TALK I

Vector spaces and categorification

§ Motivation: the stacky view of geometric structures

What is a vector bundle on a manifold M? There are two ways of thinking about it:

(1) We can think of it as an assignment of a vector space V_x to each point $x \in M$. Heuristically, you can imagine that a vector bundle is a smooth map

$$V : M \to \mathsf{Vect} : x \mapsto V_x.$$

where Vect is the category of vector spaces. But this is just intuition.

(2) Formally, a vector bundle is a big space E together with a projection $\pi : E \to M$ such that each fibre $\pi^{-1}(x)$ is a vector space, which is locally trivial...

To me, the point of stacks is to legitimise the first perspective. If you're willing to sit through the formalisation of this in terms of category theory and so on, then at the end, there is a "thing" which deserves to be thought of as a geometric object, called the stack of vector spaces. We use the same notation, Vect for both the category of vector spaces and the stack of vector spaces. Then a vector bundle over a manifold M is really the same thing as a smooth map $M \to \mathsf{Vect}$.

This perspective brings two types of objects closer together, conceptually:

(1) $\mathbb{C}$-valued functions
(2) Vector bundles (Vect-valued functions)

You might not see the value in this rewriting, and I'm not claiming that this is the 'right way' to think about vector bundles in particular, or geometric structures in general. But whatever you work on, you might have seen the practical benefit of treating different-looking things on an equal footing. This is an instance of that philosophy.

Now, here is a claim, which we will try to spend the rest of the talk exploring.

Proposition. We can think of Vect as a categorification of $\mathbb{C}$.
$\implies$ Vector bundles are a categorification of $\mathbb{C}$-valued functions.

§ Categorification and decategorification

What do we mean when we say Vect is a categorification of $\mathbb{C}$?
Categorification is a general phenomenon; here are some examples.

Example (Ingmar's lunch talk). Take the category of finite sets with isomorphisms. There is one connected component for each natural number, and the cardinality is a map $FinSet \to \mathbb{N}$. We say that $\mathbb{N}$ decategorifies $FinSet$, or equivalently that $FinSet$ categorifies $\mathbb{N}$.

Example. Start with a category X and a distinguished object $u \in X$. Then

$$\Omega_u X := \mathrm{End}_X(u)$$

is a set, called the 'loops of X (at u)'. This is an example of categorification: we take a category and produce a set.

In our setting, $\mathbb{C}$ is a set and Vect is a category, but there are some structural similarities between them: for example, they both have a multiplication, $\times$ and $\otimes$. Note that the unit for $\times$ is $1 \in \mathbb{C}$.

Observation. Notice:

- The multiplicative unit of $\otimes$ is $\mathbb{C} \in \mathsf{Vect}$.
- A $\mathbb{C}$-linear map $T : \mathbb{C} \to \mathbb{C}$ is described by a complex number, $T(1) \in \mathbb{C}$

- Adding linear maps $T + T'$ corresponds to addition of complex numbers, $(T + T')(1) = T(1) + T'(1)$.
- Composing linear maps corresponds to multiplication of complex numbers, $(T' \circ T)(1) = T(1) \times T'(1)$.

Therefore, $\mathrm{End}_{\mathsf{Vect}}(\mathbb{C})$ is isomorphic to $\mathbb{C}$ as a ring.

So, to summarise, start with Vect. Taking the endomorphisms of a distinguished object—the monoidal unit—we get a set with a multiplication which comes from composing endomorphisms.

§ Higher structures

If you are convinced of this, the question is then, can we keep going?

- What categorifies Vect?

It should be a 2-category $2\mathsf{Vect}$ whose objects are "2-vector spaces".

Definition (Informal). A 2-category is a collection of objects, and for each pair of objects, a *category* (rather than a set) of morphisms between them.

If a set has categorical level 0, and a category has categorical level 1, then a 2-category has categorical level 2. Whatever $2\mathsf{Vect}$ is, the jump $\mathsf{Vect} \rightsquigarrow 2\mathsf{Vect}$ should be conceptually similar to the jump $\mathbb{C} \rightsquigarrow \mathsf{Vect}$.

Given a 2-category X with a distinguished object $u \in X$, the loops of X at u is

$$\Omega_u X := \mathrm{End}_X(u)$$

which is a category, rather than a set, because X is a 2-category.

