# Graded manifolds and shifted symplectic geometry

Andres Klene

Contents



Lecture notes

- Carchedi `https://arxiv.org/pdf/2303.11140`
- Calaque–Ronchi `https://arxiv.org/pdf/2510.04625`
- Kotov–Salnikov `https://arxiv.org/pdf/2108.13496`
- Cueca–Maglio–Valencia `https://arxiv.org/pdf/2510.09448`

## 1. Graded commutative rings

**1.1. Ordinary story.** Let $k$ be a commutative ring. Let $G$ be an abelian group equipped with a group homomorphism $\rho : G \to k^\times$. A $k$-algebra $A$ is $G$-graded if the underlying abelian group is

$$A = \bigoplus_{i \in G} A_i$$

and $|ab| = i + j$ for homogeneous elements $a \in A_i$ and $b \in A_j$. It is $G$-commutative if furthermore $ab = \rho(i+j)ba$, where we use the $k$-algebra structure $k \to A$ to define a composite $G \to k^\times \to A^\times$.

The grading is really a map of $k$-algebras $A \to k[G]$ which doesn't need to be surjective! For $G = 1$ we have $k[G] \cong k$ so we get a canonical pointing! Since $k[-]$ is a left adjoint (to $R \mapsto R^\times$), not a right adjoint, we cannot write this any other way. A map $G \to k^\times$ induces a map $A \to k[k^\times]$. To define graded-commutative stuff you also need a representation $G \to A^\times$ or equivalently $k[G] \to A$. In some sense a "section" of the grading?!

For instance, the structure of a $G$-representation on $V$ is a group map $G \to \mathrm{End}(V)^\times$, or equivalently by the above adjunction a ring map $k[G] \to \mathrm{End}(V)$.

These data define a composite $A \to k[G] \to k$ of $k$-algebras, hence a distinguished point of $\mathrm{Spec}(A)$.

An $\mathbb{S}$-grading on an $\mathbb{E}_\infty$-ring $R$ is therefore a map $\mathbb{S} \to R^\times$...?

Example 1.1.1. Take $k = \mathbb{Z}$, so that a $k$-algebra is the same thing as a ring. First take $G = \mathbb{Z}$ and $\rho : \mathbb{Z} \to \{\pm 1\}$ to be the surjective map. Thus a $\mathbb{Z}$-commutative $\mathbb{Z}$-algebra recovers the notion of graded commutative ring, for this $\rho$.

Example 1.1.2. Take $G = \mathbb{Z}/2$ and $\rho : \mathbb{Z}/2 \to \{\pm 1\}$ to be the isomorphism. Thus a $\mathbb{Z}/2$-commutative $\mathbb{Z}$-algebra recovers the notion of supercommutative ring, for this $\rho$.

Example 1.1.3. Take $k = \mathbb{C}$ and $G = U(1)$ and $\rho : U(1) \to \mathbb{C}^\times$ to be the obvious representation. But such a notion of $U(1)$-commutative $\mathbb{C}$-algebra...

Of course, we could take different $\rho$: this would give the naive definitions of commutative graded ring when $\rho$ is the trivial representation, and

**1.2. Examples.** Here are some examples.

Example 1.2.1. Let $V$ be a finite-dimensional real vector space. The exterior algebra

$$\bigwedge^{\bullet} V := \frac{\mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \dots}{v \otimes w + w \otimes v = 0}$$

is graded by the tensor degree.

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_i) \otimes (w_1 \otimes \cdots \otimes w_j) &= (-1)^i w_1 \otimes (v_1 \otimes \cdots \otimes v_i) \otimes (w_2 \otimes \cdots \otimes w_j) \\ &= (-1)^{i+i} (w_1 \otimes w_2) \otimes (v_1 \otimes \cdots \otimes v_i) \otimes (w_3 \otimes \cdots \otimes w_j) \\ &= \ldots \\ &= (-1)^{ij} (w_1 \otimes \cdots \otimes w_j) \otimes (v_1 \otimes \cdots \otimes v_i)\end{aligned}$$

Therefore $\bigwedge^\bullet V$ is graded commutative.

Non-example 1.2.2. If $V$ is equipped with a symmetric bilinear form $\langle -, - \rangle$ then one can use this bilinear form to define the following "deformation" of $\bigwedge^\bullet V$ as follows:

$$\dot{\bigwedge} V := \frac{\mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \ldots}{v \otimes w + v \otimes w = \langle v, w \rangle}.$$

This is the Clifford algebra associated to the inner product space $(V, \langle -, -, \rangle)$. It is graded by tensor degree, but it fails to be graded-commutative.

## 2. Graded manifolds

Definition 2.0.1. A *graded manifold*

### 2.1. Equivalence with vector bundles.

Theorem 2.1.1 (Batchelor's Theorem).

### 2.2. Relation with supermanifolds.

### 2.3. Relation with derived manifolds.

## 3. Shifted symplectic geometry

### 3.1. Symplectic graded manifolds.

### 3.2. The BV formalism.

### 3.3. Lagrangians.

Theorem 3.3.1. *The intersection of two Lagrangians in an $n$-shifted manifold has a $(n-1)$-shifted structure.*

Corollary 3.3.2. *The derived critical locus has a $(-1)$-shifted structure.*

### 3.4. Transgression.

Theorem 3.4.1. *The mapping stack to an $n$-shifted manifold has an $(n-d)$-shifted structure.*

Example 3.4.2. The stack of $G$-bundles on a surface is symplectic, because $\mathbf{B}G$ is 2-shifted. $G$-bundles on the bulk form a Lagrangian submanifold.

### 3.5. AKSZ as a fully extended TFT.

## 4. Shifted Poisson structures

### 4.1. Shifted deformation quantisation.