Models for Smooth Infinitesimal Analysis

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Preface

This book appears perhaps at the wrong moment, since it goes against the mathematical tide, which nowadays seems to be moving away from abstraction and conceptualization towards concreteness and specialization. Nevertheless, we have decided to publish it, rather than wait for the turn of the tide. Our purpose is to explain infinitesimals and infinitely large integers, as they were used before their elimination by the set-theoretic trend in mathematics. Our explanation doesn't go against this trend, but tries to give a consistent reinterpretation of infinitesimals in a set-theoretic context, through the use of sheaf theory.

A set-theoretic interpretation of infinitesimals appears to have been provided already by A. Robinson and his school, with the creation of non-standard analysis, and the reader may well wonder whether we are reformulating non-standard analysis in terms of sheaves. However, one should notice that *two* kinds of infinitesimals were used by geometers like S. Lie and E. Cartan, namely invertible infinitesimals and nilpotent ones. Non-standard analysis only takes the invertible ones into account, and the claims to the effect that non-standard analysis provides an axiomatization of *the* notion of infinitesimal is therefore incorrect. This is particularly astonishing when one realizes that notions like differential form, curvature, etc., were originally based upon the notion of nilpotent infinitesimal.

The use of sheaves to model nilpotent infinitesimals is not new. In fact, nilpotent infinitesimals are used in Grothendieck's theory of schemes to handle infinitesimal structures in the context of algebraic geometry. But the theory of schemes lacks an adequate language to deal directly with nilpotent infinitesimals, in the way that non-standard analysis provides such a language (and semantics) for invertible infinitesimals.

It was the discovery of Lawvere that a Grothendieck topos may be viewed as a universe of "variable" sets, and that consequently set-theoretic language can be interpreted directly in a topos. Therefore, working with the topos built from schemes, rather than with the schemes themselves, one obtains a model for this generalized set-theory with nilpotent infinitesimals.

Lawvere also discovered that by considering smooth versions of the toposes occurring in algebraic geometry (toposes built from rings of smooth functions, rather than polynomials) one obtains models for ordinary differential geometry. In these models, infinitesimal structures of the kind used by Cartan, for instance, can be interpreted directly, and in this context Cartan's arguments are literally valid.

These ideas, dating from 1967, remained unpublished, and were taken up only in the mid-seventies. This resulted in two main lines of development. On the one hand, there was the purely axiomatic development of differential geometry with nilpotent infinitesimals, or "synthetic differential geometry". On the other hand, smooth toposes were constructed, which showed not only the consistency of the axiomatic approach, but also provided a direct connection with the classical theory of manifolds.

The emphasis of our book is on this second line of development. Our main concern has been to show that synthetic differential geometry has a clear and direct relation to the classical theory. This relation is based on the fact that, unlike non-standard analysis, synthetic differential geometry has natural models built from smooth functions and their ideals. The main novelty of our approach, with regard to both non-standard analysis and synthetic differential geometry, is precisely the construction of such mathematically natural models containing nilpotent as well as invertible infinitesimals.

We started our collaboration at the end of 1982, when Reyes was spending his sabbatical year at the University of Utrecht. The actual writing of the book took place between the fall of 1983 and the spring of 1985. During this period, the authors were able to work in close contact. Besides several shorter visits, Reyes spent the summer of 1984 at the University of Amsterdam, and Moerdijk spent the academic year 1984–85 at McGill University.

We gave courses and seminars on parts of the contents of the book at the University of Utrecht in 82–83, at the University of Montreal in 83–84, and at McGill University in 84–85. Moreover, between 1983 and 1986, the material was presented in lectures at Aarhus during the workshop on categorical methods in geometry, at Bogotá during the seminario-taller de categorías, and at the universities of Paris, Lille, Cambridge, Columbia, Rome, Milano, Parma, Warsaw, Carnegie-Mellon, Maryland, Campinas, Saõ Paulo, La Rioja (Logrono), Zaragoza and Santiago de Compostela. We would like to thank our colleagues at these institutions who made these visits

possible, for their hospitality and support.

We gratefully acknowledge the almost continuous financial support by the Netherlands Organization for Pure Research (ZWO), le Conseil de recherches en sciences naturelles et en génie du Canada, and le Ministère de l'éducation du Gouvernement du Québec. In particular, Reyes' visits to Utrecht and Amsterdam were partly supported by ZWO, and we are grateful to Dirk and Dook van Dalen, and Anne Troelstra for making these visits possible and pleasant. Moerdijk's year in Montréal was made possible by an invitation of the Groupe interuniversitaire en études catégoriques, and we would like to express our thanks to all the members of the Groupe for creating such pleasant working conditions.

During these years, those who have helped us are too numerous to be mentioned here individually. But we are specially indebted to Dana Scott. It was he who suggested the possibility of writing a monograph on models of synthetic differential geometry, who gave us advice on the organization of the book, presented it to Springer-Verlag, and provided the facilities to prepare the final text. A special word of thanks also goes to Bill Lawvere, without whose constant support we would never have been able to write this book, and to Ngo Van Quê, who had the patience to explain some analysis and differential geometry to us ignorant logicians. Moreover, we would like to thank Oscar Bruno, Marta Bunge, Eduardo Dubuc, Iole Druck, Luis Español, Alfred Frölicher, André Joyal, Anders Kock, Michael Makkai, Colin McLarty, Peter Michor, María del Carmen Mínguez, Wil van Est, and Gavin Wraith for valuable conversations and comments on parts of the manuscript.

We would also like to thank Yvonne Voorn and Lise Perreault, who have typed endless earlier versions, Roberto Minio for valuable advice on matters connected with the editing of this book, and Staci Quackenbush, who prepared this final text that you have before you.

Finally, the second author probably would have not survived this experience, had it not been for the encouragement of Marie. Not only did she give advice on a variety of matters connected with the book, but her unreasonable conviction that this project could be brought to an end, proved to be contagious.

Montréal, I.M., G.E.R.

Contents

Introduction

The theory of manifolds goes back to Riemann's lecture "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" ("On the hypotheses which lie at the foundations of geometry"), which was delivered on June 10, 1854, to the faculty of the University of Göttingen. Since there were members of the faculty who knew little mathematics, Riemann chose a rather informal style of exposition to make his lecture intelligible. In part one of this lecture, he set himself the task of "constructing the concept of a multiply extended quantity from general notions of quantity", a task he regards as being of a "philosophical nature, where difficulties lie more in the concepts than in the construction ...."

On the basis of this lecture alone, it seems nearly impossible to determine what form such a construction should take, and hence we cannot know how far Riemann had advanced towards the accomplishment of his task. For a modern reader, however, it is very tempting to regard his efforts as an endeavor to define a "manifold", and it is precisely the clarification of Riemann's ideas, as understood by his successors, which led gradually to the notions of manifold and Riemannian space as we know them today.

In this context it is important to notice that Riemann himself pointed out in his lecture the existence of "manifolds in which the fixing of positions requires not a finite number, but either an infinite sequence or a continuous manifold of numerical measurements. Such a manifold form, for instance, the possibilities for a function in a given region, the possible shapes of a solid figure, etc." This quotation reveals already a first limitation of the theory of manifolds in its modern guise:

The category $\mathbb{M}$ *of* C^∞ *-manifolds and* C^∞ *-maps between them is not cartesian closed. In particular, the space of* C^∞ *-maps between two manifolds is not necessarily a manifold.*

The need for a cartesian closed category of smooth spaces and smooth maps has repeatedly been pointed out in connection to physics. We mention the following considerations, due to Lawvere (1980): The motion of a certain body B (for example, a 0-dimensional system of particles, a 1-dimensional elastic cord, a 2-dimensional flexible shell, a 3-dimensional solid) is often represented by a map

$$q: T \times B \to E,$$

where T is (the 1-dimensional space to measure) time, and E is the ordinary flat 3-dimensional space. Thus, the motion may be thought of as assigning to a couple (time, particle of B) the corresponding place in E during the motion.

For other purposes, however, it is useful to consider motion as a map

$$\overline{q}: B \to E^T$$

which assigns to each particle of B its path through E, where E^T is the space of (smooth) paths. The action of the vector space V of translations of the flat space E allows us to define a map

$$(\dot{\ }): E^T \to V^T$$

using Newton's notation. By composing with $\overline{q}$ we obtain a new map which, in turn, gives us (by adjunction) the velocity map

$$v: T \times B \to V$$

of the motion q.

Still another way of considering motion is necessary for some purposes, namely as a map

$$\overline{\overline{q}}: T \to E^B$$

which assigns to a time the (smooth) placement of the body in space at that time. Letting μ be the mass distribution of B, we obtain a map (by convexity of E)

$$\frac{1}{\mu(B)} \int_B (\ ) d\mu: E^B \to E,$$

which assigns to each placement of B the corresponding position of the center of mass. Once again, composing this map with $\overline{\overline{q}}$, we obtain a new map

$$T \longrightarrow E$$

giving for each time the center of mass of the systems in motion at that time.

The various connections between these ways of regarding motion should be expressed precisely by the adjunctions available from the cartesian closed structure of a category of smooth spaces and smooth maps. In the words of Lawvere: "The E,B,T transforms (i.e., the adjunctions) are *more* (at least as) fundamental as any particular determination of the objects as "consisting" of points, opens, paths, etc., and indeed any such determination which does not admit these transformations is ultimately of only specialized interest".

The second limitation of the theory of manifolds may briefly be formulated as follows:

The category $\mathbb{M}$ *of* C^∞ *-manifolds lacks finite inverse limits. In particular, pullbacks of manifolds are generally not manifolds.*

This implies that curves and algebraic varieties, of the kind already studied by Descartes, are not manifolds. The trouble here is that algebraic varieties may have (and usually do have) singularities, whereas manifolds cannot. As a consequence of this exclusion, one finds that, despite many interactions, differential geometry and algebraic geometry follow their separate ways, and the methods of one cannot, without violence, be applied to the other.

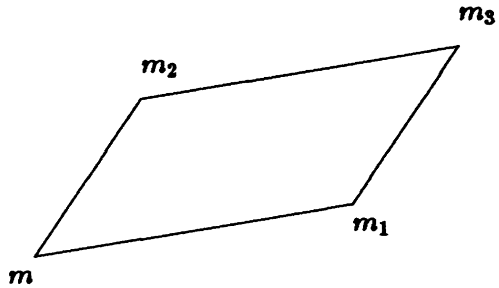
A limitation of the theory of manifolds of a different nature is:

The absence of a convenient language to deal explicitly and directly with structures in the "infinitely small".

We mention here a rather technical example, a theorem due to Ambrose, Palais and Singer, which will be discussed in Chapter V. This theorem asserts the equivalence between symmetric connections and sprays on a manifold. Connections and sprays are operations on infinitesimal structures, and one would like to show their equivalence directly. However, an appropriate language to make a direct comparison is lacking, and one first has to transform these "infinitesimal" structures into "local" ones by integration. The comparison is then possible, since it is at this "local" level that the language of the classical theory of manifolds is adequate. Finally, one returns to the original "infinitesimal" structures by some limit process, inverse to integration.

This detour should not be necessary if one had a convenient language for infinitesimals at one's disposal.

Despite the absence of such a language for infinitesimals, geometers like G. Darboux, S. Lie and E. Cartan often used "synthetic" reasoning in their work. We shall illustrate this style of reasoning with an example taken from E. Cartan (1928). After stating the formulas of Green, Stokes and Ostrogradsky, Cartan (loc. cit, p. 207) continues: "The operation which allows us to construct such formulas may be described in a very simple way. Let us first consider the case of a simple integral $\overline{\omega}(d)$ [where $\overline{\omega}(d)$ is a differential 1-form, and d is a symbol of differentiation] taken along a closed circuit (C). Let (S) be a (part of a) surface limited by (C), in n-dimensional space. Let us introduce in (S) two symbols of differentiation d_1, d_2 which are interchangeable [i.e., they commute], and let us divide (S) into the corresponding network of infinitely small parallelograms. If m is the vertex of one of these parallelograms (cf. figure)



and if m_1 and m_2 are the vertices obtained from the operations d_1 and d_2, we have

$$\int_m^{m_1} \overline{\omega} = \overline{\omega}(d_1), \quad \int_m^{m_2} \overline{\omega} = \overline{\omega}(d_2)$$

$$\int_{m_1}^{m_3} \overline{\omega} = \int_m^{m_2} \overline{\omega} + d_1 \int_m^{m_2} \overline{\omega} = \overline{\omega}(d_2) + d_1\overline{\omega}(d_2)$$

$$\int_{m_2}^{m_3} \overline{\omega} = \overline{\omega}(d_1) + d_2\overline{\omega}(d_1);$$

therefore the integral $\overline{\omega}$ taken along the boundary of the parallelogram is equal to $\overline{\omega}(d_1) + (\overline{\omega}(d_2) + d_1\overline{\omega}(d_2)) - (\overline{\omega}(d_1) + d_2\overline{\omega}(d_1)) - \overline{\omega}(d_2) = d_1\overline{\omega}(d_2) - d_2\overline{\omega}(d_1)$. The expression in the second member is nothing else but the bilinear covariant of $\overline{\omega}$ [i.e., in modern language, the exterior derivative]. For instance, if Pdx is a term of $\overline{\omega}$,

$$d_1(Pd_2x) - d_2(Pd_1x) = d_1Pd_2x - d_2Pd_1x = (dPdx)$$

[in modern notation: $dP \wedge dx$]. We obtain, thus, the Stokes' formula

$$\int Pdx + Qdy + Rdz = \int\int dPdx + dQdy + dRdz,$$

which may be extended to any number of variables".

Let us remark that this way of defining the exterior derivative by circulation along an infinitesimal parallelogram, obtaining Stokes' theorem as a byproduct, is quite popular (and rightly so!) among physicists and engineers, who keep on using this kind of reasoning.

The "symbol of differentiation" occurring in this quotation may seem rather mysterious. Let us quote again from Cartan (loc. cit, p. 179), where the sense of this notion is elucidated for Riemann spaces: "Let us consider two different systems of differentiation d and δ. The quantities du^i may be considered as products of an "infinitely small" constant parameter α by functions $\xi^i(u^1, ... u^n)$ (which are either determined or left undetermined):

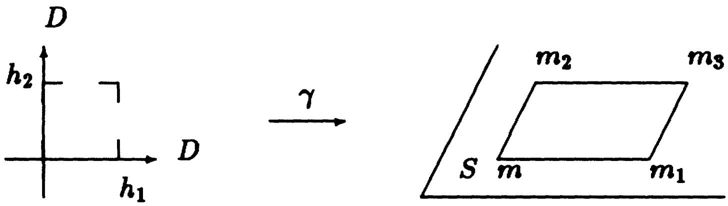
$$du^i = \alpha\xi^i(u^1, ..., u^n).$$

Similarly,

$$\delta u^i = \beta\eta^i(u^1, ..., u^n).$$

Let m be an arbitrary point with coordinates (u^i) of the Riemann space; let m_1 be the point with coordinates $(u^i + du^i)$ and m_2 the point with coordinates $(u^i + \delta u^i)$. The vector $\overrightarrow{mm_1}$ defines an elementary displacement d; the vector $\overrightarrow{mm_2}$ an elementary displacement δ."

Following this explanation, we shall interpret these notions as follows (see Chapter IV): A symbol of differentiation (on S) is a map $d: D \to S$, where D is the set of first-order infinitesimals, i.e. $D = \{h \in R | h^2 = 0\}$. Two such symbols d, δ commute if there is some $\gamma: D \times D \to S$ such that $\gamma(h, 0) = d(h)$ for all $h \in D$ and $\gamma(0, h) = \delta(h)$ for all $h \in D$. A differential 1-form (on S) is a map $\omega: S^D \times D \to R$. Cartan defines the exterior derivative of ω, i.e., a differential 2-form $d\omega: S^{D\times D} \times D \times D \to R$ via the circulation along the infinitisimal parallelogram $(\gamma, h_1, h_2) \in S^{D\times D} \times D \times D$. In fact, (γ, h_1, h_2) may be pictured as follows:



with $m = \gamma(0,0)$, $m_1 = \gamma(h_1, 0)$, $m_2 = \gamma(0, h_2)$, $m_3 = \gamma(h_1, h_2)$.

The circulation is simply

$$\int_m^{m_1}\omega+\int_{m_1}^{m_3}\omega-\int_{m_2}^{m_3}\omega-\int_m^{m_2}\omega$$

where these infinitesimal integrals are defined by

$$\int_m^{m_1}\omega=\omega([h\mapsto\gamma(h,0)],h_1)=\omega(d_1,h_1)$$

$$\int_m^{m_2}\omega=\omega([h\mapsto\gamma(0,h)],h_2)=\omega(d_2,h_2)$$

$$\int_{m_1}^{m_3}\omega=\omega([h\mapsto\delta(h_1,h),h_2)$$

$$\int_{m_2}^{m_3}\omega=\omega([h\mapsto\gamma(h_1,h_2),h_1).$$

To continue Cartan's argument, we make the blunt assumption (which is a consequence of the so-called Kock-Lawvere axiom, see II.2.4) that any function $f\colon D\to R$ may be developed in Taylor series to obtain $f(h)=f(0)+hf'(0)$.

By applying this formula to $f(h_1)=\omega([h\mapsto\gamma(h_1,h)],h_2)$, we obtain $\int_{m_1}^{m_3}\omega=\int_m^{m_2}\omega+h_1\cdot f'(0)$, which is Cartan's formula, but for the notation. To complete the definition of the exterior derivative, we let $d\omega(\gamma,h_1,h_2)=$ circulation of ω along (γ,h_1,h_2). Using infinitesimal integrals, we may thus write, letting $\partial(\gamma,h_1,h_2)$ be the circuit (C),

$$\int_{(\gamma,h_1,h_2)}d\omega=\int_{\partial(\gamma,h_1,h_2)}\omega$$

which is the infinitesimal version of Stokes' theorem. From here, as shown in detail in Chapter IV, we can derive the usual, finite version of Stokes' theorem. How this theorem relates to the classical one will be explained later on in this introduction (and more extensively in Chapter IV).

Rather than multiplying the examples of this kind, we now give a different one coming from theoretical physics. Although several "mathematical" definitions of "generalized measures" have been given (the theory of distributions, operator calculus, etc.), it seems fair to say that physicists keep on thinking of a distribution as the operation of integrating against a "real function", be it with rather pathological properties. A typical example is the function δ corresponding to the Dirac distribution. In Schiff (1968), p.56, the properties of the δ function are described as follows:

"Thus the limit of this function (e.g. $\frac{\sin gx}{\pi x}$) as $g\longrightarrow\infty$ has all the properties of the δ function: it becomes infinitely large at $x = 0$, it has unit integral, and the infinitely rapid oscillations as $|x|$ increases mean that the entire contribution to an integral containing this function comes from the infinitesimal neighborhood of $x = 0$."

We wish to point out the following features of these arguments. First of all, in the first argument no mention is made of atlases and coordinates, although manifolds are mentioned; in other words, this argument is intrinsic, and proceeds by directly manipulating geometric objects, namely differential forms and infinitesimal parallelograms. Secondly, infinitesimals are freely used, making the notion of limit unnecessary (for the particular purpose at hand). Notice, however, that in the first argument the infinitesimals must be *nilpotent*. Obviously, such infinitesimals will not do to define the δ function–for this one needs the notion of *invertible* infinitesimals, and correspondingly, of *infinitely large* reals, to make the integral of δ add up to 1.

As a final illustration of the need for an adequate language to deal with infinitesimal structures, we would like to mention the following quotation taken from the preface to S. Lie's article (1876) (also quoted in Kock (1981)): "The reason why I have postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by synthetic considerations. But I soon realized that, as appropriate [zweckmässig] the synthetic method is for discovery, as difficult it is to give a clear exposition on synthetic investigations, which deal with objects that till now have almost exclusively been considered analytically. After long vacillations, I have decided to use a half-synthetic, half-analytic form. I hope my work will serve to bring justification to the synthetic method besides the analytic one."

In this book, we will describe an approach to analysis and differential geometry, *smooth infinitesimal analysis*, which avoids the three limitations of the category of manifolds discussed above. The basic ideas of this approach are mainly due to F. W. Lawvere, and can be seen to originate from the work of C. Ehresmann, A. Weil, and A. Grothendieck. The aim is to construct categories of spaces, the so-called *smooth toposes*, which contain the category of manifolds (or more precisely, there is a full and faithful embedding of the category of C^∞-manifolds into each of these smooth toposes). Moreover, in each of these smooth toposes *inverse limits* of spaces and *function spaces* can be adequately constructed, in particular *infinitesimal spaces* like the ones needed in (our interpretation of)

Cartan's arguments, e.g., the space D of first-order infinitesimals.

The construction of these smooth toposes proceeds in two steps: one first embeds the category of manifolds $\mathbb{M}$ in the category $\mathbb{L}$ of "loci", a category of formal varieties. This new category has finite inverse limits and contains infinitesimal spaces, but function spaces can generally not be constructed in $\mathbb{L}$. As a second step, therefore, $\mathbb{L}$ is endowed with a natural Grothendieck topology, and the resulting topos $\mathrm{Sh}(\mathbb{L})$ of sheaves on $\mathbb{L}$ for this topology is the required extension of $\mathbb{L}$ in which function spaces with good properties can be constructed,

$$\mathbb{M} \subset \mathbb{L} \subset \mathrm{Sh}(\mathbb{L}).$$

This construction and variants thereof will be discussed in detail in this book, and at this stage we just sketch the idea of the extension of the category $\mathbb{M}$ of manifolds to the category $\mathbb{L}$ of loci.

To motivate the definitions, let us recall the functorial approach to algebraic geometry, as exposed in Demazure & Gabriel (1970), for example. An (algebraic) locus such as $S^1 = \{(x,y)|x^2+y^2=1\}$ is identified with a functor $S^1: C \to \mathit{Sets}$, where C is the category of commutative rings; S^1 associates with a ring A the set $S^1(A) = \{(a,b) \in A^2 | a^2 + b^2 = 1\}$, and with a ring homomorphism $A \xrightarrow{f} B$ the obvious restriction $S^1(f): S^1(A) \to S^1(B)$, sending (a,b) to $(f(a), f(b))$. As morphisms between one such locus, i.e. a functor $C \to \mathit{Sets}$, and another, one takes simply the natural transformations. Besides the usual "spaces" such as the sphere S^1, the line R given by $R(A) =$ the underlying set of A, etc., one also has "infinitesimal loci". For example, the locus $D = \{x \in R | x^2 = 0\}$, i.e. $D(A) = \{a \in A | a^2 = 0\}$, plays the rôle of the space of first-order infinitesimals. In fact, the category of algebraic loci is simply the dual (or opposite) of the category of finitely generated commutative rings: a ring $A = \mathbb{Z}[X_1, ..., X_n]/(p_1, ..., p_k)$ corresponds to the locus $\ell(A) = \{x \in R^n | p_1(x) = .. = p_k(x) = 0\}$, i.e. to the functor

$$B \mapsto \mathrm{Hom}(A,B) \cong \{\underline{b} \in B^n | p_1(\underline{b}) = .. = p_k(\underline{b}) = 0\}.$$

In our case, the category of commutative rings is replaced by that of C^∞*-rings*. A C^∞-ring is a ring A in which we can interpret every C^∞-function $\mathbb{R}^m \longrightarrow \mathbb{R}$ as an operation $A^m \longrightarrow A$ (and not just polynomial functions, as in the case of commutative rings), and a map between two such C^∞-rings is a ring homomorphism which preserves this additional structure, a "C^∞*-homomorphism*". The category $\mathbb{L}$ is simply the dual of the category of finitely generated C^∞-rings, and for a given such C^∞-ring A, the corresponding locus–

an object of $\mathbb{L}$–is denoted by $\ell(A)$.

Any manifold M is represented as an object of $\mathbb{L}$ via the C^∞-ring of smooth functions on M, $C^\infty(M)$. Furthermore, we have infinitesimal spaces such as $D = \ell(C^\infty(\mathbb{R})/(x^2))$, and $\Delta = \ell(C_0^\infty(\mathbb{R}))$ where $C_0^\infty(\mathbb{R})$ is the C^∞-ring of germs at 0 of smooth functions on $\mathbb{R}$, which will play the rôle of ***first-order infinitesimals*** and ***infinitesimals*** respectively, as will be shown later on in detail. An important space of infinitesimals is the locus $\mathbb{I} = \ell(C_0^\infty(\mathbb{R} - \{0\})$, the ring of restrictions to $\mathbb{R} - \{0\}$ of the germs at 0; $\mathbb{I}$ plays the rôle of the set of ***invertible infinitesimals***. We also have such loci as $\ell(C^\infty(\mathbb{N})/K)$, where $C^\infty(\mathbb{N}) = \mathbb{R}^{\mathbb{N}}$ is the ring of smooth functions on the natural numbers, and K is the ideal of eventually vanishing functions; this locus will act as the set of ***infinitely large natural numbers***.

When a smooth topos like $\mathrm{Sh}(\mathbb{L})$ is described in this way, namely as a category of "spaces" which extends the usual category of manifolds, its close relation to the classical theory is clear. But the structure of these spaces, being sheaves on $\mathbb{L}$, is rather complicated, and the synthetic arguments described earlier can only be interpreted in a very round-about way.

However, and this is a crucial aspect of our whole approach, a smooth topos can also be regarded as a "universe of sets", inside which one can describe constructions and give arguments in a purely set-theoretical language, so that much of the complexity of the structures used is no longer explicitly there. There is one limitation, however, to the use of set-theoretical arguments and constructions when applied in this new context: they should be constructive, and no use of the axiom of choice or the law of the excluded middle can be made.

Regarding the topos $\mathrm{Sh}(\mathbb{L})$ in this way, synthetic arguments like Cartan's can be carried out almost word by word in $\mathrm{Sh}(\mathbb{L})$. Furthermore, this point of view enables us to apply many of the classical definitions, constructions and (constructive!) proofs literally to this more general category of spaces, without ever making explicit that we are really dealing, ***not*** just with ***sets of points***, but with ***sheaves*** on $\mathbb{L}$.

To give a simple example, Cartan's argument for Stokes' theorem is constructively valid, and–working in $\mathrm{Sh}(\mathbb{L})$ as a universe of sets–it applies to an arbitrary "set", i.e., to any object of $\mathrm{Sh}(\mathbb{L})$. When one now "decodes" this set-theoretic way of looking at the sheaves on $\mathbb{L}$, one obtains the usual form of Stokes' theorem for manifolds, as we will explain in detail in Chapter IV. (In fact, one obtains a more general result, including a form of Stokes' theorem for spaces of C^∞–functions from one manifold to another, since function spaces of manifolds exist as sheaves on $\mathbb{L}$.)

So a smooth topos like $\mathrm{Sh}(\mathbb{L})$ can be looked at from two different points of view: an "*external*" one, where one regards the topos as a category of sheaves extending the category of manifolds, and a so-called "*internal*" one, where one interprets the set-theoretic language in $\mathrm{Sh}(\mathbb{L})$ and one works inside $\mathrm{Sh}(\mathbb{L})$, regarding it as a "universe of sets". It is the change of point of view that explains the relation between "synthetic arguments" and what these arguments prove about "classical" structures, like manifolds, spaces of C^∞-functions, etc. This change of point of view, made possible by the interpretation of the set-theoretic language in a smooth topos, is a continuously recurring theme in this book, and many examples will be described in detail.

In fact, the interplay between these points of view is an important aspect of the theory of Grothendieck toposes as a whole, and we take this opportunity to digress, and speculate on this phenomenon in a somewhat philosophical vein.

Topos theory has brought to light and given the means to exploit a complementarity (or duality) principle between *logic* and *structure*. A mathematical theory T (e.g., differential geometry) is usually specified by two components: (1) the type of structure S to which the notions of the theory belong; (2) the canonical interpretation $I: S \to \mathit{Sets}$, which gives the set-theoretical interpretation of the structures in question. The first component is given by listing some axioms and definitions in a fixed language, whereas the second one is obtained via set-theoretic or Tarski's semantics. We shall suggestively express this by an equation

$$T = S + I.$$

Topos theory offers the possibility of considering interpretations of a type of structures S into any topos, and not only in *Sets*. The interpretations are obtained via sheaf semantics, rather than by the "tautological" Tarski-semantics. The complimentarity principle asserts that, by enlarging the notion of interpretation in this way, no component is uniquely determined by T. Indeed, several choices of S and I are possible for one and the same theory T. Once that a component has been chosen, however, the other is determined by the "equation" $T = S + I$. So if one complicates the interpretation I from set-theoretic to sheaf semantics, this will give a corresponding simplification of the kind of structures S, and one may thus exploit the principle by choosing to simplify either one of the components I

and S, depending on the specific properties of T under consideration. In particular, given a theory T, there is no reason to fix one particular decomposition S, I once and for all. At the same time, however, the principle implies that a simplification of one of the components will *have* to result in a complication of the other, and one cannot expect to be able to simplify, say, the structures S, while keeping as interpretation the "trivial" Tarski semantics.

There is one important point, familiar from logic, and already mentioned above in the special context of our smooth toposes: it is rather awkward to work with the sheaf-theoretic interpretation I directly, since a lot of irrelevant structure has to be carried along. A more efficient way to proceed is to regard the corresponding component S as a formalized theory, with a certain "underlying logic" coming from the properties of I. In the case of sheaf semantics, this underlying logic will be constructive: all set-theoretical constructions and arguments can be performed in the formalized theory, provided the axiom of choice and the excluded middle are not used. Thus, rather than working with sheaves all the time, we can proceed as if we work with sets until a final result is reached, and only then explicitate the interpretation. It is thus only the formulation of this final result that we actually interpret, so as to provide its "classical content."

The "synthetic reasoning" as examplified in the beginning can be seen as a particular case of working in a formalized theory S. Regarded in this way, one can provide at the same time both a rigorous mathematical justification for these synthetic arguments, and explain their relation to the classical theory (or more precisely, their relation to T as it is classically decomposed into $S + I$, where I is Tarski's semantics). As noted above, one cannot expect this relation to the classical theory to be more direct, since simplification of the S-component as provided by "synthetic reasoning" should force a complication of the interpretation and its underlying logic, as is indeed the case by the very fact that many synthetic arguments are *inconsistent* with classical logic.

To conclude this introduction, we would like to outline the contents of this monograph.

In the first chapter, entitled "C^∞-rings", we introduce and investigate the notion of a C^∞-ring, which is fundamental for the whole book. These C^∞-rings, which were already mentioned above, are obtained from rings of smooth functions on manifolds by dividing by ideals and by taking filtered colimits. They play a rôle in this book which is similar to the rôle played by commutative rings in algebraic geometry. Much of the material in this chapter has already appeared in the literature, although the presentation has been streamlined, and many alternative proofs have been given. It contains all the background in analysis needed for the rest of the book.

In the second chapter, "C^∞-rings as variable spaces," we consider C^∞-rings as geometric objects. As already hinted at above, this is done by considering the dual (opposite) of the category of finitely generated C^∞-rings, the category $\mathbb{L}$ of "loci" or "formal C^∞-varieties." This category $\mathbb{L}$ contains the usual manifolds as well as infinitesimal spaces, and is closed under finite inverse limits, but $\mathbb{L}$ lacks function spaces in general. This motivates the introduction of the category $Sets^{\mathbb{L}^{op}}$ of presheaves on $\mathbb{L}$. This category contains $\mathbb{L}$ as a full subcategory, and is in fact a Grothendieck topos. This means that function spaces and inverse limits can be constructed in $Sets^{\mathbb{L}^{op}}$, and more generally, that set-theoretical constructions and arguments can be performed inside $Sets^{\mathbb{L}^{op}}$. This feature of Grothendieck toposes is a continuously recurring theme, which we will illustrate in this chapter by means of some simple examples of synthetic calculus in $Sets^{\mathbb{L}^{op}}$.

The category of presheaves on $\mathbb{L}$ is not very suitable as a model, since the embedding $\mathbb{M} \hookrightarrow \mathbb{L} \hookrightarrow Sets^{\mathbb{L}^{op}}$ of manifolds does not preserve the "good" colimits of $\mathbb{M}$ such as open covers. $Sets^{\mathbb{L}^{op}}$ is really only discussed for didactical purposes, being a very simple example of a Grothendieck topos. In chapter III we will introduce two models for synthetic calculus, called $\mathcal{G}$ and $\mathcal{F}$, for which the embedding of manifolds does preserve open covers. To this end, the notion of a Grothendieck topology and the corresponding notion of a sheaf make their first appearance, and the theme of interpreting set-theoretical constructions and arguments in a topos is given a new turn when sheaf-semantics is gradually introduced via examples. These examples often take the form of preservation results of the type: the embedding of manifolds into a given model maps compact manifolds to compact spaces in the model, etc.

These preservation results play a crucial rôle in the next chapter, "Cohomology and integration," where we will formulate and prove De Rham's theorem in the models of Chapter III. Here the presence of infinitesimal spaces is exploited to develop the theory of differential forms, in a way which captures directly the geometric intuitions underlying the synthetic arguments discussed in the beginning. For example, Stokes' theorem can now rigorously be proved in a way which follows the argument of Cartan (1928) almost word by word.

Finally, as an illustration of the complementarity principle between logic and structure, we "translate" the internal version of De Rham's theorem back into classical language, so as to obtain not only the usual version of this result for classical manifolds, but also a more general theorem about sheaves of forms and cohomology depending on an extra smooth parameter.

We consider Chapter V, "Connections on microlinear spaces," and Chapter VI, "Models with invertible infinitesimals," as the main chapters of this book. The former is mainly synthetic, whereas the other is model-theoretic. (In other words, these two chapters stand at opposite ends of the complementarity principle.)

In Chapter V, the notion of a microlinear space is introduced. Roughly, a microlinear space is a space which behaves, with respect to maps from infinitesimal spaces into it, as if it had local coordinates. The class of microlinear spaces of the topos $\mathcal{G}$ (and also of $\mathcal{Z}$) contains all the manifolds (i.e. the images of the embedding $\mathbb{M} \longrightarrow \mathcal{G}$), but also much more: it is closed under inverse limits and exponentiation, so as to include spaces with singularities and spaces of smooth functions. Among other things, we will give a very intuitive proof of the Gauss-Bonnet theorem in dimension 2, just by adding infinitesimal angles in various ways. And we will prove the theorem of Ambrose, Palais and Singer on connections and sprays for an arbitrary microlinear space. The latter result, when translated into the classical language of sheaves, gives not only the usual version for manifolds, but also a corresponding result for connections and sprays on spaces of smooth functions.

In Chapter VI, we discuss two toposes which do not only contain nilpotent infinitesimals, but also invertible ones; the need for both types of infinitesimals was already pointed out in the beginning of this introduction. The first model, $\mathcal{Z}$, is a smooth analogue of the Zariski topos. We study some elementary properties of this topos as a model for infinitesimal analysis, and we prove some preservation properties of the embedding $\mathbb{M} \longrightarrow \mathcal{Z}$ of the category of manifolds into this topos. For example, this embedding preserves (countable) open covers, compactness, etc. A crucial step in obtaining these results is the appropriate definition of notions such as "countable," "finite," "compact," etc. in the context of $\mathcal{Z}$, by using the object of "smooth natural numbers" N, rather than the natural numbers object of $\mathcal{Z}$. Finally, we modify $\mathcal{Z}$ by "forcing" the existence of invertible infinitesimals, which are in some sense "only partially" present in $\mathcal{Z}$. This results in the "Basel topos" $\mathcal{B}$, which we regard as the natural model of smooth infinitesimal analysis. All results valid for $\mathcal{Z}$ will be extended to this new topos $\mathcal{B}$.

Taking this topos $\mathcal{B}$ as the natural model, we try in the final chapter, Chapter VII, to develop an axiomatic system for smooth infinitesimal analysis. It will be shown that earlier results about the models–then proved by model-theoretic means, using classical logic and sheaf theory–can in fact be derived from these axioms, using intuitionistic logic. As another example, we will show how this axiomatization allows us to give a treatment of distributions which is, we believe, closer to the intuitions of physicists than the usual approaches such as the classical theory of distributions (as exposed in Schwartz (1973)) or operator calculus (Mikusiński (1983)). Furthermore, we will prove a "transfer principle", which says that for a rather large class of statements, validity in the model $\mathcal{G}$ is equivalent to validity in $\mathcal{B}$.

This ends the summary of the contents of the book. As the reader will have noted, we have not aimed at giving an exhaustive or even systematic development of any of the themes touched upon. By taking examples from a variety of subjects, we have rather tried to suggest the wide applicability of the methods discussed, hoping that others, who are more competent than us in the specific fields of application, will bring these methods to fruition. After all, in the words of Poincaré, "it is the man and not the method who solves a problem".

Chapter 1

C^∞-Rings

In this chapter we will introduce the rings that will play a rôle in this book which is similar to the rôle played in algebraic geometry by ordinary commutative rings with 1. These are the so-called C^∞-rings. The general notion of C^∞-ring as such does not occur in classical analysis and differential topology. However, the main examples of C^∞-rings occupy a central position in singularity theory and related subjects.

The only prerequisites for this chapter are some basic commutative algebra (the relevant facts can be found in Chapters 1, 2 and 3 of Atiyah and MacDonald (1969), for example), and familiarity with the elementary facts about C^∞-functions and C^∞-manifolds (Chapters 1 and 2 of Guillemin and Pollack (1974), referred to as GP, will do). For the notions from category theory that we use, the reader is referred to MacLane (1971).

1 C^∞-Rings

Recall that an $\mathbb{R}$-algebra is a ring A (in this book, all rings are commutative and unitary), which is equipped with a homomorphism $\mathbb{R} \longrightarrow A$. Equivalently, every map $p: \mathbb{R}^n \to \mathbb{R}^m$ which is given by an m-tuple of polynomials $(p_1, \ldots, p_m)$ with real coefficients can be interpreted as a map $A(p): A^n \to A^m$, in such a way that projections, composition and identity are preserved: $A(\pi_i) = \pi_i, A(\mathrm{id}) = \mathrm{id}$, $A(p \circ q) = A(p) \circ A(q)$.

A *C^∞-ring* is a ring in which we cannot only interpret all polynomial maps but all smooth maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$: for each smooth $f: \mathbb{R}^n \to \mathbb{R}^m$ there is a corresponding map $A(f): A^n \to A^m$, and again projections, composition and identity maps are preserved. (Of course it suffices to specify $A(f)$ for the case $m = 1$.)

Just as a homomorphism of $\mathbb{R}$-algebras is defined as a homomorphism of rings which preserves the interpretations of polynomial maps, a ***homomorphism of C^∞-rings*** is a ring homomorphism which preserves the interpretation of smooth maps. So a ring homomorphism $\varphi: A \to B$ is a homomorphism of C^∞-rings (or ***C^∞-homomorphism***) if for each smooth $f: \mathbb{R}^n \to \mathbb{R}^m$ the diagram

$$\begin{tikzcd} A^n \arrow[r, "\varphi^n"] \arrow[d, "A(f)"'] & B^n \arrow[d, "B(f)"] \\ A^m \arrow[r, "\varphi^m"'] & B^m \end{tikzcd}$$

commutes.

We may define C^∞-rings in an equivalent, but more concise way using the notion of algebraic theory in the sense of Lawvere: Let C^∞ denote the category whose objects are the spaces $\mathbb{R}^n$, $n \geq 0$, and whose arrows are all smooth functions. Then a C^∞-ring is a finite product preserving functor $A: C^\infty \to$ ***Sets***, and a C^∞-homomorphism is just a natural transformation. (Indeed, given such a functor $A: C^\infty \to$ ***Sets***, its ***underlying set*** $A(\mathbb{R})$, which we will also just write as A, has the structure of a commutative ring, since all the ring operations on $\mathbb{R}$ are smooth.)

Here are some examples of C^∞-rings. For every subset $X \subseteq \mathbb{R}^r$, the ring of smooth functions $X \longrightarrow \mathbb{R}$ has the structure of a C^∞-ring. (A function $f: X \to \mathbb{R}$ is smooth if there is an open $U \supseteq X$ and a smooth $g: U \to \mathbb{R}$ extending f.) This C^∞-ring is denoted by $C^\infty(X)$. Its C^∞-structure is simply given by composition: $\mathbb{R}^n \xrightarrow{h} \mathbb{R}^m$ is interpreted by the map $C^\infty(X)^n \longrightarrow C^\infty(X)^m$ sending $(f_1, \ldots, f_n)$ to $h \circ (f_1, \ldots, f_n)$. The construction of the ring $C^\infty(X)$ is functorial in X. That is, if $\varphi: X \to Y$ is a smooth map (i.e. if $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, φ is the restriction of a smooth function $U \longrightarrow \mathbb{R}^m$ for an open $U \supseteq X$), we obtain a C^∞-homomorphism $C^\infty(Y) \longrightarrow C(X)$ by composition with φ.

C^∞-rings of this form $C^\infty(X)$, especially when X is a manifold, will play an important rôle in the sequel. For the special case $X =$

$\mathbb{R}^n$, we have

1.1 Proposition. *The ring $C^\infty(\mathbb{R}^n)$ of smooth functions in n variables is the free C^∞-ring on n generators, these generators being the projections.*

Proof. n elements $a_1, \ldots, a_n$ of an arbitrary C^∞-ring A determine a C^∞-homomorphism $\varphi: C^\infty(\mathbb{R}^n) \to A$ with $\varphi(\pi_1) = a_1, \ldots, \varphi(\pi_n) = a_n$, by

$$\varphi(f) = A(f)(a_1, \ldots, a_n).$$

This φ is unique, since any C^∞-homomorphism $\psi: C^\infty(\mathbb{R}^n) \to A$ has to make the diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}^n)^n \arrow[r, "\psi^n"] \arrow[d, "C^\infty(\mathbb{R}^n)(f)"'] & A^n \arrow[d, "A(f)"] \\
C^\infty(\mathbb{R}^n) \arrow[r, "\psi"'] & A
\end{tikzcd}$$

commute, so chasing $(\pi_1, \ldots, \pi_n) \in C^\infty(\mathbb{R}^n)^n$ around gives

$$A(f)(\psi(\pi_1), \ldots, \psi(\pi_n)) = \psi(f \circ (\pi_1, \ldots, \pi_n)) = \psi(f). \qquad \square$$

Other examples of C^∞-rings are the rings of germs of smooth functions. Let $p \in \mathbb{R}^n$. A (smooth) *germ* is an equivalence class of smooth functions $f: U \to \mathbb{R}$ defined on some open neighborhood U of p, two such functions $f: U \to \mathbb{R}$ and $g: V \to \mathbb{R}$ being equivalent if they coincide on a neighborhood of p. The equivalence class of an $f: U \to \mathbb{R}$ is denoted by $f|_p$, the "germ of f at p". The ring of germs at p which we denote by $C_p^\infty(\mathbb{R}^n)$, is actually a C^∞-ring, and the C^∞-ring structure is again simply induced by composition, just as in the cases $C^\infty(X)$.

By the smooth version of Tietze's extension theorem (i.e., the fact that any C^∞-function $f: X \to \mathbb{R}$ can be extended to a C^∞-function $f: \mathbb{R}^n \to \mathbb{R}$, for $X \subset \mathbb{R}^n$ closed) every germ at p is the germ of a function defined on all of $\mathbb{R}^n$, so $C_p^\infty(\mathbb{R}^n)$ is a quotient of $C^\infty(\mathbb{R}^n)$,

$$C_p^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)/m^g_{\{p\}}$$

where $m^g_{\{p\}}$ is the ideal of functions having zero germ at p, i.e. vanishing in a neighborhood of p.

This makes $C_p^\infty(\mathbb{R}^n)$ into a C^∞-ring by the following proposition, which is in some sense the basic fact making the theory of C^∞-rings manageable.

1.2 Proposition. *Any ideal I (in the ordinary ring-theoretic sense) in a C^∞-ring A is a C^∞-congruence. Thus, the canonical projection $p: A \to A/I$ induces a C^∞-ring structure on A/I making p into a C^∞-homomorphism.*

Proof. We have to show that if $a_i = b_i \bmod I$ for $i = 1, \ldots, n$ and $f: \mathbb{R}^n \to \mathbb{R}$ is smooth, then $A(f)(a_1, \ldots, a_n) = A(f)(b_1, \ldots, b_n) \bmod I$. But by Hadamard's lemma there are smooth functions $g_1, \ldots, g_n: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$f(x) - f(y) = \sum_{i=1}^{n} (x_i - y_i) g_i(x, y).$$

A preserves this equation, i.e.

$$\begin{aligned} &A(f)(a_1, \ldots, a_n) - A(f)(b_1, \ldots, b_n) = \\ &\textstyle\sum_{i=1}^n (a_i - b_i) A(g_i)(a_1, \ldots, a_n, b_1, \ldots, b_n) \in I. \end{aligned}$$

□

As another example of 1.2., we have a C^∞-structure on the quotient $C^\infty(\mathbb{R}^n)/m^\infty_{\{0\}}$, where $m^\infty_{\{0\}}$ is the ideal of functions which are flat at 0 (all partial derivatives vanish at 0). Thus $m^\infty_{\{0\}}$ is the kernel of the Taylor series expansion at 0

$$T_0: C^\infty(\mathbb{R}^n) \to \mathbb{R}[[X_1, \ldots, X_n]]$$

In fact we have the following result, where $m^\infty_{\mathbb{R}^n \times \{0\}}$ is the ideal of functions which are flat on $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^m$.

1.3 Borel's Theorem. *The Taylor series gives an isomorphism*

$$C^\infty(\mathbb{R}^{n+m})/m^\infty_{\mathbb{R}^n \times \{0\}} \xrightarrow{\sim} C^\infty(\mathbb{R}^n)[[Y_1, \ldots, Y_m]].$$

Proof. We have to show that the Taylor expansion by partial derivatives with respect to y, $T_0: C^\infty(\mathbb{R}^{n+m}) \to C^\infty(\mathbb{R}^n)[[Y_1, \ldots, Y_n]]$,

$$T_0 f(x, y) = \sum_\alpha D_y^\alpha(f)(x, 0) \cdot \frac{1}{\alpha!} Y^\alpha$$

is surjective (α ranges over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, and $Y^\alpha = Y_1^{\alpha_1} \cdot \ldots \cdot Y_m^{\alpha_m}$; $|\alpha|$ stands for $\alpha_1 + \ldots + \alpha_m$; and $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_m!$). So suppose we are given $g \in C^\infty(\mathbb{R}^n)[[Y_1, \ldots, Y_m]]$,

$g = \sum_\alpha g_\alpha(x) Y^\alpha$. We need to find a smooth $f(x,y)\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with $\frac{1}{\alpha!} D_y^\alpha(f)(x,0) = g_\alpha(x)$ for all α.

Take $\varphi\colon \mathbb{R}^m \to [0,1]$ with $\varphi(y) = 1$ if $|y| \le \frac{1}{2}$, $\varphi(y) = 0$ if $|y| \ge 1$ (φ smooth, of course). We wish to put

$$f(x,y) = \sum_\beta g_\beta(x) y^\beta \varphi(t_\beta \cdot y) \tag{1}$$

for a choice of $t_\beta \in \mathbb{R}$, $1 < t_\beta$ for each β, in such a way that the sum

$$\sum_\beta D^\gamma g_\beta(x) y^\beta \varphi(t_\beta \cdot y) \tag{2}$$

is uniformly convergent for each $\gamma = (\gamma_1, \dots, \gamma_{n+m})$. For in that case f is well-defined and smooth, and we can differentiate term by term. Moreover since $\varphi(t_\beta \cdot y) = 1$ on a neighbourhood of 0, $\frac{1}{\alpha!} D_y^\alpha(f)(x,0) = g_\alpha(x)$ for all $\alpha = (\alpha_1, \dots, \alpha_m)$.

To see that such a choice of t_β can indeed be made, write the β-th term in (1) as

$$\left(\frac{1}{t_\beta}\right)^{|\beta|} g_\beta(x)(t_\beta \cdot y)^\beta \varphi(t_\beta \cdot y),$$

and put

$$\psi_\beta(y) = y^\beta \varphi(y).$$

Then ψ_β has compact support, so

$$M_\beta = \max_{|\gamma|<|\beta|} |D^\gamma(g_\beta(x) \cdot \psi_\beta(y))|$$

exists (here $\gamma = (\gamma_1, \dots, \gamma_{n+m}), \beta = (\beta_1, \dots, \beta_m)$). Since $t_\beta > 1$, we have for $|\gamma| < |\beta|$

$$\begin{aligned} |D^\gamma g_\beta(x) y^\beta \varphi(t_\beta y)| &\le \left| \left(\tfrac{1}{t_\beta}\right)^{|\beta|} D^\gamma(g_\beta(x)\psi_\beta(t_\beta y)) \right| \\ &\le \left(\tfrac{1}{t_\beta}\right)^{|\beta|} \cdot t_\beta^{|\gamma|} \cdot M_\beta < M_\beta / t_\beta \end{aligned}$$

Now choose $\varepsilon_\beta > 0$ so small that $\sum_\beta \varepsilon_\beta < \infty$, and choose $t_\beta > M_\beta/\varepsilon_\beta$. Then (2) is eventually dominated by $\sum \varepsilon_\beta$. □

We thus conclude by 1.2 that the ring of formal power series $R[[X_1, \dots, X_n]]$ is a C^∞-ring. (Exercise: use the chain-rule to give an explicit description of the C^∞-ring structure on $\mathbb{R}[[X_1, \dots, X_n]]$.)

The ring of dual numbers $\mathbb{R}[\varepsilon] = \mathbb{R}[X]/(X^2)$ is also a C^∞-ring. This follows again from 1.2, since

$$\mathbb{R}[\varepsilon] = C^\infty(\mathbb{R})/(x^2)$$

by Hadamard's lemma: if $g \in C^\infty(\mathbb{R})$, $g(x) = g(0) + xg'(0) + x^2h(x)$ for some smooth h, so modulo $(x^2), g$ is polynomial. The C^∞-structure on $\mathbb{R}[\varepsilon]$ is also easily described explicitly: if $f: \mathbb{R}^n \to \mathbb{R}$, its interpretation $\mathbb{R}[\varepsilon](f): \mathbb{R}[\varepsilon]^n \to \mathbb{R}[\varepsilon]$ is the map

$$(a_1 + b_1\varepsilon, \ldots, a_n + b_n\varepsilon) \mapsto f(a_1, \ldots, a_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) \cdot b_i$$

$\mathbb{R}[\varepsilon]$ is an example of a *Weil algebra.* As we shall see below, all Weil algebras have in fact a canonical C^∞-ring structure.

Not all the C^∞-rings $C^\infty(X)$ with $X \subseteq \mathbb{R}^n$ that we mentioned above are quotients of $C^\infty(\mathbb{R}^n)$. A (smooth) function on X extends to a function on $\mathbb{R}^n$ if and only if X is closed. (In one direction this is just smooth Tietze; and if X is not closed, choose a sequence $x_n \in X$, $x_n \to p \notin X$, and a smooth function f on X with $f(x_n) \to \infty$.) So for *closed* X we have

$$C^\infty(X) \cong C^\infty(\mathbb{R}^n)/m_X^0,$$

where m_X^0 is the ideal of functions vanishing on X. If $X \subseteq \mathbb{R}^n$ is *locally closed*, that is $X = F \cap U$ with F closed and U open (this is equivalent to X being locally compact), $C^\infty(X)$ is still finitely generated, but we have to increase the dimension and pass to $\mathbb{R}^{n+1}$. To show this, we need

1.4 Lemma. *Each open set $U \subseteq \mathbb{R}^n$ has a characteristic function, that is, U is of the form $f^{-1}(\mathbb{R} - \{0\})$ for some smooth $f: \mathbb{R}^n \to \mathbb{R}$.*

Proof. For basic open n-cubes $U = (a_1, b_1) \times \ldots \times (a_n, b_n)$ this is no problem (see GP, p. 7). If U is arbitrary, write $U = \bigcup_{m \in \mathbb{N}} U_m$, where the U_m are open n-cubes, and let f_m be a characteristic function for U_m. We will put

$$f(x) = \sum_{m=0}^{\infty} f_m(x) \cdot \varepsilon_m$$

where the $\varepsilon_m > 0$ are chosen small enough so as to make f smooth. For example, take ε_m such that $D^\alpha(\varepsilon_m \cdot f_m) \leq 2^{-m}$ for all α with $|\alpha| \leq m$ (this can be done since the f_m have compact support). Then as in the proof of 1.3, the sequence $\{f_m(x) \cdot \varepsilon_m\}$ as well as the sequences of term-by-term derivatives $\{D^\alpha(f_m(x) \cdot \varepsilon_m)\}$ are eventually dominated by $\sum 2^{-m}$, so f is well-defined and smooth. $\square$

Consequently, an open subspace $U \subseteq \mathbb{R}^n$ is diffeomorphic to a closed subspace $\hat{U} = \{(x, y) | y \cdot f(x) = 1\} \subseteq \mathbb{R}^{n+1}$, where f is a characteristic function for U. And hence any locally closed $X \subseteq \mathbb{R}^n$ is diffeomorphic to a closed subspace of $\mathbb{R}^{n+1}$, so $C^\infty(X)$ is finitely generated.

Finitely generated C^∞-rings, i.e. rings of the form $C^\infty(\mathbb{R}^n)/I$, are convenient to deal with, because C^∞-homomorphisms between such rings can be described explicitly. If

$$\Phi: C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$$

is a C^∞-homomorphism, $\Phi(\pi_1), \ldots, \Phi(\pi_n)$ give a smooth map $\varphi: \mathbb{R}^m \to \mathbb{R}^n$, and $\Phi(f) = f \circ \varphi$ (modulo the ideals). Thus C^∞-homomorphisms

$$C^\infty(\mathbb{R}^n)/I \longrightarrow C^\infty(\mathbb{R}^m)/J$$

are in 1-1 correspondence with equivalence classes of smooth maps $\varphi: \mathbb{R}^m \to \mathbb{R}^n$ such that

$$I \subseteq \varphi_*(J) = \{f \in C^\infty(\mathbb{R}^n) | f \circ \varphi \in J\},$$

while two such φ and φ' are equivalent if for $i = 1, \ldots, n$,

$$\pi_i \circ \varphi - \pi_i \circ \varphi' \in J.$$

Note that $\varphi_*(J)$ is an ideal if J is. The condition $I \subseteq \varphi_*(J)$ can equivalently be formulated as $\varphi^*(I) \subseteq J$, where $\varphi^*(I)$ is the ideal generated by $\{f \circ \varphi | f \in I\}$.

In the particular case where $I = m_X^0$ and $J = m_Y^0$ for closed $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, this correspondence comes down to a natural bijection between C^∞-homomorphisms $C^\infty(X) \longrightarrow C^\infty(Y)$ and smooth maps $Y \to X$. Using the remark on locally closed X made above, we can reformulate this as follows.

1.5 Proposition. *Let $\mathbb{E}$ be the category of locally closed subspaces of some $\mathbb{R}^n$ and smooth maps. Then the contravariant functor from $\mathbb{E}$ into finitely generated C^∞-rings*

$$X \mapsto C^\infty(X)$$

is full and faithful. □

Our next aim is to discuss limits and colimits of C^∞-rings. Suppose we are given a diagram $(A_i)_i$ of C^∞-rings, i.e. of finite product preserving functors $C^\infty \to \mathit{Sets}$ and some natural transformation between them. As easily seen, the "point-wise" constructed inverse limit

$$B : C^\infty \to \mathit{Sets}, \qquad B(\mathbb{R}^n) = \varprojlim_i A_i(\mathbb{R}^n),$$

is again finite product preserving, and is the inverse limit of $(A_i)_i$ in the category of C^∞-rings. In other words,

(1) *Inverse limits of C^∞-rings are computed as inverse limits of their underlying sets.*

A first try to construct $\varinjlim A_i$ would be to take the point-wise colimit in *Sets*, i.e. let $D: \overrightarrow{C^\infty} \to$ *Sets* be the functor $D(\mathbb{R}^n) = \varinjlim_i A_i(\mathbb{R}^n)$. D need not preserve finite products in general, but it does if the diagram $(A_i)_i$ of C^∞-rings is *directed* (since directed colimits commute with finite limits), and in this case D is the colimit of the A_i in the category of C^∞-rings, i.e.

(2) *Directed colimits of C^∞-rings are computed as colimits of their underlying sets.*

Next, let us look at binary coproducts. If A and B are C^∞-rings, we write $A \otimes_\infty B$ for the coproduct, and $A \xrightarrow{i_A} A \otimes_\infty B \xleftarrow{i_B} B$ for the canonical inclusions. Note first that it follows from the universal property defining the coproduct, that if $I \subset A$ and $J \subset B$ are ideals,

$$A/I \otimes_\infty B/J \cong (A \otimes_\infty B)/(I, J),$$

where (I, J) is the ideal generated by $i_A(I) \cup i_B(J)$. Also, since $C^\infty(\mathbb{R}^n)$ is free on n generators,

$$C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R}^m) \cong C^\infty(\mathbb{R}^n \times \mathbb{R}^m),$$

and the coproduct inclusions come from the projections $\mathbb{R}^n \xleftarrow{\pi_1} \mathbb{R}^n \times \mathbb{R}^m \xrightarrow{\pi_2} \mathbb{R}^m$. Thus

(3) $$C^\infty(\mathbb{R}^n)/I \otimes_\infty C^\infty(\mathbb{R}^m)/J \cong C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/(I, J),$$

and $(I, J) = (I \circ \pi_1 + J \circ \pi_2)$.

Since every C^∞-ring is the directed colimit of finitely generated (even finitely presented) ones ("generators and relations"), (2) and (3) enable us to compute any colimit of C^∞-rings explicitly.

For example, (2) implies that the ring of germs $C_p^\infty(\mathbb{R}^n)$ is isomorphic to $\varinjlim_{p \in U} C^\infty(U)$, the colimit over the diagram consisting of rings $C^\infty(U)$ for open neighbourhoods U of p and restriction maps $C^\infty(U) \longrightarrow C^\infty(V)$ whenever $V \subseteq U$.

The free C^∞-ring on countably many generators is the colimit of the sequence

$$C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2) \to \ldots \to C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n+1}) \to \ldots$$

the maps being induced by the projections $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ on the first n coordinates. Let us write $C^\infty(\mathbb{R}^{<\omega})$ for this colimit. So an element of $C^\infty(\mathbb{R}^{<\omega})$ is a function $f: \mathbb{R}^\omega \to \mathbb{R}$ which depends only on the first n coordinates, for some n, and moreover this dependence is smooth.

We finish this section by discussing the problem of universally inverting elements of C^∞-rings. Given a C^∞-ring A and a set Σ of elements of A, we want to construct a C^∞-homomorphism $A \xrightarrow{\eta} A\{\Sigma^{-1}\}$, where $A\{\Sigma^{-1}\}$ is the universal solution to inverting all the elements of Σ; i.e. each $\eta(a)$, $a \in \Sigma$, is invertible, and for any C^∞-homomorphism $\varphi: A \to B$ such that $\varphi(a)$ is invertible for all $a \in \Sigma$, there is a unique C^∞-homomorphism ψ making the diagram below commute.

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & A\{\Sigma^{-1}\} \arrow[d, "\psi"] \\ & B \end{tikzcd}$$

Let us sketch the construction of $A\{\Sigma^{-1}\}$. Clearly, $A\{\Sigma^{-1}\}$ can be constructed as the directed colimit of the rings $A\{\Delta^{-1}\}$ for Δ a finite subset of Σ. And $A\{\{a,b\}^{-1}\} \cong A\{(a \cdot b)^{-1}\}$, so we may restrict ourselves to the case that Δ is a singleton. Writing A as a colimit of finitely generated rings A_i (all containing a) we have

$$A\{a^{-1}\} = \varinjlim A_i\{a^{-1}\}.$$

But for finitely generated rings $C^\infty(\mathbb{R}^n)/I$ we have an explicit description: for general reasons, if $C^\infty(\mathbb{R}^n) \xrightarrow{\eta} C^\infty(\mathbb{R}^n)\{a^{-1}\}$ has been constructed, then $(C^\infty(\mathbb{R}^n)/I)\{a^{-1}\} \cong C^\infty(\mathbb{R}^n)\{a^{-1}\}/(\eta(I))$, and for the case $A = C^\infty(\mathbb{R}^n)$ we have

1.6 Proposition. *Let $a \in C^\infty(\mathbb{R}^n)$, and let $U = a^{-1}(\mathbb{R}-\{0\})$. Then*

$$C^\infty(\mathbb{R}^n)\{a^{-1}\} \cong C^\infty(U).$$

The reader may like to check directly that $C^\infty(U)$ has the universal property required of $C^\infty(\mathbb{R}^n)\{a^{-1}\}$ (hint: use partitions of unity), but we will derive this proposition from a more general result in Section 2 (cf. Corollary 2.2).

Note that it follows from 1.6 that rings of germs can be constructed by inverting elements: if $p \in \mathbb{R}^n$ and $\Sigma = \{f | f(p) \neq 0\}$, then

$$C_p^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)\{\Sigma^{-1}\}.$$

2 *Manifolds as C^∞-rings*

By a manifold we always mean a smooth manifold with a countable basis. In particular, every manifold is paracompact and can be embedded as a (closed) subspace of $\mathbb{R}^m$ for some m.

A C^∞-ring A is ***finitely presented*** if it is isomorphic to one of the form $C^\infty(\mathbb{R}^m)/I$ for some m and some ***finitely generated*** ideal I. Our first aim in this section is to show that if M is a manifold, $C^\infty(M)$ is a finitely presented C^∞-ring. We begin by taking a closer look at rings of the form $C^\infty(U)$, where $U \subseteq \mathbb{R}^n$ is open.

2.1 Lemma. *Let M be a manifold, and suppose the smooth functions $g_1, \ldots, g_n : M \to \mathbb{R}$ are independent, i.e. for each common zero point $x \in (g_1, \ldots, g_n) = \{x | g_i(x) = 0,\ i = 1, \ldots, n\}$ the linear map $(dg_{1x}, \ldots, dg_{nx})$: $T_x(M) \to \mathbb{R}^n$ is a surjection. Then the ideal $(g_1, \ldots, g_n)$ coincides with $m^0_{Z(g_1,\ldots,g_n)}$.*

Proof. One inclusion is clear. For the other, let us first note that for an arbitrary function $h \in C^\infty(M), h \in (g_1, \ldots, g_n)$ iff for each $x \in M$ there is an open neighbourhood U of x such that $h|U \in (g_1, \ldots, g_n)|U$ (the ideal in $C^\infty(U)$ generated by the $g_i|U$). To see this, suppose we indeed have a cover of M by open U's such that $h|U = \varphi_U^1 \cdot g_1|U + \ldots + \varphi_U^n \cdot g_n|U$ for some smooth φ_U^i. We may assume that this cover is locally finite, so there is a partition of unity $\{\rho_U\}_U$ subordinate to it. Then

$$\begin{aligned} h = \textstyle\sum_U \rho_U \cdot h &= \textstyle\sum_U \rho_U \cdot (\varphi_U^1 \cdot g_1 + \ldots + \varphi_U^n \cdot g_n) \\ &= \textstyle\sum_{i=1}^n g_i \cdot \sum_U \rho_U \cdot (\varphi_U^1 + \ldots + \varphi_U^n) \in (g_1, \ldots, g_n). \end{aligned}$$

(Later on, we will express this by saying that $(g_1, \ldots, g_n)$ is a "germ determined" ideal.)

Now take $h \in C^\infty(M)$ with $h|Z(g_1, \ldots, g_n) = 0$, and choose $x \in M$. If $x \notin Z(g_1, \ldots, g_n)$, then trivially there is a neighbourhood U of X such that $h|U \in (g_1, \ldots, g_n)|U$. If $x \in Z(g_1, \ldots, g_n)$, then by hypothesis $g = (g_1, \ldots, g_n): M \to \mathbb{R}^n$ is a submersion at x, so by the local submersion theorem (see GP, p. 20) there exist local coordinates $U \xrightarrow{\varphi} M$ and $V \xrightarrow{\psi} \mathbb{R}^n$, where $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open neighbourhoods of the origin, $\varphi(0) = x$, $\psi(0) = g(x)$, such that

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\varphi"'] & \mathbb{R}^n \arrow[d, "\psi"] \\ U \arrow[r, "p"'] & V \end{tikzcd}$$

commutes, where $p(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$. We claim that $h|\varphi(U) \in (g_1, \ldots, g_n)|\varphi(U)$. Indeed, replacing h by $h \circ \varphi = k$, it suffices to note that if $k: U \to \mathbb{R}$ and $h|\{0\} \times \mathbb{R}^{m-n} = 0$, then $h \in (\pi_1, \ldots, \pi_n)$. But this is just Hadamard's lemma again: we can write

$$k(x_1, \ldots, x_m) = k(0, \ldots, 0, x_{n+1}, \ldots, x_m) + \sum_{i=1}^{n} x_i v_i(x_1, \ldots, x_m)$$

for some smooth v_i, and the first term vanishes. □

An immediate consequence of this lemma is Proposition 1.6 of the previous section. For purely algebraic reasons (cf. 1.1), we have $C^\infty(\mathbb{R}^n)\ \{a^{-1}\} \cong C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1)$, so it suffices to state

2.2 Corollary. *Let $U \subseteq \mathbb{R}^n$ be open, and let a be a characteristic function for U. Then*

$$C^\infty(U) \cong C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1).$$

Proof. As we have seen in Section 1 (see 1.4 and the remarks following),

$$C^\infty(U) \cong C^\infty(\hat{U}) \cong C^\infty(\mathbb{R}^{n+1})/m^0_{\hat{U}}.$$

But by 2.2, $m^0_{\hat{U}} = (y \cdot a(x) - 1)$. □

2.3 Theorem. *For every manifold $M, C^\infty(M)$ is finitely presented.*

Proof. If $M \subseteq \mathbb{R}^n$, then by the ε-Neighbourhood Theorem (see GP,

p. 69-70) there is an open neighbourhood $U \subseteq \mathbb{R}^n$ of M and a smooth retraction $U \xrightarrow{r} M$. Thus $C^\infty(M)$ is a retract of $C^\infty(U)$. But just as in ordinary commutative algebra, one can show easily that retractions of finitely presented C^∞-rings are finitely presented. □

Coproducts of C^∞-rings of the form $C^\infty(M)$, for M a manifold, are easily calculated. We do this in two steps. For open subspaces of some $\mathbb{R}^n$ we have

2.4 Lemma. *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. Then*

$$C^\infty(U) \otimes_\infty C^\infty(V) \cong C^\infty(U \times V).$$

Proof. This follows immediately from the universal properties defining the coproduct and the rings $C^\infty(U)$ and $C^\infty(V)$ (see 1.6). Let $a(x_1): \mathbb{R}^n \to \mathbb{R}$ and $b(x_2): \mathbb{R}^m \to \mathbb{R}$ be characteristic functions for U and V. Then

$$\begin{aligned}
&C^\infty(U) \otimes_\infty C^\infty(V)\\
&\cong C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x_1) - 1) \otimes_\infty C^\infty(\mathbb{R}^{m+1})/(z \cdot b(x_2) - 1)\\
&\cong C^\infty(\mathbb{R}^{n+m+2})/(y \cdot a(x_1) - 1, z \cdot b(x_2) - 1)\\
&\cong C^\infty(\mathbb{R}^{n+m})\{a(x_1)^{-1}, b(x_2)^{-1}\}\\
&\cong C^\infty(\mathbb{R}^{n+m})\{(a(x_1) \cdot b(x_2))^{-1}\}\\
&\cong C^\infty(U \times V),
\end{aligned}$$

since $a(x_1) \cdot b(x_2)$ is a characteristic function for $U \times V$. □

2.5 Proposition. *Let M_1 and M_2 be manifolds. Then*

$$C^\infty(M_1) \otimes_\infty C^\infty(M_2) \cong C^\infty(M_1 \times M_2).$$

Proof. This follows from 2.4 for purely categorical reasons. Let $M_1 \subseteq \mathbb{R}^{n_1}$ and $M_2 \subseteq \mathbb{R}^{n_2}$. As in the proof of 2.3 there are open neighbourhoods $U_1 \supseteq M_1$ and $U_2 \supseteq M_2$, and retractions $r_1: U_1 \to M_1$ and $r_2: U_2 \to M_2$. The maps r_1 and r_2 induce C^∞-homomorphisms $\eta_j: C^\infty(M_j) \to C^\infty(U_j)$, and we have restriction homomorphisms $\rho_j: C^\infty(U_j) \to C^\infty(M_j)$, with $\rho_j \circ \eta_j = \mathrm{id}$. Similarly, from the maps $M_1 \times M_2 \leftrightarrows U_1 \times U_2$ we have

$$C^\infty(U_1 \times U_2) \underset{\eta}{\overset{\rho}{\rightleftarrows}} C^\infty(M_1 \times M_2)$$

with $\rho\eta = 1$. So we get a diagram

$$\begin{tikzcd}
C^\infty(U_1) \arrow[r, "i_{U_1}"] \arrow[d, shift right, "\rho_1"'] & C^\infty(U_1 \times U_2) \arrow[d, shift right, "\rho"'] & C^\infty(U_2) \arrow[l, "i_{U_2}"'] \arrow[d, shift right, "\rho_2"'] \\
C^\infty(M_1) \arrow[r, "i_{M_1}"'] \arrow[u, shift right, "\eta_1"'] & C^\infty(M_1 \times M_2) \arrow[u, shift right, "\eta"'] & C^\infty(M_2) \arrow[l, "i_{M_2}"] \arrow[u, shift right, "\eta_2"']
\end{tikzcd}$$

(the horizontal inclusions being induced by the projections). Now from the fact that the upper horizontal part is a coproduct (2.4) the same follows for the lower part: consider C^∞-homomorphisms $\varphi_1: C^\infty(M_1) \to A$ and $\varphi_2: C^\infty(M_2) \to A$, A any C^∞-ring, and find a unique $\xi: C^\infty(U_1 \times U_2) \to A$ such that $\xi \circ i_{U_j} = \varphi_j \circ \rho_j$. Let $\varsigma = \xi \circ \eta$. Then $\varsigma \circ i_{M_1} = \xi \circ \eta \circ i_{M_1} = \xi \circ i_{U_1} \circ \eta_1 = \varphi_1 \circ \rho_1 \circ \eta_1 = \varphi_1$, and similarly $\varsigma \circ i_{M_2} = \varphi_2$.

An equally easy diagram argument shows that this ς is the unique one with $\varsigma \circ i_{M_j} = \varphi_j$. □

If $M \xrightarrow{f} N$ is a smooth map of manifolds and Z is a submanifold of N such that f is transversal to Z (written $f \pitchfork Z$), then $f^{-1}(Z)$ is a submanifold of M. This situation is "preserved" by the contravariant functor $M \mapsto C^\infty(M)$ of manifolds into C^∞-rings:

2.6 Proposition. *Let $f: M \to N$ be transversal to the submanifold $Z \subseteq N$. Then from the pullback diagram of manifolds*

$$\begin{tikzcd}
Z \arrow[r, tail] & N \\
f^{-1}(Z) \arrow[r, tail] \arrow[u] & M \arrow[u, "f"']
\end{tikzcd}$$

we obtain a pushout diagram of C^∞-rings

$$\begin{array}{ccc} C^\infty(N) & \longrightarrow & C^\infty(M) \\ \downarrow & & \downarrow \\ C^\infty(Z) & \longrightarrow & C^\infty(f^{-1}(Z)) \end{array}$$

In the proof of 2.6 we will use the following

2.7 Lemma. *Let $B = C^\infty(\mathbb{R}^k)/(f_1, \ldots, f_p)$ be a finitely presented C^∞-ring, and let $\{W_\alpha\}$ be a cover of $Z(f_1, \ldots, f_p)$ by open sets (in $\mathbb{R}^k$). Then*

$$B \cong \varprojlim B_{\alpha_1,\ldots,\alpha_n},$$

where for each finite set $\{\alpha_1, \ldots, \alpha_n\}, W_{\alpha_1,\ldots,\alpha_n} = W_{\alpha_1} \cap \ldots \cap W_{\alpha_n}$ and $B_{\alpha_1,\ldots,\alpha_n} = C^\infty(W_{\alpha_1,\ldots,\alpha_n})/(f_1|W_{\alpha_1,\ldots,\alpha_n}, \ldots, f_p|W_{\alpha_1,\ldots,\alpha_n})$, and the inverse limit is taken over the diagram of restrictions $B_{\alpha_1,\ldots,\alpha_n} \to B_{\alpha'_1,\ldots,\alpha'_m}$ for $\{\alpha_1, \ldots, \alpha_n\} \subset \{\alpha'_1, \ldots, \alpha'_m\}$.

Proof. It suffices to show that if we are given elements $g_\alpha \in C^\infty(W_\alpha)$ which are compatible in the sense that for all α, β,

$$g_\alpha|W_{\alpha\beta} - g_\beta|W_{\alpha\beta} \in (f_1|W_{\alpha\beta}, \ldots, f_p|W_{\alpha\beta})$$

then there exists a unique (modulo $(f_1, \ldots, f_p)$) $g \in C^\infty(\mathbb{R}^k)$ such that for each α

$$g|W_\alpha - g_\alpha \in (f_1|W_\alpha, \ldots, f_p|W_\alpha).$$

By adding $\mathbb{R}^k - Z(f_1, \ldots, f_p)$ to the cover $\{W_\alpha\}_\alpha$ and going to an appropriate refinement, we may assume that the $\{W_\alpha\}$ form a locally finite cover of $\mathbb{R}^k$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{W_\alpha\}$, and let $g = \sum_\alpha \rho_\alpha g_\alpha$. To show that one has $g|W_\alpha - g_\alpha \in (f_1|W_\alpha, \ldots, f_p|W_\alpha)$, it suffices as in the proof of Lemma 2.1 to prove that each point $x \in W_\alpha$ has a neighbourhood $V_x \subset W_\alpha$ such that

$$g|V_x - g_\alpha|V_x \in (f_1|V_x, \ldots, f_p|V_x).$$

But if V_x is chosen such that only $\rho_{\alpha_1}, \ldots, \rho_{\alpha_n}$ do not vanish on V_x, then $\sum_{i=1}^n \rho_{\alpha_i} = 1$ on V_x, so

$$g|V_x - g_\alpha|V_x = \sum_{i=1}^n \rho_{\alpha_i}(g_{\alpha_i} - g_\alpha) \in (f_1|V_x, \ldots, f_p|V_x)$$

by compatibility of the family $\{g_\alpha\}$. To show that g is unique, suppose g' also satisfies the requirements. Then $(g - g')|W_\alpha \in (f_1|W_\alpha, \ldots, f_p|W_\alpha)$ for each α, hence by the argument of the first part of the proof of lemma 2.1, $g - g' \in (f_1, \ldots, f_p)$. $\square$

Proof of Proposition 2.6. We claim that Lemma 2.7 implies that it suffices to show that there exists an open cover $\{U_\alpha\}$ of N such that for all finite $A = \{\alpha_1, \ldots, \alpha_n\}$, the diagram

$$(*)\qquad \begin{array}{ccc} C^\infty(U_A) & \xrightarrow{-\circ f} & C^\infty(V_A) \\ \downarrow & & \downarrow \\ C^\infty(U_A \cap Z) & \longrightarrow & C^\infty(V_A \cap f^{-1}(Z)) \end{array}$$

is a pushout, where $U_A = U_{\alpha_1} \cap \ldots \cap U_{\alpha_n}, V_A = f^{-1}(U_A)$. Indeed, suppose that this is the case, and that we are given C^∞-homomorphisms $\varphi: C^\infty(M) \to B$ and $\psi: C^\infty(Z) \to B$ such that $\varphi(g \circ f) = \psi(g|Z)$ for each $g \in C^\infty(N)$. Since $C^\infty(M)$, $C^\infty(Z)$ are finitely presented, we may without loss assume that B is also finitely presented, say $B = C^\infty(\mathbb{R}^k)/(f_1, \ldots, f_p)$. The cover $\{U_\alpha\}$ of N induces (using 1.6) an open cover $\{W_\alpha\}$ of $Z(f_1, \ldots, f_p)$ such that φ, ψ restrict to C^∞-homomorphisms

$$\varphi_A: C^\infty(V_A) \to C^\infty(W_A)/(f_1|W_A, \ldots, f_p|W_A)$$
$$\psi_A: C^\infty(U_A \cap Z) \to C^\infty(W_A)/(f_1|W_A, \ldots, f_p|W_A).$$

Since all the squares in (*) are pushouts, we obtain unique factorizations

$$\xi_i: C^\infty(V_A \cap f^{-1}(Z)) \to C^\infty(W_A)/(f_1|W_A, \ldots, f_p|W_A),$$

which by Lemma 2.7 can be put together to produce the required

$$\xi: C^\infty(f^{-1}(Z)) \to B.$$

So let us now show that such a cover of N indeed exists.

Let Z have codimension q. By the local immersion theorem, we can cover N by neighbourhoods U_α such that $Z \cap U_\alpha$ is cut out by independent functions $g_1, \ldots, g_q: U_\alpha \to \mathbb{R}$, i.e.

$$Z \cap U_\alpha = Z(g_1, \ldots, g_q).$$

Since $f \pitchfork Z$, the functions $g_i \circ f: V_\alpha = f^{-1}(U_\alpha) \to \mathbb{R}$ are also independent, so by Lemma 2.1 we conclude that

$$\begin{tikzcd}
C^\infty(U_\alpha) \arrow[r] \arrow[d] & C^\infty(V_\alpha) \arrow[d] \\
C^\infty(U_\alpha \cap Z) \arrow[r] & C^\infty(V_\alpha \cap f^{-1}(Z))
\end{tikzcd}$$

is the same as

$$\begin{tikzcd}
C^\infty(U_\alpha) \arrow[r, "- \circ f"] \arrow[d] & C^\infty(V_\alpha) \arrow[d] \\
C^\infty(U_\alpha)/(g_1, \ldots, g_q) \arrow[r] & C^\infty(V_\alpha)/(g_1 \circ f, \ldots, g_q \circ f)
\end{tikzcd}$$

which is obviously a pushout.

Since Lemma 2.1 can similarly be applied to open subsets of some U_α, we conclude that for this cover $\{U_\alpha\}$, all the squares in (*) are pushouts. □

Two maps $M_1 \xrightarrow{f_1} N$ and $M_2 \xrightarrow{f_2} N$ are called transversal ($f_1 \pitchfork f_2$) if for each $x_1 \in M_1$ and $x_2 \in M_2$ with $f(x_1) = y = f(x_2)$, $im(df_{1x_1})$ and $im(df_{2x_2})$ span $T_y(N)$. This is equivalent to saying that the map $f_1 \times f_2 \colon M_1 \times M_2 \to N \times N$ is transversal to the diagonal $\Delta \subseteq N \times N$. Since $(f_1 \times f_2)^{-1}(\Delta)$ is the pullback of f_1 and f_2, it follows from 2.5 and 2.6 that the pullback of such a pair $M_i \xrightarrow{f_i} N$ with $f_1 \pitchfork f_2$ ("a *transversal pullback*") is mapped into a pushout of C^∞-rings. (Conversely, preservation of transversal pullbacks implies 2.5 (take N = one point) and 2.6 (take f_2 the inclusion).)

Let us summarize, then, what we have obtained. Let $\mathbb{M}$ be the category of manifolds and smooth maps.

2.8 Theorem. *The contravariant functor*

$$\mathbb{M} \to (C^\infty\text{-}rings),\ M \mapsto C^\infty(M)$$

is full and faithful (cf. 1.5), maps into finitely presented C^∞-rings, and sends transversal pullbacks to pushouts. □

3 Local C^∞-rings

Let us remind the reader of some notions from commutative algebra. A *local ring* is a non-trivial ($0 \neq 1$) ring A which has the property that for all $a, b \in A$, if $a + b = 1$ then either a or b is invertible. This is equivalent to the existence of a unique maximal ideal $m = m_A$ in A. A/m is a field, called the *residue field* of A. A homomorphism $\varphi: A \rightarrow B$ between local rings is called *local* if φ reflects invertibility: $\varphi(a)$ invertible in B implies a invertible in A, or $\varphi(m_A) \subseteq m_B$.

We will start this section by taking a look at local C^∞-rings. A very important example of a local C^∞-ring is the ring of germs $C_p^\infty(\mathbb{R}^n)$. Another local C^∞-ring is the formal power series ring $\mathbb{R}[[X_1, \ldots, X_n]]$. This may be checked directly, but it is easier to observe that the Taylor expansion at zero (cf. 1.3) factors through $C_0^\infty(\mathbb{R}^n)$, i.e. we have a quotient map

$$C_0^\infty(\mathbb{R}^n) \twoheadrightarrow \mathbb{R}[[X_1, \ldots, X_n]].$$

and then use

3.1 Lemma. *If $\varphi: A \rightarrow B$ is a surjective homomorphism of non-trivial C^∞-rings and A is local, then so is B and φ is a local homomorphism.*

Proof. This is a purely algebraic matter: we can write $B = A/I$, and ideals in B are in 1-1 correspondence with ideals in A which contain I. So if A has a unique maximal ideal, so has A/I, and this is preserved by the projection $A \longrightarrow A/I$. □

We shall see shortly that it is not true that every finitely generated local C^∞-ring is a quotient of a ring of the form $C_p^\infty(\mathbb{R}^n)$.

Note that inverse limits of local C^∞-rings need not be local. It is easy commutative algebra to check that *directed* colimits of local C^∞-rings are again local. But it is not true that all colimits of local C^∞-rings are local:

3.2 Example. If A and B are local, $A \otimes_\infty B$ need not be. Let $\mathcal{F}$ be a non-principle ultrafilter on $\mathbb{N}$, and consider the prime ideal $I = \{f | Z(f) \in \mathcal{F}\}$ in $C^\infty(\mathbb{N})$. $C^\infty(\mathbb{N})/I$ is local, but the ring

$C^\infty(\mathbb{N})/I \otimes_\infty C^\infty(\mathbb{N})/I$ is not: write

$$\begin{aligned} C^\infty(\mathbb{N})/I \otimes_\infty C^\infty(\mathbb{N})/I &\cong C^\infty(\mathbb{N}\times\mathbb{N})/(I(x), I(y)) \\ &\cong C^\infty(\mathbb{N}\times\mathbb{N})/J, \end{aligned}$$

where $J = \{f \mid \exists A,\ B \in \mathcal{F}\ f|A \times B = 0\}$. Now choose $0 < \varepsilon < 1$ and let

$$U = \{(x,y) \in \mathbb{R}^2 \mid x > y - \varepsilon\}$$

$$V = \{(x,y) \in \mathbb{R}^2 \mid x < y + \varepsilon\},$$

and choose characteristic functions $u, v\colon \mathbb{R}^2 \to [0,1]$ for U and V, respectively. Then $u + v$ is invertible in $C^\infty(\mathbb{N}\times\mathbb{N})/J$, but neither u nor v is. For example, if $u \cdot h = 1 \bmod J$ for some h, then u would have to be invertible on a set of the form $A \times B$ with $A, B \in \mathcal{F}$, which is clearly not the case.

For rings of germs, coproducts are better behaved:

3.3 Proposition. *$C_p^\infty(\mathbb{R}^n) \otimes_\infty C_q^\infty(\mathbb{R}^m) \cong C_{(p,q)}^\infty(\mathbb{R}^{n+m})$, for every two points $p \in \mathbb{R}^n, q \in \mathbb{R}^m$.*

Proof. From the formula (3) of Section 1 for coproducts, we see that we have to show

$$\left(m_p^g, m_q^g\right) = m_{(p,q)}^g.$$

$\subseteq$ is clear. For $\supseteq$, suppose we have $f\colon \mathbb{R}^{n+m} \to \mathbb{R}$ with $f|U \times V = 0$ for open sets $U \ni p$ and $V \ni q$. Let u and v be characteristic functions for U and V. Then $u \cdot v \cdot f = 0$ in $C^\infty(\mathbb{R}^{n+m})$. But u and v are invertible in $C^\infty(\mathbb{R}^{n+m})/(m_p^g, m_q^g)$, so $f = 0$ in $C^\infty(\mathbb{R}^{n+m})/(m_p^g, m_q^g)$, i.e. $f \in (m_p^g, m_q^g)$. □

3.4 Corollary. *There exists a finitely generated local C^∞-ring which is not a quotient of a ring of germs.*

Proof. The ring $C^\infty(\mathbb{N})/I$ of 3.2 is finitely generated. If we had a surjection $C_p^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{N})/I$, then the map

$$C_p^\infty(\mathbb{R}^n) \otimes_\infty C_p^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{N})/I \otimes_\infty C^\infty(\mathbb{N})/I$$

would also be surjective. But then $C^\infty(\mathbb{N})/I \otimes_\infty C^\infty(\mathbb{N})/I$ would be local by 3.1 and 3.3, contradicting 3.2. □

We will now turn our attention to a special class of local C^∞-rings.

3.5 Definition. A local C^∞-ring A is called *pointed* if there exists a C^∞-homomorphism $A \longrightarrow \mathbb{R}$ ("a point").

Any $\mathbb{R}$-algebra map $p: A \to \mathbb{R}$ is surjective, so $\mathrm{Ker}(p)$ is a maximal ideal in A since $\mathbb{R}$ is a field. So if A is a local ring, $\mathrm{Ker}(p) = m_A$, and we conclude that every pointed local ring has $\mathbb{R}$ as its residue field. Moreover, its point $A \longrightarrow \mathbb{R}$, being isomorphic to $A \longrightarrow A/m_A$, is unique.

It is remarkable that it suffices to require the map $A \longrightarrow \mathbb{R}$ to be an $\mathbb{R}$-algebra homomorphism:

3.6 Proposition. *(a) Let $A = C^\infty(\mathbb{R}^n)/I$ be a finitely generated C^∞-ring, and let $\varphi: A \to \mathbb{R}$ be an $\mathbb{R}$-algebra map. Then φ is of the form ev_p (the evaluation at p, $\mathrm{ev}_p(f) = f(p)$) for a unique point p in the zero set $Z(I) = \cap\{f^{-1}(0) | f \in I\}$ of I.*

(b) Let A be any C^∞-ring. Every $\mathbb{R}$-algebra homomorphism $A \longrightarrow \mathbb{R}$ is a C^∞-homomorphism.

Proof. (b) follows from (a), by writing A as a colimit of finitely generated C^∞-rings. To prove (a), it suffices to find $p \in \mathbb{R}^n$, since it then follows that $p \in Z(I)$. Also, we may assume $I = (0)$ by composing with the projection $C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)/I$. So let $\varphi: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ be any $\mathbb{R}$-algebra map. $\mathbb{R}$ is a field, so $\mathrm{Ker}(\varphi)$ is a maximal ideal, and since every closed set $F \subseteq \mathbb{R}^n$ is the zero set of a smooth function (Lemma 1.4), it follows that

$$\mathcal{F} = \{Z(f) | f \in \mathrm{Ker}(\varphi)\}$$

is a maximal filter. Now if $\mathcal{F}$ contains a compact set K we are done, since then $\mathcal{F} = \{F | p \in F\}$ for some $p \in K$, and thus $\varphi(g - g(p)) = 0$ for every $g \in C^\infty(\mathbb{R}^n)$, i.e. $\varphi = \mathrm{ev}_p$. But $\mathcal{F}$ even contains an $(n-1)$-sphere: let $g = \|x\|^2 \in C^\infty(\mathbb{R}^n)$, and let $r = \varphi(g)$. Then $\varphi(g - r) = 0$, i.e. $\{x | \|x\|^2 = r\} \in \mathcal{F}$. □

As a side remark, let us note that 3.6 enables us to improve 2.8 a little:

3.7 Corollary. *The contravariant functor*

$$M \mapsto C^\infty(M)$$

is a full and faithful functor from $\mathbb{M}$ into the category of $\mathbb{R}$-algebras.

Proof. Let $H: C^\infty(M) \to C^\infty(N)$ be an $\mathbb{R}$-algebra homomorphism. By 2.8 we have to show that H is in fact a C^∞-homomorphism. If $p \in N$ then $\mathrm{ev}_p \circ H = \mathrm{ev}_{\varphi(p)}$ for some point $\varphi(p) \in M$ by 3.6, i.e. $H(f) = f \circ \varphi$. But φ is smooth, since for every $f: M \to \mathbb{R}$,

$f \circ \varphi = H(f)$ is smooth. □

For finitely presented C^∞-rings, that is rings of the form $C^\infty(\mathbb{R}^n)/I$ for a finitely generated ideal I, 3.5 doesn't give anything new, since

3.8 Proposition. *If A is a finitely presented C^∞-ring, then A is local iff A has a unique point.*

Proof. Let $A \cong C^\infty(\mathbb{R}^n)/(f_1, \ldots, f_n)$. Recall from the proof of Lemma 2.1 that for every $g \in C^\infty(\mathbb{R}^n), g \in (f_1, \ldots, f_n)$ iff there exists an open cover $\{U_\alpha\}_\alpha$ of $\mathbb{R}^n$ such that $g|U_\alpha \in (f_1|U, \ldots, f_n|U_\alpha)$ for each α. We now prove the equivalence.

($\Rightarrow$) Since A is non-trivial, it follows from the remark just made that $Z(f_1, \ldots, f_n) = \{X \in \mathbb{R}^m | f_i(x) = 0, i = 1, \ldots, n\}$ is non-empty. If $p, q \in Z(f_1, \ldots, f_n)$ and $p \neq q$, then $B = C^\infty(\mathbb{R}^m)/\{g | g(p) = 0 = g(q)\}$ is a quotient of A, hence B is local (3.1). But $B \cong \mathbb{R} \times \mathbb{R}$, a non-local ring. So $Z(f_1, \ldots, f_n)$ consists of exactly one point, i.e. A has a unique point (cf. 3.6 (a)).

($\Leftarrow$) Again by 3.6, if A has a unique point then $Z(f_1, \ldots, f_n) = \{p\}$ for some $p \in \mathbb{R}^m$. But then A is a local ring since $f(p) \neq 0$ implies that f is invertible. Indeed, let $g \in C^\infty(\mathbb{R}^n)$ be such that $f \cdot g \equiv 1$ on a neighbourhood U of p. So by the remark that we began our proof with, it follows that $f \cdot g - 1 \in (f_1, \ldots, f_n)$, i.e. $f \cdot g = 1$ in A. □

Neither of the implications of 3.8 holds for finitely generated C^∞-rings. $\Leftarrow$ fails for $C^\infty(\mathbb{R})/I$, with $I = \{f | f(0) = 0$ and support(f) is compact$\}$. And $\Rightarrow$ fails by 3.4 and the following proposition. Note that we actually proved something slightly stronger than 3.8, namely the corresponding fact for germ-determined finitely generated C^∞-rings (cf. 4.1 and 4.8 below.)

3.9 Proposition. *Any finitely generated pointed local C^∞-ring is a quotient of a ring of germs.*

Proof. Let $A = C^\infty(\mathbb{R}^n)/I$ be a pointed local C^∞-ring, i.e. we have a map $\mathrm{ev}_p : C^\infty(\mathbb{R}^n)/I \twoheadrightarrow \mathbb{R}$ for some $p \in Z(I)$. $\mathbb{R}$ is the residual field, so ev_p is a local homomorphism. Take $f \in C^\infty(\mathbb{R}^n)$ with $f|U = 0$ for some neighbourhood U of p. Let $g \in C^\infty(\mathbb{R}^n)$ be a function with $g(p) = 1$ and $g|(\mathbb{R}^n - U) = 0$. Then $f \cdot g = 0$ in A and g is invertible in A, so $f = 0$ in A. Thus $m^g_{\{p\}} \subseteq I$, i.e. $C^\infty_p(\mathbb{R}^n) \twoheadrightarrow A$. □

3.10 Proposition. *A sub-C^∞-ring of a pointed local C^∞-ring is local.*

Proof. Let $A \xrightarrow{p} \mathbb{R}$ be a pointed local C^∞-ring, and let $B \rightarrowtail A$. B can be written as a directed colimit of finitely generated C^∞-rings, so it suffices to prove the case where B is finitely generated, say $B = C^\infty(\mathbb{R}^n)/I$. The composite $B \rightarrowtail A \xrightarrow{p} \mathbb{R}$ is of the form ev_x for some $x \in \mathbb{R}^n$ by 3.6, and we can factor the projection $C^\infty(\mathbb{R}^n) \longrightarrow B$ through $C^\infty_x(\mathbb{R}^n)$,

$$\begin{tikzcd}
C^\infty(\mathbb{R}^n) \arrow[r, two heads] \arrow[dr] & B \arrow[r, tail] & A \arrow[r, "p"] & \mathbb{R} \\
& C^\infty_x(\mathbb{R}^n) \arrow[u] \arrow[urr, "\mathrm{ev}_x"'] & &
\end{tikzcd}$$

by the argument of 3.9. So by 3.1, B is local since $C^\infty_x(\mathbb{R}^n)$ is. □

3.11 Corollary. *Every pointed local C^∞-ring is a colimit of finitely generated local C^∞-rings.*

3.12 Corollary. *If A and B are pointed local C^∞-rings, then so is $A \otimes_\infty B$.*

Proof. By 3.11 it suffices to show this for finitely generated A and B. But in this case the assertion follows from 3.1, 3.3, and 3.9. □

An important class of C^∞-rings is the class of Weil algebras. In commutative algebra these are defined as certain $\mathbb{R}$-algebras (Definition 3.13), but they are in fact C^∞-rings (Theorem 3.17).

3.13 Definition. A *Weil algebra* is a local $\mathbb{R}$-algebra W (a local ring with an $\mathbb{R}$-algebra structure) which, regarded as an $\mathbb{R}$-vector space, is finite dimensional and can be written as $W = \mathbb{R} \oplus m$ (the first component is the $\mathbb{R}$-algebra structure, the second is the maximal ideal in W).

An example that we have mentioned already in Section 1 is the ring $\mathbb{R}[\varepsilon] = \mathbb{R} \oplus \varepsilon\mathbb{R}$, and we observed that $\mathbb{R}[\varepsilon] \cong C^\infty_0(\mathbb{R})/(x^2)$. This is an example of a ring of *jets*.

3.14 Ehresmann Jets are Weil Algebras. Let m be the maximal ideal of $C_0^\infty(\mathbb{R}^n)$. The ring

$$J_n^k = C_0^\infty(\mathbb{R}^n)/m^{k+1}$$

is called the ring of jets of order k (in n variables), or the ring of k-*jets*. We will show that every ring J_n^k is a Weil algebra. The maximal ideal $m = \{f | f(0) = 0\}$ of $C_0^\infty(\mathbb{R}^n)$ is the ideal $(x_1, \ldots, x_n)$ generated by the n projections, by Hadarmard's lemma. So $C_0^\infty(\mathbb{R}^n) = \mathbb{R} \oplus m$ and it suffices to show that J_n^k is a finite dimensional vector space. But this follows by Taylor expansion: $C_0^\infty(\mathbb{R}^n)/m^{k+1}$ is generated by the constant function 1 and all functions $\Pi_{i \in A} x_i$ with $A \subseteq \{1, \ldots, n\}$ and $|A| \leq k$. For example, for $k = 1$ we can write any $f \in C_0^\infty(\mathbb{R}^n)$ as

$$\begin{aligned} f(x) &= f(0) + x_1 g_1(x) + \ldots + x_n g_n(x) \\ &= f(0) + x_1 g_1(0) + \ldots + x_n g_n(0) + \textstyle\sum_{i=1}^n \sum_{j=1}^n x_i x_j g_{ij}(x) \end{aligned}$$

by twice applying Hadamard, and the last $\sum\sum$ vanishes modulo m^2, so $\{1, x_1, \ldots, x_n\}$ generates J_n^1.

Although it is not true that all Weil algebra are rings of jets, such rings generate the Weil algebras in some sense, as we will now show. But first we need a lemma from commutative algebra.

3.15 The Nakayama Lemma. *Let R be a ring, and $m \subset R$ an ideal such that $1 + \alpha$ is invertible for all $\alpha \in m$ (e.g. the maximal ideal of a local ring). Let A be a finitely generated R-module, and B a submodule of A. Then $A \subset B + m \cdot A$ implies $A \subset B$.*

Proof. Let $a_1, \ldots, a_n$ be generators of A. We show $a_i \in B$. Since $A \subset B + m \cdot A$, each a_i can be written as

$$a_i = b_i + \sum_{j=1}^n \alpha_i^j a_j$$

with $\alpha_i^j \in m$ and $b_j \in B$. So $b_i = \sum_j (\delta_i^j - \alpha_i^j) a_j$ (Kronecker delta). But $\det((\delta_i^j - \alpha_i^j)) = 1 + \alpha$ with $\alpha \in m$, which is invertible. So by Cramer's rule, each a_j can be written as a linear combination of b_i's□

3.16 Corollary. *Let W be a Weil algebra, $W = \mathbb{R} \oplus m$ as above. Then $m^k = 0$ for some k.*

Proof. W is a finite dimensional vector space over $\mathbb{R}$, so the sequence of subspaces

$$m \supset m^2 \supset m^3 \supset \dots$$

must become constant. But $m^{k+1} = m^k$ implies $m^k = 0$ by an application of 3.15 with $B = 0$. □

3.17 Theorem. *(Characterization of Weil algebras). Let A be an $\mathbb{R}$-algebra. Then the following are equivalent: (in each case, m denotes the maximal ideal of the relevant local ring)*

(1) *A is a Weil algebra*
(2) *A is isomorphic to an $\mathbb{R}$-algebra of the form $\mathbb{R}[X_1, \dots, X_n]/I$ for an ideal I such that for some $k, X^\alpha \in I$ for all α with $|\alpha| = k$.*
(3) *A is isomorphic to $\mathbb{R}[[X_1, \dots, X_n]]/I$ for an ideal I containing m^k for some k.*
(4) *A is a quotient of a ring of jets J_n^k.*
(5) *A is isomorphic to a ring $C_0^\infty(\mathbb{R}^n)/I$ which is finite dimensional as a real vector space.*

Proof. (1) $\Rightarrow$ (2) If $A = \mathbb{R} \oplus m$ as in Definition 3.13 and $a_1, \dots, a_n$ generate m, then A can be written as a quotient

$$\mathbb{R}[X_1, \dots, X_n] \twoheadrightarrow A,$$

$A \cong \mathbb{R}[X_1, \dots, X_n]/I$, and it follows from 3.16 that for some k, $X = X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n} \in I$ for all α with $|\alpha| = k$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) follows by Taylor expansion and Borel's theorem (1.3).

(4) $\Rightarrow$ (5) In 3.14 we have shown that J_n^k is finite dimensional, so this is clear.

(5) $\Rightarrow$ (4) Suppose $A = C_0^\infty(\mathbb{R}^n)/I$ is finite dimensional. Then the sequence

$$I + m \supset I + m^2 \supset \dots$$

eventually becomes constant, i.e. $I + m^k = I + m^{k+1} = I + m(I + m^k)$. So by the Nakayama lemma, $I + m^k \subset I$, or $m^k \subset I$.

(4) $\Rightarrow$ (1) The same argument as 3.14 shows that quotients of jets are Weil algebras. □

3.18 Corollary. *Every Weil algebra is a finitely presented C^∞-ring.*

Proof. Every ideal in $\mathbb{R}[X_1, \dots, X_n]$, or in $\mathbb{R}[[X_1, \dots, X_n]]$ is finitely generated, by the Hilbert basis theorem. □

3.19 Corollary. *Let W be a Weil algebra, and A an arbitrary C^∞-ring. Then any $\mathbb{R}$-algebra homomorphism $W \longrightarrow A$, or $A \longrightarrow W$, is a C^∞-homomorphism.*

Proof. In both directions this is a simple application of Hadamard's lemma again. We will do the case $W \longrightarrow A$. (For the other case see 3.20). Let $W = C^\infty(\mathbb{R}^n)/I$ with $m^{k+1} \subseteq I$. W is finitely generated, so we may assume that A is also finitely generated, say $A = C^\infty(\mathbb{R}^m)/J$. Let

$$\varphi: C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$$

be an $\mathbb{R}$-algebra homomorphism, and let $f_i \in C^\infty(\mathbb{R}^m)$ represent $\varphi(x_i), i = 1, \ldots, n$. By Hadamard, any $g \in C^\infty(\mathbb{R}^n)/I$ can be written as

$$g(x) = g(0) + \sum_{|\alpha|\leq k} x^\alpha g_\alpha(0)$$

for smooth functions g_α. (Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index as usual, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$.) So

$$\varphi(g) = g(0) + \sum_{|\alpha|\leq k} f^\alpha(x) g_\alpha(0)$$

where $f^\alpha(x) = f_1(x)^{\alpha_1} \cdot \ldots \cdot f_n(x)^{\alpha_n}$, i.e.

$$\varphi(g) = g \circ f$$

for $f = (f_1, \ldots, f_n): \mathbb{R}^m \to \mathbb{R}^n$. □

One half of 3.19 can be improved:

3.20 Proposition. *Let A be a C^∞-ring, and let F be a formal algebra, i.e. a C^∞-ring of the form $\mathbb{R}[[X_1, \ldots, X_n]]/I$ for some ideal I. Then any $\mathbb{R}$-algebra homomorphism $A \longrightarrow F$ is a C^∞-homomorphism.*

Proof. The proof is most easily explained by introducing the Krull-topology (or m-topology): any local ring L can be topologized by

$$x_n \to x \textit{ iff } \forall k \exists n \forall \ell \geq n \; x_\ell - x \in m^k,$$

where m is the maximal ideal of L.

Observe that a local $\mathbb{R}$-algebra homomorphism $B \xrightarrow{\varphi} C$ of local $\mathbb{R}$-algebra is continuous (by definition $\varphi(m_B) \subseteq m_C$, so $\varphi(m_B^k) \subseteq m_C^k$).

Also, any $\mathbb{R}$-algebra map between pointed local $\mathbb{R}$-algebras (i.e. a map commuting with the points) is a local homomorphism, hence continuous.

We now proceed as follows. By writing A as a directed colimit of finitely generated C^∞-rings, it suffices to prove the proposition for finitely generated A, say $A = C^\infty(\mathbb{R}^n)/I$. In this case we have a diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^n)/I \arrow[r, "\varphi"] & F \arrow[r, "p"] & \mathbb{R} \\ C^\infty(\mathbb{R}^n) \arrow[u, "r"] \arrow[r, "s"'] & C^\infty_x(\mathbb{R}^n) \arrow[u, "\psi"'] & \end{tikzcd}$$

where p is the canonical point of F, and the factorization of $\varphi \circ r$ as $\psi \circ s$ exists since F is local.

Now since r and s are surjective, φ is a C^∞-homomorphism iff $\varphi \circ r = \varphi \circ s$ is, iff ψ is. To show that ψ is a C^∞-homomorphism, let $\theta_0 : C^\infty(\mathbb{R}^n) \to F$ be the C^∞-homomorphism defined by $\theta_0(\pi_i) = \psi_0(\pi_i)(i = 1, \ldots, n)$, where $\psi_0 = \psi \circ s$. θ_0 can be factored as $\theta \circ s$ for a (the) C^∞-homomorphism $\theta: C^\infty_x(\mathbb{R}^n) \to F$, since if $f|U \equiv 0$ for a neighbourhood U of x, then if g is a characteristic function for U (cf. Lemma 1.4) then $f \cdot g \equiv 0$, so $\theta_0(f) \cdot \theta_0(g) = \theta_0(f \cdot g) = 0$. But $\theta_0(g)$ is invertible in F, so $\theta_0(f) = 0$. Now θ and ψ coincide on polynomials, and these are dense in $C^\infty_x(\mathbb{R}^n)$. So $\theta = \psi$ since F is Hausdorff (because $\cap_k m^k = \{0\}$ in F). □

As a final consequence of 3.17, we have

3.21 Corollary. (a) *If W and W' are Weil algebras, then so is $W \otimes_\infty W'$, and this is the same as the tensor product of W and W' as $\mathbb{R}$-algebras.*

(b) *If $A \subset W$ and W is a Weil algebra, then so is A.*

Proof. (a) can be checked easily from the formula (3) for coproducts in Section 1, or can be concluded from 3.19 (check the universal properties). Part (b) follows from 3.10. □

3 Appendix: Real Closed Rings and Weil Algebras

In this appendix we will make some further remarks on local C^∞-rings, which require more algebraic preliminary knowledge. The results that follow will not be used in the rest of the book.

Let us first recall some definitions from commutative algebra. A field K is called *real closed* if

(i) K is totally ordered
(ii) $\forall x \in K (x > 0 \Rightarrow \exists y \; x = y^2)$
(iii) every polynomial of odd degree has a root.

An equivalent definition of real closed is: K is *formally real* (i.e. -1 is not a sum of squares) and has no proper algebraic formally real extension.

A local ring A is called *Henselian* if for every monic polynomial $p(t)$ with coefficients in A, any simple root in the residue field k_A can be lifted to A, i.e.

$$\forall \alpha \in k_A (p(\alpha) = 0 \text{ and } p'(\alpha) \neq 0 \text{ in } k_A \Rightarrow \exists a \in A \; p(a) = 0 \text{ and } \pi(a) = \alpha)$$

where $\pi: A \to k_A$.

We remark that the lifted root is necessarily unique: if given a simple root $\alpha \in h_A$, both a and b are liftings, then we can write (by the Euclidean algorithm) $p(b) = p(a) + (b-a)[p'(a) + (b-a)q(a,b)]$ with $q \in A[t_1, t_2]$, and $p(a) = p(b) = 0$ while $a - b \in m_A$ since they both lift α, so $p'(a) + (b-a)q(a,b)$ is invertible since $p'(a)$ is, and hence $a = b$.

A local ring A is called *separably real closed* if A is Henselian and k_A is real closed.

3.22 Theorem. *Every local C^∞-ring is separably real closed.*

We shall prove this theorem for finitely generated C^∞-rings only (for the general case see the comments and references to this chapter in Appendix 5). The proof is really an application of the following extension of the implicit function theorem.

3.23 Lemma. *Let $H \subset \mathbb{R}^n$ be a closed set, and let $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth map. Suppose we are given a smooth $r: H \to \mathbb{R}$ such that*

$$\forall x \in H : f(r(x), x) = 0 \quad \text{and} \quad \frac{\partial f}{\partial t}(r(x), x) \neq 0.$$

Then there exist an open $U \supset H$ and a smooth $s: U \to \mathbb{R}$ such that

$s|H = r$ and $\forall x \in U f(s(x), x) = 0$.

Proof. By writing $H = H_1 \cup H_2$, where $H_1 = \{x \in H | \frac{\partial f}{\partial t}(r(x), x) > 0\}$ and $H_2 = \{x | \frac{\partial f}{\partial t}(r(x), x) < 0\}$, and working on each of the closed sets H_1, H_2 separately, we may assume $\frac{\partial f}{\partial t}(r(x), x) > 0$ on all of H. Moreover, by definition r is the restriction of a smooth function $r: V \to \mathbb{R}$ on an open neighbourhood $V \supset H$, and by choosing V small enough we may assume $\frac{\partial f}{\partial t}(r(x), x) > 0$ on all of V.

Since $\frac{\partial f}{\partial t}(r(x), x): V \to \mathbb{R}$ is smooth, we can find an open cover $\{V_\alpha\}_\alpha$ of V and $\delta_\alpha > 0$ such that for each α,

$$(1) \qquad \forall x \in V_\alpha \forall t \in \mathbb{R}(|r(x) - t| \leq \delta_\alpha \Rightarrow \frac{\partial f}{\partial t}(t, x) > 0)$$

Without loss we may assume that $\{V_\alpha\}_\alpha$ is locally finite, and we can take a partition of unity $\{\rho_\alpha\}_\alpha, \rho_\alpha: V \to \mathbb{R}$, subordinate to the cover $\{V_\alpha\}_\alpha$. Let $\alpha(x) = \sum_\alpha (r(x) - \delta_\alpha \rho_\alpha(x)), \beta(x) = \sum_\alpha (r(x) + \delta_\alpha \cdot \rho_\alpha(x))$ Then $\alpha, \beta: V \to \mathbb{R}$ are smooth, $\alpha < r < \beta$, and

$$(2) \qquad \forall x \in V \forall t \in \mathbb{R}(\alpha(x) \leq t \leq \beta(x) \Rightarrow \frac{\partial f}{\partial t}(t, x) > 0).$$

Moreover, by shrinking V a little further if necessary, we may assume that

$$(3) \qquad \forall x \in V f(\alpha(x), x) < 0 < f(\beta(x), x).$$

It follows from (2) and (3) that for fixed $x \in V$, $f(-, x)$ has exactly one zero in $[\alpha(x), \beta(x)]$, say $s(x)$. By the usual implicit function theorem, $s: V \to \mathbb{R}$ is smooth, while clearly $s|H = r$. $\square$

Proof of theorem 3.22. As said, we will prove this for finitely generated C^∞-rings only. Let $A = C^\infty(\mathbb{R}^m)/I$ be a finitely generated local C^∞-ring. So $k_A = C^\infty(\mathbb{R}^m)/J$, where J is the maximal ideal containing I. We first note that J corresponds to a maximal (cf. 1.4) filter of closed sets. Indeed, let

$$\Phi = \{F \subset \mathbb{R}^m | \exists f \in J \; F = Z(f)\}$$

and let

$$J' = \{g \in C^\infty(\mathbb{R}^m) | Z(g) \in \Phi\}.$$

Then J is a proper ideal iff Φ is a proper filter, iff J' is a proper ideal, and $J' \supseteq J$, so $J' = J$ by maximality. Thus

$$f \in J \iff f|F = 0 \quad \text{for some} \quad F \in \Phi.$$

A is *Henselian*: Let $f(t)$ be a monic polynomial with coefficients in A, represented by

$$f(t,x) = t^n + f_1(x)t^{n-1} + \ldots + f_n(x)$$

with $f_i \in C^\infty(\mathbb{R}^m)$, and let $r(x) \in C^\infty(\mathbb{R}^m)$ represent a simple root in k_A, that is,

(i) $f(r(x),x) = 0$ on some $F \in \Phi$
(ii) $\frac{\partial f}{\partial t}(r(x),x) \neq 0$ on some $G \in \Phi$.

Applying Lemma 3.22 to $H = F \cap G \in \Phi$ yields an open $U \supset H$ and a smooth $s: U \to \mathbb{R}$ such that $f(s(x),x) = 0$ and $s|H = r$. Let V be open with $H \subset V \subset \overline{V} \subset U$, and find (by "smooth Tietze") a smooth $\tilde{s}: \mathbb{R}^m \to \mathbb{R}$ such that $\tilde{s}|V = s$. $\tilde{s}$ represents an element of A, and by Lemma 3.24 below, $f(\tilde{s}) = 0$ in A.

3.24 Lemma. *Let A, k_A be as in the proof of 3.22, and let $H \in \Phi$. Assume $g \in C^\infty(\mathbb{R}^m)$ is such that $g|V = 0$ on some open $V \supset H$. Then $g \in I$.*

Proof. Let h be a characteristic function for V. Then $g \cdot h = 0$ in $C^\infty(\mathbb{R}^m)$, hence in A. But $h \neq 0$ on V, so h is invertible in k_A, hence in A. So $g = 0$ in A, which proves the lemma. □

Proceeding with the proof of the theorem, let us show that k_A is *real closed*: (i) the order on $k_A = C^\infty(\mathbb{R}^m)/J$ is defined by $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in$ some $F \in \Phi$. This order is total since Φ is a maximal filter of closed sets.
(ii) $x > 0 \Rightarrow \exists y\ y^2 = x$: this is easy, and in fact true in any C^∞-ring. In this particular case, if $f(x) \in C^\infty(\mathbb{R}^m)$ represents an element > 0 of k_A, then there exists an $F \in \Phi$ such that $\forall x \in F f(x) > 0$. By smooth Tietze we can define a $g \in C(\mathbb{R}^m)$ such that $g(x)^2 = f(x)$ (on an open set containing F).
(iii) polynomials of odd degree have zeros: clearly, it suffices to show this for irreducible monic polynomials p. But for such p, p and p' cannot have any non-constant common factors, i.e. the resultant $\mathrm{Res}(p,p') \neq 0$. So if p is represented by

$$f(t,x) = t^n + f_1(x)t^{n-1} + \ldots + f_n(x)$$

with $f_i(x) \in C^\infty(\mathbb{R}^m)$, then we find an $F \in \Phi$ and an open $U \supset F$ such that for all $x \in U$, $\mathrm{Res}(f(t,x), \frac{\partial f}{\partial t}(t,x)) \neq 0$. In other words, for each $x \in U$, $f(-,x)$ and $\frac{\partial f}{\partial t}(-,x)$ have no common zeros. Now let $r(x)$ be the first root of the polynomial $f(t,x) \in \mathbb{R}[t]$. This root exists since $\mathbb{R}$ is real closed, and it is simple since $\frac{\partial f}{\partial t}(r(x), x) \neq 0$. Therefore by the implicit function theorem, $r: U \to \mathbb{R}$ is smooth. So any smooth $s \in C^\infty(\mathbb{R}^m)$ such that $s|F = r|F$ represents a root of $p(t)$ in k_A. □

3.25 Corollary. *Any C^∞-field is real closed.*

3.26 Corollary. *Any local C^∞-ring A is of the form $A = k_A \oplus m_A$ (as rings) with m_A the maximal ideal.*

Proof. We show that the exact sequence $0 \to m_A \to A \to k_A \to 0$ is split-exact. Consider partial sections (K, s) of π, where K is a subfield ($\mathbb{R}$-algebra) of k_A. Let (K, s) be a maximal section (Zorn).

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ K \arrow[ur, "s"] & k_A \arrow[l, phantom, "\supset"] \end{tikzcd}$$

Take $\alpha \in k_A - K$. If α is transcendental over K, we can extend s to a section on $K(\alpha) = K(x)$, contradicting maximality. And if α is algebraic over K, there is an irreducible monic polynomial f with $f(\alpha) = 0, f'(\alpha) \neq 0$. By Henselianness, α can be lifted to a root $\beta \in A, \pi(\beta) = \alpha$, and by sending α to β we obtain an extension of s to $K(\alpha)$, again contradicting maximality of s. So $K = k_A$ □

Finally, let us mention another characterization of Weil algebras.

3.27 Proposition. *Let A be a C^∞-ring. Then A is a Weil algebra iff A has at most one point $A \longrightarrow \mathbb{R}$ and is finite dimensional as a vector space.*

Proof. One implication is clear. Conversely, assume A has at most one point and is a finite dimensional vector space. Let m be a maximal ideal. Then $K = A/m$ is formally real (as any C^∞-ring) and is again finite dimensional vector space, hence an algebraic extension of $\mathbb{R}$. So $K = \mathbb{R}$ by the equivalent of real closedness given at the beginning of this appendix. Since A has at most one point, it follows that m is the unique maximal ideal of A, i.e., A is local. Hence A

satisfies 3.17 (a) by 3.8 (Alternative, $0 \to m \to A \to \mathbb{R} \to 0$ is split exact, so $A \cong \mathbb{R} \oplus m$.) □

4 Ideals of Smooth Functions

In this section we will discuss several types of ideals in the rings $C^\infty(\mathbb{R}^n)$, that is, several types of finitely generated C^∞-rings. The definitions, however, make sense for arbitrary C^∞-rings:

4.1 Definition. Let A be a C^∞-ring.

(a) A is *point determined* if it can be embedded into a power $\Pi_{i\in I}\mathbb{R}$ of $\mathbb{R}$ (a "direct product of points").

(b) A is *near-point determined* if it can be embedded into a direct product of Weil algebras.

(c) A is *closed* if it can be embedded into a direct product of formal algebras (C^∞-rings of the form $\mathbb{R}[[X_1,\ldots,X_n]]/I$).

(d) A is *germ determined* if it can be embedded into a direct product of pointed local rings.

If A is finitely generated, we can rephrase these conditions entirely in terms of closure properties of the ideal I, where $A = C^\infty(\mathbb{R}^n)/I$. It follows that the conditions are independent of the representation: if M and N are manifolds and $C^\infty(M)/I$ and $C^\infty(N)/J$ are isomorphic C^∞-rings, then I satisfies the relevant condition if and only if J does. Recall that $Z(I) = \cap\{f^{-1}(0) | f \in I\}$.

4.2 Theorem. *Let A be a C^∞-ring of the form $C^\infty(M)/I$, where M is a manifold, say $M \subseteq \mathbb{R}^p$. Then*

(a) *A is a point determined iff for all $f \in C^\infty(M)$:*

$$\forall x \in Z(I) f(x) = 0 \Rightarrow f \in I$$

(b) *A is near-point determined iff for all $f \in C^\infty(M)$:*

$$\forall x \in Z(I) \forall k \in \mathbb{N}\; T_x(f) \in (T_x(I) + m^k) \Rightarrow f \in I$$

where m is the maximal ideal of $R[[X_1,\ldots,X_n]]$, $\dim \mathrm{M} = n$, *and T_x is the Taylor series of f at x (using local coordinates around x).*

(c) *A is closed iff for all $f \in C^\infty(M)$:*

$$\forall x \in Z(I) T_x(f) \in T_x(I) \Rightarrow f \in I$$

(d) *A is germ determined iff for all $f \in C^\infty(M)$*

$$\forall x \in Z(I) f|_x \in I|_x \Rightarrow f \in I.$$

If I has one of the properties stated at the right hand sides of the equivalences, we also say that I is a *point determined ideal*, etc.

4.3 Remark. Note that it follows from 4.2 that for an ideal I of any of these four types, to show that $f \in I$ it suffices to show that $f|U \in (I|U)$ for some open neighbourhood U of $Z(I)$.

Also note that for ideals of any of these types, we have a

Nullstellensatz: $1 \in I$ iff $Z(I) = \emptyset$.

Proof of Theorem 4.2. (a) ($\Rightarrow$) take $f \in C^\infty(M)$ and suppose $f \notin I$. Then $f \neq 0$ so there is a map $\varphi: A \to \mathbb{R}$ of C^∞-rings such that $\varphi(f) \neq 0$. But $\varphi = \mathrm{ev}_x$ for some $x \in Z(I)$ by 3.6, so $f(x) \neq 0$.

($\Leftarrow$) $A \xrightarrow{\varphi} \Pi_{x \in Z(I)} \mathbb{R}$, $\varphi(f) = \{f(x)\}_{x \in Z(I)}$ is injective.

(d) ($\Rightarrow$) Again take $f \in C^\infty(M)$ with $f \notin I$. By hypothesis there exists a map $A \xrightarrow{\varphi} B$ into a pointed local C^∞-ring B such that $\varphi(f) \neq 0$. Since A is finitely generated, we may assume that B is, i.e. by 3.9 and 3.10 $B \cong C_0^\infty(\mathbb{R}^n)/J$ for some ideal J. As in the proof of 3.9 we can show that if

$$C^\infty(M)/I \xrightarrow{\varphi} C_0^\infty(\mathbb{R}^n)/J \xrightarrow{\mathrm{ev}_0} \mathbb{R}$$

corresponds to the point $p \in Z(I) \subseteq M$, then φ can be factored as

$$\begin{tikzcd} C^\infty(M)/I \arrow[rr, "\varphi"] \arrow[dr] & & C_0^\infty(\mathbb{R}^n)/J \\ & C_p^\infty(M)/I|_p \arrow[ur] & \end{tikzcd}$$

So if $\varphi(f) \neq 0, f|_p \neq 0$.

($\Leftarrow$) As in (a), this is easy: the map which takes the germ at each point,

$$A \to \prod_{x \in Z(I)} C_x^\infty(I), \quad f \to \{f|_x\}_x$$

is again injective.

(b) and (c) are proved similarly. $\Leftarrow$ is again easy, and we only do $\Rightarrow$ for (c):

If $f \in C^\infty(M)$ and $f \notin I$, then there exists a formal algebra $\mathbb{R}[[X_1,\ldots,X_m]]/J$ and a C^∞-homomorphism

$$\varphi: C^\infty(M)/I \to \mathbb{R}[[X_1,\ldots,X_m]]/J$$

with $\varphi(f) \neq 0$. As in the proof of (d) $\Rightarrow$, $\mathrm{ev}_0 \circ \varphi$ corresponds to a unique $x \in Z(I) \subset C^\infty(M)$ and we obtain a commutative diagram

$$\begin{tikzcd}
C^\infty(M)/I \arrow[r, "\varphi"] \arrow[d, "T_x"'] \arrow[dr, "ev_x"] & \mathbb{R}[[X,\ldots,X_m]] \arrow[d, "ev_0"] \\
\mathbb{R}[[X,\ldots,X_n]]/T_x(I) \arrow[r] & \mathbb{R}
\end{tikzcd}$$

But (cf. the proof of 4.6), $T_x(I) = \bigcap_{k\in\mathbb{N}}(T_x(I) + m^k)$. Hence $T_x(\varphi) \notin (T_x(I) + m^k)$ for some $k \in \mathbb{N}$. □

4.4 Remark. $C^\infty(M)$ carries a natural topology, the *Fréchet topology*, or the topology of uniform convergence of a function and all its derivatives on compacta, making $C^\infty(M)$ into a Fréchet space. The basic open neighbourhoods of a function $f \in C^\infty(M)$ are the sets (using local coordinates for M)

$$V(f,\varepsilon,n,K) = \{g \in C^\infty(M) | \sup_{x\in K, |\alpha|\leq n} |D^\alpha g(x) - D^\alpha f(x)| < \varepsilon\}$$

for $\varepsilon > 0$, $n \in \mathbb{N}$ and $K \subseteq M$ compact.

The ideals in $C^\infty(M)$ which are closed for this topology are precisely the closed ideals introduced in 4.2 (c), by the following theorem which we state without proof. (For a proof, see e.g. Malgrange (1966).)

Whitney's Spectral Theorem. *Let $I \subseteq C^\infty(M)$ be an ideal, and $\bar{I}$ its closure in the Fréchet topology. Then*

$$f \in \bar{I} \Longleftrightarrow \forall x \in Z(I) T_x(f) \in T_x(I). \qquad \square$$

The following proposition is obvious.

4.5 Proposition. *For any ideal $I \subseteq C(M)$,*

*I is point determined $\Rightarrow$ I is near-point determined $\Rightarrow$
$\Rightarrow$ I is closed $\Rightarrow$ I is germ determined.* □

The middle one of these implications can be reversed.

4.6 Proposition. *An ideal $I \subseteq C^\infty(M)$ is closed iff it is near-point determined.*

Proof. It suffices to show that if I is an ideal in $\mathbb{R}[[X_1, \ldots, X_n]]$, and m is the maximal ideal in $\mathbb{R}[[X_1, \ldots, X_n]]$, then $I = \cap_k (I + m^k)$. But from the Nakayama lemma (3.15), it follows that if R is any local Noetherian ring with maximal ideal m_R, then $\cap_k m_R^k = (0)$ (apply 3.15 with $A = \cap_k m_R^k$, $B = (0)$). Hence if I is any ideal in R, R/I is still local and Noetherian, so $\cap_k (I + m_R^k) = I$. The case $R = \mathbb{R}[[X_1, \ldots, X_n]]$ now proves the proposition. (Algebraists express the above by saying that a local Noetherian ring is a Zariski ring.) □

4.7 Examples and Notation. Let $X \subseteq \mathbb{R}^n$. We have the following ideals in $C^\infty(\mathbb{R}^n)$:

$$\begin{aligned} m_X^0 &= \{f \in C^\infty(\mathbb{R}^n) | f|X = 0\} \\ m_X^\infty &= \{f \in C^\infty(\mathbb{R}^n) | D^\alpha f|X = 0, \text{ all } \alpha\} \\ m_X^g &= \{f \in C^\infty(\mathbb{R}^n) | \exists \text{ open } U \supseteq X \; f|U = 0\} \end{aligned}$$

m_X^0 is point determined, m_X^∞ is closed or near-point determined, and m_X^g is germ determined. These ideals also show that the other two implications in 4.5 cannot be reversed. For instance, in the case $X = \{0\}$ we have that m_X^g is not closed, and m_X^g is not point determined. Note, by the way, that $m_X^0 = m_X^\infty$ if $X \subset \overline{\mathrm{Int}(X)}$.

A family of functions $f_\alpha \in C^\infty(M)$ is called *locally finite* if every $x \in M$ has a neighbourhood U_x such that $f_\alpha | U_x = 0$ for all but finitely many indices α. If $\{f_\alpha\}_\alpha$ is a locally finite family, we have a smooth function $\Sigma_\alpha f_\alpha$ defined by pointwise addition. A partition of unity is a locally finite family $\{f_\alpha\}$ such that $\Sigma_\alpha f_\alpha = 1$. Germ determined ideals are precisely the ideals that are not only closed under finite sums, but also under locally finite ones:

4.8 Proposition. *Let $I \subseteq C^\infty(M)$ be an ideal. Then the following are equivalent:*

(i) *I is germ determined*
(ii) *If $\{f_\alpha\}_\alpha \subseteq I$ is a locally finite family then $\Sigma_\alpha f_\alpha \in I$,*

(iii) *If $\{\rho_\alpha\}_\alpha$ is a partition of unity, and for each $\alpha, f_\alpha \in I$, then* $\Sigma \rho_\alpha \cdot f_\alpha \in I$.

Proof. (i) $\Rightarrow$ (ii) Choose $x \in Z(I)$, and let U_x be a neighbourhood of x such that all the f_α but $f_{\alpha_1}, \ldots, f_{\alpha_n}$ vanish on U_x. Then $\Sigma_\alpha f_\alpha|U_x = (f_{\alpha_1} + \ldots + f_{\alpha_n})|U_x \in (I|U_x)$. So certainly $(\Sigma_\alpha f_\alpha)|_x \in I|_x$. Hence by (i), $\Sigma_\alpha f_\alpha \in I$.

(ii) $\Rightarrow$ (iii) clear.

(iii) $\Rightarrow$ (i) Suppose that $f \in C^\infty(M)$ and $f|_x \in I|_x$ for all $x \in Z(I)$, or equivalently for all $x \in M$. In other words, there is an open cover $\{U_\alpha\}$ of M such that for each α, $f|U_\alpha = g_\alpha|U_\alpha$ for some $g_\alpha \in I$, and without loss we may take $\{U_\alpha\}$ to be locally finite. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then

$$f = \sum_\alpha \rho_\alpha \cdot f = \sum_\alpha \rho_\alpha \cdot g \in I. \qquad \square$$

4.9 Corollary. *If $I \subseteq C^\infty(M)$ is a germ determined ideal, then so is (I, h) for every $h \in C^\infty(M)$. In particular, finitely generated ideals are germ determined.*

Proof. Using 4.8 (iii), let $f_\alpha = g_\alpha + h \cdot \varphi_\alpha \in (I, h)$ with $g_\alpha \in I$, and let $\{\rho_\alpha\}$ be a partition of unity. Then

$$\sum_\alpha \rho_\alpha \cdot f_\alpha = \sum_\alpha \rho_\alpha g_\alpha + h \cdot \sum_\alpha \rho_\alpha \varphi_\alpha \in (I, h). \qquad \square$$

4.10 Remark. The implication finitely generated $\Rightarrow$ germ determined in 4.9 cannot be reversed. Even a point determined ideal need not be finitely generated. In fact we will see below (cf. 4.19) that for example $m^\infty_{[0,1]} \subseteq C^\infty(\mathbb{R})$ is not even countably generated.

Observe on the other hand that 2.1 says that an ideal generated by finitely many independent functions is point determined.

It follows from the description in 4.2 that for any ideal I in $C^\infty(\mathbb{R}^n)$, say, there is a smallest germ determined ideal $\tilde{I}$ containing I, defined by

$$f \in \tilde{I} \Leftrightarrow \forall x \in Z(I) f|_x \in I|_x.$$

It is clear that $Z(I) = Z(\tilde{I})$, so $1 \in \tilde{I}$ iff $Z(I) = \phi$, and the operation $C^\infty(\mathbb{R}^n)/I \longrightarrow C^\infty(\mathbb{R}^n)/\tilde{I}$ may collapse a non-trivial ring to the trivial one with $1 = 0$.

In fact this operation $I \mapsto \tilde{I}$ is *functorial*, as follows easily from the explicit description of C^∞-homomorphisms between finitely generated C^∞-rings that we gave in Section 1: if $G: \mathbb{R}^m \to \mathbb{R}^n$ induces a C^∞-homomorphism

$$C^\infty(\mathbb{R}^n)/I \longrightarrow C^\infty(\mathbb{R}^m)/J$$

then the same G induces a map

$$C^\infty(\mathbb{R}^n)/\tilde{I} \longrightarrow C^\infty(\mathbb{R}^m)/\tilde{J},$$

for $f \in \tilde{I}$ implies $f \circ G \in \tilde{J}$. Indeed, if $\{U_\alpha\}$ is a cover of $\mathbb{R}^n$ and $f|U_\alpha = h_\alpha|U_\alpha$ with $h_\alpha \in I$, then by assumption $h_\alpha \circ G \in J$ and $f \circ G|G^{-1}(U_\alpha) = (h_\alpha \circ G)|G^{-1}(U_\alpha)$. Or in the notation of Section 1, $G_*(I) := \{f | f \circ G \in J\}$ and

$$(G_*(J))^\sim \subseteq G_*(\tilde{J}),$$

so $I \subseteq G_*(J)$ implies $\tilde{I} \subseteq G_*(\tilde{J})$. Thus we obtain a functor from the category of finitely generated C^∞-rings to the category of germ determined finitely generated C^∞-rings, which is actually left adjoint to the inclusion.

Similarly, each ideal I is contained in a smallest closed ideal $\bar{I}$ and a smallest point determined ideal $m^0_{Z(I)}$, and we obtain left adjoints to the inclusions of categories. We will come back to this in the next section.

The properties of ideals listed in 4.2 are in general not very well behaved with respect to coproduct of C^∞-rings. For example, the ring $C^\infty(\mathbb{R}) \otimes_\infty C_0^\infty(\mathbb{R})$ is not germ determined (so even the C^∞-tensor product of a germ determined ideal and a point determined one need not be germ determined). For,

$$C^\infty(\mathbb{R}) \otimes_\infty C_0^\infty(\mathbb{R}) = C^\infty(\mathbb{R} \times \mathbb{R})/I,$$

where I is the ideal generated by $\{f \circ \pi_2 | f \in m^g_{\{0\}}\}$. If $g \in I$, $g = \Sigma_{i=1}^n f_i(x)\varphi_i(x,y)$ and g vanishes on a tubular neighbourhood $\mathbb{R} \times (-\varepsilon, \varepsilon)$ of the x-axis. But if I were germ determined, a function vanishing on *any* neighbourhood of $Z(I) = \mathbb{R} \times \{0\}$ would have to be in I (cf. 4.3). (To be explicit, take the neighbourhood $U = \cup\{(n-1, n+1) \times \left(-\frac{1}{|n|+1}, \frac{1}{|n|+1}\right) | n \in Z\}$ of $\mathbb{R} \times \{0\}$, take an open V with $\mathbb{R} \times \{0\} \subset V \subset \overline{V} \subset U$, and take $f \in C^\infty(\mathbb{R}^2)$ with $Z(f) = \overline{V}$ (cf. 1.4). Then $f \notin I$, but $\forall x \in Z(I) f|_x = 0 \in I|_x$.)

There is one class of ideals, however, that is well behaved as regards coproducts, namely the class of ***flat ideals***, i.e. ideals of the form

$$m_X^\infty = \{f \in C^\infty(\mathbb{R}^n) | T_x(f) = 0, \quad \text{all} \quad x \in X\}$$

for a closed $X \subseteq \mathbb{R}^n$. The following theorem will be of fundamental importance in the next chapter.

4.11 Theorem. *Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ be closed sets and let $(f_k)_k$ be a sequence of functions such that $\forall k\ f_k \in m^\infty_{X\times Y}$. Then there are non-negative functions $\varphi \in m^\infty_X, \psi \in m^\infty_Y$ with $Z(\varphi) = X, Z(\psi) = Y$ and such that*

$$f_k \in (\varphi(x) + \psi(y)) \cdot m^\infty_{X\times Y} \quad \text{for all} \quad k.$$

4.12 Corollary. *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be closed. Then as ideals in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$,*

$$(m^\infty_X, m^\infty_Y) = m^\infty_{X\times Y}.$$

In fact, Theorem 4.11 is a generalization of the following result from Tougeron (1972):

4.13 Corollary. *Let $X \subseteq \mathbb{R}^n$ be closed and let $(f_k)_k$ be a sequence of function in $m^\infty_X \subseteq C^\infty(\mathbb{R}^n)$. Then there exists a non-negative function $\varphi \in m^\infty_X$ with $Z(\varphi) = X$ and such that $f_k \in \varphi \cdot m^\infty_X$ for each k.*

This corollary follows from 4.11 by putting $m = 0$, $Y = \{*\} = \mathbb{R}^m$. For the proof of Theorem 4.11, we need the following lemma (cf. Tougeron (1972), p. 77).