Principle. This should also be seen as an instance of decategorification. Whatever decategorification is supposed to be, it decreases the categorical level by one:

- it sends categories to sets
- it sends 2-categories to categories
- ...

Categorification increases the categorical level by one.

There are various candidates for a good 2-category of 2-vector spaces. I will now give one of them. First, I have to define an algebraic notion:

Definition. Let A and B be $\mathbb{C}$-algebras (vector spaces with a multiplication). An (A, B)-bimodule is a vector space M, which is a left A-module and a right B-module.

A map between (A, B)-bimodules M and N is a map of vector spaces $M \to N$ which preserves the left A-action and the right B-action.

Write ${}_A\mathsf{Mod}_B$ for the category of (A, B)-bimodules.

Definition. Let $2\mathsf{Vect}_{\mathsf{Alg}}$ be the following 2-category:

- objects are associative $\mathbb{C}$-algebras, denoted A and B...
- between any two objects A and B, rather than having a set of morphisms, we have a category of morphisms: that category is ${}_A\mathsf{Mod}_B$.

To reiterate, a morphism $A \to B$ is not a map of algebras; rather, it is an (A, B)-bimodule. A 2-morphism (morphism between morphisms) $M \to N$ is a map of bimodules.

Proposition. $2\mathsf{Vect}_{\mathsf{Alg}}$ has a multiplication, and the loops at the monoidal unit

$$\Omega_{\mathbb{1}}(2\mathsf{Vect}_{\mathsf{Alg}})$$

is isomorphic to Vect.

Proof. The multiplication on $2\mathsf{Vect}_{\mathsf{Alg}}$ is given by the tensor product of algebras, and the unit object is the $\mathbb{C}$-algebra $\mathbb{C}$. Then $\Omega_{\mathbb{C}}(2\mathsf{Vect}_{\mathsf{Alg}})$ is the category of $(\mathbb{C}, \mathbb{C})$-bimodules, which is just the category of vector spaces. □

TALK II

Principal bundles and representations

§ Delooping

Last time, we taked about looping as an example of decategorification. What I didn't mention explicitly was the reverse idea.

Let G be a group. Define the category $\mathbf{B}G$:

- objects: just one, $\bullet$
- arrows: elements of G
- composition: group multiplication $G \times G \to G$

Notice that $\Omega\mathbf{B} \cong G$. The category $\mathbf{B}G$ is called the delooping of G.

Proposition. A functor $\rho : \mathbf{B}G \to \mathsf{Vect}$ is the same thing as a complex representation of G.

Proof. A functor sends objects to objects and arrows to arrows. Since $\mathbf{B}G$ has one object, this amounts to picking a vector space $V = \rho(\bullet) \in \mathsf{Vect}$, together with a function $G \to \mathrm{End}_{\mathsf{Vect}}(V)$. The fact that functors preserve composition means that this function is a representation. □

Today, we will take some steps towards formalising the idea of a "geometric object" of vector spaces.

§ Smooth categories

Last time, we gave a heuristic definition of category.

Slightly more formally, a (small) category C consists of a set of objects and a set of arrows between them. There are two maps

$$\text{source}, \text{target} : \ \mathrm{Arrows}(C) \rightrightarrows \mathrm{Objects}(C),$$

which send an arrow to its source or its target, and there is a map

$$\mathrm{id} : \ \mathrm{Objects}(C) \to \mathrm{Arrows}(C)$$

which sends an object x to the identity arrow id_x.

A functor $C \to D$ is a pair of functions

$$\mathrm{Objects}(C) \to \mathrm{Objects}(D) \qquad \mathrm{Arrows}(C) \to \mathrm{Arrows}(D)$$

which are compatible with these maps.

Definition. A "smooth category" is a category where the sets $\mathrm{Objects}(C)$ and $\mathrm{Arrows}(C)$ are manifolds, and these maps are all smooth. (This has a similar vibe as the definition of a Lie group.)

A smooth functor $C \to D$ is a functor such that $\mathrm{Objects}(C) \to \mathrm{Objects}(D)$ and $\mathrm{Arrows}(C) \to \mathrm{Arrows}(D)$ are smooth.

Example. Remember that whenever we have a set, we can view it as a (pretty boring) category, which has only identity arrows.

