between Banach spaces is *Fredholm* if $\ker D$ and $\operatorname{coker} D$ are finite-dimensional. The *index* of D is defined to be the difference

$$\operatorname{index} D := \dim \ker D - \dim \operatorname{coker} D$$

of these dimensions. A smooth map between Banach manifolds is *Fredholm* if its differential is a Fredholm operator.

Strictly speaking, $\bar{\partial}_E$ is not actually Fredholm, since its domain and codomain are (non-Banach) spaces of smooth sections. Thus we describe a way of completing these spaces—the framework of Sobolev spaces—which works well with another good property of elliptic operators, called elliptic regularity. We state this property for reference before defining the notation therein.

Theorem 2.8 (Elliptic regularity). *Let Σ be a Riemann surface, M an almost-complex manifold, ℓ a positive integer, J a C^ℓ-smooth almost-complex structure on M and $p > 2$ a real number. Every J-holomorphic curve in $W^{1,p}(\Sigma, M)$ is also in $W^{\ell+1,p}(\Sigma, M)$. In particular, if J is smooth then every J-holomorphic curve $u : \Sigma \to M$ is smooth.*

Proof. See [MS12, Proposition 3.1.10]. □

We only have space to describe some relevant facts, whose proofs can be found in [MS12, Appendix B]. Let Σ be a surface and M a manifold. For a positive integer ℓ and real number $p > 2$, write $W^{\ell,p}(\Sigma, M)$ for the Sobolev space of functions $u : \Sigma \to M$ which, in local coordinate charts, have their first ℓ weak derivatives in L^p. As mentioned above, this is the completion of $C^\infty(\Sigma, M)$ Since $\ell p > \dim_{\mathbb{R}} \Sigma$, it can be shown that every such function is continuous, and in fact of regularity C^ℓ. The tangent space at $u \in W^{\ell,p}(\Sigma, M)$ is the space $W^{\ell,p}(u^*TM)$ of $W^{\ell,p}$-regular sections of u^*TM.

We will enlarge the base space for the section $\bar{\partial}_J$ to be the subset

$$\mathcal{B} = \{u \in W^{1,p}(\mathbb{P}^1, \mathbb{P}^2) : u_*[\mathbb{P}^1] = [\mathbb{P}^1]\}$$

of $W^{1,p}$ functions which represent the positive generator of $H_2(\mathbb{P}^2; \mathbb{Z})$. Just as in the smooth setting, consider the vertical differential

$$D_u \colon W^{1,p}(u^*TM) \to L^p(\Lambda^{0,1}\mathbb{P}^1 \otimes_{\mathbb{C}} u^*TM)$$

as a differential operator between Sobolev spaces of sections. It is clear that $\mathcal{B}$ contains the our original guess $\{u \in C^\infty(\mathbb{P}^1, \mathbb{P}^2) : u_*[\Sigma] = [\mathbb{P}^1]\}$, but at first glance it may seem that this enlargement may be too big, in that the zero set of $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ might contain extra unwanted functions which are not smooth.

2.4 Real Cauchy–Riemann operators

Note that this Subsection borrows greatly from [MS12, Appendix C].

We can see from the discussion preceding ($\star$) that the complex-antilinear part of D_u is a compact operator, and the complex-linear part is a holomorphic structure on u^*TM.

Definitions 2.9. A *smooth complex linear Cauchy–Riemann operator* is a smooth, complex-linear operator

$$D : \Omega^0(E) \to \Omega^{0,1}(E)$$

satisfying the Leibniz rule $D(f\xi) = f(D\xi) + (\bar{\partial} f)\xi$ for complex functions f, just like the Dolbeault operator in §2.2. Given $p > 2$. a *real linear Cauchy–Riemann operator of class L^p* on a complex vector bundle $E \to \mathbb{P}^1$ is an operator of the form

$$D + \alpha$$

where D is a smooth complex linear Cauchy–Riemann operator on E and α is an L^p-section of the bundle $\Lambda^{0,1}\mathbb{P}^1 \otimes \mathrm{End}_{\mathbb{R}}(E)$. These operators automatically satisfies the Leibniz rule for real functions.

Example 2.10. The discussion at the start of this chapter implies that D_u is a real linear Cauchy–Riemann operator (of class L^p). When $J = I$ then $D_u = \bar{\partial}_{u^*TM}$ is itself smooth and complex-linear, and the α-part vanishes.

One might ask for the advantage of defining these operators, instead of working directly with D_u. The upshot is that, in the next section, we will study D_u by decomposing it into operators between subspaces of its domain and codomain.

3 Transversality

Recall (from Section 1) that u^*TM splits as $\operatorname{im} \mathrm{d}u \oplus \nu_u$. The diagram ($\star$) implies that the line bundle $\operatorname{im} \mathrm{d}u$ is D_u-invariant, so D_u

$$D_u : W^{1,p}(\operatorname{im} \mathrm{d}u) \oplus W^{1,p}(\nu_u) \to L^p(\Lambda^{0,1}\mathbb{P}^1 \otimes_{\mathbb{C}} \operatorname{im} \mathrm{d}u) \oplus L^p(\Lambda^{0,1}\mathbb{P}^1 \otimes_{\mathbb{C}} \nu_u)$$

can be block-triangularised (abusing notation slightly) as

$$\begin{pmatrix} D_u|_{\operatorname{im} \mathrm{d}u} & K \\ 0 & D_u|_{\nu_u} \end{pmatrix}$$

for some compact operator $K : W^{1,p}(\nu_u) \to L^p(\Lambda^{0,1}\mathbb{P}^1 \otimes_{\mathbb{C}} \operatorname{im} \mathrm{d}u)$. It follows that $D_u|_{\operatorname{im} \mathrm{d}u}$ and $D_u|_{\nu_u}$ are both real linear Cauchy–Riemann operators themselves. This is useful because operators on line bundles are much easier to make conclusions about.