4.14 Lemma. *There are constants $C_r, r = (r_1, \ldots, r_n) \in \mathbb{N}^n$, such that for each compact $X \subseteq \mathbb{R}^n$ and each $\varepsilon > 0$ there is an $\alpha_\varepsilon \in C(\mathbb{R}^n)$ with the following properties*

(1) *$0 \leq \alpha_\varepsilon \leq 1, \alpha_\varepsilon(x) = 0$ on a neighbourhood of X, and $\alpha_\varepsilon(X) = 1$ outside the neighbourhood $B(X,\varepsilon) = \{x \in \mathbb{R}^n | d(x,X) < \varepsilon\}$.*
(2) *for every $x \in \mathbb{R}^n$ and $r \in \mathbb{N}^n$, $|D^r\alpha_\varepsilon(x)| \leq C_r \cdot \varepsilon^{-|r|}$ (recall that $|r| = r_1 + \ldots + r_n$).*

Proof. Fix a smooth function $\varphi: \mathbb{R}^n \to [0,1]$ with $\varphi(x) = 0$ if $|x| \geq \frac{1}{8}$ and $\int \varphi = 1$, and let

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$$

for $\varepsilon > 0$. Given a compact set, find a smooth function $\tilde{\alpha}: \mathbb{R}^n \to [0,1]$ with $\tilde{\alpha}(x) = 1$ if $d(x,X) \leq \varepsilon/2$, and $\tilde{\alpha}(x) = 0$ if $d(x,X) \geq 3\varepsilon/4$, and define $\alpha = \alpha_\varepsilon$ by

$$\alpha(x) = 1 - \int \varphi_\varepsilon(x-y)\tilde{\alpha}(y)dy.$$

Then $\alpha(x)$ is a well-defined C^∞-function on $\mathbb{R}^n$, and it is clear that $\alpha(x) = 0$ if $d(x, X) \leq \varepsilon/4$, while $\alpha(x) = 1$ if $d(x, X) \geq \varepsilon$. Moreover, for a given $r = (r_1, \ldots, r_n), |D^r\alpha(x)| = |\int D^r_x\varphi_\varepsilon(x-y)\tilde{\alpha}(y)dy| \leq |\int_{\mathrm{supp}(\varphi_\varepsilon)} D^r\varphi_\varepsilon(t)dt| \leq \left(\frac{1}{4}\varepsilon\right)^r \sup|D^r\varphi_\varepsilon(t)| \leq \varepsilon^{-|r|}4^{-n}\sup|D^r\varphi(t)|$ (the last $\leq$ by the chain-rule). □

For the key lemma, we leave the realm of C^∞-functions temporarily. Given a compact $X \subseteq \mathbb{R}^n$, let $L^\infty_X(\mathbb{R}^n)$ be the set of continuous functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall\varepsilon > 0\ \forall s \in \mathbb{N}\ \exists\delta > 0\ \forall y \in \mathbb{R}^n (d(y, X) \leq \delta \Rightarrow |f(y)| \leq \varepsilon \cdot d(y, X)^s).$$

The following is now immediate (use Hadamard's lemma for (i)):

4.15 Lemma. (i) $m^\infty_X \subseteq L^\infty_X$

(ii) *If $f \in L^\infty_X$, then $|f|^{\frac{1}{p}} \in L^\infty_X$ for $p = 1, 2, \ldots$* □

The key lemma is

4.16 Lemma. *Assume $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are compact. Let $(f_k)_k$ be a sequence of elements of $L^\infty_{X\times Y}$. Then, there are non-negative functions $\varphi \in m^\infty_X, \psi \in m^\infty_Y$ with $Z(\varphi) = X, Z(\psi) = Y$, such that*

$$\frac{f_k(x,y)}{\varphi(x) + \psi(y)} \longrightarrow 0 \quad \textit{whenever} \quad d((x,y), X \times Y) \longrightarrow 0.$$

Proof. By definition of $L^\infty_{X\times Y}$ (with $\varepsilon = 1$) we find for each $\ell \in \mathbb{N}$ a $\delta(\ell) > 0$ such that for each $k \leq \ell$, and all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$(d((x,y), X\times Y) \leq \delta(\ell) \Rightarrow |f_k(x,y)| \leq d((x,y), X \times Y)^{2\ell}).$$

Let $\mu: \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that

$$\frac{2}{2^{\mu(\ell)}} \leq \delta(\ell), \text{ for each } \ell,$$

and let $S : \{i \in \mathbb{N} | i \geq \mu(1)\} \longrightarrow \mathbb{N}$ be defined by

$$S(i) = \ell \mathit{iff}\ \mu(\ell) \leq i < \mu(\ell + 1).$$

By lemma 4.14 (with $\varepsilon = 2^{-i}$), there are $\alpha_i(x) \in C^\infty(\mathbb{R}^n)$ and $\beta_i(y) \in C^\infty(\mathbb{R}^m)$ with $0 \leq \alpha_i \leq 1, 0 \leq \beta_i \leq 1, \alpha_i \equiv 0$ on a neighbourhood of $X, \beta_i \equiv 0$ on a neighbourhood of Y, and

$$\alpha_i(x) = 1 \quad \text{if} \quad d(x, X) \geq 2^{-(i+1)},$$

$$\beta_i(y) = 1 \quad \text{if} \quad d(y, Y) \geq 2^{-(i+1)},$$

while furthermore for each $r \in \mathbb{N}^n, s \in \mathbb{N}^m$,

$$|D^r\alpha_i(x)| \le 2^{-i|r|}C_r, \ \ |D^s\beta_i(y)| \le 2^{-i|s|}C_s.$$

Now define

$$\varphi(x) = \sum_{i \ge \mu(1)} 2^{-is(i)}\alpha_i(x), \ \ \psi(y) = \sum_{i \ge \mu(1)} 2^{-is(i)}\beta_i(y).$$

Then φ and ψ are smooth non-negative functions with $\varphi \in m_X^\infty$, $\psi \in m_Y^\infty$, and $Z(\varphi) = X, Z(\psi) = Y$. (Just check the uniform convergence of these series as well as their derivatives, using the bounds of the derivatives of α_i, β_i).

To check the conclusion of the lemma, let $k \in \mathbb{N}$ and ε, $0 < \varepsilon < 1$, be given. Then for some $\ell \ge k$,

$$\frac{2}{2^{\mu(\ell)}} < \varepsilon < 1.$$

Assume that $d((x,y), X \times Y) \le 2^{-\mu(\ell)}$. Then $d(x,X) \le d((x,y), X \times Y) \le 2^{-\mu(\ell)}$, and we may choose a (unique) $i > \mu(\ell)$ such that $2^{-(i+1)} < d(x,X) \le 2^{-i}$. Similarly, let t be such that $\mu(\ell+t) \le i \le \mu(\ell+t+1)$, i.e. $S(i) = \ell+t$. Then $\alpha_i(x) = 1$, so

$$\begin{aligned}\varphi(x) + \psi(y) \ge \varphi(x) &\ge 2^{-i(\ell+t)} \cdot \alpha_i(x) = 2^{-i(\ell+t)} \ge d(x,X)^{\ell+t} \\ &\ge 2^{-(\ell+t)}d((x,y), X \times Y)^{\ell+t}.\end{aligned}$$

On the other hand, we may without loss (the other case is symmetric) assume that $d(y,Y) \le d(x,X)$, so

$$\begin{aligned}d((x,y), X \times Y) &\le d(x,X) + d(y,Y) \le 2d(x,X) \le 2^{1-i} \\ &\le 2^{1-\mu(\ell+t)} \le \delta(\ell+t),\end{aligned}$$

and this implies that $|f_k(x,y)| \le d((x,y), X \times Y)^{2(\ell+t)}$, so

$$\frac{|f_k(x,y)|}{\varphi(x)+\psi(y)} \le 2^{\ell+t}d((x,y), X \times Y)^{\ell+t} \le \left(\frac{2}{2^{\mu(t)}}\right)^{\ell+t} < \varepsilon. \qquad \square$$

Proof of Theorem 4.11. (i) We first consider the case where X and Y are compact. Given $f_k \in m_{X\times Y}^\infty$, define

$$g_{k,r,p}(x,y) = |D^r f_k(x,y)|^{\frac{1}{p}}.$$

By Lemma 4.15, $g_{k,r,p} \in L_{X\times Y}^\infty$, and we can use Lemma 4.16 to obtain $\varphi \in m_X^\infty$, $\psi \in m_Y^\infty$ with $Z(\varphi) = X$, $Z(\psi) = Y$, and moreover

$$\frac{g_{k,r,p}(x,y)}{\varphi(x)+\psi(y)} \longrightarrow 0 \quad \text{whenever} \quad d((x,y), X \times Y) \longrightarrow 0.$$

Now define functions h_k on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$h_k(x,y) = \begin{cases} \frac{f_k(x,y)}{\varphi(x)+\psi(y)} & \text{if } (x,y) \notin X \times Y \\ 0 & \text{if } (x,y) \in X \times Y. \end{cases}$$

Then any partial derivative of h_k involves sums of terms whose absolute values are

$$\frac{|D^r f_k(x,y) D^s(\varphi+\psi)(x,y)|}{(\varphi(x)+\psi(y))^p} = \left(\frac{g_{k,r,p}(x,y)}{\varphi(x)+\psi(y)}\right)^p |D^s(\varphi+\psi)(x,y)|,$$

and we conclude that h_k is smooth. So

$$f_k(x,y) = (\varphi(x)+\psi(y)) \cdot h_k(x,y) \quad \text{with} \quad h_k \in m^\infty_{X \times Y},$$

and the theorem is proved for the special case where X and Y are compact.

(ii) For the general case where X and Y are just closed, let $\{U_i\}$ and $\{V_i\}$ be locally finite covers of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively by bounded open sets, and let $\{\rho_i\}, \{\nu_j\}$ be partitions of unity subordinate to these covers. We define

$$X_i = X \cap \operatorname{supp}(\rho_i), \;\; Y_i = Y \cap \operatorname{supp}(\nu_j).$$

As in the case (i), we can find for each i, j functions $\varphi_i \in m^\infty_{X_i}$ and $\varphi_j \in m^\infty_{Y_j}$ with $Z(\varphi_i) = X_i$, $Z(\psi_j) = Y_j$, such that for each k, r, p

$$(*) \quad \frac{g_{k,r,p}(x,y)}{\varphi_i(x)+\psi_j(y)} \longrightarrow 0 \quad \text{if} \quad d((x,y), X_i \times Y_j) \longrightarrow 0.$$

Now define

$$\varphi(x) = \sum_i \rho_i(x)\varphi_i(x), \quad \psi(y) = \sum_j \nu_j(y)\psi_j(y).$$

Clearly $\varphi(x) \in m^\infty_X, \psi(y) \in m^\infty_Y$, and $Z(\varphi) = X$, $Z(\psi) = Y$. So it remains to show that for each k the function

$$h_k(x,y) = \begin{cases} \frac{f_k(x,y)}{\varphi(x)+\psi(y)} & \text{if } (x,y) \notin X \times Y \\ 0 & \text{if } (x,y) \in X \times Y \end{cases}$$

is smooth and flat on $X \times Y$. So suppose (x_n, y_n) is a sequence in $\mathbb{R}^n \times \mathbb{R}^m$ coverging to $(x,y) \in X \times Y$. Then there are neighbourhoods $U \ni x$ and $V \ni y$ such that for some $i, j, \rho_i > 0$ on $\overline{U}$ and $\nu_j > 0$ on $\overline{V}$, and we may assume that $x_n \in U$, $y_n \in V$, all n. $\overline{U}$ and $\overline{V}$ are compact, so there is a lower bound $m > 0$ for both $\rho_i|\overline{U}$ and $\nu_j|\overline{V}$. Then

$$\varphi(x_n) + \psi(y_n) \geq m \cdot (\varphi_i(x_n) + \psi_j(y_n))$$

and $(x, y) \in U \times V \cap X \times Y \subset X_i \times Y_j$, so $(*)$ above implies that

$$\frac{g_{k,r,p}(x_n, y_n)}{\varphi(x_n) + \psi(y_n)} \longrightarrow 0.$$

Thus it follows as before that all partial derivatives of h_k are continuous. $\square$

4.17 Remark. There is an alternative proof of Corollary 4.12 from case (i) by using 4.13 to conclude first that (m_X^∞, m_Y^∞) is germ-determined: if $\{\theta_\alpha(x, y)\}_\alpha$ is a countable partition of unity and each $\theta_\alpha \cdot f \in (m_X^\infty, m_Y^\infty)$ for some $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, then we may find $\varphi_\alpha \in m_X^\infty$, $\psi_\alpha \in m_Y^\infty$ such that $\theta_\alpha \cdot f \in (\varphi_\alpha, \psi_\alpha)$, and from 4.13, we conclude that $\theta_\alpha \cdot f \in (\varphi(x), \psi(y))$ for some $\varphi \in m_X^\infty, \psi \in m_Y^\infty$. But this ideal is germ determined, so $f \in (\varphi, \psi) \subseteq (m_X^\infty, m_Y^\infty)$.

To prove 4.12, suppose $f(x, y) \in m_{X \times Y}^\infty$. Then using the notation of case (ii) above, case (i) implies that for each i, j

$$f(x, y) \in (\varphi_i(x) + \psi_j(y)),$$

so $\rho_i(x)\nu_j(y)f(x, y) \in (\rho_i(x)\varphi_i(x), \nu_j(y)\psi_j(y))$, and since for each i, j, $\rho_i(x)\varphi_i(x) \in m_X^\infty$, $\nu_j(y)\psi_j(y) \in m_Y^\infty$, we find

$$\rho_i\nu_j f \in (m_X^\infty, m_Y^\infty).$$

The latter ideal is germ determined, so $f \in (m_X^\infty, m_Y^\infty)$.

4.18 Corollary. *The C^∞-ring*

$$\mathbb{R}[[X_1, \ldots, X_n]]/I \otimes_\infty \mathbb{R}[[Y_1, \ldots, Y_m]]/J$$

is canonically isomorphic to $\mathbb{R}[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]]/(I, J)$.

Proof. By the general formula $A/I \otimes_\infty B/J \cong A \otimes_\infty B/(I, J)$ (page 26) it suffices to show that

$$\mathbb{R}[[X_1, \ldots, X_n]] \otimes_\infty \mathbb{R}[[Y_1, \ldots, Y_m]] \cong \mathbb{R}[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]].$$

But this follows by applying Corollary 4.12 with $X = \{0\} \subset \mathbb{R}^n$, $Y = \{0\} \subseteq \mathbb{R}^m$, and Borel's theorem 1.3. $\square$

4.19 Corollary. *Let $X \underset{\neq}{\subset} \mathbb{R}^n$ be closed. Then m_X^∞ is not countably generated.*

Proof. By 4.13, $m_X^\infty \subseteq m_X^\infty \cdot m_X^\infty$, and m_X^∞ would be finitely generated (even principal) if it were countably generated. But in that case it would follow from the Nakayama Lemma (3.15) that $m_X^\infty = (0)$.□

4.20 Remark. (a) Applying 4.12 to the case where $Y = \mathbb{R}^m$ we have that as ideals in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\pi_1^*(m_X^\infty) = m_{X\times\mathbb{R}^m}^\infty$$

(recall that $\pi_1^*(m_X^\infty)$ is the ideal generated by $\{f \circ \pi_1 | f \in m_X^\infty\}$). So $\pi_1^*(m_X^\infty)$ is closed. We don't know in general whether $\pi_1^*(I)$ is a closed ideal in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ when I is closed in $C^\infty(\mathbb{R}^n)$.

(b) If $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are compact, then we have the following equality of ideals in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$(m_X^g, m_Y^g) = (m_{X\times Y}^g).$$

This can be proved along the lines of 3.3, but it also follows from 4.12. Indeed, if $f \in m_{X\times Y}^g$, i.e. $f|U \equiv 0$ for some $U \supseteq X \times Y$, then since X and Y are compact, we can find open $V \supset X$, $W \supset Y$ such that $V \times W \subseteq U$. By 4.12, $(m_{\overline{V}}^\infty, m_{\overline{W}}^\infty) = (m_{\overline{V}\times\overline{W}}^\infty)$ so $f \in (m_{\overline{V}}^\infty, m_{\overline{W}}^\infty) \subseteq (m_X^g, m_Y^g)$.

Note that this equality is false if X and Y are not both compact (cf. the example preceding 4.11).

Chapter II

C^∞-Rings as Variable Spaces

In this chapter we will consider the geometric aspect of C^∞-rings. As a first step, we will introduce the category $\mathbb{L}$ of loci or formal C^∞-varieties. These are just the duals of finitely generated C^∞-rings. This category $\mathbb{L}$ contains the usual category of manifolds, but also other useful objects such as infinitesimal spaces. However, function spaces can in general not be constructed in $\mathbb{L}$. This is only possible when the domain space is sufficiently small (the dual of a Weil algebra).

In order to have all function spaces available, we will, in Section 2, extend the category $\mathbb{L}$ to the category $\mathit{Sets}^{\mathbb{L}^{op}}$ of presheaves on $\mathbb{L}$, the so-called smooth functors. This larger category has good function spaces and convenient exactness properties. Our viewpoint will be to regard $\mathit{Sets}^{\mathbb{L}^{op}}$ as a generalized set-theoretic universe, where—intuitively—every set is a smooth space (and the old sets are embedded as discrete spaces). By way of some simple examples, we will try to explain how set-theoretic constructions and arguments can be carried out in $\mathit{Sets}^{\mathbb{L}^{op}}$. This will enable us to do analysis "inside" this richer universe. After having discussed some basic properties of "the smooth line" in $\mathit{Sets}^{\mathbb{L}^{op}}$, we will, in Section 3, turn to some more complicated and more interesting properties of this universe.

As will be pointed out, the universe $\mathit{Sets}^{\mathbb{L}^{op}}$ has also many undesirable or even pathological properties, and it is mainly introduced as a first step towards the "better worlds" which will be considered in later chapters.

1 The Category $\mathbb{L}$ of Loci

In this section we will try to stress the "*geometric*" aspect of C^∞-rings, rather than the algebraic one, by studying the category of *loci*, or *formal C^∞-varieties*, which is simply the opposite category of the category of finitely generated C^∞-rings. Consequently, some of the things we will say here are simply dual to results of earlier sections. But nevertheless, it is important to become familiar with the geometric way of looking at C^∞-rings, since in later chapters it is as "smooth *spaces*", i.e. as geometric objects, that C^∞-rings are going to appear in the various toposes; we will study toposes in which the category of C^∞-rings is *contravariantly* embedded.

Let us introduce some notation. In Section I.1, we defined the category of finitely generated C^∞-rings and C^∞-homomorphisms, denoted by $(C^\infty\text{-rings})_{\mathrm{fg}}$. We define the category $\mathbb{L}$ of *loci* (sometimes called *formal C^∞-varieties*) as

(1) $\mathbb{L}$ is the opposite category of $(C^\infty - \mathit{rings})_{fg}$.

So the objects of $\mathbb{L}$ are "the same" as those of $(C^\infty - \mathit{rings})_{fg}$, but the arrows are reversed. To avoid confusion about whether a C^∞-ring is regarded as an object of $(C^\infty - \mathit{rings})_{fg}$ or as an object of $\mathbb{L}$, we write $\ell(A)$, an object of $\mathbb{L}$, for the locus corresponding to a given C^∞-ring A. So ℓA is the *formal dual* of A, and $\mathbb{L}$-maps $\ell B \longrightarrow \ell A$ are C^∞-homomorphisms $A \longrightarrow B$. Let us once more repeat the explicit description of maps in $\mathbb{L}$:

(2) If $B = C^\infty(\mathbb{R}^n)/J$, $A = C^\infty(\mathbb{R}^m)/I$, a map $\ell B \longrightarrow \ell A$ is an equivalence class of smooth functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ with the property that $f \in I \Rightarrow f \circ \varphi \in J$, while φ is equivalent to φ' if componentwise, $\varphi_i - \varphi_i' \in J (i = 1, \ldots, m)$.

It is important to observe that the reverse operation of "deleting ℓ" can be described entirely within $\mathbb{L}$, i.e. the C^∞-ring is recoverable from its locus ℓA, since $A \cong \mathit{hom}_{\mathbb{L}}(\ell A, \ell C^\infty(\mathbb{R}))$.

For the object $\ell C^\infty(\mathbb{R})$ of $\mathbb{L}$ we will also write R. So for the n-fold product we have by I.1.1,

$$R^n \cong \ell C^\infty(\mathbb{R}^n).$$

If ℓA is an arbitrary object of $\mathbb{L}$, with $A = C^\infty(\mathbb{R}^n)/I$, we will also write suggestively

$$\ell A = \{x \in R^n | f(x) = 0 \quad \text{all} \quad f \in I\},$$

i.e. ℓA is the *"formal zero-set"* of I, a subobject of R^n in $\mathbb{L}$, as opposed to the "real" zeroset $Z(I)$, which is a subobject of $\mathbb{R}^n$ in $\mathbb{E}$ (cf. I.1.5). (Later on we will see that this notation $\{x \in R^n | f(x) = 0,$ all $f \in I\}$ is not just suggestive. ℓA is *really* the zero set of I, not in the category of sets, but in an extended universe where spaces have "more" points.)

In fact, $\mathbb{E}$ also appears as a dual, namely (and this is not a definition)

(2) $\mathbb{E}$ is equivalent to the opposite category of the category of finitely generated point determined C^∞-rings.

So no new symbol is needed for this opposite category. For the other cases we introduce the following notation.

(3) $\mathbb{G}$ is the opposite category of the category of germ determined finitely generated C^∞-rings.

(4) $\mathbb{F}$ is the opposite category of the category of closed finitely generated C^∞-rings.

If A is a C^∞-ring of the appropriate sort, we denote its dual, an object of $\mathbb{G}$ or $\mathbb{F}$, again by $\ell(A)$.

We now have a sequence of full subcategories

$$\mathbb{E} \hookrightarrow \mathbb{G} \hookrightarrow \mathbb{F} \hookrightarrow \mathbb{L},$$

and these inclusions all have right adjoints, since as we saw in Section 4 (following 4.10), there exist left adjoints at the level of C^∞-rings.

We introduce some more *notation*:

(a) the right adjoint $\mathbb{L} \longrightarrow \mathbb{E}$ to the inclusion is called γ, as are its restrictions to $\mathbb{G}$ and $\mathbb{F}$. So

$$\gamma: \mathbb{L} \to \mathbb{E}, \quad \gamma\ell(C^\infty(\mathbb{R}^n)/I) = Z(I).$$

(b) the right adjoint $\mathbb{L} \longrightarrow \mathbb{G}$, as well as its restriction to $\mathbb{F}$, is called λ. So

$$\lambda: \mathbb{L} \to \mathbb{G}, \quad \lambda(\ell(C^\infty(\mathbb{R}^n)/I)) = \ell(C^\infty(\mathbb{R}^n)/\tilde{I}).$$

(c) the right adjoint $\mathbb{L} \longrightarrow \mathbb{F}$ is called κ. So

$$\kappa: \mathbb{L} \to \mathbb{F}, \quad \kappa(\ell(C^\infty(\mathbb{R}^n)/I)) = \ell(C^\infty(\mathbb{R}^n)/\overline{I}).$$

For the record, we state

1.1 Proposition. *The full inclusions* $\mathbb{E} \subset \mathbb{F} \subset \mathbb{G} \subset \mathbb{L}$ *are inclusions* *of coreflective subcategories, i.e. all inclusions have right adjoints.* □

Since the inclusions are full, we have isomorphisms

$$\gamma(X) \cong X \text{ in } \mathbb{E}, \quad \gamma(\ell A) \cong \ell A \text{ in } \mathbb{G}, \quad \kappa(\ell B) \cong \ell B \text{ in } \mathbb{F}$$

which are natural in X, ℓA, ℓB respectively. (Watch out: to the left of the isomorphism signs, X stands for the image of X under the embedding $\mathbb{E} \hookrightarrow \mathbb{L}$, ℓA for the image under $\mathbb{G} \hookrightarrow \mathbb{L}$, ℓB for the image under $\mathbb{F} \hookrightarrow \mathbb{L}$.)

Also, for $\ell A \in \mathbb{L}$ we have canonical monomorphisms

$$\gamma(\ell A) \hookrightarrow \ell A, \quad \lambda(\ell A) \hookrightarrow \ell A, \quad \kappa(\ell A) \hookrightarrow \ell A \text{ in } \mathbb{L}.$$

(Again, $\gamma(\ell A)$ stands for the image of $\gamma(\ell A)$ under $\mathbb{E} \hookrightarrow \mathbb{L}$, etc.)

The right adjoints can be used to describe inverse limits in $\mathbb{G}$ or in $\mathbb{F}$ (in $\mathbb{E}$, you already know how to do it: just inverse limits of topological spaces): first compute the limit in $\mathbb{L}$, then coreflect back into $\mathbb{G}$ or $\mathbb{F}$. So for example, if ℓA, ℓB are objects of $\mathbb{G}$, their product in $\mathbb{G}$ is

$$\ell A \times_{\mathbb{G}} \ell B = \lambda(\ell A \times_{\mathbb{L}} \ell B) = \lambda(\ell(A \otimes_\infty B)).$$

An important advantage of passing to the opposite category is the manifolds now become correctly, i.e. covariantly, embedded. We can copy 2.8 as

1.2 Proposition. *The functor* $\mathbb{M} \xrightarrow{s} \mathbb{L}$, $M \mapsto \ell C^\infty(M)$, *is full and faithful, and preserves transversal pullbacks.*

1.3 Closed Subloci. If $A = C^\infty(\mathbb{R}^n)/I$ and $B = C^\infty(\mathbb{R}^n)/J$ with $J \supset I$, there is a quotient map $A \twoheadrightarrow B$, hence a monomorphism of loci $\ell B \rightarrowtail \ell A$. If we have an inclusion (isomorphic to one) of this form, we will say that ℓB is a *closed sublocus* of ℓA. Closed subloci of a fixed locus ℓA behave like closed subspaces of a topological space, in the sense that arbitrary meets and finite joins of closed subloci are closed. For meets this is clear: if $\{\ell B_\alpha \rightarrowtail \ell A\}_\alpha$ is a collection of closed subloci of ℓA with $A = C^\infty(\mathbb{R}^n)/I$, and $B_\alpha = C^\infty(\mathbb{R}^n)/J_\alpha$, then $B = C^\infty(\mathbb{R}^n)/(\cup_\alpha J_\alpha)$ gives a largest sublocus of $\ell B \rightarrowtail \ell A$ contained in every ℓB_α, $\ell B = \cap_\alpha \ell B_\alpha$. (More precisely, one has to show that if $\varphi: \ell C \to \ell A$ is any map of $\mathbb{L}$ and φ factors through all the ℓB_α, then φ factors through ℓB, which is easy.)

For finite sups, take two closed subloci $\ell B_1 \rightarrowtail \ell A$ and $\ell B_2 \rightarrowtail \ell A$, say $B_i = C^\infty(\mathbb{R}^n)/J_i$. Then $\ell B = \ell C^\infty(\mathbb{R}^n)/(J_1 \cap J_2)$ is the smallest subobject of ℓA containing ℓB_1 and ℓB_2, i.e. $\ell B = \ell B_1 \cup \ell B_2$, because the diagram

$$\begin{tikzcd}
\ell B_1 \cap \ell B_2 \arrow[r] \arrow[d] & \ell B_1 \arrow[d] \\
\ell B_2 \arrow[r] & \ell B
\end{tikzcd}$$

is a pushout in $\mathbb{L}$ (and also a pullback, but this is irrelevant here). Let us give the details: we have to show that the dual square

$$\begin{tikzcd}
C^\infty(\mathbb{R}^n)/(J_1 + J_2) & C^\infty(\mathbb{R}^n)/J_1 \arrow[l] \\
C^\infty(\mathbb{R}^n)/J_2 \arrow[u] & C^\infty(\mathbb{R}^n)/(J_1 \cap J_2) \arrow[l] \arrow[u]
\end{tikzcd}$$

is a pullback. So take two C^∞-homomorhisms $C^\infty(\mathbb{R}^m)/K \xrightarrow{\Phi} C^\infty(\mathbb{R}^n)/J_1$ and $C^\infty(\mathbb{R}^m)/K \xrightarrow{\Psi} C^\infty(\mathbb{R}^n)/J_2$, represented by smooth $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ and $\psi: \mathbb{R}^n \to \mathbb{R}^m$ respectively, such that $\varphi_i - \psi_i \in J_1 + J_2$ $(i = 1, \ldots, m)$, say $\varphi_i - \psi_i = h_i + k_i$ with $h_i \in J_1$, $k_i \in J_2$. Then $\varphi_i - h_i$ also represents Φ, and $\varphi_i - h_i = \psi_i + k_i$ also represents Ψ. In other words, we may assume $\varphi = \psi$. But then if $f \in K$ we have $f \circ \varphi = f \circ \psi \in I_1 \cap I_2$. Summarizing,

Proposition. *Finite sups and arbitrary meets of closed subloci of a given locus ℓA are again closed.*

1.4 Open Subloci. If $A = C^\infty(\mathbb{R}^n)/I$ and U is an open subspace of $\mathbb{R}^n$, we have a restriction map $A \longrightarrow C^\infty(U)/(I|U)$, or a map of loci $\ell(C^\infty(U)/(I|U)) \longrightarrow \ell A$. We denote this dual also by

$$\ell A \cap s(U) = \ell(C^\infty(U)/(I|U))$$

(s is the functor $\mathbb{M} \hookrightarrow \mathbb{L}$ of 1.2). This notation is justified, since $\ell A \cap s(U) \to \ell A$ is a monomorphism in ℓ, and we have a pullback diagram

$$\begin{tikzcd}
\ell A \cap s(U) \arrow[r, tail] \arrow[d] & \ell A \arrow[d] \\
s(U) \arrow[r, tail] & \mathbb{R}^n
\end{tikzcd}$$

in $\mathbb{L}$. Indeed, the corresponding dual diagram of C^∞-rings is a pushout, as follows immediately from the universal properties (i.e., $(C^\infty(\mathbb{R}^n)/I)\{a^{-1}\} \cong (C^\infty(\mathbb{R}^n)\{a^{-1}\})/\eta(I)$, as we stated at the end of Section 1.

We have just seen that closed subloci behave nicely with respect to arbitrary infs and finite sups. The dual situation for open subloci is not quite as nice, unfortunately. For finite sups and meets there is no problem. The smallest sublocus of ℓA containing $\ell A \cap s(U)$ and $\ell A \cap s(V)$ is $\ell A \cap s(U \cup V)$, so is again open (this is really Lemma I.2.7), and the meet of $\ell A \cap s(U)$ and $\ell A \cap s(V)$ is $\ell A \cap s(U \cap V)$. But arbitrary sups of open subobjects need not be open. For example, consider $\ell A = \ell(C^\infty(\mathbb{R}) \otimes_\infty C_0^\infty(\mathbb{R}))$ (cf. the example preceding I.4.11). Let $U_n \subseteq \mathbb{R} \times \mathbb{R}$ be the open $(-n, n) \times \mathbb{R}$. Then one can check that $\bigvee_{n \in \mathbb{N}}(\ell A \cap s(U_n))$ is the subobject $\ell C^\infty(\mathbb{R} \times \mathbb{R})/m^g_{\{0\}}$ of ℓA, which is actually a *closed* subobject.

Sups of open subobjects are much better behaved in the category $\mathbb{G}$ of germ determined loci, as we will show and exploit in chapter III.

We want to single out one statement which follows trivially from the fact that finite unions of opens are well behaved, i.e. that $(\ell A \cap s(U)) \cup (\ell A \cap s(V)) = \ell A \cap s(U \cup V)$:

1.5 Corollary. *Let $\ell A \in \mathbb{L}$, say $A = C^\infty(\mathbb{R}^n)/I$, so $\ell A \rightarrowtail R^n$. If $U, V \subseteq R^n$ are open and $\ell A \subset s(U \cup V)$ as subobjects of R^n, then $\ell A \cap s(U) = 0$ implies $\ell A \subset s(V)$.* □

Closed and open subloci have some properties similar to closed and open subspaces of normal spaces. We list some of these properties for subloci of R^n.

1.6 Proposition. *(Tietze extension theorem for loci). If $\ell A \rightarrowtail \ell B$ is closed, then any map $\ell A \longrightarrow R$ in $\mathbb{L}$ can be extended to a map $\ell B \longrightarrow R$ in $\mathbb{L}$.*

Proof. Trivial, since R is the dual of the free C^∞-ring on one generator. □

1.7 Lemma. *Let $\ell A \subset R^n$ be a closed sublocus, say $A = C^\infty(\mathbb{R}^n)/I$, and $s(U) \subset R^n$ an open one. Then $\ell A \subset s(U)$ iff there is a finitely generated ideal $I_0 \subset I$ such that $Z(I_o) \subset U$.*

Proof. ($\Rightarrow$) $\ell A \subset s(U)$ means that we have a commutative diagram of C^∞-rings

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[rr] \arrow[dr] & & C^\infty(\mathbb{R}^n)/I \arrow[dl, "\Phi"] \\ & C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1) & \end{tikzcd}$$

where a is a characteristic function for U, and Φ is induced by $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$. Thus $\varphi_{n+1} \cdot a(\varphi_1, \ldots, \varphi_n) - 1 \in I$, so if we put $I_0 = (\varphi_{n+1} \cdot a(\varphi_1, \ldots, \varphi_n) - 1)$ we have $I_0 \subset I$, and $Z(I_0) \subset U$, since the corresponding diagram of zero sets (apply the functor $\gamma\colon \mathbb{L} \to \mathbb{E}$)

$$\begin{tikzcd} \mathbb{R}^n & Z(I_0) \arrow[l] \arrow[dl, "\varphi"] \\ \hat{U} \arrow[u, "\pi"] & \end{tikzcd}$$

commutes (as usual, $\hat{U} = \{(x, y) \in \mathbb{R}^{n+1} | y \cdot a(x) = 1\}$).

($\Leftarrow$) If $Z(I_0) \subset U$ for some finitely generated $I_0 \subset I$, then $f|U = g|U \Rightarrow f - g \in I_0$ since finitely generated ideals are germ determined (I.4.9).

So the quotient mapping

$$C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)/I_0$$

factors through the restriction $C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(U)$, in other words $\ell(C^\infty(\mathbb{R}^n)/I_0) \subseteq s(U)$. So a fortiori $\ell A \subseteq s(U)$. $\square$

1.8 Proposition. *(a) s preserves closure of open subloci, thus $\ell(C^\infty(\mathbb{R}^n)/m^0_{\overline{U}})$ is the smallest closed sublocus of R^n containing $s(U)$.*

(b) (normality) Let $\ell A \subset R^n$ be closed and let $s(U) \subset R^n$ be

open. If $\ell A \subset s(U)$ then there exist an open $s(V) \subset R^n$ and a closed $\ell B \subset R^n$ such that $\ell A \subset s(V) \subset \ell B \subset s(U)$.

Proof. (a) Suppose $s(U) \subset \ell A \subset R^n$, where $A = C(\mathbb{R}^n)/I$. Then the restriction $C(\mathbb{R}^n) \longrightarrow C(U)$ factors through $C(\mathbb{R}^n)/I$, so if $f \in I$ then $f|U = 0$. But then also $f|\overline{U} = 0$, i.e. $I \subseteq m^0_{\overline{U}}$.

(b) This follows immediately from (a) and 1.7: let $\ell A \subset s(U)$ and write $A = C^\infty(\mathbb{R}^n)/I$. Take a finitely generated $I_0 \subset I$ with $Z(I_0) \subset U$, and apply normality of $\mathbb{R}^n$ to obtain an open V with $Z(I_0) \subset V \subset \overline{V} \subset U$. Now let $B = C^\infty(\mathbb{R}^n)/m^0_{\overline{V}}$. □

1.9 Proposition. *(normality) (a) If ℓA, $\ell B \subset R^n$ are closed and $\ell A \cap \ell B = 0$, then there are disjoint opens U, $V \subseteq \mathbb{R}^n$ with $\ell A \subset s(U)$, $\ell B \subset s(V)$ (and hence by 1.2, $s(U) \cap s(V) = 0$).*

(b) If ℓA, $\ell B \subset R^n$ and $\ell A \cap \ell B = 0$, then there is a morphism $R^n \xrightarrow{f} R$ in $\mathbb{L}$ with $f|\ell A \equiv 1$, $f|\ell B \equiv 0$.

Proof. (b) follows from (a), 1.8 (b) and external normality of $\mathbb{R}^n$ (Urysohn's lemma). For (a), write $A = C^\infty(\mathbb{R}^n)/I$ and $B = C(\mathbb{R}^n)/J$. $\ell A \cap \ell B = 0$ implies by Lemma 1.7 that there are finitely generated $I_0 \subset I$ and $J_0 \subset J$ with $Z(I_0) \cap Z(J_0) = \emptyset$. So by external normality, we find disjoint opens $U \supset Z(I_0)$ and $V \supset Z(J_0)$. Then $\ell A \subset s(U)$ and $\ell B \subset s(V)$, by 1.7. □

1.10 Some Infinitesimal Loci. If $p \in \mathbb{R}$, we can form the sublocus $\Delta_p = \cap\{s(U) | p \in U \text{ open} \subseteq \mathbb{R}\} \cong \ell C^\infty_p(\mathbb{R})$, called the germ of R at p. Similarly, if $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, we have the germ of R^n at p,

$$\Delta_p = \cap_{p \in U} s(U) = \ell C^\infty_p(\mathbb{R}^n).$$

Note that for $p \in \mathbb{R}^n$, $\Delta_p = \Delta_{p_1} \times \ldots \times \Delta_{p_n}$.

If ℓA is an arbitrary locus, where $A = C^\infty(\mathbb{R}^n)/I$, a *point of ℓA* is, of course, a map $1 \xrightarrow{p} \ell A$ in $\mathbb{L}$, where $1 = R^0 = \ell C^\infty(\mathbb{R}^0)$ is the one-point locus. Points of ℓA are in 1-1 correspondence with ordinary points of $Z(I) \subset \mathbb{R}^n$. If p is a point of ℓA, $\Delta_p \cap \ell A$ (where $\Delta_p \subset R^n$) is by definition the *germ of ℓA at p*.

Thus, whereas the usual category of manifolds $\mathbb{M}$ is too small to contain spaces which are germs of manifolds at some point, these "very small submanifolds" do exist in $\mathbb{L}$. And if M is a manifold, the germ of a function $f: M \to \mathbb{R}$ at point p as defined in Section 1 now simply becomes the restriction of f to the germ of M at p. More precisely, if M is a closed subspace of $\mathbb{R}^n$, so $s(M) \subset R^n$, germs $f|_p$

of smooth functions $M \longrightarrow \mathbb{R}$ are in 1-1 correspondence with maps $\Delta_p \cap s(M) \longrightarrow R$, and $\Delta_p \cap s(M) \cong \ell C_p^\infty(M) = \cap\{s(U) | p \in U \text{ open} \subset M\}$ in $\mathbb{L}$.

If ℓA is any locus, $A = C^\infty(\mathbb{R}^n)/I$, an ***infinitesimal sublocus*** of ℓA is by definition a sublocus contained in $\ell A \cap \Delta_p$ for some point p of R^n.

Let us write Δ for $\Delta_0 = \cap_{n\in\mathbb{N}}\ s\left(-\frac{1}{n}, \frac{1}{n}\right) \subset R$. Some important infinitesimal subloci of R at 0, i.e. subloci contained in Δ, are

$$D = \ell(C^\infty(\mathbb{R})/(x^2)) = \{x \in R | x^2 = 0\}$$

and more generally

$$D_k = \ell(C^\infty(\mathbb{R})/(x^{k+1})) = \{x \in R | x^{k+1} = 0\}.$$

Similarly, in higher dimensions we have

$$D(n) = D_1(n) \subset D_2(n) \subset D_3(n) \subset \dots$$

where

$$\begin{aligned} D_k(n) &= \ell(C_0^\infty(\mathbb{R}^n)/m^{k+1}) \\ &= \{x \in R^n | x^\alpha = 0,\ \text{ all } \alpha \text{ with } |\alpha| = k+1\}. \end{aligned}$$

These are all contained in $\Delta^n \subset R^n$.

All these infinitesimal subloci of R^n are *pointed*, i.e. the point $1 \xrightarrow{0} R^n$ factors through each $D_k(n)$. An example of an infinitesimal sublocus of R at 0 which does not have any points is the "***locus of invertible infinitesimals***"

$$\begin{aligned} \mathbb{I} &= \ell(C^\infty(\mathbb{R} - \{0\})/(m^g_{\{0\}} | \mathbb{R} - \{0\})) \\ &= \cap_{n>0} s(\left(-\tfrac{1}{n}, \tfrac{1}{n}\right) - \{0\}). \end{aligned}$$

Just as in the case of germs, k-jets of maps just become ***restrictions*** in $\mathbb{L}$. If $R^n \xrightarrow{f} R$ is a map of $\mathbb{L}$, f corresponds to a smooth function $F: \mathbb{R}^n \to \mathbb{R}$, and the k-jet $j_n^k(F)$ of F, i.e. the equivalence class of $F|_0$ modulo m^{k+1}, corresponds to the restriction $D_k(n) \xrightarrow{f} R$.

Note that by working within a parametrizable neighbourhood, we can define similar subloci, not only of R^n at 0, but of $s(M)$ at p, where p is a point of a manifold M. And jets of maps $s(M) \longrightarrow R$ are just the appropriate restrictions in $\mathbb{L}$.

We conclude this section by discussing some function-loci, or ***exponentials*** in $\mathbb{L}$. If ℓA and ℓB are loci, we wish to define a locus of maps from ℓB to ℓA, i.e. a locus $\ell(A)^{\ell(B)}$ for which there is a natural

1-1 correspondence

$$\frac{\ell C \to \ell A^{\ell B}}{\ell B \times \ell C \to \ell A}$$

of maps in $\mathbb{L}$. Such a locus $\ell A^{\ell B}$ need not exist, but if ℓB is small enough, such an object $\ell A^{\ell B}$ can be formed in $\mathbb{L}$.

For example, for the object $D = \ell(C^\infty(\mathbb{R})/(x^2))$, we may define for any locus ℓA, $A = C^\infty(\mathbb{R}^n)/I$, the *tangent-locus* $T(\ell A)$ of ℓA, by

$$T(\ell A) = \ell\left(C^\infty(\mathbb{R}^n \times \mathbb{R}^n)/(I(x), \{\sum y_i \frac{\partial f}{\partial x_i} | f \in I\})\right),$$

and we have

1.11 Proposition. *$T(\ell A)$ is (isomorphic to) the exponential $\ell(A)^D$ in $\mathbb{L}$, i.e. there is a natural 1-1 correspondence*

$$\frac{\ell C \longrightarrow T(\ell A)}{\ell C \times D \longrightarrow \ell A}$$

of maps in $\mathbb{L}$ (and hence the definition of $T(\ell A)$ as given above is independent of the representation chosen for A).

Proof. Take a C^∞-ring $C = C^\infty(\mathbb{R}^n)/J$. Maps $\ell C \longrightarrow T(\ell A)$ are represented by smooth $\varphi(z)\colon \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$, and maps $\ell C \times D = \ell(C^\infty(\mathbb{R}^m \times \mathbb{R})/(J(z), x^2)) \to \ell A$ by smooth $\psi(z,x)\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$. We claim that the correspondence is given by

$$\varphi \mapsto \tilde{\varphi}, \ \tilde{\varphi}(z,x) = \varphi_1(z) + x \cdot \varphi_2(z)$$

where $\varphi_1, \varphi_2\colon \mathbb{R}^m \to \mathbb{R}^n$ are the components of φ, and

$$\psi \mapsto \hat{\psi}, \ \hat{\psi}(z) = < \psi(z,0), \frac{\partial \psi}{\partial x}(z,0) >$$

where $\frac{\partial \psi}{\partial x} = < \frac{\partial \psi_1}{\partial x}, \ldots, \frac{\partial \psi_n}{\partial x} >$. We have to show two things:

(a) These operations are inverse to each other: one way round $\hat{\tilde{\varphi}}(z) = < \varphi_1(z), \varphi_2(z) > = \varphi(z)$. And the other way, $\tilde{\hat{\psi}}(z,x) = \psi(z,0) + x \cdot \frac{\partial \psi}{\partial x}(z,0)$, so by Hadamard $\psi(z,x) - \tilde{\hat{\psi}}(z,x)$ is of the form $x^2 u(z,x)$ for some u, i.e. ψ and $\tilde{\hat{\psi}}$ are equivalent maps of loci.

(b) φ is a map of the relevant loci iff ψ is:

($\Rightarrow$) suppose φ is a map, i.e. for all $g\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$g(x,y) \in (I(x), \{\sum y_i \frac{\partial f}{\partial x_i} | f \in I\}) \Rightarrow g \circ \varphi \in J.$$

Take $f \in I$. We have to show $f\psi \in (J, x^2)$. But

$$\begin{aligned} f(\psi(z,x)) &= f(\varphi_1(z) + x\varphi_2(z)) \\ &= f(\varphi_1(z)) + x \cdot \sum \varphi_{2i}(z)\tfrac{\partial f}{\partial x_i}(\varphi_1(z)) + x^2 u(\varphi_1(z), x\varphi_2(z)) \end{aligned}$$

(for some u, by Hadamard)

$$= f(x) \circ \varphi + x \cdot \left(\sum y_i \tfrac{\partial f}{\partial x_i}\right) \circ \varphi + x^2 u(\ldots),$$

and the first two terms are in J by hypothesis, so $f \circ \psi \in (J, x^2)$;

($\Leftarrow$) suppose ψ is a map, i.e. for all $f : \mathbb{R}^n \to \mathbb{R}$,

$$f \in I \Rightarrow f \circ \psi \in (J, x^2), \quad \text{i.e.} \quad f(\varphi_1(z) + x\varphi_2(z)) \in (J, x^2).$$

We have to show that if $g \in \{I(x), \sum y_i \frac{\partial f}{\partial x_i} | f \in I\}$, then $g \circ \varphi \in J$. Taking a generator $f \in I$ of this first ideal, we have by hypothesis that there are $g_j \in J$ such that $f \circ \psi$ can be written as

$$\begin{aligned} f(\varphi_1(z) + x\varphi_2(z)) &= \textstyle\sum_j g_j(z) u_j(z,x) + x^2 v(z,x) \\ &= \textstyle\sum_j g_j(z)(u_j(z,0) + x \cdot \tilde{u}_j(z,0)) + x^2 v(z,x). \end{aligned}$$

Writing $f(\varphi_1(z) + x\varphi_2(z)) = f(\varphi_1(z)) + x \cdot \sum_{i=1}^n \varphi_{2i}(z) \frac{\partial f}{\partial x_i}(\varphi_1(z)) +$ $x^2(\ldots)$, and substituting $x = 0$ we find

$$(f \circ \varphi)(z) = f(\varphi_1(z)) = \sum g_j(z) \cdot u_j(z, o) \in J.$$

Hence for all x,

$$x \cdot \sum_{i=1}^n \varphi_{2i}(z) \cdot \frac{\partial f}{\partial x_i}(\varphi_1(z)) + x^2(\ldots) = x[\sum_j g_j(z) \cdot \tilde{u}_j(z,o) + x \cdot v(z,x)]$$

so cancelling x, and then taking $x = 0$, we obtain

$$\sum_{i=1}^n \varphi_{2i}(z) \frac{\partial f}{\partial x_i}(\varphi_1(z)) = \sum_j g_j(z) \cdot \tilde{u}_j(z,o) \in J$$

or,

$$\left(\sum_{i=1}^n y_i \frac{\partial f}{\partial x_i}\right) \circ \varphi \in J. \qquad \square$$

In order to justify the name tangent-locus, we will show that this is really the usual tangent space of a manifold M in the case that $A = s(M)$.

1.12 Proposition. *Let $M \in \mathbb{M}$ be a manifold. Then*

$$s(M)^D \cong s(TM),$$

where TM is the total space of the tangent bundle of M. Moreover,

s maps the projection $TM \longrightarrow M$ to the map $\mathrm{ev}_0\colon s(M)^D \to s(M)$. *So a tangent vector at a point $p \in M$ is the same thing as a point $1 \xrightarrow{v} s(M)^D$ in $\mathbb{L}$ such that $\mathrm{ev}_0 \circ v = p$, or equivalently, a map $D \xrightarrow{v} s(M)$ such that $1 \xrightarrow{0} D \xrightarrow{v} s(M) = p$.*

Proof. Let M be a closed submanifold of $\mathbb{R}^p$, say. Writing M as a retract of an open U, $M \subset U \subset \mathbb{R}^p$, we find as in I.2.3 that we may assume that $C^\infty(M) \cong C^\infty(\mathbb{R}^n)/(f_1, \ldots, f_m)$, where M is identified with a closed subspace of $\mathbb{R}^n$, and $(f_1, \cdots, f_m) = m_M^0$. By 1.11 above,

$$(1) \quad s(M)^D \cong \ell(C^\infty(\mathbb{R}^n \times \mathbb{R}^n)/(m_M^0, \{\sum y_i \frac{\partial f}{\partial x_i}(x) | f(x) \in m_M^0\})),$$

and since $m_M^0 = (f_i, \ldots, f_m)$ in $C^\infty(\mathbb{R}^n)$, also

$$\begin{aligned}&(m_M^0(x), \{\textstyle\sum y_i \frac{\partial f}{\partial x_i}(x) | f \in m_M^0\}) = \\ &(f_1(x), \ldots, f_m(x), \textstyle\sum y_i \frac{\partial f_1}{\partial x_i}(x), \ldots, \sum y_i \frac{\partial f_m}{\partial x_i}(x)),\end{aligned}$$

in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. TM is a closed submanifold of $\mathbb{R}^n \times \mathbb{R}^n$, and we claim that for any $g(x, y)\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

(2)
$$g(x,y)|TM = 0 \Leftrightarrow g(x,y) \in \left(m_M^0(x), \{\sum y_i \frac{\partial f}{\partial x_i}(x) | f \in m_M^0\}\right).$$

Since both m_{TM}^0 and the ideal on the right-hand side of (2) are germ-determined (the latter is in fact finitely generated, as we just noted), it suffices to find an open cover $\{U_\alpha\}$ of $\mathbb{R}^n$ such that (2) holds on each of the pieces $U_\alpha \times \mathbb{R}^n$. But by the local immersion theorem, there is an open cover $\{U_\alpha\}$ of $\mathbb{R}^n$ such that the inclusion $U_\alpha \cap M \subset M$ is diffeomorphic (equivalent) to $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. So it suffices to show (*) for the special case $M = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Then $m_M^0 = (x_{k+1}, \ldots, x_n)$ and $TM = (\mathbb{R}^k \times \{0\}) \times (\mathbb{R}^k \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^n$. So clearly $g \in (m_M^0(x), \{\sum y_i \frac{\partial f}{\partial x_i} | f \in m_M^0\} \Rightarrow g|TM = 0$. Conversely, if $g|TM = 0$ then by Hadamard, we have $g(x,y) = g(x_1, \ldots, x_k, 0, \ldots, 0, y_1, \ldots, y_k, 0, \ldots, 0) + \sum_{i=k+1}^n x_i h_i(x,y) + \sum_{j=k+1}^n y_j k_j(x,y)$. Now the first term vanishes, $x_i \in m_M^0(x)$, and $y_j = \sum_{i=1}^n y_i \frac{\partial f}{\partial x_i}(x)$ for $f \equiv x_j \in m_M^0(x)$, so $g \in (m_M^0(x), \{\sum y_i \frac{\partial f}{\partial x_i} | f \in m_M^0(x))$.

The other assertions in the proposition are clear. $\square$

The last two propositions about exponentiation can be generalized to arbitrary Weil algebras, i.e. to closed subloci of some $D_k(n)$.

1.13 Theorem. *Let W be a Weil algebra. Then $\ell W \in \mathbb{L}$ is exponentiable, i.e. for any $\ell A \in \mathbb{L}$ the exponential $\ell A^{\ell W}$ exists in $\mathbb{L}$.*

Proof. We will give an argument which is simpler than the proof of 1.11 for the case $\ell W = D$, but less explicit.

Let $A = C^\infty(\mathbb{R}^n)/I$. Then ℓA is the joint equalizer of the maps

$$R^n \underset{0}{\overset{\{f : f \in I\}}{\rightrightarrows}} R$$

in $\mathbb{L}$, and since $(-)^{\ell W}$ preserves all inverse limits, ℓA can be constructed as the joint equalizer

$$\ell A^{\ell W} \longrightarrow (R^n)^{\ell W} \underset{0^{\ell W}}{\overset{\{f^{\ell W} : f \in I\}}{\rightrightarrows}} R^{\ell W}$$

so it suffices to show that each $(R^n)^{\ell W}$ can be constructed in $\mathbb{L}$. And since $(R^n)^{\ell W} \cong (R^{\ell W})^n$ (if these exist), we may assume $n = 1$, i.e. we are left with the task of showing that $R^{\ell W}$ exists in $\mathbb{L}$.

Let ℓB be any locus, $B = C^\infty(\mathbb{R}^m)/J$, and suppose that $W = C_0^\infty(\mathbb{R}^\ell)/(m^k + J)$ for some ideal J. Then an element of $B \otimes_\infty W$ can be represented by a smooth function $F(z, w) : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}$ of the form

$$F(z, w) = \sum_{|\alpha| \le k} w^\alpha D^\alpha F(z, 0),$$

i.e. F corresponds to a $j(k, \ell)$ – tuple of elements of B (here $j(k, \ell) = \sharp\{\alpha = (\alpha_1, \ldots, \alpha_\ell) \,||\alpha| \le k\}$). The ideal J may create some further linear dependencies among the $w^\alpha, |\alpha| \le k$, but it is clear that as $\mathbb{R}$-vector spaces

$$B \otimes_\infty W \cong \bigoplus_{i=1}^{\dim W} B$$

where $\dim W$ is the dimension of W as a real vector space. Thus we have 1-1 correspondences

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\ell B \times \ell W \xrightarrow{f} R}{C^\infty(\mathbb{R}) \xrightarrow{f} B \otimes_\infty W}}{f \in B \otimes_\infty W}}{\langle f_1, \ldots, f_{\dim W}\rangle \in \oplus_{i=1}^{\dim W} B}}{\langle f_i : \ell B \to R\rangle_{i=1}^{\dim W}}}{f : \ell B \to R^{\dim W}}$$

so the exponential $R^{\ell W}$ can be constructed as the product $R^{\dim W}$ in $\mathbb{L}$. □

1.14 Remark. Just as in the case of 1.12, one can show that for a manifold M, the exponential $s(M)^{D_k(n)}$ corresponds under the embedding $s : \mathbb{M} \hookrightarrow \mathbb{L}$ to the bundle of k-jets in n variables on M. And more generally, $s(M)^{\ell W}$ is isomorphic to $s(N)$ for some manifold N which has the structure of a vector bundle over M with the bundle projection corresponding to the evaluation $\mathrm{ev}_0 : s(N) = s(M)^{\ell W} \to s(M)$. These "generalized jet-bundles" over M have been introduced by A. Weil, and are called *prolongations of M*.

To be able to conclude that the exponentials of the form $\ell A^{\ell W}$ also exist in $\mathbb{G}$ and $\mathbb{F}$, we need

1.15 Lemma. *Let $A = C^\infty(\mathbb{R}^n)/I$ be a finitely generated C^∞-ring, and let W be a Weil algebra. If A is germ determined, then so is $A \otimes_\infty W$. And if A is closed, then so is $A \otimes_\infty W$.*

Proof. Let k be the vector space dimension of W. We have seen in the proof of 1.13 that at the level of real vector spaces,

$$A \otimes_\infty W \cong A \oplus \ldots \oplus A \quad (k \text{ times}).$$

So if $A \longrightarrow \Pi_{i=1} B_i$ is an injective homomorphism of C^∞-rings, so is $A \otimes_\infty W \longrightarrow \Pi_{i \in I} B_i \otimes_\infty W$. Since A is germ determined (resp. closed) iff A can be embedded in a product of pointed local C^∞-rings (resp. Weil algebras), it suffices to show that if B is pointed local then so is $B \otimes_\infty W$, and if B is a Weil algebra, then so is $B \otimes_\infty W$. But this is clear (cf. 3.10, 3.18). □

1.16 Corollary. *Let W be a Weil algebra. Then ℓW is exponentiable in $\mathbb{G}$ and $\mathbb{F}$.*

Proof. This follows from 1.13, 1.15 and the fact that $\mathbb{G}$ and $\mathbb{F}$ are coreflective subcategories of $\mathbb{L}$. For example, in the case of $\mathbb{G}$, the exponential $\ell A^{\ell W}$ of a locus $\ell A \in \mathbb{G}$ can be constructed as $\lambda(i(\ell A)^{\ell W})$, where $\lambda\colon \mathbb{L} \to \mathbb{G}$ is the right adjoint of the inclusion functor $i : \mathbb{G} \hookrightarrow \mathbb{L}$. Indeed, we have for any other $\ell B \in \mathbb{G}$ the following natural bijections

$$\begin{array}{ll}
\underline{\quad B \to \lambda(i(lA)^{lW}) \text{ in } G \quad} & \\[1ex]
\underline{\quad i(lB) \to i(lA)^{lW} \text{ in } L \quad} & (\text{since } i \dashv \lambda) \\[1ex]
\underline{\quad i(lB) \times lW \to i(lA) \text{ in } L \quad} & (\text{by } 1.13) \\[1ex]
\underline{\quad i(lB \times lW) \to i(lA) \text{ in } L \quad} & (\text{by } 1.15) \\[1ex]
\quad lB \times lW \to lA \text{ in } G &
\end{array}$$

□

2 *Smooth Functors as a Basis for Synthetic Calculus*

Although the category $\mathbb{L}$ of loci has good exactness properties (e.g. $\mathbb{L}$ has all finite inverse limits), and contains *covariantly* the category $\mathbb{M}$ of smooth manifolds in such a way that transversal pullbacks in $\mathbb{M}$ are preserved by the inclusion functor $\mathbb{M} \hookrightarrow \mathbb{L}$, there is one respect in which it is deficient, namely, it lacks good function spaces. Only duals of Weil algebras have been seen to be exponentiable in $\mathbb{L}$.

In this section, we shall enlarge $\mathbb{L}$ to the category $\mathit{Sets}^{\mathbb{L}^{\mathrm{op}}}$ of covariant set-valued functors, or *presheaves* on $\mathbb{L}$, the smooth functors of the title, and prove that this larger category has good function spaces and good exactness properties.

Notice that by the Yoneda embedding

$$Y : \mathbb{L} \hookrightarrow \mathit{Sets}^{\mathbb{L}^{\mathrm{op}}}, Y(\ell A) = \mathbb{L}(-, \ell A),$$

$\mathbb{L}$ may be identified with a full subcategory of $\mathit{Sets}^{\mathbb{L}^{\mathrm{op}}}$, and we usually just write ℓA for $Y(\ell A)$. Furthermore, the resulting inclusion preserves $\varprojlim$ and exponentials which exists in $\mathbb{L}$. We obtain an embedding of $\mathbb{M}$ into $Sets^{\mathbb{L}^{op}}$, simply by composing the embedding s of proposition 1.2 with the Yoneda embedding. The resulting embedding will also be denoted by s. For the record,

2.1 Proposition. *The functor* $s\colon \mathbb{M} \to Sets^{\mathbb{L}^{op}}$, $M \mapsto \mathbb{L}(-, sM)$, *is full and faithful and preserves transversal pullbacks of* $\mathbb{M}$. □

As said in the introduction to this chapter, our viewpoint is to regard $Sets^{\mathbb{L}^{op}}$ as a ***generalized set-theoretic universe*** whose generalized sets, or ***"variable" sets*** are the smooth functors. This universe contains the ordinary or ***"constant" sets*** as constant functors. The embedding will be denoted by

$$\Delta : Sets \hookrightarrow Sets^{\mathbb{L}^{op}}, \Delta(S)(\ell A) = S \quad \text{for all} \quad \ell A \in \mathbb{L}.$$

Δ has a right adjoint Γ, called the ***global sections functor***, which evaluates *a smooth* functor at the one-point locus $1 = s(\{*\}) = \ell(C^\infty(\mathbb{R})/(x))$,

$$\Gamma\colon Sets^{\mathbb{L}^{op}} \to Sets, \ \Gamma(F) = F(1).$$

The usual set theoretic constructions of new objects from given ones may be performed within $Sets^{\mathbb{L}^{op}}$, using set theoretic rather than sheaf theoretic language. For example, if F and G are objects of $Sets^{\mathbb{L}^{op}}$, then there is a function space object G^F in $Sets^{\mathbb{L}^{op}}$ behaving exactly like the "set" of functions from F to G, and we can construct objects (generalized sets) of the form $\{x \in F | \varphi(x)\}, \{S \subseteq F | \psi(S)\}$, etc. Moreover, the usual set theoretical arguments remain valid, provided they are ***constructive***. (Alternatively, intuitionistic logic is to be used for carring out constructions and arguments inside the universe $Sets^{\mathbb{L}^{op}}$.)

Throughout this section, we shall explain this language and its logic, mainly by examples. In Appendix 1 we will discuss the general notion of "forcing" which gives precise rules to interpret the languages under consideration in Grothendieck toposes – categories of which $Sets^{\mathbb{L}^{op}}$ is a particular and simple example. (Readers who are not familiar with forcing and Grothendieck toposes are recommended to read Appendix 1 with the particular examples of this section and of Chapter III at hand.)

As said above, we view a smooth functor $F \in Sets^{\mathbb{L}^{op}}$ as a "variable set". An ***element of F at stage $\ell A \in \mathbb{L}$*** is, by definition, an element of (the set) $F(\ell A)$. Using the Yoneda embedding (to identify ℓA with $Y(\ell A)$), such an element is simply a map $\xi\colon \ell A \to F$ in $Sets^{\mathbb{L}^{op}}$. Notice that, given a map $\ell B \xrightarrow{\alpha} \ell A$ in $\mathbb{L}$ and an element ξ

of F at stage ℓA, we obtain by composition a new element $\xi \circ \alpha$ of F at stage ℓB. The element $\xi \circ \alpha$ will often be called *the restriction* of ξ *along* α, and will also be denoted by $\xi|\alpha$ (or sometimes when we need to distinguish F from other presheaves, by $F(\alpha)(\xi)$).

In a similar vein, we view a morphism $F \xrightarrow{\eta} G$ in $Sets^{\mathbb{L}^{op}}$, i.e. a natural transformation, as a map of "variable sets" which sends elements of F at stage $\ell A \in \mathbb{L}$ to elements of G at the same stage, in such a way that restrictions are preserved. Thus, η sends a commutative diagram

$$\begin{tikzcd}
\ell B \arrow[dd, "\alpha"'] \arrow[dr, "\xi'"] & \\
& F \\
\ell A \arrow[ur, "\xi"'] &
\end{tikzcd}$$

to a commutative diagram

$$\begin{tikzcd}
\ell B \arrow[dd, "\alpha"'] \arrow[dr, "\eta(\xi')"] & \\
& G \\
\ell A \arrow[ur, "\eta(\xi)"'] &
\end{tikzcd}$$

With these preliminaries out of way, we can easily show that $Sets^{\mathbb{L}^{op}}$ has good function spaces and good exactness properties. (These are general properties of any Grothendieck topos, cf. Appendix 1.)

2.2 Proposition. *$Sets^{\mathbb{L}^{op}}$ is a cartesian closed category.*

Proof. This is essentially obvious, once the meaning is understood: $Sets^{\mathbb{L}^{op}}$ has finite $\varprojlim$ and exponentials. In fact, $Sets^{\mathbb{L}^{op}}$ has arbitrary (small) inverse limits, and these are simply computed pointwise. That is, if $F\colon I \to Sets^{\mathbb{L}^{op}}$ is a functor on a small category I, i.e. a small diagram of functors $(F_i)_{i\in I}$, we let

$$(\varprojlim_{i\in I} F_i)(\ell A) = \varprojlim_{i\in I} F_i(\ell A) \quad (\text{in } Sets)$$

for each $\ell A \in \mathbb{L}$. Notice that $\varprojlim_i F_i$ is indeed a presheaf on $\mathbb{L}$.

To complete the definition, we let $\varprojlim_i F_i \xrightarrow{\pi_j} F_j$ be the natural transformation defined by the obvious projections (in the category *Sets*) $\varprojlim_i F_i(\ell A) \to F_j(\ell A)$ (for $\ell A \in \mathbb{L}$). The universal property is now easily verified.

As a special case, if F, $G \in \mathit{Sets}^{\mathbb{L}^{\mathrm{op}}}$, the canonical projections of the product

$$F \xleftarrow{\pi_1} F \times G \xrightarrow{\pi_2} G$$

induce a bijection between elements of $F \times G$ at stage ℓA and pairs consisting of an element of F and one of G at that stage. We shall write a single line for this bijection. Thus

$$\frac{\ell A \to F \times G}{\ell A \to F, \ell A \to G}$$

Turning to exponentials, we define for $F, G \in \mathit{Sets}^{\mathbb{L}^{\mathrm{op}}}$ the functor G^F by

$$G^F(\ell A) = \mathit{Sets}^{\mathbb{L}^{\mathrm{op}}}(\ell A \times F, G) = \begin{array}{l}\text{natural transformations}\\ \text{from } \ell A \times F \text{ to } G.\end{array}$$

G^F can be made into a presheaf by defining restrictions along an $\alpha: \ell B \to \ell A \in \mathbb{L}$ in the obvious way. Using our previous notation, we have

$$\frac{\ell A \to G^F}{\ell A \times F \to G}$$

by the very definition of G^F.

Note that there is an *evaluation map*

$$G^F \times F \xrightarrow{\mathrm{ev}} G$$

in $\mathit{Sets}^{\mathbb{L}^{\mathrm{op}}}$ defined by $\mathrm{ev}_{\ell A}(\eta, \xi) = \eta \circ (\mathrm{Id}_{\ell A}, \xi)$ for each $\ell A \in \mathbb{L}$, where $\eta: \ell A \times F \to G$ and $\xi: \ell A \to F$. The universal property of the exponential may be formulated as follows: ev induces a (natural) bijection between natural transformations $H \to G^F$ and natural transformations $H \times F \to G$. By extending the previous notation, we write

$$\frac{H \to G^F}{H \times F \to G}$$

The verification of this universal property is left to the reader. □

If $F: \mathbb{L}^{\mathrm{op}} \to \mathit{Sets}$ is a presheaf on $\mathbb{L}$, a subpresheaf of F is just a subfunctor of F, i.e. a presheaf S with $S(\ell A) \subseteq F(\ell A)$ for every $\ell A \in \mathbb{L}$, and with the same restrictions as F. Such subfunctors act as "subsets" of F. Given F, we can construct the "variable" *powerset* of F in $Sets^{\mathbb{L}^{op}}$ by considering not only such subfunctors S, but "subfunctors at arbitrary stages":

2.3 Proposition. *$Sets^{\mathbb{L}^{op}}$ has power objects. That is, given an $F \in Sets^{\mathbb{L}^{op}}$, there is a $\mathcal{P}(F) \in Sets^{\mathbb{L}^{op}}$ such that we have a natural bijection between natural transformations σ and subfunctors S (for any $G \in Sets^{\mathbb{L}^{op}}$):*

$$\frac{G \xrightarrow{\sigma} \mathcal{P}(F)}{S \subseteq G \times F}$$

Proof. Define $\mathcal{P}(F)(\ell A) =$ the set of subfunctors of $\ell A \times F$. $\mathcal{P}(F)$ has the structure of a presheaf on $\mathbb{L}$: given $\ell B \xrightarrow{\alpha} \ell A \in \mathbb{L}$ and $S \in \mathcal{P}(F)(\ell A)$, define $S|\alpha = (\alpha \times \mathrm{Id}_F)^{-1}(S)$. In other words, for any $(\beta, \xi) \in \ell B \times F(\ell C)$, i.e. $\ell C \xrightarrow{\beta} \ell B$ in $\mathbb{L}$ and $\xi \in F(\ell C)$,

$$(\beta, \xi) \in (S|\alpha)(\ell C) \Leftrightarrow (\alpha \circ \beta, \xi) \in S(\ell C).$$

The asserted correspondence between the natural transformations $G \xrightarrow{\sigma} \mathcal{P}(F)$ and the subfunctors $S \subseteq G \times F$ is given by

$$(\varsigma, \xi) \in S(\ell A) \Leftrightarrow (\mathrm{Id}_{\ell A}, \xi) \in \sigma_{\ell A}(\varsigma)$$

for $\ell A \in \mathbb{L}, \varsigma \in G(\ell A), \xi \in F(\ell A)$. □

Let us recall some of the loci (identified with smooth functors via the Yoneda embedding) that were introduced in the preceding section, and that will play an important rôle in the sequel:

The smooth line or the reals:

$$R = \ell C^\infty(\mathbb{R}) = s(\mathbb{R})$$

The point:

$$1 = \ell(C^\infty(\mathbb{R})/(x)) = s(\{*\}) = \{X \in R | x = 0\}$$

The first-order infinitesimals:

$$D = \ell(C^\infty(\mathbb{R})/(x^2)) = \{x \in R | x^2 = 0\}$$

The k^{th}-order infinitesimals:

$$D_k = \ell(C^\infty(\mathbb{R})/(x^{k+1})) = \{x \in R | x^{k+1} = 0\}$$

The infinitesimals:

$$\begin{aligned} \Delta &= \ell(C^\infty(\mathbb{R})/m^g_{\{0\}}) \\ &= \{x \in R \mid f(x) = 0, \text{ all } f \in m^g_{\{0\}}\} \\ &= \textstyle\bigcap_{n=1}^\infty s\left(-\frac{1}{n}, \frac{1}{n}\right) \end{aligned}$$

The positive reals:

$R_{>0} = \ell(C^\infty(\mathbb{R}^2)/(y\chi_{>0}(x) - 1)) = \ell C^\infty(\mathbb{R}_{>0}) = s(\mathbb{R}_{>0})$, where $\chi_{>0}(x) \neq 0$ iff $x > 0$.

The non-negative reals:

$R_{\geq 0} = \ell(C^\infty(\mathbb{R})/m^\infty_{\geq 0})$, where $m^\infty_{\geq 0} = m^\infty_{[0,\infty)}$, is the ideal of functions flat on $[0,\infty)$. Similarly we have $R_{\geq a}$ and $R_{\leq b}$, for any $a, b \in \mathbb{R}$.

The smooth interval:

$$[a,b] = \ell(C^\infty(\mathbb{R})/m^\infty_{[a,b]}) = R_{\geq a} \cap R_{\leq b}$$

The infinitesimal interval:

$$[0,0] = \ell(C^\infty(\mathbb{R})/m^\infty_{\{0\}}) = \ell\mathbb{R}[[X]]$$

Note that $[0,0]$ is quite different from the point 1. Besides these loci, we will consider smooth functors like:

Curves:

$$R^R$$

Functions on the unit interval:

$$R^{[0,1]}$$

Infinitesimal curves:

$$R^D$$

These examples can be multiplied at will. To simplify later developments, we shall now give explicit descriptions of the elements of these smooth functors at a given stage $\ell A \in \mathbb{L}$, say $A = C^\infty(\mathbb{R}^n)/I$. This provides a kind of *dictionary* which expresses our fundamental notions such as "reals" (elements of R), "infinitesimals", and the like in terms of the classical notions of C^∞-functions and their ideals.

(1) A *real* at stage ℓA is an equivalence class $f(x)$ mod I, where $f \in C^\infty(\mathbb{R}^n)$, that is an element of A. So a representative of such a real is an element $f(x) \in \mathbb{R}$ depending smoothly on the parameter

$x \in \mathbb{R}^n$, a "smoothly *varying* real" so to speak. (So not only is R is a "*variable* set" in the sense that its elements depend on the stage under consideration, but also the elements of R are themselves varying.)

(2) The "*ring structure*" of $R : 0, 1, +$ and $\cdot$ at stage ℓA correspond to

$$0 = 0 \bmod I, 1 = 1 \bmod I$$

$$(f \bmod I) + (g \bmod I) = (f + g) \bmod I$$

$$(f \bmod I) \cdot (g \bmod I) = (f \cdot g) \bmod I$$

Note that by the Yoneda embedding, a real at stage ℓA is the same as a map $\ell A \longrightarrow R = s(\mathbb{R})$ in $\mathbb{L}$ (or in $Sets^{\mathbb{L}^{op}}$), and given two such maps a and b, $a + b$ is the map $\ell A \xrightarrow{(a,b)} R \times R \xrightarrow{+} R$, where $+$ is the image under s of $+ : \mathbb{R}^2 \to \mathbb{R}$. Similarly for $\cdot$. Or in terms of the C^∞-rings of Chapter I, reals at ℓA correspond to elements of A as we have just seen, and the ring structure of R at stage ℓA corresponds precisely, to the ring structure of A. (In this sense, R is a ring object in $Sets^{\mathbb{L}^{op}}$ which "contains" all the finitely generated C^∞-rings in *Sets*, i.e. R is the "generic" C^∞-ring.)

(3) A *first-order infinitesimal* (an element of $D \subset R$) at stage ℓA is a class $f \bmod I$ with $f^2 \in I$.

(4) A k^{th}*-order infinitesimal* (an element of $D_k \subset R$) at stage ℓA is a class $f \bmod I$ such that $f^{k+1} \in I$.

(5) An *infinitesimal* (i.e. an element of Δ) at stage ℓA is a class $f \bmod I$ such that for every $\varphi \in m^g_{\{0\}} \subseteq C^\infty(\mathbb{R})$, $\varphi \circ f \in I$; equivalently (cf. Lemma 1.7) a class $f \bmod I$ such that for every open neighbourhood $U \subset \mathbb{R}$ of 0, $f^{-1}(U) \supset Z(I_0)$ for some finitely generated $I_0 \subset I$.

(6) A *positive real* at stage ℓA is a class $f(x) \bmod I$ such that for some $g \in C^\infty(\mathbb{R}^n), g(x) \cdot \chi_{>0}(f(x)) - 1 \in I$. (Here $\chi_{>0}$ is a characteristic function for $\{x \in \mathbb{R} | x > 0\}$, cf. I.1.4.). Again using an argument as in Lemma 1.7, this is equivalent to a class $f(x) \bmod I$ such that for some finitely generated $I_0 \subset I$, $Z(I_0) \subset f^{-1}(\mathbb{R}_{>0})$.

(7) A *non-negative real* at stage ℓA is a class $f \bmod I$, $f \in C^\infty(\mathbb{R}^n)$, such that for every $\rho \in m^\infty_{\geq 0} \subseteq C^\infty(\mathbb{R})$, $\rho \circ f \in I$.

(8) A *real in the unit interval* at stage ℓA is a class $f \bmod I$ such

that for every $\rho \in m^{\infty}_{[0,1]}, \rho \circ f \in I$.

(9) A *function* (from R to R) at stage ℓA is a class $F(x,t) \bmod \pi^*(I)$ where $F \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and π is the projection $\mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ (recall that $\pi^*(I)$ is the ideal generated by $\{f \circ \pi | f \in I\}$). Thus, a representative F of a function in R^R is a function $F(x,-) \in \mathbb{R}^{\mathbb{R}}$ depending smoothly on the parameter $x \in \mathbb{R}^n$.

(10) A *function defined on the unit interval* $[0,1] \subset R$ at stage ℓA is a class $F(x,t) \bmod (I, m^{\infty}_{[0,1]})$.

(11) An *infinitesimal curve* in R at stage ℓA is a class $F(x,t)$ mod $(I,(t^2))$.

(12) The *evaluation* of a function at a real at stage ℓA is the class $F(x,f(x)) \bmod I$, where $F(x,t) \bmod \pi^*(I)$ is the function and $f(x) \bmod I$ is the real.

In the sequel, we will occasionally extend this dictionary, when analysing how basic notions in the universe $Sets^{\mathbb{L}^{op}}$ and other universes to be introduced are represented from an "external", classical point of view. Often however, this task will be left to the reader.

This dictionary of basic notions is incomplete as far as logic is concerned. Indeed, as mentioned before, the logic of $Sets^{\mathbb{L}^{op}}$ is not classical and so we need to extend our dictionary to cover logical connectives, quantifiers, etc.

Before going into this matter, let us try to formulate some basic properties of the loci under consideration and see where the problem lies.

(i) "R is a commutative ring with unit".

What do we mean by this statement? In (1), (2) of the dictionary above we have seen that $R(\ell A) = A$ (the underlying set of the C^∞-ring A), and that the natural transformations

$$1 \underset{1}{\overset{0}{\rightrightarrows}} R \qquad R \times R \underset{\cdot}{\overset{+}{\rightrightarrows}} R$$

are given at stage ℓA by the ring structure of A.

Of course, part of the meaning of (i) is "the operation $\cdot$ is commutative", or using logical notation,

$$\forall x, y \in R\, (x \cdot y = y \cdot x).$$

We shall interpret this statement as follows: "for every couple of elements of R at an arbitrary stage ℓA, $\ell A \xrightarrow{a} R$ and $\ell A \xrightarrow{b} R$, the element $a \cdot b$ coincides with the element $b \cdot a$ at this stage ℓA", and we abbreviate the conclusion to

$$\ell A \Vdash (x \cdot y = y \cdot x)[a, b],$$

or more briefly, as $\ell A \Vdash a \cdot b = b \cdot a$. With this notation, our statement is interpreted as "for every pair of elements (a, b) of R at stage ℓA, $\ell A \Vdash x \cdot y = y \cdot x[a, b]$", and a moments reflection convinces us that this just means that the product operation of the ring A is commutative.

As a second example, consider

(ii) "0 is the only real killed by multiplication with all first-order infinitesimals".

The meaning of this statement is more involved, but we can let the logic do the work for us and reformulate (ii) as

(ii′) $\forall x \in R\ (\forall y \in D(x \cdot y = 0) \to x = 0)$.

Following the previous example and the $\Vdash$-notation just introduced, (ii′) is interpreted as "for every element of R at a stage ℓA, $\ell A \xrightarrow{a} R$, we have $\ell A \Vdash \forall y \in D(x \cdot y = 0) \longrightarrow x = 0[a]$." The question is, how do we interpret this last $\Vdash$-clause?

Before answering this question, let us observe that (ii) can also be formulated as

(ii″) "the exponential adjoint $R \longrightarrow R^D$ of the (restriction of) the product $R \times D \dot{\to} R$ is injective"

(that is, if the function $y \mapsto x \cdot y$ is identically 0, then $x = 0$). Since, as can easily be checked, the component of the natural transformation $R \longrightarrow R^D$ at stage ℓA is given by the canonical inclusion $A \longrightarrow A[\varepsilon]$, (ii″) is obviously true.

However, coming back to our earlier question, we find that if we interpret $\ell A \Vdash \forall y \in D(x \cdot y = 0) \longrightarrow x = 0\ [a]$ as "for every element of D at stage $\ell A \ldots$", then this need not hold for all stages ℓA, i.e. (ii′) is false! So what went wrong?

To understand the nature of the difficulty, let us repeat that our point of view is that $Sets^{\mathbb{L}^{op}}$ should be a generalized set-theoretic universe where set-theoretical construction can be performed. In particular, if $\varphi(x)$ is a formula like $\forall y \in D x \cdot y = 0$ where the free variable x ranges over R, we should be able to construct $\{x \in R | \varphi(x)\}$ as a subobject of R in $Sets^{\mathbb{L}^{op}}$, i.e. as a *functor*. To achieve this, we define $\ell A \Vdash \varphi(x)[a]$, or $\ell A \Vdash \varphi(a)$, in such a way that the map $\{x \in R | \varphi(x)\}: \mathbb{L}^{\mathrm{op}} \to \mathit{Sets}$ defined by

$$\{x \in R | \varphi(x)\}(\ell A) = \{\ell A \xrightarrow{a} R |\ \ell A \Vdash \varphi(a)\}$$

is a (contravariant) *functor*, indeed a subfunctor of R.

The trouble we ran into is now simply this: interpreting $\ell A \Vdash \forall y \in D(x \cdot y = 0) \to x = 0\ [a]$ in the simple minded way suggested above does not make $\{x \in R | \forall y \in D(x \cdot y = 0) \longrightarrow x = 0\}$ into a functor. To obtain a functor we have to interpret the quantifier $\forall$ and the connective $\to$ as follows: for $a: \ell A \to R$ (i.e., $a \in A$),

$\ell A \Vdash \forall y \varphi(x, y)[a]$ iff for every $\ell B \xrightarrow{f} \ell A$ in $\mathbb{L}$ and for every element h of D at stage $\ell B, \ell B \Vdash \varphi(x, y)[a|f, h]$

(recall $a|f = a \circ f$, or in terms of C^∞-rings, $a|f = f(a)$ where $a \in A$ and $f: A \to B$), and

$\ell A \Vdash \varphi(x) \to \psi(x)[a]$ iff for every $\ell B \xrightarrow{f} \ell A$ in $\mathbb{L}$, if $\ell B \Vdash \varphi(x)[a|f]$ then also $\ell B \Vdash \psi(x)[a|f]$.

Putting together these clauses, $\ell A \Vdash \forall y \in D(x \cdot y = 0) \to x = 0[a]$ is now interpreted as: for every $\ell B \xrightarrow{f} \ell A$, if $\ell B \Vdash \forall y \in D(x \cdot y = 0)[a|f]$ then $\ell B \Vdash x = 0[a|f]$. To see that this is true, let us suppose that $\ell B \Vdash \forall y \in D(x \cdot y = 0)[b]$, where $b = a|f \in B$. Now consider the stage $\ell C = \ell B \times D = \ell B[\varepsilon]$ and let p_2 be the projection $\ell B \times D \longrightarrow D$. By the interpretation of $\forall$, $\ell B \Vdash \forall y \in D(x \cdot y = 0)[b]$ implies that $\ell B \times D \Vdash x \cdot y = 0[b|p_1, p_2]$, where p_1 is the projection $\ell B \times D \longrightarrow \ell B$. But this say that in $B[\varepsilon]$, $b \cdot \varepsilon = 0$. Hence $b = 0$, showing that (ii′) is true. (ii′) is a consequence of the so-called *Kock-Lawvere axiom* for synthetic derivation:

(iii) *"Any infinitesimal curve is uniquely a straight line"*

or more explicitly,

(iii′) $\forall \alpha \in R^D \exists!(x, y) \in R \times R \forall z \in D\ \alpha(z) = x + y \cdot z$.

(iii′) is interpreted as: for every element f of R^D at an arbitrary stage ℓA, $\ell A \Vdash \exists!(x, y) \in R \times R\ \forall z\ (\alpha(z) = x + y \cdot z)[f]$. This first of all raises the question of how to handle the existential quantifier $\exists$, which is easily solved by noting that the naive interpretation leads to a functor, i.e. the general clause is

$\ell A \Vdash \exists y \in R \varphi(x, y)[a]$ iff there is an element b of R at stage ℓA such that $\ell A \Vdash \varphi(x, y)[a, b]$.

To check (iii′), we first prove that

$$\ell A \Vdash \exists x, y \in R\, \forall z \in D\, (\alpha(z) = x + y \cdot z)[f].$$

f is an element of R^D at stage ℓA, so according to the dictionary, we can write $A = C^\infty(\mathbb{R}^n)/I$ say, and $f = F(x,t) \bmod (I(x), t^2)$. By Hadamard's lemma, there is a smooth function $G \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ such that

$$(*) \qquad F(x,t) = F(x,0) + t\frac{\partial F}{\partial t}(x,0) + t^2 G(x,t).$$

Let $a = F(x,0) \bmod I$ and $b = \frac{\partial F}{\partial t}(x,0) \bmod I$. We claim that $\ell A \Vdash \forall z \in D\ (\alpha(z) = x + y \cdot z)[f, a, b]$. According to the clause for the universal quantifier, we need to check that for all $\ell B \xrightarrow{\varphi} \ell A$ and all $\ell B \xrightarrow{c} D$,

$$(**) \qquad \ell B \Vdash \alpha(z) = x + y \cdot z[f|\varphi, a|\varphi, b|\varphi, c].$$

In particular, this must be true for $\ell B = \ell A \times D$, with φ and c the projections p_1 and p_2. On the other hand, if (**) holds for this particular choice, $\ell B = \ell A \times D, \varphi = p_1, c = p_2$, then it holds for any $\ell B \xrightarrow{\varphi} \ell B$ and $\ell B \xrightarrow{c} D$: indeed, φ and c give a map $(\varphi, c)\colon \ell B \to \ell A \times D$ and the restrictions of p_1 and p_2 along this map are precisely φ and c, so this is clear from the functoriality of $\Vdash$ (that has been built into the clause for $\forall$). One expresses this by saying that "*p_2 is the generic element of D*".

Having reduced our task to showing that (**) holds for p_1 and p_2, we can unwind the definitions involved to find that we need to verify

$$F(x,t) = F(x,0) + t\frac{\partial F}{\partial t}(x,0) \bmod (I, t^2)$$

which follows from (*).

To complete the proof of (iii′) we still need to check uniqueness of a and b, that is

$$\ell A \Vdash \forall u, v \in R(\forall z \in D(u + v \cdot z = x + y \cdot z) \to u = x \wedge v = y)[a, b]$$

which can be proved exactly as (ii′).

Let us point out that there is another reformulation of (iii). A "straight line" is given by a couple of "reals", i.e. an element of $R \times R$. So (iii) says that the map

$$\alpha\colon R \times R \to R^D,$$

which as a natural transformation can be described by the components

$$\alpha_{\ell A}: A \times A \to A[\varepsilon],\ \alpha_{\ell A}(a,b) = a + \varepsilon b,$$

is an isomorphism, and this can be checked straightforwardly. (In fact, regarding α as a map in $\mathbb{L}$, α is just the isomorphism we get from Proposition 1.12 and the fact that $T(\mathbb{R}) = \mathbb{R} \times \mathbb{R}$.)

The Kock-Lawvere axiom (iii) or (iii$'$) enables us to define the derivative of a function $\alpha \in R^R$ by reasoning inside $Sets^{\mathbb{L}^{op}}$. As stated above and in Appendix 1, the interpretation of the language via $\Vdash$ makes this reasoning valid, provided we use *constructive* arguments only. So, given $\alpha \in R^R$, define for $x \in R, \alpha'(x)$ to be the unique element of R such that

$$\forall y \in D\ \alpha(x+y) = \alpha(x) + y \cdot \alpha'(x).$$

(Translated into classical language, this means that if $\alpha \in R^R$ is given at stage $\ell A, A = C^\infty(\mathbb{R}^n)/I$ by a class $F(x,t)$ mod $\pi^*(I), \alpha'$ is given at the same stage by the class $\frac{\partial F}{\partial t}(x,t)$ mod $\pi^*(I)$).

As a next example, consider the statement

(iv) "The strict order relation $R_{>0}$ and the preorder relation $R_{\geq 0}$ are compatible with the ring structure of R, and with each other (i.e. a positive real is non-negative)".

Let us check, after reformulation, a few of the statements involved in (iv). For instance, take

$$\forall x, y(x > 0 \wedge y > 0 \to x + y > 0).$$

According to our interpretation, we need to check that for all stages ℓA and for all elements $\ell A \xrightarrow{a} R$, $\ell A \xrightarrow{b} R$ at ℓA,

$$\ell A \Vdash a > 0 \wedge b > 0 \to a + b > 0,$$

that is, for all $\ell B \xrightarrow{f} \ell A$ in $\mathbb{L}$, if $\ell B \Vdash (a|f > 0 \wedge b|f > 0)$ then also $\ell B \Vdash (a+b)|f > 0$. By changing the notation, we may take $\ell A = \ell B, f = \mathrm{Id}_{\ell A}$. So assume $\ell A \Vdash a > 0 \wedge b > 0$, with $A = C^\infty(\mathbb{R}^n)/I$, $a = f \bmod I, b = g \bmod I$. As pointed out in (6) of the dictionary, this gives by Lemma 1.7 that there exists a finitely generated $I_o \subset I$ such that $Z(I_o) \subset f^{-1}(\mathbb{R}_{>0}) \cap g^{-1}(\mathbb{R}_{>0})$. Hence $Z(I_o) \subset (f+g)^{-1}(\mathbb{R}_{>0})$, so $\ell A \Vdash a + b > 0$.

It is much harder to check that

$$\ell A \Vdash (x \geq 0 \wedge y \geq 0 \to x + y \geq 0)[a,b]$$

for arbitrary a, b at a stage ℓA. By changing notation as above we need to prove that $\ell A \Vdash a \geq 0 \wedge b \geq 0$ implies $\ell A \Vdash a + b \geq 0$. Now by (7) of the dictionary, $\ell A \Vdash a \geq 0 \wedge b \geq 0$ means that $\forall \rho \in m^{\infty}_{\mathbb{R}_{\geq 0}}$, $\rho(f(x)) \in I$ and $\rho(g(x)) \in I$, where $a = f \bmod I, b = g \bmod I$. From this we have to conclude

$$\forall \rho \in m^{\infty}_{\mathbb{R}_{\geq 0}}(f(x) + g(x)) \in I.$$

So let $\rho \in m^{\infty}_{\mathbb{R}_{\geq 0}}$ and consider $\mu(u,v) = \rho(u+v) \in m^{\infty}_{\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}}$. By Corollary 4.12, we can write $\mu(u,v) = F(u,v)\theta(u) + G(u,v)\psi(v)$, with θ, $\psi \in m^{\infty}_{\mathbb{R}_{\geq 0}}$. Substituting $f(x)$ for u and $g(x)$ for v yields that $\rho(f(x) + g(x)) \in I$.

The remaining cases of (iv) are quite similar to these two and are omitted (the properties are listed in theorem 2.4 below).

As another example involving 4.12, let us consider the so-called *Integration Axiom*

(v) $\forall \alpha \in R^{[0,1]} \exists ! \beta \in R^{[0,1]} (\beta' \equiv \alpha \wedge \beta(0) = 0)$.

(The derivative β' is well-defined by the Kock-Lawvere axiom (iii), since $\ell A \Vdash \forall x \in [0,1] \forall y \in D\; x + y \in [0,1]$, for every $\ell A \in \mathbb{L}$.)

To see that (v) holds, let f be an element of $R^{[0,1]}$ at stage ℓA, that is $f = F(x,t) \bmod (I(x), m^{\infty}_{[0,1]})$ according to the dictionary. Define

$$G(x,t) = \int_0^t F(x,u)du, \quad g = G(x,t) \bmod (I, m^{\infty}_{[0,1]})$$

Then g is the element of $R^{[0,1]}$ at stage ℓA that we need for β in (v). The only problem really is to check that g is well defined. In fact, it suffices to show that

$$F(x,t) \in m^{\infty}_{[0,1]}(t) \cdot C^{\infty}(\mathbb{R}^n \times \mathbb{R}) \Rightarrow G(x,t) \in m^{\infty}_{[0,1]}(t) \cdot C^{\infty}(\mathbb{R}^n \times \mathbb{R}).$$

But $G = 0$ on $\mathbb{R}^n \times [0,1]$, and this implies (once again by Corollary 4.12) that $G \in m^{\infty}_{[0,1]} \cdot C^{\infty}(\mathbb{R}^n \times \mathbb{R})$.

We point out that there is an alternative approach to the integration axiom (v), for which the reader is referred to Appendix 3.

Given α as in (v), we write $\beta(t) = \int_0^t \alpha(t)dt$ for the unique β satisfying $\beta(0) = 0, \beta' \equiv \alpha$.

Integration respects the order-relations $<$ and $\leq$, in the sense that the following two properties hold:

(vi) $\forall \alpha \in R^{[0,1]}((\forall t \in [0,1]\alpha(t) > 0) \rightarrow \int_0^1 \alpha(t)dt > 0)$,

(vii) $\forall \alpha \in R^{[0,1]}((\forall t \in [0,1]\alpha(t) \geq 0) \rightarrow \int_0^1 \alpha(t)dt \geq 0)$.

The proof of (vi) is easy: let $f = F(x,t) \bmod (I(x), m^{\infty}_{[0,1]})$ be an element of $R^{[0,1]}$ at stage $\ell A, A = C^{\infty}(\mathbb{R}^n)/I$, so that $\beta(t) = \int_0^t \alpha(t)dt$

is represented by $G(x,t) = \int_0^t F(x,u)du$, as in the proof of (v) above. If $\ell A \Vdash \forall t \in [0,1] f(t) > 0$, then we have in particular for the "generic" element of $[0,1]$, the projection $\ell A \times [0,1] \xrightarrow{p_2} [0,1]$, that $\ell A \times [0,1] \Vdash f(p_2) > 0$. Since $\ell A \times [0,1]$ is the locus corresponding to the C^∞-ring $C^\infty(\mathbb{R}^n \times \mathbb{R})/(I(x), m^\infty_{[0,1]})$, this means, as we have seen earlier, that for a finitely generated $J \subset (I(x), m^\infty_{[0,1]})$

$$Z(J) \subset \{(x,t) | F(x,t) > 0\}.$$

Let $I_0 \subset I$ be a finitely generated ideal such that $J \subset (I_0(x), m^\infty_{[0,1]})$. Then $\forall x \in Z(I_0) \forall t \in [0,1]\ F(x,t) > 0$, so by definition of $G(x,t)$, $\forall x \in Z(I_0) G(x,1) > 0$, i.e. by (6) of the dictionary,

$$\ell A \Vdash \int_0^1 \alpha(t)dt > 0,$$

with f as value for α, thus proving (vi).

For (vii), we need to do more work. The argument is based on the following refinement of a special case of I.4.13.

Lemma. *Let $(f_n)_n$ be a sequence of functions in $m^\infty_{\mathbb{R}_{\geq 0}} \subseteq C^\infty(\mathbb{R})$. Then there exists a smooth function $\varphi \in m^\infty_{\mathbb{R}_{\geq 0}}$ such that*

(i) *$\varphi(t) > 0$ if $t < 0$, $\varphi'(t) < 0$ for $t < 0$, and $\varphi''(t) > 0$ for $t < 0$,*
(ii) *$f_n \in \varphi \cdot m^\infty_{\mathbb{R}_{\geq 0}}$, for each n.*

Proof of lemma. By I.4.13, we can write the l-th derivative $f_n^{(l)}$ of f_n as a product of k functions in $m^\infty_{\mathbb{R}_{\geq 0}}$,

$$f_n^{(l)} = \Pi_{i=1}^k g_n^{l,k,i}, \quad g_n^{l,k,i} \in m^\infty_{\mathbb{R}_{\geq 0}},$$

for each l, n, and $k \in \mathbb{N}$. And again by I.4.13, there is a $\psi \in m^\infty_{\mathbb{R}_{\geq 0}}$ with $\psi(t) > 0$ for $t < 0$ such that $\frac{d^2}{dt^2}\left(g_n^{l,k,i}\right) \in \psi \cdot m^\infty_{\mathbb{R}_{\geq 0}}$. Now let

$$\varphi(t) = \int_0^t \left(\int_0^u \psi(w)dw\right) du.$$

Then (i) clearly holds. For (ii), we have to show that the function $h_n(t)$ defined by

$$h_n(t) = \begin{cases} f_n(t)/\varphi(t), & \text{for } t < 0 \\ 0, & \text{otherwise} \end{cases}$$

is smooth. But any derivative of h_n can be written as a sum of terms of the form $\left(f_n^{(l)}(t) \cdot \varphi^{(r)}(t)\right) / \varphi(t)^k$; such a term is equal to

$\Pi_{i=1}^{k}\left(g_n^{l,k,i}(t)/\varphi(t)\right)\cdot\varphi^{(r)}(t)$, and each of the quotients $g_n^{l,k,i}(t)/\varphi(t)$ tends to zero as $t \longrightarrow 0$, by two applications of l'Hôpital's rule. This shows that h_n is smooth, and proves the lemma. $\square$

Proof of (vii). First note that if φ is a convex function as in (i) of the lemma, then $\varphi(\alpha x + (1-\alpha)y) \le \alpha\varphi(x) + (1-\alpha)\varphi(y)$ (where $x, y \in \mathbb{R}$ and $0 \le \alpha \le 1$), as is well-known and easily checked. By induction, it follows that $\varphi\left(\sum_{i=1}^{n}\alpha_i x_i\right) \le \sum_{i=1}^{n}\alpha_i\varphi(x_i)$, for $x_i \in \mathbb{R}$, $0 \le \alpha_i \le 1$, $\sum\alpha_i = 1$, and hence by taking Riemann sums,

$$\varphi\left(\int_0^1 k(t)dt\right) \le \int_0^1 \varphi(k(t))dt$$

for any (smooth) function k.

To prove (vii), take $f \in R^{[0.1]}$ at stage ℓA, represented by $F(x,t)$ mod $(I(x), m^{\infty}_{[0,1]})$ as in the proofs of (v) and (vi), and assume that $\ell A \Vdash \forall t \in [0,1] f(t) \ge 0$. Considering "the generic t", given by the projection $\ell A \times [0,1] \xrightarrow{p_2} [0,1]$, this means by (7) of the dictionary that

$$(*)\qquad \begin{array}{c}\rho(F(x,t)) \in (I(x), m^{\infty}_{[0,1]}) \textit{ for all}\\ \rho \in m^{\infty}_{\mathbb{R}_{\ge 0}} \subseteq C^{\infty}(\mathbb{R}).\end{array}$$

We have to show that $\ell A \Vdash \int_0^1 f \ge 0$, i.e. that

$$(**)\qquad \rho\left(\int_0^1 F(x,t)dt\right) \in I(x), \textit{ for all } \rho \in m^{\infty}_{\mathbb{R}_{\ge 0}}.$$

To see that (**) holds, take any $\rho \in m^{\infty}_{\mathbb{R}_{\ge 0}}$, and write $\rho^{(l)}(t) = \Pi_{i=1}^{k} g^{l,k,i}(t)$, for functions $g^{l,k,i} \in m^{\infty}_{\mathbb{R}_{\ge 0}}$, using I.4.13. By the lemma above, there exists a function $\varphi \in m^{\infty}_{\mathbb{R}_{\ge 0}}$ as in (i) of the lemma with all $g^{l,k,i} \in \varphi \cdot m^{\infty}_{\mathbb{R}_{\ge 0}}$. Now define $h\colon \mathbb{R}^n \to \mathbb{R}$

$$h(x) = \begin{cases} \rho\left(\int_0^1 F(x,t)dt\right) / \int_0^1 \varphi(F(x,t))dt, & \text{if } \int_0^1 F(x,t)dt < 0 \\ 0, & \text{otherwise.}\end{cases}$$

h is well-defined, since as was noted in the beginning of the proof $0 < \varphi\left(\int_0^1 F(x,t)dt\right) \le \int_0^1 \varphi(F(x,t))dt$ when $\int_0^1 F(x,t)dt > 0$. h is smooth, since a partial derivative $D^{\alpha}h(t)$ can be written as a sum of terms of the following form (for some smooth function $A(x)$)

$$\rho^{(l)}\left(\int_0^1 F(x,t)dt\right) A(x) / \left(\int_0^1 \varphi(F(x,t))dt\right)^k,$$

and we can rewrite the absolute value of this as

$$\left|\Pi_{i=1}^k \left(g^{l,k,i}\left(\int_0^1 F(x,t)dt\right) / \int_0^1 \varphi(F(x,t))dt\right) A(x)\right| =$$
$$= \left|\left(\Pi_{i=1}^k \varphi\left(\int_0^1 F(x,t)dt\right) / \int_0^1 \varphi(F(x,t))dt\right) \tilde{A}(x)\right| \leq \left|\tilde{A}(x)\right|,$$

where $\tilde{A}(x) = A(x) \cdot \Pi_{i=1}^k \rho_i \left(\int_0^1 F(x,t)dt\right)$ with $\rho_i \in m^\infty_{\mathbb{R}_{\geq 0}}$, by choice of φ. So $|\tilde{A}(x)| \longrightarrow 0$ if $|\int_0^1 F(x,t)dt| \longrightarrow 0$. By definition of h, (**) follows, and the proof of (vii) is complete. $\square$

Now that we have seen some particular examples of the interpretation of the language in $Sets^{\mathbb{L}^{op}}$, let us summarize the $\Vdash$ -clauses for the logical connectives. (This is the special case of Appendix 1 where the Grothendieck topology is the minimal one, i.e. only isomorphisms cover.) The x_i are variables ranging over elements of smooth functors F_i, the a_i are the values to be assigned to these variables. Moreover, $\ell A \Vdash \varphi(x_1, \ldots, x_n)[a_1, \ldots, a_n]$ will be written as $\ell A \Vdash \varphi(a_1, \ldots, a_n)$.

$\ell A \Vdash \varphi(a_1, \ldots, a_n) \wedge \psi(a_1, \ldots, a_n)$ iff both $\ell A \Vdash \varphi(a_1, \ldots, a_n)$ and $\ell A \Vdash \psi(a_1, \ldots, a_n)$

$\ell A \Vdash \varphi(a_1, \ldots, a_n) \vee \psi(a_1, \ldots, a_n)$ iff either $\ell A \Vdash \varphi(a_1, \ldots, a_n)$ or $\ell A \Vdash \psi(a_1, \ldots, a_n)$

$\ell A \Vdash \exists x_0 \in F_0 \varphi(x_0, a_1, \ldots, a_n)$ iff there is an $\ell A \xrightarrow{a_0} F$ such that $\ell A \Vdash \varphi(a_0, a_1, \ldots, a_n)$

$\ell A \Vdash \varphi(a_1, \ldots, a_n) \to \psi(a_1, \ldots, a_n)$ iff for every $\ell B \xrightarrow{f} \ell A$ in $\mathbb{L}$, if $\ell B \Vdash \varphi(a_1|f, \ldots, a_n|f)$ then $\ell B \Vdash \psi(a_1|f, \ldots, a_n|f)$

$\ell A \Vdash \forall x_0 \in F_0 \varphi(x_0, a_1, \ldots, a_n)$ iff for every $\ell B \xrightarrow{f} \ell A$ in $\mathbb{L}$ and every $\ell B \xrightarrow{b} F_0$, $\ell B \Vdash \varphi(b, a_1|f, \ldots, a_n|f)$

If $x_1, \ldots, x_n$ are variables ranging over smooth functors $F_1, \ldots, F_n$ as above, $\{(x_1, \ldots, x_n) \in F_1 \times \ldots \times F_n | \varphi(x_1, \ldots, x_n)\}$ defines a subfunctor of $F_1 \times \ldots \times F_n$ by

$(a_1, \ldots, a_n) \in \{(x_1, \ldots, x_n) \in F_1 \times \ldots \times F_n | \varphi(x_1, \ldots, x_n)\}(\ell A)$ iff $\ell A \Vdash \varphi(a_1, \ldots, a_n)$,

for all $\ell A \in \mathbb{L}$, and all $a_1 \in F_1(\ell A), \ldots, a_n \in F_n(\ell A)$.

Note that the logic now provides a precise mathematical interpretation of the "suggestive" notation that was used in 1.10 and at other places. For instance, we can now *prove* that $D = \{x \in R | x^2 = 0\}$, or that $1 = \{x \in R | \forall y \in D x \cdot y = 0\}$ (cf. (ii′) above).

One says that a sentence φ (a statement without free variables) *holds in (is valid in, is true in)* $Sets^{\mathbb{L}^{op}}$ if for every $\ell A \in \mathbb{L}$, $\ell A \Vdash \varphi$ (equivalently, by functoriality of $\Vdash$, if $1 \Vdash \varphi$). This is written as $Sets^{\mathbb{L}^{op}} \models \varphi$. So we have the following theorem.

2.4 Theorem. *The following statements are valid in* $Sets^{\mathbb{L}^{op}}$:

(i) *R is a commutative ring with 1.*

(ii) *(cancellation of infinitesimals)* $\forall x \in R(\forall y \in D(x \cdot y = 0) \to x = 0)$

(iii) *(Kock-Lawvere axiom)* $\forall \alpha \in R^D \exists!(x,y) \in R \times R \forall z \in D \alpha(z) = x + y \cdot z$

(iv) *$<$ and $\leq$ on R are compatible with the ring structure, that is, they are transitive and reflexive, $0 < 1$, and*

$$\forall x, y \in R(x > 0 \wedge y > 0 \to x \cdot y > 0 \wedge x + y > 0)$$

$$\forall x, y \in R(x \geq 0 \wedge y \geq 0 \to x \cdot y \geq 0 \wedge x + y \geq 0)$$

$$\forall x \in R(x^2 > 0 \leftrightarrow \exists y \in R(x \cdot y = 1))$$

$$\forall x \in R(x > 0 \to x \geq 0)$$

$$\forall x \in R(x^n = 0 \to 0 \leq x \leq 0)$$

(v) *(Integration axiom)* $\forall \alpha \in R^{[0,1]} \exists! \beta \in R^{[0,1]}(\beta' = \alpha \wedge \beta(0) = 0)$

(vi) $\forall \alpha \in R^{[0,1]}((\forall t \in [0,1] \alpha(t) > 0) \to \int_0^1 \alpha(t) dt > 0)$

(vii) $\forall \alpha \in R^{[0,1]}((\forall t \in [0,1] \alpha(t) \geq 0) \to \int_0^1 \alpha(t) dt \geq 0)$

□

3 *Some Topological Properties of the Smooth Line*

In the context of variable sets, it is no longer reasonable to require the line R to be a field. But in $Sets^{\mathbb{L}^{op}}$ the situation is even worse: R is not even a *local ring*, i.e.

(1) $Sets^{\mathbb{L}^{op}} \not\models \forall x, y \in R\ (x + y \text{ invertible } \to x \text{ invertible } \vee y \text{ invertible}).$

The reason for this ultimately lies in the fact that the embedding

$s : \mathbb{M} \hookrightarrow \mathit{Sets}^{\mathbb{L}^{op}}$ does not preserve (finite) open covers, as we shall see later on. For example, although $\mathbb{R} = (\leftarrow, 1) \cup (0, \rightarrow)$ in *Sets*,

$$\mathit{Sets}^{\mathbb{L}^{op}} \not\models \forall x \in R(x < 1 \vee x > 0).$$

As we said in the introduction to this chapter, this is one of the undesirable features of $\mathit{Sets}^{\mathbb{L}^{op}}$ which make it necessary to replace $\mathit{Sets}^{\mathbb{L}^{op}}$ by one of the more complicated universes to be introduced in later chapters.

Related to the fact that R is not a local ring is the fact that R is not *Archimedean*. To make this more precise, we need to construct the natural numbers or the integers in the universe $\mathit{Sets}^{\mathbb{L}^{op}}$. The ordinary natural numbers $\mathbb{N} \in \mathit{Sets}$ give a constant presheaf $\mathbb{N}$,

$$\mathbb{N}(\ell A) = \mathbb{N}$$

with all restrictions identity mappings, and it can be shown that $\mathbb{N}$ plays the rôle of "the" natural numbers in $\mathit{Sets}^{\mathbb{L}^{op}}$. In more technical terms, the constant presheaf $\mathbb{N}$ is the *natural numbers object* (nno) of the topos $\mathit{Sets}^{\mathbb{L}^{op}}$ (cf. Appendix 1).

$\mathbb{N}$ is in a natural way embedded in R, by the natural transformation with components at stage $\ell A, A = C^\infty(\mathbb{R}^n)/I$ say,

$$\mathbb{N} = \mathbb{N}(\ell A) \hookrightarrow R(\ell A) = A$$

sending n to the corresponding constant function (mod I) in A.

By saying that R is not Archimedean we mean that

(ii) $\mathit{Sets}^{\mathbb{L}^{op}} \not\models \forall x \in R\, \exists n \in \mathbb{N}\; x < n$.

Indeed, (ii) would imply that the embedding $s : \mathbb{N} \hookrightarrow \mathit{Sets}^{\mathbb{L}^{op}}$ preserves the cover $\mathbb{R} = \bigcup_{n \in \mathbb{N}}(-n, n)$, which is not the case. So as with R not being a local ring, the problem lies in the fact that s does not preserve open covers.

Another perhaps undesirable property of $\mathit{Sets}^{\mathbb{L}^{op}}$ is that the smooth unit interval $[0, 1] \subset R$ is not compact,

(iii) $\mathit{Sets}^{\mathbb{L}^{op}} \not\models [0, 1]$ is compact.

This is logically quite a bit more complicated than the statement we have considered so far, since it involves open covers, i.e. families of open subsets of R in $\mathit{Sets}^{\mathbb{L}^{op}}$, so we will need to use the power object of Proposition 2.3 and quantify over $\mathcal{P}(\mathcal{P}(R))$.

The *topology* on R is the *order topology*, i.e. the set of opens is the object $\{U \subset R | \forall x \in U \exists y, z \in R(y < x < z \wedge \forall x' \in R(y < x' < z \wedge x' \in U))\}$, a subfunctor of $\mathcal{P}(R)$. So if U is a subset of R at stage ℓA (a subfunctor of $\ell A \times R$, cf. the proof of 2.3), then $\ell A \Vdash$ " U is open" if and only if

$$\ell A \Vdash \forall x \in U \exists y, z \in R(y < x < z \wedge \forall x'(y < x' < z \to x' \in U)).$$

Let us write $\mathcal{O}(R)$ for the subfunctor of $\mathcal{P}(R)$ consisting of open "subsets" of R. We wish to analyse what it means for an $\ell A \in \mathbb{L}$ to force that $[0,1]$ is compact, i.e.

$$\text{(iii}'\text{)}\ \ell A \Vdash \forall \mathcal{U} \subset \mathcal{O}(R)([0,1] \subset \cup\mathcal{U} \to \exists \text{ finite } \mathcal{U}' \subset \mathcal{U}\ \ [0,1] \subset \cup\mathcal{U}'),$$

where $\mathcal{U} \subset \mathcal{O}(R)$ of course stands for $\mathcal{U} \in \mathcal{P}(\mathcal{O}(R))$, the powerobject of 2.3. (Don't compute $\mathcal{P}(\mathcal{O}(R))(\ell A)$!) We can now mechanically unwind this and let the logical interpretation do the work. The antecedent is, of course, interpreted as $\forall x \in [0,1] \exists U \in \mathcal{U}\ x \in U$. For the consequent we use the natural number object again, and rewrite it as

$$\exists n \in \mathbb{N} \exists f : n \to \mathcal{U} \forall x \in [0,1] \exists i < n\ x \in f(i),$$

where $\exists f : n \to \mathcal{U}$ stands for $\exists f \in \mathcal{U}^{\{y \in \mathbb{N} | y < n\}}$ (here $\{y \in \mathbb{N} | y < n\}$ denotes the obvious subobject of the constant presheaf $\mathbb{N}$, and $\mathcal{U}^{\{y \in \mathbb{N} | y < n\}}$ is the function space as in 2.2).

To prove (iii), we have to find an ℓA such that (iii$'$) does not hold. For this purpose, we let ℓA be the locus of positive invertible infinitesimals (cf. 1.10)

$$\mathbb{I}_{>0} = \{x > 0 | \ \forall n \in \mathbb{N}\ x < \frac{1}{n}\} \subset R.$$

So $\mathbb{I}_{>0} = \ell(C^\infty(\mathbb{R}_{>0})/(m^g_{\{0\}}|\mathbb{R}_{>0})), (m^g_{\{0\}}|\mathbb{R}_{>0})$ being the ideal of functions $\mathbb{R}_{>0} \to \mathbb{R}$ vanishing on some interval $(0, \varepsilon)$. At this stage $\mathbb{I}_{>0}$, we have an infinitesimal $\delta > 0$ given by the embedding in $\mathbb{L}$,

$$\delta : \mathbb{I}_{>0} \hookrightarrow R.$$

So at stage $\mathbb{I}_{>0}$, we can construct the open cover

$$\mathcal{U} = \{(x - \delta, x + \delta) | x \in [0,1]\}$$

of $[0,1]$, and it is intuitively clear and easy to check formally that this cover cannot have a finite subcover. Since this is the first time we deal with power objects, let us spell out the details. Suppose to the contrary that for some $n \in \mathbb{N}$,

$$\mathbb{I}_{>0} \Vdash \exists f : n \to \mathcal{U} \forall x \in [0,1] \exists i < n\ x \in f(i).$$

Since $\mathbb{N}$ is a constant presheaf, an $f \in \mathcal{U}^{\{0,\ldots,n-1\}}$ at stage $\mathbb{I}_{>0}$ corresponds to n elements $U_0, \ldots, U_{n-1}$ of $\mathcal{U}$, i.e. to n intervals

$(a_i - \delta, a_i + \delta)$ with $a_i \in R(\mathbb{I}_{>0}), i = 0, \ldots, n-1$. Thus we conclude

$\mathbb{I}_{>0} \Vdash \forall x \in [0,1](x \in (a_0 - \delta, a_0 + \delta) \vee \ldots \vee x \in (a_{n-1} - \delta, a_{n-1} + \delta))$.

So by considering the "generic element" π_2 of $[0,1]$ at stage $\mathbb{I}_{>0} \times [0,1]$, we find an $i < n$ such that

$$\mathbb{I}_{>0} \times [0,1] \Vdash a_i|\pi_1 - \delta|\pi_1 < \pi_2 < a_i|\pi_1 + \delta|\pi_1.$$

In particular, by restricting along the "points" $t: \mathbb{I}_{>0} \to \mathbb{I}_{>0} \times [0,1]$ corresponding to reals $t \in [0,1] \subset \mathbb{R}$ (in *Sets*), we get

(*) $\forall t \in [0,1] \subset \mathbb{R} : \mathbb{I}_{>0} \Vdash a_i - \delta < t < a_i + \delta$

(where the second t denotes the constant $\mathbb{L}$-map $\mathbb{I}_{>0} \longrightarrow R$ corresponding to $t \in \mathbb{R}$). a_i is the "germ at 0" of a C^∞-function $g_i: \mathbb{R}_{>0} \to \mathbb{R}$, and (*) gives

$$\forall t \in [0,1] \exists \varepsilon > 0 \forall y \in (0, \varepsilon)\; g_i(y) - y < t < g_i(y) + y,$$

which is clearly impossible. This proves (iii).

In view of the fact that $[0,1]$ is not compact, the following theorem may seem quite surprising.

3.1 Theorem. *In $Sets^{\mathbb{L}^{op}}$ it holds that every function from $[0,1]$ to R is uniformly continuous, i.e.*

$$Sets^{\mathbb{L}^{op}} \Vdash \forall f \in R^{[0,1]}\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in [0,1]\, (|x - y| < \delta \to |f(x) - f(y)| < \epsilon)$$

(although the absolute value does not exist in $Sets^{\mathbb{L}^{op}}$, we can use it as an obvious shorthand).

Proof. Choose such f and ε at stage $\ell A \in \mathbb{L}$, i.e. $\ell A \times [0,1] \xrightarrow{f} R$ and $\ell A \xrightarrow{\varepsilon} R_{>0}$, represented by $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \bmod (I(x), m^0_{[0,1]})$ and $E: \mathbb{R}^n \to \mathbb{R} \bmod I$ (where $A = C^\infty(\mathbb{R}^n)/I$). By Lemma 1.7, we can find a finitely generated ideal $I_0 \subset I$ such that at $\ell A_0 \supseteq \ell A$, $A_0 = C^\infty(\mathbb{R}^n)/I_0$, E still represents a positive element of R, i.e. ε extends to a map $\ell A_0 \xrightarrow{\varepsilon} R_{>0}$. We will now work with f as a map $\ell A_0 \times [0,1] \longrightarrow R$.

Choose any $\mu > 0$, and consider f as a map $\ell A_0 \times [-\mu, 1+\mu] \longrightarrow R$ (still represented by F). By continuity of F and compactness of $[-\mu, 1+\mu]$, we find for each $x \in Z(I_0)$ a $\delta_x > 0$ and a neighbourhood U_x such that

$$\forall y \in \mathbb{R}^n\, \forall s, t \in [-\mu, 1+\mu] : y \in U_x \wedge |s - t| < \delta_x \to |F(y,s) - F(y,t)| < E(x).$$

And by a partition of unity argument, we find an open neighbourhood V of $Z(I_0)$ and a smooth $D: \mathbb{R}^n \to \mathbb{R}$ such that $D > 0$ on V and

$$\text{(i)} \qquad \begin{array}{l} \forall x \in V\ \forall s,t \in [-\mu, 1+\mu]\ (|s-t| < D(x) \to \\ |F(x,s) - F(x,t)| < E(x)). \end{array}$$

Now D corresponds to an element δ of R at ℓA_0 with $\ell A_0 \Vdash \delta > 0$, and we claim that

$$\ell A_0 \Vdash \forall x, y \in [0,1]\ (|x-y| < \delta \to |f(x) - f(y)| < \varepsilon).$$

To prove the claim, take $\ell B \xrightarrow{g} \ell A_0$ and α, $\beta: \ell B \to [0,1]$, represented by $a, b: \mathbb{R}^m \to \mathbb{R}$ respectively, such that $\ell B \Vdash |\alpha - \beta| < \delta|g$. Say $B = C^\infty(\mathbb{R}^n)/J$. As before we find a finitely generated $J_0 \subset J$ such that already at ℓB_0,

$$\ell B_0 \Vdash |\alpha - \beta| < \delta \circ g$$

where $B_0 = C^\infty(\mathbb{R}^m)/J_0$, and since $J \supseteq g^*(I_0) = \{\varphi \circ g | \varphi \in I_0\}$, we may without loss assume that $J_0 \supseteq g^*(I_0)$ (since $g^*(I_0)$ is finitely generated).

Also, since $\ell B \Vdash 0 \leq \alpha, \beta \leq 1$ and $(-\mu, 1+\mu) \subset R$ corresponds to a finitely presented C^∞-ring, we may enlarge J_0 if necessary such that

$$\ell B_0 \Vdash -\mu < \alpha, \beta < 1 + \mu.$$

We now need to show that

$$\ell B_0 \Vdash |(f|g)(\alpha) - (f|g)(\beta)| < \delta \circ g.$$

Since J_0 is finitely generated, it suffices to check this at the points of $Z(J_0)$, i.e. we need to show

$$\text{(ii)}\ \forall y \in Z(J_0)\ |F(g(y), a(y)) - F(g(y), b(y))| < \delta(g(y)).$$

But if $y \in Z(J_0)$, then $a(y), b(y) \in (-\mu, 1+\mu)$ and $g(y) \in Z(I_0)$ by the properties of J_0, so (ii) follows from (i).

(Alternatively, it suffices to prove this for the *generic* pair α, β with $|\alpha - \beta| < \delta$ at ℓA_0, i.e. for the two projections π_y, π_z at ℓB, where $B = C^\infty(W)/(I_0(x))$ and $W = \{(x,y,z) \in \mathbb{R}^{n+2} | |y-z| < \delta(x)\}$.) □

The explanation for the fact that despite the lack of compactness we still get *uniform* continuity comes from the existence of Lebesgue numbers in $Sets^{\mathbb{L}^{op}}$:

3.2 Theorem. *In* $Sets^{\mathbb{L}^{op}}$, *every open cover of* $[0,1]$ *has a Lebesgue number, i.e.*

$$Sets^{\mathbb{L}^{op}} \models \ \forall \mathcal{U} \subset \mathcal{O}(R)\ ([0,1] \subset \cup\mathcal{U} \rightarrow \\ \exists \delta > 0 \forall x \in [0,1] \exists U \in \mathcal{U}\ (x-\delta, x+\delta) \subset U).$$

Rather than proving theorem 3.2 directly, we will derive it from a stronger principle of "compactness":

$$\text{(CMP)}\ \forall S \subset R \times R\ (\forall x \in [0,1] \exists \varepsilon > 0\ \{x\} \times (-\varepsilon, \varepsilon) \subset S \rightarrow \\ \exists \varepsilon > 0\ [0,1] \times (-\varepsilon, \varepsilon) \subset S).$$

Note that this axiom is false classically! It is true, however, in $Sets^{\mathbb{L}^{op}}$ (Theorem 3.4), and this implies that every open cover of $[0,1]$ in $Sets^{\mathbb{L}^{op}}$ has a Lebesgue number:

3.3 Lemma. *Consider the following three statements:*

(i) *every open cover of* $[0,1] \subset R$ *has a Lebesgue number*
(ii) *every neighbourhood* W *of* $[0,1] \times \{0\}$ *in* $[0,1] \times R$ *contains a tubular neighbourhood.*
(iii) *CMP as above.*

Then (iii) $\Rightarrow$ *(ii)* $\Leftrightarrow$ *(i). The proof is completely constructive, so these implications also hold in* $Sets^{\mathbb{L}^{op}}$.

Proof. (iii) $\Rightarrow$ (ii) is clear. For (ii) $\Rightarrow$ (i), let $\mathcal{U}$ be an open cover of $[0,1]$ in R, and let

$$W = \cup\{\{x\} \times (-\delta, \delta) | \exists U \in \mathcal{U}\ (x-\delta, x+\delta) \subset U\}.$$

Then W is a neighbourhood of $[0,1]$, for if $x \in [0,1]$, there exists a $U \in \mathcal{U}$ and a $\delta > 0$ with $(x-\delta, x+\delta) \subset U$. Hence

$$(x - \frac{1}{2}\delta, x + \frac{1}{2}\delta) \times (-\frac{1}{2}\delta, \frac{1}{2}\delta) \subseteq W.$$

Thus by (ii),

$$\exists \lambda > 0\ [0,1] \times (-\lambda, \lambda) \subset W.$$

Clearly λ is a Lebesgue number for $\mathcal{U}$.

Conversely, for (i) $\Rightarrow$ (ii), let $[0,1] \subset W \subset [0,1] \times R$, and let $\mathcal{U}_x = \{(x-\delta, x+\delta) | (x-\delta, x+\delta)^2 \cap ([0,1] \times R) \subset W\}$. Then $\mathcal{U} = \bigcup_{x \in [0,1]} \mathcal{U}_x$ is an open cover of $[0,1]$, so it has a Lebesgue number $\lambda > 0$. Then

$$\forall x \in [0,1] \exists y \in [0,1] \exists \delta > 0\ (x-\lambda, x+\lambda) \subset (x-\delta, x+\delta) \in \mathcal{U}_y,$$

so $\forall x \in [0,1](x-\lambda, x+\lambda)^2 \cap ([0,1]\times R) \subset W$. Thus $[0,1]\times(-\lambda,\lambda) \subset W$.

□

3.4 Theorem. *CMP is valid in* $\mathit{Sets}^{\mathbb{L}^{op}}$, *i.e.*

$$\mathit{Sets}^{\mathbb{L}^{op}} \models \quad \forall S \subset R \times R\, (\forall x \in [0,1] \exists \varepsilon > 0 \{x\} \times (-\varepsilon, \varepsilon) \subset S \to \exists \varepsilon > 0\, [0,1] \times (-\varepsilon,\varepsilon) \subset S).$$

Proof. Suppose S is an element of $P(R \times R)$ at stage ℓA, where $A = C^\infty(\mathbb{R}^n)/I$, such that $\ell A \Vdash \forall x \in [0,1] \exists \varepsilon > 0\ \{x\} \times (-\varepsilon,\varepsilon) \subset S$. Then in particular

$$\ell A \times [0,1] \Vdash \exists \varepsilon > 0\ \{\pi_2\} \times (-\varepsilon,\varepsilon) \subset S$$

so there exists a map $\varepsilon : \ell A \times [0,1] \to R_{>0}$ in $\mathbb{L}$, represented by $E: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ say, such that

(1) $\ell A \times [0,1] \Vdash \{\pi_2\} \times (-\varepsilon,\varepsilon) \subset S$.

By Lemma 1.7 there exists a finitely generated ideal $I_0 \subset I$ such that E induces a map $\ell(C^\infty(\mathbb{R}^n)/I_0) \times [0,1] \longrightarrow R_{>0}$. Therefore $\forall x \in Z(I_0) \forall t \in [0,1]\ E(x,t) > 0$. Then by compactness of $[0,1]$ and a partition of unity argument (as in the proof of Theorem 3.1) we find a smooth $D: \mathbb{R}^n \to \mathbb{R}$ such that

(2) $\forall x \in Z(I_0) \forall t \in [0,1]\ 0 < D(x) < E(x,t)$.

Therefore D represents an element δ of R at stage ℓA, and $\ell A \Vdash \delta > 0$. We claim that $\ell A \Vdash [0,1] \times (-\delta,\delta) \subset S$, i.e.

(3) $\ell A \Vdash \forall x \in [0,1]\ \{x\} \times (-\delta,\delta) \subset S$.

We need only consider the generic element of $[0,1]$, the projection π_2 at $\ell A \times [0,1]$; i.e. we have to show that

$$\ell A \times [0,1] \Vdash \{\pi_2\} \times (-\delta,\delta) \subset S.$$

But this is immediate from (1), since (2) implies that $\ell A \times [0,1] \Vdash (\delta | \pi_2) < \varepsilon$. □

3.5 Remark. Theorem 3.2 can also be proved directly by arguments similar to the ones used in the proof of Theorem 3.4. Along the same lines, we can show the validity in $\mathit{Sets}^{\mathbb{L}^{op}}$ of the analogue of the principle (CMP) for any object $s(M)$, M a compact manifold. Or, analogous to 3.2, we can show that Lebesgue numbers exist for open covers of $s(M)$ in $\mathit{Sets}^{\mathbb{L}^{op}}$, M a compact manifold. And similarly, Theorem 3.1 can be generalized by replacing $[0,1]$ and R by $s(M)$ and $s(N)$, where M is a compact manifold and N is an arbitrary manifold.

As we have seen above, notions like "open cover" or "compactness" are just as complex in the internal language of $Sets^{\mathbb{L}^{op}}$ as they are in classical analysis – the defining formulas are literally the same. However, if, using the $\Vdash$-interpretation, we want to phrase in the external, classical language what it means to be an internal (in $Sets^{\mathbb{L}^{op}}$) open cover of $[0,1]$ for example, the outcome is rather complex. A nice example of such a difference in "conceptual complexity" of a different kind is given by the notion of distribution with compact support. Classically, a distribution with compact support on a manifold M is an $\mathbb{R}$-linear mapping

$$C^\infty(M) \xrightarrow{\mu} \mathbb{R}$$

which is continuous with respect to the Fréchet topology on $C^\infty(M)$. In $Sets^{\mathbb{L}^{op}}$, however, it is just an R-linear map $R^{s(M)} \longrightarrow R$. (Note that in $Sets^{\mathbb{L}^{op}}$, $R^{s(M)}$ and in general R^F for any smooth functor F, has a natural R-algebra structure. Reasoning inside the set-theoretic universe $Sets^{\mathbb{L}^{op}}$, R^F is the *set of functions from F to R*, so the R-algebra structure can be defined as the usual "pointwise" one. In more categorical terms, for example addition $+: R^F \times R^F \to R^F$ is the exponential transpose of the composition $R^E \times R^E \times E \xrightarrow{\sim} (R \times R)^E \times E \xrightarrow{\mathrm{ev}} R \times R \xrightarrow{+} R$.)

3.6 Proposition. *There is a natural one-to-one correspondence between distributions with compact support on a manifold M and R-linear mappings $R^{s(M)} \longrightarrow R$ in $Sets^{\mathbb{L}^{op}}$.*

Proof. In one direction, suppose $\mu: C^\infty(M) \to \mathbb{R}$ is a distribution with compact support on M. μ induces a natural transformation $\tilde{\mu}: R^{s(M)} \to R$ as follows. If $\ell A \in \mathbb{L}$, with $A = C^\infty(\mathbb{R}^n)/I$ say, an $f \in R^{s(M)}(\ell A)$ gives an $\mathbb{L}$-map $\ell A \times s(M) \xrightarrow{f} R$, i.e. a class $F \bmod \pi_1^*(I)$, where $F: \mathbb{R}^n \times M \to \mathbb{R}$ (cf. proposition I.2.5). Let $\tilde{\mu}_{\ell A}(f)$ be the class of the function

$$\mu(F): \mathbb{R}^n \to \mathbb{R},\ \mu(F)(x) = \mu(F(x,-)).$$

μ is continuous and linear, and it commutes with partial derivatives of F along the coordinates of $\mathbb{R}^n$ ($\frac{\partial \mu(F)}{\partial x_i} = \mu\left(\frac{\partial F}{\partial x_i}\right)$, etc.), so $\mu(F)$ is smooth. Moreover, if $F \in \pi_1^*(I)$, i.e. $F(x,m) = \Sigma_i G_i(x,m) \cdot \varphi_i(x)$ where $\varphi_i \in I$, then $\mu(F) = \Sigma_i \mu(F_i) \cdot \varphi_i$, so $\tilde{\mu}_{\ell A}(f)$ is well defined and does not depend on the representing function F. That $\tilde{\mu}$ is a natural transformation and is R-linear can be checked very easily.

Conversely, if $\varphi: R^{s(M)} \to R$ is an R-linear map in $Sets^{\mathbb{L}^{op}}$, we define

$$\mu = \Gamma\varphi: \Gamma(R^{s(M)}) \to \Gamma R$$

to be the restriction of φ to global sections (i.e. to elements at stage 1). Indeed, $\Gamma(R) = \mathbb{R}$ and $\Gamma(R^{s(M)}) = Sets^{\mathbb{L}^{op}}(s(M), R) = C^\infty(M)$ by Proposition 2.1. It is clear that μ is $\mathbb{R}$-linear, so it remains to show that μ is continuous for the Fréchet topology.

First note that if $g: \mathbb{R} \times M \to \mathbb{R}$ is smooth, then $g \in R^{s(M)}(R)$, and by naturality of φ,

$$\mu(g_t) = \varphi_R(g)(t)$$

(where $g_t(-) = g(t, -)$), i.e. $\mu(g_t)$ depends smoothly on t. So μ is a path-smooth $\mathbb{R}$-linear form on $C^\infty(M)$, and it suffices to prove the following lemma. □

3.7 Lemma. *Let $\mu: E \to \mathbb{R}$ be an $\mathbb{R}$-linear form on a Fréchet vectorspace E, and assume that μ is path-smooth, i.e. for any C^∞-function $g(t): \mathbb{R} \to E$, $\mu(g(t)): \mathbb{R} \to \mathbb{R}$ is also C^∞. Then μ is continuous.*

Proof of Lemma. Suppose μ is not continuous. Then we have a sequence of points $e_n \in E$ with $\lim_{n\to\infty} e_n = 0$ and $\mu(e_n) = 1$ for each n. Let the topology of E be defined by an increasing sequence of semi-norms $p_m, m \in \mathbb{N}$. Since $\lim e_n = 0$, we can assume (extracting a subsequence of $\{e_n\}$ if necessary) that

$$p_n(e_n) \leq (n^2)^{-n}.$$

Let $\kappa: \mathbb{R} \to \mathbb{R}$ be a smooth function with $\kappa(0) = 1$ and $\mathrm{supp}(\kappa) \subset [-1, 1]$. For any $n \in \mathbb{N}$, let

$$\kappa_n(t) = \kappa(n^2(t - \frac{1}{n})), \ \forall t \in \mathbb{R}.$$

Then for $n \geq 3, \kappa_n$ is a smooth function with support contained in $\left[\frac{1}{n} - \frac{1}{2(n+1)}, \frac{1}{n} + \frac{1}{2(n+1)}\right]$, so since these intervals are disjoint, the function $g: \mathbb{R} \to E$,

$$g(t) = \sum_{n\geq 3} \kappa_n(t) e_n$$

is well defined. Moreover, $g(t)$ is smooth. It suffices to check this for

$t = 0$. But for any k,

$$g^{(k)}(t) = \sum (n^2)^k \kappa^{(k)}(n^2(t - \frac{1}{n})) \cdot e_n,$$

so for a semi-norm p_m, we have

$$p_m(g^{(k)}(t)) \leq A_k(n^2)^{k-n}$$

where $A_k = \sup\{\kappa^{(k)}(t) | t \in \mathbb{R}, t \in \operatorname{supp}(x_n) \text{ for some } n \geq m\}$. The second term converges to 0 as n increases. Therefore, as $g^{(k)}(0) = 0$, the function $g^{(k)}(t)$ is also continuous at $t = 0$. Thus $g(t)$ is smooth. But then $\mu(g(0)) = \mu(0) = 0$ while $\mu(g(\frac{1}{n})) = \mu(e_n) = 1$, which contradicts the fact that $\mu(g(t))$ is a continuous function of t. This proves the lemma.