Let M be a manifold. We can turn it into a smooth category by taking $\mathrm{Objects} = M$ and $\mathrm{Arrows} = M$; there are only identity arrows.

Example (Čech nerve). Here is a more interesting example. It's a confusing example of a category, so we'll try to go slow. Again let M be a manifold, but this time, take an open cover $\mathcal{U} = \{U_i : i \in I\}$, so that $M = \bigcup_i U_i$. Let $\check{C}_{\mathcal{U}}$ be the smooth category:

- Manifold of objects:
$$\mathrm{Objects}(\check{C}_{\mathcal{U}}) = \coprod_i U_i$$
An object is $x \in U_i$ for some i. Because x can be a member of multiple different open sets, we write the corresponding object of our category as (x, i).
- Manifold of arrows:
$$\mathrm{Arrows}(\check{C}_{\mathcal{U}}) = \coprod_{i,j} U_i \cap U_j$$
An arrow $x \in U_i \cap U_j$ is written $(x; i, j)$. It is an arrow $(x, i) \to (x, j)$. There are two smooth maps
$$\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i$$
which send (x, i, j) to (x, i) and (x, j) respectively.
- composition of (x, i, j) and (x, j, k) is (x, i, k)

Remark. This reduces to the previous example in the case that the open cover is just M itself.

Example. Let G be a Lie group. (For example, a finite group, or a matrix group.) Then $\mathbf{B}G$ is a smooth category.

Proposition. A smooth functor $\check{C}_U \to \mathbf{B}G$ is the same thing as $\boxed{}$

Proof. Let $g : \check{C}_U \to \mathbf{B}G$ be a smooth functor. Nothing interesting happens on objects: there is a smooth map $\coprod_i U_i \to \bullet$. But on arrows, we need to send $(x, i) \xrightarrow{(x,i,j)} (x, j)$ to a morphism $\bullet \to \bullet$ in $\mathbf{B}G$, i.e. an element $g(x, i, j) \in G$. The fact that functors preserve composition means that
$$g(x, j, k) \cdot g(x, i, j) = g(x, i, k).$$
Writing $g(-; i, j) : U_i \cap U_j \to G$ as g_{ij}, we arrive at the following data:

- An open cover $M = \bigcup_i U_i$
- Smooth functions $g_{ij} : U_i \cap U_j \to G$
- Satisfying the equation $g_{jk} g_{ij} = g_{ik}$

If you were paying attention at my talk last semester, you will recognise this as the transition data for a principal G-bundle on M!

$\Longrightarrow$ $\boxed{}$ = a principal G-bundle on M. □

If we make our cover sufficiently fine, then every principal G-bundle arises in this way.

Question. Why does this happen? It is roughly for two reasons, which I will sketch:

(1) The smooth category $\check{C}_{\mathcal{U}}$ is morally a sort of resolution of the manifold M. (For the experts, by which I mean the members of the reading group, it is a cofibrant replacement).
(2) The category $\mathbf{B}G$ is 'equivalent' to the category of G-torsors. This is because all G-torsors are isomorphic, and any one of them has G worth of automorphisms.

So the smooth functor $\check{C}_{\mathcal{U}} \to \mathbf{B}G$ is to be thought of as describing a smooth map $M \to \{G\text{-torsors}\}$.

Remark. Actually, we can push this a little further. You may know that if I have a principal G-bundle, then, given an complex G-representation, we can form the associated bundle, which is a vector bundle. How does this look in our language?

Let $\check{C}_{\mathcal{U}} \to \mathbf{B}G$ be a principal G-bundle, and $\rho : \mathbf{B}G \to \mathsf{Vect} : \bullet \mapsto \mathbb{C}^n$ be a representation. The associated vector bundle is simply the composition

$$\begin{tikzcd} \check{C}_{\mathcal{U}} \arrow[r, "g"] \arrow[rr, bend right, "h=\rho\circ g"'] & \mathbf{B}G \arrow[r, "\rho"] & \mathsf{Vect} \end{tikzcd}$$

Why is this? Composing with ρ means we have the following data:

- An open cover $M = \bigcup_i U_i$
- Smooth functions $h_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$
- Satisfying the equation $h_{jk} h_{ij} = h_{ik}$

which are the transition functions for a vector bundle.