3.1 Automatic transversality

In particular, we have the following crucial result.

Theorem 3.1 (Special case of [MS12, Theorem 3.1.10(iii)]). *Let D be a real linear Cauchy–Riemann operator of class $W^{1,p}$ on a complex line bundle $E \to \mathbb{P}^1$.*

(i) Suppose $c_1(E) < 0$. Then D is injective.

(ii) Suppose $c_1(E) > -2$. Then D is surjective.

This was first claimed in [Gro85, 2.1.C_1], but went unproved for over a decade until it was finally shown by Hofer–Lizan–Sikarov in [HLS98] using the following key idea.

Lemma 3.2 ([HLS98, Lemma]). *Every $\xi \in \ker D$ also lies in the kernel of some smooth complex Cauchy–Riemann operator (which is allowed to depend on ξ).* □

This reduces the problem to the smooth complex-linear case, where it is is significantly easier as the situation becomes completely algebro-geometric. We shall prove the result in the case where D is the Dolbeault operator $\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$.

Proof when $D = \bar{\partial}_E$. Note first that $\bar{\partial}_E$ induces a holomorphic structure on E by the discussion in Example 2.2, so we can take $E = \mathcal{O}(c_1(E))$; see Example 1.3.

(i) By Dolbeault's theorem (Theorem 2.3), the kernel of $\bar{\partial}_E$ is the space of global sections of E. Since $c_1(E)$ is negative, this space is trivial, so $\bar{\partial}_E$ is injective.

(ii) Similarly, we have $\operatorname{coker} \bar{\partial}_E = H^1(\mathbb{P}^1, \mathcal{O}(c_1(E)))$, which by Serre duality is isomorphic to

$$H^0(\mathbb{P}^1, T^*\mathbb{P}^1 \otimes \mathcal{O}(c_1(E))^*) = H^0(\mathbb{P}^1, \mathcal{O}(-2 - c_1(E))).$$

Since $-2 - c_1(E)$ negative, this space is trivial, so $\bar{\partial}_E$ is surjective. □

Together with the block-triangulation described above, this is enough to prove the following.

Definition 3.3. An almost-complex structure J is *regular* if the section $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ is transverse to the zero section $u \mapsto (u, 0)$; that is, if D_u is surjective for all $u \in \bar{\partial}_J^{-1}(0)$.

Corollary 3.4 (Automatic transversality). *Every $J \in \mathcal{J}(\omega_{\mathbb{P}^2})$is regular.*

Proof. The (first) Chern numbers of $\operatorname{im} \mathrm{d}u$ and ν_u are 2 and 1 respectively. Thus by part (ii) of the previous theorem, each of these—and by extension D_u itself—is surjective. □

Automatic transversality is another dramatic feature of four dimensions, which does not persist in higher dimensions. It evidently rests on the fact that the quotient $u^*TM/(\operatorname{im} \mathrm{d}u)$ is a line bundle, so one can apply Theorem 3.1 directly. However, the argument survives if one asks for a suitable filtration u^*TM consisting of D_u-invariant sub-bundles; see [MS12, Lemma 3.3.2].

Remark 3.5 (Transversality is generic). Transversality is not always as automatic as in Corollary 3.4. Fix a positive integer ℓ. Sard's Theorem states that the set of critical values of a C^ℓ map $f : M \to N$ between finite-dimensional manifolds (that is, points $y \in N$ such that $\mathrm{d}_x f$ is not surjective for some $x \in f^{-1}(y)$) has measure zero whenever $\ell > \dim M - \dim N$. One shouldn't readily expect regularity to be so generic in the infinite-dimensional setting, but since Fredholm operators only fail to be surjective by a finite number of dimensions, it is perhaps not very surprising that Sard's Theorem extends to this case. In particular, the Sard–Smale Theorem [Sma65] states that the set of critical values of a C^ℓ Fredholm map $F : X \to Y$ between seperable Banach manifolds is meagre whenever $\ell > \operatorname{index}(F)$.

Remark 3.6. As established by Remark 3.5, if we find ourselves at a non-regular almost-complex structure, we can normally perturb it to a regular one. Unfortunately, this is not always possible if we also want to preserve other structure such as equivariance. This motivates the framework of Kuranishi spaces, which aims to systematically describe the geometry of moduli spaces cut out by non-transverse sections.

3.2 Regular almost-complex structures

Definition 3.7. The *universal moduli space (of lines)* is the space $\widetilde{\mathscr{M}}(\mathscr{J}(\omega_{\mathbb{P}^2}))$ of all pairs (u, J) where $J \in \mathscr{J}(\omega_{\mathbb{P}^2}))$ and $u \in \widetilde{\mathscr{M}}(J)$.

Since we proved that D_u is surjective for all $u \in \widetilde{\mathscr{M}}(\mathscr{J}(\omega_{\mathbb{P}^2}))$, it would follow in the finite-dimensional setting—from the implicit function theorem—that $\widetilde{\mathscr{M}}(J)$ is a smooth manifold for all J. However, the domain[5] and codomain of π are infinite-dimensional, so we require the following infinite-dimensional version of the ordinary implicit function theorem, which we state without proof.