To complete the proof of the theorem, it now suffices to observe that $\Gamma\tilde{\mu} = \mu$ and that $(\widetilde{\Gamma\varphi}) = \varphi$. $\square$

Chapter III

Two Archimedean Models for Synthetic Calculus

In chapter II, we introduced the category of smooth functors $\mathit{Sets}^{\mathbb{L}^{op}}$. This category has good function spaces, infinitesimal spaces, convenient exactness properties, and it contains the usual category of manifolds $\mathbb{M}$. Furthermore, the embedding $\mathbb{M} \hookrightarrow \mathit{Sets}^{\mathbb{L}^{op}}$ preserves the *good* limits in $\mathbb{M}$, namely tranversal pullbacks. Nevertheless, $\mathit{Sets}^{\mathbb{L}^{op}}$ has pathological properties: the *smooth line* R, which is a commutative ring with unit, is not even a local ring. Moreover, R is not Archimedean. From a somewhat different viewpoint, one can say that, besides some *good* limits, $\mathbb{M}$ also has *good* colimits, such as open covers. The trouble with $\mathit{Sets}^{\mathbb{L}^{op}}$ is that these covers are not preserved by the embedding $\mathbb{M} \hookrightarrow \mathit{Sets}^{\mathbb{L}^{op}}$.

In this chapter, we describe and contrast two models for synthetic calculus in which R is local and Archimedean, i.e. in which the sentences

$$\begin{array}{ll} \neg 0 = 1 & \\ \forall x \in R & (x \text{ invertible } \vee (1-x) \text{ invertible}) \\ \forall x \in R & \exists n \in \mathbb{N}\ x < n \end{array}$$

are valid. Furthermore, the inclusion of $\mathbb{M}$ in either of them preserves arbitrary open covers, thus eliminating the difficulties connected with $\mathit{Sets}^{\mathbb{L}^{op}}$.

It is here that the notion of a Grothendieck topology (on $\mathbb{L}$), designed to handle precisely these problems, makes its appearance. To put it in a nutshell: we cut down our universe $\mathit{Sets}^{\mathbb{L}^{op}}$ to those functors which believe that open covers of $\mathbb{M}$ are covers in $\mathit{Sets}^{\mathbb{L}^{op}}$, in a sense to be explained below. A technical problem remains, however. We want our topology to be *sub-canonical*. This implies, essentially, that statements about our universes correspond in a direct way to statements of classical analysis about rings of smooth functions and their ideals. For this reason we have to consider certain subcategories of $\mathbb{L}$ ($\mathbb{G}$ and $\mathbb{F}$ in the present case), and Grothendieck topologies on those. (Later on, some models based on the whole category $\mathbb{L}$ will be considered, see Chapter VI.)

1 The Topos $\mathcal{G}$ of Germ-Determined Ideals and its Logic

In this section we will introduce a ***set theoretic universe***, or more precisely a Grothendieck topos, which is not just a category of *Set*-valued functors on loci, as the universe $Sets^{\mathbb{L}^{op}}$, but a subcategory consisting of those functors which are sheaves, in a sense to be explained. As a consequence of restricting our attention to sheaves, we will have to modify our interpretation of the set theoretic language. This makes the present universe in some respects more difficult to handle than our ***preparatory*** universe $Sets^{\mathbb{L}^{op}}$, but the advantages of this more complicated approach will become apparent in the course of this chapter, and in later chapters.

Rather than taking *Set*-valued functors on $\mathbb{L}$, we will consider only the smaller category $\mathbb{G}$ introduced in Section II.1. Recall that the objects of $\mathbb{G}$ are the duals $\ell(C^\infty(\mathbb{R}^n)/I)$ of finitely generated C^∞-rings, where I is a germ-determined ideal (and loci isomorphic to such), and that morphisms of $\mathbb{G}$

$$\ell(C^\infty(\mathbb{R}^n)/I) \longrightarrow \ell(C^\infty(\mathbb{R}^m)/J)$$

are equivalence classes of smooth functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ with the property that $J \subseteq \varphi_*(I) = \{f \in C^\infty(\mathbb{R}^n) | f \circ \varphi \in I\}$. As we have seen in II.1, $\mathbb{G}$ contains all manifolds, and all (duals of) Weil algebras. $\mathbb{G}$ is not closed under taking open subloci (see II.1.4). That is, if $I \subset C^\infty(\mathbb{R}^n)$ is germ-determined and U is an open subset of $\mathbb{R}^n$, $(I|U)$ need not be a germ-determined ideal in $C^\infty(U)$. (For example, let $I = m^g_{\{0\}} \subset C^\infty(\mathbb{R})$, $U = \{t | t > 0\}$). However, applying the coreflection λ, we get $\lambda(\ell(C^\infty(\mathbb{R}^n)/I) \cap s(U)) = \ell(C^\infty(U)/(I|U)^\sim)$, which is (isomorphic to) an object of $\mathbb{G}$.

In $\mathbb{G}$, we specify some families of morphisms with common codomain as ***covering families***, or ***covers***, as follows. $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell A\}_\alpha$ is a covering family in $\mathbb{G}$ iff

(i) for every α there exists a $b_\alpha \in A$ and a commutative diagram

$$\begin{tikzcd} \ell A_\alpha \arrow[rr, "\sim"] \arrow[dr, "f_\alpha"'] & & \lambda\ell(A\{b_\alpha^{-1}\}) \arrow[dl, "g_\alpha"] \\ & \ell A & \end{tikzcd}$$

where g_α is the canonical map $\lambda\ell(A\{b_\alpha^{-1}\}) \hookrightarrow \ell A\{b_\alpha^{-1}\} \hookrightarrow \ell A$

(ii) $\{\gamma\ell A_\alpha \xrightarrow{\gamma f_\alpha} \gamma\ell A\}_\alpha$ is a surjective family of topological spaces, i.e. for every point $x \in \gamma\ell A$ there is an α such that $x = \gamma f_\alpha(y)$ for some $y \in \gamma\ell A_\alpha$.

1.1 Lemma. *The covering families as described by (i) and (ii) define a Grothendieck topology on* $\mathbb{G}$.

Proof. By the definition of a Grothendieck topology (see Appendix 1) we have to check the following three conditions:

1. isomorphisms cover: this is clear.

2. stability under pullback, i.e. if $\{\ell A_\alpha \longrightarrow \ell A\}_\alpha$ is a cover and $\ell B \xrightarrow{f} \ell A$, then the family of pullbacks $\{\ell A_\alpha \times_{\ell A} \ell B \longrightarrow \ell B\}_\alpha$ in $\mathbb{G}$ is again a cover: we may assume that $\ell A_\alpha \longrightarrow \ell A$ is $\lambda\ell(A\{b_\alpha^{-1}\}) \longrightarrow \ell A$. Since λ preserves pullbacks (being right adjoint to the inclusion $\mathbb{G} \hookrightarrow \mathbb{L}$), it follows from the universal property of $A\{b_\alpha^{-1}\}$ that

$$\begin{tikzcd} \ell A_\alpha \arrow[r, tail] & \ell A \\ \lambda\ell(B\{f(b_\alpha^{-1})\}) \arrow[u] \arrow[r, tail] & \ell B \arrow[u] \end{tikzcd}$$

is a pullback in $\mathbb{G}$ (where we write f also for the C^∞-homomorhism $A \longrightarrow B$), so condition (i) is verified. Condition (ii) immediately follows from the fact that γ, being a right adjoint, preserves pullbacks. Indeed, by assumption $\gamma\ell A_\alpha \subset \gamma\ell A$ and $\gamma\ell A = \cup_\alpha \gamma\ell A_\alpha$, hence $\gamma\ell B = \cup_\alpha (\gamma f)^{-1}(\ell A_\alpha) = \cup_\alpha \gamma(\ell A_\alpha \times_{\ell A} \ell B)$.

3. Stability under composition, i.e. if $\{\ell A_\alpha \longrightarrow \ell A\}_\alpha$ is a cover

and for each α, $\{\ell A_{\alpha\beta} \longrightarrow \ell A_\alpha\}_\beta$ is a cover (β running over some index set depending on α) then so is the family of composites $\{\ell A_{\alpha\beta} \longrightarrow \ell A\}_{\alpha,\beta}$: We may assume that $\ell A_\alpha = \lambda\ell(A\{b_\alpha^{-1}\})$ for some $b_\alpha \in A$, and, since λ preserves pullbacks and $\lambda\ell(A\{b_\alpha^{-1}\}) \hookrightarrow \ell A\{b_\alpha^{-1}\}$ corresponds to a surjection of C^∞-rings, that $\ell A_{\alpha\beta} = \lambda\ell((A_\alpha\{b_\alpha^{-1}\})\{c_{\alpha\beta}^{-1}\})$ for some $c_{\alpha\beta} \in A\{b_\alpha^{-1}\}$. From the explicit description of $A\{(\cdot)^{-1}\}$ given in Chapter I, it is clear that $(A_\alpha\{b_\alpha^{-1}\})\{c_{\alpha\beta}^{-1}\} \cong A\{d_{\alpha\beta}^{-1}\}$ for some $d_{\alpha\beta} \in A$. So condition (i) holds. Now (ii) just means that if $\gamma\ell A_\alpha = \cup_\beta \gamma\ell A_{\alpha\beta}$ for each α, and $\gamma\ell A = \cup_\alpha \gamma\ell A_\alpha$ then $\gamma\ell A = \cup_{\alpha\beta}\gamma\ell A_{\alpha\beta}$, which is clear. □

We shall usually work with the following equivalent description of this Grothendieck topology on $\mathbb{G}$.

1.2 Lemma. *Let $\ell A \in \mathbb{G}$, say $A = C^\infty(U)/I$ for an open subset $U \subseteq \mathbb{R}^n$ and a germ-determined ideal I. Then the covering families of ℓA are precisely the families (isomorphic to ones) of the form*

$$\{\ell(C^\infty(U_\alpha)/(I|U_\alpha)^\sim) \hookrightarrow \ell(C^\infty(U)/I)\}_\alpha$$

where $\{U_\alpha\}$ is a cover of $Z(I)$ by open sets in U.

Proof. $\gamma\ell A = Z(I)$, and moreover if $\varphi_\alpha : U \to \mathbb{R}$ is a characteristic function for U_α, then $\ell(C^\infty(U_\alpha)/(I|U_\alpha)^\sim) \cong \lambda\ell(A\{\varphi_\alpha^{-1}\})$, so it is clear that such families coming from a cover $\{U_\alpha\}$ of $Z(I)$ satisfy conditions (i) and (ii). The converse is equally obvious. □

This Grothendieck topology makes $\mathbb{G}$ into a *site*, which we also denote by $\mathbb{G}$. Our universe $\mathcal{G}$ is now defined as *the topos of sheaves on* $\mathbb{G}$. Some explanation may be in order: a sheaf on $\mathbb{G}$ is a functor

$$F: \mathbb{G}^{\mathrm{op}} \to \mathit{Sets}$$

with the following property. For any covering family $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell A\}_\alpha$ in $\mathbb{G}$, and any family of elements $x_\alpha \in F(\ell A_\alpha)$ which are compatible in the sense that for each pullback in $\mathbb{G}$

$$\begin{tikzcd}
\ell A_\alpha \arrow[r, "f_\alpha"] & \ell A \\
\ell A_\alpha \underset{\ell A}{\times} \ell A_\beta \arrow[u, "p_\alpha"] \arrow[r, "p_\beta"] & \ell A_\beta \arrow[u, "f_\beta"']
\end{tikzcd}$$

we have that $x_\alpha | p_\alpha = x_\alpha | p_\beta \in F(\ell A_\alpha \times_{\ell A} \ell A_\beta)$, there is a unique $x \in F(\ell A)$ (the *join* of the x_α) such that $x | f_\alpha = x_\alpha$ for all α. (Here we use $|$ as in Chapter II. In the sequel, we will often just write x for $x | f_\alpha$ whenever it is clear that we speak about elements of $F(\ell A_\alpha)$ rather than of $F(\ell A)$.)

So $\mathcal{G}$ is the category of sheaves and natural transformations, which is a full subcategory of $Sets^{\mathbb{G}^{op}}$. A reason for restricting our attention to germ-determined loci lies in the following lemma.

1.3 Lemma. *The Yoneda embedding $\mathbb{G} \overset{Y}{\hookrightarrow} Sets^{\mathbb{G}^{op}}$ factors through $\mathcal{G}$. In other words, all representable functors $\mathbb{G}(-, \ell B)$ are sheaves. (One calls a site with this property subcanonical, see Appendix 1).*

Proof. Let $\{\ell A_\alpha \hookrightarrow \ell A\}_\alpha$ be a cover, and let $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell B\}_\alpha$ be a compatible family of elements of $\mathbb{G}(-, \ell B)$. We may assume that $A = C^\infty(\mathbb{R}^n)/I$, and hence by lemma 1.2 that $A_\alpha = C^\infty(U_\alpha)/(I|U_\alpha)^\sim$, where $Z(I) \subseteq \cup_\alpha U_\alpha$. Let $B = C^\infty(\mathbb{R}^m)/J$. Although in general a map $\ell(C^\infty(U)/I) \longrightarrow \ell(C^\infty(V)/J)$ in $\mathbb{G}$ need not come from a smooth function $U \longrightarrow V$ ($U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ open), this is true if $V = \mathbb{R}^m$ since $C^\infty(\mathbb{R}^m)$ is free on m generators. So f_α must be induced by a smooth $F_\alpha : U_\alpha \to \mathbb{R}^m$, and compatibility means that for each α, β,

$$(*) \qquad \pi_i \circ (F_\alpha - F_\beta) | U_\alpha \cap U_\beta \in (I | U_\alpha \cap U_\beta)^\sim$$

for each $i = 1, \ldots, m$. Let $\{\rho_\alpha : U \to [0,1]\}_\alpha$ be a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$ of $U = \cup_\alpha U_\alpha$ (replacing $\{U_\alpha\}$ by a refinement if necessary, we may without loss assume that it is a locally finite cover), and let

$$F = \sum \rho_\alpha \cdot F_\alpha : U \to \mathbb{R}^m.$$

Note that since $Z(I) \subset U$ and I is germ-determined, the restriction map $C^\infty(\mathbb{R}^n)/I \to C^\infty(U)/(I|U)$ is an isomorphism. Moreover for the same reason, if $g \in J$ then $g \circ F_\alpha \in (I|U_\alpha)^\sim$ for each α, hence $g \circ F \in (I|U)$. Thus F determines a morphism $\ell A \xrightarrow{f} \ell B$. We have to check that $f|\ell A_\alpha = f_\alpha$, and that f is the unique one with this property. These two facts are checked similarly, and we only do the first. Choose α, and consider a point $x \in U_\alpha$. Let $V \subset U_\alpha$ be a neighbourhood of x, meeting U_β only for finitely many β, say $\beta_1, \ldots, \beta_k$. By compatibility (*), we may choose V so small that $\pi_i \circ \rho_{\beta_j}(F_\alpha - F_{\beta_j}) \in (I|U_{\beta_j} \cap V)$ for $i = 1, \ldots, m; j = 1, \ldots, k$. Hence since $\mathrm{supp}(\rho_{\beta_j}) \subset U_{\beta_j}$, $\pi_i \circ \rho_{\beta_j}(F_\alpha - F_{\beta_j}) \in (I|V)^\sim$. Therefore on V,

$$\pi_i \circ (F_\alpha - F) = \pi_i \circ \sum_{j=1}^{k} (\rho_{\beta_j} F_\alpha - \rho_{\beta_j} F_{\beta_j}) \in (I|V)^\sim.$$

This holds for a neighbourhood V of each $x \in U_\alpha$, so we conclude $\pi_i \circ (F_\alpha - F) \in (I|U_\alpha)^\sim$, i.e. $f|\ell A_\alpha = f_\alpha$. (Notice that germ-determined is only used for uniqueness.) □

When we compose $\mathbb{G} \xrightarrow{Y} \mathcal{G}$ with the embedding $\mathbb{M} \xrightarrow{s} \mathbb{G}$ we obtain the following corollary, which expresses that manifolds appear in $\mathcal{G}$ as they do in *Sets*. It goes without saying that this is of importance if one wishes to do differential geometry inside $\mathcal{G}$.

1.4 Corollary. *There is a full and faithful embedding of the category of manifolds into the universe $\mathcal{G}$, denoted by*

$$\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{G}.$$ □

As in Chapter II, we will regard $\mathcal{G}$ as a set theoretic universe, where the *sets* have elements at various stages: $x \in F(\ell A)$ is an element of F at stage ℓA. Parallel to propositions II.2.2 and 3 we have the following proposition, which shows that we can perform set-theoretic constructions on sheaves.

1.5 Proposition. *$\mathcal{G}$ is a topos; that is to say, $\mathcal{G}$ is a cartesian closed category, and $\mathcal{G}$ has powerobjects.*

Proof. (i) (Small) inverse limits in $Sets^{\mathbb{G}^{op}}$ are computed *pointwise* (see the proof of II.2.2), and from this it is easy to see that inverse limits of sheaves are again sheaves. So inverse limits in $\mathcal{G}$ are computed as in $Sets^{\mathbb{G}^{op}}$.

(ii) The same is actually true for exponentials, for whenever E and $F: \mathbb{G}^{op} \to Sets$ are sheaves then so is F^E (where F^E is defined as in II.2.2). Indeed, let $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell A\}_\alpha$ be a cover in $\mathbb{G}$, and suppose we are given a compatible family of natural transformations

$$\tau_\alpha : \ell A_\alpha \times E \to F.$$

So for each pullback square of the form (*) (above lemma 1.3),

$$\begin{tikzcd} \ell A_\alpha \times_{\ell A} \ell A_\beta \times E \arrow[r, "p_\alpha \times E"] \arrow[d, "p_\beta \times E"] & \ell A_\alpha \times E \arrow[d, "\tau_\alpha"] \\ \ell A_\beta \times E \arrow[r, "\tau_\beta"] & F \end{tikzcd}$$

commutes. Now define $\tau : \ell A \times E \to F$ as follows. Given $\ell B \xrightarrow{g} \ell A$, $x \in E(\ell B)$, make pullbacks in $\mathbb{G}$

$$\begin{tikzcd} \ell A_\alpha \arrow[r, "f_\alpha"] & \ell A \\ \ell B_\alpha \arrow[r, "f'_\alpha"] \arrow[u, "g_\alpha"] & \ell B \arrow[u, "g"] \end{tikzcd}$$

Then $\{\ell B_\alpha \xrightarrow{f'_\alpha} \ell B\}_\alpha$ is a cover, and $\tau_{\ell B_\alpha}(x|f'_\alpha) \in F(\ell B_\alpha)$ defines a compatible family for this cover, so there is a unique $y \in F(\ell B)$ with $y|f'_\alpha = \tau_{\ell B_\alpha}(x|f'_\alpha)$. Let $\tau_{\ell B}(g, x) = y$ for this y. From the fact that F is a sheaf, it follows that τ is natural, and that τ is the unique element of $F^E(\ell A)$ such that $\tau|f_\alpha = \tau_\alpha$. (Note that we never used that E is a sheaf.) As in II.2.2, the evaluation induces a natural bijection (any $G \in \mathcal{G}$)

$$\frac{G \longrightarrow F^E}{G \times E \longrightarrow F}$$

which means, by definition, that F^E is indeed the exponential in $\mathcal{G}$.

(iii) Given a sheaf F, the *powersheaf* $\mathcal{P}(F)$ is the object of $\mathcal{G}$ such that there is a natural bijection

$$\frac{G \xrightarrow{\quad\sigma\quad} \mathcal{P}(F)}{\text{subsheaves } S \text{ of } G \times F}$$

for each sheaf G. Of course, a *subsheaf* is a subfunctor which is itself a sheaf.

The powersheaf can be constructed in $\mathcal{G}$ as in the proof of II.2.3, but replacing *subfunctor* by *subsheaf*. Thus, we define

$$\mathcal{P}(F)(\ell A) = \text{the set of subsheaves of } \ell A \times F,$$

with restrictions as in II.2.3: If $S \in \mathcal{P}(F)(\ell A)$ and $\ell B \xrightarrow{f} \ell A$, then $S|f$ is defined by

$$(g, x) \in S|f \leftrightarrow (f \circ g, x) \in S$$

for any $\ell C \xrightarrow{g} \ell B$ in $\mathbb{G}$ and $x \in F(\ell C)$. Then $S|f$ is a subsheaf of $\ell B \times F$. We leave it to the reader to check that $\mathcal{P}(F)$ is a sheaf. The natural bijection mentioned above is given by the same formula as in II.2.3:

$$(y, x) \in S(\ell A) \leftrightarrow (\mathrm{id}_{\ell A}, x) \in \sigma_{\ell A}(y)$$

for $y \in G(\ell A)$, $x \in F(\ell A)$, $\ell A \in \mathbb{G}$. □

The inclusion functor $\mathcal{G} \hookrightarrow Sets^{\mathbb{G}^{op}}$ has a left adjoint

$$a : Sets^{\mathbb{G}^{op}} \to \mathcal{G},$$

the so-called ***associated sheaf functor***, or ***sheafification functor***; for $F \in Sets^{\mathbb{G}^{op}}$, $a(F)$ is called the sheaf associated to F, or the sheafification of F. (All this is part of the general theory of Grothendieck toposes; see Appendix 1 and the references given there.)

Consequently, the restriction of the global sections functor from II.2,

$$\Gamma\colon \mathcal{G} \to Sets, \quad \Gamma(F) = F(1)$$

has a left adjoint, the composite $Sets \xrightarrow{\Delta} Sets^{\mathbb{G}^{op}} \xrightarrow{a} \mathcal{G}$. In this case, $a \circ \Delta$ can easily be described explicitly (see appendix 1 for the general case). If S is a set, $a \circ \Delta(S)(\ell A) = \mathrm{Cts}(\gamma \ell A, S)$, i.e. the set of continuous functions $Z(I) \longrightarrow S$, where $A = C^\infty(\mathbb{R}^n)/I$ and S is given the discrete topology. We will usually just write $\Delta: Sets \to \mathcal{G}$ for this composite $a \circ \Delta$. Sheaves of the form $\Delta(S)$ are called *constant sheaves.*

Γ also has a right adjoint $B: Sets \to \mathcal{G}$, defined by

$$B(S)(\ell A) = S^{\gamma \ell A}, \text{ the set of all functions } \gamma\ell A \longrightarrow S.$$

For $\ell B \longrightarrow \ell A$ in $\mathbb{G}$, the restriction is defined in the obvious way. If $S \xrightarrow{f} T$ is a function in *Sets*, $B(f): B(S) \to B(T)$ is defined by composition. Note that $B(S)$ is indeed a sheaf: if $\{p_\alpha \in B(S)(\ell A_\alpha)\}_\alpha$ is a compatible family for a cover $\{\ \ell A_\alpha \longrightarrow \ell A\ \}$, then $p_\alpha: \gamma\ell A \to S$, and by compatibility (and leftexactness of γ) p_α and p_β coincide on $\gamma\ell A_\alpha \cap \gamma\ell A_\beta$. So there is a unique function $p: \gamma\ell A = \cup_\alpha \gamma\ell A_\alpha \to S$ extending all the p_α. The correspondence

$$\frac{\Gamma F \xrightarrow{\ \varphi\ } S \text{ in } Sets}{F \xrightarrow{\ \tau\ } B(S) \text{ in } \mathcal{G}}$$

is given by

$$\varphi = \tau_1$$
$$\tau_{\ell A}(x)(t) = \varphi(x|t)$$

where $\ell A \in \mathbb{G}$, $x \in F(\ell A), t \in \gamma\ell A$, and on the righthand side t stands for the corresponding map $1 \longrightarrow \ell A$ in $\mathbb{G}$ (corresponding to the C^∞-homomorphism $A \to \mathbb{R}$ defined by the evaluation at t).

So we have the following proposition, which implies that Γ preserves all limits and colimits. This fact will be of considerable importance later on (see Chapter IV).

1.6 Proposition. *There are adjoint functors*

$$\begin{tikzcd} Sets \arrow[r, shift left=2, "\Delta"] \arrow[r, shift right=2, "B"'] & \mathcal{G} \arrow[l, "\Gamma" description] \end{tikzcd}, \qquad \Delta \dashv \Gamma \dashv B$$

$\square$

Next, we wish to give an interpretation of the language of $\mathcal{G}$, just as we did for $Sets^{\mathbb{L}^{op}}$ in §II.2. The basic notions are interpreted just as for $Sets^{\mathbb{L}^{op}}$ (see the list (1)–(12) in §II.2.) This is possible by lemma 1.3 and proposition 1.5. However, if we interpret the logical connectives as summarized at the end of Section II.2, we cannot show that if $x_1, \ldots, x_n$ are variables ranging over sheaves $F_1, \ldots, F_n \in \mathcal{G}$, the *subfunctor*

$$\{(x_1, \ldots, x_n) \in F_1 \times \ldots \times F_n | \varphi(x_1, \ldots, x_n)\}$$

as defined in II.2 is a *subsheaf*. We run out of the universe $\mathcal{G}$. This problem can be avoided by a rather obvious change in the inductive clauses, which we now indicate. The idea behind it is to sheafify the subfunctor involved whenever we run out of the universe $\mathcal{G}$. As it stands, the clauses may look rather abstract, but the examples that will be given throughout this chapter should easily make the reader familiar with this interpretation, and demonstrate its usefulness.

So, as in Chapter II.2, we give inductive clauses for

$$\ell A \Vdash \varphi(a_1, \ldots, a_n),$$

where $\varphi(x_1, \ldots, x_n)$ is a formula with variables x_i ranging over sheaves $F_i \in \mathcal{G}$, and the a_i are the values of these variables, $a_i \in F_i(\ell A)$. The clauses for $\wedge, \rightarrow, \forall$ are as before, but for $\vee$ and $\exists$ they are new. Moreover, there is a clause for negation $\neg$.

$\ell A \Vdash \varphi(a_1, \ldots, a_n) \wedge \psi(a_1, \ldots, a_n)$ iff both $\ell A \Vdash \varphi(a_1, \ldots, a_n)$ and $\ell A \Vdash \psi(a_1, \ldots, a_n)$

$\ell A \Vdash \varphi(a_1, \ldots, a_n) \vee \psi(a_1, \ldots, a_n)$ iff there is a cover $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell A\}_\alpha$ in $\mathbb{G}$ such that for each α, either

$$\ell A_\alpha \Vdash \varphi(a_1|f_\alpha, \ldots, a_n|f_\alpha), \text{ or } \ell A_\alpha \Vdash \psi(a_1|f_\alpha, \ldots, a_n|f_\alpha)$$

$\ell A \Vdash \exists x_0 \in F_0 \varphi(x_0, a_1, \ldots, a_n)$ iff there is a cover $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell A\}_\alpha$ in $\mathbb{G}$ such that for each α there is a $b_\alpha \in F_0(\ell A_\alpha)$ with $\ell A_\alpha \Vdash \varphi(b_\alpha, a_1|f_\alpha, \ldots, a_n|f_\alpha)$

$\ell A \Vdash \varphi(a_1, \ldots, a_n) \rightarrow \psi(a_1, \ldots, a_n)$ iff for each $\ell B \xrightarrow{f} \ell A$ in $\mathbb{G}$, if $\ell B \Vdash \varphi(a_1|f, \ldots, a_n|f)$ then also $\ell B \Vdash \psi(a_1|f, \ldots, a_n|f)$

$\ell A \Vdash \forall x_0 \in F_0 \varphi(x_0, a_1, \ldots, a_n)$ iff for each $\ell B \xrightarrow{f} \ell A$ in $\mathbb{G}$ and every $b \in F_0(\ell B), \ell B \Vdash \varphi(b, a_1|f, \ldots, a_n|f)$

$\ell A \Vdash \neg \varphi(a_1, \ldots, a_n)$ iff for every $\ell B \xrightarrow{f} \ell A$ in $\mathbb{G}$, if $\ell B \Vdash \varphi(a_1|f, \ldots, a_n|f)$ then $B = 0$ (i.e. B is the trivial ring).

One can now show by induction on φ that

(i) (*functoriality*) if $\ell B \xrightarrow{f} \ell A$ and $\ell A \Vdash \varphi(a_1, \ldots, a_n)$ then also $B \Vdash \varphi(a_1|f, \ldots, a_n|f)$;

(ii) (*local character*) if $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell A\}_\alpha$ is a cover, and for each α, $\ell A_\alpha \Vdash \varphi(a_1|f_\alpha, \ldots, a_n|f_\alpha)$, then $\ell A \Vdash \varphi(a_1, \ldots, a_n)$.

Consequently, the ***subfunctor*** $\{(x_1, \ldots, x_n)|\varphi(x_1, \ldots, x_n)\}$ of $F_1 \times \ldots \times F_n$ defined by

$$(a_1, \ldots, a_n) \in \{(x_1, \ldots, x_n)|\varphi(x_1, \ldots, x_n)\}(\ell A) \leftrightarrow \ell A \Vdash \varphi(a_1, \ldots, a_n)$$

(indeed a subfunctor by (i)) is in fact a ***subsheaf*** of $F_1 \times \ldots \times F_n$ (by (ii)).

As in Chapter II.2, we define "***φ is valid in $\mathcal{G}$***" for a sentence φ, denoted by $\mathcal{G} \models \varphi$, as

$$\mathcal{G} \models \varphi \text{ iff } 1 \Vdash \varphi \text{ iff for each } \ell A \in \mathbb{G}, \ell A \Vdash \varphi.$$

The fact that we can consistently define subsheaves of this form $\{(x_1, \ldots, x_n)|\varphi(x_1, \ldots, x_n)\}$ means that such ***set-theoretic*** constructions can be performed within $\mathcal{G}$. All this is part of the general theory of Grothendieck toposes, which not only says that these generalized ***sets*** can be constructed, but that the usual set theoretic arguments remain valid, provided they are constructive and explicit (i.e. they are intuitionistically correct; see Appendix 1). This important meta-theorem which enables us to transfer the constructive results of classical analysis immediately to universes like $\mathcal{G}$, will be exploited throughout the book.

In $\mathcal{G}$, we can develop analysis in a ***synthetic*** way, based on the ***line*** R as in II.2,

$$R = s(\mathbb{R}) = \mathbb{G}(-, \ell C^\infty(\mathbb{R}))$$

(cf. 1.3, 1.4). Just as for $Sets^{\mathbb{L}^{op}}$, R is a commutative ring object in $\mathcal{G}$. R has a canonical order $<$ in $\mathcal{G}$, which is defined as in Chapter II, i.e. if $a, b \in R(\ell A)$ where $A = C^\infty(\mathbb{R}^n)/I$, and a and b are represented by smooth functions $a(x), b(x) : \mathbb{R}^n \to \mathbb{R}$, then

(1) $\ell A \Vdash a < b$ iff there is a finitely generated $I_o \subset I$ such that $\forall x \in Z(I_0)\ a(x) < b(x)$

(see (6) of the dictionary in II.2). When I is germ-determined, this is equivalent to

(2) $$\ell A \Vdash a < b \Leftrightarrow \forall x \in Z(I)\ a(x) < b(x)$$

(for (2) $\Rightarrow$ (1), take a smooth $\varphi: \mathbb{R}^n \to \mathbb{R}$ such that $Z(I) \subset V \subset Z(\varphi) \subset \{x|a(x) < b(x)\}$ for some open V. Then $\varphi \in I$.) The non-strict pre-order relation $\leq$ defined by

(3) $\quad \ell A \Vdash a \leq b$ iff for each $\rho \in m^{\infty}_{\geq 0} \subset C^{\infty}(\mathbb{R}), \rho \circ (b-a) \in I$

for A, a, b as above, cannot be simply reformulated in terms of $Z(I)$.

The *"set" of natural numbers* $\mathbb{N}_{\mathcal{G}}$ in $\mathcal{G}$ is the sheaf

$$\mathbb{N}_{\mathcal{G}} = \Delta(\mathbb{N}).$$

So elements of $\mathbb{N}_{\mathcal{G}}(\ell A)$ are continuous functions $Z(I) \longrightarrow \mathbb{N}$. We will usually write just $\mathbb{N}$ for the sheaf $\mathbb{N}_{\mathcal{G}}$, when it is clear whether we mean $\mathbb{N} \in \mathit{Sets}$ or $\mathbb{N} \in \mathcal{G}$. Notice that $\mathbb{N}_{\mathcal{G}} \cong s(\mathbb{N})$. Similarly, the *integers* in $\mathcal{G}$ are interpreted as $\mathbb{Z}_{\mathcal{G}} = \Delta(\mathbb{Z}) \cong s(\mathbb{Z})$, for which we just write $\mathbb{Z}$. And the *rationals* are interpreted as $\mathbb{Q}_{\mathcal{G}} = \Delta(\mathbb{Q})$, denoted just $\mathbb{Q}$. (This interpretation of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ is the one consistent with our definition of the *internal logic* of $\mathcal{G}$ by means of $\Vdash$; see appendix 1). By applying s to $\mathbb{Z} \hookrightarrow \mathbb{R}$, $\mathbb{Z}_{\mathcal{G}}$ is canonically embedded in R. More explicitly, if $A = C^{\infty}(\mathbb{R}^n)/I$, $\ell A \in \mathbb{G}$, and $p \in \mathbb{Z}(\ell A)$, i.e. p is a continuous function $Z(I) \longrightarrow \mathbb{Z}$, then since $\mathbb{Z}$ is discrete, there is a smooth function $\overline{p}: \mathbb{R}^n \to \mathbb{R}$ with $\forall x \in Z(I) \exists$ neighbourhood $V \ni x$ such that $\overline{p}|V$ is constant, with value $p(x)$, and this $\overline{p}$ is uniquely determined modulo I. This defines a natural transformation $\mathbb{Z} \hookrightarrow R$. Similarly, we can define a natural transformation $\mathbb{Q} \hookrightarrow R$.

R looks a lot better in $\mathcal{G}$ than it does in $\mathit{Sets}^{\mathbb{L}^{op}}$:

1.7 Proposition. *In $\mathcal{G}$, the following are valid*

(i) *R is a commutative ring with 1*
(ii) *$<$ and $\leq$ are compatible with the ringstructure (as in II.2.4)*
(iii) *R is a local ring; i.e.*

$$\begin{aligned} \mathcal{G} &\models \neg 0 = 1 \\ \mathcal{G} &\models \forall x, y \in R (x + y \in U(R) \to x \in U(R) \vee y \in U(R)) \end{aligned}$$

where $U(R) = \{z \in R | z$ is invertible$\}$.

(iv) *R is an Archimedean ring, i.e.*

$$\mathcal{G} \models \forall x \in R \quad \exists n \in \mathbb{N} \quad x < n.$$

Proof. (i) and (ii) are proved just as in II.2 (the new interpretation of $\vee, \exists$, and $\neg$ is not involved). (iii) $\mathcal{G} \models \neg 0 = 1$: by the definition of $\neg$, we need to show that if $\ell A \Vdash 0 = 1$, then A is trivial. But if $A = C^{\infty}(\mathbb{R}^n)/I$, then $\ell A \Vdash 0 = 1$ iff $1 \in I$, so this is clear. For the other clause, take $\ell A \in \mathbb{G}$, $A = C^{\infty}(\mathbb{R}^n)/I$, and $a, b \in R(\ell A)$ represented by $a(x), b(x): \mathbb{R}^n \to \mathbb{R}$, and assume $\ell A \Vdash a + b$ is invertible. Then in particular by functoriality of $\Vdash$, $1 \Vdash a|p + b|p$ is invertible for each point $p \in Z(I)$ (regarded as a map $1 \xrightarrow{p} \ell A$ in $\mathbb{G}$). So $a(p)+b(p) \neq 0$. Let $V = \{x \in \mathbb{R}^n | a(x) \neq 0\}$, $W = \{x \in \mathbb{R}^n | b(x) \neq 0\}$. Then $Z(I) \subseteq V \cup W$, and $a|V$ is invertible in $A_V = C^\infty(V)/(I|V)^\sim$, $b|W$ is invertible in $A_W = C^\infty(W)/(I|W)^\sim$, i.e. $\ell A_V \Vdash a \in U(R), \ell A_W \Vdash b \in U(R)$. Since $\{\ell A_V \hookrightarrow \ell A, \ell A_W \hookrightarrow \ell A\}$ is a cover of ℓA (lemma 1.2), we conclude that $\ell A \Vdash (a \in U(R) \vee b \in U(R))$. (iv) We have to show that for each $a \in R(\ell A), \ell A \Vdash \exists m\ a < m$. It suffices to consider the *generic* a, which is the identity at stage $\ell A = R$ (since every other $a \in R$ is the restriction of this *generic element*). But $\{\ell C^\infty(\leftarrow, m) \hookrightarrow R\}_m$ is a cover, and $\ell C^\infty(\leftarrow, m) \Vdash \mathrm{id} < m$. So $R \Vdash \exists m\ \mathrm{id} < m$. This proves (iv). □

1.8 Remark. To prove (iii), we could have taken just the *generic pair* of elements a, b with $\ell A \Vdash a + b \in U(R)$, rather than an arbitrary pair. This generic pair is the pair $a = \pi_1, b = \pi_2$ at stage $\ell A = \ell C^\infty(\mathbb{R}^2 - \{0\})$. The assertion that $\ell C^\infty(\mathbb{R}^2 - \{0\}) \Vdash (\pi_1 \in U(R) \vee \pi_2 \in U(R))$ means that $\ell C^\infty(\mathbb{R}^2 - \{0\})$ is covered by the loci $\ell C^\infty((\mathbb{R} - \{0\}) \times \mathbb{R})$ and $\ell C^\infty(\mathbb{R} \times (\mathbb{R} - \{0\}))$. So to prove that $\mathcal{G} \models$ *R is an Archimedean local ring* we only need the following covers (recall $s : \mathbb{M} \hookrightarrow \mathbb{G}$ as in 1.4):

(a) the empty family covers the trivial locus;
(b) $\{s(\leftarrow, m) \hookrightarrow s(\mathbb{R})\}_{m>0}$ covers $s(\mathbb{R})$;
(c) $\{s(\mathbb{R} \times (\mathbb{R} - \{0\})) \hookrightarrow s(\mathbb{R}^2), s((\mathbb{R} - \{0\}) \times \mathbb{R}) \hookrightarrow s(\mathbb{R}^2)\}$ is a cover.

The Grothendieck topology on $\mathbb{G}$ as described in 1.2 is the *smallest* Grothendieck topology on $\mathbb{G}$ satisfying the condition (a)–(c). In this sense, this topology is precisely the one *forcing* R to be an Archimedean local ring (see Moerdijk & Reyes (1984a)).

In the *constructive context* of analysis inside $\mathcal{G}$ we cannot expect R to be a field in the sense that $\mathcal{G} \models \forall x \in R(x = 0 \vee x \in U(R))$, but it gets close: the following property holds.

1.9 Proposition. *In $\mathcal{G}$, R is a field in the following sense:*

$$\begin{aligned}\mathcal{G} \models\ & \forall x_1, \ldots, x_n \in R(\neg(x_1 = 0 \wedge \ldots \wedge x_n = 0) \to \\ & (x_1 \in U(R) \vee \ldots \vee x_n \in U(R))).\end{aligned}$$

Proof. Let $\ell A \xrightarrow{a} R^n$, $a = (a_1, \ldots, a_n)$ be an n-tuple of reals at stage ℓA, such that $\ell A \Vdash \neg(a_1 = 0 \wedge \ldots \wedge a_n = 0)$. Say $A = C^\infty(\mathbb{R}^m)/I$, and

a_i is represented by $a_i(x)\colon \mathbb{R}^m \to \mathbb{R}$. Then $C^\infty(\mathbb{R}^m)/(I, a_1, \ldots, a_n)$ is trivial (recall from I.4.9 that $(I, a_1, \ldots, a_n)$ is germ-determined if I is), or equivalently,

$$A(I) \cap \bigcap_{i \leq n} Z(a_i(x)) = \emptyset.$$

So $Z(I) \subseteq U_{a_1(x)} \cup \ldots \cup U_{a_n(x)}$. Since $\ell A \cap s(U_{a_i(x)}) \Vdash a_i \in U(R)$, it follows that $\ell A \Vdash a_1 \in U(R) \vee \ldots \vee a_n \in U(R)$. $\square$

1.10 Infinitesimal Spaces; Kock-Lawvere Axiom. As in $Sets^{\mathbb{L}^{op}}$, we can define some *infinitesimal subspaces* of R^n, such as

$$\begin{aligned} D &= \{x \in R \mid x^2 = 0\} \\ D_k(n) &= \{x_1, \ldots, x_n) \in R^n \mid x_{i_1} \cdot \ldots \cdot x_{i_{k+1}} = 0, \text{ any } k+1\text{–tuple} \\ &\quad\; i_1, \ldots, i_{k+1}\} \end{aligned}$$

which correspond under the Yoneda embedding to the loci defined in II.1.10. Similarly, we have the *set of infinitesimals* $\Delta = \bigcap_{n>0}\left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq R$, which can be defined within $\mathcal{G}$ as

$$\Delta = \{x \in R \mid \forall n \in \mathbb{N}_{\mathcal{G}}(-\frac{1}{n+1} < x < \frac{1}{n+1})\},$$

corresponding to the locus $\ell C^\infty(\mathbb{R})/m^g_{\{0\}}$, and the interval $[0,0]$ corresponding to the locus $\ell C^\infty(\mathbb{R})/m^\infty_{\{0\}}$. $[0,0]$ properly contains the *set of nilpotent elements*

$$D_\infty = \{x \in R \mid \exists n \in \mathbb{N}\ x^n = 0\}.$$

It is easy to see that

$$\begin{aligned} (*) \quad \mathcal{G} \models \forall x \in R(x \in U(R) \quad &\longleftrightarrow \quad \neg x = 0 \\ &\longleftrightarrow \quad x < 0 \vee x > 0). \end{aligned}$$

In $\mathcal{G}$, Δ is the set of reals which are *almost* zero in the intuitionistic sense:

$$\Delta = \{x \in R \mid \neg\neg x = 0\} \text{ in } \mathcal{G}.$$

To see this, one can show for a real $a \in R(\ell A)$ that $\ell A \Vdash a \in \Delta \leftrightarrow \neg a \in U(R)$. But it also follows from Proposition 1.7 and (*) by reasoning *intuitionistically* inside $\mathcal{G}$: If $x > 0$ then by Archimedeanness $\exists n\ \frac{1}{n} < x$, so $x \notin \Delta$. Thus $\Delta \subset \{x \mid \neg\neg x = 0\}$. Conversely, if $x = 0$ then $\forall n > 0\left(-\frac{1}{n} < x < \frac{1}{n}\right)$, so if $\neg\neg x = 0$ then $\forall n > 0 \neg\neg\left(-\frac{1}{n} < x < \frac{1}{n}\right)$. But then also $\forall n > 0\left(-\frac{1}{n} < x < \frac{1}{n}\right)$. For choose $n > 0$. Since R is local, $x < \frac{1}{n} \vee \frac{1}{n+1} < x$. Since

$\frac{1}{n+1} < x \longrightarrow \neg x < \frac{1}{n+1}$ and we know $\neg\neg x < \frac{1}{n+1}$, it follows that $x < \frac{1}{n}$. Similarly $-\frac{1}{n} < x$.

As in $Sets^{\mathbb{L}^{op}}$, we have

$$D_k = D_k(1) \subset D_\infty \subset [0,0] \subset \Delta \text{ in } \mathcal{G},$$

and these inclusions are proper (recall $[0,0] = \ell(C^\infty(\mathbb{R})/m^\infty_{\{0\}})$.

The Kock-Lawvere axiom (by which differentiation is defined)

$$R^D \cong R \times R,$$

holds in $\mathcal{G}$, as well as its generalizations saying that in $\mathcal{G}$ it is valid that any map in $R^{D_k(n)}$ is given by a unique polynomial in n variables of total degree $\leq k$. This can be proved just as the simple form of the Kock-Lawvere axiom for $Sets^{\mathbb{L}^{op}}$. Also, we have that in $\mathcal{G}$

$$s(M)^D \cong s(TM),$$

since the Yoneda embedding preserves any exponentials that exist in $\mathbb{G}$, and the isomorphism is true in $\mathbb{G}$ (see II.1.12 – 1.16).

Finally, we note that we can integrate in $\mathcal{G}$.

1.11 Proposition. *(Integration axiom)*

$$\mathcal{G} \models \forall f \in R^{[0,1]} \exists! g \in R^{[0,1]} (g' \equiv f \wedge g(0) = 0)$$

(g' makes sense, since $\mathcal{G} \models \forall x, y \in R(x \in D \wedge y \in [0,1] \rightarrow x + y \in [0,1])$.)

Proof. This is similar to the case of $Sets^{\mathbb{L}^{op}}$ (see II.2.4), but with $(I(x), m^\infty_{[0,1]}(t))^\sim$ instead of $(I(x), m^\infty_{[0,1]}(t))$. More precisely, we have to show that, given $f \in R^{[0,1]}$ at stage ℓA, where $A = C^\infty(\mathbb{R})/I$, and $f = F(x,t): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ mod $(I(x)), m^\infty_{[0,1]})^\sim, G(x,t) = \int_0^t F(x,u)du$ is well-defined. So suppose $F(x,t) \in (I(x), m^\infty_{[0,1]}(t))^\sim$. Then for each $(x_0, t_0) \in Z(I) \times [0,1]$ there is a neighbourhood $U \times V$ of (x_0, t_0) such that F can be written as

$$(*) \quad F(x,t) = \sum \varphi_i(x,t) a_i(x) + \sum \psi_j(x,t) b_j(t) \text{ on } U \times V$$

with $a_i \in I, b_j \in m^\infty_{[0,1]}$, and $\varphi_i, \psi_j \in C^\infty(U \times V)$. For fixed x_o, finitely many of these V's cover $[0,1]$, say $V_1, \ldots, V_k$. Let $\rho_1(t), \ldots, \rho_k(t)$ be a partition of unity subordinate to $V_1, \ldots, V_k$. Then for $U_{x_0} = U_1 \cap \ldots \cap U_k \ni x_0, V_{x_0} = V_1 \cup \ldots \cup V_k \supset [0,1]$, we have for all

$(x,t) \in U_{x_0} \times V_{t_0}$,

$$F(x,t) = \sum_{s=1}^{k} \sum_{i} \rho_s(t)\psi_i^s(x,t)a_i^s(x) + \sum_{s} \sum_{j} \psi_j^s(x,t)\rho_s(t)b_j^s(t),$$

(where a_i^s is the a_i in (*) for $U_s \times V_s$, etc.). Thus, we may assume that equation (*) holds on a neighbourhood $U \times V$ of $\{x_0\} \times [0,1]$. Then for all $x \in U$,

$$G(x,t) = \sum_i a_i(x) \int_0^t \varphi_i(x,u)du + \sum_j \int_0^t \psi_j(x,u)b_j(u)du.$$

As in the case of $Sets^{\mathbb{L}^{op}}$, it follows from I.4.12 (with U for $\mathbb{R}^n$, V for $\mathbb{R}^m$) that $\int_0^t \psi_j(x,u)b_j(u)du \in (m^{\infty}_{[0,1]}|V)\cdot C^{\infty}(U)$. So $G(x,t)|U \times V \in (I|U, m^{\infty}_{[0,1]}|V)$. This holds for each $x_0 \in Z(I)$, and therefore $G(x,t) \in (I, m^{\infty}_{[0,1]})^{\sim}$. $\square$

1.12 Proposition. (i) $\mathcal{G} \models \forall \alpha \in R^{[0,1]}((\forall t \in [0,1]\alpha(t) > 0) \rightarrow \int_0^1 \alpha(t)dt > 0)$,

(ii) $\mathcal{G} \models \forall \alpha \in R^{[0,1]}((\forall t \in [0,1]\alpha(t) \geq 0) \rightarrow \int_0^1 \alpha(t)dt \geq 0)$.

Proof. The proof of (i) is exactly the same as for $Sets^{\mathbb{L}^{op}}$ (cf. II.2.4 (iv)). For (ii), one slightly modifies the proof of the case for $Sets^{\mathbb{L}^{op}}$ (II.2.4(vii)); instead of showing that (*) implies (**) in the proof of (vii) in Section II.2, one now shows that if $\rho(F(x,t)) \in (I(x), m^{\infty}_{[0,1]})^{\sim}$ for all $\rho \in m^{\infty}_{R\geq 0}$, then also $\rho(\int_0^1 F(x,t)dt) \in I(x)$ for all $\rho \in m^{\infty}_{R\geq 0}$, where I is germ-determined. The modification is just like the one in the proof of 1.11 above. $\square$

2 The topos $\mathcal{F}$ of closed ideals

In the topos $\mathcal{G}$ that has been introduced in the previous section, the order $\leq$ on R is rather hard to handle: we need the complicated result I.4.12 to prove that $\leq$ is compatible with the ring structure, and to show that the integration axiom holds in $\mathcal{G}$. In the present section, we will consider the topos $\mathcal{F}$ of sheaves on the site $\mathbb{F}$ of duals of closed (or near-point determined) C^{∞}-rings. In many respects, $\mathcal{F}$ has properties similar to those of $\mathcal{G}$, but there are some differences, notably in the treatment of $\leq$.

Recall that (up to isomorphism) the objects of $\mathbb{F}$ are duals of C^∞-rings $C^\infty(\mathbb{R}^n)/I$ where I is a *closed* ideal. The morphisms of $\mathbb{F}\ell C^\infty(\mathbb{R}^n)/I \to \ell C^\infty(\mathbb{R}^m)/J$ are equivalence classes of smooth functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $J \subseteq \varphi_*(I)$. The inclusion $\mathbb{F} \hookrightarrow \mathbb{L}$ has a right adjoint κ:

$$\mathbb{F} \underset{\hookrightarrow}{\overset{\kappa}{\longleftarrow}} \mathbb{L}.$$

We equip $\mathbb{F}$ with a *Grothendieck topology* defined as for the topos $\mathbb{G}$: $\{\ell A_\alpha \xrightarrow{f_\alpha} \ell A\}_\alpha$ is a covering family in $\mathbb{F}$ iff

(i) for every α there exists a $b_\alpha \in A$ and a commutative diagram

$$\begin{tikzcd}
\ell A_\alpha \arrow[rr, "\sim"] \arrow[dr, "f_\alpha"'] & & \kappa\ell(A\{b_\alpha^{-1}\}) \arrow[dl] \\
& \ell A &
\end{tikzcd}$$

where $\kappa\ell(A\{b_\alpha^{-1}\}) \to A$ is the canonical map;

(ii) $\{\gamma\ell A_\alpha \xrightarrow{\gamma f_\alpha} \gamma\ell A\}_\alpha$ is a surjective family of topological spaces.

Just as for $\mathbb{G}$, one can show that this indeed defines a Grothendieck topology on $\mathbb{F}$, and that this is exactly the *open cover topology* in the sense expressed by the following lemma.

2.1 Lemma. *If $\ell(C^\infty(U)/I)$ is an object of $\mathbb{F}$, then its covering families are precisely the families*

$$\{\ell(C^\infty(U)_\alpha/\overline{(I|U_\alpha)}) \hookrightarrow \ell(C^\infty(U)/I)\}$$

where $\{U_\alpha\}_\alpha$ is an open cover of $Z(I)$ in U. □

The category $\mathcal{F}$ is the topos of sheaves on this site $\mathbb{F}$. That is, $\mathcal{F}$ is the category of functors $\mathbb{F}^{\mathrm{op}} \to$ *Sets* which satisfy the property of existence of unique joins for compatible families as described for $\mathbb{G}$. $\mathcal{F}$ is a topos (analogous to 1.5; see also Appendix 1), and all representables are sheaves:

2.2 Lemma. *The site $\mathbb{F}$ is subcanonical. Consequently, the category of manifolds $\mathbb{M}$ is fully and faithfully embedded in $\mathcal{F}$. This embedding is again denoted by $s\colon \mathbb{M} \hookrightarrow \mathcal{F}$.*

Proof. Completely analogous to the one for $\mathcal{G}$ (see 1.3, 1.4). □

2.3 Proposition. *There are adjoint functors*

$$\begin{tikzcd} \mathit{Sets} \arrow[r, shift left=2, "\Delta"] \arrow[r, shift right=2, "B"'] & \mathcal{F} \arrow[l, "\Gamma"'] \end{tikzcd}, \qquad \Delta \dashv \Gamma \dashv B$$

Proof. As for $\mathcal{G}$. □

We can interpret our set-theoretic language in $\mathcal{F}$, where the basic notions and the inductive clauses for the logical connectives are as described for $\mathcal{G}$. This defines a notion of ***internal validity*** of a sentence φ in $\mathcal{F}$, written as $\mathcal{F} \models \varphi$. So we can do analysis in $\mathcal{F}$ based on the line $R = \mathbb{F}(-, \ell C^\infty(\mathbb{R}))$. R carries an order $<$ defined by the same equivalence as for $\mathcal{G}$:

$$\ell A \Vdash a < b \Leftrightarrow \forall x \in Z(I) \quad a(x) < b(x),$$

where $\ell A = \ell(C^\infty(\mathbb{R}^n)/I) \in \mathbb{F}$, and $a, b \in \ell A(R)$ are represented by $a(x), b(x) \colon \mathbb{R}^n \to \mathbb{R}$. The natural numbers at stage ℓA are the continuous (or locally constant) functions $Z(I) \to \mathbb{N}$, i.e. the sheaf $\mathbb{N}_{\mathcal{F}}$ of natural numbers in $\mathcal{F}$ is given by

$$\mathbb{N}_{\mathcal{F}}(\ell A) = \mathrm{Cts}(\gamma \ell A, \mathbb{N}).$$

2.4 Proposition. *(i) In $\mathcal{F}$, it holds that R is an Archimedean local ring, with order $<$ compatible with the ring structure.*

(ii) In $\mathcal{F}$, R is a field in the sense of 1.9.

Proof. (i) as for $\mathcal{G}$, see 1.7. The proof of (ii) is as that of proposition 1.9, noting that if I is closed and $1 \in C\ell(I, a_1, \dots, a_n)$ ($C\ell$ for closure), then $Z(I, a_1, \dots, z_n) = \emptyset$, so $1 \in (I, a_1, \dots, a_n)$ since this ideal is germ-determined (here we use the same notation as in 1.9). □

We define the order $\leq$ in $\mathcal{F}$ in the same way as for $\mathcal{G}$. So if $a \in \ell A(R)$ is represented by $a(x) \colon \mathbb{R}^n \to \mathbb{R}$ ($A = C^\infty(\mathbb{R}^n)/I$, I closed), then

$$\ell A \Vdash 0 \leq a \Leftrightarrow \forall \rho \in m^\infty_{[0,\to)} \rho(a(x)) \in I.$$

2.5 Lemma. *Let $X \subset \mathbb{R}^n$ be closed. In $C^\infty(\mathbb{R}^n), m^\infty_X$ is the closure of the ideal m^g_X.*

Proof. Obvious from Whitney's spectral theorem (I.4.4). □

In $\mathcal{F}$, we define Δ as $\cap_{n>0}\ell C^\infty(-\frac{1}{n},\frac{1}{n}) = \{x \in R | \forall n > 0 \left(-\frac{1}{n} < x < \frac{1}{n}\right)\}$ as before. Δ is the object of *infinitesimals.*

2.6 Proposition. *The following are valid in $\mathcal{F}$.*

(i) $\forall x \in R\colon x \leq 0 \leftrightarrow \neg x > 0 \leftrightarrow \forall \varepsilon > 0 x < \varepsilon$;
(ii) $[0,0] = \Delta = \{x | \neg\neg x = 0\}$.

Proof. (i) If x is a real at stage ℓA, we have by Archimedeanness that $\ell A \models \forall \varepsilon > 0\ x < \varepsilon$ iff for all $n \in \mathbb{N}$ $\ell A \Vdash x < \frac{1}{n+1}$. So the equivalence $\mathcal{F} \models \forall x \in R(x \leq 0 \leftrightarrow \forall \varepsilon > 0\ x < \varepsilon)$ is just a reformulation of 2.5 for the case $X = (\leftarrow, 0] \subset \mathbb{R}$. Clearly, $x \leq 0 \to \neg x > 0$ is valid. Conversely, reasoning in $\mathcal{F}$, suppose $\neg x > 0$ and choose $\varepsilon > 0$. Then $x < \varepsilon$ or $x > 0$ since R is a local ring and $<$ is compatible. Thus $x < \varepsilon$. (ii) is immediate from (i). □

2.7 Corollary. *In $\mathcal{F}$, the order relation $\leq$ on R is compatible with the ring structure.*

Proof. We could use I.4.12 as before, but in fact the assertion now follows much more easily from 2.6 by reasoning inside F. Consider, for example, the implication $x \leq 0 \wedge y \leq 0 \to x + y \leq 0$. Suppose $x \leq 0 \wedge y \leq 0$, and take $\varepsilon > 0$. Then $x < \frac{1}{2}\varepsilon \wedge y < \frac{1}{2}\varepsilon$, so $x + y < \varepsilon$ by compatibility of $<$. Thus $x + y \leq 0$. □

The *Kock-Lawvere axiom* $R^D \cong R \times R$ and its generalizations hold in $\mathcal{F}$. This is proved just as for $\mathcal{G}$. Recall that

$$D_\infty = \cup_k D_k = \{x \in R | \exists n \in \mathbb{N}\ x^n = 0\}.$$

The following proposition does not hold in $\mathcal{G}$.

2.8 Proposition. *Let M be a manifold. In $\mathcal{F}$, the restriction map*

$$s(M)^{\Delta} = s(M)^{[0,0]} \xrightarrow{\rho} s(M)^{D_\infty}$$

is an isomorphism. ("In $\mathcal{F}$, manifolds believe that all infinitesimals are nilpotent".)

Proof. It suffices to consider the case $M = \mathbb{R}$. First we show that $\rho\colon R^{[0,0]} \to R^{D_\infty}$ is monic. Note that if W is a Weil algebra and

$\ell W \xrightarrow{\varphi} R$ is a map in $\mathbb{L}$, say with $\varphi(0) = 0$, then (by Hadamard's lemma) for some $k \in \mathbb{N}$, φ can be factored as

$$\begin{tikzcd} \ell W \arrow[r, "\varphi"] \arrow[dr] & R \\ & D_k \arrow[u] \end{tikzcd}$$

Now let $f \in R^{[0,0]}(\ell A)$, i.e. $f: \ell A \times [0,0] \to R$ where we write $\ell A = \ell(C^\infty(\mathbb{R}^n)/I) \in \mathbb{F}$ and $\times$ denotes the product in $\mathbb{F}$, and suppose $\ell A \Vdash f|D_\infty = 0$, i.e. the composite $\ell A \times D_k \hookrightarrow \ell A \times [0,0] \xrightarrow{f} R$ is identically zero, for every $k \in \mathbb{N}$. If W is any Weil algebra, and $\ell W \xrightarrow{\varphi} \ell A \times [0,0]$ is any map in $\mathbb{L}$, then by the remark just made, φ factors as

$$\begin{tikzcd} \ell W \arrow[r, "\varphi"] \arrow[dr] & \ell A \times [0,0] \arrow[r, "f"] & R \\ & \ell A \times D_k \arrow[u] \arrow[ur] & \end{tikzcd}$$

so $f \circ \varphi = 0$. By near-point determinedness, we conclude that $f = 0$ as an element of the ring $C^\infty(\mathbb{R}^n \times \mathbb{R})/C\ell(I(x), m^\infty_{\{0\}}(t))$. So $\ell A \Vdash f = 0$.

To see that ρ is epic, we make a further analysis of $R^{D_\infty}(\ell A)$ and $R^{[0,0]}(\ell A)$. Clearly,

$$R^{[0,0]}(\ell A) \cong C^\infty(\mathbb{R}^n \times \mathbb{R})/\overline{(I(x), m^\infty_{\{0\}}(t))}.$$

On the other hand, an element f of $R^{D_\infty}(\ell A)$ is the same as an element of $\varprojlim_k A[[t]]/(t^{k+1}) = A[[t]]$. Writing $A[[t]]$ as an inverse limit like this, it follows immediately from lemma II.1.15 that $A[[t]]$ is closed if A is. Furthermore, by Borel's theorem (see I, 1.3) the canonical map

$$C^\infty(\mathbb{R}^n \times \mathbb{R}) \to A[[t]]$$

is a surjection, so we conclude that

$$A[[t]] \cong C^\infty(\mathbb{R}^n \times \mathbb{R}) / \bigcap_k (I(x), t^{k+1}).$$

Since $(I(x), t^{k+1})$ is closed, it is clear that $\overline{(I(x), m^\infty_{\{0\}})} \subset (I(x), t^{k+1})$ for each k. So the component $\rho_{\overline{A}}$ is just the canonical map

$$C^\infty(\mathbb{R}^n \times \mathbb{R}) / \overline{(I(x), m^\infty_{\{0\}})} \to C^\infty(\mathbb{R}^n \times \mathbb{R}) / \bigcap_k (I(x), t^{k+1}),$$

which is a surjection of C^∞-rings. A fortiori

$$\ell A \Vdash \forall f \in R^{D_\infty} \exists g \in R^{[0,0]} \rho(g) = f. \qquad \square$$

2.9 Remark. Given the analyses of the exponentials R^{D_∞} and $R^{[0,0]}$ in the second part of the proof, we see that 2.8 can be reformulated algebraically as: for any closed ideal $I \subseteq C^\infty(\mathbb{R}^n)$, we have the following equality of ideals in $C^\infty(\mathbb{R}^n \times \mathbb{R})$,

$$\overline{(I(x), m^\infty_{\{0\}})} = \bigcap_k (I(x), t^{k+1}).$$

2.10 Proposition. *In $\mathcal{F}$, the following hold:*

(i) $\forall f \in R^{[0,1]} \exists! g \in R^{[0,1]} (g' = f \,\&\, g(0) = 0)$; *and writing* $\int_0^x f(t)dt$ *for this* $g(x)$:

(ii) $\forall f \in R^{[0,1]} ((\forall t \in [0,1] f(t) > 0) \to \int_0^1 f(t)dt > 0)$

(iii) $\forall f \in R^{[0,1]} ((\forall t \in [0,1] f(t) \geq 0) \to \int_0^1 f(t)dt \geq 0)$

Proof. (i) Again, this could be proved using I.4.12 as for $\mathcal{G}$, but there is a much easier proof. As before, we have to show that if $I \subseteq C^\infty(\mathbb{R}^n)$ is a closed ideal and $F(x,t) \in \overline{(I(x), m^\infty_{[0,1]})} \subseteq C^\infty(\mathbb{R}^n \times \mathbb{R})$, then also $G(x,t) = \int_0^t F(x,u)du \in \overline{(I, m^\infty_{[0,1]})}$. But by proposition 2.6, $\overline{(I, m^\infty_{[0,1]})} = \overline{(I, m^g_{[0,1]})}$ and integration is continuous, so we may assume that $F(x,t) \in (I, m^g_{[0,1]})$. But then clearly also $G(x,t) \in (I, m^g_{[0,1]})$. (ii) is proved just as for $\mathcal{G}$ and $Sets^{\mathbb{L}^{op}}$, and (iii) follows immediately from (ii), by 2.6(i) above. $\square$

A striking difference between $\mathcal{F}$ and $\mathcal{G}$ is the following density result.

2.11 Theorem. *Theorem Let $X \xrightarrow{\alpha} Y$ be a morphism of $\mathcal{F}$. Then $R^\alpha \colon R^Y \to R^X$ is a monomorphism in F iff $\Gamma(R^\alpha) \colon \mathcal{F}(Y,R) \to \mathcal{F}(X,R)$ is a monomorphism in Sets.*

Proof. $\Rightarrow$ is clear, since Γ has a left adjoint. For $\Leftarrow$, suppose $\Gamma(R^\alpha)$ is mono, and take $\ell A \in \mathbb{F}$. Suppose $u, v \in R^Y(\ell A)$ and $R^\alpha_{\ell A}(u) = R^\alpha_{\ell A}(v)$, i.e. $u, v\colon \ell A \times Y \to R$ and the composites

$$\ell A \times X \xrightarrow{1\times\alpha} \ell A \times Y \underset{v}{\overset{u}{\rightrightarrows}} R$$

are equal. We show that $u = v$. Take an element (f, y) of $\ell A \times Y$ at stage $\ell B \in \mathbb{F}$, i.e. $\ell B \xrightarrow{(f,y)} \ell A \times Y$. B is closed, so to show that $u\circ(f,y) = v\circ(f,y)$ it suffices to prove that $u\circ(f,y)\circ\varphi = v\circ(f,y)\circ\varphi$ for every $\ell W \xrightarrow{\varphi} \ell B$ in $\mathbb{F}$, where W is a Weil algebra. By assumption, $u\circ(f\varphi\times Y)\circ(W\times\alpha) = v\circ(f\varphi\times Y)\circ(W\times\alpha)$

$$\begin{array}{ccccc}
\ell A\times X & \xrightarrow{\;A\times\alpha\;} & \ell A\times Y & \underset{v}{\overset{u}{\rightrightarrows}} & R \\
\uparrow{\scriptstyle f\varphi\times X} & & \uparrow{\scriptstyle f\varphi\times Y} & & \\
\ell W\times X & \xrightarrow{\;\ell W\times\alpha\;} & \ell W\times Y & &
\end{array}$$

so by exponential adjointness, $u_0\circ\alpha = v_0\circ\alpha$,

$$X \xrightarrow{\alpha} Y \underset{u_0}{\overset{v_0}{\rightrightarrows}} R^{\ell W}$$

where u_0 is the transposed of $\ell W \times Y \xrightarrow{f\varphi\times Y} \ell A\times Y \xrightarrow{u} R$, and similarly for v_0. But $R^{\ell W} \cong R^n$ for some n (see I, 1.13), so by composing with each of the n projections $R^{\ell W} \to R$ and using that $\Gamma(R^\alpha)\colon \mathcal{F}(Y,R) \rightarrowtail \mathcal{F}(X,R)$, we conclude that $u_0 = v_0$. Hence $u\circ(f\varphi\times Y) = v\circ(f\varphi\times Y)$, and therefore we have $u\circ(f,y)\circ\varphi = v\circ(f,y)\circ\varphi$, which was to be shown. $\square$

2.12 Corollary. *Let $M \xrightarrow{\varphi} N$ be a smooth map of manifolds. Then $\varphi(M)$ is dense in N iff $R^{s(\varphi)}\colon R^{s(N)} \to R^{s(M)}$ is a monomorphism in $\mathcal{F}$.*

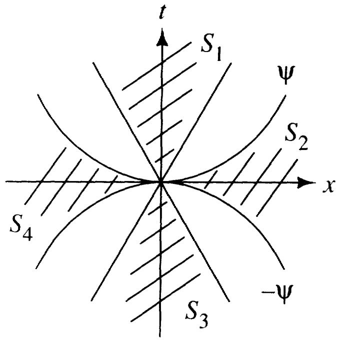
2.13 Corollary. *In $\mathcal{F}$, the following are valid:*

(i) $\forall f \in R^{R_{\geq 0}}(\forall t \in R f(t^2) = 0 \to f = 0)$
(ii) $\forall f \in R^R(\forall t \in U(R) f(t) = 0 \to f = 0)$

(recall that $U(R) = \{x \in R | x$ is invertible$\} = \{x \in R | x \neq 0\}$).

In fact, 2.13(i) is already true in $Sets^{\mathbb{L}^{op}}$ (see Appendix 3, but the present proof for $\mathcal{F}$ is much easier). Not so for (ii), however, which fails in $\mathcal{G}$, as the following example shows

2.14 Example. Let $t = \psi(x)$ be a flat function $\mathbb{R} \to \mathbb{R}, \psi \geq 0, Z(\psi) = \{0\}$, as in the picture below (a Dutch windmill).



The shaded area $S = S_1 \cup S_2 \cup S_3 \cup S_4$ is closed, hence there is a smooth function $F(x,t)$ such that $S = F^{-1}(0)$ (see I.1.4). F is flat at $(0,0)$, hence $F(x,\psi(x))$ is flat at 0. Take a function $\varphi(x) \in m^{\infty}_{\{0\}}$ which is flatter than $F(x,\psi(x))$, i.e. $F(x,\psi(x)) \notin (\varphi(x))$, again with $\varphi \geq 0, Z(\varphi) = \{0\}$, and let $f = F(x,t) \bmod (\varphi(x))$ (here $(\varphi(x))$ as ideal in $C^\infty(\mathbb{R} \times \mathbb{R})$). We claim that in $\mathcal{G} = \mathrm{Sh}(\mathbb{G})$, with $A = C^\infty(\mathbb{R})/(\varphi)$,

(i) $\ell A \Vdash \forall t \in U(R) f(t) = 0$,
(ii) $\ell A \Vdash \forall t \in D_\infty f(t) = 0$,
(iii) $\ell A \not\Vdash f \equiv 0$.

Since $(\varphi(x))$ is a germ-determined ideal in $C^\infty(\mathbb{R} \times \mathbb{R})$, it suffices for (i) to note that for each $(x,t) \in Z(\varphi(x)) = \{0\} \times \mathbb{R}$ with $t \neq 0$, F vanishes on a neighbourhood of (x,t), which is clear from the part $S_1 \cup S_3$ of S. (ii) just means that $\frac{\partial^n F}{\partial t^n}(x,0) = 0$, which is clear from the part $S_2 \cup S_4$ of S. Finally, if $\ell A \Vdash f \equiv 0$, i.e. $F(x,t) \in (\varphi(x))$, say $F(x,t) = H(x,t)\varphi(x)$, then $F(x,\varphi(x)) \in (\varphi(x)) \subset C^\infty(\mathbb{R})$, contradicting the choice of φ.

2.15 Remark on Integration. It should be noted that the validity of the integration axiom in $\mathcal{F}$, which we proved in 2.10, can also be proved easily along the lines of Appendix 3. For in that proof, the difficult part was precisely to show the validity of 2.13(i) in $Sets^{\mathbb{L}^{op}}$, which is easy in $\mathcal{F}$.

3 The Topology of Manifolds in $\mathcal{F}$ and $\mathcal{G}$

In this section, we will consider some topological properties of objects $s(M)$, where M is a manifold, and s is the embedding $\mathbb{M} \hookrightarrow \mathcal{F}$ or $\mathbb{M} \hookrightarrow \mathcal{G}$. The proofs are virtually the same for $\mathcal{F}$ and for $\mathcal{G}$, and we will just choose one of them in each case. Accordingly, $\models$ and $\Vdash$ refer either to the interpretation in $\mathcal{F}$, or the one in $\mathcal{G}$, or both.

As in $Sets^{\mathbb{L}^{op}}$, topological properties of R in $\mathcal{F}$ and $\mathcal{G}$ refer to the *order topology* on R, i.e. the subsheaf $\mathcal{O}(R)$ of $\mathcal{P}(R)$ defined by the condition that for each $S \in \mathcal{P}(R)(\ell A)$, $S \in \mathcal{O}(R)(\ell A)$ iff $\ell A \Vdash \forall x \in S \exists a, b \in R(a < x < b \wedge \forall x'(a < x' < b \rightarrow x' \in S))$. This indeed defines an internal topology on R, in the sense that it is valid in $\mathcal{F}$ and $\mathcal{G}$ that $\mathcal{O}(R)$ is closed under binary intersections and arbitrary unions, i.e.

$$\models (\forall U, V \in \mathcal{O}(R)\; U \cap V \in \mathcal{O}(R)) \text{ and}$$
$$\models (\forall \mathcal{U} \in \mathcal{P}(\mathcal{O}(R)) \cup \mathcal{U} \in \mathcal{O}(R)).$$

Reasoning as in the argument for Archimedeanness of R in $\mathcal{F}$ and $\mathcal{G}$, it can be shown that rationals are dense, i.e. in $\mathcal{F}$ and $\mathcal{G}$,

$$\models \forall a, b \in R(a < b \rightarrow \exists q \in \mathbb{Q}\; a < q < b).$$

where $\mathbb{Q}$ denotes the sheaf of internal rationals, i.e. $\Delta(\mathbb{Q}) = \mathbb{Q}_{\mathcal{F}}$, respectively $\mathbb{Q}_{\mathcal{G}}$. So the order topology $\mathcal{O}(R)$ on R has a basis of rational intervals. If (p, q) is a rational interval at stage ℓA, i.e. $p, q \in \mathbb{Q}(\ell A)$ with $\ell A \Vdash p < q$, then there is a cover $\{\ell A_\alpha \hookrightarrow \ell A\}_\alpha$ of ℓA and there are *external* (i.e. in *Sets*) rationals $p_\alpha < q_\alpha$ such that $\ell A_\alpha \Vdash p = p_\alpha \wedge q = q_\alpha$. So on a cover of ℓA the interval (p, q) is an external rational interval. Thus, $\mathcal{O}(R)$ is the smallest subsheaf of $\mathcal{P}(R)$ which is an *internal* topology on R and contains all subsheaves of R of the form $s(p, q)$, i.e. all rational intervals coming from outside. It is also clear that every open subset U of $\mathbb{R}$ in *Sets* defines an internal open $s(U)$ of R. So $\mathcal{O}(R)$ is also the internal topology on R generated by the subsheaves (global sections of $\mathcal{P}(R)$) $s(U)$, U open $\subseteq \mathbb{R}$.

If M is a manifold, we can internally define a topology on $s(M)$ by regarding $s(M)$ as being built up inside the topos by an atlas of pieces of $s(R^n)$. The embedding s can also be defined for manifolds with boundary, and if M is a manifold with boundary, $s(M)$ can also be built up by an atlas inside $\mathcal{F}$ or $\mathcal{G}$, allowing closed half-spaces defined by the internal order $\leq$ on R. Thus there is also in internal topology on manifolds with boundary. This internal topology on

$s(M)$ (M with or without boundary) again coincides with the topology generated by (i.e. having as a basis internally) the opens coming from *outside*, from *Sets*, just as for the case of R. So, we can take the following as a definition (or as a proposition, if one starts from the internal order topology).

3.1 Definition. *The topology in $\mathcal{F}$ or $\mathcal{G}$ on a manifold $s(M)$ is the smallest internal topology containing all the subsheaves of the form $s(U)$, U an open subset of M. This topology coincides with the topology defined inside $\mathcal{F}$ or $\mathcal{G}$ by the order $<$ on R.*

What 3.1 means in practice is that if $S \in P(sM)(\ell A)$, i.e. S is a subsheaf of $s(M)$ at stage $\ell A(\ell A \in \mathbb{F}$ or $\ell A \in \mathbb{G})$, then $\ell A \Vdash$ (S *is open*) iff for all $\ell B \xrightarrow{f} \ell A$ and all $b \in s(M)(\ell B)$, i.e. $b: \ell B \to s(M)$, $\ell B \Vdash b \in (S|f)$ iff for some cover $\{\ell B_\alpha \xrightarrow{g_\alpha} \ell B\}_\alpha$ and some open subspaces $U_\alpha \subset M$ in *Sets*

$$\ell B_\alpha \Vdash b|g_\alpha \in s(U_\alpha) \subset (S|fg_\alpha), \text{ for each } \alpha.$$

3.2 Proposition. *The full embeddings $s: \mathbb{M} \hookrightarrow \mathcal{F}$ and $s: \mathbb{M} \hookrightarrow \mathcal{G}$ preserve tranversal pullbacks and open covers.*

Proof. The preservation of transversal pullbacks is clear from II.1.2. Preservation of open covers means that if $\{U_\alpha\}_\alpha$ is an open cover of M, then the subsheaf $\mathcal{U}$ of $\mathcal{O}(s(M))$ (in $\mathcal{F}$ or $\mathcal{G}$) generated by the global sections $s(U_\alpha)$ (so $\mathcal{U}$ is the smallest subsheaf of $\mathcal{O}(sM)$ satisfying the conditions $1 \Vdash s(U_\alpha) \in \mathcal{U}$, for each α) is an internal cover, i.e. $\Vdash \forall x \in s(M) \exists U \in \mathcal{U}\, x \in U$. It suffices to prove this for the generic $x \in s(M)$, which is the identity at stage $s(M)$. But $s(M) \Vdash \exists U \in \mathcal{U}\ \mathrm{id} \in U$, since $s(U_\alpha) \Vdash \mathrm{id} \in s(U_\alpha) \in \mathcal{U}$ by definition of $\mathcal{U}$, and $\{s(U_\alpha) \hookrightarrow s(M)\}_\alpha$ is a cover in the sites $\mathbb{F}$ and $\mathbb{G}$. □

Such open covers which come from outside are very easy to handle. Arbitrary internal open covers may be much more complicated (try to describe the sheaf $\{\mathcal{U} \in P(\mathcal{O}(R)) | \cup \mathcal{U} = R\}$ of open covers of R!). However, it will turn out that a consistent use of *generic* elements will enable us to prove the properties we need. One such property is compactness. As in $Sets^{\mathbb{L}^{op}}$, we say that $s(M)$ is *compact* if $\models$ *every open cover of $s(M)$ has a finite subcover*, i.e. $\models \forall \mathcal{U} \subset \mathcal{O}(sM)(\cup \mathcal{U} = sM \to \exists \mathcal{U}' \subset \mathcal{U}(\cup \mathcal{U}' = sM \wedge \mathcal{U}'$ is finite)), where *$\mathcal{U}'$ is a finite subset of $\mathcal{U}$* means that $\exists n \in \mathbb{N}\ \exists f: \mathbb{N} \to \mathcal{U}\ \mathcal{U}' = \{f(m) | m < n\}$.

3.3 Theorem. *The embeddings $\mathbb{M} \xrightarrow{s} \mathcal{F}$ and $\mathbb{M} \xrightarrow{s} \mathcal{G}$ preserve com-*

pactness, i.e. if M is a compact manifold in Sets, then it is valid in $\mathcal{F}$ and $\mathcal{G}$ that $s(M)$ is compact.

Proof. To fix notation, let us prove the case of $\mathcal{F}$. Suppose $\mathcal{U}$ is a subsheaf of $\mathcal{O}(sM)$ at the stage ℓA, where $A = C^\infty(\mathbb{R}^n)/I$, and $\ell A \Vdash \cup \mathcal{U} = s(M)$. Then for the *generic* element $\ell A \times s(M) \xrightarrow{\pi_2} s(M)$ it must hold that

$$\ell A \times s(M) \Vdash \exists U \in (\mathcal{U}|\pi_1)\ \pi_2 \in U.$$

Thus, we find an open cover of $Z(I) \times M$, say $\{V_\alpha\}_\alpha$, and for each α an U'_α such that for $B_\alpha = C^\infty(V_\alpha)/\overline{(I|V_\alpha)}$,

$$\ell B_\alpha \Vdash \pi_2 \in U'_\alpha \in \mathcal{U}$$

(where we should really have written $\mathcal{U}|(\ell B_\alpha \hookrightarrow \ell A \times s(M) \to \ell A)$ instead of just $\mathcal{U}$—we shall frequently use this simplification of notation). By definition 1.1. of $\mathcal{O}(sM)$, we may assume that

$$\ell B_\alpha \Vdash \pi_2 \in s(U_\alpha) \subset U'_\alpha$$

for some external open $U_\alpha \subset M$, provided we choose the cover $\{V_\alpha\}$ sufficiently fine. Since the V_α's cover $Z(I) \times M$ and M is compact, we can find for each $p \in Z(I)$ a neighbourhood W_p and finitely many α's, say $\alpha_p^1, \ldots, \alpha_p^{m_p}$, such that $W_p \times M \subset V_{\alpha_p^1} \cup \ldots \cup V_{\alpha_p^{m_p}}$. Let $A_p = C^\infty(W_p)/\overline{(I|W_p)}$. Then $\{\ell A_p \hookrightarrow \ell A\}_{p \in Z(I)}$ is a cover of ℓA, and

$$\ell A_p \times s(M) \Vdash \pi_2 \in s(U_{\alpha_p^1}) \vee \ldots \vee \pi_2 \in s(U_{\alpha_p^{m_p}}).$$

By genericity, it then follows that

$$(*) \quad \ell A_p \Vdash \forall x \in s(M)(x \in sU_{\alpha_p^1} \vee \ldots \vee x \in sU_{\alpha_p^{m_p}}).$$

Moreover, if we choose the V_α to be of the form $V_\alpha^1 \times V_\alpha^2$, then the projection $\ell B_{\alpha_p^i} \to \ell A_p$ has a right inverse (choose any point in V_α^2), hence from $\ell B_\alpha \Vdash s(U_\alpha) \subset U'_\alpha \in \mathcal{U}$ we conclude by restriction along this right inverse $\ell A_p \hookrightarrow \ell B_{\alpha_p^i}$ that $\ell A_p \Vdash \exists U \in \mathcal{U} s(U_{\alpha_p^i}) \subset U$, for each $i = 1, \ldots, m_p$. In combination with (*), we conclude

$$\ell A_p \Vdash \exists U_1, \ldots, U_{m_p} \in \mathcal{U} s(M) \subseteq U_1 \cup \ldots \cup U_{m_p}.$$

Hence since the ℓA_p cover ℓA,

$$\ell A \Vdash \exists \textit{ finite } \mathcal{U}' \subset \mathcal{U} s(M) \subseteq \cup \mathcal{U}'. \qquad \square$$

3.4 Remark. Let $M \subset \mathbb{R}^n$ be a manifold, with the metric inherited from $\mathbb{R}^n$. Then $s(M) \subset R^n$ is an internal metric space, and it follows from 3.3 that if M is compact, it holds in $\mathcal{F}$ and $\mathcal{G}$ that every open cover of $s(M)$ has a Lebesgue number $\lambda > 0$ for the internal metric. (Since the absolute value does not exist in $\mathcal{G}$ or $\mathcal{F}$ as a map $R \to R$, we interpret ***internal metric*** as a ***relation*** $\{(x,y,r)|d(x,y) < r\}$ rather than a function $d(x,y)$.)

Alternatively, we could have concluded this from a principle CMP as in Chapter II (see II.3.2-3.5). The principle

$$CMP\colon \forall S \subset s(M) \times R(\forall x \in M \exists \varepsilon > 0\{x\} \times (-\varepsilon, \varepsilon) \subset S \to \exists \varepsilon > 0$$
$$s(M) \times (-\varepsilon, \varepsilon) \subset S)$$

is valid in $\mathcal{F}$ and $\mathcal{G}$. This follows either by adapting the proof of theorem 3.3 or by an argument similar to II.3.4.

3.5 Theorem. *Let M and N be manifolds. In $\mathcal{F}$ and $\mathcal{G}$ it is valid that all functions $s(M) \to s(N)$ are continuous.*

Proof. We prove the case of $\mathcal{G}$. The ***all*** in this theorem is of course interpreted internally, i.e. we consider elements of $s(N)^{s(M)}$, the "*set*" of all functions inside the topos, at arbitrary stages. So take any $F \in s(N)^{s(M)}(\ell A)$, where $A = C^\infty(\mathbb{R}^n)/I$. So F comes as a map $\ell A \times s(M) \to s(N)$ in $\mathbb{G}$. F need not come from a smooth map $\mathbb{R}^n \times M \to N$, but by writing N as a retract of an open neighbourhood in some $\mathbb{R}^k$ (see GP), it follows that F can at least be represented by a smooth map (also called F) $W \times M \xrightarrow{F} N$, where $W \subset \mathbb{R}^n$ is an open neighbourhood of $Z(I)$. We will prove that $\ell A \Vdash \forall p \in s(M) \forall U \in \mathcal{O}(sM)(F(p) \in U \to \exists V \in \mathcal{O}(sM)(p \in V \wedge F(V) \subset U)$. So choose $\ell B \xrightarrow{g} \ell A$ and an element $\ell B \xrightarrow{p} s(M)$ of $s(M)$ at stage ℓB. It suffices to consider basic open neighbourhoods of $F(p)$ at stage ℓB, i.e. neighbourhoods of the form $s(U)$, U open in N. So take such a U, and assume $\ell B \Vdash (F|g)(p) \in s(U)$, i.e. we have a factorization

$$\begin{tikzcd}
\ell B \arrow[r, "{(g,p)}"] \arrow[drr] & \ell A \times s(M) \arrow[r, "F"] & s(N) \\
 & & s(U) \arrow[u]
\end{tikzcd}$$

If $B = C^\infty(\mathbb{R}^m)/J$, g comes from a map $\mathbb{R}^m \to \mathbb{R}^n$, and (by writing

M as a retract of an open neighbourhood in some $\mathbb{R}^k$) we may assume that p comes from a smooth $p: W' \to M$, where $W' \subset \mathbb{R}^m$ is an open neighbourhood of $Z(J)$, and by choosing W' small enough, without loss $g(W') \subset W$. Now for each $y \in Z(J)$, $F \circ (g,p)(y) \in U \subset N$. Let $V_y \subset M$ be a neighbourhood of $p(y)$ and $O_y \subset W'$ be a neighbourhood of y, such that

$$p(O_y) \subset V_y \text{ and } \forall z \in O_y \forall x \in V_y F(g(z), x) \in U.$$

Write $B_y = C^\infty(O_y)/(J|O_y)^\sim$. Then

$$\ell B_y \Vdash p \in s(V_y) \wedge (F|g)(s(V_y)) \subset s(U).$$

Hence since $\{\ell B_y \hookrightarrow \ell B\}_{y \in Z(J)}$ is a cover of ℓB,

$$\ell B \Vdash \exists V \in \mathcal{O}(xM)(p \in V \wedge (F|g)(V) \subset s(U)). \qquad \square$$

We will see later on in this section that theorem 3.5 also follows immediately from an *axiom of continuous choice*, see 3.11.

Using the interpretation of the set theoretic language in $\mathcal{F}$ and $\mathcal{G}$, we can define *internal partitions of unity* on a manifold $s(M)$ as indexed subsheaves of $[0,1]^{s(M)}$ defined analogous to the classical case of *Sets*. More precisely, if $I \in \mathcal{F}$ or $\mathcal{G}$ is the *index set*, a partition of unity indexed by I is a map $I \xrightarrow{\rho} [0,1]^{s(M)}$, $\rho(i) = \rho_i$, such that it is valid in the topos that $\{\rho_i | i \in I\}$ is locally finite and has sum 1, i.e.

$$(*) \forall x \in s(M) \exists U \in \mathcal{O}(sM)(x \in U \wedge \exists n \in \mathbb{N}\ \exists f: \{0, \dots, n-1\} \rightarrowtail I$$
$$(\forall i \in I(\exists k < n\ i = f(k) \vee \rho_i | U \equiv 0) \wedge \rho_{f(0)} + \dots + \rho_{f(n-1)} | U \equiv 1)).$$

(Note that *finite* is interpreted here not in the way we did it for compactness, but as *finite and decidable* (see Appendix 1). We cannot just say here $\exists f: \{0, \dots, n-1\} \to I$, not necessarily injective, with $\rho_{f(0)} + \cdots \rho_{f(n-1)} | U \equiv 1$, etc. We may sum the same ρ_i twice here, and this cannot automatically be avoided by throwing such i out, i.e. by constructing from f a $g: \{0, \dots, k-1\} \to I$ which is injective and has the same image as f. The problem is that I need not be *decidable*, i.e. $\forall i, j \in I(i = j \vee i \neq j)$ need not be valid. However, in most of the cases we consider, the set I can be chosen to be decidable, so we do not have to be as careful as this remark suggests we should.)

If $\mathcal{U}$ is an internal open cover of $s(M)$, and $\rho: I \to [0,1]^{s(M)}$ is a partition of unity on $s(M)$, we say that $\rho = \{\rho_i\}_i$ is *subordinate to* $\mathcal{U}$ if

$$\forall i \in I\ \exists U \in \mathcal{U}\ \forall x \in s(M)(x \in U \vee \rho_i(x) = 0)$$

is valid in the topos.

Despite the fact that the ***intuitionistic logic*** of the universes $\mathcal{F}$ and $\mathcal{G}$ forces us to be very careful, partitions of unity are fairly well behaved. First of all, every external partition of unity $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ on a manifold M gives rise to an internal one, indexed by the constant sheaf $\Delta(\mathcal{A})$ (this sheaf has decidable equality).

3.6 Proposition. *The embeddings $s\colon \mathbb{M} \hookrightarrow \mathcal{F}$ and $s\colon \mathbb{M} \hookrightarrow \mathcal{G}$ preserve partitions of unity. More precisely, if $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ is a partition of unity on a manifold M, the induced map $\Delta(\mathcal{A}) \to [0,1]^{s(M)}$ is an internal partition of unity (in $\mathcal{F}$ or $\mathcal{G}$).*

Proof. The proofs are the same for $\mathcal{F}$ and $\mathcal{G}$. To show that (*) is valid, it suffices to consider the generic element $\gamma =$ identity: $s(M) \to s(M)$ at stage $s(M)$, i.e. show that

$$\begin{aligned}\text{(i)}\quad & s(M)\Vdash\exists U \in \mathcal{O}(sM)(\gamma \in U \wedge \exists n\exists f\colon\{0,\ldots,n-1\} \rightarrowtail I\\ & \forall i \in I(\exists k < n\ i = f(k)\vee\rho_i|U \equiv 0)\wedge\rho_{f(0)}+\ldots+\rho_{f(n-1)}|U \equiv 1)\end{aligned}$$

where $I = \Delta(\mathcal{A})$. Let $\{U_\xi\}$ be an open cover of M such that on U_ξ, all but the n_ξ functions $\rho_{\alpha_\xi^1},\ldots,\rho_{\alpha_\xi^{n}\xi}$ vanish. Let $f_\xi\colon\{0,\ldots n_{\xi-1}\} \to \mathcal{A}$ enumerate these. f_ξ is also an internal function (apply Δ), so obviously $s(U_\xi)\Vdash\gamma \in s(U_\xi)$ and

$$s(U_\xi)\Vdash\forall i \in \Delta(\mathcal{A})(\exists k < n_\xi\ i = f_\xi(k) \vee \rho_i|s(U_\xi) \equiv 0),$$

while moreover

$$s(U_\xi)\Vdash\rho_{\alpha_\xi^1}(\gamma) + \ldots + \rho_{\alpha_\xi^n\xi}(\gamma) = 1,$$

so by genericity of γ and the definition of f_ξ,

$$s(U_\xi)\Vdash\forall y \in s(U_\xi)\rho_{f(0)}(y) + \ldots + \rho_{f(n_\xi-1)}(y) = 1.$$

Since $\{s(U_\xi) \hookrightarrow s(M)\}_\xi$ is a cover in the site ($\mathbb{F}$ or $\mathbb{G}$), it follows that $s(M)\Vdash\exists U \in \mathcal{O}(sM)(\ldots)$ as in (i). □

As a consequence of 3.6, partitions of unity on $s(M)$ exist which are subordinate to a given open cover of $s(M)$ which comes from outside, i.e. which is generated by $\{s(U)|U \in \mathcal{U}\}$ for an open cover $\mathcal{U}$ of M in *Sets*. For internal covers, we need a more complicated argument.

3.7 Theorem. *Let M be a manifold. In $\mathcal{F}$ and $\mathcal{G}$, it is valid that partitions of unity subordinate to any open cover of $s(M)$ exist.*

Proof. Again, the proof is the same for $\mathcal{F}$ and $\mathcal{G}$. To fix notation, we prove the case of $\mathcal{F}$. Let $\mathcal{U}$ be an open cover of $s(M)$ at stage $\ell A \in \mathcal{F}, A = C^\infty(\mathbb{R}^n)/J$. So $\mathcal{U} \in P(\mathcal{O}(sM))(\ell A)$ and $\ell A \Vdash \forall x \in s(M)\ \exists U \in \mathcal{U}\ x \in U$. In particular, applying this to the generic x, the projection $\ell A \times s(M) \xrightarrow{\pi_2} s(M)$, we find a cover $\{O_\alpha \times V_\alpha\}_\alpha$ of $Z(J) \times M$ by opens in $\mathbb{R}^n \times M$ such that if $\ell A_\alpha = \ell A \cap s(O_\alpha)$, i.e. $A_\alpha = C^\infty(O_\alpha)/\overline{(J|O_\alpha)}$, we have

$$(*) \qquad \ell A_\alpha \times s(V_\alpha) \Vdash \pi_2 \in U_\alpha \in \mathcal{U}_\alpha, \text{ for some } U_\alpha,$$

and since the opens of the form $s(W)$, W open $\subset M$, form a basis for the topology on $s(M)$, we can choose this cover $\{O_\alpha \times V_\alpha\}_\alpha$ so fine that

$$\ell A_\alpha \times s(V_\alpha) \Vdash \pi_2 \in s(W_\alpha) \subset U_\alpha \in \mathcal{U}.$$

Let $\{W_i\}_{i \in I}$ be a locally finite refinement of this cover $\{O_\alpha \times V_\alpha\}$, so $W = \bigcup_{i \in I} W_i \supset Z(J) \times M$. By external paracompactness, there exists a partition of unity $\{\rho_i : \mathbb{R}^n \times M \to [0,1]\}_{i \in I}$ subordinate to $\{W_i\}$, i.e. $\mathrm{supp}(\rho_i) \subset W_i$ and $\sum \rho_i(x) = 1$ for all $x \in W$. $\{\rho_i\}_{i \in I}$ determines an element ρ of $([0,1]^{s(M)})^{\Delta(I)}(\ell A)$, i.e. a map $\rho: \Delta(I) \times \ell A \times s(M) \to [0,1]$ in $\mathcal{F}$, in the obvious way. We claim that this ρ is an internal partition of unity subordinate to $\mathcal{U}$ at stage ℓA. That is,

(1) $\ell A \Vdash \forall i \in \Delta I\ \exists U \in \mathcal{U}\ \forall x \in s(M)(x \in U \vee \rho_i(x) = 0)$

(2) $\ell A \Vdash \forall x \in s(M)\ \exists nbd N \ni x \exists i_1, \ldots, i_n \in \Delta I \forall j \in \Delta I$

$$(j = i_1 \vee \ldots \vee j = i_n \vee \forall y \in N \rho_j(y) = 0)$$

(3) $\ell A \Vdash \sum \rho_i = 1$.

To prove (1), choose $i \in I$. Then there exists an α with $\mathrm{supp}(\rho_i) \subset W_i \subset O_\alpha \times V_\alpha$. So $\ell A \times s(M) \Vdash (\pi_2 \in s(V_\alpha) \vee \rho_i(\pi_2) = 0)$. Since π_2 is generic, $\ell A \Vdash \forall x \in s(M)(x \in s(V_\alpha) \vee \rho_i(x) = 0)$. Also,

$$\ell A_\alpha \times s(V_\alpha) \Vdash \pi_2 \in s(W_\alpha) \subset U_\alpha \in \mathcal{U},$$

so $V_\alpha \subset W_\alpha$, and thus

$$\ell A_\alpha \Vdash \exists U \in \mathcal{U} s(V_\alpha) \subset \mathcal{U},$$

since the projection $\ell A_\alpha \times s(V_\alpha) \to \ell A_\alpha$ has a right inverse. The ℓA_α form a cover of ℓA, so

$$\ell A \Vdash \exists U \in \mathcal{U} s(V_\alpha) \subset U.$$

Thus

$$\ell A \Vdash \exists U \in \mathcal{U} \forall x \in s(M)(x \in U \vee \rho_i(x) = 0).$$

For (2), consider the generic element x of $s(M)$, i.e. the projection $\ell A \times s(M) \xrightarrow{\pi_2} s(M)$. Let $\{E_\xi \times N_\xi\}_\xi$ be a cover of W by open sets in $\mathbb{R}^n \times M$ such that each $E_\xi \times N_\xi$ meets only finitely many W_i, say $W_{i_\xi^1}, \ldots, W_{i_\xi^n \xi}$. Write $\ell A_\xi = \ell A \cap s(E_\xi)$. Then

$$\ell A_\xi \times s(N_\xi) \Vdash \pi_2 \in s(N_\xi),$$

and for $j \in I, j \notin \{i_\xi^n, \ldots, i_\xi^{n_\xi}\}$ implies that $\text{supp}(\rho_j) \cap E_\xi \times N_\xi = \emptyset$, so

$$\ell A_\xi \Vdash \forall y \in s(N_\xi) \rho_j(y) = 0.$$

A fortiori

$$\ell A_\xi \times s(N_\xi) \Vdash \forall y \in s(N_\xi) \rho_j(y) = 0.$$

Thus (by 3.9 of Appendix 1)

$$\ell A_\xi \times s(N_\xi) \Vdash \forall j \in \Delta I (j = i_\xi^1 \vee \ldots \vee j = i_\xi^n \xi \vee \forall y \in s(N_\xi) \rho_j(y) = 0).$$

Since the objects $\ell A_\xi \cap s(N_\xi)$ of $\mathbb{F}$ cover $\ell A \times s(M)$, it follows that

$$\begin{aligned} \ell A \times s(M) \Vdash \quad & \exists\, nbd\; N \ni \pi_2\; \exists i^1, \ldots, i^n \in \Delta I\; \forall j \in \Delta I \\ & (j = i^1 \vee \ldots \vee j = i^n \vee \forall y \in N\; \rho_j(y) = 0). \end{aligned}$$

Since π_2 is generic (2) holds.

(3) is now clear from the proof of (2), since $\ell A_\xi \times s(N_\xi) \Vdash \rho_i^n \xi(\pi_2) + \ldots + \rho_{i_\xi}^{n_\xi}(\pi_2) = 1$. This completes the proof. □

Next we look at connectedness. Classically, a manifold M is connected if for any two opens $U, V \subset M, U \cup V = M$ implies $U \cap V \neq \emptyset$. This is equivalent to the following form of *chain-connectedness*: M is *chain-connected* if for ever open cover $\mathcal{U}$ of M, any two points $x, y \in M$ can be connected by a $\mathcal{U}$-chain, i.e. there are $U_1, \ldots, U_n$ in $\mathcal{U}$ with $x \in U_1, y \in U_n$, and $U_i \cap U_{i+1}$ is nonempty for each $i = 1, \ldots, n-1$. In the constructive context of $\mathcal{F}$ and $\mathcal{G}$, however this notion of chain-connectedness is stronger than the usual form. For easy reference, let us state the definitions explicitly. (Note that *non-empty* is replaced by the positive, and therefore constructively more manageable *inhabited*, cf. Appendix 1.3.7.)

3.8 Definition. Let X be a topological space in a Grothendieck topos (say $\mathcal{F}$ or $\mathcal{G}$).

(i) X is called *chain-connected* if the following sentence is valid in the topos:

$$\forall \mathcal{U} \subset \mathcal{O}(X)(X = \cup \mathcal{U} \to \forall x, y \in X\; \exists n\; \exists f\colon \{1, \ldots, n\} \to \mathcal{U}$$

$$(x \in f(1) \wedge y \in f(n) \wedge \forall i < n \, \exists z \in X z \in f(i) \cap f(i+1)))$$

(ii) X is called *connected* if the sentence

$$\forall U, V \in \mathcal{O}(X) \quad (\exists x\, x \in U \cap \exists x\, x \in V \wedge U \cup V = X \to \exists x \in X\, x \in U \cap V)$$

is valid in the topos.

As said, the implication (i) $\Rightarrow$ (ii) is constructively valid, and therefore it holds in any topos.

3.9 Proposition. *Let X be a topological space which is inhabited, i.e. $\exists x\, x \in X$. The following are equivalent.*

(i) *X is chain-connected*
(ii) *for every set S, every continuous function $X \to S$ (S with the discrete topology) is constant.*
(iii) *every continuous function $X \to \mathcal{P}(1)$ ($\mathcal{P}(1)$ discrete) is constant.*

These equivalences are constructively valid, so they hold in any Grothendieck topos (where *set* means object of the topos). Note that $\mathcal{P}(1) = \mathcal{P}(\{*\})$ can be rather complicated in this context, and certainly it is not just a two-element set.

Proof. One has to be careful to avoid the pitfalls of intuitionistic or constructive reasoning, and therefore we will give the argument in detail.

(i) $\Rightarrow$ (ii) Suppose $X \xrightarrow{f} S$ is continuous. Since X is inhabited, there is an $x_0 \in X$. Let $s_0 = f(x_0)$. Now consider the open cover $\{f^{-1}(s) | s \in S\}$ of X. If x is any point of X, there are $s_1, \ldots, s_n$ such that $x_0 \in f^{-1}(s_1), x \in f^{-1}(s_n)$, and some $z_i \in f^{-1}(s_i) \cap f^{-1}(s_{i+1})$. Thus, $s_1 = f(x_0) = s_0$, $f(x) = s_n$ and $s_i = f(z_i) = s_{i+1}$. So $s_n = s_0$, i.e. $f(x) = s_0$. Therefore $\forall x \in X\, f(x) = s_0$, i.e. f is constant.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i) Let $\mathcal{U}$ be an open cover of X, and let $x_o \in X$. Define a function $X \xrightarrow{f} \mathcal{P}(1)$ by

$$* \in f(x) \text{ iff there are } U_1, \ldots, U_n \in \mathcal{U} \text{ with } x_o \in U_1, x \in U_n, \text{ and for each } i < n, \exists z_i \in U_i \cap U_{i+1}.$$

We claim that f is continuous, i.e. locally constant. Suppose $x \in X$, and choose $V \in \mathcal{U}$ with $x \in V$. Then for any $y \in V$, $f(x) = f(y)$. For if $* \in f(x)$, then there is a chain $U_1, \ldots, U_n$ from x_0 to x as indicated, and hence $U_1, \ldots, U_n$, V is a chain from x_0 to y. Thus

$* \in f(y)$. So $f(x) \subseteq f(y)$. Similarly $f(y) \subseteq f(x)$. By (iii), f is constant. Since $f(x_0) = \{*\}$, we must have $\forall x \in X f(x) = \{*\}$. So (i) holds.

3.10 Theorem. *The embeddings $s\colon \mathbb{M} \hookrightarrow \mathcal{F}$ and $s\colon \mathbb{M} \hookrightarrow \mathcal{G}$ preserve connectedness. More precisely, if M is a connected manifold, then in $\mathcal{F}$ and $\mathcal{G}$ it is valid that $s(M)$ is chain-connected.*

Proof. We prove the case of $\mathcal{G}$. Suppose $\mathcal{U}$ is an open cover at stage $\ell A \in \mathbb{G}, A \in C^\infty(\mathbb{R}^n)/I$. As in the proof of 3.7, this means that there is a cover of $Z(I) \times M$ by opens $\{O_\alpha \times V_\alpha\}$ in $\mathbb{R}^n \times M$ such that

(i) $\ell A_\alpha \times s(V_\alpha) \Vdash \pi_2 \in s(W_\alpha) \subset U_\alpha \in \mathcal{U}$

for some open $W_\alpha \subset M$ and some U_α, where $A_\alpha = C^\infty(O_\alpha)/(I|O_\alpha)^\sim$. (Note that from $\ell A_\alpha \times s(V_\alpha) \Vdash \pi_2 \in s(W_\alpha)$ it follows that $V_\alpha \subset W_\alpha$, so we could just as well have taken $W_\alpha = V_\alpha$.) To prove that $\ell A \Vdash (\forall x, y \in X\ \exists U_1, \ldots, U_n \in \mathcal{U}(x \in U_1 \wedge y \in U_n \wedge \forall i < n\ \exists z \in U_i \cap U_{i+1}))$ it suffices to consider the *generic pair* $x = \pi_2$, $y = \pi_3$ at stage $\ell A \times s(M) \times s(M)$. Choose $p \in Z(I)$, and consider

$$\mathcal{U}_p = \{V_\alpha | p \in O_\alpha\}.$$

$\mathcal{U}_p$ covers M, so for fixed $r \in M, t \in M$, there are $\alpha_1, \ldots, \alpha_n$ with $r \in V_{\alpha_1}$, $t \in V_{\alpha_n}$ and $V_{\alpha_i} \cap V_{\alpha_{i+1}} \neq \emptyset (i < n)$. Let $O_p = O_{\alpha_1} \cap \ldots \cap O_{\alpha_n}, A_p = C^\infty(O_p)/(I|O_p)^\sim$. Then clearly

(ii) $\ell A_p \times s(V_{\alpha_1}) \times s(V_{\alpha_n}) \Vdash \pi_2 \in s(V_{\alpha_1}) \wedge \pi_3 \in s(V_{\alpha_n}) \wedge \forall_{i<n} \exists z \in sM\ z \in s(V_{\alpha_i}) \cap s(V_{\alpha_{i+1}})$.

Also $\ell A_p \subset \ell A_{\alpha_i}$ for $i = 1, \ldots, n$, and $\ell A_\alpha \times s(V_\alpha) \to \ell A_\alpha$ covers in $\mathbb{G}$, so by (i),

(iii) $\ell A_p \times s(V_{\alpha_1}) \times s(V_{\alpha_n}) \Vdash \exists U \in \mathcal{U} s(V_{\alpha_i}) \subset U$, for $i = i, \ldots, n$.

Since p, r, t were arbitrary, (ii) and (iii) imply

$$\ell A \times s(M) \times s(M) \Vdash \exists n \in \mathbb{N}\ \exists U_1, \ldots, U_n \in \mathcal{U}(\pi_2 \in U_1 \wedge \pi_3 \in U_n \wedge$$
$$\wedge \forall_{i<n} \exists z \in sM\ z \in U_i \cap U_{i+1}).$$

Since the pair π_2, π_3 is generic,

$$\ell A \Vdash \forall x, y \in sM\ \exists n \in \mathbb{N}\ \exists U_1, \ldots, U_n \in \mathcal{U}(x \in U_1 \wedge y \in U_n \wedge$$
$$\wedge \forall_{i<n} \exists z \in sM z \in U_1 \cap U_{i+1}). \qquad \square$$

We will see shortly that $s\colon \mathbb{M} \hookrightarrow \mathcal{F}$ and $s\colon \mathbb{M} \hookrightarrow \mathcal{G}$ map connected manifolds to indecomposable (*unzerlegbar*, see 3.13) objects in $\mathcal{F}$ and

$\mathcal{G}$ respectively. In fact, this is a corollary of the following principle of a more logical nature. (Principles of this type will be familiar to readers with some background in intuitionistic analysis.)

3.11 Theorem. *Let M be a manifold, and let T be a set, $\Delta(T)$ the corresponding constant sheaf in $\mathcal{F}$ or $\mathcal{G}$. Then in both $\mathcal{F}$ and $\mathcal{G}$, the following axiom of choice is valid*

$$\forall R \in \mathcal{P}(sM \times \Delta T)(\forall x \in sM\ \exists t \in \Delta T\ R(x,t) \rightarrow$$

$$\rightarrow \exists \text{ open cover } \mathcal{U} \text{ of } sM\ \forall U \in \mathcal{U}\ \exists t \in \Delta T \forall x \in U R(x,t).$$

Strange as this axiom may seem to some readers, it is very useful. Before proving theorem 3.11, we mention some consequences.

3.12 Corollary. *(Open Refinement Theorem). Let M be a manifold. In $\mathcal{F}$ and $\mathcal{G}$ it is valid that every countable (i.e. indexed by $\mathbb{N}_{\mathcal{F}}$, $\mathbb{N}_{\mathcal{G}}$ respectively) cover of sM has an open refinement. In set theoretic language*

$$\models \forall F \in \mathcal{P}(sM)^{\mathbb{N}}(\forall x \in sM\ \exists n\ x \in F(n) \rightarrow$$

$$\rightarrow \forall x \in sM\ \exists U \in \mathcal{O}(sM)\ \exists n\ x \in U \subseteq F(n)).$$

Proof. Clear from 3.11, since $\mathbb{N}_{\mathcal{F}} = \Delta(\mathbb{N}), \mathbb{N}_{\mathcal{G}} = \Delta(\mathbb{N})$. □

3.13 Corollary. *The embeddings $s\colon \mathbb{M} \hookrightarrow \mathcal{F}$ and $s\colon \mathbb{M} \hookrightarrow \mathcal{G}$ map connected manifolds to indecomposable objects, i.e. if M is a connected manifold then*

$$\Vdash \forall A, B \in \mathcal{P}(sM)(\exists x\ x \in A \wedge \exists x\ x \in B \wedge A \cup B = sM \rightarrow \exists x\ x \in A \cap B).$$

(In intuitionistic analysis, indecomposable is often defined by the weaker statement $\forall A, B \in \mathcal{P}(sM)(A \cup B = sM \wedge A \cap B = \emptyset \rightarrow A = sM \vee B = sM)$.)

Proof. Apply 3.12 to the two-element cover and use connectedness (3.10). □

Note that for the special case $M = [0,1]$, 3.13 also follows from the integration axiom.

From 3.11, we obtain a very simple proof of theorem 3.5.

3.14 Corollary. *Let (X, δ) be a metric space in $\mathcal{F}$ or $\mathcal{G}$ having a dense subset D which is a constant sheaf, and let M be a manifold. Then in $\mathcal{F}$ or $\mathcal{G}$, all functions $s(M) \rightarrow X$ are continuous. In particular,*

$$\models \forall f \in s(N)^{s(M)}\ f \text{ is continuous}$$

where N is another manifold, and $\models$ denotes validity in $\mathcal{F}$ or $\mathcal{G}$.

Proof. Recall that D is dense if $\models \forall U \in \mathcal{O}(X)\ \exists d \in D d \in U$. For the first assertion, apply 3.11 to the predicates $R_n(x,d) = \delta(f(x),d) < 2^{-n}$ for a given function $f \in s(M)^X$. For the case of $f \in s(N)^{s(M)}$, observe that $\Delta(N)$, the constant sheaf corresponding to the set of points of N, is dense in $s(N)$. □

Proof of 3.11. To fix notation, we prove the case of $\mathcal{G}$. Suppose $R \in \mathcal{P}(sM \times \Delta T)(\ell A)$, i.e. R is a subsheaf of $\ell A \times s(M) \times \Delta(T)$, and $\ell A \Vdash \forall x \in sM\ \exists t \in \Delta T\ R(x,t)$. In particular, for the generic $\ell A \times sM \xrightarrow{\pi_2} sM$, $\ell A \times sM \Vdash \exists t \in \Delta T\ R(\pi_2, t)$. Therefore, there exists a cover $O_\alpha \times V_\alpha$ of $Z(I) \times M$ by open subsets of $\mathbb{R}^n \times M$, and for each α a $t_\alpha \in T$ such that if we write $A = C^\infty(\mathbb{R}^n)/I$, $A_\alpha = C^\infty(O_\alpha)/(I|O_\alpha)^\sim$,

$$\ell A_\alpha \times s(V_\alpha) \Vdash R(\pi_2, t_\alpha).$$

By genericity of π_2 (and since t_α exists at stage ℓA_α already, in fact $t \in T = \Delta(T)(1)$),

$$\ell A_\alpha \Vdash \forall y \in s(V_\alpha) R(y, t_\alpha),$$

so certainly

$$\ell A_\alpha \times s(V_\alpha) \Vdash \pi_2 \in S(V_\alpha) \wedge \forall y \in s(V_\alpha) R(y,t).$$

Since $\{\ell A_\alpha \times s(V_\alpha) \hookrightarrow \ell A \times s(M)\}_\alpha$ is a cover in $\mathbb{G}$,

$$\ell A \times s(M) \Vdash \exists U \in \mathcal{O}(sM)\ \exists t \in \Delta(T)(\pi_2 \in U \wedge \forall y \in U R(y,t)).$$

Hence by genericity of π_2 again

$$\ell A \Vdash \forall x \in sM\ \exists U \in \mathcal{O}(sM)\ \exists t \in \Delta(T)(x \in U \wedge \forall y \in U\ R(y,t)).\ \square$$

Chapter IV

Cohomology and Integration

The main emphasis in this monograph is on models of synthetic differential geometry, and it is not our purpose to give a systematic treatment of the synthetic theory. It seems nevertheless worthwhile to give some illustrations of synthetic differential geometry at work. Accordingly, the purpose of this chapter, and even more so of the next, is to actually develop some differential geometry in the synthetic context. The treatment is self-contained; in particular, the first three sections of this chapter do not presuppose familiarity with the corresponding classical theory. The models discussed in the previous chapter will play a crucial rôle when it comes to comparing the synthetic framework with the classical one.

In the first section, we will present the theory of differential forms, and the De Rham cohomology of a *smooth space* (an object of some topos, like $\mathcal{F}$ or $\mathcal{G}$). It will be seen that the presence of infinitesimal objects like D will enable us to express the underlying geometric intuitions more directly than can be done in the classical context.

In the second section, we will describe the singular homology of a smooth space. Here infinitesimal structures do not really come in, and consequently the synthetic context cannot help to simplify the exposition.

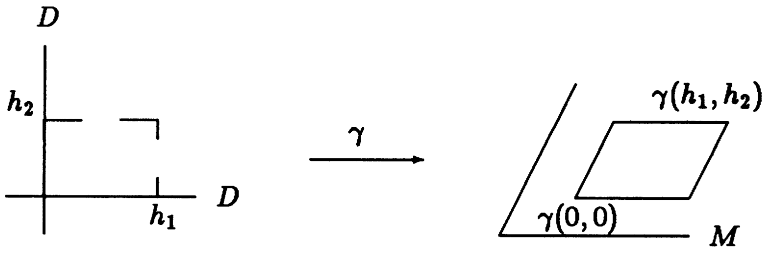
De Rham's theorem states that the integration map induces an isomorphism between the De Rham cohomology and the dual of the singular homology of a manifold. In section 3, we will show that De Rham's theorem holds for manifolds in $\mathcal{F}$ and $\mathcal{G}$, and we will show that the De Rham cohomology and the isomorhism of De Rham's theorem for a manifold in $\mathcal{F}$ or $\mathcal{G}$ are in some sense *the same* as classically, i.e. in *Sets*.

As an illustration of the relation between proving things in (models for) synthetic differential geometry and proving things classically, we will in a next section *translate* some of the results for $\mathcal{F}$ and $\mathcal{G}$ into the language of classical differential geometry. In this way, we immediately obtain some results about differential forms and cohomology depending smoothly on some extra parameters.

In an appendix, we present Weil's version of the De Rham isomorphism, which is also valid in the synthetic context.

1 De Rham Cohomology

In classical differential geometry the De Rham complex of a manifold is built up from differential forms and exterior differentiation. In the context of synthetic differential geometry, these building blocks can be defined for any object M, since all objects are *smooth spaces.* Thus to define these notions, let M be any smooth space, i.e. an object of some unspecified model. An ***infinitesimal n-cube on M*** is an element of $M^{D^n} \times D^n$, i.e. an $n+1$-tuple $(\gamma, h_1, \ldots, h_n)$. For the case $n = 2$, we may picture this as



The *object of infinitesimal n-chains*, $C_n(M)$, is the free R-module generated by the infinitesimal n-cubes on M. So an element of $C_n(M)$ is a formal linear combination

$$\sum_{i=1}^{p} a_i(\gamma_i, h_1^i, \ldots, h_n^i)$$

where $a_i \in R$ and $(\gamma_i, h_1^i, \ldots, h_n^i) \in M^{D^n} \times D^n$.

An *n-form on M* is a map

$$M^{D^n} \times D^n \xrightarrow{\omega} R, \ (\gamma, h_1, \ldots, h_n) \mapsto \int_{(\gamma, h_1, \ldots, h_n)} \omega$$

assigning a number (a *size*, like length, area, volume, etc.) to every infinitesimal n-cube, subject to the following conditions:

(1) *homogeneity:* $\omega(a \cdot_i \gamma, h_1, \ldots, h_n) = a \cdot \omega(\gamma, h_1, \ldots, h_n)$, where $a \cdot_i \gamma: D^n \to M$ is defined by

$$a \cdot_i \gamma(x_1, \ldots, x_i, \ldots, x_n) = \gamma(x_1, \ldots, ax_i, \ldots, x_n),$$

for every $a \in R$ and infinitesimal n-cube $(\gamma, h_1, \ldots, h_n)$.

(2) *alternation:* $\omega(\sigma\gamma, h_1, \ldots, h_n) = \mathrm{sgn}(\sigma) \cdot \omega(\gamma, h_{\sigma(1)}, \ldots, h_{\sigma(n)})$, where σ is any permutation of $\{1, \ldots, n\}$, and $\sigma\gamma$ is γ composed with the coordinate permutation induced by σ, i.e.

$$\sigma\gamma(x_1, \ldots, x_n) = \gamma(x_{\sigma(1)}, \ldots, x_{\sigma(n)});$$

$\mathrm{sgn}(\sigma)$ is the signature of σ.

(3) *degeneracy:* $\omega(\gamma, h_1, \ldots, 0, \ldots, h_n) = 0$.

The *object of n-forms* on M is denoted by $\Lambda^n(M)$.

Note that by the Kock-Lawvere axiom, i.e. $R^D \cong R \times R$, and the degeneracy condition, each n-form ω on M can be written as

$$\omega(\gamma, h_1, \ldots, h_n) = h_1 \cdot \ldots \cdot h_n \cdot \tilde{\omega}(\gamma)$$

for a unique map $\tilde{\omega}: M^{D^n} \to R$. This map $\tilde{\omega}$ satisfies the homogeneity condition ($\tilde{\omega}(a \cdot_i \gamma) = a \cdot \tilde{\omega}(\gamma)$) and is alternating ($\tilde{\omega}(\sigma\gamma) = \mathrm{sgn}(\sigma)\tilde{\omega}(\gamma)$). Thus we obtain a 1–1 correspondence between elements $\omega \in \Lambda^n(M)$ and alternating homogeneous map $\tilde{\omega}: M^{D^n} \to R$, and we will often identify the two.

If $\omega: M^{D^n} \times D^n \to R$ is an n-form on M, we will write

$$\int_{(-)} \omega: C_n(M) \to R$$

for the unique R-linear map extending ω.

Taking the boundary of an infinitesimal n-cube defines an R-linear *boundary operator*

$$\partial: C_{n+1}(M) \to C_n(M)$$

given by the formula

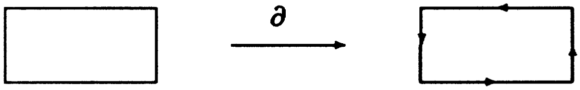
$$\partial(\gamma, h_1, \ldots, h_{n+1}) = \sum_{i=1}^{n+1} \sum_{\alpha=0,1} (-1)^{i+\alpha} F_{i\alpha}(\gamma, h_1, \ldots, h_n).$$

where $F_{i\alpha}(\gamma, h_1, \ldots, h_{n+1})$ is the infinitesimal n-cube

$$([(x_1, \ldots, x_n) \mapsto \gamma(x_1, \ldots, \alpha \cdot h_i, \ldots, x_n], h_1, \ldots, \widehat{h}_i, \ldots, h_{n+1}).$$

Thus, for example, if $\gamma: D^2 \to R^2$ is the embedding, then

$$\partial(\gamma, h_1, h_2) = \\ (\gamma(-, 0), h_1) + (\gamma(h_1, -), h_2) - (\gamma(-, h_2), h_1) - (\gamma(0, -), h_2).$$



We observe that spelling out the definition of ∂ yields that

$$\partial \circ \partial = 0.$$

If we put $C_n(M) = (0)$ for $n < 0$ then we obtain a so-called *(differential) complex*. In general, a complex A (of R-modules) is a sequence

$$\ldots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \ldots \qquad (n \in \mathbb{Z})$$

or

$$\ldots \to A_{n-1} \xrightarrow{d_n} A_n \xrightarrow{d_{n+1}} A_{n+1} \to \ldots \qquad (n \in \mathbb{Z})$$

of R-modules and R-linear maps, such that $\partial_n \partial_{n+1} = 0$, or $d_{n+1} d_n = 0$. (Usually, the subscripts on ∂ and d are omitted.) If A and B are complexes, a *map of complexes*, or a *chain map* $f: A \to B$ is a sequence of R-linear maps $f_n: A_n \to B_n$ which preserve the structure of the complex, i.e. commute with the ∂'s, or the d's. (Again, we suppress subscripts on f.)

Given this terminology, the construction of the complex $C.(M) = \{C_n(M)\}$ is (covariantly) functorial in M: a map $M \xrightarrow{f} N$ induces R-linear maps

$$f_*: C_n(M) \to C_n(N)$$

defined on generators by composition, i.e. $f_*(\gamma, h_1, \ldots, h_n) = (f \circ \gamma, h_1, \ldots, h_n)$, and since

$$f_*(\partial(\gamma, h_1, \ldots, h_{n+1})) = \partial(f_*(\gamma, h_1, \ldots, h_{n+1}))$$

this yields a map of complexes.

The boundary operator $C_{n+1}(M) \xrightarrow{\partial} C_n(M)$ enables us to define an R-linear map $\Lambda^n(M) \xrightarrow{d} \Lambda^{n+1}(M)$, called the *exterior differentiation map*, by putting for each n-form $\omega: M^{D^n} \times D^n \to R$,

$$\int_{(\gamma, h_1, \ldots, h_{n+1})} d\omega = \int_{\partial(\gamma, h_1, \ldots, h_{n+1})} \omega.$$

This is well-defined, since as is easily checked, $d\omega: M^{D^{n+1}} \times D^{n+1} \to R$ is indeed homogeneous and alternating, and satisfies the degeneracy condition. Moreover, since $\partial^2 = 0$, we find that $d^2 = 0$. Observe that the defining equation for d is *"'Stokes' theorem for infinitesimal* *n-chains*". Below, we will see how to prove the usual form of Stokes' theorem for *big* n-chains.

Again, the construction of $\Lambda^n(M)$ is (contravariantly) functorial in M: a map $f: M \to N$ induces R-linear maps

$$f^*: \Lambda^n(N) \to \Lambda^n(M)$$

by composition: if ω is an n-form on N and $(\gamma, h_1, \ldots, h_n)$ is an infinitesimal n-chain on M, then

$$f^*(\omega)(\gamma, h_1, \ldots, h_n) = \omega(f \circ \gamma, h_1, \ldots, h_n),$$

and we extend to $\Lambda^n(N)$ by linearity. Thus by definition,

$$\int_{f_*(\gamma, h_1, \ldots, h_n)} \omega = \int_{(\gamma, h_1, \ldots, h_n)} f^*(\omega).$$

The f^* together (for each n) give a chain map $f^*: \Lambda^{\cdot}(N) \to \Lambda^{\cdot}(M)$, since

$$d(f^*\omega) = f^*(d\omega).$$

We remark here that if M is R^n (or more generally a manifold in the classical sense) we obtain the usual notions of form and exterior differentiation. This point will be proved in 3.7 below, where a comparison is made between the classical approach and the (model theory of the) synthetic approach.

The *De Rham complex* of R-modules (and R-linear maps) of an arbitrary object M is the sequence

$$\ldots \to \Lambda^{n-1}(M) \xrightarrow{d} \Lambda^n(M) \xrightarrow{d} \Lambda^{n+1}(M) \to \ldots$$

where $\Lambda^n(M)$ is defined above for $n \geq 0$, and $\Lambda^n(M) = (0)$ for $n < 0$. The *De Rham cohomology* R-modules of M are defined, as in the classical case, by

$$H^n(M) = F^n(M)/E^n(M).$$

where $F^n(M) = \mathrm{Ker}(\Lambda^n(M) \xrightarrow{d} \Lambda^{n+1}(M))$ ("the *closed n-forms*") and $E^n(M) = \mathrm{Im}(\Lambda^{n-1}(M) \xrightarrow{d} \Lambda^n(M))$ ("the *exact n-forms*"). (Note that $E^n(M) \subset F^n(M)$ since $d^2 = 0$.)

If $f: M \to N$, then by naturality of d, $f^*: \Lambda^n(N) \to \Lambda^n(M)$ maps closed forms on N to closed forms on M, and exact forms on N to exact ones on M, so we obtain a map $f^* = H^n(f): H^n(N) \to H^n(M)$, making $H^n(-)$ into a contravariant functor.

Let us recall the integration-axiom stated in II.2.4, and shown to be valid in $Sets^{\mathbb{L}^{op}}$ (chapter II), and in $\mathcal{F}$ and $\mathcal{G}$ (chapter III):

$$\forall f \in R^{[0,1]} \exists! g \in R^{[0,1]} (g(0) = 0 \wedge g' \equiv f).$$

Given f, this g will be denoted by $g(x) = \int_0^x f(t)dt$, as usual.

Note that in the terminology of the De Rham cohomology, the integration axiom implies that

$$H^1([0,1]) = (0).$$

(Indeed, let $\tilde{\omega}: [0,1]^D \to R$ be a 1-form on $[0,1]$, and write $\tilde{\omega}(d \mapsto a + db) = \tilde{\omega}_a(b)$ (every element of $[0,1]^D$ is of the form $d \mapsto a + db$ by the Kock-Lawvere axiom). $\tilde{\omega}_a: R \to R$ is homogeneous, hence it is of the form $b \mapsto f(a) \cdot b$ for a unique $f(a) \in R$. Thus $\tilde{\omega}(d \mapsto a + db) = f(a) \cdot b$. This $\tilde{\omega}$ is usually denoted by $f(x)dx$. Now it is easily seen that every 1-form on $[0,1]$ is closed, and that $f(x)dx = dg$ for $g \in R^{[0,1]}$ iff $g' \equiv f$.)

Using the integration axiom, we can define integration of a form along a *finite n-cube* $\gamma: I^n \to M$ by the formula

$$\int_\gamma \omega = \int_0^1 \cdots \int_0^1 \tilde{\omega}\,((h_1, \ldots, h_n) \mapsto \gamma(t_1 + h_1, \ldots, t_n + h_n))dt_1 \ldots dt_n.$$

Just as for the infinitesimal chains, we define the *object of finite n-chains*, $\Gamma_n(M)$, as the free R-module generated by the set of maps $I^n \to M$, and an R-linear boundary operator $\partial: \Gamma_{n+1}(M) \to \Gamma_n(M)$. These definitions are again functorial in the obvious way: an $f: M \to N$ induces by composition R-linear maps

$$f_*: \Gamma_n(M) \to \Gamma_n(N), \qquad f_*(\gamma) = f \circ \gamma$$

for each n, and these form a chain map, since

$$\partial f_*(\gamma) = f_*(\partial\gamma).$$

The integral

$$\int : \Gamma_n(M) \times \Lambda^n(M) \to R$$

is R-linear in each variable separately, while moreover (as for infinitesimal chains) we again have by definition

(1) $\int_{f_*\gamma} \omega = \int_\gamma f^*\omega.$

Less trivial is the extension of Stokes' identity, which we used to define exterior differentiation, from infinitesimal n-chains to finite n-chains:

1.1 Stokes' Theorem. *Let M be any object, and let $\gamma \in \Gamma_{n+1}(M)$, $\omega \in \Lambda^n(M)$. Then*

$$\int_\gamma d\omega = \int_{\partial\gamma} \omega.$$

Before we give the proof of Stokes' theorem, let us point out that the usual rules of differentiating under the integral sign follow easily from the integration axiom. For example, if $f(x,y): I^2 \to R$, then for all $a \in I$,

(2) $\int_o^a \frac{\partial f}{\partial y}(x,y)dx \equiv \frac{\partial}{\partial y} \int_0^a f(x,y)dx.$

Indeed, to prove this it suffices, by the synthetic definition of $\frac{\partial}{\partial y}$, to show that for all $d \in D$, and $b \in I$, we have $d \cdot \int_0^a \frac{\partial f}{\partial y}(x,b)dx = (\int_0^a f(x,b+d)dx - \int_0^a f(x,b)dx)$, and this is clear from linearity of $\int_0^a$, since $f(x,b+d) - f(x,b) = d \cdot \frac{\partial f}{\partial y}(x,b)$ (again by definition of $\frac{\partial f}{\partial y}$).

Proof of 1.1. For ease of notation, we only do the case $n = 1$; the general case is completely analogous. So choose $\gamma \in \Gamma_2(M), \omega \in \Lambda^1(M)$. By linearity, it suffices to prove Stokes' identity for generators γ and ω, i.e. $\gamma: I^2 \to M$ and $\omega: M^D \times D \to R$. By pulling back along γ (i.e. replacing ω by $\gamma^*(\omega)$) we may assume, by (1) above, that $M = I^2$, and $\gamma = \mathrm{id}$. Now define two functions $f, g: I^2 \to R$ by the integration axiom:

$$\begin{aligned} g(x,y) &= \int_0^x \int_0^y d\omega((d_1,d_2) \mapsto (t_1+d_1, t_2+d_2))dt_1dt_2 \\ f(x,y) &= \int_0^x \omega(d \mapsto (t+d,o))dt + \int_0^y \omega(d \mapsto (x,t+d))dt \\ &\quad - \int_0^x \omega(d \mapsto (t+d,y))dt - \int_0^y \omega(d \mapsto (0,t+d))dt. \end{aligned}$$

Then Stokes' identity says that $g(1,1) = f(1,1)$. We will prove that $f \equiv g$. By the integration axiom, it suffices to show for every $x \in I$ that

$$f(x,0) = g(x,0), \text{ and } \forall y \in I \tfrac{\partial f}{\partial y}(x,y) = \tfrac{\partial g}{\partial y}(x,y).$$

Again applying the integration axiom, now in the y-coordinate, we only have to show

$$f(0,0) = g(0,0), \quad \forall x \in I: \frac{\partial f}{\partial x}(x,0) = \frac{\partial g}{\partial x}(x,0)$$

and for all $y \in I$:

$$\frac{\partial f}{\partial y}(0,y) = \frac{\partial g}{\partial y}(0,y), \quad \forall x \in I \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 g}{\partial x \partial y}(x,y).$$

Now it is clear from the definition that $f(x,y) = 0 = g(x,y)$ whenever $x = 0$ or $y = 0$, so the first three of these four equations hold. For the fourth, it suffices to show for each $x, y \in I$,

$$\forall h,k \in D \quad h \cdot k \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y) = h \cdot k \cdot \frac{\partial^2 g}{\partial x \partial y}(x,y).$$

To see this, we simply compute: as for the right-hand side,

$$\begin{aligned} & h \cdot k \cdot \tfrac{\partial^2 g}{\partial x \partial y}(x,y) = \\ & h \cdot k \cdot \tfrac{\partial}{\partial x}|_{(x,y)} \textstyle\int_0^x \tfrac{\partial}{\partial y} \int_0^y d\omega((d_1,d_2) \mapsto (t_1 + d, t_2 + d_2))dt_1 dt_2 \\ = \; & h \cdot k \cdot \tfrac{\partial}{\partial x}|_{(x,y)} \textstyle\int_0^x d\omega((d_1,d_2) \mapsto (t_1 + d_1, y + d_2))dt_1 \\ = \; & h \cdot k \cdot d\omega((d_1,d_2) \mapsto (x + d_1, y + d_2)). \end{aligned}$$

Since x, y do not occur both in the first and last terms of the definition of $f(x,y)$, these second derivatives vanish, and hence for the left-hand side we have

$$\begin{aligned} h \cdot k \cdot \; & \tfrac{\partial^2 f}{\partial x \partial y}(x,y) = h \cdot k \cdot \left(\tfrac{\partial^2}{\partial x \partial y}|_{(x,y)} \left(\textstyle\int_0^y \omega(d \mapsto (x, t+d))dt - \right.\right. \\ & \left.\left. \textstyle\int_0^x \omega(d \mapsto (t+d, y))dt \right)\right) = \\ & = h \cdot k \left(\tfrac{\partial}{\partial x}|_x \omega(d \mapsto (x, y+d)) - \tfrac{\partial}{\partial y}|_y \omega(d \mapsto (x+d, y)) \right) \\ & = k \cdot [\omega(d \mapsto (x+h, y+d)) - \omega(d \mapsto (x, y+d))] \\ & = h \cdot [\omega(d \mapsto (x+d, y+k)) - \omega(d \mapsto (x+d, y))] \\ & = h \cdot k \cdot d\omega((d_1,d_2) \mapsto (x+d_1, y+d_2)), \end{aligned}$$

where the last equation holds by definition of exterior differentiation in terms of *Stokes for infinitesimal cubes.* □

We now check the three *axioms* for a cohomology theory, namely the homotopy invariance (or Poincaré lemma), the Mayer-Vietoris sequence, and the disjoint-union lemma.

