Hertford College Studentship:
Rotational diffusions on generalised Sierpiński gaskets

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This report summarises work done in the last four weeks of my six-week studentship at Hertford College. (The first two weeks were mostly spent reading prerequisite material, learning the basics of Mathematica, and exploring different directions the project might have gone in.) I was supervised by Professor Ben Hambly, whose guidance is greatly appreciated.

1. Background

My studentship project was an investigation into a specific construction in the field of stochastic analysis, a branch of mathematics that studies random processes.

The main objects of study were random walks; these are processes which model how a particle might move around randomly within a space. (Continuous spacetime random walks are often called diffusions.) I was looking at random walks defined on *fractals*, self-similar geometric objects which are characterised by their "roughness". A classical example, and a central one to this research, is the Sierpiński gasket SG_2 (pictured at the top-left of Figure 1). In the 90s, the mathematician Takashi Kumagai—a frequent collaborator of my supervisor—constructed a family of rotationally-symmetric random walks, named *p-stream diffusions*, on SG_2. Here, p is a parameter which controls the direction and magntiude of the rotation. Walks on a gasket SG_2 are defined as (renormalised) limits of walks on graph approximations $\mathrm{SG}_2^{(k)}$, pictured in Figure 1. This convergence is conditional on the sequence having an all-important property called **decimation-invariance**.

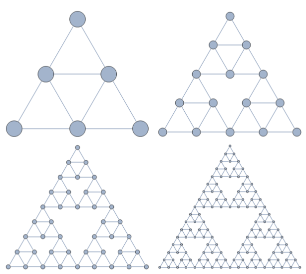

FIGURE 1. The approximations $\mathrm{SG}_2^{(1)}, \mathrm{SG}_2^{(2)}, \mathrm{SG}_2^{(3)}, \mathrm{SG}_2^{(4)}$ of SG_2.

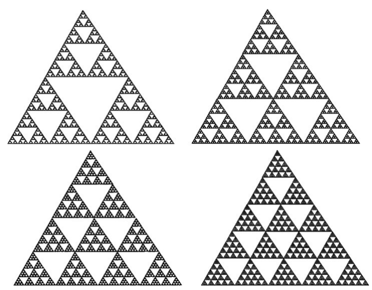

FIGURE 2. The generalised gaskets SG_2, SG_3, SG_4 and SG_5.

Since Kumagai's series of papers almost 30 years ago, the subfield has been relatively quiet. The aim of my project was to investigate the potential extension of Kumagai's work to a larger group of fractals $\{\mathrm{SG}_3, \mathrm{SG}_4, \dots\}$ named the *generalised Sierpiński gaskets* because of their similarity with the original gasket SG_2; a few are pictured in Figure 2. (Commonly, SG_n is called the *level-n* gasket.) It turns out that his construction of p-stream walks does not directly work on generalised gaskets, so we need to introduce another parameter q which controls rotation at the boundary. Decimation-invariance imposes a relationship between the walk parameters p and q, which was the focus of my research.

2. Research

A few years ago, an undergraduate student—also supervised by my supervisor Ben Hambly—undertook a very similar project. He studied the cases $n = 3, 4, 5$ and, together with Ben, posed some tentative conjectures on the relationship between p and q for general n.

The first thing I did was write an algorithm in Mathematica which generates the Markov transition matrix for arbitrary n. This allowed me to collect much more data than was previously possible. I then analysed the observed asymptotic distribution of the p/q relationship, concluding that one of the main conjectures posed by Hambly and his previous student was false. This was my first major result, and arguably the most important one of the project.

The falsity of the conjecture made the problem harder (and also slightly less interesting, because one potential application no longer existed). I spent the next two weeks trying to explain why the conjecture was false in various ways: combinatorial, algebraic, probabilistic. Some notable ideas were developed, but nothing came close to a rigorous proof. Regardless, I made some progress on the other question asked by the previous student. I have posed a conjecture of my own, about certain polynomials which encode information about p and q.

The last two weeks were spent looking at the problem from a different point of view; via harmonic methods. The idea was to see if it was possible to generalise some energy-minimisation framework from symmetric walks to non-symmetric walks. I proved that this was not possible, which was expected but still slightly disappointing.

3. Comments

This project was an open-ended investigation into some unanswered questions. It was therefore not expected that I produce enough original research to publish a paper. Despite this, it was a very valuable opportunity, because I got to experience what real research is like: I made empirical observations, I was stuck for many days at a time, I talked to other mathematicians, etc. It felt like real research, not like an undergraduate mini-project specifically crafted to be doable.

In this sense it was a better experience than a reading course, in which I would have easily achieved the intended aims but would not have gained the same real research experience. This was the main reason I chose this project; it didn't require any heavy prerequisites (like a project on geometry, the field that I'm interested in long-term, would). I enjoyed the process a lot, and it has confirmed my intention to pursue research by applying for a PhD.