Theorem 3.8 ([MS12, Theorem A.3.3])**.** *Let ℓ be a positive integer and $F : X \to Y$ a C^ℓ map between Banach manifolds. If $y \in Y$ is a regular value of F, then $f^{-1}(y)$ is a C^ℓ Banach manifold and the tangent space $T_x F^{-1}(y)$ is given by* $\ker \mathrm{d}_x f$. □

Note that the manifolds in question are Banach. We have already enlarged individual moduli spaces from smooth to Sobolev regularity, but to apply the infinite-dimensional implicit function theorem to the projection π, we must also enlarge the manifold $\mathscr{J}(\omega_{\mathbb{P}^2})$. To do this, consider C^ℓ almost-complex structures; that is, C^ℓ-smooth sections of $\operatorname{End}(T\mathbb{P}^2)$ for some integer $\ell \geqslant 1$.

[5]A priori, $\widetilde{\mathscr{M}}(\mathscr{J}(\omega_{\mathbb{P}^2}))$ could be finite-dimensional; it turns out this cannot happen because π is Fredholm.

The space of C^ℓ tame almost-complex structures on $(\mathbb{P}^2, \omega_{\mathbb{P}^2})$ is denoted by $\mathcal{J}^\ell(\omega_{\mathbb{P}^2})$. We have

$$\mathcal{J}(\omega_{\mathbb{P}^2}) = \bigcap_{\ell \geqslant 1} \mathcal{J}^\ell(\omega), \quad \text{so} \quad \widetilde{\mathcal{M}}(\mathcal{J}(\omega_{\mathbb{P}^2})) = \bigcap_{\ell \geqslant 1} \widetilde{\mathcal{M}}(\mathcal{J}^\ell(\mathbb{P}^2)).$$

The upshot of all this is the following technical result which first appeared in [McD87], which (again) we will not prove.

Proposition 3.9. *The space $\widetilde{\mathcal{M}}(\mathcal{J}^\ell(\omega_{\mathbb{P}^2}))$ is a separable C^ℓ-Banach manifold.* □

3.3 The moduli space is a manifold

Proposition 3.9 and the implicit function theorem has the following consequence.

Theorem 3.10. *Let $J \in \mathcal{J}(\omega_{\mathbb{P}^2})$. The moduli space $\widetilde{\mathcal{M}}(J)$ of J-holomorphic lines is a smooth manifold of dimension* index D_u.

Note that this does not exclude the possibility that $\widetilde{\mathcal{M}}(J)$ is empty.

Proof. The differential of π at some $(u, J) \in \widetilde{\mathcal{M}}(\mathcal{J}^\ell(\omega_{\mathbb{P}^2}))$,

$$\mathrm{d}_{(u,J)}\pi : T_{(u,J)}\widetilde{\mathcal{M}}(\mathcal{J}^\ell(\omega_{\mathbb{P}^2})) \to T_J\mathcal{J}^\ell(\omega_{\mathbb{P}^2}),$$

is simply the linear projection $(\xi, \zeta) \mapsto \zeta$, so we have $\ker \mathrm{d}_{(u,J)}\pi \cong \ker D_u$. Now, since D_u is surjective, the linear map

$$\iota : T_J\mathcal{J}^\ell(\omega_{\mathbb{P}^2}) \to \operatorname{im} D_u : \zeta \mapsto \zeta \circ \mathrm{d}u \circ j$$

is well-defined and annihilates the image of $\mathrm{d}_{(u,J)}\pi$. Thus the cokernel of $\mathrm{d}_{(u,J)}\pi$ is zero. (In general this argument gives $\dim \operatorname{coker} \mathrm{d}_{(u,J)}\pi \leqslant \dim \operatorname{coker} D_u$.) Now, by the implicit function theorem, each individual moduli space $\pi^{-1}(J) \subseteq \widetilde{\mathcal{M}}(\mathcal{J}^\ell)$ is a submanifold of dimension index D_u. □

Remark 3.11. The proof above essentially proceeds by identifying the (co)kernel of $\mathrm{d}\pi_{(u,J)}$ with that of D_u. There is thus no coincidence of vocabulary: J is a regular value of π if it is regular in the sense of Definition 3.3.

In light of this, the dimension of $\widetilde{\mathcal{M}}(J)$ is given by the following formula.

Theorem 3.12 (Index Theorem)**.** *Let D be a real linear Cauchy–Riemann operator of class $W^{1,p}$ on a complex line bundle $E \to \mathbb{P}^1$. Then D is Fredholm with index given by*

$$\operatorname{index} D = 2(\operatorname{rank}_{\mathbb{C}}(E) + c_1(E)).$$

In particular, the index of D_u does not depend on the p used for the Sobolev setup.

Proof. This follows immediately from the Atiyah–Singer Index theorem [Pal66]. However, we can deduce it from the analysis we have already done, because D differs by a compact operator from a smooth complex linear Cauchy–Riemann operator. It is well-known (see for example [Whi22, Theorem 12.7]) that compact pertubations of Fredholm operators are Fredholm with the same index. Thus, because every complex Cauchy–Riemann operator on E is the Dolbeault operator for the holomorphic structure it induces on E, the Fredholm index of D is given by the calculations of Example 2.4. □

Example 3.13. In the case $D = D_u$ for some $u \in \bar{\partial}_J^{-1}(0)$, we have $E = u^*T\mathbb{P}^2$, so

$$\operatorname{index} D_u = 2(2+3) = 10.$$

In general, the index of a Cauchy–Riemann operator computes the *virtual dimension* of the moduli space it cuts out. When J is regular, this coincides with the actual dimension of $\widetilde{\mathscr{M}}(J)$ as a manifold.