1.2 Poincaré Lemma. *The De Rham cohomology of R^n is the same as that of a one-point space $\{*\}$:*

$$H^q(R^n) = H^q(\{*\}) = \begin{cases} R & \text{if } q = 0 \\ (0) & \text{if } q \neq 0 \end{cases}.$$

We shall derive this lemma from the following result.

1.3 Theorem. *Let $F: I \times M \to N$ be a homotopy from F_0 to F_1. Then for each n there is an R-linear map*

$$K_n: \Lambda^n(N) \to \Lambda^{n-1}(M)$$

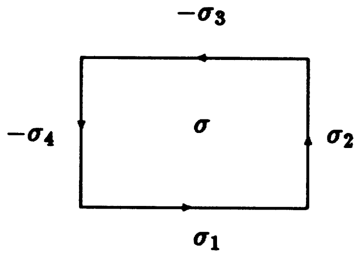
such that for every $\omega \in \Lambda^n(N)$,

$$F_1^*(\omega) - F_0^*(\omega) = K_{n+1}d\omega + dK_n\omega.$$

Proof. Define $K_n(\omega)$ for $\omega \in \Lambda^n(N)$ by setting for each $\gamma \in M^{D^{n-1}}$

$$K_n(\omega)(\gamma) = \int_0^1 \omega((h_1, \ldots, h_n) \mapsto F_{t+h_1}(\gamma(h_2, \ldots, h_n)))dt.$$

It is easy to check that $K_n(\omega): M^{D^{n-1}} \to R$ is homogeneous and alternating, so this defines an $(n-1)$-form $K_n(\omega) \in \Lambda^{n-1}(M)$. For notational purposes, let us assume $n = 2$, and take any infinitesimal 2-cube $(\sigma, d_1, d_2) \in M^{D^2} \times D^2$. Denote the four terms in the boundary of σ by σ_i, so $\partial(\sigma, d_1, d_2) = (\sigma_1, d_1) + (\sigma_2, d_2) - (\sigma_3, d_1) - (\sigma_4, d_2)$



We have to show that

(*) $\int_{(\sigma,d_1,d_2)}(F_1^*(\omega) - F_0^*(\omega) - (K_3(d\omega) + dK_2\omega)) = 0.$

To this end, define for $\omega \in \Lambda^n(M)$, $\gamma \in M^{D^{n-1}}$

$$K_n^x(\omega)(\gamma) = \int_0^x \omega((h_1, \ldots, h_n) \mapsto F_{t+h_1}(\gamma(h_2, \ldots, h_n)))dt$$

(so $K_n^1 = K_n$), and let

$$G(x) = \int_{(\sigma,d_1,d_2)} (F_x^*(\omega) - F_o^*(\omega) - (K_3^x d\omega + dK_2^x\omega)).$$

To prove (*), i.e. $G(1) = 0$, it suffices by the integration axiom to show that $G(0) = 0$, and $\forall x \in I$ $G'(x) = 0$, i.e. $\forall d \in D$ $\forall x \in I G(x+d) - G(x) = 0$. Now $G(0) = 0$ is clear. For the other equation, fix $x \in I$ and $d_o \in D$. To compute $G(x + d_o) - G(x)$, that is

(**) $\int_{(\sigma,d_1,d_2)} F_{x+d_o}^*(\omega) - F_x^*(\omega) - K_3^{x+d_o}(d\omega) + K_3^x(d\omega) -$
$dK_2^{x+d_o}(\omega) + dK_2^x(\omega),$

we define an infinitesimal 3-cube (ρ, d_o, d_1, d_2) by

$$\rho(h_1, h_2, h_3) = F_{x+h_1}(\sigma(h_2, h_3)).$$

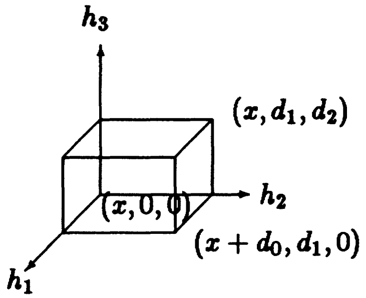
$\int_{(\rho,d_o,d_1,d_2)} d\omega$ can now be computed in two ways. The first is directly:

$$\begin{aligned} \int_{(\rho,d_0,d_1,d_2)} d\omega &= d_2 \cdot d_1 \cdot d_0 \cdot d\omega((h_1, h_2, h_3) \mapsto F_{x+h_1}(\gamma(h_2, h_3))) \\ &= d_2 \cdot d_1(K_3^{x+d_o}(d\omega)(\sigma) - K_3^x(d\omega)(\sigma)) \\ &= \int_{(\sigma,d_1,d_2)} K_3^{x+d_o}(d\omega) - K_3^x(d\omega). \end{aligned}$$

The other way to compute it is by the definition of $d\omega$ (*infinitesimal Stokes*):

$$\int_{(\rho,d_0,d_1,d_2)} d\omega = \int_{\partial(\rho,d_0,d_1,d_2)} \omega = \int_f \omega - \int_{ba} \omega + \int_\ell \omega - \int_r \omega + \int_t \omega - \int_{bo} \omega$$

(where $\partial\rho = (f - ba) + (\ell - r) + (t - bo)$, f refers to the restriction of ρ to *the front* of the infinitesimal cube below, ba to *the back*, etc.)



Now ℓ is the infinitesimal 2-cube

$$([(h_1, h_3) \mapsto \rho(h_1, 0, h_3) = F_{x+h_1}(\sigma(0, h_3))], d_0, d_2),$$

so

$$\begin{aligned} \int_\ell \omega &= d_0 \cdot d_2 \omega((h_1, h_2) \mapsto F_{x+h_1}(\sigma(0, h_3))) \\ &= d_2(K_2^{x+d_0}(\omega)(\sigma_4) - K_2^x(\omega)(\sigma_4)).reges - 04b.tex \end{aligned}$$

Similarly,

$$\begin{aligned} \int_r \omega &= d_2(K_2^{x+d_o}(\omega)(\sigma_2) - K_2^x(\omega)(\sigma_2)) \\ \int_t \omega &= d_1(K_2^{x+d_o}(\omega)(\sigma_3) - K_2^x(\omega)(\sigma_3)) \\ \int_{bo} \omega &= d_1(K_2^{x+d_o}(\omega)(\sigma_1) - K_2^x(\omega)(\sigma_1)). \end{aligned}$$

Moreover, clearly,

$$\int_f \omega = \int_{(F_x+d_0)_*(\sigma,d_1,d_2)} \omega = \int_{(\sigma,d_1,d_2)} (F_{x+d_o})^*(\omega)$$

and

$$\int_{ba} \omega = \int_{(\sigma,d_1,d_2)} F_x^*(\omega).$$

Putting all these together, we find

$$\begin{aligned} \textstyle\int_{\partial(\rho,d_0,d_1,d_2)} d\omega &= \textstyle\int_{(\sigma,d_1,d_2)}(F^*_{x+d_o}(\omega) - F^*_x(\omega)) \\ &- \textstyle\int_{\partial(\sigma,d_1,d_2)} K_2^{x+d_o}(\omega) + \int_{\partial(\sigma,d_1,d_2)} K_2^x(\omega). \end{aligned}$$

Combining this with the first computation of $\int_{(\rho,d_0,d_1,d_2)} d\omega$, we conclude that the integral (**) vanishes. Hence (*) holds, and the proof is complete. □

1.4 Corollary. *(Homotopy Invariance.) Let M, N be any objects. If $M \overset{f}{\underset{g}{\rightrightarrows}} N$ are homotopic, then $H^{\cdot}(f) = H^{\cdot}(g)\colon H^{\cdot}(N) \to H^{\cdot}(M)$. In particular, if M and N are homotopy equivalent, then $H^{\cdot}(M) \cong H^{\cdot}(N)$ (whence the Poincaré lemma).* □

Let us now turn to the Mayer-Vietoris sequence. Recall that a partition of unity subordinate to a cover $\{U,V\}$ of M is a pair of maps $\rho_U, \rho_V\colon M \to R$ such that for all $x \in M, \rho_U(x) + \rho_V(x) = 1$, and moreover, for all $x \in M$

$$x \in U \text{ or } \rho_U(x) = 0, \text{ and } x \in V \text{ or } \rho_V(x) = 0.$$

1.5 Proposition. *Assume that $M = U \cup V$, where U and V are étale subobjects of M (i.e., if $\varphi\colon D^q \to M$ and $\varphi(0) \in U$, then $im(\varphi) \subseteq U$; similarly for V). If $\{U,V\}$ has a partition of unity subordinate to it, then the sequence*

$$0 \to \Lambda^q(M) \to \Lambda^q(U) \oplus \Lambda^q(V) \to \Lambda^q(U \cap V) \to 0$$

is exact (the i's-denote the inclusions, the homomorphisms are given by $\omega \mapsto (i_U^*(\omega), i_V^*(\omega))$ and $(\mu,\nu) \mapsto i^*_{U\cap V}(\nu) - i^*_{U\cap V}(\mu)$).

Proof. The fact that the first map is monic follows from the fact that $\{U,V\}$ is an étale cover.

To show exactness in the middle, let $(\mu,\nu) \in \Lambda^q(U) \oplus \Lambda^q(V)$ be such that $i^*_{U\cap V}(\nu) = i^*_{U\cap V}(\mu)$. Define

$$\omega = \rho_U \cdot \mu + \rho_V \cdot \nu,$$

where $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to $\{U, V\}$, and

$$(\rho_U \cdot \mu)(\varphi) = \begin{cases} \rho_U(\varphi(0)) \cdot \mu(\varphi) & \text{if} \quad \varphi(0) \in U \\ 0 & \text{if} \quad \rho_U(\varphi(0)) = 0 \end{cases}$$

and similarly for $\rho_v \cdot \nu$. (Notice that since U and V are étale, this definition makes sense.) Then

$$i_U^* \omega - \mu = i_U^*(\rho_U \cdot \mu) + i_U^*(\rho_V \cdot \nu) - (\rho_U|_U \cdot \mu + \rho_V|_U \cdot \mu)$$

But $i_U^*(\rho_U \cdot \mu) = \rho_U|_U \cdot \mu$, and by definition of a partition of unity and the fact that $i_{U \cap V}^*(\mu) = i_{U \cap V}^*(\nu)$, also $i_U^*(\rho_V \cdot \nu) = \rho_V|_U \cdot \mu$. Hence $i_U^* \omega = \mu$. Similarly it follows that $i_V^*(\omega) = \nu$, so the sequence is exact in the middle.

To show that the right hand map is epic, one shows similarly that any $\omega \in \Lambda^q(U \cap V)$ comes from the pair $(-\rho_V \cdot \omega, \rho_U \cdot \omega)$. □

This short exact sequence is called the *Mayer-Vietoris sequence*, and it induces a long exact sequence, as stated in the following:

1.6 Corollary. *Under the hypothesis of the preceding proposition, there is a long exact sequence*

$$\ldots H^q(U) \oplus H^q(V) \to H^q(U \cap V) \to H^{q+1}(M) \to H^{q+1}(U) \oplus H^{q+1}(V).$$

The linear map $d^*: (U \cap V) \to H^{q+1}(M)$, the so-called **Bockstein homomorphism**, *may be described by*

$$d^*[\omega] = \begin{cases} [-d(\rho_V \cdot \omega)] & \text{on} \quad U \\ [d(\rho_U \cdot \omega)] & \text{on} \quad V. \end{cases}$$

Proof. This is some elementary homological algebra. The general situation is that we are given a shortexact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of complexes (i.e. each dimension $0 \to A^q \xrightarrow{f} B^q \xrightarrow{g} C^q \to 0$ is exact). To define $d^*: H^q(C) \to H^{q+1}(A)$, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^{q-1} \arrow[r, "f"] \arrow[d, "d"] & B^{q-1} \arrow[r, "g"] \arrow[d, "d"] & C^{q-1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{q} \arrow[r, "f"] \arrow[d, "d"] & B^{q} \arrow[r, "g"] \arrow[d, "d"] & C^{q} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{q+1} \arrow[r, "f"] & B^{q+1} \arrow[r, "g"] & C^{q+1} \arrow[r] & 0
\end{tikzcd}$$

and take $[c] \in H^q(C)$, so $dc = 0$. Write $c = g(b)$ for some $b \in B^q$, and observe that $dc = dg(b) = g(db) = 0$, whence $db = f(a)$ for some $a \in A^{q+1}$. Define $d^*[c]$ to be this $[a]$. This all looks like a horrible application of the axiom of choice (which is not available in the synthetic context), but it is not, and moreover d^* is well-defined on equivalence classes. Both assertions follow from the fact that $d^*[c]$ is the *unique* equivalence class $[a]$ such that for some $b \in B^q$, $f(a) = db$ and $[g(b)] = [c]$. To see this, assume that both a and a' are candidates, i.e.

$$\begin{array}{lll} f(a) = db, & [g(b)] = [c], & \text{some} \quad b \\ f(a') = db', & [g(b')] = [c], & \text{some} \quad b'. \end{array}$$

Then since $[g(b)] = [g(b')]$, $g(b - b') = dc_o$ for some c_o. g is epi, so $c_o = g(b_o)$ for some b_o. But then $g(b - b' - db_o) = dc_o - dc_o = 0$, hence $b - b' - db_o = f(a_o)$ for some a_o. Thus $f(da_o) = db - db' - d^2b_0 = f(a) - f(a')$, and since f is mono, $da_o = a - a'$, i.e. $[a] = [a']$. Linearity of d^* is now obvious.

It remains to show that the long sequence is exact, which is easy and can safely be left to the reader. □

The last axiom is the *disjoint-union lemma*, the proof of which is obvious.

1.7 Proposition. *If $M = \amalg_\alpha M_\alpha$ is a disjoint union, then*

$$H^q(M) \cong \Pi_\alpha H^q(M_\alpha). \qquad \square$$

1.8 Infinitesimal Cubes as Finite Cubes (digression). An infinitesimal n-cube $(\gamma, h_1, \ldots, h_n) \in M^{D^n} \times D^n$ on M may canonically be regarded as a finite n-cube $I^n \to M$: let $I^n \xrightarrow{\varphi} D^n$ be the map $(t_1, \ldots, t_n) \mapsto (h_1 t_1, \ldots, h_n t_n)$, and consider the finite n-chain

$\sigma = \gamma \circ \varphi: I^n \to M$. Thus an n-form ω on M can be integrated along such an infinitesimal n-cube $(\gamma, h_1, \ldots, h_n)$ in two ways, namely as $\omega(\gamma, h_1, \ldots, h_n)$ and, by the definition based on the integration axiom, as $\int_\sigma \omega$. We will now point out that these two ways of integrating yield the same result if M has the following *extension property* (E):

(E) The canonical map $M^{D_2^n} \xrightarrow{r} M^{D^n}$ induced by the inclusion $D \hookrightarrow D_2 = \{x \in R | x^3 = 0\}$ is a retraction (i.e. there is a section i, $r \circ i = \mathrm{id}$).

Every R^n has property (E), and more generally, so do all formal manifolds (in any of the senses proposed; in particular, it is not hard to check that the Kock-Lawvere axiom implies that it is valid in the various toposes that we have considered that objects of the form $s(M)$, M a manifold in *Sets*, have property (E)). Moreover, if an object M has property (E), then so do all exponentials M^N, and all retracts of M.

Proposition. *Let M be an object having property (E), and let $(\gamma, h_1, \ldots, h_n)$ be an infinitesimal n-cube on M. Let $\sigma = \gamma\varphi: I^n \to M$, where $\varphi(t_1, \ldots, t_n) = (h_1 t_1, \ldots, h_n t_n)$ as above. Then*

$$\int_\sigma \omega = \int_{(\gamma, h_1, \ldots, h_n)} \omega$$

(on the left, actual integration; on the right, formal integration, i.e. $\omega(\gamma, h_1, \ldots, h_n)$).

Proof. By property (E), we may extend γ to $\overline{\gamma}: D_2^n \to M$. Replacing ω by $\overline{\gamma}^*(\omega)$, we may assume $M = D_2^n$, and γ

$$\begin{tikzcd}
I^n \arrow[r, "\varphi"] \arrow[dr, "\sigma"'] & D^n \arrow[r, tail, "j"] \arrow[d, "\gamma"] & D_2^n \arrow[dl, "\overline{\gamma}"] \\
& M &
\end{tikzcd}$$

is the inclusion j. So ω is an n-form on D_2^n, corresponding to $\tilde{\omega}: (D_2^n)^{D^n} \to R$. Define two functions $F, G: I^n \to R$ by

$$F(x) = \int_o^{x_1} \ldots \int_0^{x_n} \tilde{\omega}((d_1, \ldots, d_n) \mapsto \sigma(t_1 + d_1, \ldots, t_n + d_n))\, dt_1 \ldots dt_n$$
$$G(x) = \omega(j, x_1 h_1, \ldots, x_n h_n),$$

where $x = (x_1, \ldots, x_n)$. The equation in the proposition then means $F(1, \ldots, 1) = G(1, \ldots, 1)$. Since $F(x_1, \ldots, x_n) = 0 = G(x_1, \ldots, x_n)$ if $x_i = 0$ for some i, it suffices (as in the proof of Stokes' theorem) by the integration axiom to show that for $(x_1, \ldots, x_n) \in I^n$

$$(*) \quad \frac{\partial^n F}{\partial \underline{x}}(x_1, \ldots, x_n) = \frac{\partial^n G}{\partial \underline{x}}(x_1, \ldots, x_n)$$

where $\partial \underline{x} = \partial x_1 \ldots \partial x_n$. By homogeneity, we have

$$\frac{\partial^n G}{\partial \underline{x}}(x_1, \ldots, x_n) = \omega(j, h_1, \ldots, h_n) = h_1 \cdot \ldots \cdot h_n \tilde{\omega}(j).$$

On the other hand,

$$\begin{aligned} &\frac{\partial^n F}{\partial \underline{x}}(x_1, \ldots, x_n) = \\ &\quad = \tilde{\omega}((d_1, \ldots, d_n) \mapsto (h_1(x_1 + d_1), \ldots, h_n(x_n + d_n))) = \\ &\quad = h_1 \cdot \ldots \cdot h_n \tilde{\omega}((d_1, \ldots, d_n) \mapsto (h_1 x_2 + d_1, \ldots, h_n x_n + d_n)), \end{aligned}$$

the last equality by homogeneity (which can be applied since $D + D \subset D_2$). By the Kock-Lawvere axiom, the map $: D^n \to R$,

$$(u_1, \ldots, u_n) \mapsto \tilde{\omega}((d_1, \ldots, d_n) \mapsto (u_1 + d_1, \ldots, u_n + d_n))$$

is of the form

$$(u_1, \ldots, u_n) \mapsto \tilde{\omega}(j) + \sum u_i a_i + \sum u_i u_{i'} b_{ii'},$$

for some $a_i, b_{ii'}, \in R$. Hence

$$h_1 \cdot \ldots \cdot h_n \tilde{\omega}((d_1, \ldots, d_n) \mapsto (h_1 t_1 + d_1, \ldots, h_n t_n + d_n) =$$
$$= h_1 \cdot \ldots \cdot h_n(\tilde{\omega}(j) + \sum h_i^2 t_i a_i + \sum h_i^2 t_i h_i^2 b_{ii'}) = h_1 \cdot \ldots \cdot h_n \tilde{\omega}(j)$$

since all the other terms vanish. Thus (*) holds, and hence by the integration axiom $F \equiv G$. In particular, $F(1, \ldots, 1) = G(1, \ldots, 1)$ □

2 Singular Homology

Let M be a smooth space. A ***singular q-simplex*** of M is a map $\Delta_q \xrightarrow{\sigma} M$, where $\Delta_q (q \geq 0)$ is the standard q-simplex $[e_o, \ldots, e_q] = \{(x_o, \ldots, x_q) \in R^{q+1} | 0 \leq x_i \leq 1, \text{ and } \sum x_i = 1\}$ ($\{e_o, \ldots, e_q\}$ denotes the standard base of R^{q+1}). We let $S_q(M)$ be the free R-module generated by the singular q-simplices; the elements of $S_q(M)$ are called ***singular q-chains***. There is an R-linear ***boundary operator***

$$\partial = \partial_q : S_q(M) \to S_{q-1}(M)$$

defined on generators $\Delta_q \xrightarrow{\sigma} M$ by

$$\partial_q(\sigma) = \sum_{j=0}^{q} (-1)^j \sigma \circ \varepsilon_q^j,$$

where $\varepsilon_q^j : \Delta_{q-1} \to \Delta_q$ is the j-th face of Δ_q, i.e. $\varepsilon_q^j(x_0, \ldots, x_{q-1}) = (x_0, \ldots, x_{j-1}, 0, x_j, \ldots, x_q)$. Since $\partial \circ \partial = 0$ (as is easily checked), this defines a complex $S.(M)$ if we agree that $S_q(M) = (0)$ for $q < 0$. Note that this definition of $S.(M)$ is functorial in M: a map $M \xrightarrow{f} N$ induces R-linear maps $f_*: S_q(M) \to S_q(N)$ for each q, defined on generators by composition, i.e. $f_*(\sigma) = f \circ \sigma$, and this yields a chain map $f_*: S.(M) \to S.(N)$ because $\partial(f_*\sigma) = f_*(\partial\sigma)$.

As usual, we define submodules $B_q(M) = \mathrm{Im}(\partial_{q+1})$ (***boundaries***) and $Z_q(M) = \mathrm{Ker}(\partial_q)$ (***cycles***) of $S_q(M)$, and since $\partial^2 = 0, B_q(M) \subset Z_q(M)$ so we can define the q-dimensional ***singular homology R-***module of M by

$$H_q(M; R) = Z_q(M)/B_q(M).$$

Clearly, $H_q(-; R)$ is a covariant functor.

We proceed now as in the case of the De Rham cohomology by proving the three key properties, viz. the Poincaré lemma or homotopy invariance, the existence of the (longexact) Mayer-Vietoris sequence, and the disjoint union lemma.

2.1 Theorem. *Let $F: I \times M \to N$ be a homotopy from F_0 to F_1. Then for each q there is an R-linear map*

$$P = P_q : S_q(M) \to S_{q+1}(N)$$

such that for every $\sigma: \Delta_q \to M$,

$$F_{1*}(\sigma) - F_{0*}(\sigma) = \partial P_q(\sigma) + P_{q-1}(\partial\sigma).$$

Proof. We will first define a triangulation P_q of $I \times \Delta_q$, i.e. a sum

$P_q \in S_q(I \times \Delta_q)$ of maps $\Delta_{q+1} \to I \times \Delta_q$, and then for $\sigma\colon \Delta_q \to M$ we let $P_q(\sigma) \in S_{q+1}(N)$ be the composition (= sum of compositions)

$$\Delta_{q+1} \xrightarrow{P_q} I \times \Delta_q \xrightarrow{1\times\sigma} I \times M \xrightarrow{F} N.$$

Each of the maps involved in the definition of P_q will be affine, and it is useful to introduce some notation. Recall that a singular q-simplex σ on a convex subset $M \subset R^n$ is called affine if there are points $m_o, \dots, m_q \in M$ such that

$$\sigma(x_o, \dots, x_q) = x_o m_o + \dots + x_q m_q;$$

such an affine simplex is denoted by $[m_o, \dots, m_q]$. So $\partial[m_o, \dots, m_q] = \sum_{k=0}^{q}(-1)^k[m_o, \dots, \widehat{m}_k, \dots, m_q]$. If $S = [m_o, \dots, m_q]$ is an affine q-simplex, the k-th q–1-simplex occurring in ∂S will be denoted by $S(\widehat{k})$, i.e.

$$S(\widehat{k}) = [m_o, \dots, \widehat{m}_k, \dots, m_q].$$

In $I \times \Delta_q$, we distinguish the points $e_j^\alpha = (\alpha, e_j)$ for $\alpha = 0, 1, j = 0, \dots, q$. Let us write S_j for the affine q+1-simplex $[e_j^1, \dots, e_q^1, e_0^0, \dots,$ on $I \times \Delta_q (j = 0, \dots, q)$. We now define P_q by

$P_q = \sum_{j=0}^{q} -S_j,$ if q is even
$P_q = \sum_{j=0}^{q}(-1)^j S_j,$ if q is odd.

Let us verify that indeed

$$\partial P_q(\sigma) + P_{q-1}(\partial\sigma) = F_{1*}(\sigma) - F_{0*}(\sigma)$$

for every $\sigma\colon \Delta_q \to M$. From the definition of $P_q(\sigma)$ given above it is clear that it suffices to consider the *generic* case where $\sigma = [e_o, \dots, e_q] = \mathrm{id}\colon \Delta_q \to \Delta_q$, and $F\colon I \times M \to N$ is the identity $I \times \Delta_q \to I \times \Delta_q$. Thus, we verify that for each q,

$$\begin{aligned}&\partial P_q([e_o, \dots, e_q]) - \textstyle\sum_{k=0}^{q}(-1)^k P_{q-1}([e_o, \dots, \widehat{e}_k, \dots, e_q]) = \\ &= [e_o^1, \dots, e_q^1] - [e_o^o, \dots, e_q^o].\end{aligned}$$

Indeed, if q is even,

$$\partial P_q([e_o, \dots, e_q]) = -\sum_{j=0}^{q}\sum_{k=0}^{q+1}(-1)^k S_j(\widehat{k}),$$

while

$$\begin{aligned}&P_{q-1}(\partial[e_o, \dots, e_q]) = \textstyle\sum_{k=0}^{q}(-1)^k P_{q-1}([e_o, \dots, \widehat{e}_k, \dots, e_q]) \\ &= \textstyle\sum_{k=0}^{q}(-1)^k \sum_{\substack{j=0 \\ j\neq k}}^{q}(-1)^{p_j^k}[e_j^1, \dots, \widehat{e}_k^1, \dots, e_q^1, e_o^o, \dots, \widehat{e}_k^o, \dots, e_j^o]\end{aligned}$$

(where $p_j^k = j$ if $j < k$, $p_j^k = j - 1$ if $j > k$; note that only one of e_k^1, e_k^o is omitted, depending on whether $j < k$ or $k < j$)

$$\begin{aligned}
&= \textstyle\sum_{j=0}^{q} \big[\sum_{k=0}^{j-1}(-1)^k(-1)^{j-1} S_j(\widehat{\mathrm{q} - \mathrm{j} + \mathrm{k} + 1}) + \\
&\qquad \textstyle\sum_{k=j+1}^{q}(-1)^k(-1)^j S_j(\widehat{k-j})\big] \\
&= \textstyle\sum_{j=0}^{q} \big[\sum_{\ell=q-j+1}^{q}(-1)^{\ell-q+j-1}(-1)^{j-1} S_j(\hat{\ell}) + \\
&\qquad \textstyle\sum_{\ell=1}^{q-j}(-1)^{\ell+j}(-1)^j S_j(\hat{\ell})\big] \\
&= \textstyle\sum_{j=0}^{q} \big[\sum_{\ell=1}^{q}(-1)^{\ell} S_j(\hat{\ell}) \,\big], \quad \text{since } q \text{ is even.}
\end{aligned}$$

Hence

$$\begin{aligned}
\partial P_q + P_{q-1}\partial &= -\textstyle\sum_{j=0}^{q}\sum_{k=0}^{q+1}(-1)^k S_j(\hat{k}) + \sum_{j=0}^{q}\sum_{k=1}^{q}(-1)^k S_j(\hat{k}) \\
&= \textstyle\sum_{j=0}^{q}\left[s_j(\widehat{q+1}) - S_j(\hat{0})\right] \quad (q \text{ is even}) \\
&= \left[e_o^l, \ldots, e_q^l\right] - \left[e_o^o, \ldots, e_q^o\right],
\end{aligned}$$

since everything else cancels because of $S_j(\widehat{q+1}) = S_{j-1}(\hat{0})$. And if q is odd,

$$\partial P_q([e_o, \ldots, e_q]) = \sum_{j=0}^{q}(-1)^j \sum_{k=0}^{q+1}(-1)^k S_j(\hat{k}),$$

while

$$\begin{aligned}
&P_{q-1}(\partial[e_o, \ldots, e_q]) = \textstyle\sum_{k=0}^{q}(-1)^k P_{q-1}([e_1, \ldots, \hat{e}_k, \ldots, e_q]) \\
&= \textstyle\sum_{k=0}^{q}(-1)^k \sum_{\substack{j=0 \\ k\neq k}}^{q} [e_j^1, \ldots, \hat{e}_j^1, \ldots, \hat{e}_k^1, \ldots, e_q^1, e_o^o, \ldots, \hat{e}_k^0, \ldots, e_j^o] = \\
&= \textstyle\sum_{j=0}^{q} \Big[\sum_{k=0}^{j-1}(-1)^{k+1} S_j(\widehat{\mathrm{q} - \mathrm{j} + \mathrm{k} + 1}) + \\
&\qquad \textstyle\sum_{k=j+1}^{q}(-1)^{k+1} S_j(\widehat{k-j})\Big] \\
&= \textstyle\sum_{j=0}^{q} \Big[\sum_{\ell=q=j+1}^{q}(-1)^{\ell-q+j} S_j(\hat{\ell}) + \sum_{\ell=1}^{q-j}(-1)^{\ell+j+1} S_j(\hat{\ell})\Big] \\
&= \textstyle\sum_{j=0}^{q}\sum_{\ell=1}^{q}(-1)^{\ell+j+1} S_j(\hat{\ell}), \quad \text{since } q \text{ is odd,}
\end{aligned}$$

and from this it immediately follows as in the case where q is even that $\partial P_q([e_o, \ldots, e_q]) + P_{q-1}(\partial[e_o, \ldots, e_q]) = [e_o^1, \ldots, e_q^1] - [e_o^o, \ldots, e_q^o]$. This completes the proof. □

2.2 Corollary. *(Homotopy invariance) If* $M \overset{f}{\underset{g}{\rightrightarrows}} N$ *are homotopic maps, then* $H.(f;R) = H.(g;R) \colon H.(M;R) \to H.(N;R)$. *In particular, if* M *and* N *are homotopy equivalent, then* $H.(M;R) \cong H.(N;R)$.

2.3 Corollary. *(Poincaré lemma) Let* $M \subset R^n$ *be convex and in-*

habited. Then

$$H_q(M;R) = \left\{ \begin{array}{lll} R & \text{if} & q=0 \\ (0) & \text{if} & q>0 \end{array} \right\}$$

Proof. If M is a single point, this is clear; and if M is arbitrary convex, inhabited, it is contractible, hence by corollary 1 it has the same singular homology as a single point.) □

We now turn to the exactness of the Mayer-Vietoris sequence. Things are considerably more difficult here than in the case of the De Rham cohomology. Let $M = U \cup V$, and let $S_q{}^{\{U,V\}}(M)$ be the submodule of $S_q(M)$ generated by $S_q(U) \cup S_q(V)$. Then from the short exact sequence

$$0 \to S_.(U \cap V) \to S_.(U) \oplus S_.(V) \to S_.^{\{U,V\}}(M) \to 0$$

we obtain (as usual) a longexact sequence which is just like the one of Mayer-Vietoris but for the fact that the homology $H_.^{\{U,V\}}(M;R)$ of the complex $S_.^{\{U,V\}}(M)$ appears instead of $H_.(M;R)$. What is the connection between the two? To answer this question, we shall from now on assume that

1. R is Archimedean
2. Δ_q is compact, for each $q \geq 0$
3. Every finite cover of Δ_q has a (finite) open refinement (each $q \geq 0$).

2.4 Proposition. *The canonical map* $H_.^{\{U,V\}}(M;R) \to H_.(M;R)$ *induced by the inclusion* $S_.^{\{U,V\}}(M) \hookrightarrow S_.(M)$ *is an isomorphism.*

Proof. We apply assumptions 1.–3. to a special chain map

$$\mathrm{sd}^M : S_.(M) \to S_.(M)$$

viz. the *barycentric subdivision.* sd^M is natural in M, and hence completely determined by the chains $(\mathrm{sd}^{\Delta_q})_q(\mathrm{id}) \in S_q(\Delta_q)$, which are defined as follows. Slightly more generally, we define for each affine complex $\Delta_q \xrightarrow{\sigma} M$ into a convex $M \subseteq R$ a chain $\mathrm{sd}_q^M(\sigma) \in S_q(M)$ by induction on q:

$$\begin{aligned} \mathrm{sd}_o^M([m_o]) &= [m_o] \\ \mathrm{sd}_q^M([m_o,\dots,m_q]) &= (-1)^q[\mathrm{sd}(\partial[m_o,\dots,m_q]), b], \end{aligned}$$

where $b = \sum_{j=0}^q \frac{1}{q+1} m_j$ is the barycenter of $[m_o,\dots,m_q]$, and the outer brackets [ ] are interpreted as: if $\tau = \sum_{i=0}^{n} a_i[n_o^i, \dots, n_{q-1}^i]$ is a chain of affine $q-1$-simplices, then $[\tau, b]$ is the chain of affine q-simplices $\sum_{i=0}^{n} a_i[n_o^i, \dots, n_{q-1}^i, b]$. $\square$

Thus in particular, we have defined $(\mathrm{sd}^{\Delta_q})_q(\mathrm{id}) \in S_q(\Delta_q)$ for each $q \geq 0$, and as just said this determines $\mathrm{sd}_q^M(\sigma)$ for every q-simplex $\Delta_q \xrightarrow{\sigma} M$ by

$$\mathrm{sd}_q^M(\sigma) = \sigma_*((\mathrm{sd}^{\Delta_q})_q(\mathrm{id})).$$

(Note that in case σ happens to be affine, this definition of $\mathrm{sd}_q^M(\sigma)$ coincides with the one already given.) One easily checks that each $\mathrm{sd}^M: S(M) \to S(M)$ is a chain map. The proof is now completed by noting the properties of sd stated in the following three lemmas.

Lemma 1. Every singular simplex in $(\mathrm{sd}^{\Delta_q})_q{}^m(\mathrm{id})$ has diameter $\leq (^q/q+1)^m \mathrm{diam}(\Delta_q)$.

Proof. trivial induction on q. $\square$

Lemma 2. Let $M = U \cup V$. For each singular q-simplex $\sigma: \Delta_q \to M$ there is an $m \geq 0$ such that every simplex in $\mathrm{sd}^m(\sigma)$ (where sd $=$ $(\mathrm{sd}^M)_q$) factors through either U or V, i.e. $\mathrm{sd}^m(\sigma) \in S_q^{\{U,V\}}(M)$.

Proof. Since $\Delta_q = \sigma^{-1}(U) \cup \sigma^{-1}(V)$, we also have (by assumption 3 on Δ_q) that $\Delta_q = \mathrm{Int}\ \sigma^{-1}(U) \cup \mathrm{Int}\ \sigma^{-1}(V)$. From compactness of Δ_q (assumption 2) we obtain a Lebesgue number $\lambda > 0$ for this cover. Since R is Archimedean (assumption 1), there is an $m \geq 0$ such that $(^q/q+1)^m \mathrm{diam}(\Delta_q) < \lambda$. Then every simplex in $\mathrm{sd}^m(\mathrm{id})$ factors through $\mathrm{Int}(\sigma^{-1}(U))$ or through $\mathrm{Int}(\sigma^{-1}(V))$, and this implies that every simplex in $\mathrm{sd}^m(\sigma)$ factors through U or V. $\square$

Lemma 3. For every M there are R-linear maps

$$R_q = R_q^M : S_q(M) \to S_{q+1}(M)$$

(natural in M) such that for every $\sigma \in S_q(M)$,

$$\mathrm{sd}_q^M(\sigma) - \sigma = \partial R_q(\sigma) + R_{q-1}(\partial\sigma).$$

Consequently, the map $H_q(M; R) \to H_q(M; R)$ induced by the chain map sd^M is the identity.

Proof. As in the definition of sd_q^M, because of naturality in M all of R_q^M is determined by fixing $R_q^{\Delta_q}(\mathrm{id})$. This will be done by induction on q: For $q = 0$, there is only one choice $\Delta_1 \to \Delta_0$ for $R_o^{\Delta_o}(\mathrm{id})$. And if $R_{q-1}^{\Delta_{q-1}}$ is defined as is required by the lemma, consider $\mathrm{sd}_q^{\Delta_q}(\mathrm{id}) - \mathrm{id} \in S_q(\Delta_q)$: Since

$$\begin{aligned} & \partial(\mathrm{sd}_q^{\Delta_q}(\mathrm{id}) - \mathrm{id} - R_{q-1}^{\Delta_{q-1}}(\partial(\mathrm{id}))) \\ = \; & \mathrm{sd}_{q-1}^{\Delta_{q-1}}(\partial(\mathrm{id})) - \partial(\mathrm{id}) - \partial R_{q-1}^{\Delta_{q-1}}(\partial \mathrm{id})) \\ = \; & R_{q-2}^{\Delta_{q-2}}(\partial\partial \mathrm{id}) \quad \text{(by induction hypothesis)} \\ = \; & 0, \end{aligned}$$

it follows from the contractibility of Δ_q that there exists a $\sigma \in S_{q+1}(\Delta_q)$ such that

$$\partial\sigma = \mathrm{sd}_q^{\Delta_q}(\mathrm{id}) - \mathrm{id} - R_{q-1}^{\Delta_{q-1}}(\partial \mathrm{id})$$

(by the Poincaré lemma). Thus for $R_q^{\Delta_q}$ we can take this σ. (The reader may suspect that in order to obtain $R_q^{\Delta_q}(\Delta_q)$ as a function of q we have to apply the axiom of dependent choices (on q), which is not available in the synthetic context. But this is not so, since the Poincaré lemma does not merely yield the existence of a σ as above: by applying the theorem 2.1 to a fixed contraction of Δ_q we obtain an explicit description of σ!)

Putting these three lemmas together, we complete the proof of the proposition. □

2.5 Corollary. *(Mayer-Vietoris sequence). Assume that $M = U \cup V$. Then there is a long exact sequence*

$$\ldots H_q(U \cap V) \xrightarrow{\delta_*} H_q(U) \oplus H_q(V) \longrightarrow H_q(M) \to H_{q-1}(U \cap V) \ldots$$

deduced from the short exact sequence

$$0 \to S.(U \cap V) \to S.(U) \oplus S.(V) \to S.^{\{U,V\}}(M) \to 0. \qquad \square$$

As a final property of singular homology that we need, we have

2.6 Proposition. *If $M = \amalg_\alpha M_\alpha$ is a disjoint union of a family $(M_\alpha)_\alpha$ indexed by a decidable set $\{\alpha\}$, then*

$$H.(M; R) = \oplus_\alpha H.(M_\alpha; R)$$

where $\oplus_\alpha$ denotes the coproduct of the family $\{H.(M_\alpha; R)\}_\alpha$.

Proof. This follows immediately from the fact that Δ_q is indecomposable (because of the integration axiom) (i.e. if $\Delta_q = A \cup B$, A, B disjoint, then $\Delta_q = A$ or $\Delta_q = B$), and thus any map $\Delta_q \xrightarrow{\sigma} M$ factors through some M_α. □

In section 1 we have seen how the integration axiom allows us to define for any q-form ω on M the integral $\int_\gamma \omega$ along an n-chain $\gamma: I^q \to M$. From this, we can define the integral

$$\int_\sigma \omega$$

for a simplicial q-chain $\sigma: \Delta_q \to M$ in any of the standard ways. Let us quickly describe one version (which seems notationally not too involved) in more detail. For this, we temporarily replace the standard simplices $\Delta_q = [e_o, \ldots, e_q]$ by their isomorphic copies (also called Δ_q)

$$\Delta_q = \{(x_1, \ldots, x_q) \in R^q | 0 \leq x_q \leq \ldots \leq x_1 \leq 1\}.$$

Observe that the faces of this Δ_q are the maps $\varepsilon^i: \Delta_{q-1} \to \Delta_q$, $\varepsilon^o(x_1, \ldots, x_{q-1}) = (x_1, \ldots, x_{q-1}, 0), \varepsilon^i(x_1, \ldots, x_{q-1}) =$ $(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_{q-1}) (1 \leq i \leq q-1)$, and $\varepsilon^q(x_1, \ldots, x_{q-1}) =$ $(1, x_1, \ldots, x_{q-1})$.

There is an obvious (orientation preserving) projection

$$\pi_q: I^q \to \Delta_q, \quad (x_1, \ldots, x_q) \mapsto (x_1, x_1 \cdot x_2, \ldots, x_1 \cdot \ldots \cdot x_q),$$

by use of which we can define the above integral $\int_\sigma \omega$ as

$$\int_\sigma \omega = \int_{\sigma \circ \pi_q} \omega.$$

Writing out the boundary $\partial \pi_q$ as a sum of maps $I^{q-1} \to \Delta_q$ immediately gives that $\partial \pi_q = (\partial \Delta_q) \circ \pi_{q-1}$ modulo some degenerate chains $I^{q-1} \to \Delta_q$ (these are affine chains whose images have a dimension $< q-1$, so the integral over any $q-1$-form vanishes). Consequently, we obtain Stokes' theorem for simplices: if ω is any q–1-form on M and $\gamma: \Delta_q \to M$ is a simplicial q-chain, then

$$\begin{aligned} \textstyle\int_{\partial \gamma} &= \textstyle\int_{\partial \Delta_q} \gamma^*(\omega) = \int_{\partial \Delta_q \circ \pi_{q-1}} \gamma^*(\omega) && \text{(by definition)} \\ &= \textstyle\int_{\partial \pi_q} \gamma^*(\omega) = \int_{\pi_q} \gamma^*(d\omega) && \text{(by cubical Stokes')} \\ &= \textstyle\int_\gamma d\omega && \text{(again by definition).} \end{aligned}$$

Having defined $\int_\sigma \omega$ for $\omega \in \Lambda^q(M)$ and generators $\sigma \in S_q(M)$, we extend this to a map

$$\Lambda^q(M) \times S_q(M) \xrightarrow{\int} R, \quad (\omega, \sigma) \mapsto \int_\sigma \omega$$

which is R-linear in both ω and σ separately. Clearly, this integration is natural in M, in the sense that

$$\int_{f_*(\sigma)} \omega = \int_\sigma f^*(\omega).$$

Because of (the simplicial form of) Stokes' theorem the restriction of the integral to

$$F^q(M) \times Z_q(M) \xrightarrow{\int} R.$$

sends exact forms as well as boundaries to 0, and thus we may pass to quotients to obtain an R-linear map

$$\begin{array}{l} H^q(M) \xrightarrow{I} H_q(M;R)^* = \hom_R(H_q(M;R), R) \\ [\omega] \mapsto ([\sigma]) \mapsto \int_\sigma \omega). \end{array}$$

De Rham's theorem states that in *Sets*, this map I is an isomophism for every manifold M. (Of course, $H^q(M)$ is constructed differently in *Sets*; the synthetic approach that we followed in section 1 is not available in *Sets*.) In the next section, we will show that the map I is an isomorphism for manifolds in $\mathcal{F}$ and $\mathcal{G}$, and we will discuss the relation of this *De Rham theorem in toposes* to the classical case of *Sets*.

3 De Rham's Theorem in $\mathcal{F}$ and $\mathcal{G}$

The main aim of this section is to show that De Rham's theorem holds in the toposes $\mathcal{F}$ and $\mathcal{G}$ that we discussed in chapter III:

3.1 Theorem. *Let $M \in \mathbb{M}$ be a manifold. The R-linear map $H^q(sM) \to H_q(sM;R)^*$ described synthetically above is an isomorphism in $\mathcal{F}$ and in $\mathcal{G}$.*

All the arguments in this section apply to $\mathcal{F}$ in exactly the same way as they do to $\mathcal{G}$. Therefore, we will only discuss the case of $\mathcal{G}$.

Recall that we have a diagram of functors

$$\mathbb{M} \xrightarrow{\;s\;} \mathcal{G} \;\begin{array}{c} \xleftarrow{\;\Delta\;} \\ \xrightarrow[\;B\;]{\;\Gamma\;} \\ \longleftarrow \end{array}\; \mathit{Sets} \qquad \Delta \dashv \Gamma \dashv B$$

In chapter III we have seen that s is full and faithful, and that s preserves transversal pullbacks, open covers, partitions of unity, compactness, etc. Moreover, $\mathcal{G}$ is a model of synthetic differential geometry in the sense that R is a local ring satisfying the Kock-Lawvere axiom, the integration axiom, etc. Consequently, the *synthetic* results of sections IV.1 and IV.2 all hold in $\mathcal{G}$. (In section 2, the only things we used beyond the *usual* axioms of synthetic differential geometry were the assumptions of Archimedeanness of R, and compactness and the open refinement property of Δ_q (see III.3.3, III.3.12). We proved in chapter III that these assumptions hold in $\mathcal{G}$.)

To prove theorem 3.1, and to compare it with the classical De Rham, we start with two lemmas on free R-modules in $\mathcal{G}$ having a constant basis, i.e. modules in $\mathcal{G}$ of the form $\mathrm{Free}_R(\Delta X)$ for some $X \in \mathit{Sets}$ (up to isomorphism).

3.2 Lemma. *If F is a free R-module in $\mathcal{G}$ with constant basis, then every surjection $M \to F$ of R-modules in $\mathcal{G}$ splits.*

Proof. Let $F = \mathrm{Free}_R(\Delta X)$. By applying Γ we obtain a split diagram of vector spaces over $\mathbb{R}$ in Sets

$$\Gamma(M) \;\begin{array}{c} \xrightarrow{\;\Gamma(\alpha)\;} \\ \xleftarrow[\;s\;]{} \end{array}\; \mathrm{Free}_{\mathbb{R}}(X)$$

But we have canonical bijections

$$\begin{array}{cl}
\mathrm{Free}_{\mathbb{R}}(X) \xrightarrow{s} \Gamma(M) & \text{in } \mathrm{mod}_{\mathbb{R}}(\mathit{Sets}) \\ \hline
X \xrightarrow{s} \Gamma(M) & \text{in Sets} \\ \hline
\Delta X \xrightarrow{s} M & \text{in } \mathcal{G} \\ \hline
\mathrm{Free}_R(\Delta X) \xrightarrow{s} M & \text{in } \mathrm{mod}_R(\mathit{Sets})
\end{array}$$

and clearly $\alpha \circ s = \mathrm{id}$. □

3.3 Lemma. *If $F_1 \xrightarrow{\alpha} F_2$ is a homomorphism in $\mathcal{G}$ of free R-modules with constant bases, then* $\mathrm{Im}(\alpha), \ker(\alpha)$*, and* $\mathrm{Cok}(\alpha)$ *are also free with constant bases.*

Proof. Similar to that of 3.2. □

3.4 Proposition. *The class C of objects $M \in \mathcal{G}$ such that $H_q(M; R)$ is a free R-module with constant basis (for each $q \geq 0$) has the following closure properties.*

1. *C contains R^n for each $n \geq 0$.*
2. *Let $M = U \cup V$. If U, V and $U \cap V$ belong to C, then do does M.*
3. *If $M = \coprod_\alpha M_\alpha$ is a disjoint union indexed by a constant set $\{\alpha\}$, and each M_α belongs to C, then so does M.*

Proof. (1) is clear from the Poincaré lemma (corollary 2.3). For (2), it suffices, by the long Mayer-Vietoris sequence to show that if $F_1 \to F_2 \to F_3$ is an exact sequence of R-modules, and F_1, F_3 are free with constant bases, then so is F_2. But this follows immediately by applying lemmas 3.2, 3.3 to the diagram

$$\begin{tikzcd}
F_1 \arrow[rr] \arrow[dr] & & F_2 \arrow[rr] \arrow[dr] & & F_3 \\
& F_1' \arrow[ur] & & F_3' \arrow[ur] &
\end{tikzcd}$$

To prove (3), let $\{\alpha\} = \Delta(I)$. Any constant set (constant sheaf) is decidable, so it follows from proposition 2.8 that if $H_q(M_i; R) \cong \mathrm{Free}_R(\Delta X_i)$ then $H_q(M, R) \cong \mathrm{Free}_R(\Delta(\bigcup_{i \in I} X_i))$ (assuming the X_i are disjoint). □

Before we can show that the class of objects $M \in \mathcal{G}$ such that $I\colon H^q(M) \xrightarrow{\sim} H_q(M; R)^*$ has similar closure properties, we need one more lemma.

3.5 Lemma. *Let $F_1 \to F_2 \to F_3$ be an exact sequence in $\mathcal{G}$ of free R-modules with constant bases. Then the dual sequence $F_3^* \to F_2^* \to F_1^*$ is also exact.*

Proof. (Note that the classical proof of the corresponding fact for vector spaces in *Sets* uses the axiom of choice, which does not hold in $\mathcal{G}$.) Exactness of $F_1 \to F_2 \to F_3$ is equivalent to exactness of $0 \to F_1' \to F_2 \to F_3' \to 0$, where $F_1 \twoheadrightarrow F_1' \rightarrowtail F_2 \twoheadrightarrow F_3' \rightarrowtail F_3$. But by lemma 3.3, if F_1, F_2, F_3 are free on constant bases, so are F_1', F_3'. By lemma 3.2, exactness of $0 \to F_1' \to F_2 \to F_3' \to 0$ is equivalent to $F_2 \cong F_1' \oplus F_3'$. Obviously, it then follows that $F_2^* \cong F_1'^* \oplus F_3'^*$, so it suffices to show that the epi-mono factorization is preserved by dualization; more precisely, that the dual of a surjection is mono (which is clear), and that the dual of a mono is surjective. So let $F_1 \overset{\mu}{\rightarrowtail} F_2$ in $\mathcal{G}$, where $F_i = \mathrm{Free}_R(\Delta X_i)$. Then in *Sets*, there is a linear map $\lambda: \Gamma F_2 \to \Gamma F_1$, i.e. $\lambda: \mathrm{Free}_{\mathbb{R}}(X_2) \to \mathrm{Free}_{\mathbb{R}}(X_1)$, such that $\lambda \circ \Gamma\mu = \mathrm{id}$, and this map can be lifted to an R-linear map $\nu: F_2 \to F_1$ in $\mathcal{G}$ such that $\nu\mu = \mathrm{id}$. So $\mu^*: F_2^* \to F_1^*$ is epic. (More generally, if F is a free R-module with constant basis and M is any R-module in $\mathcal{G}$, then an $\mathbb{R}$-linear map $\varphi: \Gamma F \to \Gamma M$ can be lifted uniquely to an R-linear map $\Phi: F \to M$ with $\Gamma\Phi = \varphi$, as follows immediately from the adjunction $\Delta \dashv \Gamma$.) □

3.6 Proposition. *The class $\mathcal{R}$ of objects $M \in \mathcal{C} \subset \mathcal{G}$ such that $H^q(M) \xrightarrow{I} H_q(M; R)^*$ is an isomorphism (each $q \geq 0$) has the closure properties (1)–(3) as in 3.4, but with the extra assumption in (2) that $\{U, V\}$ is an étale cover of M (see 1.5).*

Proof. (1) follows from the two Poincaré lemmas, for De Rham cohomology and singular homology, (3) follows from the two disjoint union lemmas. (2) is only slightly more involved: it follows from the 5-lemma applied to the diagram obtained from the long Mayer-Vietoris sequence for De Rham cohomology, and its dual for singular homology (which is exact by 3.5):

$$\begin{array}{ccccccccc}
\cdots \to & H^q(U \cap V) & \xrightarrow{d^*} & H^{q+1}(M) & \to & H^{q+1}(U) \oplus H^{q+1}(V) & \to & H^{q+1}(U \cap V) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_q(U \cap V)^* & \xrightarrow{\delta^*} & H_{q+1}(M)^* & \to & H_{q+1}(U)^* \oplus H_{q+1}(V)^* & \to & H_{q+1}(U \cap V)^* & \to \cdots
\end{array}$$

□

Indeed, this diagram is commutative: the only nontrivial square is the one involving the Bockstein homomorphisms d^* and δ^*, and for this case we have the following

Lemma. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence o complexes $\dots \to A^q \xrightarrow{q} A^{q+1} \to \dots$ etc., and let $0 \to \mathcal{C} \xrightarrow{\beta} \mathcal{B} \xrightarrow{\alpha} \mathcal{A} \to 0$ be an exact sequence of complexes $\dots \to \mathcal{A}_{q+1} \xrightarrow{\partial} \mathcal{A}_q \to \dots$. Let $\phi^q_{A\mathcal{A}}: A^q \to \mathcal{A}^*_q$, $\phi^q_{B\mathcal{B}}, \phi^q_{C\mathcal{C}}$ be R-linear pairings such that both

(1)
$$\begin{tikzcd} A^q \arrow[r, "f^q"] \arrow[d, "\psi^q_{A\mathcal{A}}"'] & B^q \arrow[d, "\psi^q_{B\mathcal{B}}"] \\ A^*_q \arrow[r, "\alpha^*_q"] & \mathcal{B}^*_q \end{tikzcd}$$
and similarly for $\psi_B\mathcal{B}$ and $\psi_C C$,

and

(2)
$$\begin{tikzcd} A^q \arrow[r, "d"] \arrow[d, "\psi^q_{A\mathcal{A}}"'] & A^{q+1} \arrow[d] \\ A^*_q \arrow[r, "\alpha^*"] & A^*_{q+1} \end{tikzcd}$$
$\psi^{q+1}_{A\mathcal{A}}$, *and similarly for $\psi_B\mathcal{B}$ and $\psi_C C$,*

commute. Then the diagram

(3)
$$\begin{tikzcd} H^q(C) \arrow[r, "d^*"] \arrow[d, "\psi^q_{C\mathcal{C}}"'] & H^{q+1}(A) \arrow[d, "\psi^{q+1}_{A\mathcal{A}}"] \\ H_q(\mathcal{C})^* \arrow[r, "\delta^*"] & H_{q+1}(\mathcal{A})^* \end{tikzcd}$$

also commutes, where δ^* is the dual of δ_*.

Proof. Recall the definition of the Bockstein maps $d^*: H^q(C) \to H^{q+1}(A)$ and $\delta_*: H_{q+1}(\mathcal{A}) \to H_q(\mathcal{C})$: given $[c] \in H^q(C)$, we find $b \in B^q$, $a \in A^{q+1}$ such that $f(a) = db$ and $g(b) = c$ and put

$d^*[c] = [a]$, while given $[\underline{a}] \in H_{q+1}(\mathcal{A})$ we find $\underline{b} \in \mathcal{B}_{q+1}$ and $\underline{c} \in \mathcal{C}_q$ such that $\alpha(\underline{b}) = \underline{a}$ and $\partial \underline{b} = \beta(\underline{c})$ and put $\delta_*[\underline{a}] = [\underline{c}]$.

Now we compute (in the computation, we only use commutativity of 2) for $B, \mathcal{B}$, but the others are used to define 3) überhaupt): Let $[c] \in H^q(C)$, with a and b as above, and $[\underline{a}] \in H^{q+1}(\mathcal{A})$, with $\underline{b}$ and $\underline{c}$ as above. Then

$$\begin{aligned}
\phi^{q+1}_{A\mathcal{A}}(d^*[c])([\underline{a}]) &= \left[\phi^{q+1}_{A\mathcal{A}}(a)(\alpha(\underline{b}))\right] \\
&= \left[\phi^{q+1}_{B\mathcal{B}}(f(a))(\underline{b})\right] \quad \text{(by 1)} \\
&= \left[\phi^{q+1}_{B\mathcal{B}}(db)(\underline{b})\right] \\
&= \left[\phi^{q}_{b\mathcal{B}}(b)(\partial\underline{b})\right] \quad \text{(by 2)} \\
&= \left[\phi^{q}_{B\mathcal{B}}(b)(\beta\underline{c})\right] \\
&= \left[\phi^{q}_{c\mathcal{C}}(gb)(\underline{c})\right] \quad \text{(by 1)} \\
&= \left[\phi^{q}_{c\mathcal{C}}([c])(\delta_*[\underline{a}]\right) .
\end{aligned}$$

This completes the proof of the lemma, and of proposition 3.6. □

Proof of Theorem 3.1. Let $M \in \mathbb{M}$ be a manifold. We will show that $s(M) \in \mathcal{R} \subset \mathcal{C}$ (see 3.4, 3.6) by an induction on open subsets of M. Let

$$\mathcal{O} = \{U \subset M | U \text{ open}, s(U) \in \mathcal{R}\}.$$

Now assume for the moment that M has a basis $\mathcal{B}$ consisting of relatively compact open sets, which is closed under finite (non-empty) intersections and contained in $\mathcal{O}$. By clauses (2) and (3) of 3.4, 3.6, $\mathcal{O}$ then contains all finite unions of elements of $\mathcal{B}$, and is closed under disjoint (countable) unions. But then $\mathcal{O}$ must contain all the open subsets of M, and in particular M itself, for if U is an open subset of M, we may write $U = \bigcup_{n=0}^{\infty} V_n$, with each V_n relatively compact. Now construct by induction an open cover $\{W_n\}$ of U such that each W_n is a finite union of relatively compact *basic* open sets (i.e. elements of $\mathcal{B}$), hence $W_n \in \mathcal{O}$, such that

$$\overline{V}_k \subset \bigcup_{n=0}^{k} W_n \subset \bigcup_{n=0}^{k} \overline{W}_n \subset \bigcup_{n=0}^{k+1} W_n,$$

and $W_{n+2} \bigcap W_k = \emptyset$ for each $k \leq n$. Then clearly $U = \bigcup_n W_n = \bigcup_{n \text{ even}} W_n \cup \bigcup_{n \text{ odd}} W_n \in \mathcal{O}$, since $(\bigcup_{n \text{ even}} W_n) \cap (\bigcup_{n \text{ odd}} W_n) = \bigcup_{n \text{ even}} ((W_n \cap W_{n-1}) \cup (W_n \cap W_{n+1})) \in \mathcal{O}$.

To complete the proof, we only have to show that M has such a basis $\mathcal{B}$. Clearly, this holds if M is an open subspace of $\mathbb{R}^n$, since we can just take $\mathcal{B}$ to consist of finite intersections of open balls.

For an arbitrary manifold M, it thus follows that $\mathcal{O}$ contains all the subsets U of M which are diffeomorphic to some open of $\mathbb{R}^n$ (where $n = \dim M$), and hence we can take $\mathcal{B}$ to consist of all these U. □

Theorem 3.1 implies the classical version of De Rham's theorem. To see this, we will begin by showing (as we promised to do in section 1) that our notion of form does not differ from the classical one whenever the two make sense.

3.7 Proposition. *For any manifold $M \in \mathbb{M}$, Γ maps the morphism $\Lambda^q(s(M)) \xrightarrow{d} \Lambda^{q+1}(s(M))$ in $\mathcal{G}$ to the map $\Lambda^q(M) \xrightarrow{d} \Lambda^{q+1}(M)$, where the first denotes the interpretation of the synthetic definition of form and exterior differentiation in $\mathcal{G}$, while the second denotes the usual vector-space of forms and exterior differentiation map from classical differential geometry. Moreover, if $M \xrightarrow{f} N$ in $\mathbb{M}$, then similarly Γ maps $s(f)^*: \Lambda^q(sN) \to \Lambda^q(sM)$ in $\mathcal{G}$ to the usual pullback map $f^*: \Lambda^q(N) \to \Lambda^q(M)$.*

Proof. The global sections of $\Lambda^q(sM)$ are the maps $s(M)^{D^q} \to R$ in $\mathcal{G}$ such that in $\mathcal{G}$ it holds that they are homogeneous and alternating. But $s(M)^{D^q}$ is just the q-th iterate of the tangent bundle, $s(M)^{D^q} \cong s(T^q(M))$, while $R = s(\mathbb{R})$, so these are the maps $T^q(M) \to \mathbb{R}$ in $\mathbb{M}$ which are homogeneous and alternating in $\mathcal{G}$. Classically, on the other hand, $\Lambda^q(M)$ is defined as the set of maps $T(M) \times_M \ldots \times_M T(M) \to \mathbb{R}$ (q-fold fibered product) all of whose fibers $T_x(M) \times \ldots \times T_x(M) \to \mathbb{R}$ are alternating and R-linear in each variable separately. Thus, to show $\Gamma(\Lambda^q(s(M)) \cong \Lambda^q(M)$ it suffices to prove *synthetically* (hence in $\mathcal{G}$) that alternating homogeneous maps $M^{D^q} \to R$ are in 1-1 correspondence with alternating maps $M^D \times_M \ldots \times_M M^D \to R$ which are (pointwise) R-linear in each coordinate separately (where M is a manifold, so the fibers $(M^D)_x = T_x(M)$ have a vectorspace structure). In fact by local parametrization it suffices to consider the case $M = R^n$, and for ease of notation we will take $q = 2$. Suppose given a map $(R^n)^{D^2} \xrightarrow{\omega} R$ which is homogeneous and alternating. By the Kock-Lawvere axiom, each $f: D^2 \to R^n$ is given as

$$f(x, y) = \underline{a} + x \cdot \underline{b} + y \cdot \underline{c} + xy \cdot \underline{d}$$

for unique vectors $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in R^n$. To show that ω is determined by its restriction to $(R^n)^D \times_{R^n} (R^n)^D$ (consisting of such f with $\underline{d} = 0$) we show that $\omega(f)$ does not depend on $\underline{d}$. Indeed, writing $\omega_{\underline{a}}$ for the restriction of ω to the fiber over $\underline{a}$, $\omega_{\underline{a}}(\underline{b}, \underline{c}, \underline{d})$ for $\omega(f)$, we get for all

$\underline{b}, \underline{c}, \underline{d} \in R^n$

$$\omega_{\underline{a}}(\underline{b}, \underline{c}, \underline{d}) = -\omega_{\underline{a}}(\underline{c}, \underline{b}, \underline{d}) \quad \text{(alternating)}$$

and hence

$$\omega_{\underline{a}}(0, 0, \underline{d}) = 0.$$

Moreover, for fixed $\underline{c}$ and $\underline{b}$ respectively, $\omega_{\underline{a}}(\underline{b}, -, -)$ and $\omega_{\underline{a}}(-, \underline{c}, -)$ are R-linear maps $(R^n)^2 \to R$ (R-linearity follows from homogeneity, cf. V.1.5 below), so

$$\begin{aligned} \omega_{\underline{a}}(\underline{b}, \underline{c}, \underline{d}) - \omega_{\underline{a}}(\underline{b}, \underline{c}, 0) &= \omega_{\underline{a}}(\underline{b}, 0, \underline{d}) \\ &= \omega_{\underline{a}}(\underline{b}, 0, 0) + \omega_{\underline{a}}(0, 0, \underline{d}) \\ &= 0 + 0 = 0. \end{aligned}$$

The fact that Γ preserves exterior differentiation is now immediate from the fact that both $\Gamma(d)$ and the *classical* d satisfy Stokes' theorem (since Γ trivially preserves the boundary operator ∂). The case of f^* is obvious. □

Thus, the classical representation theorem that every form on M is locally of the form $\sum f(\underline{x}) dx_{i_1} \wedge \ldots \wedge dx_{i_n}$ holds in $\mathcal{G}$ for all objects of the form $s(M)$. (In fact this can also be shown directly by a synthetic argument.)

3.8 Corollary. *(Classical De Rham) For any manifold $M \in \mathbb{M}$, the canonical map*

$$\begin{aligned} &H^q(M) \to H_q(M; \mathbb{R})^* = \hom_{\mathbb{R}}(H_q(M; \mathbb{R}), \mathbb{R}) \\ &[\omega] \mapsto ([\gamma] \mapsto \textstyle\int_\gamma \omega) \end{aligned}$$

is an isomorphism.

Proof. We have already observed that the global sections functor Γ preserves the notions of form and exterior derivative (cf. the proposition above), and also, trivially, Γ preserves the notion of q-simplex and boundary of such. Thus by exactness of Γ (Γ has both adjoints) Γ also preserves $H^q(M)$ and $H_q(M; R)^*$, i.e. $\Gamma(H^q(sM) \cong H^q(M)$, $\Gamma(\hom_R(H_q(M; R), R)) \cong \hom_{\mathbb{R}}(H_q(M; \mathbb{R}), \mathbb{R})$. So the corollary follows by applying Γ to the theorem 3.1. □

From the fact that Γ preserves $H^q(M)$, $H^q(M, R)$, we immediately obtain the following two comparison results.

3.9 First Comparison Theorem. *For any manifold M and any set X*

$$H_q(M;\mathbb{R}) \cong \mathrm{Free}_{\mathbb{R}}(X) \textit{ in Sets iff } H_q(sM, R) \cong \mathrm{Free}_R(\Delta X) \text{in } \mathcal{G}.$$

Proof. The adjunctions $\Delta \dashv \Gamma \dashv B$ lift to adjunctions between $\mathrm{Mod}_R(\mathcal{G})$ and $\mathrm{Mod}_{\mathbb{R}}(\mathit{Sets})$. So in particular, Γ preserves free R-modules. Since $\Gamma\Delta X = X$, $\Leftarrow$ now follows. For $\Rightarrow$ suppose $H_q(M,\mathbb{R}) \cong \mathrm{Free}_{\mathbb{R}}(X)$. We know by 3.6 (see 3.4) that $H_q(sM, R) \cong \mathrm{Free}_R(\Delta Y)$ for some $Y \in \mathit{Sets}$. By $\Leftarrow$, $\mathrm{Free}_{\mathbb{R}}(Y) = \mathrm{Free}_{\mathbb{R}}(X)$. Hence $Y \cong X$. □

3.10 Second Comparison Theorem. *For any manifold M and any set X,*

$$H^q(M) \cong \mathbb{R}^X \textit{ in Sets iff } H^q(sM) \cong R^{\Delta X} \textit{ in } \mathcal{G}.$$

Proof. Just notice that $(\mathrm{Free}_R(\Delta X))^* \cong R^{\Delta X}$, and combine 3.9 with 3.6. □

Now that we have established the validity of De Rham's theorem for the topos $\mathcal{G}$, it is natural to ask whether De Rham's theorem holds in $\mathcal{G}$ for other cohomologies. We will briefly consider two examples of this question: the case of Čech cohomology, and the case of singular cohomology.

We quickly recall the classical verson of De Rham's theorem for Čech cohomology: Let $M \in \mathbb{M}$ be a manifold, and let $\mathcal{U} = \{U_\alpha\}$ be a *good* cover of M, that is, an open cover such that all nonempty finite intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}$ are diffeomorphic to some $\mathbb{R}^n$. Assume that the index set $\{\alpha\}$ is linearly ordered. The Čech complex (with coefficients in $\mathbb{R}$) is the complex

$$C^o(\mathcal{U},\mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U},\mathbb{R}) \xrightarrow{\delta} C^2(\mathcal{U},\mathbb{R}) \to \ldots,$$

where $C^n(\mathcal{U},\mathbb{R})$ is the vectorspace $\Pi_{\alpha_o<\ldots<\alpha_n} F^o(U_{\alpha_o\ldots\alpha_n},\mathbb{R})$ over $\mathbb{R}$ ($F^o(U_{\alpha_o\ldots\alpha_n},\mathbb{R})$ denotes the vectorspace of locally constant functions $U_{\alpha_o} \cap \ldots \cap U_{\alpha_n} \to \mathbb{R}$), and we define the boundary operator $\delta: C^n(\mathcal{U},\mathbb{R}) \to C^{n+1}(\mathcal{U},\mathbb{R})$ as follows: if $f = \{f_{\alpha_o\ldots\alpha_n}\} \in C^n(\mathcal{U},\mathbb{R})$, then

$$(\delta f)_{\alpha_o\ldots\alpha_{n+1}} = \sum_{i=0}^{n+1}(-1)^i f_{\alpha_o\ldots\widehat{\alpha}_i\ldots\alpha_{n+1}}.$$

The cohomology of this complex is called the Čech cohomology of the good cover $\mathcal{U}$, and is denoted by $H^{\cdot}(\mathcal{U}, \mathbb{R})$.

De Rham's theorem for Čech-cohomology says that in this situation there is a canonical isomorphism

$$(*) \qquad H^{\cdot}(M) \xrightarrow{\sim} H^{\cdot}(\mathcal{U}, \mathbb{R})$$

Consequently, $H^{\cdot}(\mathcal{U}, \mathbb{R})$ does not depend on the good cover $\mathcal{U}$. Another immediate corollary is that since compact manifolds have finite good covers, the De Rham cohomology of such is finite dimensional. The proof of the existence of the isomorphism (*) given by A. Weil is completely constructive and explicit, and hence is valid in the synthetic context: cf. section 5 below. Consequently, since the embedding $s\colon \mathbb{M} \hookrightarrow \mathcal{G}$ preserves the ingredients of Weil's proof (notably good open covers, and partitions of unity), we obtain the analogue of theorem 3.1 for Čech cohomology instead of the dual of singular homology.

3.11 Theorem. *For any $M \in \mathbb{M}$ and any good cover $\mathcal{U}$ of M, the canonical map*

$$H^q(sM) \to H^q(s(\mathcal{U}), R)$$

is an isomorhism in the topos $\mathcal{G}$.

3.12 Remark. As said, the proof given in the appendix is purely constructive. In this respect, theorem 3.11 differs from theorem 3.1, since the dualization involved in 3.1 forces us to use the axiom of choice somewhere. Since choice is not available in $\mathcal{G}$, we have *to take it from Sets*, as is clearly illustrated by the proof of 3.2, for example.

Turning to singular cohomology, it seems an open problem whether De Rham's theorem holds in $\mathcal{G}$, at least when we interpret singular cohomology as the cohomology of the complex

$$\ldots \to \mathrm{Hom}_R(S_q(M), R) \xrightarrow{\partial^*} \mathrm{Hom}_R(S_{q+1}(M), R) \to \ldots$$

which is the dual of the complex $\to S_{q+1}(M) \to S_q(M) \to \ldots$ of section 2. In this case, some form of the axiom of choice seems to be needed to establish the result. The problem here is that the dual of the short exact sequence

$$0 \to S_q(U \cap V) \to S_q(U) \oplus S_q(V) \to S_q{}^{\{U,V\}}(M) \to 0$$

of section 2, which is

$$0 \to R^{M^{\Delta_q}} \to R^{U^{\Delta_q}} \times R^{V^{\Delta_q}} \to R^{(U \cap V)^{\Delta_q}} \to 0,$$

is not necessarily exact: an arbitrary function $(U \cap V)^{\Delta^q} \to R$ cannot in general be extended to a function $U^{\Delta^q} + V^{\Delta^q} \to R$, so the sequence is not epic on the right.

A way of circumventing this problem in the topos $\mathcal{G}$ is to replace the sheaf $s(M)^{\Delta^q} \in \mathcal{G}$ by the constant sheaf $\Delta(M^{\Delta^q})$ (recall that Δ: *Sets* $\to \mathcal{G}$ is the constant functor). Thus, let $S_{\Delta,q}(sM)$ be the free R-module in $\mathcal{G}$ generated by $\Delta(M^{\Delta^q})$. $S_{\Delta,q}(sM)$ has a constant basis, so (from lemmas 1,2 of section 4) we get a *split* exact sequence in $\mathcal{G}$,

$$0 \to S_{\Delta,q}(s(U \cap V)) \to S_{\Delta,q}(sU) \oplus S_{\Delta,q}(sV) \to S_{\Delta,q}^{\{U,V\}}(sM) \to 0$$

and therefore its dual in $\mathcal{G}$,

$$0 \to S_{\Delta,q}^{\{U,V\}}(sM)^* \to (S_{\Delta,q}(sU) \oplus S_{\Delta,q}(sV))^* \to S_{\Delta,q}(s(U \cap V))^* \to 0$$

is exact as well. Consequently, if we let $H^q_\Delta(sM)$ denote the cohomology of the complex

$$\ldots \to S_{\Delta,q}(sM) \xrightarrow{\partial^*} S_{\Delta,q+1}(sM) \to \ldots$$

we obtain a long Mayer-Vietoris sequence:

3.13 Lemma. *Let $M = U \cup V$ in $\mathbb{M}$ as before. Then in $\mathcal{G}$ there is a long exact sequence*

$$\ldots H^q_\Delta(sM) \to H^q_\Delta(sU) \oplus H^q(sV) \to H^q_\Delta(s(U \cap V)) \to H^{q+1}_\Delta(sM) \ldots$$

Proof. As before we need to show that the restriction to $S_{\Delta,q}^{\{U,V\}}(sM)$ instead of $S_{\Delta,q}(sM)$ in the complex still gives the same cohomology. For this, we only need to observe that the proof for singular homology by barycentric subdivision dualizes, since if K is a homotopy between chain maps, then trivially so is its dual. (Recall that if $f, g: A \to B$ are chain maps, say with $A^q \xrightarrow{d} A^{q+1}$, a homotopy $K: f \simeq g$ is a sequence of maps $A^{q+1} \xrightarrow{K^q} B^q$ such that $f^q - g^q = K^q d + dK^{q-1}$.)$\square$

Now we obtain exactly as before,

3.14 Theorem. *Let $M \in \mathcal{M}$ be a manifold. Then the canonical R-linear map*

$$\begin{array}{rcl} H^q(sM) & \to & H^q_\Delta(sM) \\ [\omega] & \mapsto & [\gamma \mapsto \int_\gamma \omega] \end{array}$$

is an isomorphism.

The schizophrenic character of the isomorphism is apparent: we integrate *internal* (*variable*, in $\mathcal{G}$) forms ω over *external* (constant, from *Sets*) chains γ. This was reflected in the proof: the splitting in the lemma above comes from *Sets*, and similarly the homotopy equivalence $S_{\Delta,q}^{\{U,V\}}(M)^* \to S_{\Delta,q}(M)^*$ was brought into $\mathcal{G}$ by dualizing the usual constant homotopy equivalence $S_{\Delta,q}^{\{U,V\}} \hookrightarrow S_{\Delta,q}$ coming *from outside*, from *Sets*.

4 From Synthetic to Classical Analysis

In the introduction, we pointed out that synthetic differential geometry is inconsistent with classical logic. Nevertheless, synthetic differential geometry should be regarded as an *extension* of classical differential geometry, and not as an incompatible alternative to it. This becomes clear when we consider *models* of synthetic differential geometry. Thus, the *usual* category of classical manifolds $\mathbb{M}$ is fully and faithfully embedded in the models $\mathcal{F}$ and $\mathcal{G}$, so classical facts about manifolds (that can be stated in the language of $\mathbb{M}$) are true for manifolds, i.e. objects of the form $s(M), M \in \mathbb{M}$, in $\mathcal{F}$ and in $\mathcal{G}$. Conversely, when a result is proved synthetically, it holds in $\mathcal{F}$ and $\mathcal{G}$, and we may apply the global sections functor Γ (or a similar functor) to obtain a related, in fact often *the same*, result in *Sets*, i.e. in the classical context. For example, our synthetic proof of Stokes' theorem 1.1 implies that Stokes' theorem holds in $\mathcal{G}$. As we have seen in the previous section, $\mathcal{G} \xrightarrow{\Gamma} Sets$ *preserves* the complexes $\ldots \to \Lambda^n(sM) \xrightarrow{d} \Lambda^{n+1}(sM) \to \ldots$ and $\ldots \to \Gamma_{n+1}(sM) \xrightarrow{\partial} \Gamma_n(sM) \to \ldots$, so by applying Γ to the commutative square

$$\begin{array}{ccc}
\Lambda^n(sM) \times \Gamma_{n+1}(sM) & \xrightarrow{\mathrm{id} \times \partial} & \Lambda^n(sM) \times \Gamma_n(sM) \\
\Big\downarrow {\scriptstyle d \times \mathrm{id}} & & \Big\downarrow {\scriptstyle \int} \\
\Lambda^{n+1}(sM) \times \Gamma_{n+1}(sM) & \xrightarrow{\int} & R
\end{array}$$

we obtain the classical version of Stokes' theorem.

However, if one wishes to translate synthetic *proofs*, rather than *theorems*, into the classical language, more work needs to be done: essentially all infinitesimal structures have to be replaced by limits of finite structures. These limits are not only more complicated to work with (frequently one has to introduce coordinates), but they often force one to make *mathematical detours* in proofs. Indeed, when dealing with infinitesimal structures directly, one can often just *formally integrate*, and *actual integration* can be avoided. For example, in our proof of theorem 1.3 we worked with infinitesimal cubes σ and ρ, and avoided an application of Stokes' theorem in that way, simply by the fact that Stokes' holds by definition for infinitesimal cubes. In order to translate the proof of theorem 1.3 into a classical proof (for manifolds) one can replace σ and ρ by *finite* cubes, and apply Stokes' to $\int_\rho d\omega$. Indeed, to prove that $F_1^*(\omega) - F_o^*(\omega) = K_{n+1}(d\omega)+dK_n\omega$ (see 1.3), it suffices to show that for every *finite* cube $\gamma: I^n \to M$, $\int_\gamma(F_1^*(\omega) - F_o^*(\omega)) = \int_\gamma K_{n+1}(d\omega) + dK_n\omega$, essentially because manifolds have property (E), see 1.8.

As a useful illustration, let us actually translate the proof of 1.3 into classical analysis. We encourage the reader to compare the synthetic proof carefully with the classical one given below, and to make similar comparisons for other synthetic proofs from this chapter and the next.

4.1 The Homotopy Invariance of De Rham Cohomology.

We discuss the classical proof and work in *Sets*, with classical (smooth) manifolds. Let M be a manifold, and let $\Lambda^p(M)$ denote the (real) vectorspace of smooth p-forms on M. So an element $\omega \in \Lambda^p(M)$ is a map

$$T(M) \times_M \ldots \times_M T(M) \xrightarrow{\omega} R \quad (p\text{-fold fibered product})$$

satisfying the usual conditions. $\Lambda^o(M)$ is the set of smooth maps $M \to \mathbb{R}$, and we put $\Lambda^p(M) =$ the zero vectorspace, for $p < 0$. Exterior differentiation gives a complex

$$\ldots \to \Lambda^{p-1}(M) \xrightarrow{d^{p-1}} \Lambda^p(M) \xrightarrow{d^p} \Lambda^{p+1}(M) \to \ldots,$$

and the p^{th} De Rham cohomology space of M is the vectorspace

$$H^p(M) = \mathrm{Ker}(d^p)/\mathrm{Im}(d^{p-1})$$

of *closed p-forms modulo exact p-forms.* We write $H^{\cdot}(M)$ for the sequence $\{H^p(M)\}_p$ of vectorspaces. A smooth map $M \xrightarrow{f} N$ of manifolds induces a linear map $f^* = (f^*)^p: \Lambda^p(N) \to \Lambda^p(M)$ (by composing with the obvious map

$$T(M) \times_M \ldots \times_M T(M) \xrightarrow{df \times_M \cdots_M \times df} T(N) \times_N \ldots \times_N T(N)),$$

which commutes with exterior differentiation d. So we get a map $H^p(f): H^p(N) \to H^p(M)$ for each p; that is, we get a sequence of maps $H^{\cdot}(f): H^{\cdot}(N) \to H^{\cdot}(M)$. Homotopy invariance of De Rham cohomology is expressed by the following theorem.

Theorem. *If f and $g: M \to N$ are homotopic maps, then $H^{\cdot}(f) = H^{\cdot}(g)$. In particular, if M and N are homotopy equivalent, then $H^{\cdot}(M) \cong H^{\cdot}(N)$.*

The theorem is proved by showing that if $F: M \times I \to N$ is a (smooth) homotopy from $f = F_o$ to $g = F_1$, we can find for every closed p-form ω on N a p–1-form λ on M such that $d\lambda = F_1^*(\omega) - F_o^*(\omega)$. As usual, this immediately follows from the existence of a chain-homotopy K from F_o^* to F_1^*, i.e. a sequence of linear maps $K^p: \Lambda^p(N) \to \Lambda^{p-1}(M)$ such that for all p, all $\omega \in \Lambda^p(N)$,

(1) $F_1^*(\omega) - F_o^*(\omega) = d^{p-1}K^p\omega + K^{p+1}(d^p\omega)$.

Such a map K is defined as follows. For a p-form $\omega: TN \times_N \ldots \times_N TN \to \mathbb{R}$ on N, $K\omega = K^p\omega$ will be a map $TM \times_M \ldots \times_M TM \to R$ (p–1-fold fibered product). Now choose $(x, v_1, \ldots, v_{p-1}) \in TM \times_M \ldots \times_M TM$, $x \in M, v_i \in T_x(M)$, and let

$$g_{x,\underline{v}}: I \to TN \times_N \ldots \times_N TN$$

be the map

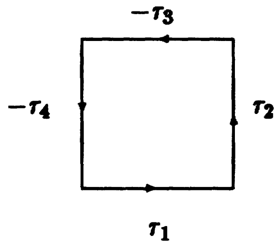
$$g_{x,\underline{v}}(t) = (F^x(t), (dF^x)_t(1), (dF_t)_x(v_1), \ldots, (dF_t)_x(v_{p-1}))$$

(Here $F^x: I \to N$ is the map $F^x(t) = F(x,t)$). Indeed, the righthand side is an element of $TN \times_N \ldots \times_N TN$ (p times): $y = F^x(t) \in N$, and $(dF^x)_t$ is a linear map $T_t(I) \to T_y(N)$, i.e. $R \to T_y(N)$, which corresponds to a vector $(dF^x)_t(1) \in T_y(N)$; also $F_t: M \to N$ defines a linear map $(dF_t)_x: T_x(M) \to T_y(N)$, so $(dF_t)_x(v_i) \in T_y(N)$. Now put

$$K\omega(x, \underline{v}) = \int_0^1 \omega(g_{x,\underline{v}}(t))dt.$$

For fixed x, $K\omega(x, -)$ is alternating and separately linear in $\underline{v}$, so $K\omega$ defines a p–1-form on N, and from the explicit definition we have given it is clear that $K\omega$ is smooth, i.e. $K\omega \in \Lambda^{p-1}(M)$. We will now verify that (1) holds. For notational convenience, we assume that $p = 2$. Let $\tau: I^2 \to M$ be any 2-chain on M, and write

$$(2) \qquad \int_\tau dK\omega = \int_{\partial\tau} K\omega = \int_{\tau_1} K\omega + \int_{\tau_2} K\omega - \int_{\tau_3} K\omega - \int_{\tau_4} K\omega$$



(where $\tau_1 = \tau(-,0), \tau_2 = \tau(1,-), \tau_3 = \tau(-,1), \tau_4 = \tau(0,-)$). We now define a 3-chain $\tau: I^3 \to M$ by

$$\rho(x_1, x_2, x_3) = F_{x_1}(\tau(x_2, x_3)),$$

and compute $\int_\rho d\omega$ in two ways.

On the one hand, by definition (writing $\underline{t} = (t_1, t_2, t_3)$, and $J_{\underline{t}}(\rho) = \left(\frac{\partial\rho}{\partial t_1}(\underline{t}), \frac{\partial\rho}{\partial t_2}(\underline{t}), \frac{\partial\rho}{\partial t_3}(\underline{t})\right)$ considered as an element of $T_p(\underline{t})(N)^3)$)

$$\begin{aligned} & \textstyle\int_\rho d\omega = \int_0^1\int_0^1\int_0^1 [\underline{t} \mapsto d\omega(\rho(\underline{t}), J_{\underline{t}}(\rho))]\, dt_1 dt_2 dt_3 = \\ (3) \quad & \textstyle= \int_0^1\int_0^1 \left[(t_2, t_3) \mapsto \int_0^1 \{t_1 \mapsto d\omega(g_{\tau(t_2,t_3)}, J_{(t_2,t_3)}(t_1)\} dt_1\right] dt_2 dt_3 \\ & \textstyle= \int_0^1\int_0^1 \left[(t_2, t_3) \mapsto K(d\omega)(\tau(t_2,t_3), J_{(t_2,t_3)}(\tau))\right] dt_2 dt_3 \\ & \textstyle= \int_\tau K(d\omega). \end{aligned}$$

On the other hand, by Stokes theorem,

$$(4) \quad \textstyle\int_\rho d\omega = \int_{\partial\rho}\omega = \int_f \omega - \int_{ba}\omega + \int_\ell \omega - \int_r \omega + \int_t \omega - \int_{bo}\omega$$

(where $\partial\rho = (f - ba) + (\ell - r) + (t - bo)$, f refers to the restriction of ρ to the front of a cube as in the picture occurring in the proof of 1.3; ba to the back, etc).

Now clearly,

$$(5) \quad \textstyle\int_f \omega = \int_{F_1 \circ \tau}\omega = \int_\tau F_1^*(\omega), \quad \text{and} \quad \int_{ba}\omega = \int_\tau F_o^*(\omega).$$

We claim that also

$$(6) \quad \textstyle\int_\ell \omega = \int_{\tau_4} K\omega, \int_r \omega = \int_{\tau_2} K\omega, \int_t \omega = \int_{\tau_3} K\omega, \int_{bo}\omega = \int_{\tau_1} K\omega.$$

Note that from (2)–(6) we get that

$$\int_\tau K(d\omega) = \int_{\partial\rho} \omega = \int_\tau F_1^*(\omega) - F_o^*(\omega) - \int_{\partial\tau} K\omega,$$

or $\int_\tau (K(d\omega) + dK\omega) = \int_\tau F_1^*(\omega) - F_o^*(\omega)$, and hence since τ is arbitrary, $F_1^*(\omega) - F_o^*(\omega) = Kd\omega + dK\omega$ (cf.(1) above).

So to complete the proof, we only need to verify (6). We will do the first equality, the others are, of course, analogous. ℓ = *left part of* $\partial\rho\colon I^2 \to N$ is the 2-chain $(s,t) \overset{\ell}{\mapsto} F_s(\tau(0,t))$, so by definition

$$\int_\ell \omega = \int_0^1 \int_0^1 \left[(s,t) \mapsto \omega(F_s(\tau(0,t)), \frac{\partial \ell}{\partial s}(s,t), \frac{\partial \ell}{\partial t}(s,t))\right] dsdt.$$

But by the chain rule, $\frac{\partial \ell}{\partial t}(s,t) := (d\ell)_{(s,t)}(0,1) = (dF_s)_{\tau(o,t)} d\tau_{(0,t)}(0,1)$. Put $x = \tau(0,t), v = (d\tau)_{(0,t)}(0,1) = \frac{\partial \tau}{\partial t}(0,t) - (d\tau_4)_t(1)$; then

$$\begin{aligned} \textstyle\int_{\tau_4} \lambda &= \textstyle\int_0^1 [t \mapsto K\omega(\tau_4(t), (d\tau_4)_t(1))]\, dt \\ &= \textstyle\int_0^1 \left[t \mapsto \int_0^1 \{s \mapsto \omega(g_{x,v}(s))\} ds\right] dt. \end{aligned}$$

So from the definition of $g_{x,v}$, it is clear that $\int_\ell \omega = \int_{\tau_4} \lambda$. This completes the proof. □

There are other ways of translating theorems that hold in $\mathcal{F}$ or $\mathcal{G}$ into results of classical analysis, than just applying the global sections functor Γ. For example, if X is a manifold (or more generally, an object of $\mathbb{E}$), the category $\mathcal{O}(X)$ of open subsets of X and inclusions is a subcategory of $\mathbb{G}$, and if $U = \bigcup_{i \in I} U_i$ in $\mathcal{O}(X)$, then $\{U_i \to U\}_{i \in I}$ is a cover in $\mathbb{G}$. Consequently, every object of $\mathcal{G}$, i.e. a sheaf $\mathbb{G}^{\mathrm{op}} \to$ *Sets*, restricts to a sheaf on X in the usual sense. In the remaining part of this section, we will state some versions of De Rham's theorem *with smooth parameters*, and show how these follow immediately from the results proved for $\mathcal{G}$ in section 3. To this end, we will give a description of the De Rham cohomology groups and the (duals of the) singular homology groups, purely in the classical terminology of sheaves of smooth modules over a space X of parameters, and independently from toposes such as $\mathcal{F}$ and $\mathcal{G}$. Those readers who are not familiar with the terminology of classical sheaf theory are referred to Godement (1964), Tennison (1975).

Let the manifold $X \in \mathbb{M}$ be our space of parameters, and let $\mathbb{R}_\infty$ be the sheaf on X of germs, of smooth real-valued functions, i.e. $\mathbb{R}_\infty(R) = C^\infty(U, \mathbb{R})$ for each open $U \subset X$, with obvious restrictions. Starting from this ringed space $(X, \mathbb{R}_\infty)$, we shall construct, for each

$M \in \mathbb{M}$, several $\mathbb{R}_\infty$-modules on X.

First of all, there is the sheaf $\underline{\Lambda}^q(M)$ on X of (smooth) q-forms on M depending (smoothly) on parameters from X:

$$\underline{\Lambda}^q(M)(U) = \text{the set of } q\text{-forms on } U \times M \text{ which locally are of the form } \textstyle\sum_{i_1 < \ldots < i_q} f_{i_1 \ldots i_q}(u, \underline{m})\, dm_{i_1} \wedge \ldots \wedge dm_{i_q}$$

(with all the functions $f_{i_1 \ldots i_q}$ smooth). Clearly, $\underline{\Lambda}^q(M)$ is indeed a sheaf on X, with obvious restrictions. Furthermore, exterior differentiation (with respect to the $\underline{m}$-variables only) defines natural transformations

$$\underline{\Lambda}^q(M) \xrightarrow{d^{q+1}} \underline{\Lambda}^{q+1}(M), \;\; d^{q+1} = \{d^{q+1}{}_U\}_U$$

for each q, thus giving rise to a sheaf complex.

We now wish to form the sheaf cohomology of this sheaf complex $\underline{\Lambda}^{\cdot}(M)$. So let us define, for each open $U \subseteq X$,

$$\begin{array}{rcl} \mathcal{F}^q(M)(U) & = & \operatorname{Ker}(d_U^{q+1}) \\ \mathcal{E}^q(M)(U) & = & \operatorname{Im}(d_U^q) \\ \mathcal{H}^q(M)(U) & = & \mathcal{F}^q(M)(U)/\mathcal{E}^q(M)(U). \end{array}$$

Fortunately, to define the sheaf cohomology we do not have to pass to the associated sheaves of $\mathcal{E}$ or $\mathcal{H}$, since

4.2 Proposition. *$\mathcal{F}^q(M), \mathcal{E}^q(M)$, and $\mathcal{H}^q(M)$ are sheaves on X, and carry a natural $\mathbb{R}_\infty$-Module structure.*

Proof. $\underline{\Lambda}^q(M)$ is a sheaf for each q, and it has an obvious $\mathbb{R}_\infty$-Module structure. This structure is inherited by $\mathcal{F}^q(M), \mathcal{E}^q(M)$ and $\mathcal{H}^q(M)$, so we only need to show that these are sheaves. For $\mathcal{F}^q(M)$, this is obvious from the fact that $\underline{\Lambda}^q(M)$ is a sheaf.

And $\mathcal{E}^q$ is a sheaf, essentially because a form which is locally exact is globally so by the existence of partitions of unity. More explicitly, if $\{U_\alpha\}_\alpha$ is an open cover of U and we are given a compatible family $\{\omega_\alpha\}, \omega_\alpha \in \underline{\Lambda}^q(M)(U_\alpha)$, such that each ω_α is of the form $d\lambda_\alpha$ for some $\lambda_\alpha \in \underline{\Lambda}^{q-1}(M)(U_\alpha)$, then if $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$, we may put

$$\begin{array}{rcl} \omega & = & \Sigma_\alpha \rho_\alpha \cdot \omega_\alpha \in \Lambda^q(M)(U) \\ \lambda & = & \Sigma_\alpha \rho_\alpha \cdot \lambda_\alpha \in \Lambda^{q-1}(M)(U) \end{array}$$

and it trivially follows that $d\lambda = \omega$.

Finally, to show that $\mathcal{H}^q(M)$ is a sheaf, choose a compatible family $[\omega_\alpha] \in \mathcal{H}^q(M)(U_\alpha)$ for a cover $\{U_\alpha\}$ of U, i.e. for some forms $\lambda_{\alpha\beta} \in$

$\underline{\Lambda}^{q-1}(M)(U_\alpha \cap U_\beta)$

$$\omega_\alpha|_{U_\alpha \cap U_\beta} - \omega_\beta|_{U_\alpha \cap U_\beta} = d\lambda_{\alpha\beta}.$$

Again take a partition of unity $\{\rho_\alpha\}$ as above, and let $\omega = \Sigma_\alpha \rho_\alpha \cdot \omega_\alpha$. We complete the proof by showing that for each β,

$$[\omega]|_{U_\beta} = [\omega_\beta] \text{ in } \mathcal{H}^q(M)(U_\beta).$$

Indeed, since $\mathcal{E}^q(M)$ has been shown to be a sheaf, it suffices to check that $\omega|_{U_\alpha \cap U_\beta} - \omega|_{U_\alpha \cap U_\beta}$ is exact on each element of the cover $\{U_\alpha \cap U_\beta\}_\alpha$ of U_β. But

$$\begin{aligned} \omega|_{U_\beta \cap U_\alpha} - \omega_\beta|_{U_\alpha \cap U_\beta} &= \textstyle\sum_\alpha \rho_\alpha \cdot \omega_\alpha|_{U_\alpha \cap U_\beta} - \sum_\alpha \rho_\alpha \cdot \omega_\beta|_{U_\alpha \cap U_\beta} \\ &= \textstyle\sum_\alpha \rho_\alpha \cdot (\omega_\alpha|_{U_\alpha \cap U_\beta} - \omega_\beta|_{U_\alpha \cap U_\beta}) \\ &= \textstyle\sum_\alpha \rho_\alpha d\lambda_{\alpha\beta} \\ &= d\textstyle\sum_\alpha \rho_\alpha \lambda_{\alpha\beta} \end{aligned}$$

□

We now define the singular homology $\mathbb{R}_\infty$-Modules, starting from the sheaf $S_q(M)$ of (smooth) simplicial q-chains which vary smoothly along the parameterspace X: $S_q(M)$ is defined to be the associated sheaf of the presheaf which assigns to an open $U \subseteq X$ the free $\mathbb{R}_\infty(U)$-module generated by the C^∞-maps $U \times \Delta_q \to M$. Thus, elements of $S_q(M)$ locally look like formal expressions of the form

$$\sum_{i=1}^{n} a_i(u)\sigma_i(u,t)$$

with both $a_i: U \to \mathbb{R}$ and $\sigma_i: U \times \Delta_q \to M$ smooth.

Observe that since every (open) subspace of M is paracompact, the process of passing from the given presheaf (which is separated) to its associated sheaf coincides with the process of closing off under partitions of unity. Thus, for example, if $\{U_\alpha\}$ covers U and for each α we are given formal expressions $\sum_{i=1}^{n_\alpha} a_i^\alpha(u)\sigma_i^\alpha(u,t)$ as elements of the presheaf over U_α, then $S_q(M)(U)$ contains an element which we may denote by

$$\sum_\alpha \sum_{i+1}^{N_\alpha} \rho_\alpha(u) \cdot a_i(u) \cdot \sigma_i^\alpha(u,t)$$

for a partition of unity $\{\rho_\alpha\}$ subordinated to $\{U_\alpha\}$.

At the presheaf level there is an obvious natural transformation induced by composition with the boundary chain $\partial: \Delta_{q-1} \to \Delta_q$, and this yields a sheaf complex

$$\to \mathcal{S}_{q+1}(M) \xrightarrow{\partial_{q+1}} \mathcal{S}_q(M) \xrightarrow{\partial_q} \mathcal{S}_{q-1}(M) \to \dots \qquad \partial_q = \{(\partial_q)_U\}_U.$$

To define the singular homology sheaves, we define presheaves $\mathcal{Z}_q(M)$, $\mathcal{B}_q(M)$ and $\mathcal{H}_q(M)$ by

$$\begin{array}{rcl} \mathcal{Z}_q(M)(U) & = & \mathrm{Ker}((\partial_q)_U) \\ \mathcal{B}_q(M)(U) & = & \mathrm{Im}((\partial_{q+1})_U) \\ \mathcal{H}_q(M)(U) & = & \mathcal{Z}_q(M)(U)/\mathcal{B}_q(M)(U). \end{array}$$

By the remark on closure under partitions of unity that we just made, we can almost literally copy the proof of the preceding proposition to show that

4.3 Proposition. *$\mathcal{Z}_q(M), \mathcal{B}_q(M)$ and $\mathcal{Z}_q(M)$ are sheaves on X, and carry a natural $\mathbb{R}_\infty$-Module structure.*

Now we are ready to formulate a more conventional form of De Rham's theorem:

4.4 Theorem. *(De Rham's theorem with parameters) The canonical $\mathbb{R}_\infty$-linear map*

$$\mathcal{H}^q(M) \xrightarrow{I} \mathcal{H}_q(M)^*$$

of $\mathbb{R}_\infty$-Modules on the ringed space $(X, \mathbb{R}_\infty)$ given by the components

$$I_U \colon \mathcal{H}^q(M)(U) \to \mathcal{H}_q(M)^*(U), \quad I_U([\omega])([\gamma]) = \int_\gamma \omega$$

is an isomorphism of sheaves. (Here $(-)^*$ denotes the dual in the category of $\mathbb{R}_\infty$-Modules. So $\mathcal{H}_q(M)^*(U)$ is the set of natural transformations from $\mathcal{H}_q(M)|_U$ to $\mathbb{R}_\infty|_U$.)

Proof. Restrict everything in the version of De Rham stated in Theorem 3.1 for the topos of sheaves on the site $\mathbb{G}$ to the subcategory of $\mathbb{G}$ consisting of open subspaces of X, and read off the different notions involved. For example, one has for an open $U \subset X$,

$$\underline{\Lambda}^q(M)(U) = \Lambda^q(M)(U)$$

where on the right-hand side, U is regarded as an object of $\mathbb{G}$. For $U = X =$ the one-point space, this is proposition 3.8. The general case is similar. More easily, one shows

$$\mathcal{S}_q(M)(U) = S_q(M)(U)$$

(again, on the left sheaves on X, on the right sheaves on $\mathbb{G}$). etc. $\square$

As an immediate consequence, we obtain the following theorem from classical analysis.

4.5 Corollary. *Let ω be a smooth X-form on M, i.e. $\omega \in \underline{\Lambda}^q(M)(X)$. If for each parameter value $x \in X$, the form $\omega(x,-) \in \Lambda^q(M)$ is exact, then there is an X-form $\alpha \in \underline{\Lambda}^{q-1}(M)(X)$ such that $\omega = d\alpha$.*

Proof. The previous theorem tells us that

$$0 \to \mathcal{E}^q(M) \to \mathcal{F}^q(M) \to \mathcal{H}_q(M)^* \to 0$$

is exact in $\mathrm{Sh}(X)$. Since $\int_\gamma \omega = 0$ for all $\gamma \in Z_q(M)(X)$, ω is locally in $\mathcal{E}^q(M)$, i.e. $\omega \in \mathcal{E}^q(M)(X)$ since this is a sheaf. □

Just as theorem 4.4 follows from theorem 3.1, theorem 3.11 implies the following result:

4.6 Theorem. *(De Rham's theorem with parameters, for Čech cohomology). Let $\mathcal{U}$ and M be as in theorem 3.11, and let $X \in \mathbb{M}$ be a space of smooth parameters. Then the canonical homomorphism of $\mathbb{R}_\infty$-Modules*

$$\mathcal{H}^q(M) \to \mathcal{H}^q(\mathcal{U}, \mathbb{R}_\infty)$$

over the ringed space $(X, \mathbb{R}_\infty)$ is an isomorphism.

Here $\mathcal{H}^q(\mathcal{U}, \mathbb{R}_\infty)$ is the cohomology of the complex $\mathcal{C}^q(\mathcal{U}, \mathbb{R}_\infty)$ of sheaves on X, $\mathcal{C}^q(\mathcal{U}, \mathbb{R}_\infty)$ being defined as the sheafproduct $\Pi_{\alpha_o<\dots<\alpha_q} \mathcal{F}^o(U_{\alpha_o\dots\alpha_q}, \mathbb{R}_\infty)$ of the $\mathbb{R}_\infty$-Modules $\mathcal{F}^o(U_{\alpha_o\dots\alpha_q}, \mathbb{R}_\infty)$, defined by setting for open $W \subset X$

$$\begin{aligned} \mathcal{F}^o(U_{\alpha_o\dots\alpha_q}, \mathbb{R}_\infty)(W) = {} & \text{smooth functions } f(x,u) \colon W \times U_{\alpha_o\dots\alpha_q} \to \mathbb{R} \\ & \text{which locally do not depend on } u \end{aligned}$$

(i.e. there are covers $\{W_\xi\}$ of W and $\{U_\eta\}$ of $U_{\alpha_o\dots\alpha_q}$ such that each $f(x,u)|W_\xi \times U_\eta$ does not depend on u.)

Similarly, we can restrict the isomorphism in $\mathcal{G}$ of theorem 3.14 to the category $\mathrm{Sh}(X)$ for $X \in \mathbb{M}$, to obtain a result with a more classical appearance, by defining a *hybrid* cohomology sheaf $\mathcal{H}^q_\Delta(M)$ on X carrying an $\mathbb{R}_\infty$-Module structure: First, we define a sheaf $S_{\Delta,q}(M)$ on X whose sections are locally of the form

$$\sum_{i=1}^{n} a_i(u)\sigma_i(t)$$

where $a_i \colon U \to \mathbb{R}$ and $\sigma_i(t) \colon \Delta_q \to \mathbb{R}$ are smooth maps. (Just as in the definition of $S_q(M)$ given above, but now with the additional requirement that $\sigma_i(u,t)$ ***locally does not depend on u***). Alternatively, $S_{\Delta,q}(M)$ is the associated sheaf of the presheaf

$$U \mapsto \text{Free } \mathbb{R}_\infty(U)\text{-module generated by } C^\infty(\Delta_q, M).$$

This gives a sheaf complex, of which we can take the dual (in the category of $\mathbb{R}_\infty$-Modules over X)

$$\ldots \to S_{\Delta,q}(M)^* \xrightarrow{\partial_q^*} S_{\Delta,q+1}(M)^* \to \ldots .$$

As before, we then show that to obtain the cohomology of this sheaf complex we may define sheaves (not just presheaves, by a partition of unity argument) $\mathcal{Z}^q_\Delta(M)$, $\mathcal{B}^q_\Delta(M)$ and $\mathcal{H}^q_\Delta(M)$ by setting for open $U \subset X$

$$\begin{array}{rcl} \mathcal{Z}^q_\Delta(M)(U) & = & \mathrm{Ker}(\partial_q^*) \\ \mathcal{B}^q_\Delta(M)(U) & = & \mathrm{Im}(\partial_{q-1}^*) \\ \mathcal{H}^q_\Delta(M)(U) & = & \mathcal{Z}^q_\Delta(M)(U)/\mathcal{B}^q_\Delta(M)(U). \end{array}$$

The restriction of the isomorphism in 3.14 now gives the following result.

4.7 Theorem. *Let $M \in \mathbb{M}$ be a manifold, and $X \in \mathbb{M}$ be the space of parameters. Then the canonical homomorphism*

$$\begin{array}{rcl} \mathcal{H}^q(M) & \to & \mathcal{H}^q_\Delta(M) \\ [\omega] & \mapsto & ([\gamma] \mapsto \int_\gamma \omega) \end{array}$$

of $\mathbb{R}_\infty$-Modules over the ringed space $(X, \mathbb{R}_\infty)$ is an isomorphism. □

If we unravel the definitions, it turns out that we obtain a result familiar in classical differential geometry: for elements $\sigma \in S_{\Delta,q}(M)^*(T)$, T an open subspace of X, we have

$$\begin{array}{ll} T \xrightarrow{\sigma} S_{\Delta,q}(M)^* & \text{in } \mathcal{G} \\ \hline S_{\Delta,q}(M) \to R^T & \text{in mod}_R(\mathcal{G}) \\ \hline \Delta(M^{\Delta^q}) \to R^T & \text{in } \mathcal{G} \\ \hline M^{\Delta^q} \to \Gamma(R^T) & \text{in } \mathit{Sets} \\ \hline M^{\Delta^q} \to C^\infty(T, \mathbb{R}) & \text{in } \mathit{Sets} \\ \hline S_{\Delta q}(M) \to C^\infty(T, \mathbb{R}) & \text{in mod}_{\mathbb{R}}(\mathit{Sets}) \end{array}$$

That is, a T-element of $\hom_R(S_{\Delta,q}(M), R) \in \mathcal{G}$, or equivalently a section of $S_{\Delta,q}(M)^*$ over T, is precisely an $\mathbb{R}^T$-valued singular cochain on M in the sense of Van Est (1958). Furthermore, a T-element of $\Lambda^q(M) \in \mathcal{G}$ is precisely a differential T-form on M of degree q in van Est's sense, i.e. an element of $\Lambda^q(M)(T)$. Thus theorem 4.7 translates into the following corollary (where X has been replaced by T).

4.8 Corollary. *The integration I is a homomorphism of the complex Ω of T-forms into the complex Σ of $\mathbb{R}^T$-valued singular co-chains on M. Furthermore,*

$$I^*: H(\Omega) \to H(\Sigma)$$

is an isomorphism. □

Note that we can deduce corollary 4.5 also from 4.8, which is a simpler result than 4.4.

5 Appendix: Weil's Version of De Rham's Theorem

Let M be a smooth space. A *good cover* of M is a cover $\{U_\alpha\}_\alpha$ such that all the finite nonempty intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}$ are isomorphic to some R^n. Fix one such cover $\mathcal{U}$, and assume that the index set $\{\alpha\}$ is linearly ordered (in the synthetic/intuitionistic sense). What follows will be a synthetic argument. Thus, intuitively, every object has a smooth structure and every function is smooth, so we do not need to assume that M is a manifold. Neither does $\mathcal{U}$ necessarily have to consist of *open* subsets of M in some sense, but we do need one assumption on $\mathcal{U}$, namely that there is a partition of unity subordinate to it (or to a refinement of $\mathcal{U}$). In particular, we assume that $\mathcal{U}$ is *pointfinite* (not necessarily neighbourhood finite, since we work synthetically) or at least that $\mathcal{U}$ has a pointfinite refinement. Thus, if $f_\alpha: U_\alpha \to V$ are maps into some R-module V, and $\{\rho_\alpha\}$ is a partition of unity subordinate to $\mathcal{U} = \{U_\alpha\}$, then $\Sigma_\alpha \rho_\alpha \cdot f_\alpha$ makes sense as a function $M \to V$.

The De Rham cohomology $H^{\cdot}(M)$ of M was defined in section 1, and the Čech cohomology $H^{\cdot}(\mathcal{U}, R)$ in section 3 (classically, but it is obvious how to define the synthetic analogue). Weil's idea for proving that $H^{\cdot}(\mathcal{U}, R) \cong H^{\cdot}(M)$ is to embed both the De Rham complex $\{\Lambda^n(M)\}$ and the Čech complex $C^n(\mathcal{U}, R)$ into a bigger complex (denoted $\ldots \to L^n \to L^{n+1} \to \ldots$ below) and show that both cohomologies are isomorphic to this bigger third cohomology.

Let $U_{\alpha_o \ldots \alpha_n} = U_{\alpha_o} \cap \ldots \cap U_{\alpha_n}$ for each sequence $\alpha_o < \ldots < \alpha_n$ of indices, and let $\partial_i = \partial_i^n : U_{\alpha_o \ldots \alpha_n} \to U_{\alpha_o \ldots \widehat{\alpha_i} \ldots \alpha_n}$ be the inclusion. Then we have a diagram

$$\Lambda^q(M) \to \Pi_{\alpha_o} \Lambda^q(U_{\alpha_o}) \rightrightarrows \Pi_{\alpha_o < \alpha_1} \Lambda^q(U_{\alpha_o \alpha_1}) \equiv\!\!\!\!\Rightarrow \ldots$$

where the first map is the obvious restriction of forms, and $\delta_i : \Pi_{\alpha_o < \ldots < \alpha_n} \Lambda^q(U_{\alpha_o \ldots \alpha_n}) \to \Pi_{\alpha_o < \ldots < \alpha_{n+1}} \Lambda^q(U_{\alpha_o \ldots \alpha_{n+1}})$ comes from pulling back a form along ∂_i^n,

$$\delta_i = \partial_i^{n^*} : \Lambda^q(U_{\alpha_o \ldots \widehat{\alpha_i} \ldots \alpha_{n+1}}) \to \Lambda^q(U_{\alpha_o \ldots \alpha_{n+1}}).$$

From this, we obtain a complex

$$\Lambda^q(M) \to \Pi_{\alpha_o} \Lambda^q(U_{\alpha_o}) \xrightarrow{\delta} \Pi_{\alpha_0 < \alpha_1} \Lambda^q(U_{\alpha_o \alpha_1}) \xrightarrow{\delta} \ldots$$

with being

$$\delta = \delta^n : \Pi_{\alpha_o < \ldots < \alpha_n} \Lambda^q(U_{\alpha_o \ldots \alpha_n}) \to \Pi_{\alpha_o < \ldots < \alpha_{n+1}} \Lambda^q(U_{\alpha_o \ldots \alpha_{n+1}})$$

as the alternating sum $\sum_{i=0}^{n+1} (-1)^i \delta_i$. Thus, δ maps a sequence $\omega = \{\omega_{\alpha_o \ldots \alpha_n}\}$ of forms to the sequence $\delta\omega = \{(\delta\omega)_{\alpha_o \ldots \alpha_{n+1}}\}$, where

$$(\delta\omega)_{\alpha_o \ldots \alpha_{n+1}} = \sum_{i=0}^{n+1} (-1)^i \delta_i(\omega_{\alpha_o \ldots \widehat{\alpha_i} \ldots \alpha_{n+1}}).$$

Indeed, precisely as in the case of the boundary operator of the singular homology complex $S_{\cdot}(M)$ we can show that $\delta^2 = 0$. So we could form its cohomology, but this is not of much use, since

5.1 Lemma. *Every sequence*

$$0 \to \Lambda^q(M) \to \Pi_{\alpha_o} \Lambda^q(U_{\alpha_o}) \xrightarrow{\delta} \Pi_{\alpha_0 < \alpha_1} \Lambda^q(U_{\alpha_0 \alpha_1}) \xrightarrow{\delta} \ldots$$

is exact.

Proof. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open cover $\mathcal{U}$, and let $K_n : \Pi_{\alpha_o < \ldots < \alpha_{n+1}} \Lambda^p(U_{\alpha_o \ldots \alpha_{n+1}}) \to \Pi_{\alpha_o < \ldots < \alpha_n} \Lambda^p(U_{\alpha_0} \ldots U_{\alpha_n})$ be defined by putting for $\omega = \{\omega_{\alpha_0 \ldots \alpha_{n+1}}\}$,

$$K_n(\omega)_{\alpha_o \ldots \alpha_n} = \sum_\alpha \rho_\alpha \omega_{\alpha \alpha_o \ldots \alpha_n},$$

where $\omega_{\alpha \alpha_o \ldots \alpha_n}$ is interpreted according to the following convention: if $\beta_o \ldots \beta_n$ is a sequence of indices (not necessarily increasing and possibly with repititions), and σ is a permutation of $\{0, \ldots, n\}$ then $\omega_{\beta_o \ldots \beta_n} = \operatorname{sgn}(\sigma) \cdot \omega_{\beta_{\sigma(0)} \ldots \beta_{\sigma(n)}}$. (So $\omega_{\ldots\alpha\ldots\alpha\ldots} = 0$). Then an easy calculation shows that

$$\delta K_n(\omega) + K_{n+1}(\delta\omega) = \omega,$$

whence the lemma. □

Now consider the diagram

$$(*) \qquad \begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Lambda^1(M) \arrow[r, "r"] \arrow[u, "d"] & \Pi_{\alpha_0}\Lambda^1(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \Pi_{\alpha_0<\alpha_1}\Lambda^1(U_{\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & \Lambda^0(M) \arrow[r, "r"] \arrow[u, "d"] & \Pi_{\alpha_0}\Lambda^0(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \Pi_{\alpha_0<\alpha_1}\Lambda^0(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & F^0(M) \arrow[r] \arrow[u] & C^0(\mathfrak{U}, R) \arrow[r] \arrow[u] & C^1(\mathfrak{U}, R) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

By the lemma, all the rows except the first are exact, and by the Poincaré lemma, so are all the columns except the first ($\mathfrak{U}$ is a good cover). Let us write

$$K^{p,q} = \Pi_{\alpha_o < \ldots < \alpha_p} \Lambda^q(U_{\alpha_o \ldots \alpha_p}),$$

$\{K^{p,q}\}_{p \geq 0, q \geq 0}$ has the structure of a *double complex*: we have maps $\delta: K^{p,q} \to K^{p+1,q}$ and $d: K^{p,q} \to K^{p,q+1}$ such that $\delta^2 = 0 = d^2$ and $\delta d = d\delta$. From such a double complex we can construct an ordinary complex by summing up the codiagonals: let

$$L^n = \oplus_{p+q=n} K^{p,q}, \quad n \geq 0$$

where $\oplus$ denotes the direct sum of R-modules. Then the L^n from a complex with boundary operator

$$D: L^n \to L^{n+1}$$

defined as follows: $D: \oplus_{p+q=n} K^{p,q} \to \oplus_{p+q=n+1} K^{p,q}$ is determined by its components $D_p: K^{p,q} \to \oplus_{p+q=n+1} K^{p,q}$ ($p = 0, \ldots, n$) which are given by

$$D_p = \delta + (-1)^p d \qquad \begin{tikzcd} K^{p,q+1} \arrow[r] & L^{n+1} \\ K^{p,q} \arrow[u, "d"] \arrow[r, "\delta"] & K^{p+1,q} \arrow[u] \end{tikzcd}$$

After a quick look it will be clear that $D^2 = 0$.

Let us write $H_L^n(M)$ for the cohomology of this complex, i.e.

$$H_L^n(M) = \ker(L^n \xrightarrow{D} L^{n+1}) / \operatorname{Im}(L^{n-1} \xrightarrow{D} L^n)$$

5.2 Theorem. *$H_L^{\cdot}(M)$ is isomorphic to both the De Rham cohomology $H^{\cdot}(M)$ and the Čech cohomology $H^{\cdot}(\mathcal{U}, R)$.*

Proof. Using the exactness of the rows (except the bottom one) of the diagram (*) we will show that the maps $r\colon \Lambda^n(M) \to \Pi_{\alpha_o} \Lambda^n(U_{\alpha_o})$ induce isomorphisms $H^n(M) \xrightarrow{r} H_L^n(M)$. But the definition of the complex $\{L^n(M)\}$ is symmetric in p and q, so by *reflecting (*) in the diagonal* a completely similar argument will yield that the maps $i\colon C^n(\mathcal{U}, R) \hookrightarrow \Pi_{\alpha_o < \ldots < \alpha_n} \Lambda^o(U_{\alpha_o \ldots \alpha_n})$, induce isomorphisms $H^n(\mathcal{U}, R) \xrightarrow{i} H_L^n(M)$.

So to prove the first isomorphism, define a chainmap $r\colon \Lambda^{\cdot}(M) \to L^{\cdot}$ by

$$r^n\colon \Lambda^n(M) \to L^n, \quad \omega \mapsto \{\omega|_{U_{\alpha_o}}\} \in K^{0,n} \subset L^n.$$

(r is indeed a chainmap, since the restriction of D to $K^{0,n}$ is just $\delta + d$, so $D(r^n\omega) = \delta(r^n\omega) + dr^n(\omega) = \{d\omega|_{U_{\alpha_o}}\} = r^{n+1}d\omega$, because $\delta r^n(\omega) = 0$ by exactness of the rows.)

Thus r induces a map $r\colon H^n(M) \to H_L^n(M)$ at the level of cohomology. We claim that at this level, r is an isomorphism. *r is surjective*: Take $\phi \in L^n$, say $\phi = \sum_{p+q=n} \phi_{p,q}$ with $\phi_{p,q} \in K^{p,q}$, such that $D\phi = 0$. We have to show that ϕ differs by a boundary $D\chi$ from some $\phi' = \sum_{p+q=n} \phi'_{p,q}$ with $\phi'_{p,q} = 0$ for all p, q except $p = 0, q = n$. We do this in n steps, using the induction step which reduces a $\phi \in L^n$ with $\phi_{p,q} = 0$ for $p = k+1, \ldots, n$ to a ϕ' with $\phi'_{p,q} = 0$ for $p = k, \ldots, n$. Indeed, since for $\phi = \sum_{p+q=n} \phi_{p,q}$,

$$D\phi = \sum_{p+q=n} \delta\phi_{p,q} + (-1)^p d\phi_{p,q}$$

it follows that $\delta\phi_{n,0} = 0$ (in $K^{n+1,0}$), $d\phi_{0,n} = 0$ (in $K^{0,n+1}$) and $\delta\phi_{u,v+1} + (-1)^{u+1} d\phi_{u+1,v} = 0$ (in $K^{u+1,v+1}$, for $u + v + 1 = n$). So if $\phi_{p,q} = 0$ for $p > k$ then $\delta\phi_{k,n-k} = 0$. Hence by exactness of the rows, $\phi_{k,n-k} = \delta\psi$ for some $\psi \in K^{k-1,n-k}$. Let $\phi' = \phi - D\psi$. Then $\phi'_{p,q} = 0$ for $p \geq k$.

r is injective: Take $\omega \in F^n(M)$ such that $r(\omega) = D\phi$ for some $\phi \in L^{n-1}$. As shown above, there is a $\psi \in K^{0,n-1} \subset L^{n-1}$ such that $[\phi] = [\psi]$ in $H_L^{n-1}(M)$, so $r\omega = D\phi = D\psi$. But ψ is a sequence $\{\psi_\alpha\}$ of n–1-forms on U_α such that $\omega|_{U_\alpha} = d\psi_\alpha$, and moreover $\delta\psi = 0$ (since $r\omega = D\psi$), so by exactness of rows there is a global form λ with $\psi_\alpha = \lambda|_{U_\alpha}$ for each α, and we conclude that $d\lambda = \omega$, i.e. $[\omega] = 0$ in $H^n(M)$. □

Chapter V

Connections on Microlinear Spaces

One of the main new features of synthetic differential geometry, as opposed to classical geometry, is the existence of infinitesimal spaces such as $D, D_2, D \times D$, etc. These spaces essentially allow us to give alternative, algebraic approaches to geometric notions which are classically defined by limit processes. For example, exterior differentiation of a form is interpreted geometrically via circulation along an n-cube, a notion which involves integrals. In Chapter IV, on the other hand, we saw how to define exterior differentiation via circulation along an infinitesimal n-cube, and no actual integration was needed in this definition.

Although the main emphasis in this book is on models of synthetic differential geometry and not on the systematic development of differential geometry in the synthetic context, we believe that it is important to illustrate the use of infinitesimal spaces in synthetic reasoning, and to explain how this relates to the classical approach. As a case study, we will now try to do precisely this for the theory of connections and related notions.

In the first section, we introduce the notion of a microlinear space, a "generalized manifold". In a nutshell, a microlinear space is a space which behaves—at least with respect to maps from infinitesimal spaces into it—as if it had local coordinates. All manifolds are microlinear spaces (in a sense made precise in Section 7); moreover, the class of microlinear spaces is closed under inverse limits and exponentiation, so as to include spaces with singularities and spaces of smooth functions.

In Section 2, we prove a general version of the Ambrose-Palais-Singer theorem on the correspondence between symmetric affine connections and sprays, for an arbitrary microlinear space. As explained in Section 7, this generalizes the classical theorem for manifolds; for example, we automatically obtain a version for spaces of smooth functions between manifolds.

Section 3 is devoted to the study of vector bundles over an arbitrary microlinear base space, and affine connections on such vector bundles. In Section 4, we study the special case of the tangent bundle.

Having infinitesimals available, one can give a very intuitive proof of the Gauss-Bonnet theorem in dimension 2, simply by adding infinitesimal angles in different ways. This will be discussed in Section 5. In fact, this is an application of the notion of a connection on a principal fiber bundle as described in Section 6, rather than that of a connection on a vector bundle discussed in Section 3.

The topos-models introduced in Chapter III enable us to rephrase the results on microlinear spaces in terms of classical manifolds, and spaces of smooth functions between manifolds. Some examples of this nature will be given in Section 7.

1 *Microlinear Spaces*

The reader may have noticed that although *external* manifolds have been extensively studied as objects of the toposes $\mathcal{G}$ and $\mathcal{F}$ via the embeddings $s\colon \mathbb{M} \to \mathcal{G}$ and $s\colon \mathbb{M} \to \mathcal{F}$ (see Chapter III), nowhere did we define the notion of *internal* manifold, parallel to the notion of "internal open subset of R^n". Of course, letting the logic do the work for us, this notion is readily available: just formalize the higher-order formula "to be manifold". Such a notion, although awkward, may be useful for some purposes. But to study concepts like connections on vector bundles it turns out that, rather surprisingly, a much weaker and more manageable notion suffices. This is the notion of a microlinear object, or rather of a *microlinear space* (since we think of all objects as "spaces"), which can be defined in any universe (any topos) with a ring R satisfying a generalized version of the Kock-Lawvere axiom.

To motivate this notion, let us take a concrete case and work with the topos $\mathcal{G}$ as our universe, and the embedding $\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{G}$.

Classically, if M is a smooth manifold, then for each $x \in M$, the fiber $\pi^{-1}(x)$ of the tangent bundle $TM \xrightarrow{\pi} M$ has the structure of a vector space over $\mathbb{R}$, whose dimension is that of the manifold M. How can we recover this algebraic structure in our universe $\mathcal{G}$ when M is considered as an object of $\mathcal{G}$, i.e. the object $s(M)$? Since π becomes the map $s(M)^D \xrightarrow{\mathrm{ev}_0} s(M)$ of $\mathcal{G}$, we should be able to define maps $s(M)^D \times_{s(M)} s(M)^D \xrightarrow{+} s(M)^D$, $s(M) \xrightarrow{o} s(M)^D$, and $R \times s(M)^D \to s(M)^D$ (all *over* $s(M)$) in $\mathcal{G}$ which express that the fibers of $s(M)^D \xrightarrow{\mathrm{ev}_o} s(M)$ are R-modules. In set-theoretical terms, if $x \in s(M)$, $\alpha \in R$, and $t_1, t_2 \in s(M)^D$ with $t_1(0) = x = t_2(0)$, we have to define "inside $\mathcal{G}$" $t_1 + t_2 \in s(M)^D$ with $(t_1+t_2)(0) = x, 0 \in s(M)^D$ and $\alpha \cdot t_1 \in s(M)^D$ with $0(0) = x = (\alpha \cdot t_1)(0)$, such that they make $s(M)^D_x = \mathrm{ev}_o^{-1}(x) \subset s(M)^D$ into an R-module.

For $M = \mathbb{R}$, this is easy: by the Kock-Lawvere axiom we can write $t_i(d) = x + a_i d$ for all $d \in D (i = 1, 2)$, and define

$$\begin{aligned} (t_1 + t_2)(d) &= x + (a_1 + a_2)(d) \\ 0(d) &= x \\ \alpha \cdot t_1(d) &= x + \alpha a_1 d, \end{aligned}$$

and it is clear that $s(\mathbb{R})^D_x = R^D_x$ is an R-module.

Notice that we can reformulate the last equation as $\alpha \cdot t_i(d) = t_i(\alpha d)$, and this equation does not make use of the ring structure of $R = s(\mathbb{R})$; the same is true for the second equation. Thus, the real question is: can we reformulate the first equation in a way that does not use the ring structure of R, so as to generalize it to any object of the form $s(M)$ (and hopefully to other objects of $\mathcal{G}$)?

Let us first observe that the generalized Kock-Lawvere axiom gives a one-to-one correspondence between such pairs of tangent vectors $(t_1, t_2) \in R^D_x \times R^D_x$ and maps $D(2) \xrightarrow{t} R$ with $t(0) = x$, given by $t(d_1, d_2) = x + a_1 d_1 + a_2 d_2$. But given t, we can define $t_1 + t_2$ by $(t_1 + t_2)(d) = t(d, d)$. And in fact, we can define addition of tangent vectors in the same way for R^n, or more generally, for every object of the form $s(M), M \in \mathbb{M}$, by using local coordinates (see also 7.1 below).

Let us rephrase this definition of addition in more general terms: First of all, the inclusions $D \hookrightarrow D(2)$ given by $d \mapsto (d, 0)$ and $d \mapsto (0, d)$ induce for *any object* $M \in \mathcal{G}$ a restriction map $r: M^{D(2)} \to M^D \times_M M^D$. So we have a diagram

$$\begin{tikzcd} M^D \times_M M^D \arrow[dr] & M^{D(2)} \arrow[l, "r"'] \arrow[r, "M^\Delta"] \arrow[d] & M^D \arrow[dl] \\ & M & \end{tikzcd}$$

where $\Delta: D \to D(2)$ is the diagonal. Whenever r is an isomorphism (as for example in the case of $M = s(N)$, $N \in \mathbb{M}$) we define the addition $M^D \times_M M^D \xrightarrow{+} M^D$ by $+ = M^{\Delta} \circ r^{-1}$.

As we will see, r is an isomorphism for all microlinear spaces, and this will imply that we can introduce a fiberwise R-module structure on $M^D \to M$ for every microlinear space M.

A suggestive way of reformulating the property that r is an isomorphism in the case of R is to consider the diagram

$$(*)\qquad \begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "0"'] & D \arrow[d, "i_2"] \\ D \arrow[r, "i_1"'] & D(2) \end{tikzcd}$$

where $i_1(d) = (d, 0), i_2(d) = (0, d)$: then r is an isomorphism precisely when $R^{(-)}$ sends $(*)$ into a limit-diagram, i.e. when

$$\begin{tikzcd} R & R^D \arrow[l] \\ R^D \arrow[u] & R^{D(2)} \arrow[l] \arrow[u] \end{tikzcd}$$

is a pullback. The notion of microlinearity isolates precisely this property of $R^{(-)}$ of sending diagrams like $(*)$ into limit diagrams.

To introduce the formal definitions, we do not fix a universe, but we work *synthetically* (see the Introduction, and Chapter VII). Let R be a commutative ring with 1 which satisfies the following (generalized) *Kock-Lawvere axiom*: if $0 \in S \subset D_k(n)$ is given as a zero-set of finitely many polynomials $p_1, \ldots, p_\ell \in R[x_1, \ldots, x_n]$, each of total degree $\leq k$, i.e. $S = \{x \in D_k(n) | p_i(x) = 0, i = 1, \ldots, \ell\}$, then every function $S \to R$ is the restriction of a polynomial in $R[x_1, \ldots, x_n]$ of total degree $\leq k$, and this polynomial is unique modulo the ideal $(p_1, \ldots, p_\ell) \subset R[x_1, \ldots, x_n]$.

Such subspaces of some $D_k(n)$, given by finitely many polynomials with coefficients in R and constant term zero, will be called *infinitesimal spaces.*

1.1 Definition. (a) Let M be any object, and let $\mathcal{D}$ be a finite diagram (co-cone) of infinitesimal spaces. $\mathcal{D}$ is called an *M-colimit* if the functor $M^{(-)}$ sends $\mathcal{D}$ into a limit diagram. (Sometimes one says: "M believes that $\mathcal{D}$ is a colimit".) (b) An object M is a *microlinear space* if every R-colimit is an M-colimit.

As a first remark, we state the following closure properties.

1.2 Proposition. *(i) R is microlinear.*

(ii) Any inverse limit of microlinear spaces is again microlinear.

(iii) If M is a microlinear space and X is an arbitrary space (object, set), then M^X is microlinear.

Proof. Obvious. □

1.3 Proposition. *If M is a microlinear space, then the tangent bundle $M^D \to M$ has a natural (functorial in M) fiberwise R-module structure.*

Proof. From the preceding discussion it is clear that the map $M^{D(2)} \xrightarrow{r} M^D \times_M M^D$ described above is an isomorphism, and we define the map $M^D \times_M M^D \xrightarrow{+} M^D$ as the composite $M^\Delta \circ r^{-1}$. In other words, if $t_1, t_2 \in M_x^D$ are given, then $(t_1 + t_2)(d) = t(d,d)$, where $t = r^{-1}(t_1, t_2)$. Moreover, $0 \in M_x^D$ is defined by $0(d) = x$, $\forall d \in D$, and if $t \in M_x^D, \alpha \in R$, we define $\alpha \cdot t$ by $\alpha \cdot t(d) = t(\alpha \cdot d), \forall d \in D$. This indeed defines an R-module structure on M_x^D. We check associativity of $+$: take $t_1, t_2, t_3 \in M_x^D$, and consider the diagram

$$
\begin{tikzcd}
& D \arrow[dr, "i_1"] & & \\
& & D(2) \arrow[dr, "j_{12}"] & \\
1 \arrow[uur, "0"] \arrow[r, "0"] \arrow[ddr, "0"'] & D \arrow[ur, "i_2"] \arrow[dr, "i_1"] & & D(3) \\
& & D(2) \arrow[ur, "j_{23}"'] & \\
& D \arrow[ur, "i_2"'] & &
\end{tikzcd}
$$

where $i_1(d) = (d,0), i_2(d) = (0,d), j_{12}((d_1,d_2) = (d_1,d_2,0), j_{23}(d_1,d_2) =$

$(0, d_1, d_2)$. This is an R-colimit, so by microlinearity of M there is a unique map $s: D(3) \to M$ with $s(d,0,0) = t_1(d), s(0,d,0) = t_2(d)$, $s(0,0,d) = t_3(d)$. Thus by definition of $+$, $s(d,d,0) = (t_1 + t_2)(d)$, and therefore $s(d,d,d) = ((t_1 + t_2) + t_3)(d)$. Similarly $s(d,d,d) = (t_1 + (t_2 + t_3))(d)$. The other conditions for M_x^D to be an R-module are trivial. Furthermore, it is easy to check that the R-module structure is natural in M; i.e. if $M \xrightarrow{f} N$ is a map of microlinear spaces, then $f^D: M^D \to N^D$ gives an R-linear map $f_x^D: M_x^D \to N_{f(x)}^D$ for each $x \in M$. We leave the details to the reader. □

An important consequence of the fact that M is microlinear is the following proposition.

1.4 Proposition. *If M is a microlinear space, then all the tangent spaces of M satisfy the Kock-Lawvere axiom, in the sense that the canonical map $T_x(M) \times T_x(M) \xrightarrow{\alpha} T_x(M)^D$ given by $\alpha(u,v)(d) = u + d \cdot v$ is a bijection, for each $x \in M$.*

Proof. Let $\varphi: D \to T_x(M) = M_x^D$ be given, and let $u = \varphi(0)$. We claim the existence of a unique $v \in T_x(M)$ with $\varphi(d) = \varphi(0) + d \cdot v$. By replacing $\varphi(d)$ by $\varphi(d) - \varphi(0)$ in $T_x M$, we may without loss assume that $\varphi(0)$ is the null vector in $T_x M$. Consider $\tau: D \times D \to M$ defined by $\tau(d_1, t_2) = \varphi(d_1)(d_2)$. Since M is microlinear and

$$D \times D \underset{p_2}{\overset{p_1}{\rightrightarrows}} D \times D \xrightarrow{\cdot} D$$

is an R-coequalizer (by the generalized Kock-Lawvere axiom; here $p_1(d_1, d_2) = (d_1, 0), p_2(d_1, d_2) = (0, d_2)$), and $\tau \circ p_1 = \tau \circ p_2$, there is a unique $v \in T_x M$ such that $\tau(d_1, d_2) = v(d_1 \cdot d_2) = (d_1 \cdot v)(d_2)$. Thus $\varphi(d_1) = d_1 \cdot v$, which completes the proof. □

The previous proposition, combined with the following one, reduces in many cases linearity of an application between R-modules to just homogeneity.

1.5 Proposition. *Let $V \xrightarrow{H} W$ be a map of R-modules, and assume that W is a microlinear space which satisfies the Kock-Lawvere axiom in the sense that $W \times W \xrightarrow{\alpha} W^D$, $\alpha(w_1, w_2)(d) = w_1 + d \cdot w_2$, is an isomorphism. Then H is linear iff H is homogeneous.*

Proof. Let $H: V \to W$ be homogeneous, i.e. $H(\alpha v) = \alpha H(v)$ for

$\alpha \in R$, $v \in V$. Take $v_1, v_2 \in V$, and consider the maps

$$D(2) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} V, \begin{cases} \varphi(d_1, d_2) &= H(d_1 v_1 + d_2 v_2) \\ \psi(d_1, d_2) &= H(d_1 v_1) + H(d_2 v_2). \end{cases}$$

Since $\varphi \circ i_k = \psi \circ i_k (k = 1, 2)$ and figure (*) of preceding 1.1 is an R-pushout, it follows that $\varphi = \psi$. So $\varphi(d, d) = \psi(d, d)$, i.e. $d(H(v_1) + H(v_2)) = dH(v_1 + v_2)$ by homogeneity. Since α is an isomorphism, we have $H(v_1) + H(v_2) = H(v_1 + v_2)$. $\sqsubset$

As an application of microlinearity, we shall now construct the *Lie algebra* of a Lie group, and more generally, of a Lie monoid.

Since in the context of SDG, all sets (all objects) have smooth structures, we may define a *Lie monoid (group)* to be a monoid (group) G, where G is a microlinear space. Thus, the fiber $T_e(G)$ of $G^D \to G$ at the unit element e is an R-module (Proposition 1.3).

1.6 Theorem. *If G is a Lie monoid, then $T_e(G)$ has a natural Lie-algebra structure; that is, there is a natural bilinear Lie-bracket operation $T_e(G) \times T_e(G) \xrightarrow{[-,-]} T_e(G)$ satisfying the identities*

(i) $[t_1, t_2] + [t_2, t_1] = 0$
(ii) $[t_1, [t_2, t_3]] + [t_2, [t_3, t_1]] + [t_3, [t_2, t_1]] = 0$

(the second is the so-called Jacobi identity).

Proof. To define the Lie bracket $[-,-]$, choose $t_1, t_2 \in T_e(G)$, and define $h \colon D \times D \to D$ by

$$h(d_1, d_2) = t_2(-d_2) \circ t_1(-d_1) \circ t_2(d_2) \circ t_1(d_1),$$

where $\circ$ denotes the monoid-operation of G. Since the diagram

(1) $D \times D \underset{p_2}{\overset{p_1}{\rightrightarrows}} D \times D \overset{\cdot}{\to} D$

occuring in the proof of 1.4 is an R-pushout and G is a microlinear space, the fact that $h(d, 0) = h(d, 0) = e$ implies that there is a unique function $[t_1, t_2] \in G_e^D$ such that $\forall d_1, d_2 \in D$ $[t_1, t_2](d_1 \cdot d_2) = h(d_1, d_2)$. This defines $[-,-]$. Since $[-,-]$ is clearly homogeneous in each variable separately, we obtain from 1.4 and 1.5 that $[-,-]$ is bilinear.

To prove that (i) and (ii) hold, we first note that clearly from the definition of $+$ in $T_e(G)$,

(2) $t_1(d) \circ t_2(d) = (t_1 + t_2)(d)$,

(3) $t(-d) = (-t)(d) = t(d)^{-1}$.

Now (i) is obvious from (3) and the fact that the defining equation for $[t_1, t_2]$ shows that $h(d_1, d_2) = h(d_2, d_1)$.

For (ii), it suffices (by using (1) twice) to show that for all $d_1, d_2, d_3 \in D$,

$$([t_1, [t_2, t_3]] + [t_2, [t_3, t_1]] + [t_3, [t_1, t_2]])(d_1 d_2 d_3) = e,$$

or equivalently by (2), (writing $\underline{d} = d_1 d_2 d_3$)

(4) $[t_1, [t_2, t_3]](\underline{d}) \circ [t_2, [t_3, t_1]](\underline{d}) \circ [t_3, [t_1, t_2]](\underline{d}) = e$.

To prove (4), first notice that from the fact that

$$\begin{array}{ccc} 1 & \longrightarrow & D \\ \downarrow & & \downarrow \\ D & \longrightarrow & D(2) \end{array}$$

is an R-pushout, it follows that whenever $(d, d') \in D(2)$ and $s, r \in T_e(G)$,

(5) $s(d) \circ r(d') = r(d') \circ s(d)$.

Let us write $a = t_1(d_1), b = t_2(d_2), c = t_3(d_3)$. So (4) can be written as

(6) $(a, (b, c))(b, (c, a))(c, (a, b)) = e$,

where (x, y) is the commutator $y^{-1}x^{-1}yx$, and the monoid operation $\circ$ has been suppressed (by (3), all the elements in (6) have inverses). To compute (6), we use that by (5) we can interchange an element (x, y) with x, or y, or (u, v) provided $\{x, y\} \cap \{u, v\} \neq \emptyset$, where x, y, u, v are all one of a, b, c. To conclude the proof, then, we compute (6) using these commutativity properties: write (6) as

(7) $(b, c)^{-1}a^{-1}(b, c)a(c, a)^{-1}b^{-1}(c, a)b(a, b)^{-1}c^{-1}(a, b)c$.

Now move $(c, a)^{-1}$ to second place, and $(a, b)^{-1}$ just right of a, to get

(8) $(b, c)^{-1}(c, a)^{-1}a^{-1}(b, c)a(a, b)^{-1}b^{-1}(c, a)bc^{-1}(a, b)c$.

Rewriting bc^{-1} as $c^{-1}b(c^{-1}, b)$ and interchanging (c^{-1}, b) and (a, b) we get

(9) $(b,c)^{-1}(c,a)^{-1}a^{-1}(b,c)a(a,b)^{-1}b^{-1}(c,a)c^{-1}b(a,b)(c^{-1},b)c$.

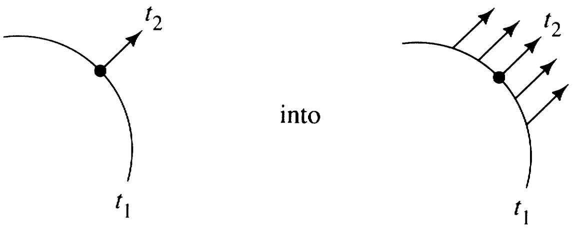
Spelling out the commutators in (9) shows that everything cancels, i.e. the expression (9) equals $e \in G$. □

2 *Connections and Sprays on a Microlinear Space*

We continue to work synthetically. If M is any space (any object), we have the canonical map

$$M^{D\times D} \xrightarrow{K} M^D \times_M M^D$$

defined by $\pi_1 K(t)(d) = t(d,0), \pi_2 K(t)(d) = t(0,d), \forall d \in D$. A *connection* on M will simply be a section of K. In pictures, a connection is a way of completing



However, we are interested only in connections which satisfy some further homogeneity conditions.

2.1 Definition. Let M be any space. An *affine connection* on M is a map

$$M^D \times_M M^D \xrightarrow{\nabla} M^{D\times D}$$

satisfying the following conditions:

(1) $\nabla(t_1,t_2)(d_1,0) = t_1(d_1), \nabla(t_1,t_2)(0,d_2) = t_2(d_2)$;
(2) $\nabla(\alpha\cdot t_1,t_2)(d_1,d_2) = \nabla(t_1,t_2)(\alpha d_1,d_2)\nabla(t_1,\alpha\cdot t_2)(d_1,d_2) = \nabla(t_1,t_2)(d_1,\alpha d_2)$.

for all $(t_1,t_2) \in M^D \times_M M^D, d_i \in D, \alpha \in R$. (Note that when M is a microlinear space, ∇ will be bilinear by 1.4, 1.5.) Moreover, ∇ is called *symmetric* (or *torsion-free*, see 4.8.1 below) if

(3) $\nabla(t_1,t_2)(d_1,d_2) = \nabla(t_2,t_1)(d_2,t_1)$.

One may think of ∇ as providing a means of transporting a tangent vector t_2 along another tangent vector t_1. To make this more explicit, we define parallel transport.

2.2 Definition. Let M be any space, and $M^D \xrightarrow{\pi} M$ its tangent bundle. A *parallel transport* on M is a function which associates with each $(t,h) \in M^D \times D$ a bijection

$$\tau_h(t,-): \pi^{-1}(t(0)) \to \pi^{-1}(t(h))$$

subject to the conditions

$$\begin{array}{rcll} \tau_0(t_1,t_2) & = & t_2 & \\ \tau_h(t_1,\lambda t_2) & = & \lambda\tau_h(t_1,t_2) & \\ \tau_h(\lambda t_1,t_2) & = & \tau_{\lambda h}(t_1,t_2) & (\text{all } \lambda \in R). \end{array}$$

We say that $\tau_h(t_1,t_2)$ is "the result of transporting t_2 parallel to itself along the curve t_1 for h seconds".

2.3 Proposition. *If τ is a parallel transport on M, then the map $\nabla: M^D \times_M M^D \to M^{D\times D}$ defined by*

$$(1)\ \ \nabla(t_1,t_2)(h_1,h_2) = \tau_{h_1}(t_1,t_2)(h_2)$$

is an affine connection on M. Conversely, if M is microlinear, then any affine connection ∇ on M gives a parallel transport τ defined by the same formula (1).

Proof. The first part is obvious. The second part really asserts that if ∇ is an affine connection on M and $t \in T_x(M)$, then for each $h \in D$ the map

$$T_xM \xrightarrow{\tau_h(t,-)} T_{t(h)}M, \quad \tau_h(t,t')(h') = \nabla(t,t')(h,h')$$

is a bijection. To see this, first define a map $\sigma(t): D_2 \to M$ by

$$\sigma(t)(d_1+d_2) = \nabla(t,t)(d_1,d_2).$$

(This is well-defined, since M is microlinear,

$$D \underset{(0,d)}{\overset{(d,0)}{\rightrightarrows}} D \times D \xrightarrow{+} D_2$$

is an R-pushout, and $\nabla(t,t)(d,0) = t(d) = \nabla(t,t)(0,d)$; σ is the spray associated to ∇, see 2.5 below.) So given $t \in T_x(M), h \in D$, we have $[d \mapsto \sigma(t)(h+d)] \in T_{t(h)}(M)$, and we claim that

$$\tau_{-h}(d \mapsto \sigma(t)(h+d),-): T_{t(h)}M \to T_xM$$

is the inverse of $\tau_h(t,-)$. One way round, take $s \in T_x M$. We have to show

(i) $\tau_{-h}(d \mapsto \sigma(t)(h+d), \tau_h(t,s)) = s$.

Now if $(h_1, h_2) \in D(2)$, we conclude from microlinearity of M and the fact that

$$\begin{array}{ccc} 1 & \longrightarrow & D \\ \big\downarrow & & \big\downarrow {\scriptstyle i_2} \\ D & \underset{i_1}{\longrightarrow} & D(2) \end{array}$$

is an R-pushout, that

$$\tau_{h_1+h_2}(t,s) = \tau_{h_2}(d \mapsto \sigma(t)(h_1+d), \tau_{h_2}(t,s)).$$

Putting $h_1 = -h_2 = h$, we obtain (i). In particular, $\tau_h(t,-)$ is always monic. The other way round, we have to show that for $r \in T_{t(h)}(M)$,

(ii) $\tau_h(t, \tau_{-h}(d \mapsto \sigma(t)(h+d), r) = r$.

Substituting $s = \tau_{-h}(d \mapsto \sigma(t)(h+d), r)$ in (i), we obtain

(iii) $\tau_{-h}(d \mapsto \sigma(t)(h+d), \tau_h(t, \tau_{-h}(d \mapsto \sigma(t)(h+d), r))) = \tau_{-h}(d \mapsto \sigma(t)(h+d), r)$.

Since $\tau_{-h}(d \mapsto \sigma(t)(h+d), -)$ is monic as we have just seen, (ii) now follows from (iii). □

Let us return to affine connections, and look at the particular case $M = R^n$. By the Kock-Lawvere axiom, a tangent vector t at $\underline{a} \in R^n$ may be written as $t(d) = \underline{a} + \underline{b}d$. Therefore by another application of the Kock-Lawvere axiom, we may write

$$\nabla(t_1, t_2)(d_1, d_2) = \underline{a} + \underline{b}_1 d_1 + \underline{b}_2 d_2 + \tilde{\nabla}_{\underline{a}}(\underline{b}_1, \underline{b}_2) d_1 d_2,$$

where $t_i(d) = \underline{a} + \underline{b}_i d (i = 1, 2)$ are tangent vectors at $\underline{a}$. Thus ∇ is completely determined by

$$\tilde{\nabla}_{\underline{a}}(-,-) \colon R^n \times R^n \to R^n$$

and equations (1)–(3) express that $\tilde{\nabla}_{\underline{a}}$ is a bilinear symmetric mapping. But any bilinear symmetric mapping $R^n \times R^n \xrightarrow{\delta} R^n$ is completely determined by its values on the diagonal: writing $\sigma(t) = \delta(t,t)$ for the quadratic form associated to δ, we may recover δ by $\delta(t_1, t_2) = \frac{1}{2}(\sigma(t_1, t_2) - \sigma(t_1) - \sigma(t_2))$, the polar form of σ.

The aim of this section is to prove a generalization of this fact for microlinear spaces. The rôle of quadratic form is played by the notion of a spray.

2.4 Definition. Let M be any space. A *spray* on M is a map $M^D \xrightarrow{\sigma} M^{D_2}$ which is a (fiberwise) homogeneous section of the natural restriction map $M^{D_2} \to M^D$ induced by the inclusion $D \hookrightarrow D_2 = \{\delta | \delta^3 = 0\}$; in other words, $M^D \xrightarrow{\sigma} M^{D_2}$ is a spray if for all $t \in M^D$, we have the following identities

(1) $\sigma(t)(d) = t(d)$ for all $d \in D$,
(2) $\sigma(\lambda t)(\delta) = \sigma(t)(\lambda\delta)$ for all $\lambda \in R, \delta \in D_2$.

For the particular case that $M = R^n$, we obtain (as before) the following explicit description of a spray

$$\sigma(t)(\delta) = \underline{a} + \underline{b}\delta + \tilde{\sigma}_{\underline{a}}(\underline{b})\delta^2, \text{ for } t(d) = \underline{a} + \underline{b}d.$$

The homogeneity condition (2) reduces to

$$\tilde{\sigma}_{\underline{a}}(\lambda\underline{b}) = \lambda^2 \tilde{\sigma}_{\underline{a}}(b)$$

which explains the "quadraticity" of σ.

The main result of this section is the following version of the Ambrose-Palais-Singer theorem.

2.5 Theorem. *Let M be a microlinear space. Then there is a natural one-to-one correspondence between symmetric affine connections ∇ on M and sprays σ on M, given by the formula*

$$\sigma(t)(d_1 + d_2) = \nabla(t,t)(d_1, d_2).$$

(σ is called the geodesic spray associated to ∇, see 4.9.)

Before starting the proof proper, we observe that the following are R-coequalizers:

(i) $D \overset{j_1}{\underset{j_2}{\rightrightarrows}} D \times D \xrightarrow{+} D_2$, where $j_1(d) = (d, 0), j_2(d) = (0, d)$;

(ii) $D(2) \times D(2) \overset{\mathrm{id}}{\underset{s}{\rightrightarrows}} D(2) \times D(2) \xrightarrow{+} D_2(2)$, where $s(d_1, d_2) = (d_2, d_1)$;

(iii) $D \times D \times D \overset{(p_1, m o p_{23})}{\underset{(p_2, m o p_{13})}{\rightrightarrows}} D \times D \xrightarrow{m} D$, where m is multiplication, and the p's are projections;

(iv) $D \times D \times D(2) \overset{(p_1, m o p_{23})}{\underset{(p_2, m o p_{23})}{\rightrightarrows}} D \times D(2) \xrightarrow{m} D(2)$;

(v) $D_2 \times D_2 \times D_2(2) \overset{(p_1, m o p_{23})}{\underset{(p_2, m o p_{23})}{\rightrightarrows}} D_2 \times D_2(2) \xrightarrow{m} D_2(2)$;

Proof of Theorem 2.5. We first describe the correspondence.

(a) From connections to sprays: Given a connection ∇, we define σ as follows. For $t \in M^D, \nabla(t,t)(d_1, d_2)$ satisfies $\nabla(t,t)(d,0) = t(d) = \nabla(t,t)(0,d)$, so using the R-coequalizer (i), we conclude that there is a unique map $\sigma(t): D_2 \to M$ satisfying

(1) $\sigma(t)(d_1 + d_2) = \nabla(t,t)(d_1, d_2)$.

(Actually we did this already in the proof of 2.3.) Clearly σ satisfies the conditions of 2.4.

Notice that we didn't use the symmetry of ∇ to define this spray σ.

(b) From sprays to connections: Here we use the R-coequalizer (v). Given a spray σ and tangent vectors $t_1, t_2 \in M_x^D$, the map

$$D_2 \times D_2(2) \to M \quad (\delta, \lambda_1, \lambda_2) \mapsto \sigma(\lambda_1 t_2 + \lambda_2 t_2)(\delta)$$

coequalizes the parallel maps in (v), by homogeneity of σ, and hence it induces a unique map $\nabla(t_1, t_2): D_2(2) \to M$ with

(2) $\nabla(t_1,t_2)(\delta\lambda_1, \delta\lambda_2) = \sigma(\lambda_1 t_1 + \lambda_2 t_2)(\delta), \forall \delta \in D_2, \qquad (\lambda_1, \lambda_2) \in D_2(2)$.

The restriction of this map to $D \times D \subset D_2(2)$ defines the map $\nabla(t_1,t_2): D \times D \to M$. ∇ is indeed a symmetric affine connection: we check 2.1 (1), i.e. $\nabla(t_1,t_2)(d_1, 0) = t_1(d_1)$. By (iii) above, it suffices to show this for the case $d_1 = \delta\lambda$ where $\delta, \lambda \in D$. But by (2) and 2.4 (2),

$$\nabla(t_1,t_2)(\delta\lambda, 0) = \sigma(\lambda t_1)(\delta) = \sigma(t_1)(\delta\lambda) = t_1(\delta\lambda).$$

The condition 2.1 (2) is equally easy, and 2.1 (3) is trivial.

(c) These operations are inverse to each other: First, given a spray σ, (b) and (a) give a new spray $\tilde{\sigma}$ which is completely determined by

$$\begin{aligned} \tilde{\sigma}(t)(\delta\lambda_1 + \delta\lambda_2) &= \nabla(t,t)(\delta\lambda_1, \delta\lambda_2) \\ &= \sigma(\lambda_1 t + \lambda_2 t)(\delta), \end{aligned}$$

where $\delta \in D_2, (\lambda_1, \lambda_2) \in D_2(2)$. By 2.4 (2), $\tilde{\sigma} = \sigma$.

The other way round is more involved. Given a symmetric affine connection ∇, (a) and (b) give a new (symmetric) connection $\tilde{\nabla}$ which is completely determined by

(3) $\tilde{\nabla}(t_1, t_2)((d_1 + d_2)\lambda_1, (d_1 + d_2)\lambda_2) = \nabla(\lambda_1 t_1 + \lambda_2 t_2, \lambda_1 t_1 + \lambda_2 t_2)(d_1, d_2)$,

where $d_i \in D, (\lambda_1, \lambda_2) \in D_2(2)$, and $(d_1 + d_2)\lambda_i \in D$. We need to show that $\tilde{\nabla} = \nabla$. To this end, fix $t_1, t_2 \in T_x M$ and let $\varphi: R^2 \to T_x M$ be the map

$$\varphi(\lambda_1, \lambda_2) = \lambda_1 t_1 + \lambda_2 t_2.$$

Now consider the composites $S = (\varphi \times \varphi \times \mathrm{id} \times \mathrm{id}) \circ \nabla$ and $\tilde{S} = (\varphi \times \varphi \times \mathrm{id} \times \mathrm{id}) \circ \tilde{\nabla}: R^2 \times R^2 \times D \times D \to M$. Clearly, we have

(4) $S(\underline{\lambda}, \underline{\mu})(d_1, d_2) = S(\underline{\mu}, \underline{\lambda})(d_2, d_1)$
(5) $S(\alpha\underline{\lambda}, \underline{\mu})(d_1, d_2) = S(\underline{\lambda}, \underline{\nu})(\alpha d_1, d_2)$
(6) $S(\underline{\lambda}, \alpha\underline{\mu})(d_1, d_2) = S(\underline{\lambda}, \underline{\mu})(d_1, \alpha d_2)$,

and similarly for $\tilde{S}$. By (5), (6) and the R-coequalizers (iii) and (iv), S is completely determined by the map $T: D(2) \times D(2) \to M$ defined by $T(d_1\underline{\lambda}, d_2\underline{\mu}) = S(\underline{\lambda}, \underline{\mu})(d_1, d_2)$. Clearly, T is symmetric, i.e. $T(\underline{\lambda}, \underline{\mu}) = T(\underline{\mu}, \underline{\lambda})$, so by the R-coequalizer (ii), $T(\underline{\lambda}, \underline{\mu})$ depends only on the sum $\underline{\lambda} + \underline{\mu} \in D_2(2)$. Consequently, $S(\underline{\lambda}, \underline{\mu})(d_1, d_2)$ depends only on the sum $d_1\underline{\lambda} + d_2\underline{\mu}$. Similarly, $\tilde{S}(\underline{\lambda}, \underline{\mu})(d_1, d_2)$ depends only on the sum $d_1\underline{\lambda} + d_2\underline{\mu}$. But let us compute $\tilde{S}(\underline{\lambda}, \underline{\mu})(d_1, d_2)$ explicitly: by (i) and (v), it suffices to consider the case where $d_1 = (\beta + \beta_2)\gamma_1, d_2 = (\beta_1 + \beta_2)\gamma_2$, with $\beta_i \in D$ and $(\gamma_1, \gamma_2) \in D_2(2)$. Then

$$\begin{aligned} &\tilde{S}(\underline{\lambda}, \underline{\mu})(d_1, d_2) = \tilde{S}(\underline{\lambda}, \underline{\mu})((\beta_1 + \beta_2)\gamma_1, (\beta_1 + \beta_2)\gamma_2) \\ &= S((\gamma_1\lambda_1 + \gamma_2\mu_1, \gamma_1\lambda_2 + \gamma_2\mu_2), (\gamma_1\lambda_1 + \gamma_2\mu_1, \gamma_1\lambda_2 + \gamma_2\mu_2))(\beta_1, \beta_2). \end{aligned}$$

Since $S(\underline{\lambda}, \underline{\mu})(d_1, d_2)$ depends only on $d_1\underline{\lambda} + d_2\underline{\mu}$, it follows that $S = \tilde{S}$. Hence $S((1,0),(0,1)) = \tilde{S}((1,0),(0,1))$, i.e. $\nabla(t_1, t_2) = \tilde{\nabla}(t_1, t_2)$.

This completes the proof. □

3 Vector Bundles and Vector Fields

As another example of the use of microlinearity, we will give a synthetic version of the covariant derivative, and prove the usual properties. Again, the context is much wider than that of classical manifolds. (For more on the relation between classical manifolds and microlinear spaces, see V.7.)

3.1 Definition. Let M be a microlinear space. A *vector bundle over* M is a map $E \xrightarrow{p} M$ of microlinear spaces, such that every fiber $E_m = p^{-1}(m)$ is equipped with an R-module structure satisfying the Kock-Lawvere axiom, i.e. the canonical map

$$H_m: E_m \times E_m \to (E_m)^D$$

given by $H_m(u,v)(d) = u + dv$ is a bijection.

3.2 Remark. It is easy to see that if $E \xrightarrow{p} M$ is a map of microlinear spaces, then each fiber E_m is also microlinear. So Proposition 1.5 is applicable in this context.

3.3 Example. If M is a microlinear space, then the tangent bundle $M^D \xrightarrow{\pi} M$ is a vector bundle over M, by 1.2 (iii) and 1.3, 1.4.

Whenever $E \xrightarrow{p} M$ is a vector bundle, we have the following diagram

(1) $E \times_M E \xrightarrow{H} E^D \xrightarrow{K} M^D \times_M E$

where $H(u,v)(d) = u + dv$, and $K(t) = \langle p \circ t, t(0) \rangle$. All the spaces in (1) carry two vector bundle structures, one coming from the tangent bundle structure given by 1.3, and one coming from the vector bundle structure of E, and the maps H and K respect these structures, as summarized in 3.4.

3.4 Linearity Properties of H and K

3.4.1 $E^D \xrightarrow{\pi} E$ is a vector bundle over E by 3.3. The R-module structure will be denoted by $\oplus$ and $\odot$.

3.4.2 E^D is also a vector bundle over M^D by $E^D \xrightarrow{p^D} M^D$: E^D and M^D are microlinear by 1.2 (iii), and its fibers satisfy the Kock-Lawvere axiom. (proof: Given $\varphi: D \to (E^D)_t = (p^D)^{-1}(t)$, $t \in M^D$, consider $\varphi_d: D \to E_{t(d)} = p^{-1}(t(d))$ defined by $\varphi_d(d') = \varphi(d')(d)$. Since $E \xrightarrow{p} M$ is a vector bundle, there are unique $u_d, v_d \in E_{t(d)}$

such that $\varphi_d(d') = u_d + d'v_d (\forall d' \in D)$. Then $\varphi(d') = u + d'v$ for all d', where $u = [d \mapsto u_d]$ and v are in $(E^D)_t$.) The module structure on $E^D \xrightarrow{p^D} M^D$ comes from pointwise applying that of E, and is denoted by $+, \cdot$. So for $s, s' \in (E^D)_t$, $(s + s')(d) = s(d) + s'(d)$ (the latter $+$ in $E_{t(d)}$), $(\alpha \cdot s)(d) = \alpha \cdot s(d)$.

3.4.3 Lemma *The $\oplus$-structure distributes over the $+$-structure in E^D whenever this makes sense. That is, if t, t', $s, s' \in E^D$ with $p \circ t = p \circ t'$, $p \circ s = p \circ s'$, $t(0) = s(0)$, $t'(0) = s'(0)$, then*

$$(t + t') \oplus (s + s') = (t \oplus s) + (t' \oplus s').$$

Proof. Recall the R-pushout that is used to define $\oplus$,

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d] & D \arrow[d, "i_2"] \\ D \arrow[r, "i_1"'] & D(2) \end{tikzcd}$$

$t \oplus s$ and $t' \oplus s'$ are computed via the maps φ, $\varphi' \colon D(2) \to E$ given by $\varphi \circ i_1 = t, \varphi \circ i_2 = s, \varphi' \circ i_1 = t', \varphi' \circ i_2 = s'$. Define $\psi \colon D(2) \to E$ by $\psi(d_1, d_2) = \varphi(d_1, d_2) + \varphi'(d_1, d_2)$. (This makes sense since $p \circ \varphi = p \circ \varphi'$.) Then $\psi \circ i_1 = t + t'$ and $\psi \circ i_2 = s + s'$ so substituting $d_1 = d_2 = d$ gives the equation we want. $\square$

3.4.4 $E \times_M E$ is microlinear (1.2 (ii)), and it is a vector bundle over E via the projection $E \times_M E \xrightarrow{p_1} E$, with R-module structure (again denoted by circled $+, \cdot$) defined by

$$(u, v) \oplus (u, v') = (u, v + v'),\ \alpha \odot (u, v) = (u, \alpha v).$$

3.4.5 $E \times_M E$ is also a vector bundle over M, via the canonical map $E \times_M E \to M$. The R-module structure is defined and denoted as

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2),\ \alpha \cdot (u_1, v_1) = (\alpha u_1, \alpha u_2).$$

Clearly, if all E_m's satisfy the Kock-Lawvere axiom, then so do all products $E_m \times E_m$.

3.4.6 $M^D \times_M E$ is a vector bundle over E, via the projection $M^D \times_M E \xrightarrow{p_2} E$, the corresponding structure being given by

$$(t,v) \oplus (t',v) = (t \oplus t', v), \ \alpha \odot (t,v) = (\alpha t, v)$$

(on the right of $=$, $t \oplus t'$ and αt come from the R-module structure of M^D, as in 1.3).

3.4.7 $M^D \times_M E$ is a vector bundle over M^D via the projection $M^D \times_M E \xrightarrow{p_1} M^D$; the corresponding structure is

$$(t,v) + (t,v') = (t, v+v')\ , \ \alpha \cdot (t,v) = (t, \alpha v).$$

3.4.8 Proposition.

(1) *H is linear as a map of vector bundles over E, i.e. linear with respect to the circled $\oplus$-structure. Moreover, H is linear with respect to the $+$-structure (this makes sense, although E^D is not a vector bundle over M, since $p^D \circ H = 0$ in M^D).*

(2) *K is linear, both as a map of vector bundles over E (w.r.t. $\oplus$), and as a map of vector bundles over M^D (w.r.t. $+$).*

(3) *The sequence of linear maps*

$$0 \to E \times_M E \xrightarrow{H} E^D \xrightarrow{K} M^D \times_M E$$

over E is exact, i.e. H is 1–1 and $\mathrm{Im}(H) = \mathrm{Ker}(K)$.

Proof. (1), (2) By 1.4, 1.5 we only need to check homogeneity, which is obvious in each case. (3) That H is monic is part of saying that the fibers of $E \xrightarrow{p} M$ satisfy the Kock-Lawvere axiom. If $(u,v) \in E \times_M E$, then $KH(u,v) = (d \mapsto p(u) = p(v), u)$, which is the 0 for the $\oplus$-structure on $M^D \times_M E$. Conversely, if $t \in E^D$ and $K(t) = 0$, then $p \circ t$ is constant, i.e. $t\colon D \to E_{p(t(0))}$. Applying the Kock-Lawvere axiom to t, we find unique u, v with $t = H(u,v)$. □

3.4.9 Remark. Such vectors $t \in E^D$ with $K(t) = 0$ are called *vertical* vectors.

This completes 3.4.

We now generalize the definition of affine connection to vector bundles, and introduce the connection maps.

3.4.10 Definition. Let $E \xrightarrow{p} M$ be a vector bundle. An *affine connection* on $E \xrightarrow{p} M$ (or briefly, on E) is a map $\nabla\colon M^D \times_M E \to$

E^D which splits K and is linear w.r.t. both vector bundle structures $\oplus$ over E and $+$ over M^D. In other words (using 1.4, 1.5 again) ∇ should satisfy the following identities:

$$p \circ \nabla(t,v)(d) = t(d), \nabla(t,v)(0) = v$$
$$\nabla(\alpha t, v)(d) = (\alpha \odot \nabla(t,v))(d) = \nabla(t,v)(\alpha d)$$
$$\nabla(t, \alpha v)(d) = (\alpha \cdot \nabla(t,v))(d) = \alpha \cdot (\nabla(t,v)(d)).$$

If ∇ is an affine connection on E, we obtain a split-exact sequence of vector bundles over E (i.e. w.r.t. $\oplus$) from 3.4.8 (3):

$$(*) \qquad 0 \to R \times_M E \overset{H}{\rightarrowtail} E^D \underset{\nabla}{\overset{K}{\rightleftarrows}} M^D \times_M E \to 0.$$

So H splits as well. Explicitly, $K \circ (\mathrm{id}_{E^D} \ominus \nabla K) = 0$, so there is a map $C_1 \colon E^D \to E \times_M E$ defined by

$$H \circ C_1 = \mathrm{id}_{E^D} \ominus \nabla K.$$

C_1 is a map of vector bundles over E, i.e. $p_1 \circ C_1 = \pi$. The other component $p_2 \circ C_1$ is the so-called *connection map* associated to ∇, and denoted by

$$C \colon E^D \to E.$$

Note that since $C_1 \circ H = \mathrm{id}$, we get $C \circ \nu = \mathrm{id}_E$, where $\nu \colon E \to E^D$ is the map defined by $\nu(e)(d) = d \cdot e$.

The vectors in E^D which are in the image of ∇ are called the *horizontal vectors*. So (*) provides a decomposition of each vector in E^D in a "horizontal" and a "vertical" component (cf. 3.4.9).

This completes 3.5.

Intuitively, C measures the extent to which a tangent vector $t \in E^D$ coincides with the parallel transport of $t(0)$ along $p \circ t$ given by ∇. To define C by formulating this intuitive idea more directly, we would have to be able to "transport $t(h)$ *back* to $t(0)$", which is not possible in general. However, in many cases this can be done, and a more direct description of C can be given. (See e.g. V.3.9-11 and V.4.7.)

3.6 Proposition. *The connection map $C \colon E^D \to E$ associated to an affine connection ∇ satisfies both linearity laws:*

(1) *If $(t,t') \in E^D \times_{M^D} E^D$ then $C(t+t') = C(t) + C(t')$*
(2) *If $(t,t') \in E^D \times_E E^D$ then $C(t \oplus t') = C(t) + C(t')$*

(the two $+$'s on the right of $=$ denote the addition in $E_{pt(0)} = E_{pt'(0)}$).

Proof. (2) just comes from the fact that (*) in 3.5 is exact for the $\oplus$-structure. For (1), take $t, t' \in E^D, p \circ t = p \circ t'$. Since $C(t) + C(t') = p_2(C_1(t) + C_1(t'))$ and H is 1–1 and preserves $+$ (3.4.8(1)), it suffices to show that $HC_1(t + t') = HC_1(t) + HC_1(t')$. But writing $HC_1 = \mathrm{id}_{E^D} \ominus \nabla K$, this is just a special case of Lemma 3.4.3 (with $\ominus$ in place of $\oplus$). $\square$

3.7 Vector Fields. Let $E \xrightarrow{p} M$ be a vector bundle. An *E-vector field* on M is a section of p, i.e. a map $M \xrightarrow{Y} E$ with $p \circ Y = \mathrm{id}_M$. We usually write Y_m for $Y(m)$. Let $\mathcal{X}(p)$ or $\mathcal{X}(E)$ denote the set of E-vector fields on M. $\mathcal{X}(E)$ has a natural R^M-module structure, just by pointwise applying the R-module structure of the fibers of E.

In the particular case where $E \xrightarrow{p} M$ is the tangent bundle $M^D \xrightarrow{\pi} M$, we write $\mathcal{X}(M)$, rather than $\mathcal{X}(M^D)$ or $\mathcal{X}(\pi)$, and "vector field" for "M^D-vector field". If $X: M \to M^D$ is a vector field on M, the following notation is used: $X_m \in T_m(M)$ for $X(m), X(d, -)$ or X_d for $X(-)(d): M \to M$, so $X_d(m) = X_m(d) = X(m)(d) = X(d, m)$.

3.8 Covariant Differentiation. Let $E \xrightarrow{p} M$ be a vector bundle, and let ∇ be an affine connection on E, with connection map C. Let $Y: M \to E$ be an E-vector field on M, and let $X: M \to M^D$ be a vector field on M. Denote by

$$Y \cdot X: M \to E^D$$

the composite $M \xrightarrow{X} M^D \xrightarrow{Y^D} E^D$. The *covariant derivative of Y along X* is the E-vector field

$$\nabla_X Y = C \circ (Y \cdot X): M \to E.$$

Intuitively, $(\nabla_X Y)_m$ measures to which extent $(Y \cdot X)_m: D \to E$ is ∇-parallel along $X_m: D \to M$. We can make this geometric interpretation more explicit, as in Proposition 3.11. But first a lemma.

3.9 Lemma. *Let M be a microlinear space, and let $X: M \to M^D$ be a vector field on M. Then X can be uniquely extended to $\tilde{X}: M \to M^{D_2}$ in such a way that the flow equation*

$$\tilde{X}_{d_1+d_2}(m) = X_{d_2}(X_{d_1}(m))$$

is satisfied.

Proof. Take $m \in M$, and consider the function $\varphi: D \times D \to M$, $\varphi(d_1, d_2) = X_{d_2}(X_{d_1}(m))$. Since

$$D \underset{\langle \mathrm{id},0\rangle}{\overset{\langle 0,\mathrm{id}\rangle}{\rightrightarrows}} D \times D \xrightarrow{+} D_2$$

is an R-coequalizer, M is microlinear, and $\varphi(d,0) = \varphi(0,d)$ because $X_0 = \mathrm{id}_M$, we conclude that there is a unique function $\tilde{X}_{(-)}(m): D_2 \to M$ with

$$\tilde{X}_{d_1+d_2}(m) = X_{d_2}(X_{d_1}(m)), \text{ for all } d_1, d_2 \in D.$$

□

3.10 Remark. Similarly to 3.9, we may "formally integrate" X to a map $D_\infty \times M \to M$ extending X, and satisfying the flow equation.

3.11 Proposition. *Let $E \xrightarrow{p} M$ be a vector bundle, ∇ an affine connection on E, and $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(E)$, as in 3.8. Then $\nabla_X Y$ is uniquely determined by the following identity:*

$$\forall h \in D: \nabla(d \mapsto \tilde{X}_m(h+d), Y_{X_m(d)})(-h) - Y_m = h \cdot (\nabla_X Y)_m.$$

In other words, $\nabla_X Y$ is computed by transporting $Y_{X_m(h)}$ back along X_m and subtracting Y_m from the result.

Proof. $\nabla_X Y$ is defined by

$$[(Y \cdot X)_m \ominus \nabla(X_m, Y_m)](h) = Y_m + h \cdot (\nabla_X Y)_m, \forall h \in D$$

($\ominus$ in E^D, $+$ in E_m), so we only need to show

$$(*)\quad [(Y \cdot X)_m \ominus \nabla(X_m, Y_m)](h) = \nabla(d \mapsto \tilde{X}_m(d+h), Y_{X_m(h)})(-h).$$

But this is immediate from microlinearity of E: Define $\varphi: D(2) \to E$ by

$$\varphi(d_1, d_2) = \nabla(d \mapsto \tilde{X}_m(d + d_1), Y_{X_m(d_1)})(-d_2).$$

Then $\varphi(d_1, 0) = (Y \cdot X)_m(d_1)$ and $\varphi(0, d_2) = \nabla(X_m, Y_m)(-d_2)$, so by definition of tangential subtraction $\ominus$, we have

$$((Y \cdot X)_m \ominus \nabla(X_m, Y_m))(h) = \varphi(h,h),$$

which gives (*). □

3.12 Directional Derivatives of Functions. Let M be a (microlinear) space, $X \in \mathcal{X}(M)$, and let $f: M \to R$. The *directional derivative of f* (in the direction of X) is the function $X(f): M \to R$ defined as follows. For $m \in M$, $f \circ X_m : D \to R$ can be written as

$$(f \circ X_m)(d) = f(m) + h \cdot X(f)(m)$$

for a unique $X(f)(m) \in R$, by the Kock-Lawvere axiom. This defines $X(f): M \to R$. It is easy to check that $X(-): R^M \to R^M$ is linear and satisfies Leibniz' rule, i.e.

$$\begin{aligned} X(\alpha \cdot f) &= \alpha \cdot X(f) \\ X(f + g) &= X(f) + X(g) \\ X(f \cdot g) &= f \cdot X(g) + g \cdot X(f). \end{aligned}$$

We now prove the usual properties of covariant differentiation in the general context of microlinear spaces.

3.13 Theorem. *Let $E \xrightarrow{p} M$ be a vector bundle, ∇ an affine connection on E, and let $X_i: M \to M^D$ and $Y, Z: M \to E$ be vector fields on M; furthermore, let $f: M \to R$. Then*

(1) $\nabla_{X_1 \oplus X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$
(2) $\nabla_{f \cdot X} Y = f \cdot \nabla_X Y$
(3) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
(4) $\nabla_X (f \cdot Y) = f \cdot \nabla_X Y + X(f) \cdot Y$ *("Koszul's law")*

($+$ refers to addition in E, $\oplus$ to addition in M^D.)

Proof. Since $(Y \cdot (X_1 \oplus X_2))_m = (Y \cdot X_1)_m \oplus (Y \cdot X_2)_m$ (the latter $\oplus$ in E^D), $(Y \cdot (fX))_m = f(m) \odot (Y \cdot X)_m$ ($\odot$ in E^D), and $((Y+Z) \cdot X)_m = (Y \cdot X)_m + (Z \cdot X)_m$ (the latter $+$ in E^D), (1)–(3) are obvious from the linearity properties of the connection map C (see 3.6). For (4), we use 3.11. Choose $h \in D$ and write $t \in M^D, t(d) = \tilde{X}(d + h, m)$. Then by 3.11, with $f \cdot Y$ for Y,

$$h \cdot \nabla_X (f \cdot Y)_m = \nabla(t, f(X_m(h)) Y_{X(m)})(-h) - g(m) Y_m.$$

On the other hand, $f(X_m(h)) = f(m) + h \cdot X(f)(m)$ by definition of $X(f)$. So using homogeneity and 3.11, we get

$$\begin{aligned} h \cdot \nabla_X (f \cdot Y)_m &= (f(m) + hX(f)(m)) \nabla(t, Y_{X_m(h)})(-h) - f(m) Y_m \\ &= f(m) \cdot h \cdot (\nabla_X Y)_m + hX(f)(m) \nabla(t, Y_{X_m(h)})(-h). \end{aligned}$$

Since $\nabla(t, Y_{X_m(h)})(-h) = Y_m + h(\nabla_X Y)_m$ and $h^2 = 0$, the last term is equal to

$$h \cdot (f(m)(\nabla_X Y)_m + X(f)(m) Y_m).$$

Since this holds for all $h \in D$, we conclude by the Kock-Lawvere axiom for E_m that $\nabla_X (fY)_m = f(m)(\nabla_X Y)_m + X(f)(m) Y_m$. □

3.14 Curves and Vector Fields. Let $E \xrightarrow{p} M$ be a vector bundle, and let ∇ be an affine connection on E with connection map C. Let $X: R \to E$ be a curve in E, i.e. a curve of vectors. The *derivative* of X is the curve $\frac{DX}{dt}: R \to E$ defined by $\frac{DX}{dt}(x) = C(d \mapsto X(x+d))$. Note that $p \circ X = p \circ \frac{DX}{dt}: R \to M$. By definition, we have the following identity, where we write $t(d) = X(x+d), t \in E^D$:

(1) $\forall h \in D\ [t \ominus \nabla(p \circ t, X(x))](h) = X(x) + h \cdot \frac{DX}{dt}(x)$.

Obviously $p \circ t$ can be extended to a map $D_2 \to M$, so we don't need an analogue of Lemma 3.9 to prove the following geometric interpretation of $\frac{DX}{dt}$, similar to 3.11. The proof of (2) below is as the proof of 3.11.

Proposition. *$\frac{DX}{dt}(x)$ is uniquely determined by the following identity:*

(2) $\forall h \in D: \nabla(d \mapsto p \circ \tilde{t}(h+d), X(x+h))(-h) - X(x) = h \cdot \frac{DX}{dt}(x)$,

where $\tilde{t}: D_2 \to E$, $\tilde{t}(\delta) = X(x+\delta)$. In other words, $\frac{DX}{dt}$ is computed by transporting $X(x+h)$ back along $p \circ X$ to the point x, and subtracting $X(x)$ from the result. □

Proposition. *Let $X, Y: R \to E$ be curves of vectors as above, and let $f: R \to R$. Then*

(3) $\frac{D(X+Y)}{dt} = \frac{DX}{dt} + \frac{DY}{dt}$

(4) $\frac{D(f \cdot X)}{dt} = f' \cdot X + f \cdot \frac{DX}{dt}$

(5) $\frac{D(X \circ f)}{dt}(x) = f'(x) \frac{DX}{dt}(f(x))$

Proof. (3) is obvious from the definition and 3.6. The proof of (4) is analogous to 3.14 (4), using (2) above rather than 3.11. (5) is clear from the fact that $[d \mapsto X(f(x+d))] = f'(x) \odot [d \mapsto X(f(x)+d)]$ in E^D, and linearity of $C: E^D \to E$.

This proves the proposition, and completes 3.14. □

4 The Tangent Bundle of a Microlinear Space

In the preceding section we described some notions related to a vector bundle $E \xrightarrow{p} M$, equipped with an affine connection $M^D \times_M E \xrightarrow{\nabla}$

E^D. We will not restrict ourselves to the special case where $E \xrightarrow{p} M$ is the tangent bundle $M^D \xrightarrow{\pi} M$ (see 3.3). So E^D is now the *iterated tangent bundle* $(M^D)^D \cong M^{D\times D}$, which carries two vector bundle structures over M^D, and $M^D \times_M E$ is just $M^{D(2)}$ since M is microlinear. One of the tiresome aspects of replacing E by M^D is that we have to keep the two copies of M^D apart, which is notationally rather messy. Let us fix some notation.

4.1 The Iterated Tangent Bundle. Let M be a microlinear space. $(M^D)^D$ is identified with $M^{D\times D}$ via $\varphi(d_1)(d_2) = \varphi(d_2, d_1)$ (so the *right hand* D of $(M^D)^D$ is the *left hand* D of $M^{D\times D}$.) $(M^D)^D = M^{D\times D}$ has two projections onto M^D, namely

$$(M^D)^D \xrightarrow{\pi_{M^D}} M^D \text{ and } (M^D)^D \xrightarrow{(\pi_M)^D} M^D.$$

So for $\tau \in M^{D\times D}$,

$$\pi_{M^D}(\tau)(d) = \tau(0,d), \text{ and } (\pi_M)^D(\tau)(d) = \tau(d,0).$$

The first map π_{M^D} makes $(M^D)^D$ into a vector bundle over M^D, with the general R-module structure of a tangent bundle, as in 3.4.1, denoted by $\oplus, \odot$. So for $\tau, \tau' \in M^{D\times D}$, $\lambda \in R$,

$$\begin{aligned}(\lambda \odot \tau)(d_1, d_2) &= \tau(\lambda d_1, d_2)\\ (\tau \oplus \tau')(d_1, d_2) &= (\tau(-, d_2) + \tau'(-, d_2))(d_1),\end{aligned}$$

i.e. $(\tau \oplus \tau')(d_1, d_2)$ is computed by fixing d_2, and then adding $\tau(-, d_2)$ and $\tau'(-, d_2)$ in M^D, and $(\lambda \odot \tau)(d_1, d_2)$ is computed by fixing d_2 and computing $\lambda \cdot (\tau(-, d_2))$ in M^D.

The second map $(\pi_M)^D$ makes $(M^D)^D$ into a vector bundle over M^D, with R-module structure given by "fixing d_1" (as in 3.4.2), and denoted by $+, \cdot$; so

$$\begin{aligned}(\lambda \cdot \tau)(d_1, d_2) &= \tau(d_1, \lambda d_2)\\ (\tau + \tau')(d_1, d_2) &= (\tau(d_1, -) + \tau'(d_1, -))(d_2).\end{aligned}$$

We have a twist-map $\sum$ induced by composition with $D \times D \to D \times D$, $(d_1, d_2) \mapsto (d_2, d_1)$, and $\sum$ is a map of vector bundles over M^D

$$\begin{tikzcd} M^{D\times D} \arrow[rr, "\Sigma"] \arrow[dr, "\pi_{M^D}"'] & & M^{D\times D} \arrow[dl, "(\pi_M)^D"] \\ & M^D & \end{tikzcd}$$

The map $K: M^{D\times D} \to M^{D(2)}$ (i.e. the map K from Section 3) is just the restriction map induced by $D(2) \subset D \times D$ for this special case $E = M^D$. M^D acts on the fibers of K, as explained in 4.2–4.4.

4.2 Translation Spaces. Recall that if $(V, +, 0)$ is an abelian group and A is an inhabited set (i.e. $\exists x\ x \in A$), then an *affine structure*, or *translation-space structure*, or *torsor structure* of A over V is a free and transitive action of V on A, denoted by

(1) $V \times A \xrightarrow{\overset{\bullet}{+}} A.$

Thus, for each pair $\tau_1, \tau_2 \in A$, there is a unique $t \in V$ with $t \overset{\bullet}{+} \tau_1 = \tau_2$, and this t is written as $\tau_2 \overset{\bullet}{-} \tau_1$. So the free and transitive action can equivalently be described by two operations

(2) $V \times A \xrightarrow{\overset{\bullet}{+}} A\,,\ \ A \times A \xrightarrow{\overset{\bullet}{-}} V$

satisfying the following identities

(3.1) $0 \overset{\bullet}{+} \tau = \tau$
(3.2) $(t_1 + t_2) \overset{\bullet}{+} \tau = t_1 \overset{\bullet}{+} (t_2 \overset{\bullet}{+} \tau)$
(3.3) $(t \overset{\bullet}{+} \tau) \overset{\bullet}{-} \tau = t$
(3.4) $(\tau_2 \overset{\bullet}{-} \tau_1) \overset{\bullet}{+} \tau_1 = \tau_2.$

Notice that if we fix $\tau_0 \in A$, we may identify A with V via

$$V \to A\,,\ t \mapsto t \overset{\bullet}{+} \tau_0.$$

Under this identification, $\overset{\bullet}{+}$ corresponds to $+$ and $\overset{\bullet}{-}$ to $-$. As a consequence, we have the following useful "metarule": To check a meaningful equation involving $\overset{\bullet}{+}, \overset{\bullet}{-}, +, -$, it suffices to drop all the dots and check the equation in the theory of abelian groups.

4.3 Example. Consider the restriction map $R^{D\times D} \xrightarrow{K} R^{D(2)}$ described in 4.1. By the Kock-Lawvere axiom, elements τ of $R^{D\times D}$ are of the form $\tau(d_1,d_2) = a + b_1d_1 + b_2d_2 + cd_1d_2$, and elements s of $R^{D(2)}$ are of the form $a + b_1d_2 + b_2d_2$; with this notation, $K(\tau) = s$. Fix $s \in R^{D(2)}$, with $s(0) = a$ say. Then $R_a^D = T_a(R)$ acts on $R_a^{D\times D} = \{\tau \in R^{D\times D} | \tau(0,0) = a\}$ by

$$t \overset{\bullet}{+} \tau = a + b_1d_1 + b_2d_2 + (b+c)d_1d_2,$$

where t and τ are given by $t(d) = a + bd, \tau(d_1,d_2) = a + b_1d_1 + b_2d_2 + cd_1d_2$. This action actually is over K, i.e. $K(t \overset{\bullet}{+} \tau) = K(\tau)$, and for each $s \in R^{D(2)}$ it makes the fiber $K^{-1}(s)$ into a translation space over $T_{s(0)}(R)$. (The corresponding operation $\overset{\bullet}{-}$ is given by $(t_1 \overset{\bullet}{-} t_2)(d) = a + (c_1 - c_2)d$, where t_i is given by $t_i(d) = a + b_1d_1 + b_2d_2 + c_id_1d_2$.) We generalize this in 4.4.

4.4 The Translation Space Structure on $M^{D\times D} \to M^{D(2)}$. Let M be a microlinear space, and $x \in M$. Fix $s \in M_x^{D(2)}$, i.e. $s \in M^{D(2)}$ with $s(0) = x$, and consider the fiber $K^{-1}(s)$ of the map $M^{D\times D} \xrightarrow{K} M^{D(2)}$. By microlinearity of M, we can define a natural action of $T_x(M)$ on $K^{-1}(s)$, denoted by

$$M_x^D \times K^{-1}(s) \xrightarrow{\overset{\bullet}{+}} K^{-1}(s),$$

by using the R-pushout

(1)
$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "0"'] & D \arrow[d, "\varepsilon"] \\ D\times D \arrow[r, "\varphi"'] & (D\times D)\vee D \end{tikzcd}$$

where $(D\times D)\vee D = \{(d_1,d_2,e) \in D^3 | d_i^2 = e^2 = d_ie = 0 (i = 1,2)\}$, and $\varphi(d_1,d_2) = (d_1,d_2,0)$, $\varepsilon(d) = (0,0,d)$. (Using the generalized Kock-Lawvere axiom, it is easy to see that this is an R-pushout.) Indeed, given $t \in T_x(M)$ and $\tau \in M^{D\times D}$, there is by (1) a unique map $(D\times D)\vee D \xrightarrow{f} M$ with $f(d_1,d_2,0) = \tau(d_1,d_2)$, $f(0,0,d) =$

$t(d)$. Define $t \overset{\bullet}{+} \tau$ by

$$(t \overset{\bullet}{+} \tau)(d_1, d_2) = f(d_1, d_2, d_1 d_2).$$

To see that this actually defines an action, one has to check (3.1) and (3.2) given in 4.2 above. (3.1) is clear. For (3.2), i.e. $(t_1 + t_2) \overset{\bullet}{+} \tau = t_1 \overset{\bullet}{+} (t_2 \overset{\bullet}{+} \tau)$, we first note that

$$(2) \qquad \begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "0"'] & D \arrow[d, "j_1"] \\ (D \times D) \vee D \arrow[r, "j_2"'] & (D \times D) \vee D(2) \end{tikzcd}$$

is an R-pushout, where

$$(D \times D) \vee D(2) = \{(d_1, d_2, e_1, e_2) \in D^4 | d_i^2 = d_i e_j = e_i^2 = e_1 e_2 = 0\}$$

and $j_1(d) = (0,0,d,0), j_2(d_1, d_2, e) = (d_1, d_2, 0, e)$. Now by definition, $((t_1 + t_2) \overset{\bullet}{+} \tau)(d_1, d_2) = g(d_1, d_2, d_1 d_2)$, where $(D \times D) \vee D \xrightarrow{g} M$ is the unique map with $g(0,0,e) = (t_1 + t_2)(e), g(d_1, d_2, 0) = \tau(d_1, d_2)$. Let $f: (D \times D) \vee D \to M$ be the map used to define $t_2 \overset{\bullet}{+} \tau$ as above, i.e. $f(d_1, d_2, 0) = \tau(d_1, d_2)$, $f(0,0,d) = t_2(d)$. By (2), we can define a unique

$$h: (D \times D) \vee D(2) \to M$$

with $h(d_1, d_2, 0, e) = f(d_1, d_2, e)$, $h(0,0,d,0) = t_1(d)$. Then $h(0,0,0,e) = t_2(e)$, so $h(0,0,e,e) = (t_1 + t_2)(e)$ by definition of $+$ on $T_x(M)$ (see 1.3). Thus by uniqueness of g, we have $g(d_1, d_2, e) = h(d_1, d_2, e, e)$, all $(d_1, d_2, e) \in (D \times D) \vee D$. On the other hand, h can be used to define $t_1 \overset{\bullet}{+} (t_2 \overset{\bullet}{+} \tau)$: since $h(0,0,e,0) = t_1(e)$, $h(d_1, d_2, 0, d_1 d_2) = (t_2 \overset{\bullet}{+} \tau)(d_1, d_2)$, we must necessarily have $(t_1 \overset{\bullet}{+} (t_2 \overset{\bullet}{+} \tau))(d_1, d_2) = h(d_1, d_2, d_1 d_2, d_1 d_2)$. We conclude that $t_1 \overset{\bullet}{+} (t_2 \overset{\bullet}{+} \tau) = (t_1 + t_2) \overset{\bullet}{+} \tau$.

So the fibers of TM act on the fibers of K, or briefly, TM acts on $M^{D \times D} \to M^{D(2)}$. We claim that this action is free and transitive.

To show this, we define an operation

$$\overset{\bullet}{-}: M^{D\times D} \times_{M^{D(2)}} M^{D\times D} \to M^{D\times D},$$

called *strong difference*, such that equations (3.3) and (3.4) of 4.2 hold. We need the following R-pushout,

$$(3) \qquad \begin{tikzcd} D(2) \arrow[r] \arrow[d] & D\times D \arrow[d, "\psi"] \\ D\times D \arrow[r, "\varphi"'] & (D\times D)\vee D \end{tikzcd}$$

where $\varphi(d_1, d_2) = (d_1, d_2, 0)$ as in diagram (1) above, and $\psi(d_1, d_2) = (d_1, d_2, d_1 d_2)$; the two maps $D(2) \to D\times D$ both are the inclusion. If $\tau_1, \tau_2 \in M^{D\times D}$ are in the same fiber of K, then there is a unique $(D\times D)\vee D \xrightarrow{f} M$ with $f\varphi = \tau_1, f\psi = \tau_2$, and we define $\tau_2 \overset{\bullet}{-} \tau_1 \in M^D$ by

$$(\tau_2 \overset{\bullet}{-} \tau_1)(d) = f(0,0,d).$$

It is easy to check that this definition of $\tau_2 \overset{\bullet}{-} \tau_1$ satisfies (3.3) and (3.4) of 4.2. We leave details to the reader.

We summarize 4.4 as follows.

Theorem. $M^{D\times D} \xrightarrow{K} M^{D(2)}$ *has a natural translation space structure over* M^D. *More precisely, for each* $x \in M^D$ *and each* $s \in M_x^{D(2)}$, $K^{-1}(s)$ *is a translation space over* $T_x(M)$ *by the operations*

$$T_x M \times K^{-1}(s) \xrightarrow{\overset{\bullet}{+}} K^{-1}(s), \; K^{-1}(s) \times K^{-1}(s) \xrightarrow{\overset{\bullet}{-}} T_x M$$

defined above.

This completes 4.4.

Several geometric notions occuring in Section 3 can be directly expressed in terms of the operation $\overset{\bullet}{-}$ of strong difference in the case where $E = M^D$. Before we go into this, we mention some useful properties of $\overset{\bullet}{+}$ and $\overset{\bullet}{-}$.

4.5 Some Further Properties of the Translation Structure. First, we have the following homogeneity properties, which can be straightforwardly proved from the definitions. (Recall the two different R-module structures on $M^{D\times D}$ discussed in 4.1.)

4.5.1 Proposition. *Let $s \in M_x^{D(2)}, t \in M_x^D, \tau, \tau_1, \tau_2 \in M_x^{D\times D}$ as above, with $K(\tau_1) = s = K(\tau_2)$. Then for any $\lambda \in R$,*

$$\lambda \cdot (\tau_2 \overset{\bullet}{-} \tau_1) = (\lambda \cdot \tau_2) \overset{\bullet}{-} (\lambda \cdot \tau_1) = (\lambda \odot \tau_2) \overset{\bullet}{-} (\lambda \odot \tau_1).$$

Moreover

$$\begin{aligned} \lambda \odot (t \overset{\bullet}{+} \tau) &= \lambda \cdot t \overset{\bullet}{+} \lambda \odot \tau \\ \lambda \cdot (t \overset{\bullet}{+} \tau) &= \lambda \cdot t \overset{\bullet}{+} \lambda \cdot \tau. \end{aligned}$$

□

Both $\overset{\bullet}{+}$ and $\overset{\bullet}{-}$ have symmetry properties, as stated in 4.5.2.

4.5.2 Proposition. *Let $\sum : M^{D\times D} \to M^{D\times D}$ be the twist map of 4.1, i.e. $\sum(\tau)(d_1, d_2) = \tau(d_2, d_1)$. Then for any $t \in M_x^D, \tau_1, \tau_2 \in M_x^{D\times D}$ with $K(\tau_1) = K(\tau_2)$,*

$$\begin{aligned} \tau_2 \overset{\bullet}{-} \tau_1 &= \textstyle\sum(\tau_2) \overset{\bullet}{-} \sum(\tau_1), \\ \textstyle\sum(t \overset{\bullet}{+} \tau_1) &= t \overset{\bullet}{+} \textstyle\sum(\tau_1). \end{aligned}$$

Proof. Clear from the defining R-pushouts used in 4.4 □

4.6 Vector Fields and Strong Difference. Let M be a microlinear space. Recall that a vector field on M is a map $X: M \to M^D$ with $\pi \circ X = \mathrm{id}$, i.e. $X(m)(0) = m$, all $m \in M$. By exponential adjointness, we may alternatively regard a vector field as a map $D \times M \xrightarrow{X} M$ with $X(0, m) = m$ for all m, or as a map $D \xrightarrow{X} M^M$ satisfying $X(0) = \mathrm{id}_M$. The interpretation $D \times M \to M$ corresponds to the notion of "infinitesimal flow", while the interpretation $D \to M^M$ corresponds to the notion of "an infinitesimal deformation of the identity map on M". (We will identity these viewpoints notationally, and write $X(m)(d), X(d)(m), X(d, m), X(m, d), X_d(m), X_m(d)$ all to denote the same thing.)

Let $\mathcal{X}(M)$ denote the space of vector fields on M. In other words, $\mathcal{X}(M)$ is the tangent space of the Lie monoid M^M at the identity. We saw in 1.6 that the group structure on $\mathcal{X}(M)$ (coming from its R-module structure as a tangent space) coincides with the monoid-structure. So for any vector field X

(1) $X_d \circ X_{-d} = \mathrm{id}_M$, all $d \in D$

(2) $X_{d_1} \circ X_{d_2} = X_{d_1+d_2}$, all $(d_1, d_2) \in D(2)$.

Moreover, if $X, Y \in \mathcal{X}(M)$ then by 1.6(5),

(3) $X_{d_1} \circ Y_{d_2} = Y_{d_2} \circ X_{d_1}$ if $(d_1, d_2) \in D(2)$.

1.6 also shows that $\mathcal{X}(M)$ has a Lie-algebra structure, with Lie-bracket given by

(4) $[X, Y](d_1, d_2) = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$.

It is of interest to note that the Lie-bracket can be expressed in terms of strong difference:

Proposition. *Let X, Y be vector fields on M, where M is microlinear, and let $m \in M$. Then*

(5) $[X, Y]_m = \sum(\overline{X} \cdot Y_m) \overset{\bullet}{-} (Y \cdot \overline{X})_m,$

where $\overline{X}$ stands for $-X$, i.e. $\overline{X}(d) = X(-d)$.

Proof. First note that (5) makes sense, i.e. $K\sum(\overline{X} \cdot Y)_m = K(Y \cdot \overline{X})_m$, by (3) above. Now define $f\colon (D \times D) \vee D \to M$ by

$$f(d_1, d_2, e) = (Y_{d_2} \circ [X, Y]_e \circ X_{-d_1})(m).$$

Then $f(d_1, d_2, 0) = Y_{d_2} \circ \overline{X}_{d_1}(m) = (Y \cdot \overline{X})_m(d_1, d_2)$, and $f(d_1, d_2, d_1 \cdot d_2) = \overline{X}_{d_1} \circ Y_{d_2}(m) = \sum(\overline{X} \cdot Y)_m(d_1, d_2)$. So from the definition of $\overset{\bullet}{-}$ we find

$$\begin{aligned} (\textstyle\sum(\overline{X} \cdot Y)_m \overset{\bullet}{-} (Y \cdot \overline{X})_m(d_1 \cdot d_2) &= f(0, 0, d_1 \cdot d_2) \\ &= [X, Y](d_1 \cdot d_2). \end{aligned}$$

The result now follows from the R-pushout (1) in the proof of 1.6$\square$

4.7 Connections and Strong Difference. Let M be a microlinear space, and ∇ and affine connection on M (see 2.1, or 3.5 where $E = M^D$). In 3.5 we defined the connection map C associated to a connection ∇. In the case $E = M^D, C$ can be described in terms of strong difference. To see this, recall that in the case $E = M^D$ the map $C\colon M^{D \times D} \to M^D$ is defined by the equation (1), where $(t_1, t_2) = K(\tau)$:

(1) $(\tau \ominus \nabla(t_1, t_2))(d_1, d_2) = (t_2 + d_1 \cdot C(\tau))(d_2)$

(so on the left, $\ominus$ is tangential addition in d_1, keeping d_2 fixed, as in 4.1; $+$ on the right is just tangential addition in $T_x M$). We want to show that $C(\tau)$ can be replaced by $\tau \overset{\bullet}{-} \nabla(t_1,t_2)$ in (1). Now $\tau \overset{\bullet}{-} \nabla(t_1,t_2)$ is defined via the unique map $(D\times D)\vee D \xrightarrow{f} M$ with $f(d_1,d_2,0) = \nabla(t_1,t_2)(d_1,d_2)$, $f(d_1,d_2,d_1d_2) = \tau(d_1,d_2)$, namely as $(\tau \overset{\bullet}{-} \nabla(t_1,t_2))(d) = f(0,0,d)$. Since $f(0,d_2,0) = t_2(d_2)$, we get

(2) $f(0,d_2,d_1d_2) = (t_2 + d_1(\tau \overset{\bullet}{-} \nabla(t_1,t_2)))(d_2)$.

Define $g\colon D(2)\times D \to M$ by

$$g(x,y,z) = f(x-y,z,xz)$$

(this makes sense, since if $(x,y,z)\in D(2)\times D$ then $(x-y,z,xz)\in (D\times D)\vee D$). Then $g(x,0,z) = f(x,z,xz) = \tau(x,y)$, and $g(0,y,z) = f(-y,z,0) = \nabla(t_1,t_2)(-y,z)$, so we must have

(3) $g(d_1,d_1,d_2) = (\tau \ominus \nabla(t_1,t_2))(d_1,d_2)$

by definition of $\ominus$. But $g(d_1,d_1,d_2) = f(0,d_2,d_1d_2)$, so (3) and (2) give

(4) $(\tau\ominus\nabla(t_1,t_2))(d_1,d_2) = t_2 + d_1(\tau \overset{\bullet}{-} \nabla(t_1,t_2))(d_2)$.

Since $C(\tau)$ is defined by (1), we conclude that the connection map C and the operation of strong difference are related as in the following proposition.

4.7.1 Proposition. *Let M be a microlinear space, and ∇ an affine connection on M with connection map $C\colon M^{D\times D}\to M^D$. Let $\tau\in M^{D\times D}$, $K(\tau) = (t_1,t_2)$. Then*

(5) $C(\tau) = \tau \overset{\bullet}{-} \nabla(t_1,t_2)$.

Consequently, if $\sigma\in M^{D\times D}$ with $K(\tau) = K(\sigma)$, then

(6) $\tau \overset{\bullet}{-} \sigma = C(\tau) - C(\sigma)$;

in particular, $C(\tau)$–$C(\sigma)$ does not depend on the connection ∇ on M. □

By Proposition 4.7.1, we can define the covariant derivative (see 3.8) in terms of $\overset{\bullet}{-}$. If X and Y are vector fields on M and $m\in M$, we have

(7) $(\nabla_X Y)_m = (Y\cdot X)_m \overset{\bullet}{-} \nabla(X_m,Y_m)$.

This provides an alternative way of calculating the properties of $\nabla_X Y$, by using translation space identities. For example, in this way we can prove Koszul's law 3.13 (4), repeated as (8) below: for $\varphi \in R^M, X, Y \in \mathcal{X}(M)$,

(8) $\nabla_X(\varphi Y) - \varphi \nabla_X Y = X(\varphi) \cdot Y$

Proof. By (7) and 4.5.1, we can rewrite the left hand side of (8) at a point m as

(9) $(\varphi Y \cdot X)_m \overset{\bullet}{-} \nabla(X_m, \varphi(m) \cdot Y_m) - (\varphi(m)(Y \cdot X)_m \overset{\bullet}{-} \nabla(X_m, \varphi(m)Y_m))$.

By general translation space properties (see the metarule in 4.2) (9) is equal to

(10) $(\varphi Y \cdot X)_m \overset{\bullet}{-} \varphi(m)(Y \cdot X)_m$,

since this makes sense, i.e. $K(\varphi Y \cdot X)_m = K(\varphi(m)(Y \cdot X)_m)$. Now (10) is calculated via the map $(D \times D) \vee D \xrightarrow{f} M$ with $f(d_1, d_2, d_1 d_2) = (\varphi Y \cdot X)_m(d_1, d_2)$, and $f(d_1, d_2, 0) = \varphi(m) \cdot (Y \cdot X)_m(d_1, d_2)$. But the f defined by

(11) $f(d_1, d_2, e) = Y_{X_{d_1}(m)}(X(\varphi)_m \cdot e + \varphi(m) d_2)$

satisfies these conditions: Clearly $f(d_1, d_2, 0) = (Y \cdot X)_m(d_1, \varphi(m) d_2) = \varphi(m) \cdot (Y \cdot X)_m(d_1, d_2)$, while on the other hand

$$\begin{aligned} f(d_1, d_2, d_1 d_2) &= Y_{X_m(d_1)}(d_2(\varphi(m) + X(\varphi)_m d_1)) \\ &= Y_{X_m(d_1)}(d_2 \cdot \varphi(X_m(d_1))) \\ &= (\varphi(X_m(d_1)) \cdot Y_{X_m(d_1)})(d_2) \\ &= (\varphi Y \cdot X)_m(d_1, d_2). \end{aligned}$$

We conclude that (10) is equal to the function $d \mapsto f(0, 0, d)$, which is $X(\varphi) \cdot Y_m$, proving (8).

4.8 Torsion. Another immediate consequence of Proposition 4.7.1 is that an affine connection is torsion free iff it is symmetric. Let M be a microlinear space with an affine connection ∇, as in 4.7. The *torsion* $T: M^{D \times D} \to M^D$ of ∇ is defined by

(1) $T(\tau) = C(\tau) - C(\sum \tau)$,

where $\tau \in M^{D \times D}$, $\sum$ as in 4.1. ∇ is called *torsion free* if $T = 0$, i.e. $T(\tau) = 0$ for all τ. Expressing C in terms of strong difference as in 4.7, we get the following alternative description of the torsion operator:

(2) $T(\tau) = (\tau \overset{\bullet}{-} \nabla(t_1, t_2)) - (\sum \tau \overset{\bullet}{-} \nabla(t_2, t_1))$

So the following statement follows from 4.5.2.

4.8.1 Proposition. *Let ∇ be an affine connection on M, as above. Then ∇ is torsion free iff ∇ is symmetric (cf. 2.1).* □

T induces an operation on vector fields. If X and Y are vector fields on M, then we define a new vector field $T(X,Y)$ by

(3) $T(X,Y) = C(Y \cdot X) - C(\Sigma(Y \cdot X))$.

Expressing C and the Lie-bracket in terms of strong difference (sec. 4.6), a simple translation space argument yields the following formula.

4.8.2 Proposition. *Let ∇ be an affine connection on M, with torsion T as above, and let $X, Y \in \mathcal{X}(M)$. Then*

(4) $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

Proof. By (3) and 4.6 (5) we can rewrite (4) as

(5) $(Y \cdot X \dot{-} \nabla(X,Y)) - (\Sigma(Y \cdot X) \dot{-} \nabla(Y,X)) = (Y \cdot X \dot{-} \nabla(X,Y)) - (X \cdot Y \dot{-} \nabla(Y,X)) - [X,Y]$.

Computing this at the base point $m \in M$ we can reduce this by the "metarule" of 4.2 to

(6) $-\Sigma(Y \cdot X)_m = -(X \cdot Y)_m - [X,Y]_m$.

But by 4.6 (5), $\Sigma(Y \cdot X)_m \dot{-} (X \cdot Y)_m = [\overline{Y}, X]_m$, so to prove (6) it suffices to show that $[\overline{Y}, X]_m = [X,Y]_m$, which is clear. □

4.9 Geodesics. Let M be a microlinear space, with an affine connection ∇ on it. In Section 2, we defined a spray $\sigma: M^D \to M^{D_2}$, the so-called *geodesic* spray associated to ∇. We briefly explain this terminology.

Let $a: R \to M$ be a curve in M. The "velocity field" of a is the curve of vectors

$$\dot{a}: R \to M^D, \quad \dot{a}(t)(d) = a(t+d).$$

Iterating this once more, we get

$$\ddot{a}: R \to M^{D \times D}, \quad \ddot{a}(t)(d_1, d_2) = a(t + d_1 + d_2).$$

The "acceleration field" of a is the curve of tangent vectors

$$\frac{D\overset{\bullet}{a}}{dt} = C \circ \ddot{a} \colon R \to M^D,$$

where C is the connection map associated to ∇ (see 3.14), a is called a *geodesic curve* (with respect to ∇) if $\nabla(\overset{\bullet}{a}, \overset{\bullet}{a}) = \ddot{a}$. Intuitively, this says that when we translate a small piece $a|(t+D) = \overset{\bullet}{a}\ (t)$ of a parallel (in the sense of ∇) to itself, along itself, we stay on this curve a; in other words a is a "straight" line, straight in the sense of ∇. Since $K(\ddot{a}) = (\overset{\bullet}{a}, \overset{\bullet}{a})$, we can express that a is geodesic equivalently by

$$\frac{D\,\overset{\bullet}{a}}{dt} = C \circ \ddot{a} = \ddot{a} \overset{\bullet}{-} \nabla(\overset{\bullet}{a}, \overset{\bullet}{a}) = 0.$$

Note that $\ddot{a}(t) \colon D \times D \to M$ is symmetric, so we can regard $\ddot{a}$ alternatively as a function $\ddot{a} \colon R \to M^{D_2}$ (since

$$D \times D \underset{(p_2, p_1)}{\overset{\mathrm{id}}{\rightrightarrows}} D \times D \xrightarrow{+} D_2 z$$

is an R-coequalizer.) If $\sigma \colon M^D \to M^{D_2}$ is a spray (see 2.4) a curve $a \colon R \to M$ is called an *integral curve* for σ if $\sigma \circ \overset{\bullet}{a} = \ddot{a} \colon R \to M^{D_2}$. If σ comes from an affine connection ∇ by the formula $\sigma(t)(d_1 + d_2) = \nabla(t,t)(d_1, d_2)$ as in 2.5, it is clear that a curve $a \colon R \to M$ is geodesic for ∇ iff it is an integral curve for σ. It follows from the correspondence in 2.5 that for each affine connection ∇ there is a symmetric affine connection $\tilde{\nabla}$ which has the same geodesic curves.

5 *The Gauss-Bonnet Theorem in Dimension 2*

In this section we will obtain an infinitesimal as well as a local version of the theorem of Gauss-Bonnet in dimension 2. As we said in the introduction, it will turn out that this theorem follows simply by considering some infinitesimal angles. Before we get to that stage, however, we introduce the necessary definitions.

Let M be a microlinear space of dimension 2, in the sense that each tangent space $T_x(M)$ is a free R-module of dimension 2, i.e. there exists a linear isomorphism $T_x(M) \xrightarrow{\sim} R^2$. We say that $t \in T_x(M)$ is *regular* if its image by some (equivalently, every) linear isomorphism $T_x(M) \to R^2$ is a vector in R^2 with at least one invertible component. The space of *(oriented) rays* of $T_x(M)$, denoted by $\mathrm{Rays}(T_x(M))$ is defined as the quotient $\mathrm{Reg}(T_x(M))/\sim$, where $\mathrm{Reg}(T_x(M))$ is the space of regular vectors in $T_x(M)$, and $t_1 \sim t_2$ iff $\exists \lambda > 0\ t_2 = \lambda t_1$.

We further postulate a free and transitive action

$$SO(2,R) \times \mathrm{Rays}(T_x(M)) \overset{\cdot}{\longrightarrow} \mathrm{Rays}(T_x(M))$$

where $SO(2,R)$ is the group of orthogonal 2×2-matrices with coefficients in R, i.e. matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a, b \in R$ with $a^2+b^2=1$. Intuitively, this action gives a *rotation*-structure on $\mathrm{Rays}(T_x(M))$: for $A \in SO(2,R), t \in \mathrm{Rays}(T_xM), A \cdot t$ is the result of rotating t over "the angle corresponding to A", and if $A \cdot t_1 = t_2$, A represents "the angle" between t_1 and t_2.

Finally, we assume that we are given a notion of *action-preserving parallel transport of rays*: for each $t \in T_x(M)$ and $h \in D$ we have a *bijection*

$$\tau_h(t,-)\colon \mathrm{Rays}(T_x(M)) \to \mathrm{Rays}(T_{t(h)}M)$$

satisfying

(1) $A \cdot \tau_h(t,r) = \tau_h(t, A\cdot r)$;
(2) $\tau_h(\lambda t, r) = \tau_{\lambda h}(t,r)$, and $\tau_0(t,r) = r$,

for $A \in SO(2,R), r \in \mathrm{Rays}(T_xM), \lambda \in R$. Intuitively, the parallel transport of rays preserves angles.

Following E. Cartan (1928), we define the *curvature form*

$$K\colon M^{D\times D} \times D \times D \to SO(2,R)$$

of the parallel transport as follows. Given $\gamma \in M^{D\times D}$ and $(h_1,h_2) \in D^2, K(\gamma,h_1,h_2)$ is the unique matrix $A \in SO(2,R)$ with

(3) $A \cdot r = \tau_{h_2}^{-1}(\gamma_4, \tau_{h_1}^{-1})(\gamma_3, \tau_{h_2}(\gamma_2, \tau_{h_2}(\gamma_1, r)))$,

where r is *any* ray in $T_{\gamma(0,0)}(M)$ (here, as in Chapter IV, $\gamma_1(d) = \gamma(d,0)$, $\gamma_2(d) = \gamma(h_1,d), \gamma_3(d) = \gamma(d,h_2), \gamma_4(d) = \gamma(0,d)$). Notice that K does not depend on r, since parallel transport preserves the action.

So what K does is simply this: given an infinitesimal 2-cube (γ,h_1,h_2), it transport any ray r along $\partial(\gamma,h_1,h_2)$ and it measures "the angle" between the result and the original ray r.

As defined above, K is not really a 2-form. However, by the Kock-Lawvere axiom we immediately conclude that

(4) $K(\gamma,h_1,h_2) = \begin{pmatrix} 1 & -bh_1h_2 \\ bh_1h_2 & 1 \end{pmatrix}$

for a unique $b \in R$; i.e. K determines a unique map

(5) $\tilde{K}: M^{D^2} \to R,\ \tilde{K}(\gamma) = b$

by (4). It is trivial to check that $\tilde{K}$ defines a 2-form (using bijectivity of $\tau_h(t, -)$ and the homogeneity condition (2)).

From now on, we assume the existence of a field of rays

(6) $X: M \to \mathrm{Rays}(TM),\ X_m \in \mathrm{Rays}(T_m M)$.

Given such a field X, the *connection form* $\tilde{\varphi}_X$ is defined as follows. First of all, define a map

$$\varphi_X: M^D \times D \to SO(2, R)$$

by letting $\varphi_X(\gamma, h)$ be the unique matrix $A \in SO(2, R)$ satisfying $A \cdot \tau_h(\gamma, X(\gamma(0))) = X(\gamma(h))$, where $\gamma \in M^D, h \in D$.

Once again, it is immediate to check that

$$\varphi_X(\gamma, h) = \begin{pmatrix} 1 & -bh \\ bh & 1 \end{pmatrix}$$

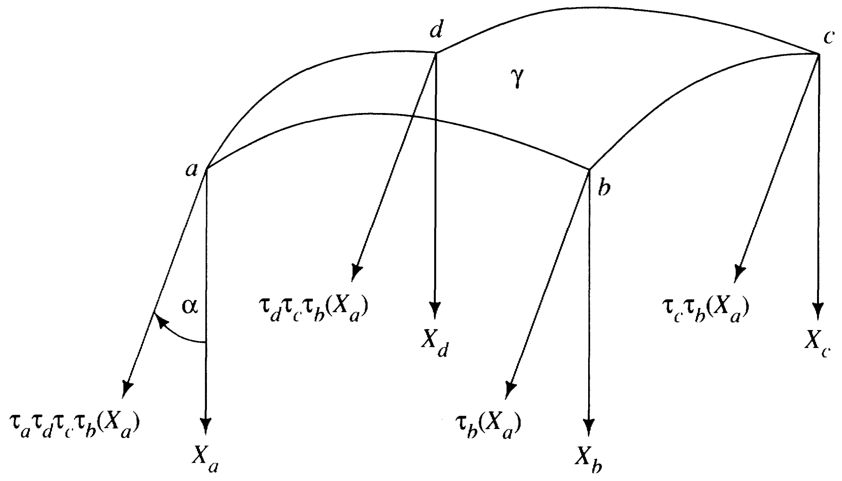
for a unique $b \in R$, so we obtain a map $\tilde{\varphi}_X: M^D \to R$ by setting $\tilde{\varphi}_X(\gamma) = b$ for this unique b. It is clear that $\tilde{\varphi}_X$ is a 1-form.

5.1 Theorem. *(Infinitesimal version of Gauss-Bonnet) Under the hypotheses described above,*

$$d\tilde{\varphi}_X = -\tilde{K}.$$

Proof. Since τ preserves angles, this is obvious from the definition, by taking $X(\gamma(0,0))$ as the r in the definition of $\tilde{K}$. $\square$

A picture may be helpful. Let a, b, c, d denote the corners of γ, and write $\tau_b(X_a)$ for $\tau_{h_1}(\gamma_1, X_a)$, etc.



Now α is the angle given by $\tilde{K}(\gamma)$. This angle can also be computed by summing the angles between X_b and $\tau_b(X_a)$, X_c and $\tau_c(X_b)$, X_d and $\tau_d(X_c)$, and X_a and $\tau_a(X_d)$—but this sum is the angle given by $d\tilde{\varphi}_X$, except that the sign is wrong.

To state the local version, we need some further functions. The *geodesic curvature* is the map

$$K_g\colon \mathrm{Reg}(M_{[0,1]}) \times [0,1] \times D \to SO(2, R),$$

where $\mathrm{Reg}(M^{[0,1]}) = \{\gamma \in M^{[0,1]} | \forall t\ \dot{\gamma}(t) = [d \mapsto \gamma(t+d)]$ is a regular vector in $T_{\gamma(t)}M\}$ denotes the space of regular curves, as follows. Given $\gamma \in \mathrm{Reg}(M^{[0,1]})$, $t \in [0,1], h \in D, K_g(\gamma, t, h)$ is the unique matrix $A \in SO(2, R)$ such that

$$(7)\quad [\dot{\gamma}(t+h)] = A \cdot \tau_h(\gamma|_{D(t)}, [\dot{\gamma}(t)]),$$

where $[\ldots]$ denotes the ray defined by the regular vector $\ldots$and $D(t) = \{t + h | h \in D\}$. Once again, given γ and t there is a unique $b \in R$ with $K_g(\gamma, t, h) = \begin{pmatrix} 1 & -bh \\ bh & 1 \end{pmatrix}$ for all $h \in D$, and K_g, gives rise to a map

$$(8)\quad \tilde{K}_g\colon \mathrm{Reg}(M^{[0,1]}) \times [0,1] \to R$$

by defining $\tilde{K}_g(\gamma, t) = b$ for this unique b.

So informally, $\tilde{K}_g(\gamma, t)$ measures the angle between the velocity

$\dot\gamma(t+h)$ at $t+h$ and the velocity $\dot\gamma(t)$ at t, by first transporting $\dot\gamma(t)$ along γ to the point $t+h$; but all this in terms of rays, i.e. velocity-directions, rather than velocities themselves.

Finally, we define a map θ,

$$\theta\colon \mathrm{Reg}(M^{[0,1]}) \times [0,1] \to SO(2,R)$$

by letting $\theta(\gamma,t)$ be the unique matrix $A \in SO(2,R)$ with

$$A \cdot X(\gamma(t)) = [\dot\gamma(t)],$$

i.e. θ measures the angle between $\dot\gamma(t)$ and $X(\gamma(t))$. This time, A is just of the form $A = \begin{pmatrix} a(t) & -b(t) \\ b(t) & a(t) \end{pmatrix}$ where $a^2(t) + b^2(t) \equiv 1$, and we cannot regard θ as an R-valued map. Nevertheless, the rate of change $\dot\theta$ may be viewed as such a function, since from $\frac{d}{dt}(a^2+b^2) \equiv 0$ we get

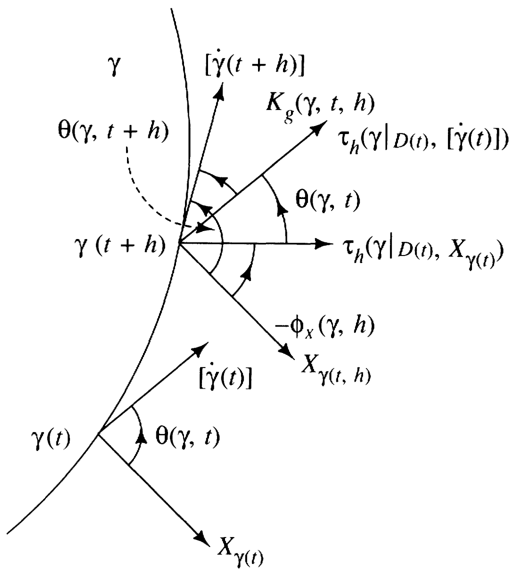
$$\theta(t+h)\theta(t)^{-1} = \begin{pmatrix} 1 & (-a'(t)b(t) + a(t)b'(t))h \\ (a'(t)b(t) - a(t)b'(t))h & 1 \end{pmatrix}.$$

Consequently, we can define

$$\text{(9)}\quad \dot\theta\colon \mathrm{reg}(M^{[0,1]}) \times [0,1] \to R, \quad \dot\theta(\gamma,t) = a'(t)b(t) - a(t)b'(t)$$

where a, b are as before.

The following picture shows all the functions that we introduced to measure angles:

(10) 

It is clear that $\theta(\gamma,t+h) = K_g(\gamma,t,h) \cdot \theta(\gamma,t) \cdot -\varphi_X(\gamma,h)$, since both matrices send $X(\gamma(t+h))$ into $[\dot{\gamma}(t+h)]$. Rephrasing this in terms of R-valued functions, we get

(11) $\dot{\theta}(t) + \tilde{\varphi}_X(\gamma|_{D(t)}) = \tilde{K}_g(\gamma, t)$.

Consider now a 2-chain $c\colon [0,1]^2 \to M$ such that each c_i is a regular curve (as usual, $c_1(t) = c(t,0), c_2(t) = c(1,t), c_3(t) = c(t,1), c_4(t) = c(0,t)$, so $\partial c = c_1 + c_2 - c_3 - c_4$). By integrating (11) along ∂c, the infinitesimal version 5.1 of Gauss-Bonnet together with Stokes' theorem IV.1.1 immediately gives the following result.

5.2 Theorem. *(Local version of Gauss-Bonnet) Using the notions introduced above, the following identity holds.*

$$\int_{\partial c} d\theta = \int_c K + \int_{\partial c} K_g. \qquad \square$$

The structure that we imposed on a "2-microlinear" space M to derive Theorem 5.2, namely that of the action

$$SO(2,R) \times \mathrm{Rays}(TM) \to \mathrm{Rays}(TM),$$

which enables us to deal with "angles", is verified when M is a microlinear space of dimension 2 having a "conformal Riemannian structure". Recall that a Riemannian metric is a map $g\colon TM \times_M TM \to R$ such that for each $x \in M$, $g_x\colon T_xM \times T_xM \to R$ is symmetric, bilinear and non-degenerate (in the sense that $g_x(u,v) \geq 0$, and $g_x(u,u) > 0$ iff $u \in \mathrm{Reg}(T_xM)$). Two such metrics g and g' are *equivalent* if $g' \equiv \lambda \cdot g$, i.e. $g'_x = \lambda(x) g_x$, for some $\lambda\colon M \to R_{>0}$. A conformal Riemannian structure is just an equivalence class G of metrics. If G is such a structure, we can define the action $SO(2) \times \mathrm{Rays}(T_xM) \to \mathrm{Rays}(T_xM)$ by letting $A \cdot r$ be the ray generated by $A\binom{v_1}{v_2}$, where $v \in r$, and $v = (v_1, v_2)$ for some orthonormal basis with respect to g_x, where $g \in G$. It is easy to check that this is well-defined, i.e. does not depend on the choices of $v \in r$, $g \in G$, and the basis. (Note that the Gramm-Schmidt procedure to construct an orthonormal basis is constructive and explicit, and is therefore available in the synthetic context.)

6 Ehresmann Connections on a Principal Fiber Bundle

In Section V.3 we introduced (affine) connections on vector bundles. If $E \xrightarrow{p} M$ is a vector bundle, a connection ∇ on E was defined as a map

$$M^D \times_M E \xrightarrow{\nabla} E^D$$

which enables us to transport a vector $v \in E$ *parallel* to, or *horizontal* over a given tangent vector $t \in M^D$, so we had the equations

$$\nabla(t,v)(0) = v, \; p \circ \nabla(t,v) = t;$$

moreover, this transport was required to be linear (or homogeneous, cf. 1.5) in t,

$$\nabla(\alpha t, v)(d) = \nabla(t,v)(\alpha d),$$

and to preserve the structure of E, i.e. to be linear in v,

$$\nabla(t, \alpha v)(d) = \nabla(t,v)(\alpha d).$$

The transport associated to ∇ can explicitly be written as $\tau_d(t,v)$, "transport v along t for d seconds", so by definition

$$\nabla(t,v)(d) = \tau_d(t,v).$$

A similar notion of transport was used in Section 5, but instead of linearly transporting vectors, we transported rays in an "angle-preserving" way. This is a special case of the general concept of a connection on a principal fiber bundle. In this section, we briefly describe this notion.

6.1 Principal Fiber Bundles. Let G be a Lie group (i.e. a microlinear space with a group structure). A *principal fiber bundle* with group G is a surjective map

$$B \xrightarrow{p} M$$

of microlinear spaces, equipped with an action of G which is free and transitive on fibers:

$$\begin{tikzcd} G \times B \arrow[rr, "\bullet"] \arrow[dr] & & B \arrow[dl] \\ & M & \end{tikzcd}$$

We write the action on the left. So the following hold: $p(g \cdot b) = p(b)$, $e \cdot b = b$, $(gh) \cdot b = g \cdot (h \cdot b)$ (which say that G acts on the fibers; e denotes the unit element of G), and moreover

$$\forall b, b' \in B \; p(b) = p(b') => \exists! g \in G \; g \cdot b = b'$$

(which says that the action is free and transitive). Thus, each fiber $B_m = p^{-1}(m)$ is isomorphic to G: given any $b \in B_m$, there is an isomorphism

$$\sigma_b : B_m \to G$$

defined by letting $\sigma_b(x)$ be the unique $g \in G$ with $g \cdot b = x$, i.e.

(1) $\sigma_b(x) \cdot b = x$.

It follows that

(2) $\sigma_{g \cdot b}(g \cdot x) = g \; \sigma_b(x) g^{-1}$.

(There is no canonical choice for an isomorphism σ_b in general. In fact, a section $s: M \to B$ makes the fiber bundle trivial, i.e. $B \xrightarrow{p} M$ is isomorphic to $M \times G \to M$ by the map $B \to M \times G, b \mapsto (p(b), \sigma_{sp(b)})$.

6.2 Ehresmann Connections. Let $B \xrightarrow{p} M$ be a principal fiber bundle with group G as in 6.1, and consider the diagram

$$\begin{tikzcd} B^D \arrow[rr, "K"] \arrow[dr, "\pi_B"'] & & M^D \underset{M}{\times} B \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

where $K(t) = (p \circ t, t(0))$ (just as the map K of Section 3). B^D and $M^D \times_M B$ both have a vector bundle structure over B, corresponding to the tangent bundle structures $B^D \to B$ and $M^D \to M$. We denote the corresponding R-module operations on the fibers by $\ominus, \odot$ (this is parallel to 3.4.1, 3.4.6). An (Ehresmann) *connection* on $B \xrightarrow{p} M$ is a section of K

$$M^D \times_M B \xrightarrow{\nabla} B^D$$

which is a map of vector bundles over B, and which preserves the G-action. In other words, ∇ satisfies the following identities:

(1) $\nabla(t,b)(0) = b, p \circ \nabla(t,b) = t$
(2) $\nabla(\alpha t, b)(d) = \nabla(t,b)(\alpha d)$, all $\alpha \in R, d \in D$
(3) $\nabla(t, g \cdot b)(d) = g \cdot (\nabla(t,b)(d))$, all $g \in G$.

From a connection ∇ on $B \xrightarrow{p} M$ we can construct a 1-form with values in the Lie-algebra $\underline{g} = T_e(G)$ of G (cf. 1.6). The definition is completely analogous to that of the connection map C given in 3.5. By linearity of ∇ and K (over B), we find that for $t \in B^D$, $K(t \ominus \nabla K t) = 0$, i.e. we can write

$$t \ominus \nabla K t \colon D \to B_m, \text{ where } m = p(t(0)).$$

(In other words, $t \ominus \nabla K t$ is a *vertical* vector. Just as in 3.5, the vectors $t \in B^D$ in the image of ∇ are called *horizontal*.) Using the identification of B_m with G via $\sigma_{t(0)}$, we obtain a map

$$\begin{aligned}&\omega(t) \colon D \to G\\ &\omega(t)(d) = \sigma_{t(0)}((t \ominus \nabla K t)(d)).\end{aligned}$$

Since $\omega(t)(0) = e$, this gives a map

$$\omega \colon B^D \to \underline{g} = T_e(G).$$

Let us rewrite the conditions (2) and (3) in terms of ω: (2) says that

(2′) $\omega(\alpha \odot t) = \alpha\omega(t)$,

i.e. ω is homogeneous. So ω is a *1-form with values in* $\underline{g}$. G acts pointwise on B^D, and (3) enables us to express $\omega(g \cdot t)$ in terms of $\omega(t)$. Writing $\delta(t)$ for $t \ominus \nabla K(t)$, we have $\delta(g \cdot t) = g \cdot \delta(t)$ by (3), hence

$$\begin{aligned}\omega(g \cdot t)(d) &= \sigma_{g \cdot t(0)}(g \cdot \delta(t)(d))\\ &= g\sigma_{t(0)}(\delta(t)(d))g^{-1} \qquad \text{(by 6.1 (2))}\\ &= g(\omega(t)(d))g^{-1}.\end{aligned}$$

Recall that G acts on $T_e(G) = G_e^D$ by conjugation; this defines a map $\mathrm{Ad} \colon G \to \mathrm{Aut}(\underline{g})$ defined by $\mathrm{Ad}(g)(t)(d) = g\, t(d) g^{-1}$. So we can rewrite $\omega(g \cdot t)(d) = g(\omega(t)(d))g^{-1}$ as

(3′) $\omega(g \cdot t) = \mathrm{Ad}(g)\omega(t)$.

If $b \in B_m$ and $s \in \underline{g} = T_e(G)$, we have a canonical vertical vector $s \cdot b \in B^D$, $(s \cdot b)(d) = s(d) \cdot b$. Since $K(s \cdot b) = (0, b)$, it follows that

(4) $\omega(s \cdot b) = s$ for all $s \in \underline{g}, b \in B$.

(*proof:* Addition of vertical vectors in B_m based at b corresponds to the multiplication in G under σ_b, just as 1.6 (2), (3).)

An Ehresmann connection on $B \xrightarrow{p} M$ is often defined directly as a $\underline{g}$-valued 1-form on B satisfying (3′), (4).

6.3 Geometric Interpretation of ω. Intuitively, what the map $\omega: B^D \to G_e^D$ does is the following. Given $t \in B^D$ and $h \in D$, we may transport $t(0)$ horizontally over $p \circ t$ to $B_{pt(h)}$, and then compare it with $t(h)$, i.e. find the unique $g_h \in G$ with $g_h \cdot \nabla(p \circ t, t(0))(h) = t(h)$. More formally, we claim that for all $(t, h) \in B^D \times D$,

(1) $\omega(t)(h) = \sigma_{\nabla(p \circ t, t(0))(h)} t(h)$,

which expresses this intuitive interpretation.

To prove (1), write τ_h for horizontal transport over $p \circ t$ during h seconds; i.e. for $b \in B_{t(0)}$ we have by definition

(2) $\tau_h(b) = \nabla(p \circ t, b)(h) \in B_{t(h)}$.

Now define $f: D(2) \to B$ by

$$f(d_1, d_2) = (\sigma_{\tau_{(d_1 - d_2)}(t(0))})^{-1} \sigma_{\tau_{d_1}(t(0))}(t(d_1)) \in B_{t(d_1 - d_2)}.$$

Then $f(d_1, 0) = (\sigma_{\tau_{d_1}(t(0))})^{-1} \sigma_{\tau_{d_1}(t(0))}(t(d_1)) = t(d_1)$, and

$$\begin{aligned} f(0, d_2) &= (\sigma_{\tau_{-d_2}(t(0))})^{-1} \sigma_{\tau_0(t(0))}(t(0)) = (\sigma_{\tau_{-d_2}(t(0))})^{-1}(e) \\ &= \tau_{d_2}(t(0)) = \nabla K t(-d_2). \end{aligned}$$

So by definition of $\ominus$, $f(h, h) = (t \ominus \nabla K(t)(h)$, and hence

$$\omega(t)(h) = \sigma_{t(0)} f(h, h) = \sigma_{\tau_h(t(0))}(t(h)),$$

which is (1).

Notice, by the way, that if $p \circ t: D \to M$ can be extended to a map $D_2 \to M$, then $\tau_h: B_{t(0)} \to B_{t(h)}$ is a bijection for each $h \in D$. This is proved just as in 2.3, or 3.9-11. The condition that $p \circ t$ can be extended is fulfilled, for example, when M has a connection as in Section 2 (or weaker, when M has property (E) of IV.1.8), or if $p \circ t$ comes from a vector field.

6.4 The Curvature Form. Let $B \xrightarrow{p} M$ be a principal fiber bundle with group G, equipped with an Ehresmann connection ∇, and denote by $\omega: B^D \to \underline{g}$ the associated 1-form satisfying (3′), (4) as in 6.3. One may define a 2-form

$$d\omega: B^{D\times D} \to \underline{g}$$

just as for R-valued forms in Chapter IV. More precisely, given $\gamma \in B^{D\times D}$ and $(h_1, h_2) \in D \times D$, we define

$$(1)\quad d\omega(\gamma, h_1, h_2) = \omega(\gamma_1, h_1) + \omega(\gamma_2, h_2) - \omega(\gamma_3, h_1) - \omega(\gamma_4, h_2)$$

(where $\partial\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$ as before, i.e. $\gamma_1 = \gamma(-,0), \gamma_2 = \gamma(h_1,-), \gamma_3 = \gamma(-,h_2), \gamma_4 = \gamma(0,-)$). Here $+$ and $-$ refer to the R-module structure of $\underline{g} = T_e(G)$. As we saw in 1.6, this coincides with the group structure induced by G; see (2), (3) in the proof of 1.6.) Since $T_e(G)$ satisfies the Kock-Lawvere axiom (1.4), we conclude that there is a unique $d\omega(\gamma) \in \underline{g}$ such that $d\omega(\gamma, h_1, h_2)$ as defined in (1) satisfies

$$(2)\quad d\omega(\gamma, h_1, h_2) = h_1 \cdot h_2 d\omega(\gamma).$$

This defines $d\omega: B^{D\times D} \to \underline{g}$.

We next introduce two $\underline{g}$-valued 2-forms on B. For the first, define a "*horizontal component*" mapping

$$h: B^{D\times D} \to B^{D\times D}$$

by

$$h(\gamma)(d_1, d_2) = \nabla(p\gamma(d_1, -), \nabla(p\gamma(-,0), \gamma(0))(d_1))(d_2),$$

or in the notation of transport,

$$h(\gamma)(d_1, d_2) = \tau_{d_2}(p\gamma(d_1, -), \tau_{d_1}(p\gamma(-,0), \gamma(0))).$$

Notice that $p \circ \gamma = p \circ h(\gamma)$. The 2-form $D\omega$ on B is defined by

$$D\omega: B^{D\times D} \to \underline{g},\ D\omega(\gamma) = d\omega(h(\gamma)).$$

The other $\underline{g}$-valued 2-form on B is defined as follows. For $\gamma \in B^{D\times D}, (h_1, h_2) \in D \times D$, let

$$(4)\quad \overline{\Omega}(\gamma, h_1, h_2) = \omega(\gamma_4)(h_2)^{-1}\omega(\gamma_3)(h_1)^{-1}\omega(\gamma_2)(h_2)\omega(\gamma_1)(h_1).$$

Then $\overline{\Omega}(\gamma, h_1, h_2) = e \in G$ if $h_1 = 0$ or $h_2 = 0$, so by the microlinearity of G (cf. the R-coequalizer (i) just below 2.5) we may write

$$(5)\quad \overline{\Omega}(\gamma, h_1, h_2) = \overline{\Omega}(\gamma)(h_1 \cdot h_2)$$

for a unique function $\overline{\Omega}(\gamma): D \to G$, i.e. $\overline{\Omega}(\gamma) \in \underline{g}$. This defines a

$\underline{g}$-valued 2-form

$$\overline{\Omega}: B^{D\times D} \to \underline{g}.$$

Intuitively, what $\overline{\Omega}$ does is the following (cf. 6.3): given a $\gamma: D \times D \to B$ and $(h_1, h_2) \in D \times D$, it transports the element $\gamma(0)$ along $\partial(\gamma, h_1, h_2)$, and it measures in terms of G the difference between the result of this translation, call it $\tau(\gamma(0))$, and $\gamma(0)$, i.e. $\sigma_{\gamma(0)}\tau(\gamma(0))$.

In the next proposition, we assume that $B^{D\times D} \to M^{D\times D}$ is a surjection. (This is valid in the models since $(-)^{D\times D}$ has a right-adjoint; cf. Appendix 4.)

6.4.1 Proposition. *$\overline{\Omega}(\gamma)$ depends only on $p \circ \gamma \in M^{D\times D}$. So by surjectivity of $B^{D\times D} \to M^{D\times D}$ it induces a unique $\underline{g}$-valued 2-form Ω on M making the following diagram commute*

$$\begin{tikzcd} B^{D\times D} \arrow[r, "\overline{\Omega}"] \arrow[d] & \underline{g} \\ M^{D\times D} \arrow[ur, "\Omega"'] & \end{tikzcd}$$

Moreover, $\overline{\Omega}$ coincides with $D\omega$, i.e. for $\gamma \in B^{D\times D}$

(6) $\overline{\Omega}(\gamma) = \Omega(p\gamma) = D\omega(\gamma) = d\omega(h\gamma).$

This 2-form on M is called the curvature of the Ehresmann connection.

Proof. We will use that ω satisfies the equation (1) of 6.3. First, we claim that if $(\gamma, h_1, h_2) \in B^{D\times D} \times D \times D$, then for *any* $b \in B_m$ (where $m = p\gamma(0)$) we have

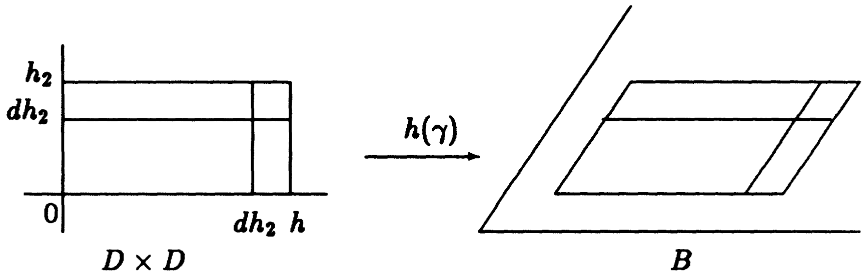
(7) $\overline{\Omega}(\gamma, h_1, h_2) = \sigma_{b_{12}}(b_{34}),$

where b_{12} is the result of transporting b first along (γ_1, h_1) and then along (γ_2, h_2), i.e. $b_{12} = \nabla(p\gamma_2, \nabla(p\gamma_1, b)(h_1))(h_2)$, and similarly $b_{34} = \nabla(p\gamma_3, \nabla(p\gamma_4, b)(h_2))(h_1)$. The proof is the same as that of 5.1, as illustrated by the picture following 5.1. This shows that $\overline{\Omega}(\gamma)$ depends only on $p \circ \gamma$.

Next, to see that $d\omega(h(\gamma)) = \overline{\Omega}(\gamma)$ is simply a matter of unwinding the definition of $d\omega$. (The reader should draw the right picture and see that it is really immediate that this identity holds.) Choose $h_1, h_2 \in D$. Then for $d \in D$, by (1) above and the structure of T_eG (cf. 1.6), we get that $d\omega(\gamma, h_1, h_2): D \to G$ is given by

$$(8)\ d\omega(h(\gamma), h_1, h_2)(d) = \omega(h(\gamma)_4)(dh_2)^{-1}\omega(h(\gamma)_3)(dh_1)^{-1}$$
$$\omega(h(\gamma)_2)(dh_2)\omega(h(\gamma)_1)(dh_1)$$

where $h(\gamma)_1 = h(\gamma)(-,0), h(\gamma)_2 = h(\gamma)(h_1,-)$, etc. as before.



By definition of $h(\gamma), h(\gamma)_1, h(\gamma)_2$ and $h(\gamma)_4$ are horizontal, so ω of these terms vanishes, i.e. (8) becomes

$$(9)\quad \begin{aligned} d\omega(h(\gamma), h_1, h_2)(d) &= \omega(h(\gamma)_3)(dh_1)^{-1} \\ &= \sigma_{h(\gamma)_3(dh_1)}\nabla(p(\gamma_3), h(\gamma)_3(0))(dh_1). \end{aligned}$$

But

$$h(\gamma)_3(dh_1) = \nabla(p\gamma(h_1,-), \nabla(p\gamma(-,0), \gamma(0))(dh_1))(h_2),$$

and

$$\nabla(p\gamma_3, h(\gamma)_3(0))(dh_1) = \nabla(p\gamma(-,h_2), \nabla(p\gamma(0,-), \gamma(0))(h_2))(dh_1),$$

so by the description of $\overline{\Omega}$ given in the first part of the proof, it is clear that

$$(10)\ d\omega(h(\gamma), h_1, h_2)(d) = \overline{\Omega}(\gamma, dh_1, h_2)$$

(i.e. the left hand side is computed by circulation of $\gamma(0)$ along $\partial(p\gamma, dh_1, h_2)$, which is clear anyway). In terms of maps $B^{D\times D} \to \underline{g}$, (10) says that $d\omega(h(\gamma)) = \overline{\Omega}(\gamma)$.

This completes the proof. □

6.5 The Case of the Gauss-Bonnet Theorem. The case of the Gauss-Bonnet theorem discussed in Section 5 fits into the general context of connections on principal fiber bundles. In section 5, we considered the bundle of rays $\mathrm{Rays}(TM) \to M$ with a section $M \xrightarrow{X} \mathrm{Rays}(TM)$, and with group $SO(2,R)$. (As pointed out in 6.1, X gives a trivialization $\mathrm{Rays}(TM) \cong M \times SO(2,R)$, so $\mathrm{Rays}(TM)$ is microlinear.) In Section 5, we defined the curvature form K directly from a transport of rays. In the case that this transport comes from an Ehresmann connection on $\mathrm{Rays}(TM) \to M$, K coincides with the curvature form Ω of 6.4, as is clear from the description of Ω given in 6.3.

7 *Back to Classical Manifolds*

In this final section, we will compare the synthetic notions discussed in this chapter with the corresponding classical ones. In this way, by interpreting in the topos-models of Chapter III and then "translating" back into the language of classical manifolds, we obtain classical versions of the synthetic results. It should be pointed out that the classical results obtained in this way are often more general than those found in the literature. For example, the Ambrose-Palais-Singer theorem is generalized to a class of spaces which includes spaces with singularities, and spaces $C^\infty(M,N)$ of smooth functions from one smooth manifold M to another N. In this context, it is interesting to observe that the original proof of this theorem proceeds via local integration, and hence does not generalize immediately.

7.1 Embedding of Manifolds in the Toposes $\mathcal{F}$ and $\mathcal{G}$.

In Chapter III we have discussed the embeddings

$$\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{F}, \ \mathbb{M} \overset{s}{\hookrightarrow} \mathcal{G}$$

of the category of smooth manifolds into the two archimedean models $\mathcal{F}$ and $\mathcal{G}$. Let us recall some elementary properties. (Here the embedding s and the global section functor Γ refer to either $\mathcal{F}$ or $\mathcal{G}$.)

(i) s is a full and faithful functor which preserves transversal pullbacks.
(ii) Γ preserves inverse limits (since it has a left adjoint Δ, the constant sheaf functor).
(iii) $\Gamma s(M) \cong M$ for all $M \in \mathbb{M}$.
(iv) $\Gamma(s(N)^{s(M)}) \cong C^\infty(M,N)$ for all $M, N \in \mathbb{M}$, as follows from (i). (This extends to manifolds with boundary; for example

$$\Gamma(s(M)^{[0,1]}) \cong C^\infty([0,1], M) \text{ for all } M \in \mathbb{M}.) \text{ etc.}$$

In particular, properties of manifolds which may be expressed by commutativity of diagrams in $\mathbb{M}$ hold classically precisely when they hold in $\mathcal{F}$ or $\mathcal{G}$. We shall use this fact tacitly in the sequel.

There is a well-defined full subcategory of *microlinear spaces* in $\mathcal{F}$ or $\mathcal{G}$, which is *closed under inverse limits and exponentation*, by 1.2. Moreover, it contains all the manifolds, as stated in the following proposition.

Proposition. *For all $M \in \mathbb{M}$, $s(M)$ is a microlinear space (in $\mathcal{F}$ or $\mathcal{G}$). In fact, all representables $Y(\ell A)$ are microlinear (where Y is the Yoneda embedding $\mathbb{F} \hookrightarrow \mathcal{F}$ or $\mathbb{G} \hookrightarrow \mathcal{G}$).*

$$\begin{tikzcd} R \times T(M) \arrow[rr, "\cdot"] \arrow[dr] & & T(M) \arrow[dl] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} T(M) \underset{M}{\times} T(M) \arrow[rr, "+"] \arrow[dr] & & T(M) \arrow[dl] \\ & M & \end{tikzcd}$$

is sent by s to the (internal) vector bundle structure obtained canonically from the microlinearity of $s(M)$.

Proof. For definiteness, let us work in $\mathcal{G}$. Since $R = s(\mathbb{R})$ is microlinear, it follows that $R^n = s(\mathbb{R}^n)$ is microlinear for every n. But if $\ell A \in \mathbb{G}$, say $A = C^\infty(\mathbb{R}^n)/I$, then ℓA is the joint coequalizer of the maps

$$R^n \underset{0}{\overset{\{s(f): f \in I\}}{\rightrightarrows}} R$$

in $\mathcal{G}$. Hence by 1.2 (ii), ℓA (or more precisely $Y(\ell A)$) is microlinear.

As for the tangent bundle structure, cover a given manifold M by a family of open subspaces $\{U_\alpha\}_\alpha$ of M, each diffeomorphic to $\mathbb{R}^n$ ($n = \dim M$). Then we have corresponding open covers

$$\begin{aligned} T(M) &= \textstyle\bigcup_\alpha T(U_\alpha) \\ T(M) \times_M T(M) &= \textstyle\bigcup_\alpha T(U_\alpha) \times_{U_\alpha} T(U_\alpha). \end{aligned}$$

Since s preserves open covers and transversal pullbacks (such as $TM \times_M TM$), we find that in $\mathcal{F}$ of $\mathcal{G}$

$$s(M)^D = \bigcup_\alpha s(U_\alpha)^D, s(M)^D \times_{S(M)} s(M)^D = \bigcup_\alpha s(U_\alpha)^D \times_{s(U_\alpha)} s(U_\alpha)^D,$$

by II.1.2.

This reduces our problem to proving that $s(\cdot), s(+)$ coincide with the operations obtained from microlinearity for the case $\mathbb{M} = \mathbb{R}^n$, and this is clear. (cf. the computation in Section 1.) □

Proposition. *(generalized Kock-Lawvere axiom) The generalized Kock-Lawvere axiom stated in Section 1 holds in $\mathcal{F}$ and $\mathcal{G}$. Explicitly, if $0 \in S \subset D_k(n)$ is the zero set (in the topos) of finitely many polynomials $p_1, \ldots, p_\ell \in R[x_1, \ldots, x_n]$, each of total degree $\leq k$, i.e. $S = \{x \in D_k(n): p_i(x) = 0, i = 1, \ldots, \ell\}$, then every $f \in R^S$ is the restriction of a map in R^{R^n} given by a polynomial of total degree $\leq k$. Furthermore, this polynomial is unique modulo the ideal $(p_1, \ldots, p_\ell) \subset R[x_1, \ldots, x_n]$.*

Proof. To fix notation, let us work in $\mathcal{G}$. So we have to show that given $k, m, n \in \mathbb{N}$, $\mathcal{G} \models [\forall f_1, \ldots, f_m \in R[x_1, \ldots, x_n]((\forall \alpha \in \mathbb{N}^n |\alpha| > k \to X^\alpha \in (f_1, \ldots, f_m)) \to \forall h \in R^{Z(f)} \exists p \in R[x_1, \ldots, x_n]$ of total degree $\leq k(\forall x \in Z(f)\ h(x) = p(x) \wedge p$ is unique mod $(f_1, \ldots, f_m)))]$, where $Z(f) = \{\underline{x} \in R^n | f_1(\underline{x}) = \ldots = f_m(\underline{x}) = 0\}$ as an object of $\mathcal{G}$.

This is a simple matter of unwinding the forcing clauses: suppose $(f_1, \ldots, f_m)$ are given at stage ℓA, and $\ell A \Vdash \forall \alpha \in \mathbb{N}^n(|\alpha| > k \to x^\alpha \in (f_1, \ldots, f_m))$. Since $R[x_1, \ldots, x_n](\ell A) = A[x_1, \ldots, x_n], f_1, \ldots, f_m$ can be identified with elements $F_1, \ldots, F_m$ of $A[x_1, \ldots, x_n]$, giving a Weil A-algebra (in *Sets*) $A[x_1, \ldots, x_n]/(F_1, \ldots, F_m)$ with $x^\alpha \in (F_1, \ldots, F_m)$ for $|\alpha| > k$ (use partitions of unity).

We have to show $\ell A \Vdash \forall h \in R^{Z(f)}(\ldots)$. So take $\ell B \to \ell A$ and $h \in R^{Z(f)}(\ell B)$. By replacing A by B we may assume $A = B$, to simplify notation. Consider the maps

$$\begin{tikzcd} \ell W \arrow[r, "\eta"] \arrow[d, "q"] & \ell A \\ R^n & \end{tikzcd}$$

in $\mathbb{G}$: η corresponds to the canonical map $A \xrightarrow{\eta} W$, and $\underline{q} = (q_1, \ldots, q_n)$ where q_i is the projection $\ell W \to R$ given by the C^∞-homomorhism $C^\infty(\mathbb{R}) \to W$ which sends the generator id to x_i. Then $\underline{q} = (q_1, \ldots, q_n) \in R^n(\ell W)$ and $\ell W \Vdash f_i|\eta(\underline{q}) = 0 (i = 1, \ldots, m)$. So $h_{\ell W}(\underline{q}) \in R(\ell W) = W$, i.e. $h_{\ell W}(\underline{q})$ is a polynomial $p(x_1, \ldots, x_n)$ mod $(F_1, \ldots, F_m)$, so in particular can be taken to be of total degree $\leq k$.

This polynomial $p(x_1, \ldots, x_n)$ mod $(F_1, \ldots, F_m)$ determines h completely, since q is the *generic* element of $Z(f)$ over A, in the sense that whenever there are $R^n \xleftarrow{\underline{u}} \ell C \xrightarrow{\alpha} \ell A$ in $\mathbb{G}$ such that $\ell C \Vdash \underline{u} \in Z(f)|\alpha = Z(f|\alpha)$, then there is a unique $\ell C \xrightarrow{\beta} \ell W$ in $\mathbb{G}$ such that $\eta\beta = \alpha$, $\underline{q}\beta = \underline{u}$. From this it is clear that ℓA forces the consequent $\exists p \in R[x_1, \ldots, x_n]$ of total degree $\leq k(\ldots)$. □

Proposition 7.2 implies that the synthetic arguments using R-colimits of infinitesimal spaces, as presented earlier in this chapter, are valid in $\mathcal{F}$ and $\mathcal{G}$. We will now see what this gives us in the case of the Ambrose-Palais-Singer theorem (see Section 2).

7.3 The Ambrose-Palais-Singer Theorem. We wish to consider the classical reformulations of this theorem. First of all, we have to compare the classical notions of connection and spray with the synthetic ones.

7.3.1 Proposition. *Let $M \in \mathbb{M}$. The global sections functor Γ (from either $\mathcal{F}$ or $\mathcal{G}$ into Sets) defines a bijection between the (internal) affine connections on $s(M)$ and the (classical) affine connections on M. Furthermore, this bijection restricts to one between symmetric connections. Similarly, Γ defines a bijection between internal sprays on $s(M)$ and classical sprays on M.*

Proof. Recall that, classically, an affine connection ∇ is given as a splitting in $\mathbb{M}$ of the canonical map K shown in the diagram

$$(1) \qquad T(M) \times_M T(M) \underset{K}{\overset{\nabla}{\rightleftarrows}} T(T(M)),$$

which is linear with respect to both vector bundle structures of the total spaces in (1) over $T(M)$, given by $(p_2, \pi_{T(M)})$ and $(p_1, T(\pi^M))$ respectively. (K is the map $\langle T(\pi_M), \pi_{T(M)}\rangle$.)

Since this linearity can be expressed by commutative diagrams in $\mathbb{M}$ and s is a functor, we conclude (using $s(TM) = s(M)^D$) that $s(\nabla)$ is a (synthetic) affine connection on $s(M)$ in $\mathcal{F}$ or $\mathcal{G}$. The case of connections now follows from $\Gamma s(M) \cong M$ for every $M \in \mathbb{M}$ and the fact that s is full and faithful and preserves transversal pullbacks; see 7.1. □

Classically, a spray on M is defined as a section of $T(\pi_M)$ (or equivalently of $\pi_{T(M)}$ by symmetry),

$$T(M) \underset{T(\pi_M)}{\overset{\sigma}{\rightleftarrows}} T(T(M))$$

which is symmetric ($\Sigma \circ \sigma = \sigma$, where $T(T(M)) \xrightarrow{\Sigma} T(T(M))$ is the canonical twist map), and homogeneous. To prove that Γ and s establish a bijection between classical sprays on M and internal sprays on $s(M)$ (in the sense of 2.4), we may proceed exactly as in the case of connections, but after having noticed the following "synthetic" lemma:

Lemma. *Let M be any microlinear space. There is a natural bijection between sprays as defined in 2.4, and maps $M^D \xrightarrow{\sigma} M^{D\times D}$ which are symmetric ($\sum\circ\sigma = \sigma$, where $\sum$ is as in 4.1), homogeneous (i.e. $\sigma(\alpha\cdot t)(d_1,d_2) = \sigma(t)(\alpha d_1, \alpha d_2)$), and satisfy $\pi_{M^D}\circ\sigma = (\pi_M)^D\circ\sigma = \mathrm{id}$.*

This lemma is obvious from the R-coequalizer

$$D\times D \underset{\tau}{\overset{\mathrm{id}}{\rightrightarrows}} D\times D \xrightarrow{+} D_2, \text{ where } \tau(d_1,d_2) = (d_2,d_1).$$

This completes the proof of 7.3.1. □

Combining 2.5 and 7.3.1, we now immediately conclude the following.

7.3.2 Corollary (The classical Ambrose-Palais-Singer theorem) *Let M be a manifold. There is a natural bijection between symmetric connections $T(M)\times_M T(M) \xrightarrow{\nabla} T(M)$ on M and sprays $T(M) \xrightarrow{\sigma} T^2(M)$ on M, given by $\sigma(t) = \nabla(t,t)$.*

In fact, Theorem 2.5 is much more general than 7.3.2, since it is valid for all microlinear spaces in a model (like $\mathcal{F}$ or $\mathcal{G}$), and not just those of the form $s(M), M \in \mathbb{M}$. As an example, we will rephrase Theorem 2.5 in classical terms for the particular case of function spaces.

7.3.3 Smooth Maps of Function Spaces Let X, Y, Z, W be manifolds. A map

$$C^\infty(X,Y) \xrightarrow{F} C^\infty(Z,W)$$

is smooth if it is smooth along curves, i.e. if for any smooth map $\mathbb{R} \times X \xrightarrow{\alpha} Y$, the map $\mathbb{R} \times Z \xrightarrow{F(\alpha)} W$ defined by

$$F(\alpha)(t,z) = F(\alpha(t,-))(z)$$

is smooth. It follows from Boman's theorem (see Boman (1967)) that $\mathbb{R}$ can be replaced by any manifold; i.e. if F is smooth in the above sense and T is a manifold, then for any smooth $T \times X \xrightarrow{\alpha} Y$, $F(\alpha): T \times Z \to W$ is again smooth.

7.3.4 Connections and Sprays on Function Spaces Let M and N be manifolds. An *affine connection on* $C^\infty(M,N)$ is a smooth splitting ∇ of the map K_*

$$C^\infty(M, TN \times_N TN) \underset{K_*}{\overset{\nabla}{\leftrightarrows}} C^\infty(M, T^2N)$$

(where K_* is induced by $T^2(N) \xrightarrow{K} TN \times_N TN$ via composition), which is linear with respect to the two structures of these spaces over $C^\infty(M,TN)$ (corresponding to the two structures of $T^2(N)$ and $TN \times_N TN$ over TN).

Similarly, one defines a *spray on* $C^\infty(M,N)$ as a smooth map

$$C^\infty(M,TN) \xrightarrow{\sigma} C^\infty(M,T^2(N))$$

which splits the map $C^\infty(M,T^2N) \xrightarrow{T(\pi_M)_*} C^\infty(M,TN)$ induced by $T(\pi_M)$ via composition, and which satisfies the obvious homogeneity and symmetry conditions.

7.3.5 Corollary (Ambrose-Palais-Singer theorem for function spaces). spaces). *Let M and N be manifolds, and assume that either M is compact of $N = \mathbb{R}^p$. There is a natural bijection between symmetric connections $C^\infty(M,TN \times_N TN) \xrightarrow{\nabla} C^\infty(M,T^2N)$ on $C^\infty(M,N)$ and sprays $C^\infty(M,TN) \xrightarrow{\sigma} C^\infty(M,T^2N)$ on $C^\infty(M,N)$ given by $\sigma(f) = \nabla(f,f)$.*

7.3.5 follows from 2.5 just as 7.3.2 does, using the following lemma.

Lemma. *Let X, Y, Z, W be manifolds, and assume that X is compact or $Y = \mathbb{R}^p$. The global sections functor Γ establishes a bijection between maps $s(Y)^{s(X)} \to s(W)^{s(Z)}$ in $\mathcal{G}$ (or $\mathcal{F}$) and smooth maps*

$C^\infty(X,Y) \to C^\infty(Z,W)$ *as in 7.3.3.*

Proof of Lemma. We will only do the case where X is compact. The case where $Y = \mathbb{R}^p$ is similar but much easier.

If $s(Y)^{s(X)} \xrightarrow{\varphi} x(W)^{s(Z)}$ is a map in $\mathcal{G}$, then $F = \Gamma\varphi\colon C^\infty(X,Y) \to C^\infty(Z,W)$ is clearly smooth in the sense of 7.3.3—just apply φ to elements of $s(Y)^{s(X)}$ at stage $R = \ell C^\infty(\mathbb{R}) \in \mathbb{G}$, and use naturality of φ.

Conversely, suppose $C^\infty(X,Y) \xrightarrow{F} C^\infty(Z,W)$ is smooth. Then F induces a map $s(Y)^{s(X)} \to s(W)^{s(Z)}$ in $\mathcal{G}$ as follows. An element $\ell A \xrightarrow{\alpha} s(Y)^{s(X)}$ at stage $\overline{A} \in \mathbb{G}$ corresponds to a map $\ell A \times s(X) \xrightarrow{\alpha} s(Y)$ in $\mathbb{G}$. Write $A = C^\infty(\mathbb{R}^n)/I, X \subset \mathbb{R}^m, Y \subset \mathbb{R}^k$; then α is induced by a smooth map

$$\mathbb{R}^n \times \mathbb{R}^m \xrightarrow{\overline{\alpha}} \mathbb{R}^k.$$

Now write Y as a retract of an open U, $Y \overset{r}{\underset{\hookrightarrow}{\leftarrow}} U \subset \mathbb{R}^k$ (see GP, pp. 69–70), and let $V = \overline{\alpha}^{-1}(U)$. Then $\ell A \times s(X) \subset s(V)$, so there is a finitely generated ideal $I_o \subset I$ such that $Z(I_o) \times X \subset V$ (cf. II.1.7). By compactness of X, we find an open O, $Z(I_o) \subset O \subset \mathbb{R}^n$, such that $O \times X \subset V$. Then we have a commutative diagram

$$\begin{tikzcd}
\ell A \times s(X) \arrow[r, "\alpha"] \arrow[d] & s(Y) \\
s(O \times X) \arrow[r, "s(\overline{\alpha})"'] & s(U) \arrow[u, "s(r)"']
\end{tikzcd}$$

and we can define $\varphi_{\overline{A}}(\alpha)\colon \ell A \times s(Z) \to s(W)$ as the restriction of the map $s(O \times Z) \to s(W)$ coming from $F(\alpha)\colon O \times Z \to W$ defined by

$$F(\alpha)(t,z) = F(r \circ \overline{\alpha}(t,-))(z),$$

which is smooth by the remarks in 7.3.3.

One has to check that φ is well-defined, and natural. We leave this to the reader. It then easily follows that the processes of defining φ from a given F and conversely are inverse to each other. □

7.3.6 Remark. Note that 7.3.5 is really more general than 7.3.2, since not every affine (symmetric) connection on $C^\infty(M,N)$ comes from one on N by composition. To take a simple example, define a connection ∇ on R^R (or equivalently, on $C^\infty(\mathbb{R},\mathbb{R})$) as follows: for $f \in R^R$ and $X,Y \in T_f(R^R)$ we may write $X(x,d) = f(x) + dg(x), Y(x,d) = f(x) + dh(x)$; then

$$\nabla_f(X,Y)(x)(d_1,d_2) = f(x) + d_1 g(x) + d_2 h(x) + d_1 d_2 g(x)h(x)f'(x)$$

defines an affine symmetric connection on R^R. Since $\nabla_f(X,Y)(x)$ depends not only on $f(x)$ but also on $f'(x)$, it cannot come from a connection on R.

This completes 7.3. As another example to illustrate the relation to the classical theory, we consider the Gauss-Bonnet theorem as discussed in Section 5.

7.4 Gauss-Bonnet in Dimension 2. Let $M \in \mathbb{M}$ be a smooth 2-dimensional manifold with a conformal Riemannian structure, that is, an equivalence class of Riemannian metrics for the equivalence relation of one being a multiple of the other by a positive C^∞-function $M \to \mathbb{R}$.

In a way which exactly parallels our synthetic construction, we may define the bundle of oriented rays $\text{Rays}(TM) \xrightarrow{p} M$. This is a principal fiber bundle with group $SO(2,\mathbb{R}) \cong S^1$.

Suppose we are given an Ehresmann connection on this bundle, i.e. a 1-form

$$T(\text{Rays}(TM)) \xrightarrow{\omega} \mathbb{R} = T_{\text{id}}(SO(2,\mathbb{R}))$$

satisfying the usual classical axioms (analogous to (3′), (4) in 6.2).

The curvature form Ω (or rather $\overline{\Omega}$, cf. 6.4.1) of this connection coincides in this case with the exterior derivative of ω, since the Lie group $SO(2,\mathbb{R})$ is abelian. Furthermore, Ω factors through $T(M) \times_M T(M)$ and gives rise to a 2-form

$$T(M) \times_M T(M) \xrightarrow{\tilde{K}} \mathbb{R}.$$

It should be clear from what we did in Sections 5 and 6 that $s(\tilde{K})$–a 2-form on M in $\mathcal{G}$–, coincides with the curvature form K defined in Section 5.

Rephrasing the notion of geodesic curvature of a regular curve as described in Section 5 in classical language, we obtain the following definition. Let $[0,1] \xrightarrow{\gamma} M$ be a regular curve (i.e. $\dot\gamma(t) \in T_{\gamma(t)}(M)$ is a non-zero vector, for all $t \in [0,1]$). Fixing $t \in [0,1]$, let $\tau_t: p^{-1}(\gamma(0)) \to p^{-1}(\gamma(t))$ be the parallel transport of fibers of p along γ. Then (writing $[-]$ for the ray generated by–)

$$\tau_{t+h} \circ \tau_t^{-1}[\dot\gamma(t)] \cdot [\dot\gamma(t+h)]^{-1} = \begin{pmatrix} a(h) & b(h) \\ -b(h) & a(h) \end{pmatrix}$$

for a unique pair of functions a, b such that $a^2+b^2 \equiv 1$. In particular, $a(0) = 1, b(0) = 0$, and our matrix may be written as

$$\mathrm{Id} + h \begin{pmatrix} 0 & b'(0) \\ -b'(0) & 0 \end{pmatrix} + h^2 A(h).$$

We then define $\tilde K_g(\gamma, t) = b'(0)$, thereby obtaining a new function $\tilde K_g: \mathrm{Reg}(M^{[0,1]}) \times [0,1] \to \mathbb{R}$.

Now assume that we have a section $M \xrightarrow{X} \mathrm{Rays}(TM)$. Consider the unique matrix $A = \begin{pmatrix} a(t) & -b(t) \\ b(t) & a(t) \end{pmatrix}$ with $a^2+b^2 = 1$ such that $A \cdot X(\gamma(t)) = [\dot\gamma(t)] =$ ray generated by $\dot\gamma(t)$, and let $\dot\theta: [0,1] \to \mathbb{R}$ be the function given by $\dot\theta(t) = a'(t)b(t) - a(t)b'(t)$ (classically, one writes "symbolically" $d\theta = (a'(t)b(t) - a(t)b'(t))dt$).

Again, $\tilde K_g$ and θ correspond to the notions defined synthetically in Section 5, and we immediately conclude the following result.

7.4.1 Theorem. *(Conformal Gauss-Bonnet in dimension 2). Under the above hypotheses, we have for every smooth regular 2-chain $c: [0,1]^2 \to M$ the identity*

$$\int_{\partial c} d\theta = \int_c \tilde K + \int_{\partial c} \tilde K_g.$$

The usual local formulation of Gauss-Bonnet follows directly from 7.4.1. If M is an oriented Riemannian manifold of dimension 2 with metric g (and conformal structure corresponding to g), the Levi-Civita connection obtained from g gives rise to an Ehresmann connection on $\mathrm{Rays}(TM) \to M$ in an obvious way. One then writes the 2-form $\tilde K$ as a multiple of the area form $d\sigma$ of (M, g), i.e. $\tilde K = K d\sigma$ for a scalar K, *the Gauss curvature of* M. Furthermore, $\tilde K_g = k_g ds$ along ∂c (where ds is the length form along ∂c), for a scalar k_g, *the geodesic curvature of* ∂c. Using this, we can rewrite 7.4.1 as

7.4.2 Theorem. *(Local Gauss-Bonnet in dimension 2). If M is an orientable Riemannian manifold and $X: M \to TM$ a nowhere* *vanishing vector field, then for any regular 2-chain* $c: [0,1]^2 \to M$,

$$\int_{\partial c} d\theta = \int_c K d\sigma + \int_{\partial c} k_g ds,$$

where K is the Gauss curvature of M, and k_g is the geodesic curvature of ∂c. □

7.5 Cartan's Definition of the Riemann-Christoffel Tensor. By now it is clear that many geometric notions can be more directly described in the synthetic context. As a final example in this chapter, we show how one can define the Riemann-Christoffel tensor of a connection by "translating along an infinitesimal 2-chain". To do this for a classical manifold $M \in \mathbb{M}$, we first embed M in $\mathcal{G}$ by the functor s, then describe the tensor in $\mathcal{G}$, and finally "take it out" by applying Γ. In this way, we can follow E. Cartan's description word by word.

So let ∇ be a connection on a smooth manifold M. We define a tensor $R: T(M) \times_M T(M) \times_M T(M) \to T(M)$ which measures the curvature of ∇, by working in $\mathcal{G}$, rather than in $\mathbb{M}$.

To simplify notation, we identify $s(M)$ with M, $s(\nabla)$ with ∇, etc. We first define a map

$$\overset{\approx}{R}: (M^{D\times D} \times D \times D) \times_M M^D \to M^D.$$

Let $(\gamma, h_1, h_2) \in M^{D\times D} \times D \times D$ be any 2-chain, and let $t_3 \in \mathbb{M}^D$. Write τ for the parallel transport defined from ∇ via $\tau_{h_1}(t_1,t_2)(h_2) = \nabla(t_1,t_2)(h_1,h_2)$ as before, and let

$$\tau(\gamma,h_1,h_2,t_3) = \tau_{h_2}^{-1}(\gamma_4, \tau_{h_2}^{-1}(\gamma_3, \tau_{h_1}(\gamma_2, \tau_{h_1}(\gamma_1, t_3))))$$

be the vector obtained by transporting t_3 around $\partial(\gamma, h_1, h_2)$ (as usual, $\gamma_1 = \gamma(-,0), \gamma_2 = \gamma(h_1,-)$, etc.). Then by definition,

$$\overset{\approx}{R}(\gamma,h_1,h_2,t_3) = \tau(\gamma,h_1,h_2,t_3) - t_3 \in T_{\gamma(0,0)}(M).$$

For a fixed γ, the function $\varphi: D \times D \to T_m(M)$ $(m = \gamma(0,0))$ defined by $\varphi(h_1,h_2) = \overset{\approx}{R}(\gamma,h_1,h_2,t_3)$ induces by microlinearity of $T_m(M)$ (see 3.3) a unique $D \xrightarrow{\theta} T_m(M)$ with $\theta(h_1 \cdot h_2) = \varphi(h_1,h_2)$. On the other hand, $T_m(M)$ satisfies the Kock-Lawvere axiom (1.4), so we may write $\theta(h) = h \cdot t$ for a unique $t \in T_m(M)$. This defines a function

$$\tilde{R}: M^{D\times D} \times_M M^D \to M^D$$

determined by $\approx{R}\,(\gamma, h_1, h_2, t_3) = h_1 \cdot h_2\ \tilde{R}\,(\gamma, t_3)$.