3.4 Induced cobordism

By Lemma 1.5, given $J_0, J_1 \in \mathscr{J}(\omega_{\mathbb{P}^2})$ there is a smooth embedded path

$$J_\bullet : [0,1] \to \mathscr{J}(\omega_{\mathbb{P}^2}) : t \mapsto J_t$$

connecting them. Write $\widetilde{\mathscr{M}}(J_\bullet) = \{(u, J_t) : t \in [0,1], u \in \widetilde{\mathscr{M}}(J_t)\}$ for the natural subset of $\mathscr{M}(\mathscr{J}(\omega_{\mathbb{P}^2}))$ defined by this path.

$$\begin{tikzcd}
\widetilde{\mathscr{M}}(J_\bullet) \arrow[r, hook] \arrow[d] & \widetilde{\mathscr{M}}(\mathscr{J}(\omega_{\mathbb{P}^2})) \arrow[d, "\pi"] \\
{[0,1]} \arrow[r, hook, "J_\bullet"] & \mathscr{J}(\omega)
\end{tikzcd}$$

Since π is a submersion, by a completely analogous argument to that in the previous subsection, we can conclude that $\widetilde{\mathscr{M}}(J_\bullet)$ is a smooth manifold with boundary $\widetilde{\mathscr{M}}(J_0) \amalg \widetilde{\mathscr{M}}(J_1)$. (This is simply stated in [McD87], and as always the full details can be found in [MS12].) We record this as a Theorem.

Theorem 3.14. *Given a smooth embedded path $J_\bullet : [0,1] \to \mathscr{J}(\omega_{\mathbb{P}^2})$, the space $\widetilde{\mathscr{M}}(J_\bullet)$ is a smooth oriented cobordism between $\widetilde{\mathscr{M}}(J_0)$ and $\widetilde{\mathscr{M}}(J_1)$.* □

Remark 3.15. Intuitively, the non-regular almost-complex structures form 'codimension 1 walls' in $\mathscr{J}(\omega)$. In more general settings, where not all almost-complex structures are regular, one must also ensure that the path $J_\bullet$ is transverse to these walls. This is always possible, and one also has an analogous Sard–Smale result for the genericity of these paths (cf. Remark 3.5).

4 Compactness

On its own, the cobordism of Theorem 3.14 is useless for establishing existence of pseudo-holomorphic lines because it is noncompact (recall: $\widetilde{\mathscr{M}}(J)$ is non-compact if it is non-empty, since it contains $\mathrm{PSL}_2(\mathbb{C})$). For example, every manifold is cobordant to $\varnothing$ via the product with $[0, 1)$. However, the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\widetilde{\mathscr{M}}(J)$ is free and proper [MS12, §6.1] so the quotient cobordism

$$\mathscr{M}(J_\bullet) = \widetilde{\mathscr{M}}(J_\bullet)/\mathrm{PSL}_2(\mathbb{C})$$

is also a manifold. This Section is dedicated to proving that $\mathscr{M}(J_\bullet)$ is indeed compact. This is enough to show that $\mathscr{M}(J_1)$ is nonempty, because $\mathbb{P}^2 = \mathscr{M}(J_0)$ is not nullcobordant; otherwise the closed 5-manifold obtained by gluing this cobordism along $\mathbb{P}^2$ would have

$$0 = \chi\big(\mathscr{M}(J_\bullet) \cup_{\mathbb{P}^2} \mathscr{M}(J_\bullet)\big) = 2\chi(\mathscr{M}(J_\bullet)) - \chi(\mathbb{P}^2),$$

which cannot hold, because $\chi(\mathbb{P}^2) = 3$ is odd. Note that the Euler characteristic of the double vanishes by Poincaré duality because it has odd dimension.

4.1 Energy and singularities

Almost nothing in this Section is simplified by specialising to $\Sigma = \mathbb{P}^1$ and $M = \mathbb{P}^2$, so we By definition, an ω-tame almost-complex structure J on a symplectic manifold (M, ω) induces a Riemannian metric $\langle -, - \rangle_J$ on M given by

$$\langle v, w \rangle_J = \frac{\omega(v, Jw) + \omega(w, Jv)}{2}.$$

Write $\|v\|_J^2$ for $\langle v, v \rangle_J$. Fix an orthonormal (with respect to j and dvol_Σ) frame $\{f, f'\}$ on $T\Sigma$. It can be seen [Oh15] that the quantity $\|\mathrm{d}u\|_J$ determined by

$$\|\mathrm{d}u\|_J^2 := \langle \mathrm{d}u\,(f), \mathrm{d}u\,(f) \rangle_J + \langle \mathrm{d}u\,(f'), \mathrm{d}u\,(f') \rangle_J$$

is well-defined independently of the choice of frame. In particular, locally in complex coordinates $s + it$ on Σ we have $\|\mathrm{d}u\|_J^2 = \|\partial_s u\|_J^2 + \|\partial_t u\|_J^2$. Define the *energy* of u by

$$E(u) := \frac{1}{2} \int_\Sigma \|\mathrm{d}u\|_J^2 \mathrm{dvol}_\Sigma.$$

Proposition 4.1. *The energy $E(u)$ depends only on the homology class $u_*[\Sigma] \in H_2(M; \mathbb{Z})$.*

Proof. In a local conformal chart, we have

$$\begin{aligned}\|\mathrm{d}u\|_J^2 \mathrm{dvol}_\Sigma &= \left(\|\partial_s u\|_J^2 + \|\partial_t u\|_J^2\right) \mathrm{d}s \wedge \mathrm{d}t \\ &= \left(\|\partial_s u + J\partial_t u\|_J^2 + \omega(\partial_s u, \partial_t u) + \omega(J\partial_s u, J\partial_t u)\right) \mathrm{d}s \wedge \mathrm{d}t \\ &= 2\omega(\partial_s u, \partial_t u)\, \mathrm{d}s \wedge \mathrm{d}t\,.\end{aligned}$$

The last equality follows because $\partial_s u + J\partial_t u = 0$. Thus we have

$$E(u) = \int_\Sigma u^*\omega.$$

Since ω is closed, it follows from Stokes' theorem that if u, v are tame pseudo-holomorphic curves representing the same homology class, then the difference $\int_\Sigma u^*\omega - \int_\Sigma v^*\omega$ vanishes. □

This means that the energy—an a priori analytic quantity—is actually topological in the setting of tame pseudo-holomorphic curves. In fact, this is where the word 'tame' comes from: one has control over the energy just by restricting homology classes. In particular, a sequence of tame pseudo-holomorphic curves which all represent the same homology class will be uniformly bounded in the $W^{1,2}$ norm.

Remark 4.2. Apart from being bounded, this implies that the harmonic energy of tame pseudo-holomorphic curves is actually *quantised* by the discrete group $H_2(M;\mathbb{Z})$. The energy of a pseudo-holomorphic curve $u : \mathbb{P}^1 \to \mathbb{P}^2$ is

$$E(u) = d\hbar,$$

where $\hbar = \int_{\mathbb{P}^1} \omega_{\mathbb{P}^2}$ is the normalisation constant of the footnote in §1.1, and d is the degree of u (in the sense already described).

Just as in ordinary complex analysis, it turns out that we can fill in removable singularities of pseudo-holomorphic curves. The precise description of 'removable' here is 'finite energy'.

Theorem 4.3 (Removable singularity theorem)**.** *If $u : \mathbb{D} \setminus \{0\} \to M$ is a J-holomorphic curve with finite energy, then u extends to a smooth map $\mathbb{D} \to M$.* □

This is an important idea of Gromov's which we unfortunately have no time to prove. However, it has the advantage of being completely familiar from complex analysis—at least when $M = \mathbb{C}$—so there is no intuition to be gained from sketching a proof.

4.2 Sphere bubbling

The following will be our main technical result for establishing compactness of cobordisms.

Theorem 4.4 ([MS12, Theorem 4.1.1]). *Let (Σ, j) be a Riemann surface without boundary, (M, J) a closed almost-complex manifold, $(J_n)_n$ a sequence of almost-complex structures on M converging (in the C^∞-topology) to J, $(\Omega_n)_n$ an increasing sequence of open sets exhausting Σ, and $(u_n : \Omega_n \to M)_n$ a sequence of J_n-holomorphic curves satisfying the uniform bound*

$$\sup_n \|\mathrm{d}u_n\|_{L^\infty(K)} < \infty$$

for all compact $K \subset \Sigma$. Then (u_n) has a subsequence converging uniformly with all derivatives to a J-holomorphic curve $u : \Sigma \to M$.

As noted in [MS94], the mean value property of harmonic functions is enough to automatically prove Theorem 4.4 in the exceptional case that $J_n = I$ is constant and integrable, and Σ is closed. However, the full proof of this result—together with the elliptic regularity result of Theorem (number)—occupies an entire appendix in [MS12]. We refer the curious reader elsewhere for a full proof. This theorem will soon be applied to $\Sigma = \mathbb{C}$, but describe a sketch of the proof in the case where Σ is compact, as in [Oh15].

Proof sketch. One uses some standard elliptic estimates to show that each $\mathrm{d}u_n$ is α-Hölder continuous for some $\alpha > 0$, which which allows us to apply the relevant Arzelà-Ascoli theorem. Thus the u_n converge in $C^{1,\alpha}$ to some u of regularity C^1, which is "weakly J-holomorphic" in the sense that

$$\int_\Sigma \langle J\, \mathrm{d}u_n\,, \nabla\xi\rangle \mathrm{dvol}_\Sigma = 0$$

for every $\xi \in \Gamma(u^*TM)$. By Theorem 2.8 (elliptic regularity), u is actually a smooth J-holomorphic curve and the convergence is C^∞. □

Remark 4.5. Gromov's proof of compactness [Gro85, Theorem 1.5.B] used a different flavour of methods (isoperimetric inequalities) than those outlined here; the work of Floer has heavily influenced the modern strategy.

Clearly this is a very powerful theorem for compactness results. However, a priori there is no reason to expect that $\|\mathrm{d}u_n\|_{L^\infty(K)}$ is bounded in n. This leads to the phenomenon of sphere bubbling, as we will now describe.

If the (u_n) all represent the same homology class, but the boundedness hypothesis of Theorem 4.4 is not met, it must be because the energy density $\|\mathrm{d}u_n\|_J^2$ is concentrating at certain points. Conformally rescaling near such a point $* \in M$ gives a sequence of maps $\tilde{u}_n := u_n/\|\mathrm{d}u_n\|_J$, defined on open neighbourhoods of $*$. Now the hypothesis of Theorem 4.4 is met, and one can apply Theorem 4.3 to conclude that both $\tilde{u}_n(z)$ and $\tilde{u}_n(1/z)$ converge; this defines a map from $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$ to M, which is called a *bubble*. Intuitively, the curves degenerate in the limit to a union of smooth surfaces in M.

Example 4.6. Consider the following simple example of degeneration. Each algebraic curve

$$\mathbb{V}_n = \{[x, y, z] \in \mathbb{P}^2 : nxy - z^2 = 0\}$$

is the image of a holomorphic embedding $u_n : \mathbb{P}^1 \to \mathbb{P}^2$. However, the sets $\mathbb{V}_n$ degenerate to $\mathbb{V}_\infty = \{xy = 0\} = \{x = 0\} \cup \{y = 0\}$, which is not the image of any holomorphic map $\mathbb{P}^1 \to \mathbb{P}^2$.

4.3 Compact cobordism

We will now see that, for energy reasons, bubbling is impossible in our context. In particular, every sequence in $\mathscr{M}(J_\bullet)$ has a convergent subsequence.