We can now define the Riemann-Christoffel tensor R simply as

$$R\colon T(M) \times_M T(M) \to T(M)$$
$$R(t_1, t_2)t_3 = \tilde{R}(\nabla(t_1, t_2))(t_3).$$

By introducing coordinates, we may assume $M = R^n$. In this case $\nabla\colon M \times R^n \to M \times R^n \times R^n \times R^n$ is completely determined by its last component ∇_4 (since K is the projection), i.e. in terms of a basis $\{e_1, \ldots, e_n\}$ of R^n,

$$\nabla_4(x, e_i, e_j)(e_\ell) = -\Gamma^\ell_{ji}(x).$$

(The Γ^ℓ_{ji} are the so-called *Christoffel symbols of the second kind*). Furthermore, a horrible coordinate computation gives

Proposition. *Let $\gamma \in M^{D\times D}$ be a 2-chain. Then $\tilde{R}(\gamma)$ depends only on $K(\gamma)$. So we have a commutative diagram*

$$\begin{tikzcd} M^{D\times D} \underset{M}{\times} M^D \arrow[r, "\tilde{R}"] \arrow[d, "K \times \mathrm{id}"'] & M^D \\ M^D \underset{M}{\times} M^D \underset{M}{\times} M^D \arrow[ur, "R"'] & \end{tikzcd}$$

Furthermore, in coordinates, the ℓ^{th} component of $\tilde{R}(\gamma, (x, e_k))$ is given by

$$R^\ell_{kji}(x) := R((x, e_i), (x, e_j))(x, e_k))_\ell$$
$$= \tfrac{\partial}{\partial x_j}\Gamma^\ell_{ki}(x) - \tfrac{\partial}{\partial x_i}\Gamma^\ell_{kj}(x) - \textstyle\sum_\alpha(\Gamma^\alpha_{kj}(x)\Gamma^\ell_{\alpha i}(x) - \Gamma^\alpha_{ki}(x)\Gamma^\ell_{\alpha j}(x)).$$

(This is the classical formula for the Riemann-Christoffel tensor.) □

We end our comparison with classical notions here. The examples of the Ambrose-Palais-Singer theorem and the Gauss-Bonnet theorem sufficiently illustrate how this comparison works in general. Of course we could go further, and translate more of Sections 1, 3, 4 in a classical language. This would show, for example, the the Lie algebra of a Lie group as defined in 1.6 corresponds via s with the classical one for manifolds (but 1.6 also works for "Lie groups" which are not manifolds, such as groups of diffeomorphisms of manifolds, etc). Similarly, the theory of "vector bundles" discussed in Section 3 includes the case of vector bundles of function spaces automatically.

Also, by taking sections at an object $X \in \mathbb{M} \subset \mathbb{G}$, rather than just global sections (the case $X = 1$), one obtains "smooth in parameter-versions" of the results, similar to what we did in Section IV.4.

We will not work all this out in detail, however: the calculations should by now be rather routine, and can safely be left to the reader.

Chapter VI

Models with Invertible Infinitesimals

We have seen in the Introduction that at least two different kinds of infinitesimals have appeared in the literature. On the one hand, there are the *nilpotent*, whose use in handling "infinitesimal" structures like jets, prolongations, connections, etc. has been extensively illustrated in the preceding two chapters. On the other hand, there are the *invertible* infinitesimals which, together with infinitely large integers, are used to analyze such notions as limits and convergence along the lines of Non-Standard Analysis, as exemplified by Robinson's book. Thus, these types of infinitesimals serve different (and complementary) purposes, and both should appear in a theory of infinitesimals that is worth its salt.

The aim of this chapter is to discuss two models of synthetic differential geometry which contain not only the "usual" nilpotent infinitesimals which are present in the models of chapter III, but also "big", invertible infinitesimals and infinitely large ("non-standard") natural numbers.

The idea of the construction of these models is simple. Recall from chapter II that the object $\mathbb{I}$ of $\mathbb{L}$ corresponding to the C^∞-ring $C^\infty(\mathbb{R} - \{0\})/(m^g_{\{0\}}|\mathbb{R} - \{0\})$ satisfies the following statements in $Sets^{\mathbb{L}^{op}}$:

$$\forall x \in \mathbb{I} \quad (x \text{ is invertible})$$
$$\forall x \in \mathbb{I} \quad (-\tfrac{1}{n} < x < \tfrac{1}{n}) \quad n = 1, 2, 3, \ldots$$

Accordingly, we called $\mathbb{I}$ the space of invertible infinitesimals (II.1.10). But $\mathbb{I}$ is not an object of $\mathbb{G}$ (or $\mathbb{F}$), and there are no invertible infinitesimals in $\mathcal{G}$ and $\mathcal{F}$. We now wish to take $\mathbb{L}$ as the category underlying our site. This, however, forces us to change the Grothendieck topology, since the "open cover topology" of chapter III is not subcanonical as a topology on $\mathbb{L}$. For the first model, the "smooth Zariski topos" $\mathcal{Z}$, we will take finite open covers only. Later on, for the

"Basel topos" $\mathcal{B}$, we will, in addition, take projections as covers. These topologies on $\mathbb{L}$ are subcanonical.

As a consequence of restricting the covers in this way, the space $N = \ell C^\infty(\mathbb{N})$ does *not* coincide with the natural number object of $\mathcal{Z}$ or $\mathcal{B}$, contrary to the situation for $\mathcal{G}$ and $\mathcal{F}$. This object N of "*smooth natural numbers*" properly contains the natural number object ("the standard natural numbers"), and it contains infinitely large numbers besides. Our global strategy will be to work with N, rather than with the natural number object, and replace the notions referring to the natural numbers (such as finiteness, Archimedeanness, compactness, etc.) throughout by their "smooth" analogues (s-finite, s-Archimedean, etc.) defined in terms of N. What this comes down to, really, is that we do not only weaken the underlying *logic* (as in the models of Chapter III, we cannot use excluded middle or choice), but also the *arithmetic*, i.e. the theory of N. In this chapter, we will demonstrate the appropriateness of this move in relation to the models $\mathcal{B}$ and $\mathcal{Z}$. In chapter VII, we will elaborate on this from a more "axiomatic" point of view.

After having introduced the topos $\mathcal{Z}$ in section 1, we describe some of the "smooth notions" and show that the natural properties hold. For example, although $[0,1]$ is not compact in $\mathcal{Z}$, it is, nevertheless, "s-compact" in the appropriate sense.

Just as the models discussed in Chapter III, $\mathcal{Z}$ is an extension of the category of manifolds, and we have a full embedding $\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{Z}$. In section 3 we study some properties of this embedding, analogous to what we did in section III.3. The proofs make essential use of some results from dimension theory, and are considerably more difficult than the corresponding ones for $\mathcal{F}$ and $\mathcal{G}$.

Section 4 constitutes another justification of our viewpoint that N should be considered as "the natural numbers", rather than the natural numbers object of the topos. We will prove that the degree of a map in $\mathcal{Z}$ is a smooth integer, and not necessarily a standard one. More generally, we show that for homology theory we obtain the expected results only if we use smooth integers everywhere, and change the basic notions, such as that of a free ring, accordingly.

We should point out that although $\mathbb{I}$ is non-empty in $\mathcal{Z}$, it is not true that $\mathbb{I}$ is inhabited, i.e. $\mathcal{Z} \not\models \exists x(x \in \mathbb{I})$. For several purposes, however, it is desirable that $\mathbb{I}$ be inhabited (cf. for example the discussion of distributions and integrals in §VII.3). Therefore, we will, in a last section, present a modification of $\mathcal{Z}$, the Basel topos $\mathcal{B}$, in which $\mathbb{I}$ is inhabited. It is this topos $\mathcal{B}$ that will serve as the natural model for the axiomatic system presented in Chapter VII. In many ways, $\mathcal{B}$ is similar to $\mathcal{Z}$, and it will suffice to just briefly indicate which modifications have to be made to extend the results about $\mathcal{Z}$ to $\mathcal{B}$.

1 A Smooth Version of the Zariski Topos

In this section, we will introduce the so-called *smooth Zariski topos*, and prove some of its basic properties as a model for synthetic calculus. This topos, denoted by $\mathcal{Z}$, is defined in a way completely analogous to the usual Zariski topos of algebraic geometry which classifies local k-algebras (k is the base ring), but with the theory of k-algebras replaced by that of C^∞-rings. (This is *not* to say that $\mathcal{Z}$ classifies local C^∞-rings, see appendix 2.)

Recall from section II.1 that the category $\mathbb{L}$ of *loci* or *formal C^∞-varieties* is the opposite of the category of finitely generated C^∞-rings and C^∞-homomorphisms. In other words, the *objects* of $\mathbb{L}$ are duals ℓA of C^∞-rings A which are (isomorphic to rings) of the form

$$A = C^\infty(\mathbb{R}^n)/I,$$

where $C^\infty(\mathbb{R}^n)$ is the ring of smooth functions $\mathbb{R}^n \to \mathbb{R}$, and I is an arbitrary ideal. Morphisms of $\mathbb{L}$ from one such dual $\ell(C^\infty(\mathbb{R}^n)/I)$ to another $\ell(C^\infty(\mathbb{R}^m)/J)$ are equivalence classes of smooth functions $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^m$ with the property that $f \in J \Rightarrow f \circ \varphi \in I$, two such functions φ and φ' being equivalent if all their components are equal modulo I, i.e. for each projection $\pi_i (i = 1, \ldots, m)$, $\pi_i \circ \varphi - \pi_i \circ \varphi' \in I$.

In $\mathbb{L}$, we specify some *finite* families of morphisms with common codomain as *covering families*, or *covers*: a family $\{\ell A_i \xrightarrow{f_i} \ell A\}_{i=1}^n$ $(n \geq 0)$ is a covering family iff there exist $a_i \in A (i = 1, \ldots n)$ such that $1 \in (a_1, \ldots, a_n)$, the ideal in A generated by $a_1, \ldots, a_n$, and for each i there is a commutative diagram

$$\begin{tikzcd}
\ell A_i \arrow[rr, "\sim"] \arrow[dr, "f_i"'] & & \ell(A\{a_i^{-1}\}) \arrow[dl, "g_i"] \\
& \ell A &
\end{tikzcd}$$

where g_i is the canonical inclusion of loci. Notice that if A is the trivial ring ($0 = 1$ in A), the empty family is a cover of A.

1.1 Lemma. *The covering families in $\mathbb{L}$ as specified above define a Grothendieck topology on $\mathbb{L}$.*

Proof. We check the three conditions of Appendix 1.

1. Isomorphisms cover: this is clear (take $n = 1, a = 1$).

2. Stability under pullback, i.e. if $\{\ell A_i \to \ell A\}_{i=1}^n$ is a cover then for any $\ell B \xrightarrow{f} \ell A$, the family $\{\ell A_i \times_{\ell A} \ell B \to \ell B\}_{i=1}^n$ is also a cover: to see this, assume $A_i = A\{a_i^{-1}\}$. $\ell B \xrightarrow{f} \ell A$ corresponds to a C^∞-homomorphism $A \xrightarrow{F} B$, and $\ell A_i \times_{\ell A} \ell B \cong \ell(B\{F(a_i)^{-1}\})$, so it suffices to remark that if $1 \in (a_1, \dots a_n)$ then also $1 \in (Fa_1, \dots, Fa_n)$.

3. Stability under composition: suppose $\{\ell A_i \to \ell A\}_{i=1}^n$ is a cover, and for each i we have a cover $\{\ell A_{ij} \to \ell A_i\}_{j=1}^{m_i}$. Assume $A_i = A\{a_i^{-1}\}$, and $A_{ij} = A\{a_i^{-1}\}\{b_{ij}^{-1}\}$. From I.1.6 it follows that $A_{ij} = A\{c_{ij}^{-1}\}$ for some $c_{ij} \in A$. Moreover, if $1 \in (a_1, \dots, a_n) \subset A$ and $1 \in (b_{i_1}, \dots, b_{im_i}) \subset A\{a_i^{-1}\}$, then $1 \in (c_{ij})_{i=1,\dots,n; j=1,\dots m_i}$, as follows again easily from the explicit description of $A\{(-)^{-1}\}$ given in I.1.6. Thus we conclude that the family $\{\ell A_{ij} \to \ell A\}_{ij}$ is a cover of ℓA. $\square$

A perhaps more practical description of this Grothendieck topology is given in the following lemma.

1.2 Lemma. *Let $\ell(C^\infty(U)/I)$ be an object of $\mathbb{L}$, where $U \subset \mathbb{R}^n$ is open, and I is any ideal. Up to isomorphism, the covering families of $\ell(C^\infty(U)/I)$ are precisely those described in one of the following three equivalent ways:*

(i) *families of the form $\{\ell(C^\infty(U_i)/(I|U_i)) \to \ell(C^\infty(U)/I)\}_{i=1}^n$, where the U_i are open subsets of U with $U_1 \cup \dots U_n = U$.*

(ii) *families of the form $\{\ell(C^\infty(U_i)/(I|U_i)) \to \ell(C^\infty(U)/I)\}_{i=1}^n$, where $U_i \subset U$ are open, and there is an open $V \subset U$ with $1 \in (I|V)$ and $U_i \cup \dots \cup U_n \cup V = U$.*

(iii) *families of the form $\{\ell(C^\infty(U_i)/(I|U_i)) \to \ell(C^\infty(U)/I)\}_{i=1}^n$, where $U_i \subset U$ are open, and there is a finitely generated ideal $I_o \subset I$ such that $Z(I_o) \subset U_1 \cup \dots \cup U_n$.*

Proof. The equivalence of (i) and (ii) follows from the fact that if $1 \in (I|V)$ then $C^\infty(U_i)/(I|U_i) \cong C^\infty(U_i \cup V)/(I|U_i \cup V)$. (ii) and (iii) are equivalent descriptions since $1 \in (I|(U - Z(I_o)))$. That all these are equivalent to the families defined above lemma 1.1 is equally easy, using I.1.6. □

This Grothendieck topology makes $\mathbb{L}$ into a *site*, again denoted by $\mathbb{L}$. The *smooth Zariski topos* is by definition the category of sheaves on $\mathbb{L}$, and natural transformations. Recall that a sheaf on $\mathbb{L}$ is a functor

$$F: \mathbb{L}^{\mathrm{op}} \to Sets$$

with the following property: for each cover $\{\ell A_i \xrightarrow{f_i} \ell A\}_{i=1}^n$, and each family of elements $x_i \in F(\ell A_i)$ which are compatible in the sense that for each pullback square

$$\begin{tikzcd}
\ell A_i \arrow[r, "f_i"] & \ell A \\
\ell A_i \times_{\ell A} \ell A_j \arrow[u] \arrow[r, "p_j"] & \ell A_j \arrow[u]
\end{tikzcd}$$

we have $x_i|p_i = x_j|p_j \in F(\ell A_i \times_{\ell A} \ell A_j)$, there is a unique $x \in F(\ell A)$ such that $x|f_i = x_i$ for each $i = 1, \ldots, n$. (See Appendix 1). $\mathcal{Z}$, the category of sheaves on $\mathbb{L}$, is a *topos*. This is proved just as III.1.5 (For a general proof see Appendix 1).

The category $\mathbb{L}$ does't change when embedded in $\mathcal{Z}$:

1.3 Lemma. *The Grothendieck topology on $\mathbb{L}$ is subcanonical. In other words, the Yoneda embedding $\mathbb{L} \xrightarrow{Y} Sets^{\mathbb{L}^{\mathrm{op}}}$, $Y(\ell B) = \mathbb{L}(-, \ell B)$, factors through $\mathcal{Z} \hookrightarrow Sets^{\mathbb{L}^{\mathrm{op}}}$.*

Proof. This is just as for $\mathcal{G}$ (cf. lemma III.1.3), and we only give a sketch. Take $\ell B \in \mathbb{L}$. To show that the functor $\mathbb{L}(-, \ell B)$ is a sheaf, let $\{\ell A_i \xrightarrow{f_i} \ell A\}_{i=1}^k$ be a cover of A, and suppose we are given a compatible family of maps $\ell A_i \xrightarrow{g_i} \ell B$. Write $B = C^\infty(\mathbb{R}^m)/J, A = C^\infty(\mathbb{R}^n)/I$. By lemma 1.2, we may assume that $A_i = C^\infty(U_i)/(I|U_i)$, where $U_1 \cup \ldots \cup U_k = \mathbb{R}$, and f_i is the canonical map. g_i is given by a smooth function $G_i: U_i \to \mathbb{R}^n$, and compatibility means that $G_i|U_j - G_j|U_i \in (I|U_i \cap U_j)$. If $(\rho_1, \ldots, \rho_k)$ is a partition of unity subordinate to the cover $U_1, \ldots, U_k$, then $G = \sum_{i=1}^k \rho_i G_i$ is the map

$\mathbb{R}^n \to \mathbb{R}^m$ giving the required map $\ell A \xrightarrow{g} \ell B$. Further details are straightforward. $\square$

Composing $\mathbb{L} \xrightarrow{Y} \mathcal{Z}$ with $\mathbb{M} \xrightarrow{s} \mathbb{L}$ of II.1.2, we obtain

1.4 Corollary. *There is a full and faithful embedding of manifolds into the topos $\mathcal{Z}$ denoted by*

$$\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{Z},$$

which preserves transversal pullbacks; explicitly,

$$s(M) = \mathbb{L}(-, \ell C^\infty(M)). \qquad \square$$

1.5 Proposition. *There are adjoint functors*

$$\begin{tikzcd} Sets \arrow[r, shift left=2, "B"] \arrow[r, shift right=2, "\Delta"'] & \mathcal{Z} \arrow[l, "\Gamma"'] \end{tikzcd}, \qquad \Delta \dashv \Gamma \dashv B$$

(Δ is the constant sheaf functor, Γ is the global sections functor).

Proof. This is proved just as for $\mathcal{G}$, see III.1.6. (The description of $\Delta(S)(\ell A)$ in terms of $g(\ell A)$ given for $\mathcal{G}$ doesn't work for $\mathcal{Z}$, but this is irrelevant. On the other hand, B can be described as in Section III.1; i.e. $B(S)(\ell A) = S^{\gamma(\ell A)}$, where $S \in Sets, \ell A \in \mathbb{L}$, is valid for the case of $\mathcal{Z}$. $\square$

We can interpret the set-theoretic language in $\mathcal{Z}$, just as we did for $\mathcal{F}$ and $\mathcal{G}$, by inductively defining $A \Vdash \varphi$, $\mathcal{Z} \models \varphi$, for sentences φ. (See also Appendix 1). Let us list some basic properties for this interpretation in $\mathcal{Z}$.

1.6 The Natural Numbers Object. As for any Grothendieck topos, the object $\Delta(\mathbb{N})$ is the ***natural numbers object*** of the topos $\mathcal{Z}$, where $\mathbb{N}$ denotes the set of natural numbers in *Sets*. In particular, $\Delta(\mathbb{N})$ satisfies full induction, and all first-order arithmetic (see Appendix 1). $\Delta(\mathbb{N})$ is the sheaf associated to the constant presheaf

$$\mathbb{L}^{\mathrm{op}} \to Sets, \quad \ell A \mapsto \mathbb{N}.$$

In the particular case of $\mathcal{Z}$, $\Delta(\mathbb{N})$ cannot be simply described as the sheaf which has as elements at a stage ℓA the locally constant functions $\gamma(\ell A) \to \mathbb{N}$, as in the case of $\mathcal{F}$ and $\mathcal{G}$ (cf. III.1, III.2). For $\mathcal{Z}$, we have the following explicit description of $\Delta(\mathbb{N})$: for $\ell A \in \mathbb{L}$, say $A = C^\infty(\mathbb{R}^n)/I$, $\Delta(\mathbb{N})(\ell A)$ is the set of equivalence classes of locally constant ***bounded*** functions α,

$$Z(I_o) \xrightarrow{\alpha} \mathbb{N},$$

where $I_o \subset I$ is a finitely generated subideal of I. Two such functions $Z(I_o) \xrightarrow{\alpha_o} \mathbb{N}$ and $Z(I_1) \xrightarrow{\alpha_1} \mathbb{N}$ are equivalent if there is a finitely generated $J, I_o \cup I_1 \subset J \subset I$, such that $\alpha_o|Z(J) \equiv \alpha_1|Z(J)$. (It is not hard to see that this indeed gives the sheafification of the constant presheaf with value $\mathbb{N}$, see Appendix 1).

As usual, we will just write $\mathbb{N}$ for the natural numbers object $\Delta(\mathbb{N}) \in \mathcal{Z}$.

1.7 The Line R. The representable $R = s(\mathbb{R}) = \mathbb{L}(-, \ell C^\infty(\mathbb{R}))$ is an object of $\mathcal{Z}$, by 1.3, 1.4. Just as for $Sets^{\mathbb{L}^{op}}$ (chapter II), and for $\mathcal{F}$ and $\mathcal{G}$ (chapter III), R has a canonical order $<$ defined as follows: for $a, b \in R(\ell A)$ represented by $a(x), b(x): \mathbb{R}^n \to \mathbb{R}$, where $A = C^\infty(\mathbb{R}^n)/I$, we have

(1) $\ell A \Vdash a < b$ iff there is a finitely generated $I_0 \subset I$ such that $\forall x \in Z(I_0)\ a(x) < b(x)$.

R is a ring-object in $\mathcal{Z}$ (just as for $Sets^{\mathbb{L}^{op}}$), and this order is clearly compatible with the ring structure of R. Note that

(2) $\mathcal{Z} \models (a \in U(R) \leftrightarrow (a > 0 \vee a < 0))$

where $U(R) = \{x \in R | x \text{ is invertible }\}$. R is also a ***local ring***, i.e.

(3) $\begin{array}{l} \mathcal{Z} \models \neg 0 = 1 \\ \mathcal{Z} \models \forall a, b \in R(a + b \in U(R) \to (a \in U(R) \vee b \in U(R))). \end{array}$

This is proved as in Remark III.1.8. (Note that the covers (a) and (c) of III.1.8 are also covers in $\mathbb{L}$). (b) of III.1.8 is not a cover, however, and accordingly, we have that R is ***not*** Archmidean in $\mathcal{Z}$, i.e.

(4) $\mathcal{Z} \not\models \forall x \in R\ \exists n \in \mathbb{N}\ x < n$.

To see this, consider the identity $R \to R$ as the generic real γ at stage R. Suppose to the contrary that $\ell C^\infty(\mathbb{R}) \Vdash \exists n \in \mathbb{N}\ x < n$. By 1.6 this means that there are a finite open cover $\{U_1, \ldots, U_k\}$ of $\mathbb{R}$ and $n_i \in \mathbb{N} (i = 1, \ldots, k)$ such that $\ell C^\infty(U_i) \Vdash \gamma < n_i$, i.e. $\forall x \in U_i\ x < n_i$. This is clearly impossible.

Consequently, the order topology on R does ***not*** coincide with the rational interval topology. When we speak about topological properties of R, we will always do this with respect to the *order topology*.

The preorder $\leq$ on R is defined just as before, i.e. for $a, b, \ell A$ as in (1),

(5) $\ell A \Vdash a \leq b$ iff $\forall \rho \in m^{\infty}_{[0,\rightarrow)} \subset C^{\infty}(\mathbb{R})\ \rho \circ (b-a) \in I$,

and this is again compatible with the ring structure of R and with $<$ (as in II.2.4). Thus, the unit interval $[0,1] = \{x \in R | 0 \leq x \leq 1\}$ in $\mathcal{Z}$ is the representable object $\ell C^{\infty}([0,1])$.

Finally, let us note that R is a field in the following sense:

(6) $\mathcal{Z} \models \forall x_1, \ldots, x_n \in R(\neg(x_1 = 0 \wedge \ldots \wedge x_n = 0) \rightarrow x_1 \in U(R) \vee \ldots \vee x_n \in U(R))$.

The proof is essentially the same as for $\mathcal{G}$, cf. III.1.9, except that one has to pass to a finitely generated subideal. If $a_1, \ldots, a_n : \mathbb{R}^m \rightarrow \mathbb{R}$ represent reals at ℓA, where $A = C^{\infty}(\mathbb{R}^m)/I$, such that $\ell A \Vdash \neg(a_1 = 0 \wedge \ldots \wedge a_n = 0)$, then $C^{\infty}(\mathbb{R}^m)/(I, a_1, \ldots, a_m)$ is trivial, hence there is a finitely generated $I_0 \subset I$ such that $Z(I_0) \cap \bigcap_{i=1}^{n} Z(a_i(x)) = \emptyset$. So $\{U_{a_1(x)}, \ldots, U_{a_n(x)}\}$ is an open cover of $Z(I_0)$, where $U_{a_i(x)} = \{x | a_i(x) \neq 0\}$, and therefore $\{\ell A \cap s(U_{a_i(x)}) \rightarrow \ell A\}_{i=1}^{n}$ covers ℓA, by lemma 1.2. Since $\ell A \cap s(U_{a_i}) \Vdash a_i \in U(R)$, we conclude that $\ell A \Vdash a_1 \in U(R) \vee \ldots \vee a_n \in U(R)$, thus proving (b).

This completes 1.7.

1.8 Some Infinitesimal Spaces. As in $Sets^{\mathbb{L}^{op}}$, $\mathcal{F}$, and $\mathcal{G}$, we have the infinitesimal subspaces $D_k(n)$ of R^n in $\mathcal{Z}$,

$$D_k(n) = \{x \in R^n | x^{\alpha} = 0 \text{ for all multi-induces } \alpha \text{ with } |\alpha| = k+1\}.$$

In R, we have $D = D_1 \subset D_2 \subset \ldots$, where $D_i = D_i(1)$, and these are all contained in $D_{\infty} = \{x \in R | \exists n \in \mathbb{N}\ x^n = 0\}$. The interval $[0,0] = \ell(C^{\infty}(\mathbb{R})/m^{\infty}_{\{0\}})$ properly contains D_{∞}. As before, we have $\mathcal{Z} \models \forall x \in R(\neg x = 0 \leftrightarrow (x > 0 \vee x < 0) \leftrightarrow x \in U(R))$, and consequently the equality

$$\Delta = \{x | \neg\neg x = 0\} = \{x | x \notin U(R)\}$$

holds in $\mathcal{Z}$. All this is just as for $\mathcal{G}$. However, unlike the case of $\mathcal{F}$ and $\mathcal{G}$, the object of *infinitesimal elements* in $\mathcal{Z}$,

$$\mathbb{\Delta} = \bigcap_{n \in \mathbb{N} - \{0\}} s\left(-\frac{1}{n}, \frac{1}{n}\right) \subset R$$

does *not* coincide with Δ; i.e. all the inclusions in the following sequence are proper in $\mathcal{Z}$:

$$D \subset D_2 \subset \ldots \subset D_\infty \subset [0,0] \subset \Delta \subset \blacktriangle.$$

To say that Δ does not coincide with $\blacktriangle$ is to say that not every element of $\blacktriangle$ is non-invertible, or equivalently, the object of ***invertible infinitesimals***

$$\mathbb{I} = \blacktriangle \cap U(R) = \{x \in \blacktriangle | x \text{ is invertible}\}$$

represented by the dual of the C^∞-ring

$$C^\infty(\mathbb{R}^*)/(m^g_{\{0\}}|\mathbb{R}^*)$$

is ***non-trivial*** (here $\mathbb{R}^* = \mathbb{R} - \{0\}$, and $m^g_{\{0\}}$ is the ideal of functions with vanishing germ at 0). It is clear that this C^∞-ring is indeed non-trivial. In fact,

$$C^\infty(\mathbb{R}^*)/(m^g_{\{0\}}|\mathbb{R}^*)) \cong C^\infty(\mathbb{R})/K,$$

where K is the ideal of functions with compact support, and the isomorphism is induced by the map $x \mapsto \frac{1}{x}$.

Of course $\mathbb{I}$ does not have any global sections, so in particular (since Γ has a right-adjoint)

(1) $\mathcal{Z} \not\models \exists x\ x \in \mathbb{I}$,

i.e. despite the non-triviality of $\mathbb{I}$, it doesn't hold in $\mathcal{Z}$ that there ***exists*** an invertible infinitesimal. (Readers having some experience with intuitionistic logic will not find this so strange, given the "strong" meaning of existence in intuitionism.) However, since the product of two non-zero objects in $\mathbb{L}$ is non-zero, in particular $\ell A \times \mathbb{I} \neq 0$ if $\ell A \neq 0$, we have

(2) $\mathcal{Z} \models \neg\neg\exists x\ x \in \mathbb{I}$.

which expresses in set-theoretic language that $\mathbb{I}$ is non-empty (again, a set being non-empty does ***not*** imply that this set has an element).

Another formula which expresses that "there are quite a lot of invertible infinitesimals" is the following cancellation property:

(3) $\mathcal{Z} \models \forall x \in R(\forall \delta \in \mathbb{I}\ x \cdot \delta = 0 \to x = 0)$

(this would be trivial if $\exists x\ x \in \mathbb{I}$ were valid). To prove (3), choose $a \in R$ at stage ℓA, say $A = C^\infty(\mathbb{R}^n)/I$ and a is represented by $a(x)\colon \mathbb{R}^n \to \mathbb{R}$, and suppose $\ell A \Vdash \forall \delta \in \mathbb{I}(a\delta = 0)$. In particular, taking the "generic" $\delta, \ell A \times \mathbb{I} \Vdash a \cdot \pi_2 = 0$. This just means that

$$a(x) \cdot y \in (I(x), m^g_{\{0\}}|\mathbb{R}^*) \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^*),$$

say

$$a(x) \cdot y = \sum_{i=1}^{k} A_i(x,y) f_i(x) + \sum_{j=1}^{\ell} B_j(x,y) g_j(y),$$

where $f_i \in I$, and each $g_j(y)\colon \mathbb{R}^* \to \mathbb{R}$ vanishes on some$(-\varepsilon_j, \varepsilon_j) - \{0\}$. Taking the partial derivative with respect to y gives that for all $x \in \mathbb{R}^n$, $y \in \mathbb{R} - \{0\}$,

$$a(x) = \sum \frac{\partial A_i}{\partial y}(x,y) f_i(x) + \sum \frac{\partial B_j}{\partial y}(x,y) g_j(y) + \sum B_j(x,y) g_j'(y).$$

Taking $y_o \in \bigcap_{j=1}(-\varepsilon_j, \varepsilon_j) - \{0\}$, we find $g_j(y_o) = 0 = g_j'(y_o)$, so $a(x) \in I$, i.e. $\ell A \Vdash a = 0$. This proves (3), and completes 1.8.

1.9 The Kock-Lawvere Axiom. The Kock-Lawvere axiom $R^D \cong R \times R$ holds in $\mathcal{Z}$, and more generally

$$s(M)^D \cong s(TM) \text{ in } \mathcal{Z},$$

since this isomorphism holds in $\mathbb{L}$ (II.1.11), and the Yoneda embedding $\mathbb{L} \hookrightarrow \mathcal{Z}$ preserves all exponentials that exist in $\mathbb{L}$. The generalized Kock-Lawvere axiom as stated in V.7.2 is also valid in $\mathcal{Z}$. The proof is just as for $\mathcal{F}$ and $\mathcal{G}$.

1.10 The Integration Axiom. This axiom holds in $\mathcal{Z}$, i.e.

$$\mathcal{Z} \models \forall f \in R^{[0,1]} \exists! g \in R^{[0,1]} (g' \equiv f \wedge g(0) = 0).$$

This follows immediately from the fact that the integration axiom holds in $Sets^{\mathbb{L}^{\mathrm{op}}}$ (II.2.4).

2 Smooth Integers

Despite the fact that $\mathcal{Z}$ is a model for the basic axioms of SDG, as we have seen in the previous section, it seems at first sight rather hard to do analysis in $\mathcal{Z}$. For example, the basic ingredients for doing some (synthetic) homology theory in the toposes $\mathcal{F}$ and $\mathcal{G}$ are the compactness of $[0,1] \subset R$ and the fact that R is Archimedean, as we have seen in chapter IV. Neither of these, however, is valid in $\mathcal{Z}$. Non-validity of Archimedeanness has already been pointed out in 1.7 above. For compactness, we have the following rather obvious counterexample.

2.1 Example. In $\mathcal{Z}$, $[0,1]$ is not compact (for the order-topology given by the order $<$). To see this, we only need to modify the argument of section II.3 slightly. Consider the generic positive invertible infinitesimal $\delta \in R$. δ is given at stage $\mathbb{I}_{>0} = \ell(C^\infty(\mathbb{R}_{>0})/(m^g_{\{0\}}|\mathbb{R}_{>0}))$ as the inclusion $\mathbb{I}_{>0} \overset{\delta}{\hookrightarrow} R$. At stage $\mathbb{I}_{>0}$, we have the open cover

$$\mathcal{U} = \{(x-\delta, x+\delta) | x \in [0,1]\}$$

of $[0,1]$. If $[0,1]$ were compact, then $\mathbb{I}_{>0} \Vdash$ "$\mathcal{U}$ has a finite subcover", i.e. we could find a finite open cover $U_1, \ldots, U_k$ of $\mathbb{R}_{>0}$ and for each $i = 1, \ldots, k$ finitely many reals $a_{ij} (j = 1, \ldots, n_i)$ at stage $\mathbb{I}_{>0} \cap s(U_i)$ such that

(1) $\mathbb{I}_{>0} \cap s(U_i) \Vdash [0,1] \subset \bigcup_{j=1}^{n_i} (a_{ij} - \delta, a_{ij} + \delta)$.

Fix one U_{i_0} with $0 \in \overline{U}_{i_0}$, and write $n = n_{i_0}$, $U = U_{i_0}, a_j = a_{i_0 j}$. Then

$$(2) \quad (\mathbb{I}_{>0} \cap s(U) \times [0,1]) \Vdash \begin{array}{l} \pi_2 \in (a_1 - \delta, a_1 + \delta) \vee \ldots \vee \\ \pi_2 \in (a_n - \delta, a_n + \delta) \end{array}$$

Let $a_j(x) \colon U \to \mathbb{R}$ represent a_j, and let

$$V_j = \{(x,y) \in U \times \mathbb{R} | y \in (a_j(x) - x, a_j(x) + x)\}.$$

Then (2) implies that $V_1, \ldots, V_n$ induce a cover of $(\mathbb{I}_{>0} \cap s(U)) \times [0,1]$, hence by 1.2 we find an $\varepsilon > 0$ such that

(3) $(U \cap (0,\varepsilon)) \times [0,1] \subseteq V_1 \cup \ldots \cup V_n$.

In particular, whenever $x \in U \cap (0,\varepsilon)$ we have $[0,1] \subset V_1^x \cup \ldots \cup V_n^x$, where $V_0^x = \{y | (x,y) \in V_j\}$. But the measure $\mu(V_1^x \cup \ldots \cup V_n^x) \leq 2xn$ takes arbitrary small values since $0 \in \overline{U \cap (0,\varepsilon)}$, a clear contradiction.

Despite the fact that $[0,1]$ is not compact, we have the following theorem.

2.2 Theorem. *In $\mathcal{Z}$ it is valid that all functions $[0,1] \to R$ are uniformly continuous; i.e.*

$$\mathcal{Z} \models \forall f \in R^{[0,1]} \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, y \in [0,1] (|x - y| < \delta \to |fx - fy| < \varepsilon).$$

Proof. Obvious from the fact that this already holds in $\mathcal{S}ets^{\mathbb{L}^{\mathrm{op}}}$, cf. II.3.1. □

As for $\mathcal{S}ets^{\mathbb{L}^{op}}$, *uniform* continuity is related to the existence of Lebesgue numbers for open covers of $[0,1]$.

2.3 Theorem. *$\mathcal{Z} \models$ every open cover of $[0,1]$ has a Lebesgue number.*

We cannot derive this immediately from the corresponding fact for $\mathcal{S}ets^{\mathbb{L}^{op}}$ (II.3.2), since (unlike the case of functions $[0,1] \to R$ of theorem 2.2) open covers in $\mathcal{Z}$ are not the same as open covers in $\mathcal{S}ets^{\mathbb{L}^{op}}$. For example, $\mathcal{Z} \models [0,1] = [0,\frac{3}{4}) \cup (\frac{1}{4},1]$, but not so for $\mathcal{S}ets^{\mathbb{L}^{op}}$. However, for *downwards closed* open covers there is no problem. (An open cover $\mathcal{U}$ is downwards closed if $U \subset V \in \mathcal{U}$ implies $U \in \mathcal{U}$.)

2.4 Lemma. *A downwards closed open cover of $[0,1]$ in $\mathcal{Z}$ is already a cover in $\mathcal{S}ets^{\mathbb{L}^{op}}$. More precisely, if $\mathcal{U}$ is a subsheaf of the sheaf of open intervals of R in $\mathcal{Z}$ at stage ℓA such that in $\mathcal{Z}$,*

$$\ell A \Vdash \forall U,V \in \mathcal{O}(R)(U \subset V \wedge V \in \mathcal{U} \to U \in \mathcal{U})$$
$$\ell A \Vdash (\mathcal{U} \text{ covers } [0,1])$$

then already $\ell A \Vdash (\mathcal{U}$ covers $[0,1])$ in $\mathcal{S}ets^{\mathbb{L}^{op}}$.

Proof. Let $\mathcal{U}$ be an open cover in $\mathcal{Z}$ at stage ℓA, and assume $\mathcal{U}$ consists of open intervals of R and is downwards closed. Thus

$$\ell A \times [0,1] \Vdash \exists U \in \mathcal{U}\ \pi_2 \in U$$

($\Vdash$ refers to $\mathcal{Z}$ all the time!), so we find a finite cover $\{W_1, \ldots, W_k\}$ of $\mathbb{R}^n \times \mathbb{R}$, and $a_i, b_i \in R$ at $\ell A_i = (\ell A \times [0,1]) \cap s(W_i)$ such that

$$\ell A_i \Vdash a_i < \pi_2 < b_i \wedge (a_i, b_i) \in \mathcal{U}.$$

It follows that there is a finitely generated ideal J_i,

$$J_i \subset (I(x), m^0_{[0,1]}(y), z \cdot \psi_i(x,y) - 1)$$

(where ψ_i is a characteristic function for W_i, i.e. $Z(\psi_i) = \mathbb{R}^n \times \mathbb{R} - W_i$) such that for $A'_i = C^\infty(\mathbb{R}^{n+2})/J_i$, $\ell A'_i \Vdash a_i < \pi_2 < b_i$. Then we find finitely generated $I_i(x) \subset I(x), M_i(y) \subset m^0_{[0,1]}(y)$, such that $J_i \subset (I_i(x), M_i(y), z\psi_i(x,y) - 1)$ for $i = 1, \ldots, k$, and by putting the I_i's and M_i's together, we get finitely generated $I' \subset I$ and $M \subset m^0_{[0,1]}$ with

$$J_i \subset (I'(x), M(y), z\psi_i(x,y) - 1), \text{ each } i = 1, \ldots, k$$

(and without loss of generality $Z(M(y)) = [0,1]$).

Now let

$$B_i = C^\infty(\mathbb{R}^{n+2})/(I'(x), M(y), Z\psi_i(x,y) - 1),$$

which is a finitely presented C^∞-ring. Then

$$\gamma(\ell B_i) = Z(I'(x), M(y), z\psi_i(x,y) - 1) = \pi^{-1}(Z(I') \times [0,1]) \cap \widehat{W}_i,$$

(where $\widehat{W}_i = Z(z\psi_i(x,y) - 1) \subseteq \mathbb{R}^{n+2}$), and since $\ell B_i \subset \ell A_i'$,

$$\ell B_i \Vdash a_i < \pi_2 < b_i,$$

which is equivalent (since B_i is finitely presented) to

$$a_i(x,y) < y < b_i(x,y)$$

for all $(x,y,z) \in \gamma(\ell B_i)$, i.e. for all $(x,y) \in Z(I') \times [0,1] \cap W_i$.

We now use the following lemma, which is just an easy application of partitions of unity.

Sublemma. Let $X \subset \mathbb{R}^n$ be closed, $\{W_1, \ldots, W_k\}$ an open cover of X, and $a_i : W_i \to \mathbb{R}$ smooth functions with $a_i > 0$ on $W_i \cap X$. Then there is a refinement $\{V_1, \ldots, V_k\}$ and a smooth $a : \mathbb{R}^m \to \mathbb{R}$ such that on V_i, $0 < a|V_i \cap X < a_i|V_i \cap X (i = 1, \ldots, k)$. □

Now apply this lemma to the cover $\{W_1, \ldots, W_k\}$ of $Z(I') \times [0,1]$ so as to find a cover $\{V_1, \ldots, V_k\}$ of $Z(I') \times [0,1]$ and functions $a, b : \mathbb{R}^{n+1} \to \mathbb{R}$ such that for each $i, V_i \subset W_i$ and

$$a_i(x,y) < a(x,y) < y < b(x,y) < b_i(x,y)$$

for $(x,y) \in V_i \cap Z(I') \times [0,1]$. The V_i's induce a cover in the site (lemma 1.2), say $\{\ell C_i \to \ell A \times [0,1]\}_i$, and we have

$$\ell C_i \Vdash a_i < a < \pi_2 < b < b_i.$$

Hence since $\mathcal{U}$ is downwards closed and $\ell A_i \Vdash (a_i, b_i) \in \mathcal{U}$,

$$\ell C_i \Vdash \pi_2 \in (a,b) \in \mathcal{U},$$

and therefore since the ℓC_i form a cover

$$\ell A \times [0,1] \Vdash \pi_2 \in (a,b) \in \mathcal{U}.$$

But this says that already *in the presheaf topos* $Sets^{\mathbb{L}^{op}}$ we force

$$\ell A \times [0,1] \Vdash \exists U \in \mathcal{U}\ \pi_2 \in U$$

since a, b no longer exist just on a cover. π_2 is generic, so $\mathcal{U}$ covers $[0,1]$ in $Sets^{\mathbb{L}^{op}}$ (at stage ℓA). □

Proof of Theorem 2.3. This now follows from 2.4 and the existence of Lebesgue numbers in $Sets^{\mathbf{L}^{op}}$ (II.3.2): Given a cover $\mathcal{U}$ at stage ℓA in $\mathcal{Z}$, let $\mathcal{U}' = \{V | \exists U \in \mathcal{U}\ V \subseteq U\}$ be the corresponding downwards closed cover. By 2.4 and II.3.2, there is a Lebesgue number δ for $\mathcal{U}'$ in $Sets^{\mathbf{L}^{op}}$. Clearly, δ is also a Lebesgue number for $\mathcal{U}'$, and hence for $\mathcal{U}$, in $\mathcal{Z}$. $\square$

Morally, the existence of Lebesgue numbers should give compactness in some sense. We can indeed obtain compactness of $[0,1]$ in $\mathcal{Z}$ by changing the notion of *finiteness*. Instead of $\mathbb{N}$, we use the object N of ***smooth natural numbers***, defined as the subobject $\{x \in Z | x \geq 0\}$ of the object Z of ***smooth integers*** $Z = \{x \in R | \sin(\pi x) = 0\}$. In the topos $\mathcal{Z}$, Z is the representable object

$$Z = \{\ell(C^\infty(\mathbb{R})/(\sin(\pi x))) \cong \ell(C^\infty(\mathbb{R})/m^0_{\mathbb{Z}}) \cong \ell C^\infty(\mathbb{Z})$$

(the second isomorphism is by I.2.1). So

$$N = \ell(C^\infty(\mathbb{R})/m^0_{\mathbb{N}}) = \ell C^\infty(\mathbb{N}).$$

This object N is different from the object $\mathbb{N}$ of ***standard*** natural numbers in $\mathcal{Z}$. N contains non-standard elements, such as the canonical inclusion $\ell(C^\infty(\mathbb{N})/T) \overset{i}{\hookrightarrow} N$, where T is the ideal of functions which vanish on a tail. This smooth natural number at stage $\ell(C^\infty(\mathbb{N})/T)$ is the ***generic infinitely large*** (smooth, non-standard) ***natural number*** of $\mathcal{Z}$.

Now if we interpret *finite* as ***a quotient of an initial segment*** $\{0, \ldots, n-1\}$ ***for some*** $n \in N$ (call this ***smooth-finite, or briefly s-finite***) we obtain s-compactness of $[0,1]$ in $\mathcal{Z}$, i.e. the validity in $\mathcal{Z}$ of the assertion that every open cover $\mathcal{U}$ of $[0,1]$ in $\mathcal{Z}$ has an s-finite refinement (rather than a ***subcover***, cf. 2.6 below). This follows immediately from the existence of a Lebesgue number (theorem 2.3), together with the fact that R is Archimedean for the smooth integers. Explicitly:

2.5 Proposition. *The following are valid in* $\mathcal{Z}$*:*

(i) *R is s-Archimedean, i.e.* $\mathcal{Z} \models \forall x \in R\ \exists n \in N\ x < n$
(ii) $[0,1]$ ***is s-compact, i.e. every open cover of*** $[0,1]$ ***has an s-finite refinement.***

Proof. As said, (ii) follows from (i) and theorem 2.3. To prove (i), it suffices to consider the generic $\gamma \in R$ at stage R given by the identity, and show that $R \Vdash \exists n \in N \ \gamma < n$. But the two opens $U_1 = \bigcup_{m \in \mathbb{Z}}(4m, 4m+3)$ and $U_2 = \bigcup_{m \in \mathbb{Z}}(4m-2, 4m+1)$ induce a cover of R in $\mathbb{L}$, and $\ell C^\infty(U_i) \Vdash \gamma < n_i$ where $n_i : U_i \to \mathbb{N}$ is any continuous function with $n_i(x) > x$. □

2.6 Remark. Ordinary compactness can be defined either by requiring that any open cover has a finite subcover, or that any open cover has a finite refinement; these two are equivalent in an arbitrary Grothendieck topos, by the axiom of finite choice

$$\begin{aligned}(1)\ & \forall P \subset \mathbb{N} \times X (\forall n < m \, \exists x \in X \ P(n,x) \to \\ & \to \exists f : \{0, \dots, m-1\} \to X \ \forall n < m \ P(n, fn))\end{aligned}$$

(which is valid in any topos). However, the axiom of *smooth-finite choice*, which is as (1), but with $\mathbb{N}$ replaced by N, does not hold. Accordingly, we cannot always pass from a finite refinement to a finite subcover, and we define s-compactness by requiring the existence of finite *refinements* only. For all practical purposes, this is sufficient.

The idea now is to replace $\mathbb{N}$ by N consistently when doing analysis inside the topos $\mathcal{Z}$. As said in the introduction, we will show in section 4 that for homology theory we obtain the correct results in $\mathcal{Z}$ only if we use smooth integers everywhere, and change the basic algebraic notions, like that of free ring, accordingly.

As a preparation to this, we will now discuss some of the most basic properties of smooth integers in the topos $\mathcal{Z}$.

We have already seen that the topological notion of compactness should really be replaced by its smooth analogue. In fact, the notion of topological space can be adjusted accordingly:

2.7 Proposition. *R is an s-topological space, i.e. the intersection of s-finitely many opens is again open (for the order topology).*

Proof. We will use this occasion to illustrate the use of the generic element of a set of the form $\{m \in N | m < p\}$, for $p \in N$ at a given stage. (This is a special case of 3.1.5, by taking $\mathbb{N}$ as a discrete manifold.)

To prove the proposition, suppose we are given a $p \in N$ and a sequence $U : \{m \in N | m < p\} \to \mathcal{O}(R)$, $U = (U_m : m < p)$, at stage $\ell A \in \mathbb{L}$, such that $\ell A \Vdash \forall m < p \ 0 \in U_m$. Let $A = C^\infty(\mathbb{R}^d)/I$. Writing $\mathbb{N}$ as a retract of an open O, $\mathbb{N} \subset O \subset \mathbb{R}$, we may assume that $\ell A \xrightarrow{p} N$ is represented by a smooth function $W \xrightarrow{p(x)} \mathbb{N}$, where $W \subset \mathbb{R}^d$ is an open which contains $Z(I_o)$ for a finitely generated $I_o \subset I$. let $W_n = p^{-1}(n)$, and let $\ell B = s(E) \cap (\ell A \times N)$, where

$$E = \bigcup_{n \in \mathbb{N}} (W_n \times \{0, \dots, n-1\}) \subset \mathbb{R}^d \times \mathbb{N}.$$

This gives maps in $\mathbb{L}$,

$$\begin{tikzcd} \ell B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & N \\ \ell A & \end{tikzcd}$$

and $\ell B \Vdash \pi_2 < p$ (more precisely, $p|\pi_1$). π_2 is the *generic element* of $\{m \in N | m < p\}$, in the sense that whenever $\ell C \xrightarrow{f} \ell A$ and $\ell C \xrightarrow{m} N$ is a smooth integer at ℓC with $\ell C \Vdash m < p|f$, there is a map g making the diagram

$$\begin{tikzcd} \ell C \arrow[dr, dashed, "g"] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "f"'] & & \\ & \ell B \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & N \\ & \ell A & \end{tikzcd}$$

commute, i.e. making m into a restriction of π_2.

By definition of the order topology, we have $\ell B \Vdash \exists \varepsilon > 0 (-\varepsilon, \varepsilon) \subset U_{\pi_2}$ so (by enlarging I_0 if necessary) we find a finite cover $\{V_1, \dots, V_k\}$ of $E \cap (Z(I_0) \times \mathbb{N})$ by opens of $\mathbb{R}^d \times \mathbb{N}$, and $\varepsilon_i : V_i \to \mathbb{R}_{>0}$, such that

$$\ell B \cap s(V_i) \Vdash (-\varepsilon_i, \varepsilon_i) \subset U_{\pi_2}.$$

By the sublemma stated in the proof of 2.4, we can actually find an $\varepsilon : E \to \mathbb{R}_{>0}$ such that $\varepsilon|(V_i \cap (Z(I_o) \times \mathbb{N})) < \varepsilon_i|(Z(I_o) \times \mathbb{N})$. Then

(1) $\ell B \Vdash 0 \in (-\varepsilon, \varepsilon) \subset U_{\pi_2}$.

Define $\delta : W = \bigcup_{n \in \mathbb{N}} W_n \to \mathbb{R}_{>0}$ to be any smooth function such that $\forall (x, k) \in E \ \delta(x) < \varepsilon(x, k)$. Then δ defines an element of $R_{>0}$ at stage ℓA, and since $\ell B \Vdash \delta|\pi_1 < \varepsilon$, (1) gives

$$\ell B \Vdash 0 \in (-\delta|\pi_1, \delta|\pi_1) \subset U_{\pi_2}.$$

So by genericity of π_2,

$$\ell A \Vdash \forall n < m\, 0 \in (-\delta, \delta) \subset U_n.$$

By using the order isomorphism $t \mapsto t - x$, it follows that

$$\begin{aligned} \mathcal{Z} \models \forall x \in R\, \forall n \in N\, \forall (U_m)_{m<n} \subset \mathcal{O}(R)(\forall m < n\; x \in U_m \to \\ \exists V \in \mathcal{O}(R)\, \forall m < n\; x \in V \subset U_m). \end{aligned} \qquad \square$$

In chapter VII, we will discuss more extensively to which extent N acts as a natural numbers object. For the purposes of this chapter, we just note that N satisfies all primitive recursive arithmetic. In fact, many objects perceive N as being the natural number object as far as definition by recursion is concerned, and this is precisely what we need to do analysis and algebra based on the smooth integers $\mathcal{Z}$. For example,

2.8 Theorem. *For any $X \in \mathcal{Z}$ and $\ell B \in \mathbb{L}$, with B finitely presented,*

$$\begin{aligned} \mathcal{Z} \models \forall f \in \ell B^X \forall g \in \ell B^{\ell B \times X} \exists! h \in \ell B^{N \times X}(\forall x \in X\; h(o,x) = f(x) \wedge \\ \wedge \forall n \in N\; \forall x \in X\; h(n+1, x) = g(h(n,x), x)). \end{aligned}$$

For the proof, we need the following lemma.

2.9 Lemma. *Let $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ be smooth with $(\varphi) = m^0_{\mathbb{N}}$, i.e. $\ell(C^\infty(\mathbb{R})/(\varphi)) \cong N$. If $I \subset C^\infty(\mathbb{R}^n)$ is finitely generated and $f \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ satisfies $f(x,n) \in I$ for all $n \in \mathbb{N}$, then $f(x,y) \in (I(x), \varphi(x))$.*

Proof. Since $(I(x), \varphi(t))$ is germ-determined by I.4.9, it is enough to check that the germ $f|_{(x_0,n)}$ is in $(I|_{x_0}, \varphi|_n)$ for each $x_0 \in Z(I)$ and $n \in \mathbb{N}$. But around (x_0, n), $f(x,y) = f(x,n) + (y-n)g(x,y)$ for some smooth g (by Hadamard), and $(y-n)|_n \in (\varphi)|_n$. $\square$

Proof of Theorem 2.8. By writing X as a colimit of representables (Appendix 1), we may assume $X = \ell A \in \mathbb{L}$. So let us write $A = C^\infty(\mathbb{R}^n)/I$ and $B = C^\infty(\mathbb{R}^m)/J$, with J a finitely generated ideal, and suppose we have maps $f \in \ell B^{\ell A}$ and $g \in \ell B^{\ell B \times \ell A}$ at stage $\ell C \in \mathbb{L}$. By replacing ℓA by $\ell A \times \ell C$, we may assume $\ell C = 1$, i.e.

$f: \ell A \to \ell B$ and $g: \ell B \times \ell A \to \ell B$ are maps in $\mathbb{L}$, represented by smooth maps $f(x): \mathbb{R}^n \to \mathbb{R}^m$ and $G(x, y): \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Define $H(t, x): \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ to be any smooth function such that $H(0, x) = F(x), H(n+1, x) = G(H(n, x), x)$. We claim that H restricts to a map $N \times \ell A \to \ell B$. Indeed, since J is finitely generated, F and G restrict to maps $\ell A_0 \to \ell B$ and $\ell B \times \ell A_0 \to \ell B$ for some $A_0 = C^\infty(\mathbb{R}^n)/I_0$, with I_0 a finitely generated ideal, $I_0 \subset I$. Now H maps $N \times \ell A = \ell(C^\infty(\mathbb{R} \times \mathbb{R}^n)/(\varphi(t), I_0(x)))$ into ℓB, by lemma 2.9. It is clear that the restriction $h: N \times \ell A \to \ell B$ of H satisfies the recursion equations.

For the uniqueness of h, suppose that we have two maps h and k satisfying the requirements, represented by H and $K: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$. Since the condition $h(o, x) = f(x)$ and $h(n+1, x) \equiv g(h(n, x), x)$ is *finitary* (the first is clear, the second says that

$$\begin{tikzcd}
N \times \ell A \arrow[r, "(+1) \times \mathrm{id}"] \arrow[d, "{(h, \pi_2)}"] & N \times \ell A \arrow[d, "h"] \\
\ell B \times \ell A \arrow[r, "g"] & \ell B
\end{tikzcd}$$

commutes, i.e. that the m differences of the components are in (φ, I); all this involves only a finite part of I), and since J is finitely generated, there is a finitely generated ideal $I_0 \subset I$ such that the restrictions h_0 and $k_0: N \times \ell A \to \ell B$, of M and K respectively, also satisfying the requirements. But then $h_0 - k_0 \in (\varphi(t), I_0(x))$ by another application of lemma 9. $\square$

2.10 Remark. Theorem 2.8 says, for example, that all manifolds perceive N as the natural numbers in some sense. This is not to say that for a manifold $M \in \mathbb{M}, s(M)^N \cong s(M)^{\mathbf{N}}$. The canonical restriction maps $s(M)^N \to s(M)^{\mathbf{N}}$ is always epic, but it fails to be monic already for $M = \mathbb{R}$, i.e. $R^N \to R^{\mathbf{N}}$ is *not* injective.

To see this, look at stage $\ell A = \ell(C^\infty(\mathbb{R})/m^g_{\{0\}})$. Let $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be any smooth function such that for all $n \in \mathbb{N}$,

$$F(x, n) = \begin{cases} 0 & \text{if} \quad |x| < 2^{-n} \\ 1 & \text{if} \quad |x| \geq 2^{-(n-1)} \end{cases} .$$

F induces a map $\ell A \times N \to R$ in $\mathcal{Z}$, and since $F(x,n) \in I(x)$ for each $n \in \mathbb{N}$, we have $\ell A \Vdash \forall n \in \mathbb{N}\ f(n) = 0$ (cf. Appendix 1). On the other hand, $\ell A \not\Vdash \forall n \in N\ f(n) = 0$, or equivalently, $F \notin (M^g_{\{0\}}(x), \varphi(y))$ where φ is as in lemma 2.9.

In other words, lemma 2.9 may fail when I is just germ-determined.

As a consequence of theorem 2.8, we can define for each sequence $p \in R^N$ (at stage ℓA say, i.e. $p: \ell A \times N \to R$) a map $N \times \ell A \xrightarrow{h} R$ with $h(0,x) = 0, h(n+1,x) = h(n,x) + p(x,n)$. Equivalently, R becomes equipped with an operation

$$N \times R^N \to R$$

which we write of course as $(n,p) \mapsto \sum_{i<n} p_i$. In other words (since N has decidable equality), we can take the sum of an *s-finite* number of elements of R. Together with the usual inverse, this gives R the structure of what we will call an *s-group* (smooth group).

A similar argument applies to any Lie group, i.e. to any group object in $\mathbb{M}$, since $s(M)$ is the dual of a *finitely presented* C^∞-ring for any $M \in \mathbb{M}$ (cf. I.2.3), so we obtain

2.11 Corollary. *The embedding $s: \mathbb{M} \hookrightarrow \mathcal{Z}$ maps Lie groups to s-groups.* □

An analogous application of theorem 2.8 shows that R is closed under s-finite products, i.e. for a smooth natural number $n \in N$ and an n-tuple $p_0, \ldots, p_{n-1} \in R$ we can define $\Pi_{i<n} p_i \in R$, satisfying the obvious s-analogues of the ring-axioms, thus making R into an *s-ring*.

The following explicit description of the s-ring structure is helpful. If $\ell A \xrightarrow{k} N$ is a smooth natural number at stage ℓA and $\ell A \times N \xrightarrow{p} R$ is a sequence of elements of R at this stage, $\sum_{i<k} p_i: \ell A \to R$ can be described as follows: write $A = C^\infty(\mathbb{R}^n)/I$, and suppose p is represented by $P: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. As in the proof of 2.7, there is a finitely generated $I_0 \subset I$ such that k can be represented by a locally constant map $U \xrightarrow{k} \mathbb{N}$, where U is an open with $Z(I_0) \subset U \subset \mathbb{R}^n$. Since $C^\infty(\mathbb{R}^n)/I \cong C^\infty(U)/I$, the map

$$Q: U \to \mathbb{R}, \quad Q(x) = \sum_{i<k(x)} P(x,i)$$

restricts to a map $g: \ell A \to R$ of $\mathbb{L}$, which is precisely $\sum_{i<k} p_i$. A similar description can be given of $\Pi_{i<k} p_i$.

As we will see in section 4, to develop some of the homology theory in Z, it will be necessary to replace all algebraic notions by their s-analogues. Thus, for example, a ***free s-group*** on a set X is defined in the obvious way using words $\sum_{i<n} x_i$ for n-tuples $x_0, \ldots, x_{n-1}$ in X, with n a ***smooth*** natural number. In this way, that object Z of smooth integers is the free s-group on one generator. Note, by the way, that Z is a sub-s-ring of R. Similarly, an R-s-module is an s-abelian s-group equipped with the obvious s-analogue of an R-module structure.

Usually, it is much harder to show the smooth analogue of a certain ring-theoretic property than to show its standard counterpart. For example, while R is a local ring object in Z by definition of the Grothendieck topology on $\mathbb{L}$, the fact that R is an s-local ring in Z is not quite as immediate:

2.12 Proposition. *R is an s-local ring in Z, i.e.*

$$Z \models \forall k \in N \, \forall (p_i)_{i<k} \in R^{\{0,\ldots,k-1\}} (\textstyle\sum_{i<k} p_i \in U(R) \rightarrow \exists i < k \; p_i \in U(R)).$$

For the proof of this proposition, we need the following result from dimension theory, which enables us to obtain a suitable *finite* cover of a certain type from an arbitrary open cover. As will become apparent in the sequel, this result is of crucial importance for proving properties of the smooth Zariski topos.

2.13 Ostrand's Theorem. *Let M be a d-dimensional manifold, and let $\mathcal{U} = \{U_m\}_{m \in \mathbb{N}}$ be a locally finite open cover of M. Then there exists an open refinement $\mathcal{V}$ of the form $\mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_{d+1}$, where each $\mathcal{V}_i$ is a family $\{V_{im}\}_{m \in \mathbb{N}}$ of pairwise disjoint open sets such that $V_{im} \subset U_m$ (all $m \in \mathbb{N}$, all $i = 1, \ldots, d+1$).*

For a proof of theorem 2.13, we refer the reader to Engelking (1978, p. 228).

Proof of Theorem 2.12. Take a C^∞-ring $A = C^\infty(\mathbb{R}^n)/I$ and a smooth natural number k at stage ℓA, i.e. $\ell A \xrightarrow{k} N$, and suppose we are given a k-tuple $(p_i : i < k)$ at stage ℓA. Since N has decidable equality, we may extend this k-tuple to an element of R^N at stage ℓA, i.e. a map $\ell A \times N \xrightarrow{p} R$, represented by a smooth map $P: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$. Moreover, as before, we represent $\ell A \xrightarrow{k} N$ by a locally constant map $k(x): U \rightarrow \mathbb{N}$, where U is an open set with $Z(I_o) \subset U \subset \mathbb{R}^n$ for a finitely generated $I_o \subset I$. So $\sum_{i<k} p_i : \ell A \rightarrow R$ is represented by

the map

$$Q: U \to \mathbb{R}, \quad Q(x) = \sum_{i<k(x)} P(x,i),$$

as we explained above. Since $\ell A \Vdash \sum_{i<k} p_i$ is invertible, we may assume by choosing I_o sufficiently big and U sufficiently small, that $\forall x \in U\ Q(x) \neq 0$.

Now consider each $U_m = k^{-1}(\{m\})$ separately, and write

$$W_{m,j} = \{x \in U_m | P(x,j) \neq 0\} \quad (j = 0, \ldots, m-1)$$

Then $U_m = \bigcup_{j=0}^{m-1} W_{m,j}$, so by Ostrand's theorem, there is a cover $\{V_m^1, \ldots, V_m^{n+1}\}$ of U_m such that each V_m^i can be written as a disjoint union $V_m^{i,0} \cup \ldots \cup V_m^{i,m-1}$ with $V_m^{i,j} \subset W_{m,j}$. (The point of applying 2.13 is, of course, that the number of V_m^i's needed is $n+1$, and does not depend on m.) Now define a finite open cover $\{V_1, \ldots, V_{n+1}\}$ of U by

$$V_i = \bigcup_{m \in \mathbb{N}} V_m^i = \bigcup_{m \in \mathbb{N}} \bigcup_{j=0}^{m-1} V_m^{i,j}.$$

This union is disjoint, so the locally constant function

$$\ell_i: V_i \to \mathbb{N}, \quad \ell_i(x) = j \text{ if } x \in V_m^{i,j}$$

defines a smooth integer at V_i, and $\ell_i < m$ on U_m, so we have $s(V_i) \cap \ell A \Vdash \ell_i < k$. Moreover, since $P(x,j) \neq 0$ on $V_m^{i,j}$, it is clear that $s(V_i) \cap \ell A \Vdash p_{\ell_i} \in U(R)$. Since $\{V_1, \ldots, V_{n+1}\}$ covers $U \supset Z(I_o)$, we conclude by lemma 1.2 that

$$\ell A \Vdash \exists \ell < k\ p_\ell \in U(R). \qquad \square$$

In the "standard" sense, R is not only a local ring object x in $\mathcal{Z}$, but even a field in the sense of 1.7(b). We don't know whether the smooth analogue of 1.7(b) is valid, i.e. whether

$$\begin{aligned} \mathcal{Z} \quad \models \quad & \forall k \in N\ \forall (p_i : i < k) \in R^{\{0,\ldots,k-1\}} (\neg \forall i < k\ p_i = 0 \to \\ & \exists i < k\ p_i \in U(R)). \end{aligned}$$

2.14 Proposition. *The order relations $\leq$ and $<$ are compatible with the ring structure on R, i.e.*

(i) $\forall p \in R^N \forall n \in N (\forall k < n\ p_k > 0 \to \sum_{k<n} p_k > 0 \wedge \Pi_{k<n} p_k > 0)$

(ii) $\forall p \in R^N \forall n \in N (\forall k < n\ p_k \geq 0 \to \sum_{k<n} p_k \geq 0 \wedge \Pi_{k<n} p_k \geq 0)$

Proof. By reasoning in Z and replacing p by p' with $p'_k = p_k$ if $k < n$, $p'_k = 1$ if $k \geq n$, it suffices to prove (i) and (ii) with $\forall k\, p_k > 0$ instead of $\forall k < n\; p_k > 0$, resp. $\forall k\, p_k \geq 0$ instead of $\forall k < n\; p_k \geq 0$. (i) follows easily from the explicit description of $\sum$ and Π given just after 2.11. For (ii), take $n \in N$ and $p \in R^N$ at stage ℓA, with $A = C^\infty(\mathbb{R}^n)/I$ say, and assume $\ell A \Vdash \forall k \in N p_k \geq 0$. As before, n and p can be represented by C^∞-functions $\mathbb{R}^n \times \mathbb{N} \xrightarrow{p(x,k)} \mathbb{N}$ and $n(x)\colon U \to \mathbb{N}$, where $U \supset Z(I_0)$ for some finitely generated $I_0 \subset I$, and since $\ell A \Vdash \forall k\, p_k \geq 0$, we have

$$(1)\ \forall \varphi \in m^\infty_{\mathbb{R}\geq 0} \subset C^\infty(\mathbb{R})\colon \varphi(p(x,k)) \in (I) \subset C^\infty(\mathbb{R}^n \times \mathbb{N}).$$

Now $\sum_{k<n} p_k \in R(\ell A)$ is represented by the function $q(x)\colon U \to \mathbb{N}$, $q(x) = \sum_{k<n(x)} p(x,k)$. Write $U = \bigcup_m U_m$, where $U_m = \{x | n(x) = m\}$. If $\varphi \in m^\infty_{\mathbb{R}\geq 0} \subset C^\infty(\mathbb{R})$, then (just as in the proof of (iv) of II.2) we can write by I.4.11

$$(2)\ \varphi \circ q(x)|U_m = \varphi\left(\textstyle\sum_{k<m} p(x,k)|U_m\right) = \textstyle\sum_{k<m} B^m_k(x)\theta^m_k(p(x,k))|U_m$$

with $\theta^m_k \in m^\infty_{\mathbb{R}\geq 0}$. By I.4.13, all the θ^m_k are a multiple of a single $\theta \in m^\infty_{\mathbb{R}\geq 0}$, so we can rewrite (2) as $\varphi \circ q|U_m = \sum_{k<m} A^m_k(x)\theta(p(x,k))$, and hence by (1) as

$$(3)\ \varphi \circ q|U_m = \textstyle\sum_{k<m}\sum_{j=1}^{s} C^m_k(x) g_j(x)$$

with $g_j \in I$. Since the U_m are disjoint, (3) clearly implies $\varphi \circ q \in (I) \subset C^\infty(U)$. □

2.15 Remark. Finally, a remark on the terminology of s-groups, s-rings, etc. This terminology is really quite misleading, since it suggests that there is a different sort of algebra, a subject like smooth algebra, s-algebra. The only reason that we have to use names like s-group, s-free s-ring, etc., is that there is already a canonical interpretation of groups, free rings, etc. in any Grothendieck topos, using the (standard) natural numbers object. If we work in a weaker theory than the full logic of toposes with a natural numbers object, say in the theory of the site $\mathbb{L}$ only, or the Cartesian closed completion of $\mathbb{L}$, we would not have a natural numbers object $\mathbb{N}$ available, and the canonical thing to use instead would be the

3 Manifolds in the Smooth Zariski Topos

In this section we will explain how some topological properties of manifolds carry over to $\mathcal{Z}$ via the embedding $\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{Z}$. Before we can do so, however, it is necessary to take a closer look at the internal topology on the objects $s(M) \in \mathcal{Z}$, for $M \in \mathbb{M}$. (The reader should compare this with the case of $\mathcal{F}$ and $\mathcal{G}$ discussed in III.3.1.)

3.1 The Internal Topology of Manifolds. Let us first look at $R = s(\mathbb{R})$, with the order topology. Since R is s-Archimedean, it follows that in $\mathcal{Z}$ the ***s-rational intervals*** $(p,q), p < q, p$ and q smooth rationals, form a basis for the internal topology. The object Q of smooth rationals is $s(\mathbb{Q})$, where $\mathbb{Q}$ is the discrete space of rational numbers in *Sets*. Thus, it is clear that if $U \subset \mathbb{R}$ is open, then $s(U) \subset s(\mathbb{R})$ is open for the order topology in $\mathcal{Z}$. But these opens $s(U)$ form an (internal) basis only for the topology generated by the *rational* intervals (as in III.3), not for the one generated by s-rational intervals.

Suppose (p,q) is an s-rational interval at stage $\ell A, A = C^\infty(\mathbb{R}^n)/I$. The two maps $\ell A \underset{q}{\overset{p}{\rightrightarrows}} Q$ are represented by a finitely generated ideal $I_o \subset I$, a countable cover $\{O_n\}_n$ of $Z(I_o)$ by pairwise disjoint open subsets of $\mathbb{R}^n$, and rationals $p_n, q_n \in \mathbb{Q}, p_n < q_n$. Moreover, for $\ell A \xrightarrow{a} R$, represented by $a(x)\colon \mathbb{R}^n \to \mathbb{R}$, we have that $\ell A \Vdash a \in (p,q)$ iff I_o can be chosen so big and the O_n so small that $p_n < a(x) < q_n$ for all $x \in O_n$. Thus, the conclusion of 3.1.1 below is clear for the case $d = 1$. The argument for the general case is entirely similar, but with s-rational intervals (p,q) replaced by s-rational cubes $(p_1,q_1) \times \ldots \times (p_d, q_d)$.

3.1.1 An Internal Basis for the Topology of R^d. We define *basic opens* of R^d to be of the following form: a basic open U at stage ℓA is given by a map $\ell A \xrightarrow{p} N$ (i.e. a smooth natural number at ℓA) and a sequence of (external) opens $U_n \subset \mathbb{R}^d, n \in \mathbb{N}$, interpreted as a subsheaf of R^d at stage ℓA (i.e. an element of $\mathcal{P}(R^d)(\ell A)$) as follows. For $\ell B \xrightarrow{f} \ell A$ and $\ell B \xrightarrow{a} R^d$, we have $\ell B \Vdash a \in U$ (or more precisely $\ell B \Vdash a \in U|f$) iff, writing $B = C^\infty(\mathbb{R}^m)/J$, there is a finitely generated $J^o \subset J$ and a disjoint open cover $\{O_k\}_k$ of $Z(J^o) \subset \mathbb{R}^m$, such that $p \circ f$ can be represented as a continuous map $\cup_k O_k \to \mathbb{N}$ with value $p_k \in \mathbb{N}$ on O_k (as in 2.9) and such that $a(x) \in U_{p_k}$ for all $x \in O_k$.

Then these basic opens indeed form a basis for the internal product topology on R^d corresponding to the order topology on R.

3.1.2 Remark. In the sequel, we will usually work with the description of opens as in 3.1.1 (see also 3.1.4 below). However, there is a more concise way of describing this basis.

In section III.3, for the case of $\mathcal{F}$ and $\mathcal{G}$, the internal topology was the smallest topology $\mathcal{O}(R^d) \subset \mathcal{P}(R^d)$ containing subsheaves of the form $s(U) \subset s(\mathbb{R}^d)$ at stage 1, U open $\subset \mathbb{R}^d$. In other words, the "constant opens", those defined at stage $\mathbb{L}$, formed a basis.

In the case of $\mathcal{Z}$, the generating opens are given at stage N. Given a sequence $\{U_n\}_{n\in\mathbb{N}}$, $s(\bigcup_{n\in\mathbb{N}}(U_n \times \{n\}))$ defines a subobject of $R^d \times N$, i.e. an element of $\mathcal{P}(R^d)(N)$. The internal topology on R^d is precisely the topology having these opens at stage N as a basis.

3.1.3 An Internal Basis for the Topology of $s(M)$. We define a basis for the internal topology on $s(M)$, where M is a manifold, in exactly the same way as for $\mathbb{R}^d$. Thus, analogous to 3.1.2, $\mathcal{O}(sM)$ is the smallest internal topology containing all the subobjects at stage N given by $s\left(\bigcup_{n\in N} U_n \times \{n\}\right) \subset s(M) \times N$, where $\{U_n\}_n$ is a sequence of (external) open subsets of M.

More explicitly, we can describe a basic open U at stage ℓA by a map $\ell A \xrightarrow{p} N$ and a sequence of opens $\{U_n\}$ in M. This defines a subsheaf $U \in \mathcal{P}(sM)(\ell A)$ as follows. For $\ell B \xrightarrow{f} \ell A$, $\ell B \xrightarrow{a} s(M)$, $\ell B \Vdash a \in U$ iff there is a finitely generated $J_o \subset J$ (where $B = C^\infty(\mathbb{R}^m)/J$) and an open $O \supset Z(J_o)$ such that $p \circ f$ is represented by a map $O \to \mathbb{N}$ with value p_k on $O_k \subset O$ say, and a is represented by $O \xrightarrow{a(x)} M$, with $a(O_k) \subset U_{p_k}$.

(Recall that by the ε-neighbourhood theorem, we can always represent a map $\ell B \xrightarrow{a} s(M)$ by a smooth $a(x){:}\, O \to M$, where $O \supset Z(J_o)$ for some finitely generated $J_o \subset J$, and O open.)

So this definition of the internal topology on $s(M)$ coincides with the internal topology on R^d defined in 3.1.1 if $M = \mathbb{R}^d$. Some more justification—making 3.1.3 into a proposition rather than a definition—will be given in 3.3.

3.1.4 Terminology. Let $\ell A \in \mathbb{L}, A = C^\infty(\mathbb{R}^n)/I$, and let M be a manifold. Suppose $\{O_m\}_m$ is a cover of $Z(I_o)$ for some finitely generated $I_o \subset I$ by disjoint open subsets of $\mathbb{R}^n$, and let $\{U_m\}_m$ be a sequence of open subsets of M. Then these two sequences represent a unique internal basic open U of $s(M)$ at stage ℓA (namely defined in 3.1.3 by the sequence $\{U_m\}_m$ and the map $\ell A \xrightarrow{p} N$ represented by $p(x)\colon \bigcup_m O_m \to \mathbb{N}$, $p(x) = m$ iff $x \in O_m$), and every internal basic open is of this form. We will say that U is *the open given by U_m over O_m*, or *by U_m at O_m*, or *with value U_m over O_m*$(m \in \mathbb{N})$.

3.1.5 Generic Elements of Basic Opens. Let $\ell A \in \mathbb{L}$ and let U be a basic open of $s(M)$ at stage ℓA, given by U_m over O_m, just as in 3.1.4. Let $V = \bigcup_{m\in\mathbb{N}}(O_m \times U_m)$, and let $B = C^\infty(V)/(I)$, where (I) is the ideal generated by the functions $V \xrightarrow{\pi_1} \bigcup_m O_m \xrightarrow{f} \mathbb{R}$, for $f \in I$. Then $\pi_1\colon V \to \mathbb{R}^n$ induces a map $\ell B \xrightarrow{\pi_1} \ell A$, and $\pi_2\colon V \to M$ gives an element $\ell B \xrightarrow{\pi_2} s(M)$ of $s(M)$ at stage ℓB. Clearly $\ell B \Vdash \pi_2 \in U$. Moreover, this is the *generic* element of U, in the usual sense: if $\ell C \xrightarrow{g} \ell A$ is any map in $\mathbb{L}$, and $\ell C \xrightarrow{x} s(M)$ is an element of $s(M)$ with $\ell C \Vdash x \in U$, then x is a restriction of π_2, i.e. there is a g making the following diagram commute:

$$\begin{tikzcd}
\ell C \arrow[dr, dashed, "g"] \arrow[drr, "x"] \arrow[ddr, "f"'] & & \\
& \ell B \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & s(M) \\
& \ell A &
\end{tikzcd}$$

This completes 3.1.5, and 3.1.

Next, we wish to say that the embedding $s\colon \mathbb{M} \hookrightarrow \mathcal{Z}$ preserves open covers, just as in III.3.2. However, since the site $\mathbb{L}$ for $\mathcal{Z}$ is defined via *finite* covers only, the statement as formulated in III.3.2: "If $\{U_\alpha\}_\alpha$ is an open cover of $\mathbb{M}$, then the subsheaf $\mathcal{U}$ of $\mathcal{P}(sM)$ generated by the conditions $1 \Vdash s(U_\alpha) \in \mathcal{U}$ is an internal cover, i.e. $\mathcal{Z} \models \forall x \in s(M) \exists U \in \mathcal{U}\ x \in U$" holds only if $\{U_\alpha\}$ is *finite*!

Let us consider *countable* covers only, which is not really a restriction since we assumed manifolds to be separable. (In fact, everything we will say extends to arbitrarily indexed covers, by defining the embedding $s\colon \mathbb{M} \hookrightarrow \mathcal{Z}$ also for (at least discrete) manifolds of arbitrary cardinality.) For a countable open cover $\{U_n\}_n$ of M, III.3.2 expresses that $\Delta(\mathbb{N}) = \mathbb{N} \xrightarrow{F} \mathcal{O}(sM)$ given by $F(n) = U_n$ defines an internal cover in $\mathcal{G}$ (or $\mathcal{F}$), i.e. $\mathcal{G} \models \forall x \in s(M) \exists n \in \mathbb{N}\ x \in F(n)$. We obtain a corresponding result for $\mathcal{Z}$, if we replace the natural numbers $\mathbb{N} = \Delta(\mathbb{N})$ by the smooth natural numbers $N = s(\mathbb{N})$. Indeed, given a sequence $\{U_n\}_n$ of opens of M covering M externally, we obtain a corresponding open U of $s(M)$ at stage N (cf. 3.1.3), and hence a map $N \to \mathcal{O}(sM)$, i.e. an internal *s-countable* (i.e. indexed by the smooth natural numbers) family of opens. We claim that this is an internal cover:

3.2 Proposition. The embedding $s\colon \mathbb{M} \hookrightarrow \mathcal{Z}$ preserves countable open covers. More precisely, if $\{U_n\}_n$ is an open cover of M, then for the corresponding s-countable family of opens $N \xrightarrow{U} \mathcal{O}(sM)$ in $\mathcal{Z}$ we have

$$\mathcal{Z} \models \forall x \in s(M)\ \exists n \in N\ x \in U_n.$$

Proof. To show that $\mathcal{Z} \models \forall x \in s(M)\ \exists n \in N\ x \in U_n$, it suffices to consider the generic $\gamma \in s(M)$ at stage $s(M)$ given by the identity, and show that $s(M) \Vdash \exists n \in N\ \gamma \in U_n$. To this end, let $\{V_k\}_k$ be a neighbourhood-finite refinement of the given cover $\{U_n\}_n$, with $V_k \subset U_{n_k}$ say. By Ostrand's theorem (2.13), there is a refinement of the form

$$\mathcal{V} = \mathcal{V}^1 \cup \ldots \cup \mathcal{V}^d,$$

$(d = \dim(M) + 1)$, where $\mathcal{V}^i = \{\tilde{V}^i_k\}_k$ is a family of pairwise disjoint opens, and $\tilde{V}^i_k \subset V_k$. Let $\tilde{V}^i = \bigcup \mathcal{V}^i$. At stage $s(\tilde{V}^i)$, we define a smooth integer p^i with value n_k over V^i_k. Then it is clear from 3.1.3 that

$$s(\tilde{V}^i) \Vdash \gamma \in U_{p^i},$$

since U_{p^i} is the internal (basic) open given by U_{n_k} over V^i_k. Since the $\tilde{V}^i$ form an open cover of M, we conclude $s(M) \Vdash \exists n \in N \gamma \in U_n$. □

3.3 Remark. We are now in a position to give some more justification for the definition 3.1.3 of the topology on $s(M)$. Synthetically, one would construct a "manifold" inside $\mathcal{Z}$ by patching together copies of R^d. If M is a manifold in *Sets*, and $\{U_n\}_n$ is a countable atlas for M with diffeomorphisms $\varphi_n\colon U_n \xrightarrow{\sim} \mathbb{R}^d$, then as in 3.2, $\mathcal{Z} \models s(M) = \bigcup_{n \in N} U_n$ (a union over *smooth* natural numbers), and, working inside $\mathcal{Z}$, we can give $s(M)$ the weak topology with respect to this s-countable cover $\{U_n | n \in N\}$, where each U_n is given the topology obtained from the internal topology on R^d and the (external) diffeomorphisms $U_n \xrightarrow{\varphi_n} \mathbb{R}^d$ $(n \in \mathbb{N})$. The resulting topology is the same:

Proposition. *The internal topology in $s(M)$ as defined in 3.1.3 coincides with the weak topology given by an s-countable atlas $\{U_n | n \in$*

$N\}$ of copies of R^d as just described.

Proof. Suppose $\{V_n\}_{n\in\mathbb{N}}$ is a sequence of opens of M, defining an internal basic open V at stage N as in 3.1.3. We need to show that V is open in the weak topology, i.e.

$$N \Vdash \forall n \in N(\varphi_n(V \cap U_n) \text{ is open in } R^d).$$

It suffices to take the generic $n \in N$, i.e. show that

$$N \times N \Vdash (\varphi_{\pi_2}(V \cap U_{\pi_2}) \text{ is open in } R^d)$$

where V really stands for $V|\pi_1$. On a one-point open (n,m) of $\mathbb{N}\times\mathbb{N}$, V is given by V_n, U_{π_2} is given by U_m, and φ_{π_2} is given by φ_m. So $\varphi_{\pi_2}(V \cap U_{\pi_2})$ is the open (by 3.1.1) of R^d at stage $N \times N$ given by $\varphi_m(V_n \cap U_m)$ over $\{(n,m)\}$, thus showing that the basic opens of 3.1.3 are open in the weak topology.

Conversely, suppose O is a subsheaf of $s(M)$ at stage $\ell A, A = C^\infty(\mathbb{R}^n)/I$, and a is a point of $s(M)$ at stage ℓA. So a can be represented by a smooth map $a(x): W \to M$, where W is an open neighbourhood of $Z(I_o)$ for a finitely generated $I_0 \subset I$, as usual. Moreover, suppose $\ell A \Vdash (a \in O \wedge O$ is open for the weak topology). So

$$\ell A \Vdash \forall n \in N(\varphi_n(O \cap U_n) \text{ is open in } R^d).$$

By 3.2, $\ell A \Vdash \exists p \in N\ a \in U_p$, so by replacing ℓA by a cover of ℓA we may without loss assume—if we choose I_0 big enough and W small enough—that there is a smooth integer p at ℓA given by a locally constant map $W \xrightarrow{p} \mathbb{N}$, such that this represents $\ell A \Vdash a \in U_p$, i.e. writing $W_n = p^{-1}(n)$,

$$\forall n \in \mathbb{N}: a(W_n) \subset U_n \subset M.$$

Since $\ell A \Vdash (\varphi_p(O \cap U_p)$ is open in $R^d)$, we also have $\ell A \Vdash (\exists$ basic open E in R^d $\varphi_p(a) \in E \subset \varphi_p(O \cap U_p))$, so, again passing to a cover of ℓA, and enlarging I_o, shrinking W if necessary, we may assume that there is a disjoint cover $\{V_{nm}\}_{n,m\in\mathbb{N}}$ of W refining $\{W_n\}$ by $V_{nm} \subset W_n$ for all n, and opens E_{nm} in R^d defining an internal basic open E of R^d at stage ℓA, with

$$\ell A \Vdash \varphi_p(a) \in E \subset \varphi_p(O \cap U_p).$$

Since $\varphi_p^{-1}(E)$ is the open in U_p given by taking the same disjoint cover $\{V_{nm}\}$, and the opens $\varphi_n^{-1}(E_{nm})$ over V_{nm}, we see that $\varphi_p^{-1}(E)$

is a basic open of $s(M)$ at stage ℓA as in 3.1.4, and clearly

$$\ell A \Vdash a \in \varphi_p^{-1}(E) \subset O,$$

showing that O is open in the topology described in 3.1.2, 3.1.3. □

3.4 Corollary. For every $M \in \mathbb{M}$, $s(M)$ is an s-topological space; i.e. the intersection of s-finitely many opens is again open.

Proof. For $M = \mathbb{R}^d$, this is proved exactly as for $M = \mathbb{R}$, see 2.7. The general case follows from the proposition in 3.3. □

A topological space X in $\mathcal{Z}$ is ***s-compact*** if every open cover of X has an s-finite refinement. More precisely, for every open cover $\mathcal{U}$ of X there exists an $n \in N$ and a map $\{m \in N | m < n\} \xrightarrow{V} \mathcal{O}(X)$ such that $X = \bigcup_{m<n} V_m$ and $\forall m < n \, \exists U \in \mathcal{U} \, V_m \subset U$. In 2.5(ii) we have seen that $[0,1] \subset R$ is s-compact. More generally, we have the following result:

3.5 Theorem. The embedding $s: \mathbb{M} \hookrightarrow \mathcal{Z}$ preserves compactness. In other words, if M is compact, then $\mathcal{Z} \models s(M)$ is s-compact, for every manifold M. (This implies the existence of a Lebesgue number, cf. VII.2.10.)

One way of proving this proposition is by using the s-compactness of $[0,1]$, along the following lines. Working in *Sets*, let M be a compact manifold, let $U_1, \ldots, U_n$ be a finite cover of M by copies of $\mathbb{R}^d$, $\varphi_i: U_i \xrightarrow{\sim} \mathbb{R}^d$. Then for some $k, M = \bigcup_{i=1}^n \varphi_i^{-1}\left([-k,k]^d\right)$. Then certainly $\mathcal{Z} \models s(M) = \bigcup_{i=1}^n \varphi_i^{-1}\left([-k-1,k+1]^d\right)$. Also, from s-compactness of $[0,1]$, it follows that $[-k-1,k+1]^d$ is s-compact, by copying the (constructively valid) classical proof that the product of two compact spaces is compact. This will show that $\mathcal{Z} \models s(M)$ is s-compact.

We will now give a more direct proof of theorem 3.5, not using s-compactness of $[0,1]$. We do this, not only because s-compactness of $[0,1]$ was rather involved so that an independent proof is worthwhile giving, but also because the following proof nicely illustrates the techniques that one uses when working with $\mathcal{Z}$.

Proof of Theorem 3.5. Let M be a compact manifold, let $\mathcal{U}$ be an open cover at stage ℓA, where $A = C^\infty(\mathbb{R}^n)/I$, and assume $\mathcal{U}$ consists of *basic* opens. Since $\ell A \times s(M) \Vdash \exists U \in \mathcal{U}\ \pi_2 \in U$, we find a finitely generated $I_o \subset I$, and a finite open cover $O_1, \dots, O_{i_o}$ of $Z(I_o) \times M$ such that, writing $\ell B_i = s(O_i) \cap \ell A \times s(M)$,

$$\ell B_i \Vdash \pi_2 \in U_i \in \mathcal{U}$$

for a basic open U_i at stage ℓB_i, for $i = 1, \dots, i_o$. By choosing I_o sufficiently big and the O_i sufficiently small, we may assume that U_i is given by an open $U_{im} \subset M$ over O_{im}, where $\{O_{im}\}_m$ is a disjoint cover of O_i (cf. 3.1.4), and since $\ell B_i \Vdash \pi_2 \in U_i$, the map $O_i \xrightarrow{\pi_2} M$ maps O_{im} into U_{im}.

$$\begin{array}{ccc} O_i & \xrightarrow{\quad \pi_2 \quad} & M \\ \cup & & \cup \\ O_{im} & \xrightarrow{\qquad\quad} & U_{im} \end{array}$$

By compactness of M, we can find an open cover $\{W_n\}_n$ of $Z(I_o)$ and for each n open subsets $E_1^n, \dots, E_{k_n}^n$ of M such that $E_1^n \cup \dots \cup E_{k_n}^n = M$, and each $W_n \times E_j^n$ is contained in some O_{im}, say

$$W_n \times E_j^n \subset O_{i_{(n,j)} m_{(n,j)}} \quad (j = 1, \dots, k_n).$$

Moreover, by passing to a suitable refinement, we may without loss assume that the cover $\{W_n\}_n$ is neighbourhood finite.

Now apply Ostrand's theorem, and obtain a refinement of $\{W_n\}_n$ of the form $\mathcal{W}^1 \cup \dots \cup \mathcal{W}^d$, where $\mathcal{W}^r$ is a family of pairwise disjoint open sets,

$$\mathcal{W}^r = \{\tilde{W}_n^r\}_{n \in \mathbb{N}}, \quad \tilde{W}_n^r \subset W_n \quad (r = 1, \dots, d).$$

Let

$$\tilde{W}^r = \bigcup_{n \in \mathbb{N}} \tilde{W}_n^r,$$

and define a smooth natural number q_r at $s(\tilde{W}^r)$, i.e. a locally constant map $\tilde{W}^r \to \mathbb{N}$, by

$$q_r | \tilde{W}_n^r = k_n.$$

Also, we can define a family of basic opens $\{V_q | 1 \le q \le q_r\}$ of $s(M)$ at stage $s(\tilde{W}^r)$, as follows. The generic $q \le q_r$ at stage $\tilde{W}^r$ is the projection $\bigcup_n \left(\tilde{W}_n^r \times \{0, \dots, k_n\}\right) \xrightarrow{\pi_2} N$ (see 3.1.5), and a family $\{V_q | 1 \le q \le q_r, q \in N\}$ of basic opens of $s(M)$ at stage $s\left(\tilde{W}^r\right)$ is the same as a single open V at stage $s\left(\bigcup_n \left(\tilde{W}_n^r \times \{0, \dots, k_n\}\right)\right)$. V

is defined as

$$V = \text{“}E_j^n \text{ over } \tilde{W}_n^r \times \{j\}\text{”}$$

(cf. 3.1.4).

Let $\ell A^r = s(\tilde{W}^r) \cap \ell A = \ell(C^\infty(\tilde{W}^r)/(I|\tilde{W}^r))$. Then q_r restricts to an element of N at stage ℓA^r, and $V = \{V_q | 1 \leq q \leq q_r\}$ restricts to a family of opens at stage ℓA^r. We claim that

(1) $\ell A^r \Vdash \forall x \in s(M)\ \exists q \in N\ (1 \leq q \leq q_r \wedge x \in V_q)$
(2) $\ell A^r \Vdash \forall q \in N (1 \leq q \leq q_r \to \exists U \in \mathcal{U} V_q \subset U)$.

Since $\{\ell A^r \to \ell A\}_{r=1}^d$ is a cover of ℓA in $\mathbb{L}$, (1) and (2) imply that

$$\ell A \Vdash \text{“}\mathcal{U} \text{ has an } s\text{-finite refinement”},$$

which would complete the proof. It thus remains to show (1) and (2).

To show (1), it suffices to consider the generic $x \in s(M)$ and prove that

(3) $\ell A^r \times s(M) \Vdash \exists q \leq q_r\ \pi_2 \in V_q$.

Fix r, and let

$$\tilde{O}_i^r = \bigcup \{\tilde{W}_n^r \times E_j^n | n \in \mathbb{N}, i_{(n,j)} = i\}.$$

Then $\tilde{W}^r \times M = \tilde{O}_1^r \cup \ldots \cup \tilde{O}_{i_o}^r$. Moreover, $\tilde{O}_i^r$ is a union of disjoint sets, so we may define a smooth integer at $s(\tilde{O}_i^r)$ by

$$q_i^r = j \text{ on } \tilde{W}_n^r \times E_j^n.$$

Then the basic open $V_{q_i^r}$ of $s(M)$ at $s(\tilde{O}_i^r)$ is the open given by E_j^r over $\tilde{W}_n^r \times E_j^n$, so it is clear that

$$s(\tilde{O}_i^r) \cap (\ell A^r \times s(M)) \Vdash \pi_2 \in V_{q_i^r}.$$

Since $\tilde{W}^r \times M = \tilde{O}_1^r \cup \ldots \cup \tilde{O}_{i_o}^r$, we conclude that (3) holds. This proves (1).

Finally, we prove (2). It suffices to take the generic $q \leq q_r$, i.e. to show

(4) $\ell C^r \Vdash \exists U \in \mathcal{U}\ V_{\pi_2} \subset U$,

where $\ell C = \ell A \times N \cap s\left(\bigcup_{n \in \mathbb{N}} (\tilde{W}_n^r \times \{1, \ldots, k_n\})\right)$. Similar to the $\tilde{O}_i^r$, define open sets $T_i^r \subset \mathbb{R}^n \times \mathbb{N}$ by

$$T_i^r = \bigcup \{\tilde{W}_n^r \times \{j\} | i_{(n,j)} = i, n \in \mathbb{N}, 1 \leq j \leq k_n\},$$

and let $\ell C_i^r = s(T_i^r) \cap \ell C^r$. Since

$$\bigcup_{i=1}^{i_o} T_i^r = \bigcup_{n \in \mathbb{N}} \left(\tilde{W}_i^r \times \{1, \ldots, k_n\}\right),$$

(4) follows if we can show that for each $i, i = 1, \ldots, i_o$

(5) $\ell C_i^r \Vdash \exists U \in \mathcal{U}\ V_{\pi_2} \subset U$.

To see that (5) holds, let

$$S_i^r = \bigcup \{\tilde{W}_n^r \times \{j\} \times E_j^n | i_{(n,j)} = i\}$$

and let

$$\ell D_i^r = s(S_i^r) \cap (\ell C_i^r \times s(M)).$$

Then there are projections

$$\begin{tikzcd} \ell D_i^r \arrow[r, "\pi_{12}"] \arrow[d, "\pi_{13}"] & \ell C_i^r \\ \ell A \times s(M) & \end{tikzcd}$$

Since $\pi_2 \colon S_i^r \to T_i^r$ obviously has a right-inverse, so does $\ell D_i^r \xrightarrow{\pi_{12}} \ell C_i^r$, and therefore it suffices to prove

(6) $\ell D_i^r \Vdash \exists U \in \mathcal{U}\ V_{\pi_2} \subset U$.

But $\ell D_i^r \xrightarrow{\pi_{13}} \ell A \times s(M)$ factors through ℓB_i and $\ell B_i \Vdash U_i \in \mathcal{U}$, where U_i is given by $U_{im} \subset M$ over O_{im}, and moreover $\pi_2(O_{im}) \subset U_{im}$, as we saw in the beginning of the proof. Restricting all this along $\ell D_i^r \xrightarrow{\pi_{13}} \ell B_i$, we find that, referring to the disjoint opens $\tilde{W}_n^r \times \{j\} \times E_j^n \subset S_i^r$, U_i is given at ℓD_i^r by $U_{im_{(n,j)}}$, over $\tilde{W}_n^r \times \{j\} \times E_j^n$, and V_{π_2} is given at ℓD_i^r by E_j^n over $\tilde{W}_n^r \times \{j\} \times E_j^n$. Since $\tilde{W}_n^r \times E_j^n \subset O_{im_{(n,j)}}$ and $\pi_2(O_{im_{(n,j)}}) \subset U_{im_{(n,j)}}$, we conclude

$$\ell D_i^r \Vdash V_{\pi_2} \subset U_i \in \mathcal{U},$$

which proves (6), and completes the proof. $\square$

Analogous to III.3.5, we have

3.6 Theorem. Let M and N be manifolds. In $\mathcal{Z}$ it holds that all functions $s(M) \to s(N)$ are continuous.

Proof. We have to show that

$$\mathcal{Z} \models \forall F \in s(N)^{s(M)} \forall x \in s(M) \forall U \in \mathcal{O}(sN)(F(x) \in U \to \\ \exists O \in \mathcal{O}(sM)(x \in O \wedge F(O) \subset U)).$$

So choose $\ell A \in \mathbb{L}, A = C^\infty(\mathbb{R}^n)/I$, and let $F: \ell A \times s(M) \to s(N)$ be a given element of $s(N)^{s(M)}$ at stage ℓA, let $\ell A \xrightarrow{p} s(M)$ be a point of $s(M)$ at stage ℓA, and let U be an open of $s(N)$ at stage ℓA, which we can assume to be a basic open. Moreover, assume $\ell A \Vdash F(p) \in U$. By writing M and N as retracts of open neighbourhoods in Euclidean space, as usual, we can find a finitely generated $I_o \subset I$ and an open $W \subset \mathbb{R}^n$ with $W \supset Z(I_o)$, such that F is represented by a smooth function

$$W \times M \xrightarrow{F} N,$$

p is represented by a smooth function

$$W \xrightarrow{p} M,$$

and U is represented by sequences of open sets $\{U_n\}_n, \{W_n\}_n$, i.e.

$$U = \text{“}U_n \text{ over } W_n\text{”}$$

as in 3.1.4, where $U_n \subset M$ and $\{W_n\}_n$ is a cover of W by pairwise disjoint open sets. Moreover, since $\ell A \Vdash F(p) \in U$, we can choose the W_n so small that $F(W_n \times p(W_n)) \subset U_n$,

$$\begin{array}{ccccc} W & \xrightarrow{(\mathrm{id}, p)} & W \times M & \xrightarrow{F} & N \\ \cup & & & & \cup \\ W_n & \longrightarrow & & \longrightarrow & U_n \end{array}$$

Now fix W_n. By continuity of F, there are a locally finite cover $\mathcal{W}_n = \{W_n^m\}_m$ of W_n and opens $O_n^m \subset M$ with $p(W_n^m) \subset O_n^m$ and $F(W_n^m \times O_n^m) \subset U_n$ (just as in III.3.5). By Ostrand's theorem (2.13), there is a refinement of $\mathcal{W}_n$ of the form

$$\mathcal{W}_n^1 \cup \ldots \cup \mathcal{W}_n^d$$

such that $\mathcal{W}_n^i = \{\tilde{W}_n^{im}\}_m$ is a family of pairwise disjoint opens, with $\tilde{W}_n^{im} \subset W_n^m$. Now let for $i = 1, \dots, d$

$$W^i = \bigcup_{m,n \in \mathbb{N}} \tilde{W}_n^{im}.$$

This is a union of pairwise disjoint open sets. Moreover, $Z(I_o) \subset W^1 \cup \ldots \cup W^d$, so if we write $\ell A^i = s(W^i) \cap \ell A$, then the $\ell A^i (i = 1, \dots, d)$ form a cover of ℓA in $\mathbb{L}$.

At W^i, define an open O^i of $s(M)$ by giving it value O_n^m over W_n^{im}. We claim that

$$\ell A^i \Vdash p \in O^i \wedge F(O^i) \subset U.$$

Indeed, since $p(\tilde{W}_n^{im}) \subset p(W_n^m) \subset O_n^m$, the first conjunct is clear. The second follows from

$$F(\tilde{W}_n^{im} \times O_n^m) \subset U.$$

This completes the proof. □

Next, we will turn to partitions of unity. As in proposition 3.2, we will restrict ourselves to countable covers whenever this is convenient. Accordingly, we consider only internal partitions of unity indexed by the smooth natural numbers N. More precisely, an (s-countable) *s-partition of unity on $s(M)$* in $\mathcal{Z}$ is a map $N \xrightarrow{\rho} [0,1]^{s(M)}$ such that the following holds in $\mathcal{Z}$:

$$(*)\qquad \begin{array}{l} \forall x \in s(M)\, \exists U \in \mathcal{O}(s(M))(x \in U \wedge \exists n \in N \\ \forall m \in N(m < n \vee \rho_m|U \equiv 0) \wedge (\sum_{m<n} p_m)|U \equiv 1). \end{array}$$

Note that this s-finite sum makes sense since R is an s-ring.

As in section III.3, we say that $\rho = \{\rho_n\}_{n \in N}$ is *subordinate* to a cover $\mathcal{U}$ of $s(M)$ if $\forall n \in N \exists U \in \mathcal{U} \forall x \in sM(x \in U \vee \rho_n(x) = 0)$ holds in $\mathcal{Z}$.

3.7 Proposition. The embedding $s\colon \mathbb{M} \hookrightarrow \mathcal{Z}$ preserves partitions of unity. More precisely, if $\{\rho_n\}_{n \in \mathbb{N}}$ is a partition of unity on a manifold M, then it is valid in $\mathcal{Z}$ that the induced map $N \xrightarrow{\rho} [0,1]^{s(M)}$ is an internal s-partition of unity. Moreover, if $\{\rho_n\}_{n \in \mathbb{N}}$ is subordinate to an open cover $\{U_n\}_{n \in \mathbb{N}}$, then in $\mathcal{Z}$, ρ is subordinate to the induced cover (cf. 3.2) $\{U_n\}_{n \in N}$.

Proof. We have to show that for the generic $\gamma \in s(M)$ at stage $s(M)$ represented by the identity,

$$(1) \quad \begin{array}{ll} s(M) \Vdash & \exists U \in \mathcal{O}(sM)(\gamma \in U \wedge \exists n \in N \\ & \forall m \in N(m < n \vee \rho_m|U \equiv 0) \wedge \sum_{m<n} p_m|U \equiv 1). \end{array}$$

Let $\mathcal{W} = \{W_k\}_{k \in \mathbb{N}}$ be a locally finite cover of M such that on W_k, all but $\rho_o, \ldots, \rho_{n_k-1}$ vanish, and (hence) $\forall x \in W_k \sum_{i<n_k} \rho_i(x) = 1$. By Ostrand's theorem, there is a refinement $\mathcal{V}$ of the form $\mathcal{V} = \mathcal{V}^1 \cup \ldots \cup \mathcal{V}^d$, where each $\mathcal{V}^i$ is a family of pairwise disjoint open sets $\{V_k^i\}_{k \in \mathbb{N}}$ with $V_k^i \subset U_k$. Let $V^i = \bigcup \mathcal{V}^i$, and define at $s(V^i)$ a smooth integer n^i by $n^i|V_k^i = n_k$, and a basic open W^i by the value W_k over V_k^i. Then the following statements are easily verified.

(2) $s(V^i) \Vdash \gamma \in W^i$
(3) $s(V^i) \Vdash \forall_m \in N(m \geq n^i \to \rho_m|W^i = 0)$
(4) $s(V^i) \Vdash \forall x \in U^i \sum_{m<n^i} \rho_m(x) = 1$

Since $M = V^1 \cup \ldots \cup V^d$, (1) follows.

The second statement of the proposition is obvious. □

3.8 Theorem. Let M be a manifold. Then it is valid in $\mathcal{Z}$ that for every open cover $\mathcal{U}$ of M there exists an (s-countable) s-partition of unity $\{\rho_n\}_{n \in N}$ subordinate to $\mathcal{U}$.

Proof. Let $\mathcal{U}$ be a cover of $s(M)$ at stage ℓA, where $A = C^\infty(\mathbb{R}^d)/I$, and assume $\mathcal{U}$ consists of basic opens. Thus

$$\ell A \times s(M) \Vdash \exists U \in \mathcal{U}\ \pi_2 \in U,$$

and this implies (as in the proof of 3.5) that there is a finitely generated subideal $I_o \subset I$, and a cover $O_1, \ldots, O_{i_o}$ of $Z(I_o) \times M$ by open subsets of $\mathbb{R}^d \times M$, such that for each $i = 1, \ldots, i_o$ there is a basic open U_i of $s(M)$ at stage $s(O_i)$, given by $U_{im} \in \mathcal{O}(M)$ over O_{im} say, where $\{O_{im}\}_m$ is a disjoint cover of O_i with the property that $\pi_2(O_{im}) \subset U_{im}$,

$$O_{im} \xrightarrow{\pi_2} U_{im};$$

moreover, $s(O_i) \cap \ell A \Vdash U_i \in \mathcal{U}$.

Now take a locally finite refinement $\{W_n\}_{n \in \mathbb{N}}$ of the cover $\{O_{im} | i = 1, \ldots, i_0, m \in \mathbb{N}\}$ of $Z(I_o) \times M$, such that for each n there is an open box $D_n \times E_n \subset \mathbb{R}^d \times M$ with

$$W_n \subset D_n \times E_n \subset O_{i_n m_n}.$$

Let $\{\rho_n : \mathbb{R}^d \times M \to [0,1]\}_{n \in \mathbb{N}}$ be a partition of unity subordinate to $\{W_n\}_n$; so

$$\begin{array}{l} \sum_n \rho_n(x) = 1 \text{ for all } x \in W = \cup_n W_n \\ \operatorname{supp}(\rho_n) \subset W_n. \end{array}$$

Then $\{\rho_n\}_n$ defines a map

$$N \times \ell A \times s(M) \xrightarrow{\rho} [0,1]$$

in $\mathcal{Z}$. We claim that $\ell A \Vdash$ "ρ is an s-partition of unity subordinate to $\mathcal{U}$", i.e.

(1) $\ell A \Vdash \forall n \in N\ \exists U \in \mathcal{U}\ \forall x \in s(M)(x \in U \vee \rho_n(x) = 0)$,
(2) $\ell A \Vdash \forall x \in s(M) \exists nbd\ V \ni x \exists n \in N\ \forall m \in N(m < n \vee \rho_m|V \equiv 0)$,
(3) $\ell A \Vdash \forall x \in s(M) \sum_{n \in N} \rho_n(x) = 1$.

To prove (1), we consider the generic $n = \pi_2$, and show

(4) $\ell A \times N \Vdash \exists U \in \mathcal{U}\ \forall x \in s(M)(x \in U \vee \rho_n(x) = 0)$.

Cover ℓA by $\ell A_i (i = 1, \ldots, i_o)$, where

$$\ell A_i = \ell A \cap s(\cup\{D_n | i_n = i\}),$$

and define basic opens E_i of $s(M)$ at $\ell A_i \times N$ by

$$E_i = U_{im_n} \text{ over } \ell A_i \times \{n\}.$$

Then

(5) $\ell A_i \times N \times s(M) \Vdash \pi_3 \in E_i \vee \rho_n(\pi_3) = 0)$,

for $\pi_3 \in E_i$ holds on $\bigcup\{D_n \times \{n\} \times E_n | i_n = i\}$, and $\mathrm{supp}(\rho_n) \subset D_n \times E_n$. By genericity of π_3, (5) gives

(6) $\ell A_i \times N \Vdash \forall x \in s(M)(x \in E_i \vee \rho_n(x) = 0)$,

So (4) follows, provided we can show that

(7) $\ell A_i \times N \Vdash E_i \in \mathcal{U}$.

To this end, consider the projection

$$\bigcup_{i_n = i} (D_n \times E_n) \xrightarrow{\pi_2} \bigcup_{i_n = i} D_n.$$

Obviously, this map *locally* has a right-inverse, defined over each open set D_n. By Ostrand's theorem applied to a locally finite refinement of the cover $\{D_n | i_n = i\}$ of $\bigcup_{i_n = i} D_n$, we find finitely many opens $S_i^1, \ldots, S_i^r$ such that

$$\bigcup_{i_n = i} D_n = \bigcup_{j=1}^{r} S_i^j,$$

and sections $t_j: S_i^j \to \bigcup_{i_n=i} D_n \times E_n$ of π_2. Since the $S_i^j (1 \leq j \leq r)$ induce a cover of ℓA_i, (7) would therefore follow if we show

(8) $\ell B_i \times N \Vdash E_i \in \mathcal{U}$

where $\ell B_i = (\ell A_i \times s(M) \cap \bigcup_{i_n=i}(D_n \times E_n)$, considered as an object over ℓA_i via the projection $\ell B_i \to \ell A_i$. But $\ell B_i \Vdash E_i = U_i \in \mathcal{U}$, since $\ell A_i \Vdash U_i \in \mathcal{U}$, and at ℓB_i, E_i and U_i are both given by the open U_{im_n} over $D_n \times E_n \times \{n\}$. Thus (8) holds, which completes the proof of (1).

(2) is easier: taking the generic $x \in s(M)$, we have to show

(9) $\ell A \times s(M) \Vdash \exists nbd\, V \ni \pi_2 \exists n \in N\ \forall n \in N (m < n \vee \rho_m | V = 0)$.

Let $\{K_n \times V_n\}_{n \in \mathbb{N}}$ be an open cover of $W = \cup_n W_n$ such that $K_n \times V_n$ meets only $W_0, \ldots, W_{k_n - 1}$, and take a neighbourhood finite refinement $\{T_m\}_m$ of $\{K_n \times V_n\}_n$, say $T_m \subset K_{n_m} \times V_{n_m}$. Applying Ostrand's theorem, $\{T_m\}_m$ has a refinement of the form

$$\mathcal{P} = \mathcal{P}^1 \cup \ldots \cup \mathcal{P}^r$$

where $\mathcal{P}^i = \{P_m^i\}_{m \in \mathbb{N}}$ is a family of pairwise disjoint opens with $P_m^i \subset T_m$. At stage $s(P^i)$, we define the basic open V^i of $s(M)$ by giving it the value V_{n_m} over P_m^i, and a smooth integer k^i with value k_{n_m} over P_m^i. Clearly

$$\begin{array}{lll} s(P^i) & \Vdash & \pi_2 \in V^i \\ s(P^i) & \Vdash & \forall m \in N (m < k^i \vee \rho_m | V^i = 0) \end{array}$$

Since $\mathcal{P}$ covers $W \supset Z(I_o) \times M$, (9) follows, thus proving (2).

(3) is now clear. □

3.9 Corollary. Every manifold is s-Lindelöf in $\mathcal{Z}$, i.e. for any $M \in \mathbb{M}$ it holds in $\mathcal{Z}$ that every open cover $\mathcal{U}$ of M has an open refinement indexed by the smooth natural numbers N.

Proof. If $\mathcal{U}$ is an open cover (at any stage), we find a partition of unity $\{\rho_n\}_{n \in N}$ (at the same stage) by theorem 3.8. By theorem 3.6, each $\rho_n^{-1}(U(R))$ is open, thus giving an open refinement of $\mathcal{U}$ indexed by N. □

We now turn to connectedness. As with compactness, the notion of chain-connectedness as defined in III.3.8 is not suitable for working in $\mathcal{Z}$. For example, the unit interval $[0,1] \subset R$ cannot be chain-connected, since this would imply compactness of $[0,1]$ (by the obvious fact that a chain of intervals from 0 to 1 must be a cover of $[0,1]$), which is false in $\mathcal{Z}$ (cf. 2.1).

We define a space X to be *s-chain connected* if for every open cover $\mathcal{U}$ of X, any two points x, y in X can be connected by a smooth-finite chain $(V_m : m \in N, m \leq n)$ refining $\mathcal{U}$; i.e. $x \in V_o, y \in V_n$, $\forall m < n\ \exists z \in V_m \cap V_{m+1}$, and $\forall m \leq n\ \exists U \in \mathcal{U} V_m \subset U$.

Thus, as with compactness, $\mathbb{N}$ is replaced by N, and to avoid s-finite choice we require a sequence *refining* $\mathcal{U}$, rather than a sequence of elements of $\mathcal{U}$.

3.10 Theorem. The embedding $\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{Z}$ maps connected manifolds to s-chain connected manifolds.

Proof. Let M be a connected manifold. Let $\mathcal{U}$ be an open cover of $s(M)$ at stage ℓA, $A = C^\infty(\mathbb{R}^n)/I$, and assume $\mathcal{U}$ consists of basic opens. As before, $\ell A \times s(M) \Vdash \exists U \in \mathcal{U} \pi_2 \in U$ implies that there is a finitely generated subideal $I_0 \subset I$ and an open cover $O_1 \cup \ldots \cup O_{i_0}$ of $Z(I_0) \times M$ by open sets in $\mathbb{R}^n \times M$ such that at $\ell B_i = s(O_i) \cap (\ell A \times sM)$ there is a basic open $U_i \in \mathcal{U}$ given by U_{im} over O_{im}, where $U_{im} \subset M$ and $\{O_{im}\}_m$ is a disjoint cover of O_i with $\pi_2(O_{im}) \subset U_{im}$. We have to show

$$
(1) \qquad \begin{aligned} &\ell A \times s(M) \times s(M) \Vdash \exists n \in N \exists (V_m : 1 \leq m \leq n)(V_m \in \\ &O(sM) \wedge \pi_2 \in V_1 \wedge \pi_3 \in V_n \wedge \forall_m < n(\exists z \in \\ &V_m \cap V_{m+1}) \wedge \forall m \leq n \exists U \in \mathcal{U} V_m \subset U) \end{aligned}
$$

To prove (1), refine the cover $\{O_{im} : i = 1, \ldots, i_o, m \in \mathbb{N}\}$ by open cubes $V_n \times W_n$, say

$$V_n \times W_n \subset O_{i_n m_n}.$$

For $p \in Z(I_o)$, let $\mathcal{W}_p = \{W_n | p \in V_n\}$. Then $\mathcal{W}_p$ covers M, so by connectedness of M, we find for any $r, t \in M$ opens $W_{n_1(p,r,t)}, \ldots, W_{n_k(p,r,t)} \in \mathcal{W}_p$ with $r \in W_{n_1(p,r,t)}, t \in W_{n_k(p,r,t)}$ and $W_{n_i} \cap W_{n_{i+1}} \neq \emptyset (i = 1, \ldots, n_{k-1})$. (The length k of this chain also depends on p, r, and t, $k = k(p,r,t)$.) Let

$$
\begin{aligned} V_{p,r,t} &= V_{n_1(p,r,t)} \cap \ldots \cap V_{n_k(p,r,t)}, \\ T_{p,r,t} &= V_{p,r,t} \times W_{n_1(p,r,t)} \times W_{n_k(p,r,t)}. \end{aligned}
$$

These sets $T_{p,r,t}$ cover $Z(I_o) \times M \times M$, so by Ostrand's theorem, we find a refinement of the form

$$
\begin{aligned} \mathcal{T} &= \mathcal{T}^1 \cup \ldots \cup \mathcal{T}^d, \\ \mathcal{T}^j &= \{T^j_m\}_{m \in \mathbb{N}}, \ T^j_m \subset T_{p_m, r_m, t_m} \end{aligned}
$$

and each $\mathcal{T}^j$ is a disjoint family of open sets. Let $T^j = \bigcup_m T^j_m$, and $\ell B^j = s(T^j) \cap (\ell A \times s(M) \times s(M))$. At ℓB^j, we have a smooth integer n^j given by the locally constant function $n^j : T^j \to \mathbb{N}$,

$$n^j | T^j_m = k_m = k(p_m, r_m, t_m)$$

(the length of the chain from r_m to t_m in $\mathcal{W}_{p_m}$). Moreover, at ℓB^j we have a sequence of opens $(V_m : 1 \le m \le n^j)$ of $s(M)$ defined as follows: The generic $m \le n^j$ over $s(T^j)$ is given by the projection

$$\bigcup_{m \in \mathbb{N}} (T^j_m \times \{1, \ldots, k_m\}) \to \mathbb{N}.$$

At $s(\bigcup_m T^j_m \times \{1, \ldots, k_m\})$, we define an open V of $s(M)$ given by

(1) $W_{n_\ell(p_m, r_m, t_m)}$ over $T^j_m \times \{\ell\} \quad (\ell = 1, \ldots, k_m)$.

Then V restricts to a sequence of basic opens of $s(M)$, $(V_m : m \in N, 1 \le m \le n^j)$ at stage ℓB^j.

Since $T^j_m \subset T_{p_m, r_m, t_m}$, it is clear that

(2) $\ell B^j \Vdash \pi_2 \in V_1 \wedge \pi_3 \in V_{n^j}$.

Moreover, since each $W_{n_1(p,r,t)}, \ldots, W_{n_k(p,r,t)}$ is a chain in $\mathcal{O}(M)$, we have

(3) $\ell B^j \Vdash \forall m \in N, 1 \le m \le n^j \; \exists z \in s(M) z \in V_m \cap V_{m+1}$.

It thus remains to show

(4) $\ell B^j \Vdash \forall m \in N \; 1 \le m \le n^j \exists U \in \mathcal{U} \; V_m \subset U$;

or equivalently by taking the "generic" m,

(5) $\ell C^j \Vdash \exists U \in \mathcal{U} \; V_{\pi_4} \subset U$,

where $\ell C^j = s(\bigcup_{m \in \mathbb{N}} (T^j_m \times \{1, \ldots, k_m\})) \cap \ell B^j \times N$, and π_4 is the projection on the last factor. To prove (5), let

$$\ell C^j_i = \ell C^j \cap s(\bigcup \{T^j_m \times \{\ell\} | i_{n_\ell(p_m, r_m, t_m)} = i\}).$$

Then $\ell C^j_1, \ldots, \ell C^j_{i_0}$ cover ℓC^j in $\mathbb{L}$, so (5) follows from

(6) $\ell C^j_i \Vdash \exists U \in \mathcal{U} \; V_{\pi_4} \subset U \quad (i = 1, \ldots, i_o)$.

For fixed i and j, let

$$\ell D^j_i = (s(M) \times \ell C^j) \cap s(\bigcup \{W_{m'} \times T^j_m \times \{\ell\} | i_n = i, \text{ and } m' = m_n\} \tag{7}$$

where $n = n_\ell(p_m, r_m, t_m)$, and consider the projection mappings

$$\begin{tikzcd} \ell D_i^j \arrow[r, "p"] \arrow[d, "q"'] & \ell C_i^j \\ \ell B_i & \end{tikzcd}$$

where p is given by the second projection $M \times (\mathbb{R}^n \times M \times M \times \mathbb{N}) \to \mathbb{R}^n \times M \times M \times \mathbb{N}$, and q is given by the first projection followed by a twist map, i.e.

$$(M \times \mathbb{R}^n) \times (M \times M \times \mathbb{N}) \xrightarrow{\pi_{12}} M \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \times M.$$

Restricting V_{π_4} of (6) along p gives V_{π_5}, i.e. (referring to (7)) the open of $s(M)$ at ℓD_i^j with value

$$W_n \text{ over } W_{m'} \times T_m^j \times \{\ell\}, n = n_{\ell(p_m, r_m, t_m)}.$$

Restricting the open U_i at ℓB_i described in the beginning of the proof along q gives the open of $s(M)$ at ℓD_i^j represented by

$$U_{im'} \text{ over } W_{m'} \times T_m^j \times \{\ell\},$$

since $V_{p_m} \times W_{m'} \subset O_{im'}$ if $m' = m_n, n = n_{\ell(p_m, r_m, t_m)}$. Since $\pi_2(O_{im'}) \subset U_{im'}$, and $\ell B_i \Vdash U_i \in \mathcal{U}$, it follows that

(8) $\ell D_i^j \Vdash V_{\pi_5} \subset U_i \in \mathcal{U}$,

and hence since $\ell D_i^j \xrightarrow{p} \ell C_i^j$ splits, we obtain (6). □

3.11 Remark. The proof of (i) ⇒ (ii) in III.3.8 uses induction, so it does not automatically give a corresponding implication "s-chain-connected ⇒ connected". Thus, one cannot directly conclude from theorem 3.10 that $s: \mathbb{M} \hookrightarrow \mathcal{Z}$ preserves connectedness. However, it is easy to prove directly that for any connected manifold $M \in \mathbb{M}$,

$$\mathcal{Z} \models \forall U, V \in \mathcal{O}(sM)(\exists x\, x \in U \wedge \exists x\, x \in V \wedge U \cup V = sM \to \exists x\, x \in U \cap V).$$

Here is a sketch: suppose U and V are internal opens at stage ℓA (not necessarily basic!) such that $\ell A \Vdash \exists x\, x \in U \wedge \exists x\, x \in V \wedge U \cup V = sM$. Then $\ell A \times s(M) \Vdash \pi_2 \in U \vee \pi_2 \in V$, so if we write $A = C^\infty(\mathbb{R}^n)/I$, there is a finitely generated ideal $I_o \subset I$ and opens $O_1, O_2 \subset \mathbb{R}^n \times M$

with $Z(I_o) \times M \subset O_1 \cup O_2$ and

$$s(O_1) \cap (\ell A \times s(M)) \Vdash \pi_2 \in U$$
$$s(O_2) \cap (\ell A \times s(M)) \Vdash \pi_2 \in V.$$

Since $\ell A \Vdash \exists x\, x \in U \wedge \exists x\, x \in V$, we may assume (choosing I_0 bigger if necessary) that for each $x \in Z(I_0), O_i \cap (\{x\} \times M) \neq \emptyset (i = 1,2)$. So by connectedness of M, the projection $O_1 \cap O_2 \xrightarrow{\pi_2} \mathbb{R}^n$ maps onto (a neighbourhood of) $Z(I_0)$. Using Ostrand's theorem, we find a finite cover $U_1, \ldots, U_k$ of $Z(I_0)$ on which π_2 splits, i.e. there are $s_i : U_i \to O_1 \cap O_2$ with $\pi_2 s_i(x) \equiv x$. This will give $\ell A \Vdash \exists x\, x \in U \cap V$.

In the Zariski smooth topos, the open refinement theorem III.3.12 holds. In fact, we can state a stronger (and more interesting) version, by replacing the natural numbers $\mathbb{N}$ by the smooth natural numbers N.

3.12 Open Refinement Theorem. Let M be a manifold. Then in $\mathcal{Z}$ it holds that every s-countable cover of M has an open refinement, i.e.

$$\mathcal{Z} \models \forall F \in \mathcal{P}(sM)^N (\forall x \in sM\, \exists n \in N\, x \in F(n) \to$$
$$\forall x \in sM\, \exists U \in \mathcal{O}(sM)\, \exists n \in N\, x \in U \subset F(n)).$$

Proof. Take $\ell A \in \mathbb{L}$, $A = C^\infty(\mathbb{R}^d)/I$, and suppose $\ell A \times N \xrightarrow{F} \mathcal{P}(sM)$ is a map in $\mathcal{Z}$ such that $\ell A \Vdash \forall x \in sM\, \exists n \in N\, x \in F(n)$. Then $\ell A \times s(M) \Vdash \exists n \in N \pi_2 \in F(n)$, so we find a finitely generated $I_0 \subset I$, a cover $O_1, \ldots, O_{i_0}$ of $Z(I_0) \times M$ by open subsets of $\mathbb{R}^d \times M$, such that each O_i can be written as a disjoint union of $\{O_{im}\}_m$, and smooth natural numbers p_i given by locally constant functions $p_i : O_i \to \mathbb{N}$ with value p_{im} on O_{im}, such that

(1) $\ell B_i \Vdash \pi_2 \in F(p_i)$,

where $\ell B_i = s(O_i) \cap \ell A \times s(M)$.

By Ostrand's theorem, we can find a refinement of the cover $\{O_{im} | i = 1, \ldots, i_o, m \in \mathbb{N}\}$ of the form

$$\mathcal{V} = \mathcal{V}^1 \cup \ldots \cup \mathcal{V}^d,$$

where each $\mathcal{V}^r$ is a family $\{V_n^r\}_n$ of pairwise disjoint open sets, such that for all n there is an open box $D_n \times E_n$ and an $O_{i_n m_n}$ with

$$V_n^r \subset D_n \times E_n \subset O_{i_n m_n}.$$

For $r = 1, \ldots, r_0, i = 1, \ldots, i_0$, define

$$W^{ri} = \bigcup \{V_n^r | i_n = i\}.$$

Then for a fixed i, the W^{ri} induce a cover $\{s(W^{ri}) \cap \ell B_i \to \ell B_i\}_{r=1}^{r_0}$ of ℓB_i. At stage $s(W^{ri})$, we can define a basic open E^{ri} of $s(M)$ with value E_n over V_n^r, and a smooth integer p^{ri} with value p_{im_n} over V_n^r. Then (for the corresponding restrictions to $s(W^{ri}) \cap \ell B_i$) we have

(2) $s(W^{ri}) \cap \ell B_i \Vdash \pi_2 \in E^{ri} \subset F(p^{ri})$.

Indeed, since $V_n^r \subset D_n \times E_n \subset O_{i_n m_n}$, it is clear that $s(W^{ri}) \cap \ell B_i \Vdash \pi_2 \in E^{ri}$. For the inclusion $E^{ri} \subset F(p^{ri})$, it suffices to consider the generic element of E^{ri}, i.e. (cf. 3.1.5) to show that

(3) $\bigcup_{i_n=i}(V_n^r \times E_n) \cap (\ell B_i \times s(M)) \Vdash \pi_3 \in F(p^{ri})$

(where p^{ri} really stands for p^{ri} restricted along the projection map $\ell B_i \times s(M) \to s(M)$). Consider the map given by interchanging the second and third factor

$$\bigcup_{i_n=i} (D_n \times E_n \times E_n) \xrightarrow{(\pi_1,\pi_3,\pi_2)} \bigcup_{i_n=i} (D_n \times E_n \times E_n).$$

This restricts to a map

$$\bigcup_{i_n=i} (D_n \times E_n \times E_n) \cap (B_i \times s(M)) \xrightarrow{\sigma} \bigcup_{i_n=i} (D_n \times E_n \times E_n) \cap (B_i \times s(M)).$$

Now p^{ri} is invariant under restriction along σ, and $\pi_3|\sigma = \pi_2$, so (1) implies

(4) $\bigcup_{i_n=i}(D_n \times E_n \times E_n) \cap \ell B_i \times s(M) \Vdash \pi_3 \in F(p^{ri})$.

Since $V_n^r \subset D_n \times E_n$, (3) follows, completing the proof of (2).

From the fact that $\{s(W^{ri}) \cap \ell B_i \to \ell A \times s(M)\}_{r=1,\dots,r_0, i=1,\dots,i_0}$ is a cover, and the genericity of π_2, we get

$$\ell A \Vdash \forall x \in s(M) \exists U \in \mathcal{O}(sM) \exists n \in N \; x \in U \subset F(n). \qquad \square$$

3.13 Remark. As at the end of section III.3, the open refinement theorem implies that $s\colon \mathbb{M} \hookrightarrow \mathcal{Z}$ maps connected manifolds to indecomposable objects. Moreover, it implies that if X is a metric space in $\mathcal{Z}$ (with metric $\delta\colon X \times X \to R$ reformulated as a relation, cf. III.3.4) and X is *s-separable*, i.e. there is a map $N \xrightarrow{F} X$ with $\{F(n) | n \in N\} \subset X$ dense, then for any $M \in \mathbb{M}$ it is valid in $\mathcal{Z}$ that all functions $s(M) \to X$ are continuous.

And similar to III.3.11, 3.13 implies an axiom of choice

$$\forall R \in \mathcal{P}(s(M) \times N)(\forall x \in sM \, \exists n \in N \; R(x,n) \to \exists \textit{ open cover } \mathcal{U} \textit{ of } s(M) \; \forall U \in \mathcal{U} \; \exists n \in N \; \forall x \in U \; R(x,n))$$

4 The Degree of a Map

As a test case for the notion of smooth integer, we will consider the degree of a map of manifolds in $\mathcal{Z}$. Suppose M is a connected compact oriented manifold. We will define the degree as a map in $\mathcal{Z}$

$$\deg\colon s(M)^{s(M)} \to R.$$

Classically, $\deg(f)$ always is an integer. In $\mathcal{Z}$, this need not be the case, but it will turn out that ***the degree is a smooth integer***:

4.1 Theorem. Let M be a connected compact oriented manifold. Then

$$\mathcal{Z} \models \forall f \in s(M)^{s(M)} \deg(f) \in Z.$$

Theorem 4.1 will follow from comparison theorems for $\mathcal{Z}$ similar to the ones proved for $\mathcal{G}$ in chapter IV (cf. IV.3.9 and IV.3.10), and accordingly we will first look at cohomology and homology of manifolds in $\mathcal{Z}$. For $\mathcal{G}$, the proofs of De Rham's theorem (IV.3.1) and the comparison theorems were based on the fact that R is Archimedean and each standard simplex $\Delta_q = \{(x_o, \ldots, x_q) \in R^{q+1} | \sum x_i = 1\}$ is compact. Although these properties both fail in $\mathcal{Z}$, we have seen in the preceding two sections that their smooth analogues hold. This will allow us to perform arguments parallel to those given in chapter IV, at least for a compact manifold, or more generally, for a manifold of finite type (i.e. a manifold with a *finite* good cover, cf. IV.5). Thus, in this section we will only consider manifolds of finite type. (For arbitrary manifolds, the situation is more complicated).

First of all, we have as an analogue of theorem IV.3.1,

4.2 De Rham's Theorem in $\mathcal{Z}$. Let M be a manifold of finite type. Then the canonical s-linear map in $\mathcal{Z}$

$$H^q(sM) \to H_q(sM; R)^*,$$

induced by integration, is an isomorphism.

Proof. (sketch) The proof is similar to that of IV.3.1, but one has to adapt the definitions, along the following lines.

$H^q(sM)$ is the internal De Rham cohomology of $S(M) \in \mathcal{Z}$, which is defined as in section IV.1. In $\mathcal{Z}$, $H^q(sM)$ is not only an R-module, but an s-module over the s-ring structure of R. $H^q(sM; R)^*$ is the dual of the singular homology module $H_q(sM, R)$. We now proceed as in chapter IV, using induction on M via the Poincaré lemma and the Mayer-Vietoris sequence, but with the following modifications.

First of all, proposition IV.3.4 does not hold in $\mathcal{Z}$: clauses (1) and (2) present no problems, but (3) is only true for the case of a *finite* (constant) index set $\{\alpha\}$. (This is related to the fact that $s: \mathbb{M} \hookrightarrow \mathcal{Z}$ does not preserve countable covers as in III.3.2; one has to pass to s-countable covers, cf. the comments preceding proposition 3.2.) However, for manifolds of *finite* type, clause (3) of IV.3.4 is irrelevant for the induction on M.

Secondly, we have to show that $S_q^{\{sU,sV\}}(sM)$ and $S_q(sM)$ have the same homology (where $\{U,V\}$ is an open cover of M), by a barycentric subdivision argument using the Lebesgue number (as in IV.2.4). But we cannot do this in $\mathcal{Z}$ unless we use smooth integers everywhere! Since the Lebesgue number is (bigger than something) of the form $\frac{1}{n}$ for some $n \in N$, we have to iterate the barycentric subdivision map $sd: S_q(sM) \to S_q(sM)$ a "non-standard" number of times (depending polynomially on n). To define these iterates sd^n for arbitrary $n \in N$, it suffices by naturality in M to define the n^{th} subdivision of the identity map $\Delta_q \xrightarrow{\mathrm{id}} \Delta_q \in S_q(\Delta_q)$ as a map

$$N \to S_q(\Delta_q), \quad n \mapsto sd^n(\mathrm{id}_{\Delta_q}).$$

This can be done by coding $S_q(\Delta_q)$ as a subobject of $({\Delta_q}^N)^Z$, and then defining the transposed map $N \times Z \times N \to \Delta_q \subset R^{q+1}$ by recursion, using theorem 2.8. (Details are tedious, but straightforward.) The number of simplices occurring in $sd^n(\mathrm{id}_{\Delta_q})$ depends arithmetically on n, so we have to define $S_q(\Delta_q)$, *not* as the free R-module generated by ${\Delta_q}^{\Delta_q}$, but as the corresponding s-free s-R-module, so as to include sums of non-standard length. And similarly for the definition of $S_q(sM)$ and $H_q(sM; R)$ occurring in the theorem.

After having replaced $\mathbb{N}$ by N systematically as just sketched, the remaining details are completely parallel to those in section IV.3. □

The theory of differential forms doesn't depend on integers (whether smooth or not), so exactly as in section IV.3 we can show

4.3 Comparison Theorem for De Rham Cohomology in $\mathcal{Z}$. Let M be a manifold of finite type. Then

$$H^q(sM) \cong \Pi_S R \text{ in } \mathcal{Z} \text{ iff } H^q(M) \cong \Pi_S \mathbb{R} \text{ in Sets},$$

where $S \in$ Sets is a (finite) set.