Theorem 4.7. *The projection* $\pi : \mathscr{M}(\mathscr{J}(\omega_{\mathbb{P}^2})) \to \mathscr{J}(\omega_{\mathbb{P}^2})$ *is proper.*

Proof. Suppose $\mathscr{K} \subseteq \mathscr{J}(\omega)$ is compact and let $([u_n], J_n) \subset \pi^{-1}(\mathscr{K})$ be a sequence in the preimage. Since $\mathscr{K}$ is compact, by passing to a subsequence we can assume J_n converges to some $J \in \mathscr{K}$.

Each J-holomorphic curve $u_n : \mathbb{P}^1 \to \mathbb{P}^2$ can be represented, via stereographic projection $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$, by a J-holomorphic curve $v_n : \mathbb{C} \to \mathbb{P}^2$ such that the map $\mathbb{C} \setminus \{0\} \to \mathbb{P}^2 : z \mapsto v_n(1/z)$ extends smoothly across 0. The stereographic projection formula implies $\|\mathrm{d}_z u_n\|_J = (1 + |z|^2)\|\mathrm{d}_z v_n\|_J$, so precomposing with an affine transformation of $\mathbb{C}$ we can assume that

$$\|\mathrm{d}_0 v_n\|_J = \sup_{z \in \mathbb{C}} \|\mathrm{d} v_n\|_J = 1.$$

We can therefore apply Theorem 4.4 to conclude that the sequence v_n converges on compact subsets of $\mathbb{C}$ to some J-holomorphic curve $v : \mathbb{C} \to \mathbb{P}^2$. By Fatou's Lemma we have

$$E(v) = E\left(\lim_{n\to\infty} v_n\right) \leqslant \liminf_{n\to\infty} E(v_n) = \hbar.$$

So not only does v has finite energy, so the map $\mathbb{C} \setminus \{0\} \to \mathbb{P}^2 : z \mapsto v(1/z)$ extends smoothly across 0, so represents a J-holomorphic curve $\mathbb{P}^1 \to \mathbb{P}^2$. The second is sharper: v has the smallest possible positive energy, so must represent the positive generator of $H_2(\mathbb{P}^2; \mathbb{Z})$.

For this to represent genuine convergence of the maps u_n (whose domains are $\mathbb{P}^1$), we require convergence around $\infty \in \mathbb{P}^1$. This means that the sequence of maps sending $z \mapsto v_n(1/z)$ should converge uniformly near zero. Indeed, if this were not the case we would get a nonconstant bubble, and the energy would be at least $2\hbar$. □

Moreover, since the projection $\mathscr{M}(J_\bullet) \to [0, 1] : ([u], J_t) \mapsto t$ is a submersion, it is in particular a Morse function with no critical points, which implies the following Corollary.

Corollary 4.8. *For all* $t \in [0, 1]$*, the manifold* $\mathscr{M}(J_t)$ *is diffeomorphic to* $\mathbb{P}^2$*.* □

5 Conclusion

So far, we have shown that the moduli space $\mathscr{M}(J)$ of unparametrised J-holomorphic lines is diffeomorphic to $\mathbb{P}^2$ for all $J \in \mathscr{J}(\omega_{\mathbb{P}^2})$. To finish the proof of Theorem 0.9 we must show that there actually exists a J-holomorphic line through every pair of points.

5.1 Proof of Theorem 0.9

Write $\widetilde{\mathscr{M}_n}(J) = \widetilde{\mathscr{M}}(J) \times \mathrm{Conf}_n(\mathbb{P}^1)$ for the space of J-holomorphic lines together with n distinct 'marked points', and $\mathscr{M}_n(J) = \widetilde{\mathscr{M}_n}(J)/\mathrm{PSL}_2(\mathbb{C})$ for the quotient space under the action

$$\phi.(u, \mathbf{z}) = (u \circ \phi^{-1}, \phi(\mathbf{z})).$$

Note that for $n \geqslant 2$, the quotient space $\mathscr{M}_n(J)$ is visibly noncompact (if it is nonempty) because the marked points can come together.

Now observe that the existence (resp. uniqueness) of a J-holomorphic line through every pair of points is equivalent to the evaluation map

$$\mathrm{ev} : \mathscr{M}_2(J) \longrightarrow \mathrm{Conf}_2(\mathbb{P}^2) : [u, (z, z')] \mapsto (u(z), u(z'))$$

being surjective (resp. injective). The following argument shows this map is a diffeomorphism.

Proof of Theorem 0.9. That ev is injective is clearly equivalent to Corollary 1.16.

To see that ev is proper, consider a sequence $[u_n, (z_n, z'_n)] \subset \mathscr{M}_2(J)$ such that $(u_n(z_n), u_n(z'_n))$ converges to $\{x, y\} \in \mathrm{Conf}_2(\mathbb{P}^2)$. Since $\mathscr{M}(J)$ is compact, by reparametrising we can extract a subsequence which converges to $(u, (z, z')) \in \widetilde{\mathscr{M}}(J) \times \mathbb{P}^1 \times \mathbb{P}^1$. Since x and y are different, the points z and z' must be different, which proves that ev is proper.