We have a similar comparison theorem for singular homology with coefficients in R, proved as in section IV.3, but necessarily using smooth integers everywhere. For our present purpose, we need a comparison theorem for singular homology with coefficients in the smooth integers Z:

4.4 Comparison Theorem for Singular Homology. Let M be a manifold of finite type. Then $H_q(sM, Z)$ is of the form $s(G)$, for a finitely generated abelian group G in Sets (regarded as a discrete manifold, cf. 2.11) and we have

$$H_q(sM, Z) \cong s(G) \text{ in } \mathcal{Z} \text{ iff } H_q(M, \mathbb{Z}) \cong G \text{ in Sets.}$$

Proof. Again, we just give a sketch; further details are as in section IV.3.

First of all, we use Mayer-Vietoris induction on a good cover of M, exactly as in §IV.3 for the case of R, to show that $H_q(sM, Z) \cong s(G)$ for a finitely generated abelian group G. (Recall the well-known facts that any such group is of the form $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_k}$, and that if two out of the three groups in a short exact sequence are finitely generated, so is the third.) For the elements of the good cover this is true by the Poincaré lemma, which remains valid in the context of $\mathcal{Z}$ (cf. the synthetic treatment of IV.2.1–3). For the induction step, suppose we have an exact sequence of abelian s-groups in $\mathcal{Z}$

$$s(A) \to s(B) \to G \to s(C) \to s(D),$$

where A, B, C, D are finitely generated abelian groups in *Sets* (regarded as discrete manifolds). Applying the functor Γ to this sequence, we obtain a short exact sequence

$$0 \to B' \to \Gamma(G) \to C' \to 0$$

in Sets, where B', C' are defined by

$$B \twoheadrightarrow B' \rightarrowtail \Gamma(G) \twoheadrightarrow C' \rightarrowtail C.$$

Hence $\Gamma(G)$ is finitely generated. Now, apply the five-lemma inside $\mathcal{Z}$ to the diagram

$$\begin{tikzcd}
s(A) \arrow[r] & s(B) \arrow[r] & G \arrow[r] & s(C) \arrow[r] & s(D) \\
s(A) \arrow[r] \arrow[u, equal] & s(B) \arrow[r] \arrow[u, equal] & s\Gamma(G) \arrow[r] \arrow[u] & s(C) \arrow[r] \arrow[u, equal] & s(D) \arrow[u, equal]
\end{tikzcd}$$

where $s\Gamma(G) \to G$ is the obvious map, and the bottom row is exact since s preserves exact sequences of discrete manifolds. Hence G is of the required form. The comparison result now follows immediately by applying Γ to the long exact Mayer-Vietoris sequence, as in §IV.3.8. □

Proof of Theorem 4.1. Using 4.3 and 4.4, we can now straightforwardly prove that the degree of a map is a smooth integer. Let M be a compact, connected oriented, q-dimensional manifold. From the corresponding clasical results and the comparison theorem, we obtain

(1) $H^q(sM) \cong R$
(2) $H_q(sM; Z) \cong Z$.

Now reason in the topos $\mathcal{Z}$, and let $f \in s(M)^{s(M)}$ (at an arbitrary stage ℓA). Recall the definition of degree: let σ be a generator of $H^q(sM)$ (the area form, say). Then from (1),

(3) $f^*(\sigma) = \lambda \cdot \sigma + d\tau$

for a unique $\lambda \in R$. This λ is by definition the degree of f, $\lambda = \deg(f)$. Now let γ be the generator of $H_q(sM; Z)$. Then from (3) we obtain by Stokes' theorem (IV.1.1)

(4) $\deg(f) \int_\gamma \sigma = \int_{f_*(\gamma)} \sigma$.

But $f_*(\gamma) = n\gamma + \partial\beta$ for a unique $n \in Z$, so $\deg(f)\int_\gamma \sigma = n\int_\gamma \sigma$, hence $\deg(f) = n \in Z$.

This proves theorem 4.1. □

4.5 Remark. We should point out that there is an alternative proof of theorem 4.1, using the results on cohomology in $\mathcal{G}$ from chapter IV, ***without*** proving similar results about $\mathcal{Z}$ first, as we did above, but instead transferring the given map $f \in s(M)^{s(M)}$ to the topos $\mathcal{G}$.

This argument seems less natural than the one given above, but it may be slightly quicker. One argues along the following lines: suppose we are given a map $f \in s(M)^{s(M)}$ in $\mathcal{Z}$, at stage ℓA say, where $A = C^\infty(\mathbb{R}^n)/I$. Thus f is a map $\ell A \times s(M) \to s(M)$ in $\mathbb{L}$, and since $s(M)$ is the dual of a finitely presented C^∞-ring, f can be extended to a map

$$f_0 \colon \ell A_0 \times s(M) \to s(M),$$

where $A_0 = C^\infty(\mathbb{R}^n)/I_0$ for a finitely generated subideal $I_0 \subset I$. ℓA_o is an object of the site $\mathbb{L}$ for $\mathcal{Z}$ as well as of the site $\mathbb{G}$ for $\mathcal{G}$, so f_0 can be regarded as a map $s(M) \to s(M)$ at stage ℓA_0 *in* $\mathcal{G}$. Now compute the degree of f_0 in $\mathcal{G}$. This will be an integer $n \in \mathbb{Z} = \Delta(\mathbb{Z})$ at stage ℓA_0 in $\mathcal{G}$ (by an argument in $\mathcal{G}$ similar to the proof of 4.1), and the assertion that $n = \deg(f_0)$ is just an equation involving $\int f_0$, as in (4) above. So $\ell A_0 \Vdash n = \deg(f_0)$ in $\mathcal{G}$ if and only if $\ell A_0 \Vdash n = \deg(f_0)$ in $\mathcal{Z}$. But $n \in \Delta(\mathbb{Z})(\ell A_0)$, where $\Delta = \Delta_{\mathcal{G}}: Sets \to \mathcal{G}$, hence n corresponds to a map $\ell A_0 \to \Delta_{\mathcal{G}}(\mathbb{Z}) = s(\mathbb{Z})$ in $\mathbb{G}$ (or equivalently, in $\mathbb{L}$), and this is precisely a *smooth* integer in $\mathcal{Z}$ at stage ℓA_0. Thus the degree of f in $\mathcal{Z}$ is a smooth integer.

This is actually an example of a more general result relating validity in $\mathcal{G}$ to validity in $\mathcal{Z}$. We will discuss this in more detail in section VII.4.

4.6 Regular Values. Classically, a way of interpreting the degree of f is by counting the inverse image $f^{-1}(p)$ of a regular value p. Regular values are dense, by Sard's theorem, so such p can always be found (see Guillemin & Pollack (1974), §I.7). A similar explanation can be given in the toposes $\mathcal{G}$ and $\mathcal{Z}$, since the existence of regular values is valid in these toposes. For $\mathcal{G}$, this is really just the stability of transversality: if $f \in s(M)^{s(M)}(\ell A)$, $\ell A \in \mathbb{G}$, f is represented by a smooth map $U \times M \xrightarrow{F} M$, where U is an open neighbourhood of a zeroset $Z(I_0)$ of some finitely generated subideal $I_0 \subset I$. If $p \in M$ is a regular value of $F(x_0, -)$ for some $x_0 \in U$, then p is also a regular value of $F(x, -)$ for all x in a neighbourhood W_{x_0} of x_0 (by stability, and compactness of M). These neighbourhoods W_{x_0}, all $x_0 \in Z(I_0)$, give a cover in $\mathbb{G}$, showing that $\ell A \Vdash \exists p \in sM$ (p is a regular value of f) in $\mathcal{G}$. The W_{x_0}, $x_0 \in Z(I_0)$, don't form a cover of ℓA in $\mathbb{L}$, however, and for the case of $\mathcal{Z}$ one has to do a little more work: Let $\ell A \in \mathbb{L}$, $f \in s(M)^{s(M)}(\ell A)$ represented by $U \times s(M) \xrightarrow{F} s(M)$ as above, with $A = C^\infty(\mathbb{R}^n)/I$ and again $U \supset Z(I_0)$ for a finitely generated $I_0 \subset I$. Arguing as we just did for $\mathcal{G}$, we can find a neighbourhood finite cover $\{W_n\}_n$ of U and $p_n \in M$ such that for all n, p_n is a regular value of each $F(x, -): M \to M$, $x \in W_n$. Let $\mathcal{V}$ be a refinement of the form $\mathcal{V} = \mathcal{V}^1 \cup \ldots \cup \mathcal{V}^d$ (Ostrand's theorem 2.13), where $\mathcal{V}^i = \{V_n^i\}_n$ is a family of disjoint open sets, with $V_n^i \subset W_n$. Let $V^i = \bigcup \mathcal{V}^i$. The $V^i (i = 1, \ldots, d)$ give a cover of ℓA in $\mathbb{L}$, by $\ell A^i = s(V^i) \cap \ell A$. At each ℓA^i we have a point p^i of $s(M)$, i.e. a map $p^i: \ell A^i \to s(M)$, induced by the function $p^i(x): V^i \to M$, $p^i(x) = p_n$ iff $x \in V_n^i$, and $\ell A^i \Vdash (p^i$ is a regular value of $f)$ in $\mathcal{Z}$. Leaving further details to the reader, we conclude that $\mathcal{Z} \models \forall f \in s(M)^{s(M)} \exists p \in s(M)$ (p is a regular value of f).

5 Forcing the Existence of Invertible Infinitesimals

Despite the fact that we have plenty of invertible infinitesimals in $\mathcal{Z}$, this topos is not really adequate to perform "internal" arguments using invertible infinitesimals, since in the logic of $\mathcal{Z}$ the existence is only expressed in a rather weak form,

$$\mathcal{Z} \models \neg\neg\exists x\; x \in \mathbb{I}.$$

The aim of this section will be to modify the topos $\mathcal{Z}$ slightly, so as to obtain a topos in which invertible infinitesimals do exist, in the sense that $\exists x\; x \in \mathbb{I}$ is valid. In all other respects, this topos will be very similar to $\mathcal{Z}$, and we will only briefly indicate to which extent the proofs of the preceding sections have to be modified for this new topos. In the next chapter, we will illustrate how the existence of invertible infinitesimals can be exploited, for example in the theory of distributions and the Dirac function.

Let $\ell A \in \mathbb{L}$ be a non-trivial locus, i.e. $1 \notin I$ where $A = C^\infty(\mathbb{R}^n)/I$. In the topos $\mathcal{Z}$, ℓA (or more precisely, its image $Y(\ell A)$ under the Yoneda embedding) can be described as a zeroset

$$\ell A = \{x \in R^n | f(x) = 0, \text{ for all } f \in I\}.$$

To say that $\mathcal{Z} \models \exists x \in R^n(x \in \ell A)$ is equivalent to saying that there is a cover $\{\ell B_i \to 1\}_{i=1}^n$ in $\mathbb{L}$ on which ℓA has a point, i.e. there are maps $\ell B_i \to \ell A$. Since 1 has only trivial covers in $\mathbb{L}$, this comes down to saying that there is a map $1 \to \ell A$, i.e. $Z(I) \subset \mathbb{R}^n$ is non-empty.

If we want $\exists x \in R^n(x \in \ell A)$ to be valid, the most economical way (in a precise sense, see Appendix 2) is to add $\{\ell A \to 1\}$ as a singleton covering family, for each non-trivial ℓA. Of course, this can only be done if the generated Grothendieck topology is still *consistent*. And in fact we want more: for example, the category of manifolds, as well as some extensions of it containing the important infinitesimal spaces, should still be fully and faithfully embedded in the resulting topos.

5.1 Definition. Let $\mathbb{B}$ be the site with underlying category $\mathbb{L}$ (now also referred to as $\mathbb{B}$), but with a finer Grothendieck topology, namely the Grothendieck topology generated by the covers of $\mathbb{L}$ as in 1.1, 1.2, together with all projections along non-trivial loci, i.e. all singleton families of the form

$$\ell A \times \ell B \to \ell A$$

where $\ell B \neq 0$ (if $\ell A \neq 0$). Let $\mathcal{B}$ be the topos of sheaves on $\mathbb{B}$. $\mathcal{B}$ is called *the Basel topos.*

Note that the generating covering families as specified for $\mathbb{B}$ do not yet form a Grothendieck topology. They are stable under pullback (for the finite open covers of $\mathbb{L}$ this was proved in lemma 1.1; for the projections this is obvious), but they are not stable under composition. For the definition of *sheaf* on $\mathbb{B}$, however, we only need a pullback-stable generating system of covers (Appendix 1), so 5.1 is sufficiently explicit to prove the following lemma.

5.2 Lemma. The Grothendieck topology on $\mathbb{B}$ is subcanonical. So the Yoneda embedding $\mathbb{L} \hookrightarrow \mathit{Sets}^{\mathbb{L}^{\mathrm{op}}} (= \mathbb{B} \hookrightarrow \mathit{Sets}^{\mathbb{B}^{\mathrm{op}}})$ factors through $\mathcal{B} \subset \mathit{Sets}^{\mathbb{B}^{\mathrm{op}}}$. In particular, there is a full and faithful embedding of manifolds $\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{B}$ which preserves transversal pullbacks.

Proof. We have to show that for any $\ell C \in \mathbb{L}$, the functor $\mathbb{L}(-, \ell C) = \mathbb{B}(-, \ell C)$ is a sheaf on $\mathbb{B}$. By Appendix 1, it suffices to consider the covers in the pullback stable system consisting of the open covers as for $\mathcal{Z}$, and the projections, as in 5.1. For open covers, this is lemma 1.3. So we only need to consider the case of a projection, say $\ell A \times \ell B \xrightarrow{\pi_1} \ell A$. So suppose $f: \ell A \times \ell B \to \ell C$ gives a compatible family for this singleton cover, i.e. $f \circ \pi_2 = f \circ \pi_3$ in the diagram

$$\begin{tikzcd}
\ell A \times \ell B \times \ell B \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\pi_3"'] & \ell A \times \ell B \arrow[r, "\pi_1"] \arrow[d, "f"] & \ell A \arrow[dl, "g"] \\
& \ell C &
\end{tikzcd}$$

we have to show that there is a unique $g: \ell A \to \ell C$ with $g\pi_1 = f$. Write $A = C^\infty(\mathbb{R}^n)/I, B = C^\infty(\mathbb{R}^m)/J$, and denote elements of $\mathbb{R}^n$ by x, elements of $\mathbb{R}^m$ by y.

Now note that it suffices to show the case $\ell C = R$. The general case $\ell C = \ell C^\infty(\mathbb{R}^k)/L$ follows, since $\ell C \subset R^k$, and a k-tuple $(f_1, \ldots, f_k): \ell A \times \ell B \to R^k$ factors through ℓC iff $h \circ (f_1, \ldots, f_k) = 0$ as a map $\ell A \times \ell B \to R$, for each $h \in L$. So suppose $\ell C = R$, and let f be represented by a smooth function $\mathbb{R}^n \times \mathbb{R}^m \xrightarrow{f} \mathbb{R}$. Since

$f \circ \pi_2 = f \circ \pi_3$, we can write

$$(1)\ f(x,y_1) - f(x,y_2) = \textstyle\sum_{i=1}^{p} \varphi_i(x,y_1,y_2)\alpha_i(x) +$$
$$+ \sum_{j=1}^{q} \psi_j(x,y_1,y_2)\beta_j(y_1) + \sum_{k=1}^{r} \chi_k(x,y_1,y_2)\gamma_k(y_2),$$

where $\alpha_i(x) \in I$ and $\beta_j(y), \gamma_k(y) \in J$. Let $p \in \mathbb{R}^m$ be a common zero of $\beta_j(y)(j = 1,\ldots,q)$ and $\gamma_k(y)(k = 1,\ldots,r)$, and let $g\colon \mathbb{R}^n \to \mathbb{R}$ be the function $g(x) = f(x,p)$. Then g induces a map $\ell A \xrightarrow{g} R$, and this is the unique one with $g \circ \pi_1 = f\colon \ell A \times \ell B \to R$. Indeed, by (1) we have

$$f(x,y) - g(x) = \textstyle\sum \varphi_i(x,y,p)\alpha_i(x) + \sum \psi_j(x,y,p)\beta_j(y) \in (I(x), J(y))$$

so $f = g \circ \pi_1$ on $\ell A \times \ell B$, the dual of $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/(I(x), J(y))$.

For uniqueness, we have to show that if $g(x)$, $h(x)\colon \mathbb{R}^n \to \mathbb{R}$ are two functions such that $g(x) - h(x) \in (I(x), J(y)) \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, then $g(x) - h(x) \in I(x)$. This is proved in the same way: write $g(x) - h(x) = \sum \varphi_i(x,y)\alpha_i(x) + \sum \psi_j(x,y)\beta_j(y)$ where $\alpha_i \in I, \beta_j \in J$, and take a common zero of the β_j's, etc. □

5.3 The Grothendieck Topology on $\mathbb{B}$. Although a pullback stable system of covering families as in 5.1 is sufficient to decide whether or not a given presheaf is a sheaf, it is not quite adequate when one considers *forcing*, since

$$\ell A \Vdash \exists x \varphi(x)$$

means that there is a cover $\{\ell B_i \to \ell A\}_i$ such that $\ell B_i \Vdash \varphi(a_i)$ for $a_i \in F(\ell B_i)$ for a given sheaf F, and there is no reason why this family $\{\ell B_i \to \ell A\}_i$ can be chosen to belong to a given pullback stable system of covering families which generates the Grothendieck topology. For the purpose of forcing, one needs stability under composition, at least in the weaker form that states that the composition of two covers in the given generating system has a *refinement* in the system (Appendix 1). Such a system is easily described for the case of $\mathbb{B}$. For a given locus ℓA, we define the *covering families* of ℓA to be the finite families

$$\{\ell B_i \xrightarrow{f_i} \ell A\}_{i=1}^{n}$$

of the following form: there is a non-trivial locus ℓB such that each f_i can be written as a composite $f_i = \pi_1 \circ g_i$

$$\begin{tikzcd} \ell B_i \arrow[r] \arrow[dr, "f_i"'] & \ell A \times \ell B \arrow[d] \\ & \ell A \end{tikzcd}$$

and $\{\ell B_i \xrightarrow{g_i} \ell A \times \ell B\}_{i=1}^n$ is an open cover of $\ell A \times \ell B$ in the sense of $\mathbb{L}$ (as defined above lemma 1.1, or as described in lemma 1.2).

Let us fix the terminology a little more explicitly: by a *cover*, or a *covering family* (in $\mathbb{B}$) we will always mean a family of this form. By an *open cover*, or $\mathbb{L}$-*cover*, we will mean a cover as defined for the site $\mathbb{L}$ in section 1. So the covers of ℓA in $\mathbb{B}$ are the open covers of $\ell A \times \ell B$, where ℓB varies over all nontrivial loci.

Analogous to lemma 1.2, we have that for $A = C^\infty(U)/I$, with U an open subset of $\mathbb{R}^n$, the cover $\{\ell B_i \to \ell A\}_{i=1}^n$ are (isomorphic to ones) of the form $\{\ell(C^\infty(W_i)/(I|W_i, J|W_i)) \xrightarrow{f_i} \ell(C^\infty(U)/I)\}_{i=1}^n$, where $J \subset C^\infty(V)$ is any proper ideal, V is an open subset of $\mathbb{R}^m$, and $\{W_1, \ldots, W_n\}$ is a cover of $Z(I_0) \times Z(J_0)$ by open subsets $W_i \subset U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ for finitely generated subideals $I_0 \subset I, J_0 \subset J$, while the f_i are just induced by the projections $W_i \xrightarrow{\pi_1} U$.

The system of covers $\{\ell B_i \xrightarrow{f_i} \ell A\}$ of the form described above is clearly stable under pullback (using the corresponding fact for $\mathbb{L}$, see lemma 1.1). It is also stable under composition in the weaker sense of requiring the existence of a refinement. To see this, suppose we are given a cover $\{\ell B_i \to \ell A\}_{i=1}^n$ in $\mathbb{B}$, and for each i a cover $\{\ell C_{ij} \xrightarrow{g_{ij}} \ell B_i\}_{j=1}^{n_i}$ in $\mathbb{B}$. So there are open covers ($\mathbb{L}$-covers) $\{\ell B_i \hookrightarrow \ell A \times \ell B\}_{i=1}^n$ and $\ell C_{ij} \hookrightarrow \ell B_i \times \ell C_i\}_{j=1}^{n_j}$ such that f_i is the composite $\ell B_i \hookrightarrow \ell A \times \ell B \xrightarrow{\pi_1} \ell A$, and g_{ij} is the composite $\ell C_{ij} \hookrightarrow \ell B_i \times \ell C_i \xrightarrow{\pi_1} \ell B_i$. Since the $\mathbb{L}$-covers are stable under pullback and composition, we have an open cover

$$\{\ell C_{ij} \hookrightarrow \ell A \times \ell B \times \ell C_i\}_{j=1}^{n_j}$$

for each $i, i = 1, \ldots, n$. Let $\ell C'_{ij} = \ell C_1 \times \ldots \times \ell C_{i-1} \times \ell C_{ij} \times \ell C_{i+1} \times \ldots \times \ell C_n$; then $\{\ell C'_{ij} \hookrightarrow \ell A \times \ell B \times \ell C_1 \times \ldots \times \ell C_n\}_{i,j}$ is an open cover, so composing with the projection on ℓA, we obtain a cover in $\mathbb{B}$,

$$\{\ell C'_{ij} \xrightarrow{h_{ij}} \ell A\}_{i,j},$$

which refines the family of composites $\{\ell C_{ij} \to \ell B_i \to \ell A\}_{i,j}$ of the two given covers, since for each $i = 1, \dots, n, j = 1, \dots, n_i$ we have a commutative diagram

$$\begin{tikzcd} \ell C_{ij} \arrow[r] & \ell B_i \arrow[r, "f_i"] & \ell A \\ \ell C'_{ij} \arrow[u, "\pi_j"] \arrow[urr, "h_{ij}"'] & & \end{tikzcd}$$

Let us quickly leaf through the description of the basic properties of $\mathcal{Z}$ given in sections 1 and 2, and see what changes have to be made for the case of $\mathcal{B}$. We have the usual adjoints

$$\mathcal{B} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} Sets, \quad \Delta \dashv \Gamma,$$

but Γ does *not* have a right adjoint for $\mathcal{B}$! (Γ does not preserve epimorphisms: for example, if ℓA is any non-trivial locus without points, $\ell A \to 1$ is an epimorphism in $\mathcal{B}$ since it is a cover in $\mathbb{B}$, but $\Gamma \ell A \to \Gamma 1$ is not, since $\Gamma \ell A = \emptyset$.)

The ***natural numbers object*** $\mathbb{N} = \Delta_{\mathcal{B}}(\mathbb{N})$ can be described exactly as for $\mathcal{Z}$, since the sheaf $\Delta_{\mathcal{Z}}(\mathbb{R})$ on $\mathbb{L}$ described in 1.6 is actually a sheaf on $\mathbb{B}$, as is easy to see.

$R = \mathbb{B}(-, R)$, which is a sheaf by lemma 5.2, is an ordered local ring in $\mathcal{B}$, with properties just as in $\mathcal{Z}$ (see 1.7). The same holds for the infinitesimal spaces as discussed for $\mathcal{Z}$ in 1.8, except that invertible infinitesimals exist in $\mathcal{B}$. Explicitly,

5.4 Proposition. Let ℓA be a non-trivial locus, $\ell A \subset R^n$. Then ℓA is inhabited in $\mathcal{B}$, i.e. $\mathcal{B} \models \exists x \in R^n x \in \ell A$. In particular, $\mathcal{B} \models \exists x \in R\, x \in \mathbb{I}$. □

Note that 1.7(3) is now trivial.

The (generalized) Kock-Lawvere axiom and the integration axiom hold in $\mathcal{B}$. The proofs are just as for $\mathcal{Z}$ (see 1.9, 1.10).

The representable object $\mathbb{B}(-, s\mathbb{N}) = s(\mathbb{N}) = N$ is a sheaf on $\mathbb{B}$, and gives the object of ***smooth natural numbers*** in $\mathcal{B}$. As for $\mathcal{Z}$, we can replace $\mathbb{N}$ by N consistently, to get the notions of s-finite, s-compact space, s-group, etc. in $\mathbb{B}$. The arithmetic of N is the same for $\mathcal{B}$ and $\mathcal{Z}$; for example, we have recursion in $\mathcal{B}$ (theorem 2.8), by which

R becomes an s-Archimedean s-local (see 1.12) s-ring in $\mathcal{B}$. (And similarly $s\colon \mathbb{M} \hookrightarrow \mathcal{B}$ maps Lie groups to s-groups in $\mathcal{B}$.) Note, by the way, that "infinitely big", or "non-standard" numbers exist in $\mathcal{B}$, similar to the existence of invertible infinitesimals: proposition 5.4 immediately gives

$$\mathcal{B} \models \exists n \in N \ \forall m \in \mathbb{N} \ m < n.$$

In section 3, we considered topological properties of objects $s(M) \in \mathcal{Z}$, while some special cases (for R, for $[0,1]$) had already been discussed in section 2 (see 2.3–7). All these results can be proved for $\mathcal{B}$, with practically identical proofs. To start with, the internal topology is described for $\mathcal{B}$ just as for the case of $\mathcal{Z}$; see 3.1. The proof of 3.2 is also valid for $\mathcal{B}$, so $s\colon \mathbb{M} \hookrightarrow \mathcal{B}$ "preserves" (countable) open covers in the sense of 3.2. As in 3.4, $s\colon \mathbb{M} \hookrightarrow \mathcal{B}$ maps manifolds to s-topological spaces. This functor preserves compactness, as in 3.5 (cf. also 2.3, 2.5 (ii)). Let us state this explicitly.

5.5 Theorem. Let $M \in \mathbb{M}$ be a compact manifold. Then in $\mathcal{B}$ it is valid that every open cover has an s-finite refinement, and (hence) a Lebesgue number.

Proof. (sketch) The proof is almost literally the same as that of 3.5. Suppose $\mathcal{U}$ is a cover of $s(M)$ at stage $\ell A \in \mathbb{B}$, and assume $\mathcal{U}$ consists of basic opens. Then $\ell A \times s(M) \Vdash \exists U \in \mathcal{U}\ \pi_2 \in \mathcal{U}$, so we find a non-trivial $\ell B \in \mathbb{L}$ and an *open* cover of $\ell B \times \ell A \times s(M)$, say $\{\ell B_i \hookrightarrow \ell B \times \ell A \times s(M)\}_{i=1}^{n}$, such that at ℓB_i there is a U_i with

$$\ell B_i \Vdash \pi_2 \in U_i \in \mathcal{U}.$$

We then proceed *exactly* as in the proof of 3.5, but with ℓA replaced by $\ell A \times \ell B$. So at the end, we find an open cover $\{\ell A^r\}$, not of ℓA but of $\ell A \times \ell B$, satisfying (1) and (2) of the proof of 3.5. But since the maps $\ell A^r \hookrightarrow \ell A \times \ell B \to \ell A$ form a cover in $\mathbb{B}$, we conclude that $\ell \Vdash$ "$\mathcal{U}$ has an s-finite refinement". □

The proofs of 3.6 and 3.7 are already valid for $\mathcal{B}$, and no modification is necessary. The proofs of 3.8–3.12 need to be modified slightly, just as for 3.5 as indicated above, in the proof of 5.5. For easy reference, we state some results explicitly.

5.6 Theorem. The embedding $s\colon \mathbb{M} \hookrightarrow \mathcal{B}$ has the following properties:

(1) In $\mathcal{B}$, all functions $s(M) \to s(N)$ are continuous (as in 3.6).
(2) s preserves partitions of unity (as in 3.7), and in $\mathcal{B}$ s-partitions of unity exist subordinate to any given open cover (as in 3.8).
(3) s maps connected manifolds to s-chain-connected (as in 3.10) and connected (as in 3.11) manifolds in $\mathcal{B}$.
(4) In $\mathcal{B}$ it holds that every s-countable cover of $s(M)$ has an open refinement (as in 3.12).

Finally, let us take a look at cohomology and degrees. Again, we will only consider manifolds of finite type. Recall that to prove the internal De Rham theorem and the comparison theorems for $\mathcal{Z}$, one proceeds as follows:

(i) $H_q(sM, R)$ is a free R-module on a constant finite basis, i.e. is of the form R^n for some $n \in \mathbb{N}$ (cf. IV.3.4).

(ii) The map $\int: H^q(sM) \to H_q(sM, R)^*$ is an isomorphism, i.e. the De Rham theorem holds in $\mathcal{Z}$ (as in IV.3.6).

(iii) Γ preserves De Rham cohomology, i.e. $\Gamma H^q(sM) \cong H^q(M)$, since Γ preserves the De Rham complex

$$\ldots \to \Lambda^q(sM) \xrightarrow{d} \Lambda^{q+1}(sM) \to \ldots,$$

and Γ is exact.

(iv) The comparison theorem for De Rham cohomology: $H^q(sM) \cong R^n$ in $\mathcal{Z}$ iff $H^q(M) \cong \mathbb{R}^n$ in *Sets*; this follows immediately from (i) and (iii), as in the proof of IV.3.9.

(v) The comparison theorem for singular homology $H_q(sM; R)$ follows from (ii), (iv), and the classical De Rham theorem in *Sets*.

The proofs of (i) and (ii) also work for $\mathcal{B}$, but (iii) is not immediate, since Γ is no longer exact (it doesn't have a right adjoint B anymore). That Γ preserves $F^q(sM)$ is proved just as in chapter IV. To conclude that $\Gamma H^q(sM) = H^q(M)$ for the case of $\mathcal{B}$, i.e. that

(1) $\Gamma(F^q(sM)/E^q(sM)) \cong \Gamma F^q(sM)/\Gamma E^q(sM)$.
(2) $\Gamma E^q(sM) \cong E^q(M)$

it suffices to show that if $\lambda \in \Gamma F^q(sM) \cong \Gamma^q(M)$ is a global $n+1$-form in $\mathcal{B}$,

$$\lambda: M^{D^{n+1}} \to R$$

and $1 \Vdash \exists \omega \in \Lambda^n(sM)\ d\omega = \lambda$, then there is a global n-form $\omega \in \Gamma\Lambda^n(sM) \cong \Lambda^n(M)$ such that $d\omega = \lambda$. So suppose $1 \Vdash \exists \omega \in \Lambda^n(sM) d\omega = \lambda$. Then there is a non-trivial locus ℓA, $A = C^\infty(\mathbb{R}^n)/I$ say, and an $\omega \in \Lambda^n(sM)(\ell A)$ with $\ell A \Vdash d\omega = \lambda: M^{D^{n+1}} \to R$. Since $M^{D^{n+1}}$ is representable, it follows that there is a finitely generated subideal $I_0 \subset I$ such that $\ell A_0 \Vdash d\omega_0 = \lambda$, where $A_0 = C^\infty(\mathbb{R}^n)/I_0$ and ω_0 is an extension of ω to ℓA_0. If p is any point of $Z(I_0)$, $1 \xrightarrow{p} A_0$, then by taking restrictions along p we get $1 \Vdash d(\omega_0|p) = \lambda$, i.e. $\omega_0|p$ is a global n-form with $d(\omega_0|p) = \lambda \in \Gamma\Lambda^{n+1}(sM)$. This shows (1) and (2). The proofs of (iv) and (v) present no extra problems for $\mathcal{B}$.

Exactness of Γ was also used in the proof for the topos $\mathcal{Z}$ of the comparison theorem for singular homology with coefficients in the smooth integers Z (cf. theorem 4.4). Although for the case of $\mathcal{B}$, Γ is not exact, it still preserves exact sequences of s-groups of the form $s(G)$, G a finitely generated abelian group in *Sets*: i.e. groups in $\mathcal{B}$ of the form $Z \oplus \ldots \oplus Z \oplus Z_{m_1} \oplus \ldots \oplus Z_{m_k}$. This follows from the following lemma, which generalizes the argument we just gave to show that Γ preserves De Rham cohomology.

5.7 Lemma. Let $\ell A, \ell B, \ell C, \ell D$ be objects of $\mathbb{B}$, with ℓA finitely presented, and let $F: \ell A^{\ell B} \to \ell C^{\ell D}$ be a map in $\mathcal{B}$. If $\mathcal{B} \models F$ is a surjection, then ΓF is a surjection.

Proof. Take $g \in \Gamma(\ell C^{\ell D})$, i.e. a map $g: \ell D \to \ell C$ in $\mathbb{B}$. Since F is a surjection, there exists a non-trivial C^∞-ring $E = C^\infty(\mathbb{R}^n)/I$ and an $f \in \ell A^{\ell B}(\ell E)$ such that $\ell E \Vdash F(f) = g$. So $f: \ell E \times \ell B \to \ell A$, and since ℓA is finitely presented, we may assume f is the restriction of a map $f_0: \ell E_0 \times \ell B \to \ell A$, where $E_0 = C^\infty(\mathbb{R}^n)/I_0$ for a finitely generated subideal $I_0 \subset I$. Taking transposed maps, $\ell E \Vdash F(f) = g \in \ell C^{\ell D}$ can be written as $\ell E \times \ell D \Vdash F(f) = g \in \ell C$. So by choosing I_0 big enough, we m assume that $\ell E_0 \times \ell D \Vdash F(f_0) = g \in \ell C$. Let $1 \xrightarrow{p} \ell E_0$ be a point of ℓE_0, and let $f_p = f_0|p$. Then $1 \times \ell D \Vdash F(f_p) = g \in \ell C$, so $1 \Vdash F(f_p) = g \in \ell C^{\ell D}$, i.e. $\Gamma F(f_p) = g$. This shows ΓF is a surjection. $\square$

As a consequence of all this, the degree of a map is a smooth integer in $\mathcal{B}$, i.e. theorem 4.1 is valid for $\mathcal{B}$.

Chapter VII

Smooth Infinitesimal Analysis

In previous chapters, notably in Chapters IV and V, we have used "synthetic arguments" at an informal level, trusting that the context would provide a meaning for the term "synthetic", and would make the arguments themselves plausible. The aim of this final chapter is to make the notion of "synthetic reasoning" explicit, by setting up an axiomatic system which is adequate to formalize the arguments from earlier chapters, as well as some others which we will present below.

There are several reasons to set up formal systems for synthetic analysis and differential geometry. Of course, the most important one is to build a rigorous framework to develop the synthetic theory; not only because of the interest this has in itself, but also because it is usually much easier to work synthetically than to work directly with the models and the corresponding notions of forcing. Moreover, having a formal system enables one to compare the synthetic theory with related approaches. These include versions of non-standard analysis and intuitionistic analysis, but also weakenings or strengthenings of the system itself. (The transfer principle that we will prove in Section 4 is related to this.) Finally, we mention the possibility of studying independence and consistency results.

The system that will be described allows the existence of invertible infinitesimals and infinitely large ("smooth") integers, as present in the toposes of Chapter VI, but absent in the Archimedean models of Chapter III. As was clear from the discussion of the models in the preceding chapter, the possibility of having invertible infinitesimals besides the usual nilpotent ones forces one to weaken not only the logic (intuitionistic logic has to be used), but also the arithmetic. This seems to be a new feature of our system, and it should be kept in mind to understand its motivation.

Accordingly, in the first section we will present a system based on the "real line" R and the "natural numbers" N. As suggested by the work done in Chapter VI, we take N (there called the set of *smooth* natural numbers) as primitive, and the basic mathematical notions refer to N, rather than to the set of "standard natural numbers" $\mathbb{N}$. Our presentation will be rather informal, and to some extent ad hoc. We have tried to state the axioms in a way that seems most comprehensible in relation to what we have done before, and that suffices for our specific purposes. In particular, we have not aimed at a "minimal" list of axioms.

In the second section, we will develop some elementary mathematics on the basis of this system. Here is may seem to the reader that we have gone into too much detail at some places. Nonetheless, this seems to be justified, since the lack of a general induction principle, and (related to this) the absence of an axiom of finite choice, make our system in some respects a rather unusual one to work in.

As an application of the presence of invertible infinitesimals, we will show in Section 3 how "improper functions", such as the δ function of Dirac, can be dealt with in our context. Although several mathematical alternatives which avoid improper functions are available (the theory of distributions, operator calculus, etc.), it seems fair to say that physicists have kept on thinking of the δ function of Dirac as a "real" function, with infinitesimal support and an infinitely large value at 0, and with the property that the total area under this "curve" is 1. Our approach captures this intuition as being literally true, and from this point of view it can thus be compared to the approach of non-standard analysis. However, it will become apparent that there are some obvious differences.

The last section is devoted to a useful transfer principle between the toposes $\mathcal{B}$ and $\mathcal{G}$. Although $\mathcal{B}$ is much harder to work with than $\mathcal{G}$, this principle tells us that for a rather large class of statements, validity in $\mathcal{B}$ is equivalent to validity in $\mathcal{G}$.

1 An Axiomatic System for Smooth Infinitesimal Analysis

In this section we will set up an axiomatic system which allows the existence of invertible infinitesimals and infinitely large natural numbers. The reader should think of the Basel topos $\mathcal{B}$ as the natural model of this system. Our presentation will be rather informal. It will be convenient to work in a "theory of variable types" as described below. The non-logically minded reader is advised to pass to 1.2 immediately, and refer to paragraph 1.1 only if necessary.

1.1 Theories of Variables Types. (after Feferman (1985)). The main features of such a theory, as opposed to the usual theories of types, are the following: (i) the types are variable, so that statements of a generality of the form "for every microlinear space ..." can be expressed directly; (ii) equations between terms of arbitrary types are admitted; and (iii) besides the usual rules for product and exponentiation, separation can be applied to form sub-types from given types—for example, $D = \{x \in R | x^2 = 0\}$ is a subtype of R. Still, every individual term is syntactically of a unique type.

Such a theory allows us to express directly statements of the form $\forall n \in N \, \exists f \in R^{\{m|m<n\}}$ (...), for example. This cannot be done in usual type theories, since $\{m|m < n\}$ is not a type. Furthermore, by (ii) we can write statements like $\forall h \in D \, \exists x \in R(x = h \wedge ...)$. Again, this is not possible if we consider D as a type in a usual type theory.

The language is specified by generating simultaneously the following three syntactic classes, together with the relation "t is of type T":

(1) individual terms $s, t \ldots$
(2) type terms $S, T, \ldots$
(3) formulas $\varphi, \psi, \ldots$

in the expected way. For example,

(1) (a) with each type term T we have an infinite list of variables of type T; (b) if s is of type S and t is of type T, then (s, t) is of type $S \times T$; (c) if s is of type S and t is of type $(S \to T)$ then $t(s)$, or ts, is of type T; etc.
(2) (a) variables X, Y are type terms; (b) if S and T are type terms, then so are $S \times T$ and $S \to T$. Furthermore, if φ is a formula, $\{x \in S | \varphi\}$ is a type term.
(3) formulas are built up from equations $t_1 = t_2$ in the usual way.

Moreover, one may specify some *individual constants* (of given types) and *type constants* in advance. We take $1, N, R$ and Ω as type constants, and $0, 1, +, \cdot, <, \leq$ as individual constants, of types $R, R, R \times R \to R$, $R \times R \to R, R \times R \to \Omega, R \times R \to \Omega$, respectively. Furthermore, $\top$ and $\perp$ are individual constants of type Ω. The intended interpretations of the type constants are "the singleton", "the natural numbers" (possibly with infinitely large elements), "the real line", and "the truth values".

As usual, we will often write T^S for $(S \to T)$, and $P(S)$ for Ω^S.

With these preliminaries out of the way, we continue to write our statements in the usual, informal way, as we did in previous chapters, just trusting that everything can be formulated in variable type theory.

We now list the axioms that will be used in the sequel. As emphasized in the introduction to this chapter, we aim at a workable list, rather than a minimal one.

1.2 Algebraic Properties of R. We postulate the algebraic structure of R as expressed by axioms (A1)–(A5).

(A1) $(R, +, \cdot, 0, 1)$ is a commutative ring with unit.

(A2) R is local, i.e.

$$\begin{cases} 0 = 1 \to \bot \\ \exists y\, x \cdot y = 1 \vee \exists y (1 - x) \cdot y = 1. \end{cases}$$

(A3) $(R, <)$ is a real Euclidean ordered local ring; i.e. $<$ is a transitive relation which is compatible with the ring structure, in the sense that

(a) $0 < 1, (0 < x \wedge 0 < y \to 0 < x + y \wedge 0 < x \cdot y)$,
(b) $\exists y (x \cdot y = 1) \leftrightarrow (0 < x \vee x < 0)$,
(c) $0 < x \to \exists y (x = y^2)$ (Euclidean property).

(Classically, the axiom (A3) expresses that the residue field of the local ring is a real Euclidean totally ordered field.)

(A4) $\leq$ is a preorder relation, compatible with the ring structure of R; i.e. $\leq$ is transitive and reflexive, and

(a) $0 \leq 1, (0 \leq x \wedge 0 \leq y \to 0 \leq x + y \wedge 0 \leq x \cdot y), 0 \leq x^2$,
(b) x nilpotent $\to 0 \leq x$.

(A5) $<$ and $\leq$ are compatible in the following sense:

(a) $x < y \to x \leq y$
(b) $x < y \wedge y \leq x \to \bot$.

1.3 Maps on Infinitesimal Spaces. Here we have just one axiom:

(A6) (Generalized Kock-Lawvere Axiom) Let $k, n, \ell \in \mathbb{N}$. If $S = \{x \in D_k(n) | p_i(x) = 0, i = 1, \ldots, \ell\}$ is the zero-set of finitely many polynomials $p_1, \ldots, p_\ell \in R[x_1, \ldots, x_n]$, each of total degree $\leq k$ and with zero constant term, then every function $S \to R$ is the restriction of a polynomial in $R[x_1, \ldots, x_n]$ of total degree $\leq k$, and this polynomial is unique modulo the ideal $(p_1, \ldots, p_\ell) \subset R[x_1, \ldots, x_n]$. (Recall that $D_k(n) = \{x \in R^n | x^\alpha = 0$, all multi-indices α with $|\alpha| = k+1\}$.)

A particular case of (A6) is the simple form of the Kock-Lawvere axiom, which is used constantly.

(A6′) $\forall f \in R^D \exists!(a,b) \in R \times R\, \forall h \in D(f(h) = a + bh)$, where D is the type term defined by $D = \{x \in R | x^2 = 0\} = D_1(1)$.

1.4 Integration Axioms. These are the axioms allowing a passage "from infinitesimal to local":

(A7) $\forall f \in R^{[0,1]} \exists! g \in R^{[0,1]}(g(0) = 0 \wedge \forall x \in [0,1] g'(x) = f(x))$, where $[0,1]$ is the type defined by $[0,1] = \{x \in R | 0 \leq x \wedge x \leq 1\}$, and $g'(x)$ is the unique b with $g(x+h) = g(x) + hg'(b)$ for all $h \in D$, cf. (A6′). The g in (A7) is of course written as $g(x) = \int_0^x f(t)dt$.

(A8) (Compatibility of $\int$ and $<$)

$$\forall x \in [0,1] f(x) > 0 \to \int_0^1 f(x)dx > 0$$

(A8′) (Compatibility of $\int$ and $\leq$)

$$\forall x \in [0,1] f(x) \geq 0 \to \int_0^1 f(x)dx \geq 0.$$

(A9) (Inverse Function Theorem) $\forall f \in R^R \forall x \in R(f'(x)$ invertible $\to \exists$ open $U \ni x, V \ni f(x)$ (f restricts to a bijection $U \xrightarrow{\sim} V$)).

1.5 Arithmetical Axioms. Before describing this group of axioms, we introduce some definitions. A *finitely presented type term* is a type term of the form

$$\{x \in R^n | f_i(x) = 0, i = 1, \ldots, k\} (k, n \in \mathbb{N}),$$

where the f_i are terms of type $R^n \to R$, without free variables (but see the remark in 1.7). A *coherent formula* is a formula whose free variables are all of a type of the form $(T \to S)$, with S and T finitely presented type terms, and which is obtained from atomic formulas $t = t'$ (where t, t' have finitely presented types) by using $\wedge, \vee, \top, \bot$, and $\exists$ over variables of finitely presented type.

We now list the axioms:

(A10) N is a subtype of R, i.e. $\forall x \in N\, \exists y \in R(x = y)$.
(A11) R is Archimedean for N, *i.e.*$\forall x \in R\, \exists n \in N\; x < n$.
(A12) (Peano axioms)

$$\begin{aligned}&0 \in N\\ &\forall x \in R(x \in N \to x + 1 \in N)\\ &\forall x \in R(x \in N \wedge x + 1 = 0 \to \bot)\end{aligned}$$

(A13) Coherent induction scheme: for every coherent formula φ,

$$\varphi(0) \wedge \forall x \in N(\varphi(x) \to \varphi(x+1)) \to \forall x \in N\varphi(x).$$

(A14) Finitely presented type recursion: for all type-terms S and T, with S finitely presented,

$$\forall f \in S^{S \times T} \forall a \in S^T \exists! g \in S^{N \times T} \forall x \in T(g(0,x) = a(x) \wedge \forall n \in N\; g(n+1, x) = f(g(n,x))).$$

(A15) Axiom of bounded search:

$$\forall P \in \mathcal{P}(N \times N)(\forall n \in N\, \exists m \in N\; P(n,m) \to \forall n_0 \in N\, \exists m_0 \in N\; \forall n \leq n_0\, \exists m \leq m_0\; P(n,m)).$$

(This axiom provides a substitute for the axiom of finite choice (cf. remark VI.2.6, which is not valid in $\mathcal{B}$ and $\mathcal{Z}$, although it is valid in $\mathcal{F}$ and $\mathcal{G}$. Some examples of the use of the axiom of bounded search will be given below; see 2.7, 2.10.)

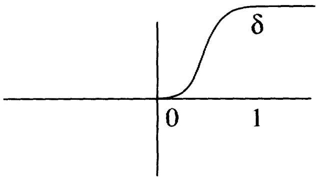
1.6 Topological Axioms. The topology $\mathcal{O}(R)$ of R is the type term denoting the order-topology of R, defined using the order relation $<$. So $\mathcal{O}(R)$ is a subtype of $\mathcal{P}(R)$. $\mathcal{O}([0,1])$ is the corresponding relative topology on $[0,1]$. ($\mathcal{O}(R)$ is indeed a topology; see 2.7!). The axioms are

(A16) $[0,1]$ is compact, in the sense that for every open cover $\mathcal{U}$ of $[0,1]$ there exists an $n \in N$ and a $V\colon \{m \in N | m < n\} \to \mathcal{O}([0,1])$ refining $\mathcal{U}$, i.e. $\forall m < n\, \exists U \in \mathcal{U}\; V_m \subset U$, such that $\forall x \in [0,1]\, \exists m < n(x \in V_m)$.
(A17) $[0,1]$ has the open refinement property; i.e.

$$\forall F \in \mathcal{P}([0,1])^N(\forall x \in R\, \exists n \in N\; x \in F_n \to \forall x \in R\, \exists U \in \mathcal{O}([0,1])\exists n \in N\; x \in U \subset F_n).$$

1.7 Special Functions. We postulate the existence of the following functions. First of all, we need a bump-function. To this end, we add an individual constant δ of type R^R, whose properties are given by the following axiom

(A18) δ is an increasing function with $x \leq 0 \to \delta(x) = 0$ and $x \geq 1 \to \delta(x) = 1$.



Furthermore, we add constants sin, cos, and exp of type R^R, so as to have the trigonometric functions and the exponential function available, and a constant of type R^R to make N a finitely presented type, together with the following axiom:

(A19) $\sin 0 = 0, \cos 0 = 1, \exp 0 = 1, (\sin x)' = \cos x, (\cos x)' = -\sin x, (\exp x)' = \exp x$ and $\eta(x) = 0 \leftrightarrow x \in N$.

(Let us remark that all the smooth functions $\mathbb{R} \to \mathbb{R}$ in *Sets* are available as maps $R \to R$ in the models, and we could add more constants to the language to denote these functions, whenever this is convenient. Note that this also extends the finitely presented types that are available; cf. 1.5. Also, we could replace the axiom for η by the axiom $\sin \pi x = 0 \wedge x \geq 0 \leftrightarrow x \in N$, where π is defined as in 3.7 below.)

1.8 Existence of Invertible Infinitesimals. To state this last group of axioms, we first need to define the *standard* or *accessible* natural numbers $\mathbb{N}$ as the following subtype of N,

$$\mathbb{N} = \{n \in N | \forall S \in \mathcal{P}(N)(0 \in S \wedge \forall n \in N(m \in S \to m+1 \in S) \to n \in S).$$

Clearly, $\mathbb{N}$ satisfies the *full induction axiom*

$$\forall P \in \mathcal{P}(\mathbb{N})(0 \in P \wedge \forall n \in \mathbb{N}(n \in P \to n+1 \in P) \to P = \mathbb{N}).$$

We may now postulate the existence of invertible infinitesimals as in the following axiom:

(A20) $\exists x \in R\ (x \text{ invertible } \wedge \forall n \in \mathbb{N}\left(-\frac{1}{n+1} < x < \frac{1}{n+1}\right))$.

By Archimedeanness for N (A11), we can equivalently formulate this axiom as $\exists n \in N\ \forall m \in \mathbb{N}(m < n)$.

As we have seen in Chapters VI and III, axiom (A20) holds in $\mathcal{B}$, but it doesn't hold in $\mathcal{F}, \mathcal{G}$, and $\mathcal{Z}$. In $\mathcal{Z}$, the weaker

(A20′) $\neg\neg\exists x \in R(x \text{ invertible } \wedge \forall n \in \mathbb{N}\left(-\frac{1}{n+1} < x < \frac{1}{n+1}\right))$

is true, and this suffices for some purposes. Even better, in $\mathcal{Z}$ we have

(A20″) $\forall x \in R(\forall y \in \mathbb{I}\; xy = 0 \rightarrow x = 0)$,

where $\mathbb{I} = \{x \in R | x \text{ is invertible } \wedge \forall n \in \mathbb{N}\left(-\frac{1}{n+1} < x < \frac{1}{n+1}\right)\}$, which is another way of saying that invertible infinitesimals are somehow present.

Furthermore, we have the following axiom:

(A21) $\forall n \in N(\forall \varepsilon \in \mathbb{I}(\varepsilon n < 1) \rightarrow n \in \mathbb{N})$.

This is actually an arithmetical axiom, since it can be written as

(A21′) $\forall n \in N(\forall x \in N - \mathbb{N}\;(x > n) \rightarrow n \in \mathbb{N})$,

i.e. "a natural number which is smaller than all non-standard natural numbers must be standard".

The axiom (A20) plays a rather special role, since it does not hold in all the models discussed in Chapters III and VI, and we will state explicitly when it is used. Note that (A21) holds vacuously if $N = \mathbb{N}$. But if N does not coincide with $\mathbb{N}$, (A21) or (A21′) says that there are a lot of non-standard numbers, in some sense.

1.9 Validity of the Axioms in the Models. Most of the axioms have been extensively discussed in earlier chapters, notably Chapters III and VI. Thus, (A1)–(A9) hold in all the models $\mathcal{F}, \mathcal{G}, \mathcal{Z}$, and $\mathcal{B}$. (For (A6), see V.7.2 (A8) and (A9) have not been mentioned explicitly, but they can be proved straightforwardly; for (A9), one may use the external inverse function theorem with an extra parameter, much as Lemma I.3.23).

(A10)–(A14) also hold in all four models. One should note that N coincides with $\mathbb{N}$ in $\mathcal{F}$ and $\mathcal{G}$, so we have classical (first-order) arithmetic and full (higher-order) induction in these toposes (cf. Appendix 1). For $\mathcal{Z}$ and $\mathcal{B}$, (A14) was discussed in VI.2.8, while (A13) will be proved below, in 4.6. (A16), (A17) have been dealt with in III.3.3, III.3.12, and VI.2.5, VI.3.5, VI.3.12, VI.5.5, VI.5.6, and (A18), (A19) are obvious. (A20), (A20′), (A20″) all fail in $\mathcal{F}$ and $\mathcal{G}$, again since $N = \mathbb{N}$ in these toposes. (A20′) and (A20″) have been seen to hold in $\mathcal{Z}$ (VI.1.8), while (A20) holds in $\mathcal{B}$ (VI.5.4).

Thus, disregarding the postponed proof of (A13) for $\mathcal{Z}$ and $\mathcal{B}$, the only axioms that have not been discussed before are (A15) and (A21). (A15) can be proved by induction in $\mathcal{F}$ and $\mathcal{G}$, since we have full induction in these cases. (A21) holds vacuously in $\mathcal{F}$ and $\mathcal{G}$. So it remains to prove the following proposition.

1.10 Proposition. *Axioms (A15) and (A21), or equivalently (A21′), hold in $\mathcal{Z}$ and $\mathcal{B}$.*

Proof. The case of (A21′), $\forall n \in N(\forall m \in N - \mathbb{N}\ (n < m) \to n \in \mathbb{N})$, is easy. First, recall that the generic $n \in N - \mathbb{N}$ is given by the inclusion $\ell C^\infty(\mathbb{N})/T \hookrightarrow \ell C^\infty(\mathbb{N}) = N$, where T is the ideal of eventually vanishing functions. Now let p be a (smooth) natural number at stage ℓA, where $A = C^\infty(\mathbb{R}^n)/I$. So p is given by a map $\ell A \to N$. Suppose $\ell A \Vdash \forall n \in N - \mathbb{N}\ (p < n)$. Then in particular $\ell A \times \ell(C^\infty(\mathbb{N})/T) \Vdash p < \pi_2$, so there are finitely generated subideals $I_0 \subset I$ and $T_0 \subset T$ such that $\ell(C^\infty(\mathbb{R}^n \times \mathbb{N})/(I_0, T_0)) \Vdash p < \pi_2$. Let m_0 be so big that $\forall f \in T_0\ \forall m \geq m_0 f(m) = 0$. Then it follows that $\forall x \in Z(I_0) \forall m \geq m_0\ p(x) < m$. So p can be represented by a bounded function $Z(I_0) \to \mathbb{N}$, and is therefore in $\mathbb{N}$ (cf. VI.1.6).

Validity of (A15) is a little more difficult. We do the case of $\mathcal{Z}$. For the case of $\mathcal{B}$, one simply adapts the argument as indicated in VI.5.5, VI.5.6. Take $\ell A \in \mathbb{L}, A = C^\infty(\mathbb{R}^d)/I$, and $P \subset N \times N$ at stage ℓA such that

(1) $\ell A \Vdash \forall n\, \exists m P(n, m)$ (n, m ranging over N).

We have to show that

(2) $\ell A \Vdash \forall n_o\, \exists m_o\, \forall n \leq n_o\, \exists m \leq m_o\ P(n, m)$.

(1) implies in particular that $\ell A \times N \Vdash \exists m \in N P(\pi_2, m)$, so there are a cover of $Z(I_0) \times \mathbb{N}$ by opens $U_1, \ldots, U_k$ of $\mathbb{R}^d \times \mathbb{N}$ (for some finitely generated $I_0 \subset I$), and continuous functions $p_i : U_i \to \mathbb{N}$ such that

(3) $(\ell A \times N) \cap s(U_i) \Vdash P(\pi_2, p_i)$.

By partitions of unity, we find a cover $\{V_1, \ldots, V_k\}$ refining the cover $\{U_1, \ldots, U_k\}$ with $V = \bigcup_{i=1}^k V_i \supseteq Z(I_o) \times \mathbb{N}$, and a C^∞-function $V \xrightarrow{a} \mathbb{R}$ such that

(4) $\forall (x, n) \in V_i\ p_i(x, n) < a(x, n)$

(cf. the sublemma in VI.2.4). Let $b : V \to \mathbb{R}$ be the smooth function defined by

(5) $b(x, n) = \sum_{m \leq n} a(x, m)$.

b represents an element of R at stage $\ell A \times N$, and we claim that

(6) $\ell A \times N \Vdash \forall n \leq \pi_2\, \exists m \leq b\ P(n, m)$.

To prove (b), take the generic $n \leq \pi_2$, i.e. let

$$\ell B = \ell A \times N \times N \cap s(\{(x,n,m) | x \in \mathbb{R}^d, m \leq n\}),$$

so that we need to show

(7) $\ell B \Vdash \exists m \leq b\, P(\pi_3, m)$.

Let $V_{ij} = \{(x,n,m) | m \leq n, (x,n) \in U_i, (x,m) \in U_j\}$. Then $\{V_{ij}\}$ covers $Z(I_0) \times \{(n,m) \in \mathbb{N} \times \mathbb{N} | n \geq m\}$, so it suffices to show

(8) $\ell B \cap s(V_{ij}) \Vdash \exists m \leq b\, P(\pi_3, m)$.

But $\ell A \times N \cap s(U_j) \Vdash P(\pi_2, p_j)$ by (3), so by restricting along $\ell B \cap s(V_{ij}) \xrightarrow{\pi_{13}} (\ell A \times N) \cap s(U_j)$ we conclude that

(9) $\ell B \cap s(V_j) \Vdash P(\pi_3, p_j \circ \pi_{13})$.

Moreover, if $(x,n,m) \in V_{ij}$, then $p_j(x,m) \leq a(x,m) \leq b(x,n)$, so (8) follows from (9). We conclude that

(10) $\ell A \Vdash \forall n_0\, \exists b \in R\, \forall n \leq n_0\, \exists m \leq b\, P(n,m)$.

Since R is Archimedean (for N), (2) follows. □

2 Some Elementary Mathematical Consequences

We shall indicate how to develop some elementary calculus and topology on the basis of the axioms listed in the preceding section. From this it will be clear that the material presented in earlier chapters (notably IV.1, V.1–6) in a "synthetic way", which at that stage didn't mean much more than that the model was left unspecified, can actually be developed in the axiomatic system of Section 1.

The presentation will be independent of the existence of invertible infinitesimals. So on the one hand, the axioms (A20) and (A21) will not be used (unless explicitly stated otherwise), while on the other hand we will allow for the possibility that N does not coincide with $\mathbb{N}$, and accordingly use a very weak system of arithmetic, with typical axioms (A13)–(A15).

2.1 Local Ring Structure of R. As a first illustration of the use of coherent induction, we note that one can now prove from the axioms that R is what we called an "s-local" ring in VI.2.12, as follows. Using recursion (A14), we can define a map

$$R^N \times N \to R,\ (x,n) \mapsto \sum_{i<n} x_i$$

satisfying the recursion equations

$$\sum_{i<0} x_i = 0,\ \sum_{i<n+1} x_i = \left(\sum_{i<n} x_i\right) + x_n,$$

as well as a map

$$R^N \times N \to R \quad (x,n) \to \Pi_{i<n} x_i$$

satisfying the corresponding equations for the product.

Proposition. *R is an ordered local ring in the extended sense that for all $x \in R^N$,*

$$\forall n \in N(\sum_{i<n} x_i > 0 \to \exists i < n(x_i > 0)).$$

Proof. The formula stated in the proposition is not coherent, but the formula

$$\varphi(n,k,x) \equiv \sum_{i<n} x_i < \frac{1}{k} \vee \exists i < n(x_i > 0)$$

(where n and k have type N, x type R) is, so we may apply induction on n to show that from the axioms in 1.2 it follows that $\forall n, k\ \forall x \varphi(n,k,x)$. So if $\sum_{i<n} x_i > 0$, we can pick $k > (\sum_{i<n} x_i)^{-1}$ by Archimedeanness (A11), and conclude that $\exists i < n(x_i > 0)$. □

2.2 Derivatives. The axioms in 1.3 contain implicitly the development of differential calculus, as the following shows. (For ease of notation, we will consider the case of functions of one variable only.) First of all, the Kock-Lawvere axiom (A6′) obviously implies

(1) $\forall f \in R^R \forall x \in R\ \exists! y \in R\ \forall h \in D\ f(x+h) = f(x) + hy.$

This unique y is denoted by $f'(x)$, the *derivative of f at x*. So (1) can be rewritten as *Taylor's formula*

(2) $\forall f \in R^R \forall x \in R\ \forall h \in D\ f(x+h) = f(x) + hf'(x).$

From the uniqueness of $f'(x)$ in (2), the usual rules follow by trivial calculations: for all $f, g \in R^R$ and $x, r \in R$,

$$
(3)\quad
\begin{aligned}
&(f+g)'(x) = f'(x) + g'(x)\\
&(r\cdot f)'(x) = r\cdot f'(x)\\
&(f\cdot g)'(x) = f(x)g'(x) + g(x)f'(x)\\
&(f\circ g)'(x) = f'(g(x))\cdot g'(x).
\end{aligned}
$$

(Note that the rules also hold for $f, g \in R^U$, where $U \subset R$ has the property that $U + D \subset U$.) Taylor formulas for functions of more variables, and of arbitrary order, follow equally easily (see Kock (1981)).

2.3 The Fermat Axiom. There is an alternative approach to differential calculus via the so-called *Fermat axiom*:

(1) $\forall f \in R^R \exists! g \in R^{R\times R} \forall x, y \in R\ f(x) - f(y) = (x-y)g(x,y)$.

(1) is provable in our system. For existence, see "Hadamard's lemma" below, in 2.4. For uniqueness, we show

2.3.1 Lemma. $\forall f \in R^R (\forall x \in R\ xf(x) = 0 \rightarrow \forall x \in R\ f(x) = 0)$.

Proof. Assume the hypothesis, and let $\lambda \in R$. We wish to show that $f(\lambda) = 0$. Let $\varphi_\lambda(x) = xf(\lambda x)$. Then $\varphi_1(x) = xf(x) = 0$, and $\varphi_1(\lambda x) = f(\lambda x) + \lambda x f(\lambda x) = \frac{\partial}{\partial x}\varphi_\lambda(x)$; so $\frac{\partial}{\partial x}\varphi_\lambda(x) = 0$ for all x. By the integration axiom, $\varphi_\lambda(x) = \varphi_\lambda(0) = 0$ for all x. In particular, $\varphi_\lambda(1) = f(\lambda) = 0$. □

On the basis of (1), we can define $f'(x) = g(x,x)$, and derive the usual properties of the derivative. Furthermore, we can prove higher order generalizations of (1), and a form of l'Hôpital's rule:

(2) $\forall f \in R^R \exists! g_k \in R^{R\times R} \forall x, y \in R$
$f(y) - f(x) = \sum_{i<k} (y-x)^i \frac{f^{(i)}(x)}{i!} + (y-x)^k g_k(y,x)$,
and $g_k(x,x) = \frac{f^{(k)}(x)}{k!}$.

(3) If $f, g \in R$, and $f^{(i)}(0) = g^{(i)}(0) = 0$ for $i = 0, \ldots, k-1$, while $f^{(k)}(x)$ is invertible, then $\exists! h \in R^R \forall x \in R\ h(x)\cdot g(x) = f(x)$. Furthermore $h(0)g^{(k)}(0) = f^{(k)}(0)$.

etc., etc.

Note that this axiom (1) is available even in cases where there are no infinitesimals, as in the topos $\mathcal{E}$ (see Appendix 2). □

2.4 Integration. To develop integral calculus, one first proves the fundamental theorem of calculus in the version of Hadamard.

2.4.1 Hadamard's Lemma. $\forall a, b \in R\ \forall f \in R^{[a,b]} \forall x, y \in [a, b]$:

$$f(y) - f(x) = (y - x)\int_0^1 f'(x + t(y - x))dt,$$

where $[a, b] = \{x \in R | a \leq x \leq b\}$.

Proof. First note that the integral makes sense, by convexity of $[a, b]$. For $x, y \in [a, b]$ given, we let $\varphi: [0, 1] \to [a, b]$ be the map $\varphi(t) = x + t(y - x)$, and compute: $f(y) - f(x) = f(\varphi(1)) - f(\varphi(0)) = \int_0^1 (f \circ \varphi)'(t)dt = \int_0^1 (y - x)(f' \circ \varphi)(t)dt = (y - x)\int_0^1 f'(x + t(y - x))dt$, using the chain rule. □

Let $f \in R^{[a,b]}$, and let $x_o \in [a, b]$. We may *define* a function $x \mapsto \int_{x_o}^x f(t)dt$ on $[a, b]$,

$$\int_{x_o}^x f(t)dt = (x - x_o)\int_0^1 f(x_o + t(x - x_o))dt.$$

Note that by convexity of $[a, b]$, $\int_{x_o}^x f(t)dt$ is defined for all $x \in [a, b]$. The use of the $\int$-sign as thus defined is compatible with earlier usage, and one can prove the integration axiom for arbitrary intervals. (We do not require $a \leq b$):

(1) $\forall a, b \in R\ \forall f \in R^{[a,b]} \left(\int_{x_o}^x f(t)dt\right)' = f(x)$.

(2) $\forall a, b \in R\ \forall f \in R^{[a,b]} \exists! g \in R^{[a,b]}(g(a) = 0 \wedge \forall x \in [a, b] g'(x) = f(x))$.

Indeed, (1) follows from the integration axiom and the properties of differentiation, and existence in (2) follows from (1). For uniqueness in (2), suppose $g(a) = 0$ and $g' \equiv 0$ on $[a, b]$. We show $\forall x \in [a, b] g(x) = g(b)$. Let $x \in [a, b]$, and define $g: [0, 1] \to R$ by $h(t) = g(x + t(b - x))$. Then $h' \equiv 0$, so h is constant by the integration axiom. Thus $g(x) = h(0) = h(1) = g(b)$.

By uniqueness in (2) and linearity of differentiation (2.2(3)), we have

(3) $\int_a^b f(x)dx$ is linear in f.

Equally easy, one shows for example

(4) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$,

(5) $\frac{\partial}{\partial y}\int_a^b f(y, x)dx = \int_a^b \frac{\partial}{\partial y} f(y, x)dx$,

(6) $\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$ (Fubini),

(7) $\int_c^d f(t)dt = \int_a^b f(\varphi(x))\varphi'(x)dx$, where $\varphi: [a,b] \to [c,d]$ satisfies $\varphi(a) = c, \varphi(b) = d$.

Notice that (3) implies additivity in the following more general sense:

(8) Let $f \in (R^{[a,b]})^N, f = (f_i)_{i \in N}$. Then

$$\forall n \in N \left(\int_a^b \sum_{i<n} f_i(x)dx = \sum_{i<n} \int_a^b f_i(x)dx \right).$$

Proof of (8). $\sum_{i<n} f_i$ is defined by recursion (A14) via $\left(\sum_{i<0} f_i\right)(x) = 0$, $\left(\sum_{i<n+1} f_i\right)(x) = \left(\sum_{i<1} f_i\right)(x) + f_n(x)$. The functions $\varphi(n) = \int_a^b \sum_{i<n} f_i(x)dx$ and $\psi(n) = \sum_{i<n} \int_a^b f_i(x)dx$ satisfy the same recursion equations by (3), so $\varphi = \psi$ by uniqueness in (A14). $\square$

2.5 Natural Numbers. We will not develop primitive recursive arithmetic systematically. The interested reader is referred to the standard references in logic (see the comments at the end). But let us note a few useful consequences of our arithmetical axioms.

Recall that a property P on a set X is decidable if $\forall x \in X(Px \vee \neg\, Px)$. A subset $U \subset X$ is *decidable* if "$x \in U$" is a decidable property. For example,

(1) N has decidable equality, i.e. $\forall x, y \in N(x = y \vee x \neq y)$.

To prove (1), it is enough (by taking $x - y$) to show that $\forall x \in N(x = 0 \vee x > 0)$. But this is a coherent formula, so we can use induction. Similarly,

(2) (N is linearly ordered) $\forall x, y(x < y \vee x = y \vee x > y)$

follows from the axioms.

Another useful consequence of the coherent induction axiom is

(3) (Decidable induction) $\forall P \in \mathcal{P}(N)(\forall n \in N(P(n) \vee \neg(P(n)) \to (P(0) \wedge \forall n \in N(P(n) \to P(n+1)) \to \forall n \in NP(n))$.

To prove (3), it suffices to note that a decidable $P \in \mathcal{P}(N)$ has a characteristic function $f_P: N \to N$, $f_P(n) = 0$ if $n \in P, f_P(n) = 1$ if $n \notin P$. So (3) reduces to

(4) $f(0) = 0 \wedge \forall n \in N(f(n) = 0 \to f(n+1) = 0) \to \forall n \in Nf(n) = 0$,

which is an instance of coherent induction, with a parameter of type

N^N.

In applying (3), it is useful to note

(5) If $P \in \mathcal{P}(N)$ is decidable, then so are $\{n|\forall k < nP(k)\}$ and $\{n|\exists k < nP(k)\}$,

as follows from the proposition in 2.1 by taking characteristic functions.

From (3) it follows that

(6) $\mathbb{N} \subset N$ is decidable iff $\mathbb{N} = N$,

so it is hard to find out whether a natural number is standard or not. One may use (A21′), or

(7) N is an end-extension of $\mathbb{N}$, i.e.

$$\forall x \in N \, \forall n \in \mathbb{N}(x < n \rightarrow x \in \mathbb{N}).$$

(7) follows by ordinary induction (on $n \in \mathbb{N}$) for the formula $\varphi(n) \equiv \forall x \in N(x < n \rightarrow x \in \mathbb{N})$ (which is not coherent).

2.6 Finite Sets and Finite Cardinals. A *finite cardinal* is a set of the form

$$[n] := \{m \in N | m < n\},$$

where $n \in N$. A set X is *finite* if there exists a surjection $[n] \rightarrow X$ for some $n \in N$, i.e. if X is a quotient of a finite cardinal. A set X is *inhabited* if $\exists x(x \in X)$.

Proposition. (i) A set X is isomorphic to a finite cardinal $[m]$ iff X is finite and has decidable equality $(\forall x, y \in X(x = y \vee x \neq y))$, and in this case m is unique.

(ii) Let $n \in N$, and let R be a strict linear order on an inhabited decidable subset $A \subset [n]$ (so in particular $\forall x, y \in A(xRy \vee x = y \vee yRx)$. Then there is a unique $m \in N$ and a unique order-isomorphism

$$\varphi\colon [m] \xrightarrow{\sim} A$$

(i.e. $\forall i, j < m(i < j \leftrightarrow \varphi(i)R\varphi(j))$).

(iii) Any finite strict linear order (X, R) is order-isomorphic to $([m], <)$, for a unique $m \in N$, and by a unique order isomorphism.

Before we prove the proposition, first a lemma.

Lemma. *Let R be a strict linear order on an inhabited decidable subset $A \subset [n]$. Then A has an R-smallest element.*

Proof. Define by recursion a function $\mu_A: N \to [n+1] \subset N$, via

$$\mu_A(0) = \begin{cases} n & \text{if} \quad 0 \notin A \\ 0 & \text{if} \quad 0 \in A, \end{cases}$$

$$\mu_A(i+1) = \begin{cases} i+1 \text{ if } i+1 \in A \text{ and } (\mu_A(i) \notin A \text{ or } (i+1)R\mu_A(i)) \\ \mu_A(i) \text{ otherwise} \end{cases}$$

Since all the properties used in the definition of μ_A are decidable, μ_A can indeed by defined by (A14). One then shows by induction (A13) that

(1) $j \leq k$ and $j \in A \to \mu_A(k) \in A$.
(2) $\forall j \leq k(j \in A \to (j = \mu_A(k) \vee \mu_A(k)Rj))$.

(2) makes sense by (1). Since A is inhabited, (1) implies that $\mu_A(n-1) \in A$. By (2), $\mu_A(n-1)$ is the smallest. □

Proof of Proposition. (iii) follows from (i) and (ii). We prove (i): $\Rightarrow$ is clear. For $\Leftarrow$, suppose $[n] \xrightarrow{p} X$ is given. Define by recursion a function $f: N \to [n+1]$, via $f(0) = 0$, and

$$f(i+1) = \begin{cases} \text{the smallest } j < n \text{ with } p(j) \notin \{p(0), \ldots, p(i)\}, \\ \text{if there is such a } j; \\ n, \text{ otherwise.} \end{cases}$$

f is well-defined by the lemma, since everything involved in decidable (using (2.5(5)). Then $f(n) = n$, so again by the lemma, there is a smallest m with $f(m) = n$. Then clearly, $p \circ f$ defines a bijection $[m] \xrightarrow{\sim} A$.

To show uniqueness of m in (i), we need to show

(1) if $\varphi: [n] \xrightarrow{\sim} [m]$, then $n = m$.

We first prove

$(2)_i$ if $\varphi: [i] \rightarrowtail [m]$ and $\forall k \in im(\varphi)\{0, \ldots, k\} \subset im(\varphi)$, then $im(\varphi) \subset [i]$,

where $im(\varphi)$ is the image of φ. $(2)_i$ is decidable (by 2.5(5), etc.), and therefore we can apply coherent induction to prove (2). $i = 1$ is clear. Suppose $(2)_i$ holds, and let $\varphi: [i+1] \to [m]$ be a 1-1 function with $\forall k \in im(\varphi)\{0, \ldots, k\} \subset im(\varphi)$. Let $\varphi(i) = j_o$, and define $\tilde{\varphi}: [i] \rightarrowtail [m]$ by $\tilde{\varphi}(k) = \varphi(k)$ if $\varphi(k) < j_o$, $\tilde{\varphi}(k) = \varphi(k) - 1$ if $\varphi(k) > j_o$. Then $\tilde{\varphi}$ satisfies the hypotheses of $(2)_i$, so $im(\tilde{\varphi}) \subset \{0, \ldots, i-1\}$.

Thus $im(\varphi) \subset [i+1]$, and $(2)_{i+1}$ is proved.

Applying $(2)_n$ to the φ in (1), we conclude $m \leq n$. By symmetry $m \leq n$. This proves (i).

For (ii), it suffices by (i) to show that for a linear order R on $[n]$, there is a unique $\varphi: [n] \to [n]$ with $i < j \Leftrightarrow \varphi(i) R \varphi(j)$. Define $\psi: N \to N$ by recursion, using the lemma above:

$$\psi(0) = \text{the } R\text{-smallest element in } [n]$$

$$\psi(i+1) = \begin{cases} \text{the R-smallest element in } [n] - \{\psi(0), \ldots, \psi(i)\}, \\ \text{if this set is inhabited;} \\ i+n, \text{ otherwise.} \end{cases}$$

ψ is well-defined, 1-1, and order-preserving. It remains to show that ψ restricts to a bijection $[n] \to [n]$. But by (i), there cannot be a surjection $[i] \to [n]$ for $i < n-1$, so $\psi(i+1)$ must be defined by the first clause, i.e. $\psi(i+1) < n$. For the same reason, $\psi(i+1) \geq n$ if $i \geq n-1$, i.e. $[n] - \{\psi(0), \ldots, \psi(n-1)\}$ is empty.

This proves the proposition. □

We now turn to some consequences of the topological axioms. As will be clear from the sequel, it is here that the lack of axioms of finite choice and (full) induction causes most problems.

First of all, we should point out that the interval topology on R is well-defined.

2.7 Proposition. *The formula $U \in \mathcal{O}(R) \leftrightarrow \forall x \in U\, \exists \varepsilon > 0 (x - \varepsilon, x + \varepsilon) \subset U$ defines a topology on R.*

Proof. The union of open sets is clearly open. For finite intersections, we have to be careful not to use finite choice! Take opens $U_0, \ldots, U_{n-1} (n \in N)$, and let $x \in \bigcap_{m<n} U_m$. So by Archimedeanness (A11), $\exists m < n\, \forall k \in N - \{0\} \left(x - \frac{1}{k}, x + \frac{1}{k}\right) \subset U_m$. By the axiom of bounded search (A15), we find an upperbound k_o for the k's needed, so $\forall m < n \left(x - \frac{1}{k_o}, x + \frac{1}{k_o}\right) \subset U_m$, i.e. $\left(x - \frac{1}{k_o}, x + \frac{1}{k_o}\right) \subset \bigcap_{m<n} U_m$. □

2.8 Metric Spaces. In the sequel, we will limit our attention to metric spaces, and concentrate on the spaces we have used so far to do some analysis

$$
\begin{aligned}
&R^n \\
&I^n = \{(x_1,\dots,x_n) \in R^n | 0 \le x_i \le 1\} \\
&S^{n-1} = \{(x_1,\dots,x_n) \in R^n | \textstyle\sum x_i^2 = 1\} \\
&B^n = \{(x_1,\dots,x_n) \in R^n | \textstyle\sum x_i^2 \le 1\} \\
&\Delta^n = \{(x_1,\dots,x_n) \in R^n | \text{each } x_i \ge 0, \text{and } \textstyle\sum x_i \le 1\}.
\end{aligned}
$$

Notice that, strictly speaking, R is not a metric space in the models, since the absolute value $d(x,y) = |x-y|$ cannot be defined. We should reinterpret the distance as a *relation*

$$d(x,y) < r$$

rather than as a function $R^2 \to R$ (or generally, $X \times X \to R$). This enables us to talk about open balls $B(x,r)$, which is the only thing needed.

2.9 The Open Refinement Property. Here, as in the sequel, X, $Y, \dots$ denote metric spaces with a given distance (as a relation, see 2.8), and $B(x,r) := \{y | d(x,y) < r\}$ for a given space with distance relation d. These open balls $B(x,r)$ form the basis for a topology, as in 2.7.

A space X is said to have the *open refinement property* if every *countable* cover of X has an open refinement, i.e.

$$\forall F \in \mathcal{P}(X)^N (\forall x\, \exists n\; x \in F_n \to \forall x\, \exists U \in \mathcal{O}(X)\, \exists n\; x \in U \subset F_n).$$

Axiom (A17) states that I has the open refinement property. We will show that most of the spaces we are interested in have the open refinement property.

2.9.1 Lemma.

(i) If X and Y have the open refinement property, then so does $X \times Y$.

(ii) If $X = \bigcup_{i \in I} \mathrm{Int}(F_i)$, and each F_i has the open refinement property, then so does X.

Proof. (i) Let $\{A_n\}_n$ be a cover of $X \times Y$. For $y \in Y$, write

$$s_y(A_n) = \{x \in X | (x,y) \in A_n\}.$$

Then $\forall y \in Y (X = \bigcup_{n \in N} s_y(A_n))$, so by the open refinement property for X,

(1) $\forall y \in Y\; \forall x \in X\; \exists k, n \in N\; B(x, 2^{-k}) \subset s_y(A_n)$.

Now let for $x \in X$ and $k, n \in N$

$$B^x_{n,k} = \{y | B(x, 2^{-k}) \times \{y\} \subset A_n\}.$$

Then (1) says that for each $x, \{B^x_{n,k} | (n,k) \in N \times N\}$ is a countable cover of Y. So by the open refinement property for Y,

(2) $\forall x \in X\ \forall y \in Y\ \exists \ell, n, k \in N\ B(y, 2^{-\ell}) \subset B^x_{n,k}$.

Since $B(y, 2^{-\ell}) \subset B^x_{n,k}$ means that $B(x, 2^{-k}) \times B(y, 2^{-\ell}) \subset A_n$, this proves (i).

(ii) is obvious. □

2.9.2 Corollary. *All manifolds have the open refinement property (manifolds with corners can be included). In particular, the spaces $I^n, R^n, \Delta^n, S^n, B^n$ listed in 2.8 all have the open refinement property.*

Proof. By (i) of the lemma, I^n has the open refinement property. Hence so do all spaces that can be covered by interiors of isomorphic copies of I^n. These include all the spaces mentioned in the corollary. (We adopt the usual definition of manifold in this "synthetic" context.) □

2.9.3 Remark. Clearly, if X has the open refinement property then all functions $X \to R$ are continuous. In particular from 2.9.2

(1) all functions $R^n \to R$ are continuous.

It may be worth noting that we do not need topological axioms to prove this:

Proof of (1) from Integration Axioms. Given $f(\underline{x}): R^n \to R$, we may write, by Hadamard's lemma (2.4.1)

$$f(\underline{x} + \underline{y}) - f(\underline{x}) = \sum_{i=1}^{n} y_i g_i(\underline{x}, \underline{y}),\ \underline{y} = (y_1, \ldots, y_n).$$

So for given $\underline{x}$, it suffices to show that each $g_i(\underline{x}, -)$ is bounded on a neighbourhood of $\underline{0}$. But every function $[0,1]^n \xrightarrow{g} R$ is bounded: we need only write

$$\begin{aligned} g(\underline{x}) - g(\underline{0}) &= \textstyle\int_0^{x_1} \cdots \int_0^{x_n} \frac{\partial^n g}{\partial \underline{x}}(\underline{x}) dx_1 \ldots dx_n \\ &< \textstyle\int_0^1 \cdots \int_0^1 [1 + \left(\frac{\partial^n g}{\partial \underline{x}}(\underline{x})\right)^2] dx_1 \ldots dx_n \end{aligned}$$

where $\partial \underline{x} = \partial x_1 \ldots \partial x_n$ and the last inequality follows from monotonicity (A8) and $a < 1 + a^2$.

Notice that this really shows that every function satisfies the Lipschitz condition. □

2.10 Compactness. Recall that by definition, a space X is compact if for every open cover $\mathcal{U}$ of X, there exists a finite open *refinement*, i.e. open sets $V_0, \ldots, V_{n-1}$ with $X = \bigcup_{m<n} V_m$ and $\forall m < n\, \exists U \in \mathcal{U}\; V \subset U$ (It does *not* follow that there also exists an open *subcover*, if we don't have finite choice.)

2.10.1 Proposition.

(i) *If X is compact, then every open cover has a Lebesgue number.*
(ii) *If X is compact and has the open refinement property, then X satisfies the principle CMP, i.e.*

$$\forall A \subset X \times R (\forall x \in X\, \exists \varepsilon > 0\, \{x\} \times (-\varepsilon, \varepsilon) \subset A \rightarrow \exists \delta > 0\, X \times (-\delta, \delta) \subset A).$$

Proof. (i) Although it may seem that one needs finite choice, the axiom of bounded search (A15) suffices: let $\mathcal{U}$ be an open cover of X, and let $\mathcal{U}' = \{B(x,\varepsilon) | x \in X, \varepsilon > 0, \text{ and } \exists U \in \mathcal{U} B(x, 2\varepsilon) \subset U\}$. Then $\mathcal{U}'$ covers X. Let $V_0, \ldots, V_{n-1}$ be a finite refinement of $\mathcal{U}'$. So $\forall i < n\, \exists x \in X\, \exists \varepsilon > 0\; V_i \subset B(x, \varepsilon) \in \mathcal{U}'$. By Archimedeanness and bounded search, $\exists k_0 \forall i < n\, \exists \varepsilon > 2^{-k_0} \exists x \in X\; V_i \subset B(x,\varepsilon) \in \mathcal{U}'$. Since the V_i cover X, it follows that 2^{-k_0-1} is a Lebesgue number for $\mathcal{U}$, i.e. $\forall x \in X\, \exists U \in \mathcal{U}\; B(x, 2^{-k_0-1}) \subset U$.

(ii) Let $A \subset X \times R$, with $\forall x \in X\, \exists \varepsilon\, \{x\} \times (-\varepsilon, \varepsilon) \subset A$. By Archimedeanness and the open refinement property, there is an open cover $\mathcal{U}$ of X such that

$$\forall U \in \mathcal{U}\, \exists n \in N\; U \times (-2^{-n}, 2^{-n}) \subset A.$$

Let $V_0, \ldots, V_{m-1}$ be a finite open refinement of $\mathcal{U}$. Then $\forall k < m \exists n \in N V_k \times (-2^{-n}, 2^{-n}) \subset A$. By bounded search, we conclude $\exists n \in N \forall k < m\; V_k \times (-2^{-n}, 2^{-n}) \subset A$, i.e. $X \times (-2^{-n}, 2^{-n}) \subset A$. □

2.10.2 Proposition.

(i) *If X and Y are compact, so is $X \times Y$.*
(ii) *If $X \xrightarrow{f} Y$ is a continuous surjection, X is compact, and Y has the open refinement property, then Y is also compact.*
(iii) *If $X = \bigcup_{i<n} \mathrm{Int} Y_i$, and each Y_i is compact, then so is X.*

Proof. (ii) and (iii) are obvious (but see Remark 2.10.3), (i) is a little tedious, since one has to avoid finite choice. Let $\mathcal{U}$ be an open cover of $X \times Y$. For $y \in Y$, write

$$A^y_\varepsilon = \{x \in X | \exists \delta > 0\, \exists U \in \mathcal{U}\ B(x,\delta) \times B(y,\varepsilon) \subset U\}.$$

Then for a given $y \in Y$, $\{A^y_\varepsilon | \varepsilon > 0\}$ covers X, so by compactness of X, there is a refinement $\{V_0, \dots, V_{n-1}\}$ which still covers X. Applying bounded search and quantifying y, we get

(1) $\forall y \in Y\ \exists \varepsilon > 0\ \exists V_0, \dots, V_{n-1}$ covering X $\forall i < n \exists U \in \mathcal{U}\ V_i \times B(y,\varepsilon) \subset U$.

Applying the existence of Lebesgue numbers (2.10.1(i)) to X and Y separately, (1) implies

(2) $\exists \varepsilon > 0\ \forall y \in Y\ \exists \delta > 0\ \forall x \in X\ \exists U \in \mathcal{U}\ B(x,\delta) \times B(y,\varepsilon) \subset U$.

Take $\varepsilon > 0$ as in (2), and let $W_0, \dots, W_{m-1}$ be a finite refinement of $\{B(y,\varepsilon) | y \in Y\}$. Although we cannot pick a point from W_i (they may be empty), we have $\forall i < m\ \exists y\ W_i \subset B(y,\varepsilon)$, and this suffices for

(3) $\forall i < m\ \exists \delta > 0\ \forall x \in X\ \exists U \in \mathcal{U}\ B(x,\delta) \times W_i \subset U$.

Again using bounded search, (3) gives

(4) $\exists \delta > 0\ \forall i < m\ \forall x \in X\ \exists U \in \mathcal{U}\ B(x,\delta) \times W_i \subset U$.

Take a δ as in (4), and let $O_1, \dots, O_{n-1}$ be a refinement of $\{B(x,\delta) | x \in X\}$. Then (4) implies that $\{O_j \times W_i | j < n, i < m\}$ is a cover of $X \times Y$ refining $\mathcal{U}$. □

2.10.3 Remark. The proof of (i) may seem unnecessarily long, but one has to be extremely careful not to use finite choice. (i) could be proved more quickly if we took compactness in the stronger sense that for every open cover of X, there is a finite refinement of the form $\{B(x_i, \delta_i) | i < n\}$, or $\{B(x_i, \delta) | i < n\}$. This, however, doesn't follow from compactness as defined above, without finite choice. (For particular spaces like I^n, however, this stronger sense does follow, as one easily checks.)

Finite choice also seems needed for the more usual versions of (ii) and (iii), obtained by deleting the requirement that Y has the open refinement property, and by replacing $X = \cup \operatorname{Int}(Y_i)$ by $X = \cup Y_i$, respectively.

2.10.4 Corollary. *Δ^n, B^n, S^n, I^n are all compact.* □

In proving the corollary, one should remember that the "usual" map $\pi_n: I^n \to \Delta^n$, $\pi_n(x_1, \ldots, x_n) = (x_1, x_1 \cdot x_2, \ldots, x_1 \cdot \ldots \cdot x_n)$ that we used in IV.2 (taking Δ^n as $\{(x_1, \ldots, x_n) | 0 \leq x_1 \leq \ldots \leq x_n \leq 1\}$) is not surjective, intuitionistically! However, Δ^n can be covered by finitely many interiors of images of I^n.

In connection to Remark 2.10.3, it may be of interest to point out that preservation properties as in 2.10.2 are proved much easier if we take CMP instead of compactness (cf. 2.10.1(ii)):

2.10.5 Proposition.

(i) *If X and Y satisfy CMP, so does $X \times Y$.*

(ii) *If $X \xrightarrow{f} Y$ is surjective and X satisfies CMP, so does Y.*

Proof. (i) Suppose $X \times Y \times \{0\} \subset A \subset X \times Y \times R$ is such that $\forall x, y\ \exists \varepsilon > 0 \{(x, y)\} \times (-\varepsilon, \varepsilon) \subset A$. By CMP for Y, we get $\forall x \in X\ \exists \varepsilon > 0 \{x\} \times Y \times (-\varepsilon, \varepsilon) \subset A$. So using CMP for X, applied to the set $B = \{(x, \varepsilon) \in X \times R_{>0} | \{x\} \times Y \times (-\varepsilon, \varepsilon) \subset A\}$, we conclude $\exists \delta > 0\ X \times (-\varepsilon, \varepsilon) \subset B$; so $X \times Y \times \left(-\frac{1}{2}\delta, \frac{1}{2}\delta\right) \subset A$.

(ii) If $A \subset Y \times R$ is such that $\forall y \in Y\ \exists \varepsilon > 0\ \{y\} \times (-\varepsilon, \varepsilon) \subset A$, then we can apply CMP for X to $A' = (f \times 1)^{-1}(A) \subset X \times R$ to conclude $\exists \delta > 0\ X \times (-\delta, \delta) \subset A$. So $Y \times (-\delta, \delta) \subset A$ since f is surjective. (Notice that we did not use continuity of f.) □

2.11 Chain Connectedness. Recall that a space X is called chain-connected if for ever open cover $\mathcal{U}$ of X and any two points $x_0, x_1 \in X_1$ there exists a chain $V_0, \ldots, V_n$ from x_0 to x_1 refining $\mathcal{U}$; i.e. $\forall i \leq n\ \exists U \in \mathcal{U}\ V_i \subset U$ and $x_0 \in V_0, x_1 \in V_n, \forall i < n\ \exists z \in X\ z \in V_i \cap V_{i+1}$. X is called path-connected if for any two points $x_0, x_1 \in X$ there is a map $I \xrightarrow{\alpha} X$ with $\alpha(0) = x_0$, $\alpha(1) = x_1$. Since all the spaces listed in 2.8 are obviously path-connected, we restrict ourselves to the following.

Proposition. *Every path-connected space is chain-connected.*

Proof. Let $\mathcal{U}$ be an open cover of a given path-connected space X, and let $x_0, x_1 \in X$. Let $I \xrightarrow{f} X$ be a path from x_0 to x_1. So $\forall t \in [0,1] \exists \delta > 0\ \exists U \in \mathcal{U}\ B(f(t), \delta) \subset U$. By CMP (2.10.1(ii)), $\exists k > 0\ \forall t \in [0,1] \exists U \in \mathcal{U}\ B\left(f(t), \frac{2}{k}\right) \subset U$. Also by compactness and the open refinement property of $I, \exists \ell > 0\ \forall t \in [0,1] f(B(t, \frac{1}{\ell})) \subset$

$B(f(t), \frac{1}{k})$ for any given $k > 0$. In particular, taking a k as above, we conclude that $(B(f(\frac{i}{\ell}), \frac{2}{k}) : i = 0, \ldots, \ell)$ is a chain from x_0 to x_1. □

2.12 Indecomposability. One often needs connectedness only in the form of indecomposability. Recall that X is *indecomposable* if $\forall A, B \in \mathcal{O}(X)\ (X = A \cup B \wedge A \cap B = \phi \to X = A \vee X = B)$.

Proposition.

(i) *I is indecomposable.*
(ii) *If X and Y are indecomposable, so is $X \times Y$.*
(iii) *If $X \xrightarrow{f} Y$ is a surjection, and X is indecomposable, so is Y.*
(iv) *If $X = \bigcup_n F_n$, and $\forall n\ \exists x \in X(x \in F_n \cap F_{n+1})$, then X is indecomposable if all the F_n are.*

Proof. (i) follows from compactness and the open refinement property: by these and 2.10.1 (i), there is an $n > 0$ with $\forall x \in I((x - \frac{1}{n}, x + \frac{1}{n}) \subset A \vee (x - \frac{1}{n}, x + \frac{1}{n}) \subset B)$. Also by decidable induction on m (2.5(3)), $\forall m > 0(\forall i \leq 2m\ \frac{i}{2m} \in A \vee \forall i \leq 2m\ \frac{i}{2m} \in B)$. (i) follows by taking $m = n$. (ii)–(iv) are obvious (for (iv), again use decidable induction). □

Remark. (i) also follows from the uniqueness in the integration axiom (A7), provided that for $I = A \cup B$ with $A \cap B = \phi$ we show $A + D \subset A$. This follows from the Kock-Lawvere axiom: the function $I \xrightarrow{\varphi} R$, $\varphi(x) = 1$ if $x \ni A$ and $\varphi(x) = 0$ if $x \in B$, is well-defined. So if $a \in A$ and $d \in D$, $\varphi(a+d) = \varphi(a) + d\varphi'(a) = 1 + d\varphi'(a)$ cannot be equal to 0; so $a + d \in A$.

This concludes our discussion of topological properties. We mention the following application.

2.13 Transversal Intermediate Value Theorem. *Let $f \in R^{[0,1]}$, and suppose 0 is a regular value, in the sense that $\forall x \in [0,1](f(x) \in U(R) \vee f'(x) \in U(R))$. If moreover $f(0), f(1) \in U(R)$, then $f^{-1}(0)$ is a strictly linearly ordered finite subset of $[0,1]$, i.e. $f^{-1}(0)$ is order-isomorphic to $[n]$ for some $n \in N$. In particular, if $f(0) < 0 < f(1)$ there is a smallest $x \in R$ with $f(x) = 0$.*

Proof. Let us first note the following. If $R \xrightarrow{g} R$ is a function with $g'(x) > 0$ for all x, then by Hadamard's lemma (2.4.1) and positivity of the integral (A8), we clearly have that g preserves and reflects the order

(1) $\forall x, y \in R(x < y \leftrightarrow g(x) < g(y))$.

Let $x \in R$. It follows that from (1) that if we choose $U \ni x$ and $V \ni g(x)$ such that $g: U \xrightarrow{\sim} V$ by (A9), then there are $s, t \in R$ with $x \in (s,t) \subset U$, $g(x) \in (g(s), g(t)) \subset V$, such that for all s', t' with $s < s' < t' < t, g$ restricts to an order-isomorphism

(2) $g: (s', t') \xrightarrow{\sim} (g(s'), g(t'))$.

As a second preliminary remark, we claim that if $f \in R^{[0,1]}$ is a function with $\forall x \in [0,1](f(x) \in U(R) \vee f'(x) \in U(R))$, then for any sequence of points $a: N \to (0,1)$, $a = (a_n)_n$, and any $\varepsilon > 0$,

(3) $\forall n \in N\, \exists \delta > 0 (0 < \delta < \varepsilon \wedge \forall k < n\; f(a_k + \delta) \in U(R))$.

(Of course, (3) would be obvious if we had finite choice.) Since $\exists \delta\ (\ldots)$ as in (3) is coherent (write $\forall k < n\; f(a_k + \delta) \in U(R)$ as $\Pi_{k<n} f(a_k + \delta) \in U(R)$), we may use induction on n. (3) is then easily proved, using the observation above ((1), (2), or the symmetric case on intervals where $f' < 0$), and the fact that R is local.

We now prove the theorem. Take $[0,1] \xrightarrow{f} R$ satisfying the hypotheses, and let $\varepsilon > 0$ be a Lebesgue number for the open cover consisting of $\{x | f(x) \in U(R)\}$ and the intervals $(s,t), s < t$, such that either $f' > 0$ on $[s,t]$ and $f: (s,t) \xrightarrow{\sim} (f(s), f(t))$ is an order-isomorphism, or $f' < 0$ on $[s,t]$ and $f: (s,t) \xrightarrow{\sim} (f(t), f(s))$ is an order-reversing isomorphism. (This is indeed a cover by (A9), and (2) above.) Let $n \in N$ be so big that $\frac{2}{n} < \varepsilon$. So for each $k < n - 1$ we have

(4) (i) $f\left[\frac{k}{n}, \frac{k+2}{n}\right] \subset U(R)$; or

(ii) $\exists s, t, s < \frac{k}{n} < \frac{k+2}{n} < t$, such that either $f' > 0$ on (s,t) and $f: ((s,t), <) \xrightarrow{\sim} ((f(s), f(t)), <)$, or $f' < 0$ on (s,t) and $f: ((s,t), <) \xrightarrow{\sim} ((f(t), f(s)), >)$.

Now we apply (2) to the sequence of points $\frac{k}{n}, 0 < k < n$, to find points $0 = a_0 < a_1 < \ldots < a_{n-2}, a_{n-1} = 1$ with $f(a_i) \in U(R)$ and each $[a_i, a_{i+1}]$ is contained in some $\left[\frac{k}{n}, \frac{k+2}{n}\right]$.

Fix an $i < n - 1$, and suppose $f' > 0$ on (s,t) with $s < a_i < a_{i+1} < t$, and $f: (s,t) \xrightarrow{\sim} (f(s), f(t))$. Since $f(a_i) < f(a_{i+1})$, either $f(a_i) < 0 < f(a_{i+1})$ in which case $\exists x \in (a_i, a_{i+1}) f(x) = 0$ since f is an order-isomorphism on (a_i, a_{i+1}); or $0 < f(a_i)$ or $f(a_{i+1}) < 0$, in which cases $f[a_i, a_{i+1}] \subset U(R)$, again since f is order-preserving. So for this fixed i,

(5) $f[a_i, a_{i+1}] \subset U(R) \vee 0 \in f[a_i, a_{i+1}]$.

A similar argument will show that (5) holds in case f is order-reversing on $[a_i, a_{i+1}] \subset$ some $\left[\frac{k}{n}, \frac{k+2}{N}\right]$ as in the second possibility of (4) (ii). In case $[a_i, a_{i+1}] \subset \left[\frac{k}{n}, \frac{k+2}{n}\right]$ as in 4 (i), (5) clearly holds. So we have proved (5) for every $i < n-1$.

Let $A = \{i < n-1 | f[a_i, a_{i+1}] \subset U(R)\}$, $B = \{i < n-1 | 0 \in f[a_1, a_{i+1}]\}$. Clearly $A \cap B = \phi$, and $A \cup B = [n-1]$ by (5). So B is a decidable subset of $[n]$. Moreover by considering a characteristic function for A, 2.1 implies that $B = \phi$ or B is inhabited. By 2.6, we conclude that B is order-isomorphic to $[m]$, for a unique $m \geq 0$. Since for $i \in B, [a_i, a_{i+1}]$ contains a unique x with $f(x) = 0$, and in fact $x \in (a_i, a_{i+1})$ since each $f(a_j) \in U(R)$, B is clearly order-isomorphic to $f^{-1}(0)$.

If $f(0) < 0 < f(1)$, then $B = \phi$ (i.e. $f[0,1] \subset U(R)$) contradicts connectedness of $[0,1]$. So B is inhabited in this case. In particular, it has a smallest element. □

2.14 Remark. In our system, one cannot hope to prove the intermediate value theorem in the form

(1) $\forall f \in R^{[a,b]}(f(a) < 0 < f(b) \rightarrow \exists f(x) = 0)$, where $a < b$.

In fact, (1) fails in all the models, even for polynomial functions. For example, consider

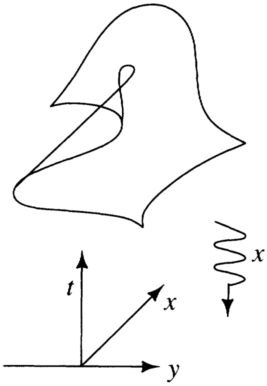
$$\mathbb{R}^2 \times \mathbb{R} \xrightarrow{F} \mathbb{R},\ F(x,y,t) = t^3 + xt + y.$$

In the models, F is interpreted as a polynomial $R \rightarrow R$ at stage $R^2 = \ell C^\infty(\mathbb{R}^2)$. But $R^2 \not\Vdash \exists t\ F(t) = 0$, since there is no continuous function $t(x,y)$ such that

$$t(x,y)^3 + t(x,y) + y = 0$$

in a neighbourhood of $x = 0, y = 0$. This is most easily seen by looking at the catastrophe map χ ("the cusp")

$$\chi\colon \{(x,y,t) \in \mathbb{R}^3 | t^3 + xt + y = 0\} \rightarrow \mathbb{R}^2, \chi(x,y,t) = (x,y).$$



It is obvious that χ has no continuous sections in any neighbourhood of $x = 0 = y$ (just go around a circle with centre $(0,0)$ in the (x,y)-plane).

We end this section with some elementary properties of infinitesimals. Again, we will keep the presentation independent from whether invertible infinitesimals exist or not.

2.15 Types of Infinitesimals. The space of ***infinitesimals*** is defined by

$$\mathbb{A} = \left\{ x \in R \middle| \forall n \in \mathbb{N} - \frac{1}{n+1} < x < \frac{1}{n+1} \right\}.$$

Two important subspaces of $\mathbb{A}$ are

$$\begin{aligned} \mathbb{I} &= \{x \in \mathbb{A} | x \text{ is invertible}\}, \\ \Delta &= \{x \in R | x \text{ is not invertible}\}. \end{aligned}$$

We also define the ring of ***accessible reals***

$$R_{\text{acc}} = \{x \in R | \exists n \in \mathbb{N}(-n < x < n)\};$$

Finally, we introduce a relation $\sharp$ on R:

$$x \sharp y \text{ iff } \exists n \in \mathbb{N} |x - y| > \tfrac{1}{n},$$

(where $|-|$ is used as an obvious abbreviation). Note that conversely $\mathbb{N}$ is definable in terms of $\sharp$ as $\mathbb{N} = \{x \in N | \frac{1}{x+1} \sharp 0\}$.

The following properties are obvious.

2.15.1 Proposition.

(i) $D \subset \Delta \subset \mathbb{A}$;
(ii) $\mathbb{A}$ *is an ideal in* R;
(iii) Δ *is an ideal in* R_{acc};

$$
\text{(iv)}\quad
\begin{aligned}
\Delta &= \{x \in R \mid \forall n \in N \left(-\tfrac{1}{n+1} < x < \tfrac{1}{n+1}\right)\}; \\
\mathbb{I} &= \{x \in \mathbb{\Delta} \mid \exists n \in N \left(x < \tfrac{1}{n+1} \vee x > \tfrac{-1}{n+1}\right)\}, \\
\mathbb{\Delta} &= \{x \in R \mid \neg(x \# 0)\}.
\end{aligned}
$$

$\square$

2.15.2 Remark. In the models, it also holds that $\Delta = \{x \mid \neg\neg x = 0\}$, but this doesn't follow from the axioms. One could add an axiom of the form $\forall x \in R(x \neq 0 \rightarrow x$ is invertible$)$, or an axiom stating that R is a field in the sense of III.1.9 and VI.1.7(6). These axioms, however, are rather different in nature than the algebraic axioms from 1.2 above, and we do not discuss them here.

2.15.3 Non-Existence of Invertible Infinitesimals By applying Hadamard's lemma (2.4.1), every $f \in R^R$ in "Δ-continuous" in the sense that

(1) $x - y \in \Delta \Rightarrow f(x) - f(y) \in \Delta$.

The analogous property for $\mathbb{\Delta}$ need not hold. In fact, by considering functions of the form $x \mapsto x/\varepsilon$, where $\varepsilon \in \mathbb{I}$, one easily shows

(2) $[\forall f \in R^R \forall x, y(x - y \in \mathbb{\Delta} \rightarrow f(x) - f(y) \in \mathbb{\Delta})] \Leftrightarrow \mathbb{I} = \phi$.

Note that, obviously,

(3) $\Delta = \mathbb{\Delta} \Leftrightarrow \mathbb{I} = \phi \Leftrightarrow R_{\mathrm{acc}} = R \Leftrightarrow N = \mathbb{N}$.

2.15.4 Proposition. *Let $f \in R^R$, and write $U(R) = \{x \in R \mid x$ is invertible$\}$.*

(i) $\forall x \in U(R)\ f(x) \in \Delta \Rightarrow \forall x \in R\ f(x) \in \Delta$
(ii) $\forall x \in R\ x \cdot f(x) \in \Delta \Rightarrow \forall x \in R\ f(x) \in \Delta$
(iii) $\forall x \in R\ f(x) \in \Delta \Rightarrow \forall x \in R\ f'(x) \in \Delta$.

Proof. (i) Choose $x_o \in R$, and suppose $f(x_o) \in U(R)$, while $\forall x \in U(R) f(x) \in \Delta$. Then $x_o \in \Delta$. Moreover, since R is local, $\forall x \in R(f(x) - f(x_o) \in U(R) \vee f(x) \in U(R))$. So by Hadamard's lemma, $\forall x \in R(x - x_o \in U(R) \vee f(x) \in U(R))$. Since $f(x) \in U(R) \rightarrow x \in \Delta$ by hypothesis, and $x_o \in \Delta$, we conclude $R = U(R) \cup \Delta$, contradicting indecomposablity (cf. 2.12).

(ii) clear from (i).

(iii) Suppose $\forall x \in R\ f(x) \in \Delta$, and take $x_o \in R$. By Hadamard, we can write $f(x_o + y) - f(x_o) = y\ g(x_o, x_o + y) \in \Delta$. So by (ii) applied to $y \mapsto g(x_o, x_o + y)$, we have $\forall y \in R\ g(x_o, x_o + y) \in \Delta$. In particular, $f'(x_o) = g(x_o, x_o) \in \Delta$. $\square$

Unlike (ii) ad (iii), (i) of the preceding proposition still holds if one replaces Δ by $\mathbb{\Delta}$. In fact, this follows from the following continuity property:

2.15.5 Proposition. *For all* $f \in R^R$, $\forall \varepsilon \in \mathbb{I}\ f(\varepsilon) \# f(0)$ *implies* $\mathbb{I} = \phi$.

Proof. Suppose $\forall \varepsilon \in \mathbb{I} f(\varepsilon) \# f(0)$, and assume $f(0) = 0$ (replace f by the function $f(x) - f(0)$). Let $\delta \in \mathbb{I}$. We will show that $R = U(R) \cup \Delta$, contradicting indecomposability of R. First of all, since R is local, $\forall x \in R \left(-\delta < x < \delta \vee x < -\frac{1}{2}\delta \vee x > \frac{1}{2}\delta\right)$, so certainly $R = \mathbb{\Delta} \cup U(R)$. In particular $\forall x \in R(f(x) \in U(R) \vee f(x) \in \mathbb{\Delta})$. But this implies $\forall x \in R(x \in U(R) \vee x \in \Delta)$ by assumption on f. $\square$

2.15.6 Corollary. Let $f \in R^R$. If $\forall x \in U(R) f(x) \in \mathbb{\Delta}$, then also $\forall x \in R\ f(x) \in \mathbb{\Delta}$.

Proof. Assume the hypothesis, and suppose to the contrary that $f(t_o) \# 0$ for some $t_o \in R$. Then necessarily $t_o \in \Delta$. So if we write $g(t) = f(t_o - t)$, we have $g(0) \# 0$ and $\forall x \in U(R)\ g(x) \in \mathbb{\Delta}$. So $\forall \varepsilon \in \mathbb{I}(g(\varepsilon) \# g(0))$, and hence by 2.15.5, $\mathbb{I} = \phi$, or equivalently, $\Delta = \mathbb{\Delta}$. But then $\forall x \in R\ f(x) \in \Delta$ by 2.15.2 (i); in particular $f(t_o) \in \Delta$, contradicting $f(t_o) \# 0$. $\square$

3 Invertible Infinitesimals and Distributions

In this section, invertible infinitesimals and infinitely large integers will be used to "represent" certain functionals, such as the Dirac functional δ defined by $\delta(f) = f(0)$, by functions, in a sense to be made precise. Once again, it is not our purpose to develop the theory systematically, but rather to give some representative examples of what can be done in the system described in the first section of this chapter, and how this relates to the classical theory via the models of Chapter VI. In this process, we touch upon several topics, such as Fourier series, improper integrals, etc. These notions can be handled straightforwardly in a context where invertible infinitesimals are available.

This section consists of two parts. In the first half, we will work purely axiomatically, on the basis of the system described in Section 1 (including the axioms (A20), (A21), now). In the second part, we will consider distributions in the models $\mathcal{B}$ and $\mathcal{Z}$, and the relation to classical distributions.

3.1 Some Notation. Recall that

$$
\begin{array}{lll}
R_{\mathrm{acc}} & = \{x \in R \mid \exists n \in \mathbb{N}(-n < x < n)\} & \text{(accessible reals)} \\
\mathbb{\Delta} & = \{x \in R \mid \forall n \in \mathbb{N}\left(-\frac{1}{n+1} < x < \frac{1}{n+1}\right)\} & \text{(infinitesimals)} \\
\mathbb{I} & = \mathbb{\Delta} \cap U(R) & \text{(invertible infinitesimals)} \\
\Delta & = \{x \in R \mid \forall n \in N\left(-\frac{1}{n+1} < x < \frac{1}{n+1}\right)\} & \text{(non-invertible infinitesimals)}
\end{array}
$$

Furthermore, we shall sometimes write

$$
\begin{array}{ll}
x > \infty \text{ for } \exists \varepsilon \in \mathbb{I}\ \varepsilon x > 1 & (x \text{ is infinitely large}) \\
x \simeq y \text{ for } x - y \in \mathbb{\Delta} & (x \text{ and } y \text{ are infinitely close}).
\end{array}
$$

3.2 Accessible Functions. A function $f \in R^{R^n}$ is *accessible* if for every multi-index α and every $x \in R_{\mathrm{acc}}$, $D^\alpha f(x) \in R_{\mathrm{acc}}$. Here are some elementary properties.

Proposition. (i) *If $f(x,t) = f(x_1, \ldots, x_{n-1}, t)$ is accessible, then so is $\int_0^1 f(x,t)dt$.*

(ii) *If f is accessible, then so are the "Fermat quotients" g_i, where*

$$f(x+y) - f(x) = \sum_{i=1}^{n} y_i g_i(x,y)$$

(this determines the g_i uniquely, see 2.3, 2.4).

(iii) *If f is accessible, then it maps accessible intervals to accessible intervals, i.e. $\forall a,b \in R_{\mathrm{acc}} \exists c,d \in R_{\mathrm{acc}} f([a,b]^n) \subset [c,d]$.*

(iv) *Every accessible function f is $\mathbb{\Delta}$-continuous in the sense that for every multi-index α,*

$$D^\alpha f(x+\varepsilon) \simeq D^\alpha f(x), \textit{ for all } x \in R^n_{\mathrm{acc}}, \varepsilon \in \mathbb{\Delta}^n.$$

Proof. (i) follows from (A8) and (A21′): if $n \in N - \mathbb{N}$, then $\forall x \in R^{n-1}_{\mathrm{acc}} \forall t \in [0,1](-n < f(x,t) < n)$. So by (A8), $-n < \int_0^1 f(x,t)dt < n$. Since n is arbitrary, $\int_0^1 f(x,t)dt \in R_{\mathrm{acc}}$ by (A21′). A similar argument applies to the α-th derivatives, by 2.4(5). (ii) follows from (i) and the formula $g_i(x,y) = \int_0^1 \frac{\partial f}{\partial x_i}(x+ty)dt$.

(iii) It suffices to take $[a,b]=[0,1]$. Then for all $x \in [0,1]^n$,

$$f(x)-f(0)=\int_0^{x_1}\cdots\int_0^{x_n}\frac{\partial^n f}{\partial x}(x)dx<\int_0^1\cdots\int_0^1(1+\left(\frac{\partial^n f}{\partial x}(x)\right)^2)dx$$

by (A8), and the fact that $a<1+a^2$ (here $\partial x=\partial x_1\ldots\partial x_n$ and $dx=dx_1\ldots dx_n$).

(iv) For $\alpha=0$, write $f(x+\varepsilon)-f(x)=\sum\varepsilon_i g_i(x,\varepsilon)$, and use (ii). For arbitrary α, a similar argument applies. □

3.3 Test Functions and Predistributions. A *test function* $f \in R^{R^n}$ is an accessible function with accessible support, i.e.

$$\exists m \in \mathbb{N}\ \forall x \in R^n (x \in [-m,m]^n \vee f(x)=0).$$

F_n denotes the space of test-functions.

A function $\varphi: R^n_{\mathrm{acc}} \to R$ defines an R-linear functional

$$\Delta_\varphi: F_n \to R$$

$$\Delta_\varphi(f)=\langle\varphi,f\rangle=\int\varphi f=\int_{-A}^{A}\cdots\int_{-A}^{A}\varphi(x)f(x)dx_1\ldots dx_n,$$

where $A \in R_{\mathrm{acc}}$ is so big that $\forall x \in R^n(f(x)=0 \vee x \in [-A,A]^n)$. Such a function $\varphi: R^n_{\mathrm{acc}} \to R$ is called a ***predistribution*** if Δ_φ is $\mathbb{A}$-*continuous* in the following sense: for all $f \in F_n$ such that $D^\alpha f(x) \simeq 0$ for all $\alpha \in \mathbb{N}^n$ and all $x \in R^n_{\mathrm{acc}}$ (or equivalently, all $x \in R^n$), also $\Delta_\varphi(f) \simeq 0$.

Remark. We note that we could equivalently take predistributions to be function $R^n \xrightarrow{\varphi} R$, on the basis of the following *extension principle*

(1) every function $R^n_{\mathrm{acc}} \to R$ can be extended to a function $R^n \to R$,

which is true in the models (see 3.13 below), although it doesn't follow from the axioms. Since we use these predistributions φ only to integrate functions of the form $\varphi \cdot f$, $f \in F_n$, however, only $\varphi | R^n_{\mathrm{acc}}$ is relevant.

The following properties are obvious.

Proposition. *The predistributions form a class which is closed under addition, multiplication by elements of F_n, derivation, integration, and tensor product (defined by $\varphi \otimes \psi(x,y)=\varphi(x)\psi(y)$).*

3.4 Distributions. Let $(R^{R^n})_a$ be the set of functions with accessible support

$$(R^{R^n})_a = \{f: R^n \to R \mid \exists m \in \mathbb{N}\ \forall x \in R^n (f(x) = 0 \vee x \in (-m,m)^n)\}.$$

A *distribution* on R^n is an R-linear map

$$(R^{R^n})_a \xrightarrow{\mu} R,$$

which is $\mathbb{A}$-continuous in the sense that

$$\forall x \in R^n\ \forall \alpha\ D^\alpha f(x) \simeq 0 \Rightarrow \mu(f) \simeq 0.$$

If f depends on more variables, say $f = f(x,y): R^n \times R^m \to R$, one writes $\mu_x(f): R^m \to R$ for the function $y \mapsto \mu(f(-,y))$.

Proposition. *Let μ be a distribution, and $f(x,y) \in (R^{R^n \times R^m})_a$.*

(i) $\frac{\partial}{\partial y_i}\mu_x(f(x,y)) = \mu_x\left(\frac{\partial f}{\partial y_i}(x,y)\right)$

(ii) *(Fubini)* $\int \mu_x f(x,y)dy = \mu_x \int f(x,y)dy$

(iii) *If ν is another distribution, with accessible support, i.e. $\forall f (f|[-A,A]^n \equiv 0 \to \nu(f) = 0)$ for some $A \in R_{\mathrm{acc}}$, then the convolution $\nu * \mu$ given by*

$$\nu * \mu = \nu_x(\mu_y(f(x+y)))$$

is well-defined, and again a distribution.

Proof. (ii) is clear from (i) and uniqueness in the integration axiom. For (i), take $n = 1$ for ease of notation, and write by linearity of μ_x,

$$\mu_x(f(x,y+d)) = \mu_x(f(x,y)) + d\mu_x\left(\frac{\partial}{\partial y}f(x,y)\right), \text{ all } d \in D.$$

Also $\forall d \in D\ \mu_x(f(x,y+d)) = \mu_x(f(x,y)) + d\frac{\partial}{\partial y}\mu_x(f(x,y))$ by definition, and (i) follows.

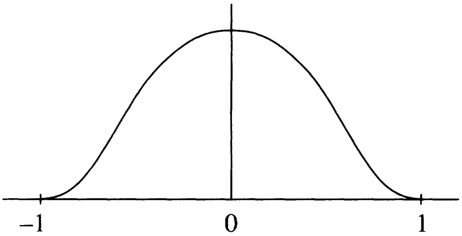
For (iii), note first that $y \mapsto f(x,y)$ has accessible support for $x \in R^n_{\mathrm{acc}}$; so $g(x) = \mu_y(f(x+y))$ is defined as a function $R^n_{\mathrm{acc}} \to R$. By (1) of 3.3, we can find a function $\tilde{g}$ with accessible support, such that $g \equiv \tilde{g}$ on $[-A,A]^n$. Then $\nu(\tilde{g})$ is defined, and does not depend on the choice of $\tilde{g}$. By abuse of notation, we write $\nu(g) = \nu_x(\mu_y(f(x+y))$ for $\nu(\tilde{g})$. It is immediate from (i) that $\nu * \mu$ is $\mathbb{A}$-continuous. □

3.5 The Dirac Distribution. This is the distribution

$$\delta: (R^{R^n})_a \to R, \delta(f) = f(0).$$

We will show that δ is, "up to an infinitesimal bit", given by a predistribution. For notational convenience, we take $n = 1$. By

$(A18)$, we can construct a bump-function $\delta(x): R \to R$



with support contained in $(-1,1)$, and with

$$\int_{-1}^{1} \delta(x)dx = 1$$

(and we can assume that δ is even and positive, when necessary). For $\varepsilon > 0$ let $\delta_\varepsilon: R \to R$ be the function

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon}\delta\left(\frac{x}{\varepsilon}\right).$$

Then $\mathrm{supp}(\delta_\varepsilon) \subset (-\varepsilon, \varepsilon)$, and $\int \delta_\varepsilon = \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon = 1$. We claim that

(1) $\forall \varepsilon \in \mathbb{I}, \varepsilon > 0\, \forall f \in F_1\ \delta(f) \simeq \int f(x)\delta_\varepsilon(x)dx$.

Proof of (1). Since f is accessible, $\forall n \in \mathbb{N}\ \forall x \in (-\varepsilon,\varepsilon)(-\frac{1}{n} < f(x) - f(0) < \frac{1}{n})$. So if we take $\delta_\varepsilon \geq 0$, we have $\frac{\delta_\varepsilon}{n} \leq \delta_\varepsilon f - \delta_\varepsilon f(0) \leq \frac{\delta_\varepsilon}{n}$ for all $n \in \mathbb{N}$. So by (A 8′),

$$\int_{-\varepsilon}^{\varepsilon} \frac{-\delta_\varepsilon(x)}{n}dx \leq \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x)f(x)dx - \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x)f(0)dx \leq \int_{-\varepsilon}^{\varepsilon} \frac{\delta_\varepsilon(x)}{n}dx.$$

Since $\int \delta_\varepsilon = 1$, we get $f(0) - \frac{1}{n} \leq \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon f \leq f(0) + \frac{1}{n}$, for all $n \in \mathbb{N}$. So $\int \delta_\varepsilon f = \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon f \simeq f(0)$.

The case of (1) for n variables is similar, using a bump function $\delta_\varepsilon: R^n \to R$. By integration by parts it follows that for all $f \in F_n$, all $\alpha \in \mathbb{N}^n$, and all $x \in R^n_{\mathrm{acc}}$

(2) $D^\alpha_x f(x) \simeq \int D^\alpha_x f(y)\delta_\varepsilon(x-y)dy$. □

We will come back to the properties of δ in 3.9 and 3.11.

3.6 Main Theorem. *("every distribution is an integral") For every distribution μ on R^n there exists a predistribution $\mu_0: R^n_{\mathrm{acc}} \to R$ such* *that for all $f \in F_n$,*

$$\mu(f) \simeq \Delta_{\mu_o}(f) := \int f(x)\mu_o(x)dx.$$

The function μ_o is said to represent the distribution μ.

Proof. Define $\mu_o(y) = \mu_x(\delta_\epsilon(x-y))$. If $f \in F_n$, then by 3.5(2) and Fubini (3.4(ii)),

$$\mu(f) \simeq \mu_x \int f(y)\delta_\epsilon(x-y)dy = \int f(y)\mu_x(\delta_\epsilon(x-y))dy = \Delta_{\mu_o}(f). \ \square$$

3.7 Polynomial Representation. We will now show that $\mu_o: R^n_{\mathrm{acc}} \to R$ in Theorem 3.6 can in fact be taken to be (the restriction of) a *polynomial* function $R^n \to R$. As a first step, we prove the following representation for δ. (Here $\frac{\pi}{2}$ is defined to be the first zero of $\cos x$ which is ≥ 0; this makes sense by 2.13.)

3.7.1 Theorem. *(Fourier integral representation of the Dirac functional). For every $n > \infty$ and every $f \in F_1$*

$$\int \frac{\sin nx}{\pi x} f(x)dx \simeq f(0).$$

Proof. First of all, we prove that

(1) $\int_0^n \frac{1-\cos y}{y^2}dy \simeq \frac{\pi}{2}$, for all $n > \infty$.

Since the function $\cos y$ is accessible, iterated application of proposition 3.2 (ii) gives $\int_0^1 \frac{1-\cos y}{y^2}dy \in R_{\mathrm{acc}}$. So, writing $\varphi(y) = \frac{1-\cos y}{y^2}$,

$$-1 < \int_0^n \varphi = \int_0^1 \varphi + \int_1^n \varphi < \int_0^1 \varphi + \left(1 - \frac{1}{n}\right) \in R_{\mathrm{acc}},$$

i.e.

(2) $\int_0^n \frac{1-\cos y}{y^2}dy \in R_{\mathrm{acc}}, \forall n \in N$.

The following formula is easily checked by taking derivatives on both sides with respect to A:

(3) $\int_0^A e^{-ax}\cos bx dx = \frac{e^{-aA}(-a\cos bA + b\sin bA)}{a^2+b^2} + \frac{a}{a^2+b^2}$.

If $a, b > 0$, then $\frac{-a\cos bA + b\sin bA}{a^2+b^2} < \frac{a+b}{a^2+b^2} < \frac{a+b}{a^2}$, so (3) gives

$$(4) \quad \begin{aligned} \frac{a}{a^2+b^2} - \frac{a+b}{a^2}e^{-aA} &< \int_0^A e^{-ax}\cos bx dx \\ &< \frac{a}{a^2+b^2} + \frac{a}{a^2+b^2}e^{-aA} \ (a, b > 0). \end{aligned}$$

Integrating twice with respect to b, between 0 and b, we obtain ("up to an infinitesimal", since (4) only holds for $b > 0$:)

(5) $b\arctan\frac{b}{a} - \frac{a}{2}\log(1+\left(\frac{b}{a}\right)^2) - \frac{1}{2}\left(\frac{b^2}{a}+\frac{b^3}{3a^2}\right)e^{-aA}$
$< \int_0^A e^{-ax}\frac{1-\cos bx}{x^2}dx$
$< b\arctan\frac{b}{a} - \frac{a}{2}\log(1+\left(\frac{b}{a}\right)^2) + \frac{1}{2}\left(\frac{b^2}{a}+\frac{b^3}{3a^2}\right)e^{-aA}$,

and change of variables ($y = bx$) will give

(6) $\arctan\frac{b}{a} - \frac{1}{2}\frac{\log(1+\left(\frac{b}{a}\right)^2)}{\left(\frac{b}{a}\right)} - \frac{1}{2}\left(\frac{b}{a}+\frac{1}{3}\left(\frac{b}{a}\right)^2\right)e^{-aA}$
$< \int_0^{bA} e^{-\frac{a}{b}y}\frac{1-\cos y}{y^2}dy$
$< \arctan\frac{b}{a} - \frac{1}{2}\frac{\log(1+\frac{b}{a})^2}{\left(\frac{b}{a}\right)} + \frac{1}{2}\left(\frac{b}{a}+\frac{1}{3}\left(\frac{b}{a}\right)^2\right)e^{-aA}$.

Now let $b \in \mathbb{I}, b > 0$, and let $a > 0$ be so small that $\frac{b}{a} > \infty$, and A so large that $bA > \infty$ and $\left(\frac{b}{a}+\frac{1}{3}\left(\frac{b}{a}\right)^2\right)e^{-aA} \simeq 0$. Then, writing $\alpha = \frac{a}{b}$, and $m > \infty$ with $\alpha m > \varepsilon \in \mathbb{I}$, the general inequality $1 - x < e^{-x} < 1$ $(x > 0)$ gives

(7) $1 - \varepsilon < 1 - \alpha y < e^{-\alpha y} < 1.$

Multiplying by $\frac{1-\cos y}{y^2}$ and integrating, we get

(8) $(1-\varepsilon)\int_0^m \frac{1-\cos y}{y^2}dy < \int_0^m e^{-\alpha y}\frac{1-\cos y}{y^2}dy < \int_0^m \frac{1-\cos y}{y^2}dy.$

Since $\int_0^m \frac{1-\cos y}{y^2}dy \in R_{\mathrm{acc}}$ (cf. (2)), (8) gives

(9) $\int_0^m \frac{1-\cos y}{y^2}dy \simeq \int_0^m e^{-\alpha y}\frac{1-\cos y}{y^2}dy,$

and since clearly $\int_0^m e^{-\alpha y}\frac{1-\cos y}{y^2}dy \simeq \int_0^{bA} e^{-\alpha y}\frac{1-\cos y}{y^2}dy$, we conclude that

(1) $\int_0^x \frac{1-\cos y}{y^2}dy \simeq \int_0^m \frac{1-\cos y}{y^2}dy \simeq \frac{\pi}{2}.$

Next, we note that it follows from (1) that

(10) for all $n > \infty$ $\int_{-n}^n \frac{\sin x}{x}dx \simeq \int_{-n}^n \left(\frac{\sin x}{x}\right)^2 dx \simeq \pi.$

Indeed, the case of $\frac{\sin x}{x}$ follow by integration by parts, since $\frac{\sin x}{x}$ is even. For the other case, one uses $1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$.

Finally, we complete the proof of the theorem. Let $f \in F_1$, with $\mathrm{supp}(f) \subset (-A, A)$ say, where $A \in R_{\mathrm{acc}}$, and write $f(x) = f(0) + xg(x)$ by Hadamard's lemma. Then by (10),

(11) $\int_{-A}^A \frac{\sin nx}{\pi x}f(0)dx = f(0)\int_{-nA}^{nA}\frac{\sin y}{\pi y}dy \simeq f(0)$ $(n > \infty)$,

since $nA > \infty$ if $A \in R_{\mathrm{acc}}$. Moreover, since g is an accessible function, $\int_{-A}^{A} \sin nx\, g(x)dx$ is infinitely close to both $\int_{-A}^{A-\frac{\pi}{n}} \sin nx\, g(x)dx$ and $\int_{-A+\frac{\pi}{n}}^{A} \sin\, nx\, g(x)dx$ $(n > \infty)$, so

(12) $2\int_{-A}^{A} \sin\, nx\, g(x)dx \simeq \int_{-A}^{A-\frac{\pi}{n}} \sin\, nx\, g(x)dx + \int_{-A+\frac{\pi}{n}}^{A} \sin nx\, g(x)dx.$

Substituting $x+\frac{\pi}{n}$ for x in the first summand, we can rewrite this sum as $\int_{-A}^{A-\frac{\pi}{n}} (g(x) - g(x+\frac{\pi}{n})) \sin nx\, dx$, which is in $\mathbb{A}$ by $\mathbb{A}$-continuity of g (3.2(iv)).

Putting (11) and (12)$\simeq 0$ together, we conclude that for $n > \infty$

$$\int_{-A}^{A} \frac{\sin nx}{x} f(x)dx = \int_{-A}^{A} \frac{\sin nx}{x} f(0)dx + \int_{-A}^{A} \sin nx\, g(x) \simeq f(0).$$

For the next corollary, we assume the extension principle (1) mentioned in the remark in 3.3 (cf. also 3.13). □

3.7.2 Corollary. *(improvement of 3.6) For any distribution μ on R^n there is a polynomial function $R^n \xrightarrow{p} R$ such that*

$$\forall f \in F_n\ \mu(f) \simeq \Delta_p(f) = \int pf.$$

Proof. Again, we only do the case $n = 1$. The usual Taylor development of $\frac{\sin nx}{x}$ gives for any $n, m \in N, \varepsilon > 0$, and $K \in R_{>0}$ a $k \in N$ such that

$$|D^{(i)} \tfrac{\sin nx}{\pi x} - D^{(i)} P_k(x)| < \varepsilon \text{ for all } x \in [-K, K], \text{ all } i < m,$$

where

$$P_k(x) = \sum_{j=0}^{k} (-1)^j \frac{n^{2j+1}}{(2j+1)!} x^{2j}.$$

In particular, choosing $n, m > \infty$ and $\varepsilon \in \mathbb{I}$ we obtain a $k \in N$ with $D^{(i)} \frac{\sin nx}{\pi x} \simeq D^{(i)} P_k(x)$ for all $x \in R_{\text{acc}}$ and all $i \in \mathbb{N}$; and more generally,

(1) $D^{(i)} \frac{\sin n(x-y)}{\pi(x-y)} \simeq D^{(i)} P_k(x-y)$, for $x, y \in R_{\text{acc}}, i \in \mathbb{N}$.

Now let $p(y)$ be the *polynomial*

$$p(y) = \int_{-n}^{n} P_k(x-y) \tilde{\mu}_0(x)dx,$$

where $\mu_o \colon R_{\text{acc}} \to R$, $\mu_o(x) = \mu_y(\delta_\varepsilon(x-y))$ as in 3.5, and $\tilde{\mu}_o$ is any extension $R \to R$ of μ_o, while n is some infinite integer (for example the same n as in (1)). Then if $f \in F_1$, with $\mathrm{supp}(f) \subset (-a, a)$ say, we have

$$\begin{array}{rcll} \int_{-a}^{a} f(y)p(y)dy & = & \int_{-a}^{a} f(y) \int_{-n}^{n} P_k(x-y)\tilde{\mu}_o(x)dxdy & \\ & = & \int_{-n}^{n} \tilde{\mu}_o(x) \int_{-a}^{a} f(y)P_k(x-y)dxdy & \text{(Fubini)} \\ & \simeq & \int_{-a}^{a} \mu_o(x)f(x)dx & (3.7.1) \\ & \simeq & \mu(f) & (3.5) \end{array}$$

$\square$

3.8 Equivalence of Predistributions. For a predistribution $\varphi\colon R^n_{\mathrm{acc}} \to R$, we are interested in the functional $\Delta_\varphi\colon F_n \to R$, rather than in the function φ itself. Thus, it is natural to introduce an equivalence relation $\sim$ on predistributions:

$$\varphi \sim \psi \text{ iff } \Delta_\varphi = \Delta_\psi, \text{ i.e. iff } \forall f \in F_n \langle \varphi, f\rangle = \langle \psi, f\rangle.$$

Proposition. *$\sim$ is an equivalence relation, compatible with addition, multiplication by elements of F_n, derivation, integration, tensor product, and convolution.*

Proof. For tensor product, let $\varphi\colon R^r_{\mathrm{acc}} \to R$ and $\nu\colon R^m_{\mathrm{acc}} \to R$ be predistributions, and assume $\varphi \sim 0$. Choose $f(x, y) \in F_{n+m}$, and let $h(y) = (\Delta_\varphi)_x(f) = \int \varphi(x)f(x, y)dx$. Then $D^\alpha h(y) = \int \varphi(x) D^\alpha_y f(x, y)dx \simeq 0$ for all α and all $y \in R^m$, so by $\mathbb{A}$-continuity of Δ_ν,

$$0 \simeq \Delta_\nu(h) = \Delta_{\varphi\otimes\nu}(f).$$

For convolution, it suffices to note that by Fubini (for integrals) $\Delta_{\varphi*\nu}(f(x)) = \Delta_{\varphi\otimes\nu}(f(x+y))$ (where we now take $m = n$ of course). $\square$

Notation. When no confusion is likely to occur, we just write

$$\varphi = \psi \text{ for } \varphi \sim \psi,$$

as is done in physics textbooks.

3.9 Elementary Properties of Dirac's δ-Function. As a first example, we show that the usual properties of the δ-function hold in our framework (cf. Schiff (1968)). Below, δ stands for a *predistribution* corresponding to the Dirac functional $\delta(f) = f(0)$ (cf. 3.5), and $=$ is used for $\sim$. The following "equations" holds:

(i) $\delta(x) = \delta(-x)$
(ii) $\delta'(x) = -\delta'(-x)$
(iii) $x\delta(x) = 0$
(iv) $x\delta'(x) = -\delta(x)$

(v) $\delta(ax) = a^{-1}\delta(x)$, for $a > 0, a \not\simeq 0$
(vi) $\delta(x^2 - a^2) = (2a)^{-1}(\delta(x-a) + \delta(x+a))$, for a as in (v)
(vii) $\int \delta(a-x)\delta(x-y)dx = \delta(a-y)$
(viii) $f(x)\delta(x-a) = f(a)\delta(x-a)$, for all $f \in F$.