It therefore suffices to show that ev is an open map. To this end, consider the evaluation map $\widetilde{\mathrm{ev}} : \widetilde{\mathscr{M}_2}(J) \longrightarrow \mathrm{Conf}_2(\mathbb{P}^2)$. The tangent space $T_u\widetilde{\mathscr{M}}(J)$ is

$$\ker D_u \oplus T_z\mathbb{P}^1 \oplus T_{z'},$$

and the differential $\mathrm{d}_{(u,(z,z'))}\widetilde{\mathrm{ev}} : \ker D_u \oplus T_z\mathbb{P}^1 \oplus T_{z'}\mathbb{P}^1 \longrightarrow T_{u(z)}\mathbb{P}^2 \oplus T_{u(z')}\mathbb{P}^2$ is given by

$$\mathrm{d}_{(u,(z,z'))}\widetilde{\mathrm{ev}}(\delta, \zeta, \zeta') = (\delta(z) + \mathrm{d}u_z\,(\zeta), \delta(z') + \mathrm{d}u_{z'}\,(\zeta')).$$

By Theorems 3.1 and 3.12, the operator $D_u|_{\nu_u}$ is surjective and Fredholm, with index

$$\mathrm{index}\, D_u|_{\nu_u} = \dim\ker D_u|_{\nu_u} = 2(\mathrm{rank}_{\mathbb{C}}(\nu_u) + c_1(\nu_u)) = 4.$$

Every nonzero $\xi \in \ker D_u|_{\nu_u}$ must have at most one root by Lemma 3.2, so the map

$$\ker D_u|_{\nu_u} \to (\nu_u)_z \oplus (\nu_u)_{z'} : \xi \mapsto (\xi(z), \xi(z'))$$

has trivial kernel. Both spaces are 4-dimensional so this is actually an isomorphism; it follows that the kernel of $\mathrm{d}_{(u,(z,z'))}\widetilde{\mathrm{ev}}$ is the tangent space of the $\mathrm{PSL}_2(\mathbb{C})$-orbit of $(u, (z, z'))$. Thus the evaluation map on the quotient,

$$\mathrm{ev} : \mathscr{M}_2(J) \to \mathrm{Conf}_2(\mathbb{P}^2)$$

is a proper injective immersion between manifolds of the same dimension, which means it is a diffeomorphism. This proves the Theorem. □

Remark 5.1. This argument fails for the analogous statement for *three* distinct points (z, z', z''), because the differential can never be surjective. Indeed, the map itself

$$\mathrm{ev} : \mathscr{M}_3(J) \to \mathrm{Conf}_3(\mathbb{P}^2)$$

cannot be onto because the domain has dimension 10 and the codomain has dimension 12, although it is injective for the same reason as for $\mathscr{M}_2$.

Our proof of Theorem 0.9 shows directly that the evaluation map $\mathrm{ev} : \mathscr{M}_2(J) \to \mathbb{P}^2 \times \mathbb{P}^2$ is a pseudocycle (see [MS12, Definition 6.4.1]). In particular, the "2-pointed genus zero Gromov–Witten invariant of $(\mathbb{P}^2, \omega_{\mathbb{P}^2})$ in the homology class $[\mathbb{P}^1]$" is well-defined:

$$\mathrm{GW}^{\mathbb{P}^2}_{[\mathbb{P}^1],2}(\mathrm{PD}[p], \mathrm{PD}[p']) = \#\{[u, (z, z')] \in \mathscr{M}_2(J) \mid u(z) = p, u(z') = p'\} = 1,$$

where c is the generator of $H^4(\mathbb{P}^2; \mathbb{Z}) \cong \mathbb{Z}$, which is Poincaré dual to the classes represented by $\{p\}$ and $\{p'\}$. This essay therefore constitutes a computation of a specific Gromov–Witten invariant without any mention of stable maps.

5.2 Neck-stretching arguments

One notable application is presented in [HWZ03], which proves that for a generic tight contact form on S^3 and a generic almost complex structure on its symplectisation $\mathbb{R} \times S^3$, one can foliate $\mathbb{R} \times S^3$ by J-holomorphic curves asymptotic to closed Reeb orbits.

The argument, explained to the author by Chris Wendl, goes as follows.

1. Embed S^3 into $\mathbb{P}^2$ as a contact hypersurface disjoint from the line at infinity $L_\infty \subset \mathbb{P}^2$, and fix any point $p_\infty \in L_\infty$
2. Consider a sequence of tame almost-complex structures $(J_n)_n \subset \mathscr{J}(\omega_{\mathbb{P}^2})$ which are

translation-invariant on increasingly long 'neck' regions $[-n, n] \times S^3$.

3. For all $p \in S$, Theorem 0.9 guarantees the existence of a unique embedded J_n-holomorphic line passing through p and p_∞.
4. These lines intersect only at p_∞, so as n goes to ∞ these lines converge to the leaves of a foliation on $\mathbb{R} \times S^3$.

This has a number of nontrivial consequences in dynamical systems; see §7 of the paper.

5.3 The original strategy

Although this essay started as a review of the ideas in [Gro85], it uses much of the modern notation, framework and proof strategy. We conclude with a description of the original perspective.

Gromov's original analytic formalism was slightly different. The proof of [Gro85, Theorem 2.3.C$_1$]—from which his version of Theorem 0.9 follows—considers the perturbed equation $\bar{\partial}_J u = g$. This corresponds to an honest Cauchy–Riemann equation $\bar{\partial}_{\widetilde{J}} \widetilde{u} = 0$, where $\widetilde{u} : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^2$ is a section of the trivial fibration and

$$\widetilde{J} = \begin{pmatrix} j & 0 \\ Jg - gj & J \end{pmatrix}$$

is an almost-complex structure on $\mathbb{P}^1 \times \mathbb{P}^2$. However, as noted in [McD87], this makes it difficult to properly interpret the bubbling phenomenon, which is probably why this formalism has fallen out of use.

Analysis aside, our proof of Theorem 0.9 differs slightly in strategy from Gromov's original one, which we now outline. Fix $p, p' \in \mathbb{P}^2$ distinct and a straight line $L \subset \mathbb{P}^2$ disjoint from $\{p, p'\}$. For $J \in \mathscr{J}(\omega_{\mathbb{P}^2})$, consider the set

$$\mathfrak{Y}(J) = \{u \in \widetilde{\mathscr{M}}(J) \mid u(0) = p, u(1) = p', u(\infty) \in L\}.$$

This is essentially a quotient $\widetilde{\mathrm{ev}}^{-1}(p, p')/\mathbb{C}* \subset \widetilde{\mathscr{M}_2}(J)/\mathbb{C}^*$, because the imposition $u(\infty) \in L$ kills the remaining action of $\mathbb{C}^* \subset \mathrm{PSL}_2(\mathbb{C})$. Gromov proceeds by showing that an appropriate homotopy $I \simeq J$ of tame almost-complex structures induces a smooth one-dimensional compact cobordism between $\mathfrak{Y}(I)$ and $\mathfrak{Y}(J)$. Because cardinality modulo 2 is a cobordism invariant, this is enough to show $\mathfrak{Y}(J)$ is non-empty (because $|\mathfrak{Y}(I)| = 1$ by Theorem 0.2).

For this to work, Gromov has to 'slightly perturb' the submanifolds $\{p\}, \{p'\}, L$ if necessary to achieve transversality. It is possible that this was the genericity issue alluded to at the end of his Example 2.4.B$''_1$, which was to be addressed in a followup paper *Pseudo-holomorphic curves in symplectic manifolds, II*. However, the sequel never appeared, so the proof outlined above remains unfinished.

References

[AL94] M. Audin and J. Lafontaine, eds. *Holomorphic curves in symplectic geometry*. Progress in Mathematics 117. Springer, 1994.

[Cho49] W-L. Chow. "On Compact Complex Analytic Varieties". In: *American Journal of Mathematics* 71 (1949), pp. 893–914.

[CW21] J. Cirici and S. O. Wilson. "Dolbeault cohomology for almost complex manifolds". In: *Advances in Mathematics* 391 (2021).

[Eli98] Y. Eliashberg. "Symplectic topology in the nineties". In: *Differential Geometry and its Applications* 9 (1998), pp. 59–88.

[Gro57] A. Grothendieck. "Sur la classification des fibrés holomorphes sur la sphère de Riemann". In: *Amer. J. Math* 76 (1957), pp. 121–138.

[Gro85] M. Gromov. "Pseudo holomorphic curves in symplectic manifolds". In: *Invent. Math* 82 (1985), pp. 304–347.

[Hat17] A. Hatcher. *Vector Bundles and K-Theory*. 2017.

[Hir66] F. Hirzebruch. *Topological Methods in Algebraic Geometry*. Third Enlarged Edition. Classics in Mathematics. Springer, 1966.

[HLS98] H. Hofer, V. Lizan, and J-C. Sikarov. "On Genericity for Holomorphic Curves in Four-Dimensional Almost-Complex Manifolds". In: *The Journal of Geometric Analysis* 7 (1998), pp. 149–159.

[Hör94] L. Hörmander. *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Classics in Mathematics. Springer, 1994.

[HWZ03] H. Hofer, K. Wysocki, and E. Zehnder. "Finite energy foliations of tight three-spheres and Hamiltonian dynamics". In: *Annals of Mathematics* 157 (2003), pp. 125–257.

[McD87] D. McDuff. "Examples of symplectic structures". In: *Invent. Math* 89 (1987), pp. 13–36.

[McD91] D. McDuff. "The local behaviour of holomorphic curves in almost-complex 4-manifolds". In: *J. Differential Geometry* 34 (1991), pp. 143–164.

[Mil58] J. Milnor. "On simply connected 4-manifolds". In: *Symposium internacional de topología algebraica International symposium on algebraic topology* (1958), pp. 122–128.

[Mor07] A. Moroianu. *Lectures on Kähler Geometry*. LMS Student Texts 69. Cambridge University Press, 2007.

[MS12] D. McDuff and D. Salamon. *J-holomorphic Curves and Symplectic Topology*. Second Edition. AMS Colloquium Publications 52. American Mathematical Society, 2012.

[MS17] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Third Edition. Oxford University Press, 2017.

[MS94] D. McDuff and D. Salamon. *J-holomorphic Curves and Quantum Cohomology*. University Lecture Series 6. American Mathematical Society, 1994.

[Oh15] Y-G. Oh. *Symplectic Topology and Floer Homology, Volume 1*. New Mathematical Monographs 28. Cambridge University Press, 2015.

[OSS80] C. Okonek, M. Schneider, and H. Spindler. *Vector Bundles on Complex Projective Spaces*. Birkhäuser, 1980.

[Pal66] R. S. Palais. *Seminar on the Atiyah–Singer Index Theorem*. Annals of Mathematics Studies 57. Princeton University Press, 1966.

[Sma65] S. Smale. "An Infinite Dimensional Version of Sard's Theorem". In: *American Journal of Mathematics* 87 (1965), pp. 861–866.

[Voi02] C. Voisin. *Hodge Theory and Complex Algebraic Geometry I*. Cambridge Studies in Advanced Mathematics 76. Cambridge University Press, 2002.

[Wel08] R. O. Wells. *Differential Analysis on Complex Manifolds*. Third Edition. Springer, 2008.

[Wen18] C. Wendl. *Holomorphic Curves in Low Dimensions*. Lecture Notes in Mathematics 2216. Springer, 2018.

[Whi22] S. White. *C4.1 Further Functional Analysis*. Lecture Notes, University of Oxford. 2022.

[Yau77] S. T. Yau. "Calabi's conjecture and some new results in algebraic geometry". In: *Proc. Nat. Acad. Sci. U.S.A.* 74 (1977), pp. 1798–1799.