Proof. Choose a representation $\delta_\varepsilon(\varepsilon \in \mathbb{I}, \varepsilon > 0)$ for δ as in 3.5, with δ_ε even. Then (i)–(viii) reduces to trivial calculations. As an example, we do (vii). Interpret both sides as functions of y, with a parameter a. If $f(y)$ is a test function, then by Fubini, $\int f(y) \int \delta(a-x)\delta(x-y)dxdy = \int \delta(a-x)(\int f(y)\delta(x-y)dy)dx = \int \delta(a-x)f(x)dx = \int \delta(a-y)f(y)dy$. so $\delta(a-x)\delta(x-y) \sim \delta(a-y)$.□

3.10 Sochozki's Formula. This formula is usually written as

(1) $\frac{i}{x+0i} = \pi\delta(x) + iP\left(\frac{1}{x}\right)$.

Here $P\left(\frac{1}{x}\right)$ is the principal value, or finite part functional, defined by

(2) $P\left(\frac{1}{x}\right)(f) = \int_{-A}^{A} g(x)dx$

where g is given by $f(x) = f(0) + xg(x)$, and $\mathrm{supp}(f) \subset (-A, A)$. (Classically, one usually defines

$$P\left(\frac{1}{x}\right)(f) = \lim_{\varepsilon \to 0}\left(\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x}dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x}dx\right),$$

but it is clear that these definitions are equivalent (or rather, infinitesimally close).) We now interpret (1) as

(3) $\frac{i}{x+\varepsilon i} \sim \pi\delta(x) + iP\left(\frac{1}{x}\right)$, for all $\varepsilon \in \mathbb{I}, \varepsilon > 0$.

To prove (3), write $\frac{i}{x+\varepsilon i} = \pi\frac{\varepsilon}{\pi(x^2+\varepsilon^2)} + \frac{ix}{x^2+\varepsilon^2}$. It suffices to show that $\frac{x}{x^2+\varepsilon^2}$ is a presentation of $P\left(\frac{1}{x}\right)$, and $\frac{\varepsilon}{\pi(x^2+\varepsilon^2)}$ one of $\delta(x)$. For the first, take a test function f, with $\mathrm{supp}(f) \subset (-A, A)$, and write $f(x) = f(0) + xg(x)$; then

$$\begin{aligned} P\left(\tfrac{1}{x}\right)(f) &:= \int_{-A}^{A} g(x)dx \simeq \int_{-A}^{A} \tfrac{x^2 g(x)}{x^2+\varepsilon^2}dx \\ &= \int_{-A}^{A} \tfrac{xf(x)}{x^2+\varepsilon^2}dx - \int_{-A}^{A} \tfrac{xf(0)}{x^2+\varepsilon^2}dx \\ &= \int_{-A}^{A} \tfrac{xf(x)}{x^2+\varepsilon^2}dx, \end{aligned}$$

since clearly $\int_{-A}^{A} \frac{x}{x^2+\varepsilon^2}dx = 0$.

To see that $\frac{\varepsilon}{\pi(x^2+\varepsilon^2)}$ represents δ, take f, g as above, and write

$$\begin{aligned}
\int_{-A}^{A} \frac{\varepsilon f(x)}{\pi(x^2+\varepsilon^2)} dx &= \frac{1}{\pi} \int_{-A}^{A} \frac{\varepsilon f(0)}{x^2+\varepsilon^2} dx + \frac{1}{\pi} \int_{-A}^{A} \frac{\varepsilon x g(x)}{x^2+\varepsilon^2} dx \\
&= \frac{1}{\pi} f(0) 2 \arctan\left(\frac{A}{\varepsilon}\right) + \frac{\varepsilon}{\pi} P\left(\frac{1}{x}\right)(g) \\
&\simeq f(0),
\end{aligned}$$

since for $\varepsilon \in \mathbb{I}, \varepsilon > 0, \arctan\left(\frac{A}{\varepsilon}\right) \simeq \frac{1}{2}\pi$, and $\frac{\varepsilon}{\pi} P\left(\frac{1}{x}\right)(g) \simeq 0$.

3.11 The Square Root of δ. As a last example, we show how the "square root of δ" may be justified in our context. To this end, we need to assume some extra properties of the bump-function δ_ε representing δ. We assume there is a predistribution $\delta_\varepsilon, \delta_\varepsilon$ even, $\delta_\varepsilon \geq 0$, with

$$\begin{gathered}
\operatorname{supp}(\delta_\varepsilon) \subset (-\varepsilon, \varepsilon) \\
\int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x) dx = 1 \\
\int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon^2 x dx = \delta_\varepsilon(0)
\end{gathered}$$

(In the models, such a δ_ε can always be found; cf. 3.14). The "square root of δ" can now be defined as

$$\delta_\varepsilon^{\frac{1}{2}}(x) = \frac{\delta_\varepsilon(x)}{\sqrt{\delta(0)}}.$$

We have the following properties:

(i) $\delta_\varepsilon^{\frac{1}{2}} \cdot \delta_\varepsilon^{\frac{1}{2}} \sim \delta_\varepsilon \quad (\text{"}\sqrt{\delta} \cdot \sqrt{\delta} = \delta\text{"})$

(ii) $\delta_\varepsilon^{\frac{1}{2}} \otimes \delta_\varepsilon^{\frac{1}{2}} = [(x,y) \mapsto \delta_\varepsilon^{\frac{1}{2}}(x) \delta_\varepsilon^{\frac{1}{2}}(y)] \sim [(x,y) \mapsto \delta_\varepsilon(x-y) \frac{\delta_\varepsilon(x)}{\delta_\varepsilon(0)}]$.

The proofs of (i) and (ii) follow immediately from the fact that $\frac{\delta_\varepsilon^2(x)}{\delta_\varepsilon(0)}$ is another representation of the distribution δ, and the "equality"

$$\delta_\varepsilon \otimes \delta_\varepsilon = [(x,y) \mapsto \delta_\varepsilon(x)\delta_\varepsilon(y)] \sim [(x,y) \mapsto \delta_\varepsilon(x-y)\delta_\varepsilon(x)].$$

It should be noted that $\delta_\varepsilon^{\frac{1}{2}}$ is *not* a classical distribution. Accordingly, this notion of "square root of δ" can not be dealt with directly in the theory of distributions.

3.12 Riemann Sums (Digression). In non-standard analysis, the main tool in the theory of integration and distribution is the fact that every integral "is" a Riemann sum. Although our integration axiom makes this representation by Riemann sums unnecessary (at least for the purposes of this chapter), it may be of interest to point out that a similar "equality" holds in our context.

Theorem. *Let $f(x,t)\colon R^n \times [0,1] \to R$, and let $K \subset R^n$ be compact.*

Then there is an $m_o \in N$ *such that for all* $\alpha \in \mathbb{N}^n$

(1) $\forall x \in K\, \forall m > m_o\; D_x^\alpha \int_0^1 f(x,t)dt \simeq D_x^\alpha \sum_{i=0}^{m-1} \frac{1}{m} f(x, \frac{i}{m})$.

Proof. We first prove a simple special case: let $g\colon [0,1] \to R$. We claim that

(2) $\exists n \in N\, \forall m > n \int_0^1 g(t)dt \simeq \sum_{i<m} \frac{1}{m} f\left(\frac{i}{m}\right)$.

To prove (2), note first that for any $m \in N - \{0\}$,

$$
\begin{aligned}
\int_0^1 g(t)dt &= \sum_{i<m} \int_{i/m}^{i+1/m} g(t)dt \\
&= \sum_{i<m} \int_{i/m}^{i+1/m} g(i/m)dt \\
&+ \sum_{i<m} \int_{i/m}^{i+1/m} (g(t) - g(i/m))dt
\end{aligned}
\tag{3}
$$

(by linearity of $\int$ and the recursion axiom A14). Now choose $\varepsilon \in \mathbb{I}, \varepsilon > 0$. By uniform continuity of g, there is an $n \in N$ such that

$$|s - t| < \frac{1}{n} \Rightarrow |g(s) - g(t)| < \varepsilon.$$

So if $m > n$, the last summand in (3) is dominated by $\sum_{i<m} \int_{i/m}^{i+1/m} \varepsilon dt$ $2\varepsilon \in \mathbb{I}$, and hence (2) follows from (3).

To prove (1), note first that by (2),

(4) $\forall k \in N\, \forall x \in R^n\, \exists m_o \in N\, \forall m > m_o\, \forall \alpha, |\alpha| < k : D_x^\alpha$
$\int_0^1 f(x,t)dt \simeq D_x^\alpha \sum_{i<m} \frac{i}{m} f(x, \frac{i}{m})$.

Now fix $k \in N - \mathbb{N}$. Then (1) follows from (4) by the following lemma. □

Lemma. *Let* $K \subset R^n$ *be a compact space, let* $\varphi(x,n)$ *be a formula in* x *and suppose* $\forall x \in R^n\, \exists n\, \forall m > n\; \varphi(x,m)$. *Then* $\exists n \in N\, \forall x \in K\, \forall m > n\; \varphi(x,m)$.

Proof. This follows easily from the open refinement property (2.9.2) and bounded search (A15). □

We now turn to the models $\mathcal{B}$ and $\mathcal{Z}$. First of all, we verify the two new "axioms" used in this section, namely the extension principle (cf. 3.3) and the existence of a function δ_ε with the properties as in 3.11.

3.13 The Extension Principle in $\mathcal{B}$ and $\mathcal{Z}$. It has to be shown that the natural restriction map

$$R^{R^n} \to R^{R^n_{\text{acc}}}$$

is a surjection in $\mathcal{B}$ and $\mathcal{Z}$. In fact, the following argument shows that it is already a surjection in $Sets^{\mathbb{L}^{op}}$. For ease of notation, we only do $n = 1$. Let $\ell A \in \mathbb{L}, A = C^\infty(\mathbb{R}^d)/I$, and assume we are given an $F \in R^{R_{\text{acc}}}(\ell A)$. So for each $n \in \mathbb{N}, F$ restricts to a map

$$\ell A \times s(-n, n) \xrightarrow{F_n} R.$$

Let $F_n(x, t): \mathbb{R}^d \times (-n, n) \to \mathbb{R}$ represent F_n. Then since all the F_n are restrictions of F, we have for $n < m$

$$F_m|\mathbb{R}^d \times (-n, n) - F_n \in (I(x)) \subset C^\infty(\mathbb{R}^d \times (-n, n)).$$

Let $(\rho_n(t))_n$ be a partition of unity of $\mathbb{R}$ with $\text{supp}(\rho_n) \subset (-n, n)$, for $n > 0$, and let $G: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be the function

$$G(x, t) = \sum_{i=1}^{\infty} \rho_i(t) F_i(x, t).$$

G represents a map $\ell A \times R \to R$, and we claim that G extends F, i.e. that the following diagram commutes in $Sets^{\mathbb{L}^{op}}$

$$\begin{tikzcd}
\ell A \times R_{\text{acc}} \arrow[r, "F"] \arrow[d] & R \\
\ell A \times R \arrow[ur, "G"'] &
\end{tikzcd}$$

To prove this, it suffices to show for each n that

(1) $F_n - G(x, t)|\mathbb{R}^d \times (-n, n) \in (I(x)) \subset C^\infty(\mathbb{R}^d \times (-n, n))$.

But if m is so big that $\rho_i(t)$ vanishes on $(-n, n)$ for $i > m$, then on $\mathbb{R}^d \times (-n, n)$,

$$F_n(x, t) - G(x, t) = \sum_{i=1}^{m} \rho_i(t)(F_n(x, t) - F_i(x, t)).$$

Since $\text{supp}(\rho_i(t)) \subset (-i, i)$ and $F_n(x, t)|\mathbb{R}^d \times (-i, i) - F_i(x, t) \in (I(x)) \subset C^\infty(\mathbb{R}^d \times (-i, i))$, it is clear that (1) holds.

Next, we verify the existence in $\mathcal{B}$ of a function $\delta_\varepsilon : R \to R$ as in 3.11.

3.14 Existence of a δ-Function in $\mathcal{B}$. It suffices to show that there is a function

$$\delta : \mathbb{R} \to \mathbb{R}$$

with $\delta \geq 0$, δ even, $\mathrm{supp}(\delta) \subset (-1, 1)$, and

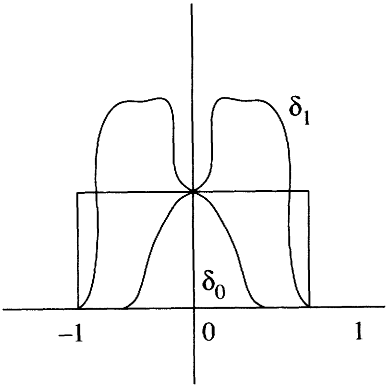
$$\int_{-1}^{1} \delta(x)dx = 1, \int_{-1}^{1} \delta(x)^2 dx = \delta(0).$$

For, given such a function, we can define

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon} s(\delta) \left(\frac{x}{\varepsilon}\right)$$

in $\mathcal{B}$, where $\varepsilon \in \mathbb{I}, \varepsilon > 0$, and $s(\delta) : R \to R$ is the map in $\mathcal{B}$ corresponding to δ.

To see that such a function δ exists, take two functions δ_o and δ_1 (both even, ≥ 0, with value 1 at 0, and support in $(-1, 1)$), such that $\int_{-1}^{1} \delta_0^2(x)dx < \int_{-1}^{1} \delta_0(x)dx$ and $\int_{-1}^{1} \delta_1^2(x)dx > \int_{-1}^{1} \delta_1(x)dx$, and assume moreover that both δ_0 and δ_1 are constant in a neighbourhood of 0.



Let $\delta_t(x) = (1 - t)\delta_0(x) + t\delta_1(x)$. Then by the mean-value theorem there exists a $t_o \in [0, 1]$ with

$$\int_{-1}^{1} \delta_{t_o}^2(x)dx = \int_{-1}^{1} \delta_{t_o}(x)dx = A, \text{ say.}$$

Now let

$$\delta(x) = \frac{1}{A} \delta_{t_o}(x).$$

Then $\int_{-1}^{1}\delta(x)dx = 1$, and $\int_{-1}^{1}\delta^2(x)dx = \frac{1}{A} = \delta(0)$.

3.15 Embedding of Distributions in $\mathcal{B}$ and $\mathcal{Z}$. We aim to show a correspondence between external distributions on $\mathbb{R}^n$, and global sections of the sheaf of internal distributions (with or without compact support).

It follows from II.3.6 that there is a bijective correspondence

$$(1) \qquad \frac{R^{R^n} \xrightarrow{\nu} R}{C^\infty(\mathbb{R}^n) \xrightarrow{\mu} \mathbb{R}}$$

between R-linear maps $R^{R^n} \to R$ in $\mathcal{B}$ and continuous $\mathbb{R}$-linear functionals $C^\infty(\mathbb{R}^n) \to \mathbb{R}$, i.e. distribution on $\mathbb{R}^n$ with compact support, in *Sets*. Recall that the correspondence was defined as follows. Given $R^{R^n} \xrightarrow{\nu} R$ in $\mathcal{B}$, one simply puts $\mu = \Gamma\nu$. Conversely, given μ, one defines a natural transformation ν by components

$$\nu_{\ell A}: R^{R^n}(\ell A) \to R(\ell A),$$
$$\nu_{\ell A}(F(x,y)) = \mu_y(F(x,y)),$$

where for $A = C^\infty(\mathbb{R}^d)/I$, $\mathbb{R}^d \times \mathbb{R}^n \xrightarrow{F(x,y)} \mathbb{R}$ represents an element F in $R^{R^n}(\ell A)$. Given an external distribution μ, we will write $s(\mu)$ for this unique R-linear map $\nu: R^{R^n} \to R$ in $\mathcal{B}$ with $\Gamma\nu = \mu$.

3.15.1 Lemma. *Let $C^\infty(\mathbb{R}^n) \xrightarrow{\mu} \mathbb{R}$ be a distribution with compact support on $\mathbb{R}^n$ (i.e. μ is continuous and linear), and let $s(\mu)$ be the corresponding R-linear map $R^{R^n} \to R$ in $\mathcal{B}$. Then $s(\mu)$ is Δ-continuous (in the sense of 3.4).*

Proof. Again, we just do the case $n = 1$. Let $A = C^\infty(\mathbb{R}^d)/I$, and let $\ell A \times R \xrightarrow{f} R$ be a function at stage ℓA, represent by $F(x,t): \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$. Suppose

(1) $\ell A \Vdash \forall t \in R\, \forall n \in \mathbb{N}\; D^{(n)}f(t) \in \Delta$.

We need to show that $\ell A \Vdash s(\mu)(f) \in \Delta$, i.e. that for each $n \in \mathbb{N} - \{0\}$, $\ell A \Vdash -\frac{1}{n} < s(\mu)(f) < \frac{1}{n}$. Since $\forall x \in R \left(\frac{1}{n+1} < x \vee x < \frac{1}{n}\right)$ is valid in $\mathcal{B}$ (R is an ordered local ring), it suffices to show that for a given $n_o \in \mathbb{N}, n_o > 0$, $\ell A \Vdash \neg(|s(\mu)(f)| > \frac{1}{n_o})$.

Suppose $\ell A \Vdash |s(\mu)(f)| > \frac{1}{n_o}$. Then there is a finitely generated subideal $I_o \subset I$ such that

(2) $\forall x \in Z(I_o)(|\mu_t(F(x,t))| > \frac{1}{n_o})$.

On the other hand, continuity of μ gives a neighbourhood

$$V = \{g: \mathbb{R} \to \mathbb{R} | \forall i < k \, \forall x \in K \, |g^{(i)}(x)| < \varepsilon\},$$

where $k \in \mathbb{N}, \varepsilon > 0, K \subset \mathbb{R}$ compact, such that $\mu(V) \subset \left(-\frac{1}{n_o}, \frac{1}{n_o}\right)$. By (1), we have

(3) $\ell A \Vdash \forall i < k \, \forall t \in \mathbb{R} |D^{(i)} f(t)| < \varepsilon,$

and from this it follows that there is a finitely generated $I_1 \subset I$ such that (3) holds with ℓA replaced by $\ell A_1, A_1 = C^\infty(\mathbb{R}^d)/I_1$; in other words, such that

(4) $\forall x \in Z(I_1) \forall t \in \mathbb{R} \, |D_t^{(i)} F(x,t)| < \varepsilon.$

(4) implies $\forall x \in Z(I_1) \, |\mu_t(F(x,t))| < \frac{1}{n_o}$ by choice of V, contradicting (2). Thus $\ell A \not\Vdash |s(\mu)(f)| > \frac{1}{n_o}$ (provided A is non-trivial).

Exactly the same argument will show that for any $\overline{B} \xrightarrow{\alpha} \overline{A}$, ($B$ non-trivial) $\ell B \not\Vdash |s(\mu)(f|\alpha)| > \frac{1}{n_o}$. So $\ell A \Vdash \neg(|s(\mu)(f)| > \frac{1}{n_o})$, as was to be shown. □

Let $C_c^\infty(\mathbb{R}^n)$ denote the set of smooth functions with compact support. There is a correspondence

$$(2) \qquad \frac{F_n \xrightarrow{\nu} R}{C_c^\infty(\mathbb{R}^n) \xrightarrow{\mu} \mathbb{R}}$$

between distributions μ in *Sets* and R-linear maps $F_n \xrightarrow{\nu} R$ in $\mathcal{B}$. This is proved just as the correspondence (1), using the following lemma.

3.15.2 Lemma. (i) *If $f \in C_c^\infty(\mathbb{R}^n)$, then $s(f): R^n \to R$ is in F_n.*

(ii) *If $f \in F_n(\ell A)$, where $A = C^\infty(\mathbb{R})/I$, then f can be represented by a function $f(x,y): \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^d$, $f(x,-) \in C_c^\infty(\mathbb{R}^n)$ (in fact $\exists N \in \mathbb{N} \, \forall x \in \mathbb{R}^d \operatorname{supp} f(x,-) \subset [-N,N]^n$).*

Proof. (i) It is clear that $s(f)$ has accessible support. To see that $s(f)$ is Δ-continuous, use uniform continuity of f to find a function $\mathbb{N} \xrightarrow{\varphi} \mathbb{N}$ with

$$\forall x,y \in \mathbb{R}^n (|x-y| < \frac{1}{\varphi(m)+1} \to |f(x) - f(y)| < \frac{1}{m+1}).$$

Then clearly $s(\varphi)\colon N \to N$ has the same property in $\mathcal{B}$, i.e.

$$\mathcal{B} \models \forall m \in N\, \forall x, y \in R^n(|x-y| < \tfrac{1}{s\varphi(m)+1} \to |s(f)(x) - s(f)(y)| < \tfrac{1}{m+1}).$$

Reasoning in $\mathcal{B}$, we note that since $s(\varphi)$ maps $\mathbb{N} \subset N$ into $\mathbb{N}$, is is clear that for $\varepsilon \in \Delta$ and $m \in \mathbb{N}$, $|f(x+\varepsilon) - f(x)| < \frac{1}{m+1}$. This being true for all $m \in \mathbb{N}$, we have $f(x+\varepsilon) - f(x) \in \Delta$.

The corresponding fact for $D^\alpha(f)$ follows, since D^α commutes with $s\colon \mathbb{M} \hookrightarrow \mathcal{B}$ (i.e. external derivatives coincide with internal ones).

(ii) We just do $n = 1$. Suppose $f\colon \ell A \times R \to R$ is an element of $F_n(\ell A)$, represented by $f(x,y)\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, say. So $\ell A \Vdash \exists n \in \mathbb{N}\, \mathrm{supp}(f) \subset (-n, n)$, i.e. there is a finite cover $\{\ell B_i \to \ell A\}_{i=1}^k$ and numbers $n_i \in \mathbb{N}$ with $\ell B_i \Vdash \mathrm{supp}(f) \subset (-n_i, n_i)$. Let $n = \max_{i \le k}(n_i)$. Then $\ell A \Vdash \mathrm{supp}(f) \subset (-n, n)$, so $\ell A \times R \Vdash f(\pi_2) = 0 \vee (-n < \pi_2 < n)$. Thus $\ell A \times s(\{y\colon |y| > n\}) \Vdash f(\pi_2) = 0$, i.e.

$$f(x,y)|\mathbb{R}^d \times \{y\colon |y| > n\} \in (I(x)),$$

so on $\mathbb{R}^d \times \{y\colon |y| > n\}$ we can write

(1) $f(x,y) = \sum_{i=1}^p A_i(x,y)\varphi_i(x)$

with $\varphi_i \in I$. Let $\rho(y)\colon \mathbb{R} \to \mathbb{R}$ be a function with $\rho(y) \equiv 1$ on $(-n-1, n+1)$, $\rho(y) \equiv 0$ on $\mathbb{R} - [-n-2, n+2]$. Let $g(x,y) = \rho(y)f(x,y)$. Then (1) implies that $g(x,y) - f(x,y) \in (I(x)) \subset C^\infty(\mathbb{R}^d \times \mathbb{R})$, so $g\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ also represents $f \in R^R(\ell A)$. Clearly g satisfies the requirements in the lemma. □

Just as in Lemma 3.15.1, one proves that the correspondence (2) yields in fact a correspondence between external distributions $C_c^\infty(\mathbb{R}^n) \xrightarrow{\mu} \mathbb{R}$ and Δ-continuous R-linear maps $F_n \xrightarrow{\mu} R$. Therefore, we may conclude the following theorem.

3.15.3 Theorem. *The global sections functor $\Gamma\colon \mathcal{B} \to \mathbf{Sets}$ induces a bijection between distributions $F_n \xrightarrow{\nu} R$ in $\mathcal{B}$ and external distributions $\Gamma\nu\colon C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$, and between distributions with compact support, i.e. R-linear maps $R^{R^n} \xrightarrow{\nu} R$ and external distributions with compact support $C^\infty(\mathbb{R}^n) \xrightarrow{\Gamma\nu} \mathbb{R}$.*

A similar correspondence holds for Z. □

3.16 Example: Functional Derivatives in $\mathcal{B}$. The following is taken verbatim from Feynman-Hibbs (1965), p. 176. "The functional

$F[x(t)]$ gives a number for each function $x(t)$ that we may choose. We may ask: How much does this number change if we make a very small change in the argument function? Thus, for small $\eta(t)$, how much is $F[x(t)+\eta(t)] - F[x(t)]$? The effect to first order in η (assuming it exists, etc.) is some linear expansion in η, say $\int K(s)\eta(s)dx$. Then $K(s)$ is called the functional derivative of F with respect to variation of the function $x(t)$ at s. It is written $\delta F/\delta x(s)$. That is, to first order,

$$F[x+\eta] = F[x] + \int \frac{\delta F}{\delta x(s)}\eta(s)dx + \ldots$$

This $\delta F/\delta x(s)$ depends on the function $x(t)$, of course, and also on the value of x. Thus it is a functional of $x(t)$, and a function of time s".

We show that functional derivatives can be defined in a mathematically rigorous way in the topos $\mathcal{B}$. The reader will notice that we use both types of infinitesimals: nilpotent and invertible ones.

Let $F\colon R^R \to R$ and $x = x(t)\colon R \to R$ be morphisms of $\mathcal{B}$. Now we reason in $\mathcal{B}$. For a "small function" η, we take $\eta(t) = hy(t)$, with $y\colon R \to R$ and $h \in D$ (i.e. $h^2 = 0$). So by the Kock-Lawvere axiom,

$$F[x + h\cdot y] - F[x] = h\mu[x,y]$$

for a unique functional μ, which is R-linear in y (since it is R-homogeneous in y, cf.V.1.5). In other words, $\mu[x,-]$ is a distribution of compact support. So by theorem 3.6,

$$\mu[x,y] \simeq \int K(x,s)y(s)ds,$$

where $K(x,s) = \mu[x,-]_t(\delta_\epsilon(t-s))$. In particular, taking $\eta = h\cdot y$ for y, we obtain the formula from the quotation above ("up to an infinitesimal"):

$$F[x+\eta] = F[x] + \mu[x,n] \simeq F[x] + \int K(x,s)\eta(s)dx.$$

4 A Transfer Principle Between the Models $\mathcal{B}$ and $\mathcal{G}$

It will be clear from the preceding chapters that it is much easier to work with $\mathcal{G}$ than with $\mathcal{B}$ (or $\mathcal{Z}$), both from an external and from an internal, logical, point of view. Externally, because the covers of

$\mathbb{G}$ reflect the topological sense of "local" properties in a direct way, while in $\mathcal{Z}$ and $\mathcal{B}$ one has finite covers only, and one constantly has to use Ostrand's theorem to transform "local" data into data "on a finite cover in the site". And internally, because we can apply induction to arbitrary properties of natural numbers in $\mathcal{G}$.

In this final section, we state and prove a principle that allows us to reduce validity in $\mathcal{B}$ to validity in $\mathcal{G}$, for a rather large class of statements. As a consequence, we obtain validity of the axiom of coherent induction (A13) in $\mathcal{B}$. (The case of $\mathcal{Z}$ is similar.)

4.1 The Language and its Interpretation. As in VII.1, we shall consider a language of variable types, but as special functions (individual constants) $R^n \to R$ we now take *all* C^∞-functions $\mathbb{R}^n \to \mathbb{R}$. As basic types we take the finitely presented types, or in terms of the models, all finitely presented loci $\ell(C^\infty(\mathbb{R}^n)/(f_1, \dots, f_m))$. In particular, $1, D$ and R are basic types. For the purposes of this section, however, we *only* consider formulas whose free variables have either a finitely presented type, or (more generally) a type of the form S^T with S and T finitely presented. The only atomic formulas are those of the form $t_1 = t_2$, where the type of t_i is finitely presented. For example, if the type of f is S^T, the type of x is T and that of y is S, then $f(x) = y$ is an atomic formula, but $f = f$ is not.

Recall that the *coherent formulas* are those obtained from atomic ones by using the logical connectives $\vee, \perp, \wedge, \top$ and the quantifier $\exists x$, where x is of finitely presented type. (So no existential quantification over function types.)

The general notion of forcing allows us to define an interpretation in $\mathcal{B}$ and $\mathcal{G}$ (in fact in each of the toposes discussed so far) for any formula of this language, starting off by interpreting the function symbols $R^n \to R$ and the finitely presented types (loci) by "themselves" in the obvious way, as we have been doing all the time. In this way, we specify the conditions under which a formula, say $\varphi(f, x, y)$, is *forced* at stage $\ell A \in \mathbb{L}$ in the topos $\mathcal{B}$, for given values F, b, c of the parameters f, x, y defined at that stage: maps

(1) $\ell A \xrightarrow{F} \ell B^{\ell C}, \ \ell A \xrightarrow{b} \ell B, \ \ell A \xrightarrow{c} \ell C$

in $\mathcal{B}$, where ℓB and ℓC are the finitely presented loci corresponding to the types of x and y, and f is of type $\ell C^{\ell B}$, say. As usual, but with $\mathcal{B}$ explicitly appearing in notation, we write

(2) $\ell A \Vdash_{\mathcal{B}} \varphi(f, x, y)[F, b, c]$.

Similarly, if $\ell A \in \mathbb{G}$, and F, b, c as in (1) are maps in $\mathcal{G}$, we have the analogous notion

(3) $\ell A \Vdash_{\mathcal{G}} \varphi(f,x,y)[F,b,x]$

of forcing in $\mathcal{G}$. We aim to compare these two notions of forcing (2) and (3). Of course this only makes sense if ℓA belongs to both $\mathbb{L}$ and $\mathbb{G}$; in particular, if ℓA is finitely presented locus.

4.2 Finite Extensions of Loci. Let $\ell A \in \mathbb{L}$ be a locus. A locus ℓB is a *finite extension* of ℓA if ℓA is a closed sublocus of ℓB (II.1.3), and ℓB is finitely presented. So up to isomorphism, we can write

(1) $A = C^\infty(\mathbb{R}^n)/I,\ B = C^\infty(\mathbb{R}^n)/I_0,$

where $I_0 \subset I$ is a finitely generated subideal. In (1), the representation $C^\infty(\mathbb{R}^n)/I$ of A depends on the one chosen for B of course, and it is not true that for a given representation $C^\infty(\mathbb{R}^n)/I$ of A, every finite extension of A is of the form $\ell(C^\infty(\mathbb{R}^n)/I_0)$ for some finitely generated subideal I_0 of I.

On the other hand, if $C^\infty(\mathbb{R}^n)/I$ is a given representation of A, any finite extension $\ell A \rightarrowtail \ell B$ corresponds to a map of the form

$$B \cong C^\infty(\mathbb{R}^{n+m})/J_0 \xrightarrow{S} C^\infty(\mathbb{R}^n)/I,$$

where S is the C^∞-homomorphism induced by composition with the "slice" $s\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+m}$, $s(x) = (x,0)$. (To see this, write B as a quotient of a free ring with an appropriate set of generators.) So if we let $I_0 = (f_1(x,0),\ldots,f_k(x,0))$, where $f_1(x,y),\ldots,f_k(x,y)$ generate J_0, then we have a commutative diagram of finite extensions

$$\begin{tikzcd} \ell A \arrow[rr, tail] \arrow[dr] & & \ell B \arrow[dl, leftarrow] \\ & \ell A_0 \arrow[ul] & \end{tikzcd}$$

where $A_0 = C^\infty(\mathbb{R}^n)/I_0$. In other words, for a given representation $C^\infty(\mathbb{R}^n)/I$ of A, *the finite extensions of the form*

$$\ell(C^\infty(\mathbb{R}^n)/I) \rightarrowtail \ell(C^\infty(\mathbb{R}^n)/I_0)$$

are cofinal among all finite extensions. From this, the following lemma is obvious.

Lemma. *The finite extensions of ℓA are directed, in the following sense: if $\ell A \rightarrowtail \ell B$ and $\ell A \rightarrowtail \ell C$ are two finite extensions, there exists a third finite extension ℓD of ℓA and a commutative diagram*

$$\begin{tikzcd}
& & \ell B \\
\ell A \arrow[urr, tail] \arrow[r, tail] \arrow[drr, tail] & \ell D \arrow[ur] \arrow[dr] & \\
& & \ell C
\end{tikzcd}$$

and in fact ℓD can be taken to be a closed sublocus of ℓB and ℓC by the maps $\ell D \to \ell B$ and $\ell D \to \ell C$, so that all the maps in the diagram are finite extensions. □

4.3 $\mathcal{B}$-Finite, $\mathcal{G}$-Finite Formulas. Let $\varphi(x_1, \dots, x_n, f_1, \dots, f_m) = \varphi(x, f)$ be a formula of the language, with free variables x_i and f_j whose types correspond to ℓB_i and $\ell C_j{}^{\ell D_j}$ respectively, with $\ell B_i, \ell C_j, \ell D_j$ finitely presented loci. φ is said to be *$\mathcal{B}$-finite* if for every $\ell A \in \mathbb{L}$, and any values of the parameters $\ell A \xrightarrow{b_i} \ell B_i$ and $\ell A \xrightarrow{F_j} \ell C_j{}^{\ell D_j}$, we have (writing $b = (b_1, \dots, b_n), F = (F_1, \dots, F_n)$)

$$\begin{gathered}\ell A \Vdash_{\mathcal{B}} \varphi(x, f)[b, F] \Leftrightarrow \text{there exist a finite extension } \ell A \rightarrowtail \ell A' \text{ and} \\ \text{extensions } \ell A' \xrightarrow{b'_i} \ell B_i, \ell A \xrightarrow{F'_j} \ell C_j^{\ell D_j} \text{ of } b_i \text{ and } F_j \text{ such that} \\ \ell A' \Vdash \varphi(x, f)[b', F'].\end{gathered}$$

We remark that it is clear from the definition of morphism in $\mathbb{L}$ that the values of the parameters can always be extended to some finitely presented locus; i.e. for any map $\ell A \xrightarrow{c} \ell B$ (or $\ell A \xrightarrow{G} \ell C^{\ell D}$) in $\mathcal{B}$, with ℓB (or ℓC and ℓD) finitely presented, there exists a finite extension $\ell A \overset{i}{\rightarrowtail} \ell A'$ and a map $\ell A' \xrightarrow{c'} \ell B$ (or $\ell A' \xrightarrow{G'} \ell C^{\ell D}$) with $c' \circ i = c$ (or $G' \circ i = G$). If we combine this observation with the lemma in 4.2, we obtain

4.3.1 Lemma. *A formula $\varphi(x, f)$ as above is $\mathcal{B}$-finite iff for every $\ell A \in \mathbb{L}$ and values $\ell A \xrightarrow{b_i} \ell B$, $\ell A \xrightarrow{F_j} \ell C_j{}^{\ell D_j}$ of x_i and f_j as above, $\ell A \Vdash \varphi(x, f)[b, F]$ iff for every finite extension $\ell A \rightarrowtail \ell A'$ and any given extension b', F' of the parameter values to $\ell A'$, there exists an "intermediate" finite extension $\ell A \rightarrowtail \ell A'' \rightarrowtail \ell A'$ such that $\ell A'' \Vdash \varphi(x, f)[b'', F'']$, where b'', F'' are the restrictions of b', F' to $\ell A''$.*

4.3.2 Proposition.

(i) *Every coherent formula is $\mathcal{B}$-finite.*
(ii) *If φ and ψ are $\mathcal{B}$-finite, then so are $\varphi \wedge \psi, \varphi \vee \psi, \top, \bot$.*
(iii) *If φ is $\mathcal{B}$-finite, then so are $\exists x\varphi$ and $\forall x\varphi$, provided that the type of x is finitely presented.*
(iv) *If φ is coherent and ψ is $\mathcal{B}$-finite, then $\varphi \to \psi$ is $\mathcal{B}$-finite.*

Proof. (i) follows from (ii) and (iii), since atomic formulas are obviously $\mathcal{B}$-finite: take for example the case $f(x) = y$, with x, y, f of types $\ell B, \ell C, \ell C^{\ell B}$ respectively, these all being finitely presented loci. Assume that $\ell A \Vdash f(x) = y[F, b, c]$ for given $\ell A \xrightarrow{F} \ell C^{\ell B}$, $\ell A \xrightarrow{b} \ell B$, $\ell A \xrightarrow{c} \ell C$. Write $A = C^\infty(\mathbb{R}^n)/I$, $B = C^\infty(\mathbb{R}^m)/J, C = C^\infty(\mathbb{R}^k)/L$, and let $\mathbb{R}^n \xrightarrow{b(x)} \mathbb{R}^m$, $\mathbb{R}^n \xrightarrow{c(x)} \mathbb{R}^k$ represent b and c, while $\mathbb{R}^n \times \mathbb{R}^m \xrightarrow{F(x,y)} \mathbb{R}^k$ represents the transposed map $\ell A \times \ell B \to \ell C$ of F. Let $I' = (\pi_1 \circ (F(x, b(x)) - c(x)), \ldots, \pi_k \circ (F(x, b(x)) - c(x)))$. Then $I' \subset I$ since $\ell A \Vdash f(x) = y[F, b, c]$, and clearly $\ell A' \Vdash f(x) = y[F', b', c']$, where $A' = C^\infty(\mathbb{R}^n)/I'$, and F', b', c' are the extensions of F, b, c which are also represented by $F(x, y), b(x), c(y)$.

(ii) and (iii) are straightforward, using generic elements for $\forall x\varphi$.

To prove (iv), we notice that $\bigvee_{i=1}^n \exists x_i \varphi_i \to \psi$ is equivalent to $\bigwedge_{i=1}^n \forall x_i(\varphi_i \to \psi)$. Since every coherent formula is equivalent to one of the form $\bigvee_{i=1}^n \exists x_i \varphi_i$ with $\varphi_i \equiv \varphi_{i1} \wedge \ldots \wedge \varphi_{ik}$ a conjunction of atomic formulas, and $(\varphi_{i1} \wedge \ldots \wedge \varphi_{ik}) \to \psi$ is equivalent to $\varphi_{i1} \to (\ldots \to (\varphi_{ik} \to \psi)) \ldots)$, it suffices to prove (iv) for the case where φ is atomic. Take for example $\varphi \equiv (f(x) = y)$, and assume

$$\ell A \Vdash (f(x) = y \to \psi(f, x, y))[F, b, c]$$

with F, b, c the values of f, x, y, just as in the proof of (i). Using representations as in the proof of (i), let $E = A/(\pi_1 \circ (F(x, b(x)) - c(x)), \ldots, \pi_k \circ (F(x, b(x)) - c(x)))$. Then $\ell E \Vdash f(x) = y[F_0, b_0, c_0]$ where F_0, b_0, c_0 are the obvious restrictions of F, b, c to $\ell E \subset \ell A$. So there exists a finite extension $\ell E \rightarrowtail \ell E'$ of ℓE such that $\ell E' \Vdash \psi(f, x, y)[F_0', b_0', c_0']$, for some extensions F_0', b_0', c_0' of F_0, b_0, c_0. By 4.2 and 4.3, we may without loss assume that $E' = C^\infty(\mathbb{R}^n)/(I', \pi_1 \circ (F(x, b(x)) - c(x)), \ldots, \pi_k \circ (F(x, b(x)) - c(x)))$ and that $F(x, y), b(x), c(x)$ also represent F_0', b_0', c_0'. But then clearly

$$\ell A' \Vdash f(x) = y \to \psi(f, x, y)[F', b', c'],$$

F', b', c' being the extensions of F_0', b_0', c_0' which are also represented by $F(x, y), b(x), c(x)$. $\square$

4.3.3 $\mathcal{G}$-Finite Formulas Using $\mathbb{G}$ and $\mathcal{G}$ rather than $\mathbb{L}$ and $\mathcal{B}$, we can define $\mathcal{G}$-finite formulas in the same way. Proposition 4.3.2 is also true for $\mathcal{G}$-finite formulas, by the same proof.

4.4 The Transfer Theorem. *Let $\mathcal{T}$ be the class of formulas $\varphi(x_1, \ldots, x_n, f_1, \ldots, f_m)$ with the property that for any finitely presented $\ell A \in \mathbb{L}$ and any values $\ell A \xrightarrow{b_i} \ell B$ and $\ell A \xrightarrow{F_j} \ell C_j^{\ell D_j}$, we have*

$$\ell A \Vdash_{\mathcal{G}} \varphi(x, f)[b, F] \Leftrightarrow \ell A \Vdash_{\mathcal{B}} \varphi(x, f)[b, F]$$

(where we use the same notation as in 4.3). Then $\mathcal{T}$ has the following properties:

(i) *$\mathcal{T}$ contains all the coherent formulas.*
(ii) *If φ and $\psi \in \mathcal{T}$, then $\varphi \wedge \psi, \top, \varphi \vee \psi, \bot \in \mathcal{T}$.*
(iii) *If $\varphi \in \mathcal{T}$ then $\forall f_i \varphi \in \mathcal{T}$; if φ is moreover $\mathcal{B}$-finite, also $\exists f_i \varphi \in \mathcal{T}$.*
(iv) *If $\varphi, \psi \in \mathcal{T}$ and φ is $\mathcal{B}$-finite as well as $\mathcal{G}$-finite, then $\varphi \to \psi \in \mathcal{T}$.*

Proof. As in 4.3.2, (i) follows from (ii) and (iii), using the obvious fact that atomic formulas belong to $\mathcal{T}$.

(ii) is obvious.

For (iii), take a formula $\forall f \varphi(f)$, where f is a variable of type $\ell C^{\ell B}$ for finitely presented loci $\ell B, \ell C$. For ease of notation, we suppress all other parameters. Assume $\ell A \Vdash_{\mathcal{G}} \forall f \varphi(f)$, with ℓA finitely presented, and consider a diagram in $\mathcal{B}$ of the form $\ell A \xleftarrow{h} \ell D \xrightarrow{F} \ell C^{\ell B}$. Clearly there exists a finite extension $\ell D' \supset \ell D$ and a commutative diagram

$$\begin{tikzcd}
\ell D \arrow[rr, "h"] \arrow[dr] \arrow[dd, "F"'] & & \ell A \\
& \ell D' \arrow[ur, "h'"'] \arrow[dl, "F'"'] & \\
\ell C^{\ell B} & &
\end{tikzcd}$$

and from $\ell A \Vdash_{\mathcal{G}} \forall f \varphi(f)$ it follows that $\ell D' \Vdash_{\mathcal{G}} \varphi(F')$ (where all other parameters in φ are restricted along h', of course). So by hypothesis, also $\ell D' \Vdash_{\mathcal{B}} \varphi(F')$, and hence $\ell D \Vdash_{\mathcal{B}} \varphi(F)$. Thus $\ell A \Vdash_{\mathcal{B}} \forall f \varphi(f)$. The other direction for the $\forall$-clause is proved in the same way.

For the $\exists$-clause, take a formula $\exists f \varphi(f)$, with φ $\mathcal{B}$-finite, and f is a variable of type $\ell C^{\ell B}$, as above. Assume $\ell A \Vdash_{\mathcal{G}} \exists f \varphi(f)$, where ℓA is finitely presented. So there is an open cover $\{U_\alpha\}_\alpha$ of $\mathbb{R}^n$, which we may without loss assume to be locally finite, and a family $\{\ell A \times s(U_\alpha) \xrightarrow{F_\alpha} \ell C^{\ell B}\}_\alpha$ of morphisms in $\mathcal{G}$ (or equivalently, in $\mathcal{B}$) such that

$$\ell A \cap s(U_\alpha) \Vdash_{\mathcal{G}} \varphi[F_\alpha], \text{ for all } \alpha.$$

By Ostrand's theorem (VI.2.13), we can find a cover $W_1 \cup \ldots \cup W_{n+1} = \mathbb{R}^n$ such that for each $i \leq n+1$, W_i can be written as a *disjoint* union

$$W_i = \bigcup_\alpha W_{i_\alpha}, \text{ with } W_{i_\alpha} \subset U_\alpha.$$

Defining $\ell A \cap s(W_i) \xrightarrow{F_i} \ell C^{\ell B}$ by glueing the $F_{i_\alpha} = F_\alpha | \ell A \cap s(W_{i_\alpha})$ together in $\mathcal{G}$, we conclude that

$$\ell A \cap s(W_i) \Vdash_{\mathcal{G}} \varphi[F_i],$$

since $\Vdash$ is local and $\{s(W_{i_\alpha}) \to s(W_i)\}_\alpha$ is a cover in $\mathbb{G}$. Since $\varphi \in \mathcal{T}$ by hypothesis, and $\ell A \cap s(W_i)$ is finitely presented, it follows that $\ell A \cap s(W_i) \Vdash_{\mathcal{B}} \varphi[f_i]$ for $i = 1, \ldots, n+1$; so $\ell A \Vdash_{\mathcal{B}} \exists f \varphi$.

Conversely, suppose $\ell A \Vdash_{\mathcal{B}} \exists f \varphi(f)$. From the description of the covers in $\mathbb{B}$ (see VI.5.3) there is a non-trivial locus ℓE and a finite open cover

$$\{\ell E_i \hookrightarrow \ell A \times \ell E\}_{i=1}^k$$

of $\ell A \times \ell E$, such that for each i there is a map $\ell E_i \xrightarrow{f_i} \ell C^{\ell B}$ with $\ell E_i \Vdash_{\mathcal{B}} \varphi[F_i]$. Since φ is $\mathcal{B}$-finite by hypothesis, it follows from 4.2 that we may without loss take ℓE, and hence the ℓE_i, to be finitely presented. So from the assumption that $\varphi \in \mathcal{T}$, it follows that $\ell E_i \Vdash_{\mathcal{G}} \varphi[F_i]$. Hence $\ell A \times \ell E \Vdash_{\mathcal{G}} \exists f \varphi$, and since $\ell A \times \ell E \to \ell A$ splits, also $\ell A \Vdash_{\mathcal{G}} \exists f \varphi$.

Finally, we prove (iv). Assume $\ell A \Vdash_{\mathcal{G}} \varphi \to \psi$, and let $\ell D \to \ell A$ be a map in $\mathcal{B}$ with $\ell D \Vdash_{\mathcal{B}} \varphi$. Since φ is $\mathcal{B}$-finite and ℓA is finite, it follows from 4.2, 4.3 that there exists a finite extension $\ell D'$ and a commutative diagram

$$\begin{tikzcd} \ell D \arrow[r] \arrow[d] & \ell A \\ \ell D' \arrow[ur] & \end{tikzcd}$$

with $\ell D' \Vdash_{\mathcal{B}} \varphi$. So by hypothesis $\ell D' \Vdash_{\mathcal{G}} \varphi$, and hence $\ell D' \Vdash_{\mathcal{G}} \psi$. Since $\psi \in \mathcal{T}$, also $\ell D' \Vdash_{\mathcal{B}} \psi$, whence $\ell D \Vdash_{\mathcal{B}} \psi$. Thus $\ell A \Vdash_{\mathcal{B}} \varphi \to \psi$.

The converse is proved similarly. □

4.5 Transfer Between $\mathcal{G}$ and $\mathcal{Z}$. Theorem 4.4 remains true if we replace $\mathcal{B}$ by $\mathcal{Z}$ throughout. The proof is simpler. In fact we get a slightly better result, since we don't need the assumption that φ is $\mathcal{Z}$-finite in the case $\exists f_i \varphi$ of 4.4 (iii), as will be clear from inspection of the proof.

4.6 Corollary. *(Coherent induction in $\mathcal{B}$ and $\mathcal{Z}$). For any coherent formula $\varphi(n, x, f)$ with free variables n of type N, and $x = (x_1, \ldots, x_k)$, $f = (f_1, \ldots, f_m)$ as in 4.3, the sentence*

$$\forall x \forall f(\varphi(0, x, f) \wedge \forall n \in N(\varphi(n, x, f) \to \varphi(n+1, x, f)) \to \forall n \in N \varphi(n, x, f))$$

is valid in $\mathcal{B}$ and $\mathcal{Z}$.

Proof. This sentence is valid in $\mathcal{G}$, since in $\mathcal{G}$ we have full induction. Moreover, the sentence is in the class $\mathcal{T}$, by 4.3.2 and the closure conditions in 4.4 (resp. 4.5). □

Appendix 1: Sheaves and Forcing

This appendix is not meant as a systematic exposition of the theory, but rather as a concise compilation of the basic notions. For more detailed information and different perspectives, the reader may wish to consult such references as Boileau and Joyal (1981), Johnstone (1977), Lambek and Scott (1986), Makkai and Reyes (1977), Osius (1975).

1 Sites

Let $\mathbb{C}$ be a category. We will assume throughout that $\mathbb{C}$ has finite limits (this condition isn't strictly necessary, but it is satisfied by all the examples in this book).

1.1 Grothendieck topologies. (a) A *sieve* on an object $C \in \mathbb{C}$ is a collection R of morphisms in $\mathbb{C}$ with codomain C, with the property that if $D \xrightarrow{f} C \in R$ then $E \xrightarrow{f \circ g} C \in R$ for any morphism $E \xrightarrow{g} D$ (D, E any objects).

(b) A *Grothendieck topology* ("first version") on $\mathbb{C}$ is a function J which assigns to each object $C \in \mathbb{C}$ a collection $J(C)$ of sieves on C, such that

(i) the maximal sieve $\{f |$ codomain $(f) = C\}$ is in $J(C)$.

(ii) (*stability*) if $R \in J(C)$ and $D \xrightarrow{f} C$ is any morphism in $\mathbb{C}$, then $f^*(R) =_{\mathrm{def}} \{E \xrightarrow{g} D | f \circ g \in R\} \in J(D)$.

(iii) (*transitivity*) if $R \in J(C)$ and S is a sieve on C such that for every $D \xrightarrow{f} C \in R$, $f^*(S) \in J(D)$, then $S \in J(C)$.

If $R \in J(C)$, we say that R is a J*-covering sieve* (or just *covering sieve* or *cover*).

(c) Let J be a Grothendieck topology on $\mathbb{C}$. If $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ is just any collection of morphisms with codomain C, then we say that $\{C_\alpha \xrightarrow{f_\alpha} C\}$ is a (J)*-cover* if the sieve generated by $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$, namely $R = \{D \xrightarrow{g} C | \exists \alpha\; g$ factors through $f_\alpha\}$ is in $J(C)$.

(d) There is another way of defining a Grothendieck topology on $\mathbb{C}$, which gives rise to exactly the same sheaves (see 2.4(c) below): A *Grothendieck topology* ("second version") on $\mathbb{C}$ is a function K which assigns to each $C \in \mathbb{C}$ a collection of families $\{C_\alpha \to C\}_\alpha$ (α running over some variable index set), called *(K-) covering families*, or *(K-) covers*, such that

(i′) $C \xrightarrow{\text{id}} C \in K(C)$

(ii′) (*stability*) if $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in K(C)$ and $D \xrightarrow{g} C$ is any morphism, then $\{C_\alpha \times_C D \xrightarrow{g_\alpha} D\}_\alpha \in K(D)$, where g_α denotes the pullback of f_α along g.

(iii′) (*transitivity*) if $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in K(C)$ and for every α we have a family $\{C_{\alpha\beta} \xrightarrow{g_{\alpha\beta}} C_\alpha\}_\beta \in K(C_\alpha)$, then the family of composites $\{C_{\alpha\beta} \xrightarrow{f_\alpha \circ g_{\alpha\beta}} C\}_{\alpha,\beta} \in K(C)$.

The elements of $K(C)$ are again called *covering families*, or *covers* of C.

(e) Any Grothendieck topology K as in (d) gives rise to a Grothendieck topology J as in (b) by setting for a sieve R on C:

$$(*) \quad R \in J(C) \Leftrightarrow \exists S \subset R \quad S \in K(C).$$

Conversely, note that if J is a Grothendieck topology, the covering families in the sense of (c) satisfy (i′)–(iii′) of (d).

(f) One may weaken condition (iii′) to (iii″), and still conclude that (*) in (e) defines a Grothendieck topology

(iii″) if $\{C_\alpha \xrightarrow{f_\alpha} C\} \in K(C)$ and for every α we have a family $\{C_\alpha\beta \xrightarrow{g_{\alpha\beta}} C_\alpha\}_\beta \in K(C_\alpha)$, then *some refinement* of $\{C_{\alpha\beta} \xrightarrow{f_\alpha \circ g_{\alpha\beta}} C\}_{\alpha,\beta}$ is in $K(C)$ (a family $\{D_\gamma \xrightarrow{g_\gamma} C\}_\gamma$ is a *refinement* of a family $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ if for every γ there is an α and an $h: D_\gamma \to C_\alpha$ with $f_\alpha \circ h = g_\gamma$).

If a function K as in (d) satisfies (i′), (ii′) and (iii″), we *still* say that K is a Grothendieck topology.

1.2 Generating Systems. (a) If $\{J_i\}_{i \in I}$ is a family of Grothendieck topologies as in 1.1(b), then the intersection $\bigcap_i J_i$ defined by

$$(\bigcap_i J_i)(C) = \bigcap_i J_i(C)$$

is again one. Consequently, if we specify a family of "covering sieves" R_ξ (R_ξ a sieve on C_ξ, say), there is always a smallest Grothendieck topology J such that $R_\xi \in J(C_\xi)$ (each ξ). This is the Grothendieck topology *generated by* the system $\{R_\xi\}_\xi$.

(b) A function B which assigns to each object $C \in \mathbb{C}$ a family of covers $B(C)$ such that

(i′) $C \xrightarrow{\text{id}} C \in B(C)$

(ii′) if $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in B(C)$ then for every $D \xrightarrow{g} C$,

$$\{C_\alpha \times_C D \to D\}_\alpha \in B(D)$$

is called a (stable) ***basis*** for a Grothendieck topology J (as in 1.1(b)) if J is the smallest Grothendieck topology such that for every sieve R on an object C,

$$R \supseteq S \in B(C) \Rightarrow R \in J(C)$$

(So for example K is a basis for J in the situation of 1.1(e)). B is called a basis for a Grothendieck topology K as in 1.1(d) if $B \subset K$ and B is a basis for the Grothendieck topology J defined from K as in (*) of 1.1(e).

If B is a basis for K or J, we also say that B *generates* K or J. The families in $B(C)$ are called ***basic covers*** of C.

1.3 Definition. A *site* is a category $\mathbb{C}$ equipped with a Grothendieck topology (as in 1.1(b) or (d)). So strictly speaking this is a pair $(\mathbb{C}, K)$ (or $(\mathbb{C}, J)$), but we will often suppress the Grothendieck topology, and speak of "a site $\mathbb{C}$".

2 Sheaves

Throughout, $\mathbb{C}$ is a category with finite limits.

2.1 Presheaves. (a) A ***presheaf on*** $\mathbb{C}$ is a functor $\mathbb{C}^{\mathrm{op}} \xrightarrow{F} \mathit{Sets}$. So F is a collection of sets $F(C)(C \in \mathbb{C})$ together with "***restriction***" functions $F(D) \xrightarrow{F(f)} F(C)$ for each $C \xrightarrow{f} D$ in $\mathbb{C}$. We often write $x|f$ for $F(f)(x)$ (where $x \in F(D)$). So

$$x|\text{id} = x$$
$$(x|f)|g = x|(f \circ g)$$

A ***morphism*** of presheaves is just a natural transformation. $\mathit{Sets}^{\mathbb{C}^{\mathrm{op}}}$ denotes the category of presheaves on $\mathbb{C}$.

(b) A ***subpresheaf*** of a presheaf F is a subfunctor $G \subset F$; i.e., for each $C \in \mathbb{C}$ we have $G(C) \subset F(C)$, and the restrictions of G are the same as those of F: for $x \in G(C) \subset F(C)$ and $D \xrightarrow{f} C$,

$G(f)(x) = F(f)(x)$.

(c) A *representable presheaf* is a presheaf of the form

$$\mathbb{C}(-,C)\colon \mathbb{C}^{\mathrm{op}} \to \mathit{Sets}$$

for some $C \in \mathbb{C}$ ($\mathbb{C}(-,C)(D) =_{\mathrm{def}} \mathbb{C}(D,C)$, and $|$ is just composition in this case). The assignment

$$C \mapsto \mathbb{C}(-,C)$$

defines a full embedding $\mathbb{C} \hookrightarrow \mathit{Sets}^{\mathbb{C}^{\mathrm{op}}}$, the so-called *Yoneda embedding*. The *Yoneda lemma* states that for any presheaf X there is a bijection between morphisms $\mathbb{C}(-,C) \xrightarrow{\tau} X$ and elements $\alpha \in X(C)$ given by $\alpha = \tau_C(\mathrm{id}_C)$, resp. $\tau_D(f) = X(f)(\alpha)$.

2.2 Definition. Some constructions of presheaves.

(a) *Limits* and *colimits* of presheaves are constructed "pointwise": if $(F_i)_i$ is a diagram of preshseaves, then

$$(\varinjlim F_i)(C) \cong \varinjlim F_i(C)$$
$$(\varprojlim F_i)(C) \cong \varprojlim F_i(C)$$

(so on the left, $\varinjlim$ and $\varprojlim$ in $\mathit{Sets}^{\mathbb{C}^{\mathrm{op}}}$, on the right in Sets).

(b) *Exponentials*: if F and G are two presheaves on $\mathbb{C}$, the exponential G^F is the presheaf defined by

$$G^F(C) = \text{the set of natural transformations } \mathbb{C}(-,C) \times F \to G$$

So an element φ of $G^F(C)$ is a function which assigns to each $D \xrightarrow{f} C$ and $x \in F(D)$ an element $\varphi_D(f,x)$—or just $\varphi(f,x)$—of $G(D)$, such that

$$\varphi(f \circ g, x|g) = \varphi(f,x)|g.$$

If $C' \xrightarrow{u} C$ and $\varphi \in G^F(C)$, then $G^F(u)(\varphi) = \varphi|u \in G^F(C')$ is defined by setting for $D \xrightarrow{g} C'$ and $x \in F(D)$,

$$(\varphi|u)_D(g,x) = \varphi_D(u \circ g, x).$$

Thus defined, $(-)^F$ is right-adjoint to $F \times (-)$, i.e., for any third presheaf H, there is a bijective correspondence between morphisms $H \xrightarrow{\sigma} G^F$ and morphisms $F \times H \xrightarrow{\tau} G$:

$$\frac{H \xrightarrow{\sigma} G^F}{F \times H \xrightarrow{\tau} G},$$

and this correspondence is natural in G and H. Given σ, one defines τ by

$$\tau_C(x, z) = \sigma_C(z)_C(\mathrm{id}_C, \alpha), \text{ for } x \in F(C), z \in H(C),$$

while conversely, given τ, one defines σ by

$$\begin{aligned} &\sigma_C(z): \mathbb{C}(-, C) \times F \to G \\ &\sigma_C(z)_D(f, x) = \tau_D(z|f, x). \end{aligned}$$

(c) *Power presheaves*: If F is a presheaf, the "presheaf of subpresheaves of F", $\mathcal{P}(F)$, is the presheaf defined by

$$\mathcal{P}(F)(C) = \text{the set of subpresheaves of } \mathbb{C}(-, C) \times F.$$

So an element S of $\mathcal{P}(F)(C)$ assigns to each D a set $S(D) \subset \mathbb{C}(D, C) \times F(D)$, such that for any $E \xrightarrow{g} D$,

$$(f, x) \in S(D) \Rightarrow (f \circ g, x|g) \in S(E).$$

If $C' \xrightarrow{u} C$ and $S \in \mathcal{P}(F)(C)$, then $S|u$ is defined by setting for $D \xrightarrow{g} C'$ and $x \in F(D)$,

$$(g, x) \in (S|u)(D) \Leftrightarrow (ug, x) \in S(D).$$

For any presheaf H, there is a bijective correspondence between subpresheaves $S \subset G \times F$ and morphisms $G \xrightarrow{\sigma} \mathcal{P}(F)$:

$$\frac{S \subset G \times F}{G \xrightarrow{\sigma} \mathcal{P}(F)}.$$

Given S, σ is defined by

$$\begin{aligned} &\sigma_C: G(C) \to \mathcal{P}(F)(C) \\ &(f, x) \in \sigma_C(y)_D \Leftrightarrow (y|f, x) \in S(D), \end{aligned}$$

for any $D \xrightarrow{f} C$, $x \in F(D), y \in G(C)$; and given τ, S is defined by

$$(y, x) \in S(C) \Leftrightarrow (\mathrm{id}_C, x) \in \sigma_C(y)(C)$$

for any $y \in G(C), x \in F(C)$.

(d) *Constant presheaves*: For any set I, there is a "constant presheaf" $\Delta(I)$, defined by $\Delta(I)(C) = I$ for all $C \in \mathbb{C}$, and $\Delta(I)(f)$ is the identity on I for all morphisms f of $\mathbb{C}$. Δ is left-adjoint to the global sections functor $\Gamma: \mathit{Sets}^{\mathbb{C}^{op}} \to \mathit{Sets}$, defined by $\Gamma(F) = F(1)$ (1 is the terminal object of $\mathbb{C}$). In other words, there is a natural bijective correspondence between morphisms $\Delta(I) \to F$ and functions $I \to \Gamma(F)$,

$$\frac{\Delta(I) \to F}{I \to \Gamma(F)}$$

2.3 Proposition. ***Every presheaf is canonically isomorphic to a colimit of representable presheaves.***

Proof. Given a presheaf F, we first construct a category $D(F)$, the ***diagram*** of F. Objects of $D(F)$ are pairs (x, C) with $x \in F(C)$, and morphisms $(x, C) \to (y, D)$ are morphisms $C \xrightarrow{f} D$ in $\mathbb{C}$ with $y|f = x$. The functor

$$D(F) \xrightarrow{\delta_F} \mathit{Sets}^{\mathbb{C}^{\mathrm{op}}}$$

assigns to (x, C) the representable presheaf $\mathbb{C}(-, C)$, and acts on morphisms by composition, in the obvious way. A simple calculation (cf. 2.3(a)) shows that $\varinjlim \delta_F \cong F$. □

2.4 Sheaves. (a) Let $\mathbb{C}$ be a site, with Grothendieck topology J as in 1.1(b). A ***sheaf*** on $\mathbb{C}$ is a presheaf F on $\mathbb{C}$ satisfying the following condition: For every $R \in J(C)$ and for every family of elements $\{x_f | f \in R\}$, $x_f \in F$ (domain (f)) such that $x_{f \circ g} = x_f|g$, there is a ***unique*** $x \in F(C)$ with the property that $x|f = x_f$ for all $f \in R$.

(b) In terms of a Grothendieck topology K on $\mathbb{C}$ as in 1.1(d), the definition of "sheaf on $\mathbb{C}$" is as follows. Let F be a presheaf on $\mathbb{C}$, and let $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ be a cover of C, i.e., $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in K(C)$. A family $\{x_\alpha \in F(C_\alpha)\}_\alpha$ is called ***compatible*** (for this cover) if for every two indices α and α', $x_\alpha|p = x_{\alpha'}|p'$, where p and p' are the projections in the pullback

$$\begin{tikzcd}
C_\alpha \times_C C_{\alpha'} \arrow[r, "p'"] \arrow[d, "p"'] & C_{\alpha'} \arrow[d, "f_{\alpha'}"] \\
C_\alpha \arrow[r, "f_\alpha"] & C
\end{tikzcd}$$

F is called a ***sheaf*** if for every cover $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in R$, and for every compatible family $\{x_\alpha \in F(C_\alpha)\}_\alpha$, there is a unique $x \in F(C)$ with $x|f_\alpha = x_\alpha$ (all α).

(c) ***Remark***: If J is obtained from K as in (*) of 1.1(e), then a presheaf F is a sheaf on $(\mathbb{C}, J)$ as in (a) iff it is a sheaf on $(\mathbb{C}, K)$ as in (b). Moreover, if B is a ***basis*** for K, then F is a sheaf on $(\mathbb{C}, K)$ iff it satisfies the condition in (b) for ***basic covers*** $\{C_\alpha \to C\}_\alpha \in B(C)$.

So as for as sheaves are concerned, we can define a site equivalently as in 1.1(b), in 1.1(d), or even just as a category with a basis B

satisfying (i′), (ii″) as in 1.2(b). (In the next section, however, we will see that *forcing* cannot be formulated directly in terms of just a basis.)

(d) The ***category of sheaves on*** $\mathbb{C}$ is the full subcategory of $Sets^{\mathbb{C}^{op}}$ whose objects are the sheaves. This category is denoted by $\mathrm{Sh}(\mathbb{C})$ (or if necessary, more explicitly by $\mathrm{Sh}(\mathbb{C}, J), \mathrm{Sh}(\mathbb{C}, K)$).

2.5 The associated sheaf functor. Let $\mathbb{C}$ be a category with a Grothendieck topology as in 2.5. The inclusion functor $\mathrm{Sh}(\mathbb{C}) \hookrightarrow Sets^{\mathbb{C}^{op}}$ has a left-adjoint

$$a\colon Sets^{\mathbb{C}^{op}} \to \mathrm{Sh}(\mathbb{C}),$$

the so-called ***associated sheaf functor***, or ***sheafification*** functor. In terms of a topology K as in 1.1(d), a is defined as follows. For a presheaf F, first construct a presheaf $F^+\colon F^+(C)$ is the set of equivalence classes of compatible families $\{x_\alpha\}_\alpha, x_\alpha \in F(C_\alpha)$, for some cover $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in K(C)$; two such families $\{x_\alpha\}_\alpha$ for $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ and $\{y_\beta\}_\beta$ for $\{D_\beta \xrightarrow{g_\beta} D\}_\beta$ are ***equivalent*** if for each α and β, $x_\alpha | p_{\alpha\beta} = y_\beta | q_{\alpha\beta}$, where $p_{\alpha\beta}$ and $q_{\alpha\beta}$ are the projections,

$$C_\alpha \xleftarrow{p_{\alpha\beta}} C_\alpha \times_C D_\beta \xrightarrow{q_{\alpha\beta}} D_\beta.$$

The restrictions in F^+ are defined on equivalence classes by

$$(\{x_\alpha\}_\alpha)|g = \{x_\alpha | p_\alpha\}_\alpha$$

where $\{x_\alpha \in F(C_\alpha)\}_\alpha$ is a compatible family for the cover $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ and

$$\begin{tikzcd} C_\alpha \times_C D \arrow[r] \arrow[d, "p_\alpha"'] & D \arrow[d, "g"] \\ C_\alpha \arrow[r, "f_\alpha"] & C \end{tikzcd}$$

is a pullback; so $\{x_\alpha | p_\alpha\}_\alpha$ is a compatible family for the cover $\{C_\alpha \times_C D \xrightarrow{p_\alpha} D\}_\alpha$. This is well-defined on equivalence classes.

F^+ is in general not a sheaf, but if we repeat this $+$-construction once more, we obtain a sheaf, and this is the associated sheaf we are looking for:

$$a(F) = (F^+)^+$$

We leave further details to the reader (they can also be found in any of the references cited above).

Notice that if F is a sheaf, then $a(F) \cong F$.

2.6 Yoneda embedding: subcanonical topologies. (a) The representable presheaves $\mathbb{C}(-,C)$ need not be sheaves. But there still is a functor

$$\mathbb{C} \xrightarrow{\varepsilon} \mathrm{Sh}(\mathbb{C})$$

which sends C to the associated sheaf $a(\mathbb{C}(-,C))$. (However, ε need no longer be full and faithful, so we get a "distorted" picture of $\mathbb{C}$ inside $\mathrm{Sh}(\mathbb{C})$.)

Since a is left-adjoint to the inclusion, the Yoneda lemma still holds: morphisms $\varepsilon(C) \to F$ are in 1–1 correspondence with elements of $F(C)$, for any sheaf F.

(b) A Grothendieck topology on $\mathbb{C}$ is called *subcanonical* if all representable presheaves are sheaves. So in this case $\mathbb{C} \xrightarrow{\varepsilon} \mathrm{Sh}(\mathbb{C})$ is simply the functor $C \mapsto \mathbb{C}(-,C)$ of 2.1(c). $\mathbb{C} \xrightarrow{\varepsilon} \mathrm{Sh}(\mathbb{C})$ is again called the Yoneda embedding, and is a full and faithful functor, when the topology on $\mathbb{C}$ is subcanonical. In this case, it is common to identify C with the sheaf $\varepsilon(C)$, and omit the ε from notation.

2.7 Constructions of sheaves. (a) Since the inclusion $\mathrm{Sh}(\mathbb{C}) \hookrightarrow Sets^{\mathbb{C}^{op}}$ has a left-adjoint, *limits* of sheaves can be calculated as limits of presheaves, i.e., "pointwise" (cf. 2.2(a)),

$$(\varprojlim F_i)(C) = \varprojlim F_i(C)$$

For *colimits*, we first compute the colimit in the category of presheaves, and then take the associated sheaf.

(b) Since a preserves colimits, we conclude from 2.3 that any sheaf is a colimit of sheaves of the form $\varepsilon(C)$ (cf. 2.6). In particular, if the Grothendieck topology is subcanonical, *every sheaf is a colimit of representable (pre-) sheaves.*

(c) If G is a sheaf, then so is the presheaf G^F defined in 2.2(b), as one easily checks. So *exponentials* in the category of sheaves are constructed just like exponentials of presheaves. And as before, we have for three sheaves F, G, and H a natural 1–1 correspondence of morphisms of sheaves

$$\frac{H \longrightarrow G^F}{F \times H \to G}$$

(d) For *powersheaves* this is not quite the same. A *subsheaf* S of a sheaf F is a subpresheaf $S \subset F$ which is itself also a sheaf. Equivalently, S satisfies the following condition: for every cover

$\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ and every $x \in F(C)$,

$$x|f_\alpha \in S(C_\alpha) \text{ for all } \alpha \Rightarrow x \in S(C).$$

The powersheaf $\mathcal{P}(F)$ of F is now defined by

$$\mathcal{P}(F)(C) = \text{the set of subsheaves of } a(\mathbb{C}(-,C)) \times F.$$

Even if the Grothendieck topology is not subcanical, we can equivalently describe $\mathcal{P}(F)$ as follows: $\mathcal{P}(F)(C)$ is the set of *subpresheaves* $S \subset \mathbb{C}(-,C) \times F$ which satisfy the following condition for every cover $\{D_\alpha \xrightarrow{f_\alpha} D\}_\alpha$, and every $D \xrightarrow{g} C$ and $x \in F(D)$:

$$(g \circ f_\alpha, x|f_\alpha) \in S(D_\alpha) \text{ for all } \alpha \Rightarrow (g,x) \in S(D).$$

We leave it to the reader to check that $\mathcal{P}(F)$ is a sheaf.

As for presheaves, there is a natural 1–1 correspondence between *subsheaves* $S \subset G \times F$ and morphisms $G \xrightarrow{\sigma} \mathcal{P}(F)$ for any two sheaves G and F:

$$\frac{G \xrightarrow{\tau} \mathcal{P}(F)}{S \subset G \times F}$$

The explicit description of this correspondence is exactly as for presheaves.

(e) *Constant sheaves*: The restriction to sheaves of the global sections functor,

$$\mathrm{Sh}(\mathbb{C}) \xrightarrow{\Gamma} \mathit{Sets}, \quad \Gamma(F) = F(1),$$

has a left-adjoint Δ: for a set I, $\Delta(I)$ now is the *associated sheaf* of the *constant presheaf* defined in 2.2(d). A sheaf of the form $\Delta(I)$ is called a *constant sheaf*. (Notice that if we write $\Delta_p(I)$ for the constant presheaf defined in 2.2(d), $\Delta_p(I)^+$ is already a sheaf, so

$$\Delta(I) = a(\Delta_p(I)) = \Delta_p(I)^+,$$

and we don't need to apply $(-)^+$ once more.)

3 Forcing

We will now explain how we can regard sheaves as sets, and interpret set-theoretic language in the context of sheaves. Throughout this section, $\mathbb{C}$ is a category with finite limits, equipped with a fixed Grothendieck topology J as in 1.1(b). Intuitively, the sheaves X on $\mathbb{C}$ are regarded as *sets*, and *elements* of X are the elements of $X(C)$

for $C \in \mathbb{C}$; we say that $x \in X(C)$ is an *element of* X *at stage* C.

3.1 Basic definitions. Recall that for each sheaf X on $\mathbb{C}$, we can form the *powersheaf* $P(X)$ which plays the role of the power-set, for n sheaves $X_1, \ldots, X_n$ we can form the *product* $X_1 \times \ldots \times X_n$ which plays the role of the cartesian product, and for two sheaves X and Y we can form the *exponential* Y^X, which plays the role of the set of functions from X to Y.

To build up the language, we need to select some basic *properties*, *functions* and *elements*. A basic n-place property P of type $X_1 \times \ldots \times X_n$ is just a subsheaf of $X_1 \times \ldots \times X_n$; a basic function f of n variables from $X_1 \times \ldots \times X_n$ to Y is just a morphism $X_1 \times \ldots \times X_n \xrightarrow{f} Y$ in $\mathrm{Sh}(\mathbb{C})$; and a basic element b of X is a morphism $1 \xrightarrow{b} X$, i.e., an element $b \in \Gamma(X)$.

(Some typical examples in this book, cf. chapters II and III, use the basic properties $<$ and $\leq$ which are subsheaves of $R \times R$, the basic functions $+, \times : R \times R \to R$, and the basic elements 0 and 1 of R.)

From these basic properties, functions and elements, we can build *formulas* and *terms* (= names for elements) in the usual way. We assume that for each sheaf X we are given a stock of free variables $x, y, z, x_1, x_2, \ldots$ ranging over X.

Each term $t(x_1, \ldots, x_n)$ with free variables *among* $x_1, \ldots, x_n$, ranging over the sheaves $X_1, \ldots, X_n$ respectively, will be interpreted as a morphism from $X_1 \times \ldots \times X_n$ to the appropriate codomain. We specify this morphism by giving the components, denoted by t_C $(C \in \mathbb{C})$. So if $a_1 \in X_1(C), \ldots, a_n \in X_n(C)$ are elements of the appropriate sheaves, we define

(1) $t_C(a_1, \ldots, a_n)$ inductively.

Moreover, for each formula $\varphi(x_1, \ldots, x_n)$ with free variables as above, we will specify for each n-tuple $a_1 \in X_1(C), \ldots, a_n \in X_n(C)$ whether φ holds for $a_1, \ldots, a_n$. We will write

(2) $C \Vdash \varphi(a_1, \ldots, a_n)$

to denote that the formula $\varphi(x_1, \ldots, x_n)$ holds for the elements a_i at stage C.

There is a standard way of going from formulas to terms in a set-theoretic context: if $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a *formula* with free variables ranging over the sheaves $X_1, \ldots, X_n, Y_1, \ldots, Y_m$, then

(3) $\{(x_1, \ldots, x_n) | \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\}$
is a *term*, which is of course meant to be interpreted as a morphism

$$Y_1 \times \ldots \times Y_m \to P(X_1, \ldots, X_n).$$

Supposing we have specified the interpretation of φ in the form (2), this morphism interpreting the term in (3), which we abbreviate as $\{\underline{x}|\varphi\}$, is defined by the components

$$\begin{array}{ll} & \{\underline{x}|\varphi\}_C : Y_1(C) \times \ldots, \times Y_m(C) \to P(X_1 \times \ldots \times \\ (4) & X_n)(C)(f, a_1, \ldots, a_n) \in \{\underline{x}|\varphi\}_C(b_1, \ldots, b_m) \text{ iff} \\ & D \Vdash \varphi(a_1, \ldots, a_n, b_1|f, \ldots, b_m|f) \end{array}$$

for $b_i \in Y_i(C), a_j \in X_j(b)$, and $D \xrightarrow{f} C$).

See also 3.2, where it is noted that this is indeed well-defined.

Let us now give the inductive clauses for the interpretation of the terms and formulas, needed in addition to (4). Let $f(x_1, \ldots, x_n)$ be a term whose free variables are among $x_1, \ldots, x_n$. The corresponding morphism $X_1 \times \ldots \times X_n \xrightarrow{f} Y$ (for the appropriate Y) is defined inductively as follows:

(5) for $t = x_i$ (some $i, 1 \leq i \leq n$), we just take the projection $X_1 \times \ldots \times X_n \to X_i$, i.e., $(x_i)_C = \pi_i : X_1(C) \times \ldots, \times X_n(C) \to X_i(C)$.

(6) for $t = b$, some basic element of some Y, the corresponding morphism $X_1 \times \ldots \times X_n \xrightarrow{b} Y$ is given by

$$b_C(a_1, \ldots, a_n) = b|(C \to 1).$$

for all $a_i \in X_i(C)$, and $C \in \mathbb{C}$ (here $C \to 1$ denotes the unique morphism from C to the terminal object 1).

(7) for $t = f(t_1, \ldots, t_n)$, where f and the t_i are terms with free variables among $x_1, \ldots, x_n$, whose interpretations are already defined as morphisms with components

$$\begin{array}{l} (t_i)_C : X_1(C) \times \ldots \times X_n(C) \to Y_i(C) \\ f_C : X_1(C) \times \ldots \times X_n(C) \to Z^{(Y_1 \times \ldots \times Y_m)}(C), \end{array}$$

then

$$t_C : X_1(C) \times \ldots \times X_n(C) \to Z(C)$$

is defined by

$$t_C(a_1, \ldots, a_n) = f_C(\mathrm{id}_C, (t_1)_C(a_1, \ldots, a_n), \ldots, (t_m)_C(a_1, \ldots, a_n)).$$

The interpretation of a *formula* $\varphi(x_1, \dots, x_n)$ (cf. (2) above) is defined inductively as follows:

(8) if $\varphi(x_1, \dots, x_n)$ is $t = t'$ for two terms t and $'$, then

$$C \Vdash \varphi(a_1, \dots, a_n) \text{ iff } t_C(a_1, \dots, a_n) = t'_C(a_1, \dots, a_n).$$

(9) if t and S are terms with free variables among $x_1, \dots, x_n$, which are interpreted as morphisms $X_1 \times \dots \times X_n \xrightarrow{t} Y$ and $X_1 \times \dots \times X_n \xrightarrow{S} P(Y)$, the interpretation of the formula $t \in S$ is specified by

$$C \Vdash t(a_1, \dots, a_n) \in S(a_1, \dots, a_n) \text{ iff } (\mathrm{id}_C, t_C(a_1, \dots, a_n)) \in S(C).$$

(10) if P is a basic property corresponding to a subsheaf P of $X_1 \times \dots \times X_n$, then $P(x_1, \dots, x_n)$ is a formula, and for $a_i \in X_i$ we have

$$C \Vdash P(a_1, \dots, a_n) \text{ iff } (a_1, \dots, a_n) \in P(C).$$

(One can of course *specify* $P \subset X_1 \times \dots \times X_n$ by giving a definition of $C \Vdash P(a_1, \dots, a_n)$ right-away, and taking this equivalence as a definition of P as a subsheaf.)

These clauses take care of the "atomic" formulas. For the logical connectives, we have the following classes:

(11) $C \Vdash \varphi \wedge \psi(a_1, \dots, a_n)$ iff $C \Vdash \varphi(a_1, \dots, a_n)$ and $C \Vdash \psi(a_1, \dots, a_n)$,

(12) $C \Vdash \varphi \vee \psi(a_1, \dots, a_n)$ iff $\exists R \in J(C)$ such that for each $D \xrightarrow{f} C \in R$, either $D \Vdash \varphi(a_1|f, \dots, a_n|f)$ or $D \Vdash \psi(a_1|f, \dots, a_n|f)$,

(13) $C \Vdash \neg \varphi(a_1, \dots, a_n)$ iff for every $D \xrightarrow{f} C$, *if* $D \Vdash \varphi(a_1|f, \dots, a_n|f)$ then $0 \in J(D)$,

(14) $C \Vdash \varphi \to \psi(a_1, \dots, a_n)$ iff for every $D \xrightarrow{f} C$, if $D \Vdash \varphi(a_1|f, \dots, a_n|f)$ then $D \Vdash \psi(a_1|f, \dots, a_n|f)$,

(15) $C \Vdash \forall y \in Y \varphi(y, a_1, \dots, a_n)$ iff for every $D \xrightarrow{f} C$ and every $b \in Y(D), D \Vdash \varphi(b, a_1, |f, \dots, a_n|f)$

(16) $C \Vdash \exists y \in Y \varphi(y, a_1, \dots, a_n)$ iff $\exists R \in J(C)$ such that for every $D \xrightarrow{f} C \in R$ there is a $b \in Y(D)$ with $D \Vdash \varphi(b, a_1|f, \dots, a_n|f)$.

This completes the description of the interpretation of terms and formulas.

3.2 Functoriality, local character. To see that (4) is well-defined, i.e., that $\{x|\varphi\}_C$ is indeed a subsheaf of $y(C) \times X_1 \times \ldots \times X_n$, we need the following two basic properties of $\Vdash$, which can be proved by induction on φ (following the classes (8)–(16)).

Functoriality: if $C \Vdash \varphi(a_1, \ldots, a_n)$ and $D \xrightarrow{f} C$, then $D \Vdash \varphi(a_1|f, \ldots, a_n|f)$

Local character: if $R \in J(C)$ and if $a_i \in X_i(C)$ are such that for each $D \xrightarrow{f} C \in R$, $D \Vdash \varphi(a_1|f, \ldots, a_n|f)$, then $C \Vdash \varphi(a_1, \ldots, a_n)$.

3.3 Other versions of Grothendieck topology. If K is a Grothendieck topology on $\mathbb{C}$ in the sense of 1.1(c), then we can define $C \Vdash \varphi(a_1, \ldots, a_n)$ inductively as in 3.1, by referring to the Grothendieck topology J (in the sense of 1.1(b)) defined from K as in 1.1(e):

$$R \in J(C) \text{ iff } \exists S \subset R \quad S \in K(C)$$

Using 3.2, one then easily checks that *if* K satisfies (i′), (ii′) and (iii′) or the weaker (iii″), then

(12′) $C \Vdash \varphi \vee \psi(a_1, \ldots, a_n)$ iff $\exists \{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in K(C)$ such that for all α, $C_\alpha \Vdash \varphi(a_1|f_\alpha, \ldots, a_n|f_\alpha)$, or $C_\alpha \Vdash \psi(a_1|f_\alpha, \ldots, a_n|f_\alpha)$

(13′) $C \Vdash \neg\varphi(a_1, \ldots, a_n)$ iff for any $D \xrightarrow{f} C$, if $D \Vdash \varphi(a_1|f, \ldots, a_n|f)$ then $\emptyset \in K(D)$.

(16′) $C \Vdash \exists y \in Y \varphi(y, a_1, \ldots, a_n)$ iff $\exists \{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \in K(C)$ such that for all α there is a $b_\alpha \in Y(C_\alpha)$ with $C_\alpha \Vdash \varphi(b_\alpha, a_1|f_\alpha, \ldots, a_n|f_\alpha)$.

In other words, we can define $C \Vdash \varphi(a_1, \ldots, a_n)$ by induction, directly in terms of K, using the clauses (8)–(16) as in 3.1 but with (12′), (13′), (16′) instead of (12), (13), (16).

3.4 Validity. A formula $\varphi(x_1, \ldots, x_n)$ is called *valid* in $\mathrm{Sh}(\mathbb{C})$ if

$$1 \Vdash \forall x_1 \in X_1 \ldots \forall x_n \in X_n \varphi(x_1, \ldots, x_n),$$

(where the free variables of φ are among the x_i, x_i ranging over the sheaf X_i; 1 is the terminal object of $\mathbb{C}$). If $\varphi(x_1, \ldots, x_n)$ is *valid* in $\mathrm{Sh}(\mathbb{C})$, we write

$$\mathrm{Sh}(\mathbb{C}) \models \varphi(x_1, \ldots, x_n).$$

So we have

$$\mathrm{Sh}(\mathbb{C}) \models \varphi(x_1, \ldots, x_n) \text{ iff for all } C \in \mathbb{C} \text{ and all } a_i \in X_i(C) (i = 1, \ldots, n)\ C \Vdash \varphi(a_1, \ldots, a_n).$$

3.5 Basic principle of validity for intuitionistic logic. With the notion of forcing as defined in 3.1 (or 3.4), all the provable formulas of *intuitionistic* logic (in the appropriate language) are valid. Without going into the details of setting up an appropriate axiomatization, we can express this informally by stating that all arguments about and constructions of *sets* which do *not* make use of the excluded middle (i.e., the axiom $\varphi \vee \neg\varphi$) or the axiom of choice ($\forall x \in X\, \exists y \in Y \varphi(x,y) \to \exists f \in Y^X\, \forall x \in X\; \varphi(x, fx)$) i.e., all arguments and constructions which are *constructive* and *explicit*, can be correctly performed in the context of *sheaves* on a site $\mathbb{C}$.

For example, a typical principle of intuitionistic set theory is the *comprehension schema*; in fact there are two, one for sets and one for functions: for any formula $\varphi(x,y)$,

$$\begin{aligned} \mathrm{Sh}(\mathbb{C}) \;&\models\; \forall y \in Y\; \exists! z \in P(X)\, \forall x \in X (x \in z \leftrightarrow \varphi(x,y)) \\ \mathrm{Sh}(\mathbb{C}) \;&\models\; \forall x \in X\; \exists! y \in Y\; \varphi(x,y) \to \exists! f \in Y^X\; \forall x \in X\; \forall y \in Y \\ &\quad\;\; (f(x) = y \leftrightarrow \varphi(x,y)) \end{aligned}$$

(recall that $\exists! z \in Z \psi(z)$ is regarded as an abbreviation of $\exists z \in Z(\psi(z) \wedge \forall z' \in Z\; \psi(z') \to z = z')$.

For some precise formulations of this basic principle ("soundness theorems for intuitionistic logic and set theory"), see the references given in the beginning of this Appendix.

3.6 The natural numbers object. This is the constant sheaf $\Delta(\mathbb{N})$. The zero-element $0 \in \mathbb{N}$ is also a basic element of the sheaf $\Delta(\mathbb{N})$, clearly, since

$$\mathbb{N} \subset \Gamma\Delta(\mathbb{N})$$

(this is an inclusion, except when the site is trivial in the sense that $\phi \in J(1)$), and the successor function $S\colon \mathbb{N} \to \mathbb{N}$ gives a morphism $\Delta(S)\colon \Delta(\mathbb{N}) \to \Delta(\mathbb{N})$. Similarly, $+$ and $\cdot$ give morphisms $\Delta(\mathbb{N}) \times \Delta(\mathbb{N}) \to \Delta(\mathbb{N})$. In other words, the *language of arithmetic* is part of the set-theoretic language of the topos $\mathrm{Sh}(\mathbb{C})$, which we can interpret as in 3.1. Under this interpretation all axioms of Peano arithmetic are valid.

We will usually just write $\mathbb{N}$ for $\Delta(\mathbb{N})$, so $\mathbb{N}$ stands for both the set of natural numbers and the corresponding constant sheaf. With this notational convention, the validity of the induction principle, for example, is formulated as

(1) $\mathrm{Sh}(\mathbb{C}) \models \forall x \in P(\mathbb{N})[0 \in x \wedge \forall y \in \mathbb{N}(y \in x \to Sy \in x) \to \forall y \in \mathbb{N}\; y \in x]$.

As said, (1) is just part of the fact that all axioms of Peano arithmetic are valid. One should keep in mind, however, that only intuitionistically correct arguments are valid in $\mathrm{Sh}(\mathbb{C})$. So in general, we conclude that all provable formulas of ***intuitionistic*** ("higher order") ***arithmetic*** are valid in $\mathrm{Sh}(\mathbb{C})$.

For formulas $\varphi(x_1,\ldots,x_n)$ of ***first-order arithmetic*** (usual classical Peano arithmetic, i.e., formulas that do not make use of powersheaves and exponentials, one can easily show by induction on φ that for every $C \in \mathbb{C}$ with $\phi \notin J(C)$:

$$C \Vdash \varphi(n_1,\ldots,n_k) \text{ iff } \varphi(n_1,\ldots,n_k) \text{ is true,}$$

where on the left, the numbers $n_i \in \mathbb{N}$ are regarded as elements of $\Delta(\mathbb{N})(C)$, in the obvious way. So for ***first-order*** formulas of arithmetic, one doesn't only get the intuitionistically provable, but all the classically provable (even: classically true) formulas as valid in $\mathrm{Sh}(\mathbb{C})$.

3.7 Finite, decidable, non-empty, and inhabited sheaves.

(a) A sheaf X is called ***finite*** if

$$\mathrm{Sh}(\mathbb{C}) \models \exists n \in \mathbb{N}\, \exists f \in X^{\{0,\ldots,n-1\}} (\forall x \in X\, \exists i < n\; x = f(i))$$

(perhaps it would be better to say "locally finite"). More generally, the property "***to be finite***" is defined for elements of $\mathcal{P}(Y)$, Y any sheaf, by setting for each $S \in \mathcal{P}(Y)(C)$, C an object of $\mathbb{C}$,

$$\begin{gathered} C \Vdash (S \text{ is finite}) \text{ iff} \\ C \Vdash \exists n \in \mathbb{N}\, \exists f \in Y^{\{0,\ldots,n-1\}} \forall s \in S\, \exists i < n\; s = f(i). \end{gathered}$$

(b) A sheaf X is called ***decidable*** if

$$\mathrm{Sh}(\mathbb{C}) \models \forall x, y \in X (x = y \vee \neg x = y)$$

For example, $\mathbb{N}$ is a decidable sheaf; in fact any constant sheaf is decidable. (And as in (a), we can define more generally for $S \in \mathcal{P}(Y)(C), C \Vdash$ "S is decidable" if $C \Vdash \forall s, t \in S(s = t \vee s \neq t)$.)

(c) Intuitonistically, ***non-empty*** is ***not*** the same thing as ***having an element***: for $S \in \mathcal{P}(Y)(C)$,

$$C \Vdash \text{"}S \text{ is non-empty"} \quad \begin{array}{l} \text{iff } C \Vdash \neg S = \phi \\ \text{iff } C \Vdash \neg \forall y \in Y \neg y \in S \end{array}$$

A set which has an element is called ***inhabited***:

$$C \Vdash \text{"}S \text{ is inhabited" iff } C \Vdash \exists y \in Y\; y \in S.$$

So "S is non-empty $\leftrightarrow \neg\neg$ (S is inhabited)" is valid in $\mathrm{Sh}(\mathbb{C})$. If we regard a sheaf X as an element of $P(X)(1)$, we get as special cases that

$$\mathrm{Sh}(\mathbb{C}) \models \text{"}X \text{ is non-empty" iff } \forall C \in \mathbb{C}\ \exists \text{ morphism } D \xrightarrow{f} C \text{ with } \phi \notin J(D) \text{ and } X(D) \neq \phi,$$

$$\mathrm{Sh}(\mathbb{C}) \models \text{"}X \text{ is inhabited" if } \exists R \in J(1) \forall D \to 1 \in R\ X(D) \neq \phi.$$

3.8 Generic elements. Let $C \in \mathbb{C}$, and let $\varepsilon(C)$ be the corresponding sheaf ($\varepsilon(C) = a(\mathbb{C}(-,C))$, see 2.6). If $\varphi(z, x_1, \ldots, x_n)$ is a formula with z ranging over $\varepsilon(C)$ and x_i over a sheaf X_i, then we have for any $D \in \mathbb{C}$ and $a_i \in X_i(D)$:

(1) $D \Vdash \forall y \varphi(y, a_1, \ldots, a_n)$ iff $D \times C \Vdash \varphi(\pi_2, a_1|\pi_1, \ldots, a_n|\pi_1)$,

where π_1 is the projection $D \times C \to D$, and π_2 denotes the element of $\varepsilon(C)(D \times C)$ corresponding to the projection $D \times C \to C$ in $\mathbb{C}$.

Proof of (1). We will only prove this in the case where the Grothendieck topology is subcanonical (cf. 2.6). $\Rightarrow$ is clear from the definition (take $f = \pi_1 \colon D \times C \to D$ in (15) of 3.1). For $\Leftarrow$, take $E \xrightarrow{f} D$ in $\mathbb{C}$, and an element $g \in \varepsilon(C)(E) = \mathbb{C}(E, C)$. According to (15) of 3.1, we have to show that

(2) $E \Vdash \varphi(g, a_1|f, \ldots, a_n|f)$.

But by assumption, $D \times C \Vdash \varphi(\pi_2, a_1|\pi_1, \ldots, a_n|\pi_1)$, and hence by functoriality (3.2) applied to $E \xrightarrow{(f,g)} D \times C$,

$$E \Vdash \varphi(\pi_2|(f,g), (a_1|\pi_1)|(f,g), \ldots, (a_n|\pi_1)|(f,g)).$$

Since $\pi_2 \circ (f,g) = g$ and $(a_i|\pi_1)|(f,g) = a_i|f$, this simply means that (2) holds.

The element $\pi_2 \in \varepsilon(C)(D \times C)$ occurring on the right-hand side of (1) is called *the generic element of C at D* (recall that we usually omit the "ε" if the topology is subcanonical).

3.9 Generating elements. Let X be a sheaf on $\mathbb{C}$, and let G be a function assigning to each object C a (possibly empty) subset $G(C) \subset X(C)$. G is said to *generate* X if the smallest subsheaf $Y \supset X$ with $G(C) \subset Y(C)$ (for all objects $C \in \mathbb{C}$) is X itself. In other words, G *generates* X iff for all $C \in \mathbb{C}$ and all $x \in X(C)$, there is a cover $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ and for each α a map $C_\alpha \xrightarrow{g_\alpha} D_\alpha$ such that $\exists y \in G(D_\alpha) y|g_\alpha = x|f_\alpha$. In other words every $x \in X$ is *locally the restriction* of something in G.

If G generates X and $\varphi(x, y_1, \ldots, y_n)$ is a formula with x ranging over X, y_i over Y_i, then it is an easy consequence of 3.2 that for any $C \in \mathbb{C}$ and $b_i \in Y_i(C)$,

$$C \Vdash \forall x \varphi(x, b_1, \ldots, b_n) \text{ iff for all pairs of morphisms } E \xleftarrow{g} D \xrightarrow{f} C \text{ and all } a \in G(E), D \Vdash \varphi(a|g, b_1|f, \ldots, b_n|f)$$

$$C \Vdash \exists x \in X \varphi(x, b_1, \ldots, b_n) \text{ iff there are a cover } \{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \text{ and morphisms } C_\alpha \xrightarrow{g_\alpha} D_\alpha \text{, such that for each } \alpha \text{ there is an } a_\alpha \in G(D_\alpha) \text{ with } C_\alpha \Vdash \varphi(a_\alpha|g_\alpha, b_1|f_\alpha, \ldots, b_n|f_\alpha).$$

In particular, if $X = \Delta(I)$ for a set I, there is for every $u \in I$ a corresponding element $\overline{u} \in \Delta(I)(C)$, and 3.2 gives

$$C \Vdash \forall x \in \Delta(I) \varphi(x, b_1, \ldots, b_n) \text{ iff for all } u \in I \; C \Vdash \varphi(\overline{u}, b_1, \ldots, b_n)$$

$$C \Vdash \exists x \in \Delta(I) \varphi(x, b_1, \ldots, b_n) \text{ iff there are a cover } \{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha \text{ and for each } \alpha \text{ a } u_\alpha \in I \text{ such that } C_\alpha \Vdash \varphi(\overline{u}, b_1|f_\alpha, \ldots, b_n|f_\alpha),$$

as follows from the fact that the elements of the form $\overline{u}$ generate $\Delta(I)$ (i.e., if $G(C) = \{\overline{u} | u \in I\} \subset \Delta(I)(C)$, G generates I).

3.10 Notational convention. We will often just write b for $b|f$, if there is no possible misunderstanding. For example, (1) of 3.8 can be written as

$$D \Vdash \forall y \; \varphi(y, a_1, \ldots, a_n) \text{ iff } D \times C \Vdash \varphi(\pi_2, a_1, \ldots, a_n),$$

since the right-hand side only makes sense if a_i stands for $a_i|g$ for some morphism $D \times C \xrightarrow{g} D$, and without any further data, the only g there is is the projection π_1.

Appendix 2: A survey of models

It would perhaps be useful to give a list of the toposes discussed in this book, and other models occurring in the literature which we have only mentioned in passing, or not at all. All these models will be presented as sheaves on a site whose underlying category is a full subcategory of the category $\mathbb{L}$ of loci. Recall that the ***objects*** of $\mathbb{L}$ are duals $\ell(C^\infty(\mathbb{R}^n)/I)$ of C^∞-rings $C^\infty(\mathbb{R}^n)/I$. The ***morphisms*** $\ell(C^\infty(\mathbb{R}^n)/I) \to \ell(C^\infty(\mathbb{R}^m)/J)$ of $\mathbb{L}$ are equivalence classes of smooth maps $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^m$ with the property that $g \in J \to g\varphi \in I$ (two such φ and φ' being equivalent if $\pi_i \circ \varphi - \pi_i \circ \varphi' \in I (i = 1, \ldots, m)$).

The other categories are described by taking only those objects $\ell(C^\infty(\mathbb{R}^n)/I)$ where I satisfies a specific condition, according to the following list:

$\mathbb{E}$ - $f \in I$ iff $f|Z(I) \equiv 0$ (i.e., $I = m^0_{Z(I)}$)

$\mathbb{F}$ - I is closed (I.4.1–2)

$\mathbb{G}$ - I is germ-determined (I.4.1–2)

$\mathbb{L}_{\mathrm{fp}}$ - I is finitely generated

$\mathbb{V}$ - I is C^∞-radical, i.e., $f \in I$ iff $\exists g \in I Z(f) = Z(g)$

$\mathbb{V}_\omega$ - I is C^∞-radical, and countably generated as a C^∞-radical ideal, in the sense that there are $g_n (n \in \mathbb{N})$ such that $f \in I$ iff $\exists_n Z(g_n) \subset Z(f)$.

$\mathbb{C}$ - $C^\infty(\mathbb{R}^n)/I$ is of the form $C^\infty(M) \otimes_\infty W$, where M is a manifold and $W \cong C^\infty(\mathbb{R}^{n-k})/J$ is a Weil algebra (I.3.13)

$\mathbb{W}$ - $C^\infty(\mathbb{R}^n)/I$ is of the form $A \otimes_\infty W$, where $\ell(A) \in \mathbb{V}$ and W is a Weil algebra

$\mathbb{W}_\omega$ - as $\mathbb{W}$, but with $\mathbb{V}_\omega$ instead of $\mathbb{V}$.

Remarks. In each case, we identify the category above with the equivalent category whose objects are $\mathbb{L}$-objects *isomorphic* to ones of the type specified in the list. $\mathbb{E}$ is just the category of closed subspaces of $\mathbb{R}^n (n \geq 0)$ and smooth maps, or since each open $U \subset \mathbb{R}^n$ is diffeomorphic to a closed $\hat{U} \subset \mathbb{R}^{n+1}$, $\mathbb{E}$ "is" also the category of locally closed subspaces of $\mathbb{R}^n (n \geq 0)$, cf.I.1.5. C^∞-radical ideals were considered in Moerdijk-Reyes (1986), Moerdijk-Quê-Reyes (1987). It is possible to give more intrinsic descriptions of categories like $\mathbb{C}$ and $\mathbb{W}$ (or at least of the associated toposes), but we will not go into this here.

Notice that there are inclusion functors as in the diagram below.

The inclusions marked with $\times$ do ***not*** admit a right adjoint; all others do, and so does the composite $\mathbb{V} \hookrightarrow \mathbb{L}$.

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, hook] \arrow[d, hook] & & \mathbb{F} \arrow[r, hook] & \mathbb{G} \arrow[dr, hook] & \\
\mathbb{V}_\omega \arrow[r, hook, "\times"] \arrow[d, hook] & \mathbb{V} \arrow[d, hook, "\times"] & \mathbb{C} \arrow[r, hook] \arrow[u, hook, "\times"] & \mathbb{L}_{\mathrm{fp}} \arrow[r, hook, "\times"] \arrow[u, hook, "\times"] & \mathbb{L} \\
\mathbb{W}_\omega \arrow[r, hook, "\times"] & \mathbb{W} \arrow[urrr, hook, "\times"'] & & &
\end{tikzcd}$$

Some of these right adjoints have been discussed in §II.1. Here, we will use r to denote *any* of them.

Each of these categories can be made into a site by equipping it with one of five Grothendieck topologies, the covers for which are

(i) isomorphisms (indiscrete or "chaotic" topology)
(ii) finite open covers
(iii) finite open covers and projections
(iv) open covers
(v) open covers and projections,

respectively. *Open covers* are of the form

$$\{r(\ell(C^\infty(U_i)/(I|U_i)) \hookrightarrow \ell(C^\infty(\mathbb{R}^n)/I)\}_i$$

where r reflects back into the category in question when necessary, and the U_i form an open cover of $\mathbb{R}^n$. *Projections* are singleton covers of the form

$$\{\ell(A) \times \ell(B) \to \ell(A)\}$$

where B is not the trivial ring (i.e., $0 \neq 1$ in B).

The following table gives the toposes corresponding to the sites that we consider. Not all combinations occur, since we restrict our attention to *subcanonical* topologies (so e.g., $\mathbb{L}$ with open covers is excluded). Furthermore, open $=$ open with projections, for all subcanonical sites considered.

	(v)		(iv)		(iii)		(ii)		(i)
$\mathbb{L}$	—		—		$\mathcal{B}$	$\hookrightarrow$	$\mathcal{Z}$	$\hookrightarrow$	$\mathit{Sets}^{\mathbb{L}^{op}}$
$\mathbb{F}$	$\mathcal{F}$	$\cong$	$\mathcal{F}$	$\hookrightarrow$	$\mathcal{F}_{\mathrm{fin}}$	$\cong$	$\mathcal{F}_{\mathrm{fin}}$	$\hookrightarrow$	$\mathit{Sets}^{\mathbb{F}^{op}}$
$\mathbb{G}$	$\mathcal{G}$	$\cong$	$\mathcal{G}$	$\hookrightarrow$	$\mathcal{G}_{\mathrm{fin}}$	$\cong$	$\mathcal{G}_{\mathrm{fin}}$	$\hookrightarrow$	$\mathit{Sets}^{\mathbb{G}^{op}}$
$\mathbb{L}_{fp}$	$\mathcal{L}[\mathrm{locAr}C^\infty]$	$\cong$	$\mathcal{L}[\mathrm{locAr}C^\infty]$	$\hookrightarrow$	$\mathcal{L}[\mathrm{loc}C^\infty]$	$\cong$	$\mathcal{L}[\mathrm{loc}C^\infty]$	$\hookrightarrow$	$\mathcal{L}[C^\infty]$
$\mathbb{C}$	$\mathcal{C}$	$\cong$	$\mathcal{C}$	$\hookrightarrow$	$\mathcal{C}_{\mathrm{fin}}$	$\cong$	$\mathcal{C}_{\mathrm{fin}}$	$\hookrightarrow$	$\mathit{Sets}^{\mathbb{C}^{op}}$
$\mathbb{V}$	—		—		$\mathcal{V}_{\mathrm{finpr}}$	$\hookrightarrow$	$\mathcal{V}_{\mathrm{fin}}$	$\hookrightarrow$	$\mathit{Sets}^{\mathbb{V}^{op}}$
$\mathbb{W}$	—		—		$\mathcal{W}_{\mathrm{finpr}}$	$\hookrightarrow$	$\mathcal{W}_{\mathrm{fin}}$	$\hookrightarrow$	$\mathit{Sets}^{\mathbb{W}^{op}}$
$\mathbb{E}$	$\mathcal{E}$	$\cong$	$\mathcal{E}$	$\hookrightarrow$	$\mathcal{E}_{\mathrm{fin}}$	$\cong$	$\mathcal{E}_{\mathrm{fin}}$	$\hookrightarrow$	$\mathit{Sets}^{\mathbb{E}^{op}}$

(The lines for $\mathbb{V}_\omega, \mathbb{W}_\omega$ are as for $\mathbb{V}, \mathbb{W}$). The horizontal arrows indicate the obvious inclusions (or equivalences) of toposes. There are also many *vertical* geometric morphisms, by the following result.

Lemma. *Let $\mathbb{A}$ and $\mathbb{B}$ be sites, and let $\mathbb{A} \xrightarrow{e} \mathbb{B}$ be a functor which has the covering lifting property (i.e., for every cover $\{B_i \xrightarrow{g_i} eA\}_i$ in $\mathbb{B}$ there is a cover $\{A_j \xrightarrow{f_j} A\}_j$ in $\mathbb{A}$ such that each $e(f_j)$ factors through some g_i)*

$$\begin{tikzcd} e(A_j) \arrow[rr] \arrow[dr] & & B_i \arrow[dl] \\ & e(A) & \end{tikzcd}$$

Then e induces a geometric morphism

$$\eta\colon \mathrm{Sh}(\mathbb{A}) \to \mathrm{Sh}(\mathbb{B})$$

given by

$$\eta^*(Y) = a(Y \circ e)$$

for a sheaf Y on $\mathbb{B}$ (a is the associated sheaf functor),

$$\eta_*(X)(B) = \mathit{Sets}^{\mathbb{A}^{op}}(\mathbb{B}(e-, B), X)$$

for a sheaf X on $\mathbb{A}$ ($\eta_*(X)$ is indeed a sheaf on $\mathbb{B}$, by the covering lifting property).

One may check that each of the inclusions in the diagram (1) above has the covering lifting property, hence giving corresponding geometric morphisms in the columns of the table. For instance, for the topology given by fnite open covers and projections, we get

$$\begin{tikzcd}
\mathcal{E}_{\text{fin}} \arrow[rr] \arrow[dd] & & \mathcal{F}_{\text{fin}} \arrow[rr] & & \mathcal{G}_{\text{fin}} \arrow[dd] \\
& \mathcal{C}_{\text{fin}} \arrow[ur] \arrow[dr] & & & \\
\mathcal{V}_{\text{fin pr}} \arrow[rr] & & \mathcal{W}_{\text{fin pr}} \arrow[rr] & & \mathcal{B}
\end{tikzcd}$$

There are more vertical geometric morphisms then just those of the type given by the lemma, due to the existence of a right adjoint r. For instance, we have geometric morphisms

$$\mathcal{G} \underset{\rho}{\overset{\gamma}{\rightleftarrows}} \mathcal{E}$$

with $\rho\gamma = \text{id}$, $\gamma^* \dashv \gamma_* = \rho^* \dashv \rho_*$ (similarly for $\mathcal{F} \rightleftarrows \mathcal{E}, \mathcal{G}_{\text{fin}} \rightleftarrows \mathcal{E}_{\text{fin}}$, etc.). For details, the reader may consult Moerdijk-Reyes (1984).

In the fourth line of the table, we have used the notation for *classifying toposes.* Recall that a C^∞-ring A in a topos $\mathcal{T}$ is a functor $C^\infty \xrightarrow{A} \mathcal{T}$ which preserves products, where C^∞ is the category of smooth maps between the spaces $\mathbb{R}^n, n \geq 0$. $A(\mathbb{R})$ is a ring object of $\mathcal{T}$, for which we also write A (as in chapter I). By the use of sheaf semantics (Appendix 1), we can give an interpretation in $\mathcal{T}$ to the statements "A is a C^∞-ring", "A is a local C^∞-ring", "A is a local Archimedean C^∞-ring". Without proof, we can then state

Proposition. $\mathcal{S}[C^\infty], \mathcal{S}[\text{loc}C^\infty], \mathcal{S}[\text{locAr}C^\infty]$ *are the classifying toposes for the theories of C^∞-rings, of local C^∞-rings and of local Archimedean C^∞-rings, respectively.*

What this means is the following (say for $\mathcal{S}[\text{loc}C^\infty]$). The underlying set functor $R \in \mathcal{S}[\text{loc}C^\infty]$ is a local C^∞-ring, and in fact that generic local C^∞-ring, in the sense that a local C^∞-ring A in a topos $\mathcal{T}$ is classified by a geometric morphisms $\mathcal{T} \xrightarrow{p} \mathcal{S}[\text{loc}C^\infty]$ with $p^*(R) = A$

$$
\begin{tikzcd}
 & \mathcal{S}[locC^\infty] \arrow[d, "p^*"] \\
C^\infty \arrow[ur, "R"] \arrow[r, "A"'] & \mathcal{T}
\end{tikzcd}
$$

More precisely, R induces an equivalence of categories between the category of geometric morphisms $\mathcal{T} \xrightarrow{p} \mathcal{S}[\mathrm{loc}C^\infty]$ and the category of local C^∞-rings in $\mathcal{T}$ (by sending p to $p^*(\mathbb{R})$). (For more information about classifying toposes, see Johnstone (1977), Makkai-Reyes (1977).)

The toposes based on $\mathbb{V}$ contain invertible infinitesimals but no nilpotent ones, and are thus more like models for constructive nonstandard analysis. The topos $\mathcal{C}$ is sometimes called the "Cahiers topos", see Dubuc (1979), Kock (1981). The models based on $\mathbb{W}$ and $\mathbb{W}_\omega$ are in some sense the simplest which contain both nilpotent and invertible infinitesimals.

The list of models given here is not exhaustive. What one would like is *subcanonical* topologies which have as many covers as possible, but are still manageable and explicitly definable. For instance, one can add the map

$$\ell(C^\infty(\mathbb{R})) \xrightarrow{t^2} \ell(C^\infty(\mathbb{R}_{\geq 0}))$$

as a cover of $\mathbb{L}$, in addition to the finite open covers and the projections (cf. also Appendix 3, theorem 4). The resulting topos is discussed in Moerdijk-Quê-Reyes (to appear).

Appendix 3: The integration axiom

We now turn to the verification of the integration axiom

(IA) $\forall f \in R^{[0,1]} \exists! g \in R^{[0,1]}(g(0) = 0 \wedge g' \equiv f)$.

We shall only sketch the argument, since details can be found in Moerdijk, Quê, Reyes (to appear), to which we will refer in this appendix as [MQR].

The idea is to establish first the "synthetic" lemma 1 below, which essentially reduces that verification to the density of squares in $R_{\geq 0}$, in the sense of part (iii) of lemma 1. (iii) in turn is a consequence of Whitney's lemma on even functions (lemma 2) and a result on flat functions (lemma 3). This proof of the integration axiom applies uniformly to all the models listed in the previous section, whereas the argument given in II.2 (based directly on theorem I.4.12) does not apply to models for which [0,1] is not representable.

Consider the following statements

(i) $\forall x \in R(x^2 \geq 0), \forall x \in R(x > 0 \to \exists y\ x = y^2)$
(ii) $\forall f \in R^R(f(x) \equiv f(-x) \to \exists F \in R^R\ f(x) \equiv F(x^2))$
(iii) $\forall f \in R^{R_{\geq 0}}(f(x^2) \equiv 0 \to f(x) \equiv 0)$
(iv) $\forall f \in R^R \exists! g \in R^R(g(0) = 0 \wedge g' \equiv f)$

Lemma 1. *Assume that R is a commutative local ring with unit, satisfying the Kock-Lawvere axiom $R \times R \xrightarrow{\sim} R^D$, and equipped with order relations $<$ and $\leq$ which are compatible with each other and with the ring structure of R (as in A1–5 of chapter VII). If (i)–(iv) hold for R, then the integration axiom holds.*

Proof. (sketch) By extending a given $f \in R^{R_{\geq 0}}$ to some $F \in R^R$ (by (i)–(iii)), we use (iv) to find a primitive G of F whose restriction to $R_{\geq 0}$ gives a primitive $g \in R^{R_{\geq 0}}$ of f. To obtain the conclusion for $f \in R^{[0,1]}$, one uses isomorphisms $[0,1) \xrightarrow{\sim} R_{\geq 0}$, $[0,1) \xrightarrow{\sim} (0,1]$. Uniqueness of g follows from (iii) and (iv).

Lemma 2. *(Whitney's lemma on even functions) Let $f(x,t): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(x,t) \equiv f(x,-t)$. Then there is a smooth $g(x,t): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $f(x,t) \equiv g(x,t^2)$.*

Proof. See Martinet (1982) or [MQR].

Lemma 3. *Let M be a smooth manifold, and let $f, g \in C^\infty(M)$. Then $f(x) \in (g(x) - t^2) \subseteq C^\infty(M \times \mathbb{R})$ iff for some $p \in C^\infty(\mathbb{R})$ which vanishes on $\mathbb{R}_{\geq 0}$, $f(x) \in (\rho(g(x))) \subseteq C^\infty(M)$.*

Proof. See [MQR].

From the last two lemmas, we obtain the following result.

Theorem 4. *The map $R \to R_{\geq 0}$ in $\mathbb{L}$ which is induced by $\mathbb{R} \xrightarrow{t^2} \mathbb{R}_{\geq 0}$ is a stable effective epimorphism.*

Proof. Let $\ell A \xrightarrow{g} R_{\geq 0}$ be any map in $\mathbb{L}$, with $A = C^\infty(\mathbb{R}^n)/I$. So g is represented by a function $g(x)\colon \mathbb{R}^n \to \mathbb{R}$ such that $\rho \circ g \in I$ whenever $\rho \in m^\infty_{\mathbb{R}_{\geq 0}}$. Consider the pullback

$$\begin{tikzcd} \ell A \arrow[r, "g"] & R_{\geq 0} \\ \ell B \arrow[u, "\pi_x"] \arrow[r, "\pi_t"] & R \arrow[u, "t^2"'] \end{tikzcd}$$

where $B = C^\infty(\mathbb{R}^n \times \mathbb{R})/(I(x), g(x) - t^2)$.

We have to show that for any $\ell B \xrightarrow{f} \ell C$ which coequalizes p_1, p_2, there is a unique h making the following diagram commute.

$$\begin{tikzcd} \ell B \times_{\ell A} \ell B \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[dr] & \ell B \arrow[r, "\pi_x"] \arrow[d, "f"] & \ell A \arrow[dl, "h"] \\ & \ell C & \end{tikzcd}$$

It suffices to consider the case $\ell C = R$. So f is represented by a smooth function $f(x,t)\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

The *existence* of such an h now becomes a statement about ideals, namely: if $f(x,t) - f(x,s) \in (I(x), g(x) - t^2, t^2 - s^2)$, then there is an h such that $f(x,t) - h(x) \in (I(x), g(x) - t^2)$. The *uniqueness* of h, on the other hand, is simply the statement: if $k(x) \in (I(x), g(x) - t^2)$, then $k(x) \in I$.

To prove existence, write

$$f(x,t) = \frac{1}{2}(f(x,t) + f(x,-t) + \frac{1}{2}(f(x,t) - f(x,-t))$$

and use lemma 2 to obtain an $\tilde{f}$ such that

$$\tilde{f}(x,t^2) = \frac{1}{2}(f(x,t) + f(x,-t)).$$

Then $f(x,t) - \tilde{f}(x,t^2) \in (I(x), g(x)-t^2)$, and we let $h(x) = \tilde{f}(x, g(x))$.

For uniqueness, write

$$k(x) = \sum A_i(x,t)\varphi_i(x) + B(x,t)(g(x) - t^2),$$

where $\varphi_i \in I$. By lemma 2 again, find $\tilde{A}_i, \tilde{B}$ with

$$k(x) = \sum \tilde{A}_i(x,t^2)\varphi_i(x) + \tilde{B}(x,t^2)(g(x) - t^2)$$

Since $\tilde{A}_i(x, g(x)) - \tilde{A}_i(x,t^2) \in (g(x) - t^2)$, we may write

$$k(x) = \sum \tilde{A}_i(x, g(x))\varphi_i(x) + C(x,t^2)(g(x) - t^2).$$

By lemma 3, there is a $\rho \in m^{\infty}_{\mathbb{R}_{\geq 0}}$ such that

$$k(x) - \sum \tilde{A}_i(x, g(x))\varphi_i(x) \in (\rho g(x))$$

Since $\rho g(x) \in I$, we conclude that $k \in I$.

Corollary 5. *The map $R \xrightarrow{t^2} R_{\geq 0}$ is also a stable effective epimorphism in $\mathbb{F}, \mathbb{G}, \mathbb{V}, \mathbb{W}(\mathbb{V}_\omega, \mathbb{W}_\omega)$.*

Proof. For $\mathbb{F}$ and $\mathbb{G}$, see [MQR]. The cases of $\mathbb{V}, \mathbb{W}, \mathbb{V}_\omega, \mathbb{W}_\omega$ are all the same, so we only consider $\mathbb{W}$ here.

From the description of pullbacks in $\mathbb{W}$, it follows that we only need to prove the following two statements, which represent existence and uniqueness, respectively (as in the proof of theorem 4), for a C^∞-radical ideal I:

$$f(x,t) - f(x,s) \in \sqrt[\infty]{}(i(x), g(x) - t^2, t^2 - s^2) \Rightarrow$$

$$\exists h(x)\ \ f(x,t) - h(x) \in \sqrt[\infty]{}(I(x), g(x) - t^2)$$

$$k(x) \in \sqrt[\infty]{}(I(x), g(x) - t^2) \Rightarrow k \in I,$$

where for an ideal J, $\sqrt[\infty]{}J = \{f | \exists g \in J\ Z(f) = Z(g)\}$ denotes the C^∞-radical of J.

The first is proved just as for $\mathbb{L}$, but the second is much simpler. Indeed, let $\rho \in m^{\infty}_{\mathbb{R}_{\geq 0}}$ be any function with $Z(\rho) = \mathbb{R}_{\geq 0}$. Since

$Z(k(x)) = Z(\sum a_i(x,t)\varphi_i(x) + b(x,t)(g(x) - t^2))$ where $\varphi_i \in I$, and $\rho g(x) \in I$, we have $Z(k(x)) \supseteq Z(\sum \varphi_i{}^2(x) + \rho g(x))$, so $k \in \sqrt[\infty]{I} = I$.

Corollary 6. ***The integration axiom holds in each of the following models:*** $\mathcal{F}, \mathcal{F}_{\text{fin}}, \mathcal{G}, \mathcal{G}_{\text{fin}}, \mathcal{B}, \mathcal{Z}, \mathcal{C}, \mathcal{C}_{\text{fin}}, \mathcal{W}, \mathcal{W}_{\text{fin}} \mathcal{W}_{\text{pr}}, \mathcal{W}_{\text{prfin}}$, ***(and similarly for the corresponding toposes with*** $\mathbb{W}_\omega$ ***instead of*** $\mathbb{W}$***),*** $\mathcal{S}[\text{locAr}C^\infty]$, $\mathcal{S}[\text{loc}C^\infty]$.

Proof. We have to check the hypotheses of lemma 1. Only two of them present problems: the compatibility of $\geq$ with $>$ and with the ringstructure of R, and (iii). For the sites big enough for $R_{\geq 0}$ to be representable, (iii) follows from corollary 5. These include $\mathcal{F}, \mathcal{F}_{\text{fin}}, \mathcal{G}, \mathcal{G}_{\text{fin}}, \mathcal{B}, \mathcal{Z}, \mathcal{W}, \mathcal{W}_{\text{prfin}}$ (and idem with $\mathbb{W}_\omega$ as underlying site). Compatibility of $\geq$ follows from I.4.12 (as in II.2) for the first six of these, and is obvious for the toposes based on $\mathbb{W}$ (or $\mathbb{W}_\omega$).

For the others, whose sites are too small to contain $R_{\geq 0}$, we prove that $R_{\geq 0}$ acts "as a representable" in the sense that a natural transformation

$$\mathbb{A}(-, \ell A) \times \mathbb{L}(i(-), R_{\geq 0}) \xrightarrow{\tau} \mathbb{A}(-, R)$$

(where $\mathbb{A} \xhookrightarrow{i} \mathbb{L}$ is the inclusion of the site in question) is induced by a morphism $\ell A \times R_{\geq 0} \to R$ in $\mathbb{L}$. (So this applies to the sites with underlying categories $\mathbb{C}, \mathbb{W}, \mathbb{L}_{\text{fp}}$). This would reduce the problem to checking both assertions for $\mathbb{L}$, and thus completes the proof.

Apply τ to the pair $(\pi_{\ell A}, t^2)$ at stage $\ell A \times R \in \mathbb{A}$, and let $g = \tau_{\ell A \times R}(\pi_{\ell A}, t^2) \colon \ell A \times R \to R$. Consider the diagram

$$\begin{tikzcd}
lA \times_{R_{\geq 0}} R \arrow[r, shift left, "\pi_{12}"] \arrow[r, shift right, "\pi_{13}"'] & lA \times R \arrow[r, "lA \times t^2"] \arrow[dr, "g"'] & lA \times R_{\geq 0} \arrow[d, "f"] \\
 & & R
\end{tikzcd}$$

where π_{12}, π_{13} is the kernel pair of $\ell A \times t^2$. We notice that $R \times_{R_{\geq 0}} R$ is an object of $\mathbb{A}$ (even if $R_{\geq 0}$ is not), in the cases where the underlying category is $\mathbb{W}$ or $\mathbb{L}_{\text{fp}}$. In these cases, $g\pi_{12} = g\pi_{13}$ by naturality of τ, and this implies, by theorem 4, that there is a unique f in $\mathbb{L}$ as indicated in the diagram, which makes the triangle commute. A further application of this theorem shows that τ comes from composing with f. The case where the underlying category of $\mathbb{A}$ is $\mathbb{C}$ actually follows by the same argument, since although $R \times_{R_{\geq 0}} R = \ell(C^\infty(\mathbb{R}^2)/(x^2 - y^2))$ is not in $\mathbb{C}$, it still follows that $g\pi_{12} = g\pi_{13}$, because $\ell A \times R \times_{R_{\geq 0}} R \in \mathbb{F}$, so it is enough to check that $g\pi_{12}u = g\pi_{13}u$ for all $\ell W \xrightarrow{u} \ell A \times R \times_{R \geq 0} R$, with W a Weil algebra.

Corollary 7. *The integration axiom also holds in the following toposes:* $\mathbf{Sets}^{\mathbb{F}^{op}}, \mathbf{Sets}^{\mathbb{G}^{op}}, \mathbf{Sets}^{\mathbb{L}^{op}}, \mathbf{Sets}^{\mathbb{C}^{op}}, \mathbf{Sets}^{\mathbb{W}_\omega^{op}}, \mathcal{S}[C^\infty]$.

Proof. Obvious from the previous results, since all the topologies are subcanonical.

To conclude this appendix, two further remarks:

(1) An appropriate reformulation of the integration axiom which doesn't make use of the existence of nilpotent infinitesimals (to define the derivative) is valid in the toposes based on $\mathbb{E}$ and $\mathbb{V}$.

(2) For the toposes whose sites are big enough to represent [0,1], a much simpler proof of the integration axiom is available. As an example, let us consider the case $\mathbf{Sets}^{\mathbb{L}^{op}}$.

The idea is to reformulate the integration axom as the existence of a constant $\int_0^1$ of sort $R^{(R^{[0,1]})}$ satisfying the following conditions:

(i) $\int_0^1 (f+g)dt = \int_0^1 f\, dt + \int_0^1 g\, dt$
(ii) $\int_0^1 af\, dt = a\int_0^1 f\, dt \quad (a \in R)$
(iii) $\int_0^1 f' dt = f(1) - f(0)$.

In terms of this constant, we can obtain the usual integration axiom by defining

$$\int_a^b f\, dt = (b-a)\int_0^1 f(a + t(b-a))dt.$$

The verification of the existence of such an operation $\int_0^1 : R^{[0,1]} \to R$ in the models presents no difficulties: let $f : \ell A \times [0,1] \to R$ be a function at stage ℓA, i.e., an equivalence class of some $F(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ modulo $(I(x), m^\infty_{[0,1]}(t))$, where $A = C^\infty(\mathbb{R}^n)/I$. Define

$$\int_0^1 f\, dt = \int_0^1 F(x,t)dt \mod I,$$

which clearly satisfies (i)–(iii), provided it is well defined. By linearity of $\int_0^1$, well-definedness comes down to showing that if $F(x,t) \in$

$(I(x), m^{\infty}_{[0,1]}(t))$, then $\int_0^1 F(x,t)dt \in I(x)$. But if $F(x,t) = \sum a_i(x,t)\varphi_i(x) + \sum b_j(x,t)\mu_j(t)$ with $\varphi_i \in I$ and $\mu_j \in m^{\infty}_{[0,1]}$, then $\int_0^1 F(x,t)dt = \sum \varphi_i(x) \int_0^1 a_i(x,t)dt \in I$.

Appendix 4: The amazing right adjoint

Let $\mathbb{C}$ be a site, as in appendix 1. (We assume that the topology on $\mathbb{C}$ is subcanonical, and that $\mathbb{C}$ has finite $\varprojlim$.) An object $A \in \mathbb{C}$ defines a functor

$$(-)^A : \mathrm{Sh}(\mathbb{C}) \to \mathrm{Sh}(\mathbb{C}), \quad X \mapsto X^A$$

by exponentiation, which preserves limits. When does $(-)^A$ have a right adjoint, i.e., a functor $(-)_A$ such that there is a natural 1–1 correspondence

(1)
$$\frac{X^A \to Y}{X \to Y_A}$$

Obviously, this can only be the case if $(-)^A$ preserves colimits (any functor having a right adjoint must preserve colimits). This is also enough, for in this case we can define

(2)
$$Y_A(C) = \mathrm{Sh}(\mathbb{C})(C^A, Y).$$

Then Y_A is indeed a sheaf, for if $\{C_\alpha \xrightarrow{f_\alpha} C\}_\alpha$ is a covering sieve in $\mathbb{C}$ and $\{C_\alpha^A \xrightarrow{g_\alpha} Y\}$ is a compatible family of elements of $Y_A(C_\alpha)$, then there is a unique $C^A \xrightarrow{g} Y$ with $g \circ f_\alpha^A = g_\alpha$, since $C = \varinjlim C_\alpha$ in $\mathrm{Sh}(\mathbb{C})$ and hence $C^A = \varinjlim C_\alpha^A$. The correspondence (1) now holds if X is representable, by definition (2). The general case follows, again since $(-)^A$ preserves colimits, because every sheaf is a colimit of representables (cf. 2.7(b) of Appendix 1).

Lemma. $(-)^A : \mathrm{Sh}(\mathbb{C}) \to \mathrm{Sh}(\mathbb{C})$ *preserves colimits iff every cover* $\{D_\alpha \xrightarrow{g_\alpha} C \times A\}_\alpha$ *in* $\mathbb{C}$ *has a refinement of the form* $\{C_\beta \times A \xrightarrow{h_\beta \times \mathrm{id}} C \times A\}_\beta$ *for some cover* $C_\beta \xrightarrow{g_\beta} C\}_\beta$ *(i.e., every* $h_\beta \times \mathrm{id}$ *factors through some* g_α*).*

Proof. In the examples, we will only make use of $\Leftarrow$, and we will leave the proof of $\Rightarrow$ as an exercise to the reader. For $\Leftarrow$, let $X = \varinjlim X_i$ be a colimit of sheaves. To show that $X^A = \varinjlim {X_i}^A$, it is enough to show that there is a natural 1–1 correspondence

(1)
$$\frac{C \to X^A}{C \to \varinjlim {X_i}^A}$$

where $C \in \mathbb{C}$. But given the assumption, (1) is a simple consequence of the way colimits of sheaves are computed (2.7 (b) of Appendix 1): indeed, a map $C \xrightarrow{f} (\varinjlim X_i)^A$ naturally corresponds to a map $C \times A \xrightarrow{\hat{f}} \varinjlim X_i$, and such a map is (as an equivalence class) represented by a compatible family of maps $D_\alpha \xrightarrow{\hat{f}_\alpha} X_{i_\alpha}$, for some cover $\{D_\alpha \xrightarrow{g_\alpha} C \times A\}_\alpha$; by assumption, then, it can also be represented as a compatible family of maps $C_\beta \times A \xrightarrow{\hat{f}_\beta} X_{i_\beta}$ for some finer cover $\{C_\beta \times A \xrightarrow{h_\beta \times \mathrm{id}} C \times A\}_\beta$. The corresponding maps $\{C_\beta \xrightarrow{f_\beta} X_i^A\}_\beta$ represent an element $C \xrightarrow{\tilde{f}} \varinjlim X_i{}^A$. It is straightforward to check that $f \mapsto \tilde{f}$ defines a 1–1 correspondence as required in (1).

Examples. Let $A = C^\infty(\mathbb{R}^n)/I$ be a C^∞-ring, and suppose $x \in \mathbb{R}^n$ is such that $\ell A \subset \ell C^\infty(U)$ for every open neighbourhood $U \subset \mathbb{R}^n$ of x (i.e., $\exists f \in I \; f^{-1}(0) \subset U$, cf. I.1.7). Then the functor

$$X \mapsto X^{\ell A}$$

has a right adjoint, in each of the toposes discussed in this book (for $\mathcal{G}$ and $\mathcal{F}$, we assume that I is germ-determined, resp. closed, of course).

To see this, suppose $\ell B = \ell(C^\infty(\mathbb{R}^m)/J)$ is another locus, and $\{\ell C_\alpha \to \ell B \times \ell A\}_\alpha$ is a cover in the site of one of these toposes. Then ℓC_α is of the form $\ell(B \otimes_\infty A) \cap s(V_\alpha)$ for some open cover $\{V_\alpha\}_\alpha$ of $\mathbb{R}^m \times \mathbb{R}^n$. Let $U_\alpha = \{y \in \mathbb{R}^m | (x, y) \in V_\alpha\}$, and let $\ell D_\alpha = \ell B \cap s(U_\alpha)$. Then the ℓD_α form a cover of ℓB, and for each α we have $s(U_\alpha) \times \ell A \subset sV_\alpha$, hence also $\ell C_\alpha \times \ell A \subset \ell D_\alpha$, and the condition of the lemma is satisfied.

Consequently, if Z is any of the *infinitesimal spaces* with a point, like D, $D_k(n)$, $\overline{W}$ (W any Weil algebra), $\mathbb{\Delta} = \cap_{n>0} \ell C^\infty\left(-\frac{1}{n}, \frac{1}{n}\right)$, or any germ $\ell C_p^\infty(M) = \ell(C^\infty(M)/m_p^g)$, then $(-)^Z$ has a right adjoint. (Notice, however, that the argument does not apply to the space $\mathbb{I}$ of invertible infinitesimals, cf. II.1.10).

An exception to these general observations is the topos defined by adding the map $R \xrightarrow{t^2} R_{\geq 0}$ as a cover (which we mentioned at the end of appendix 2), as is pointed out in Moerdijk-Quê-Reyes (to appear).

Appendix 5: Comments, References and Further Developments

The aim of this appendix is to help the reader by making comments on the significance of some results, by pointing out further developments, and by directing him or her to the literature for further information or alternative viewpoints. We do not desire, nor are we able, to give detailed attribution of credits and priorities for the various results in this book, and whenever possible we shall just refer to a main reference where the reader can satisfy his curiosity. When such a reference is not available, we try to give what we believe to be the source of the result in question. As far as terminology is concerned, we have tried to avoid descriptions of the type "the Moerdijk envelop of the Reyes topos", in favour of more informative ones. (Some exceptions occur, however, when this goes against long established tradition, such as in the case of Weil algebras.)

The notion of C^∞-ring was introduced by Lawvere in 1967 (see Lawvere (1979)), and studied by several people as a tool to construct models for Synthetic Differential Geometry (henceforth SDG); see Kock (1981). Although this general notion of C^∞-ring does not occur as such in classical analysis and differential geometry, the main examples of C^∞-rings occupy a central position in singularity theory and related subjects, as will be clear from such books as Malgrange (1966), Martinet (1982), and Tougeron (1972). For instance, the local algebras of Malgrange (1966), originally defined as quotients of rings of germs of smooth functions, are precisely the pointed local finitely generated C^∞-rings (cf. proposition I.3.9). Other examples of local C^∞-rings are the formal algebras, i.e., the quotients of rings of formal power series. A particular kind of formal algebras, the Weil algebras, were introduced by A. Weil (1953) in order to deal with nilpotent infinitesimals in differential geometry. In this paper, Weil used these algebras to define tangent bundles, Ehresmann's jet bundles (Ehresmann (1953)), and more general *prolongations* of manifolds in a purely intrinsic way.

Ideals of smooth functions have been extensively studied in the classical literature (see the books by Malgrange, Tougeron, Martinet). The closed ideals in particular play a major role, due to important results like Whitney's spectral theorem (I.4.4), or the result stating that $(f_1, \ldots, f_k) \subset C^\infty(\mathbb{R}^m)$ is closed if $f_1, \ldots, f_k$ are analytic functions on $\mathbb{R}^m$ (Malgrange (1966); Lojasiewicz (1959) for the case $m = 1$).

Given the role of these examples of C^∞-rings in the classical literature, it is not surprising that although the statements of several of the results in this chapter seem new, most of their proofs are either known or easily derivable from known ones.

Much of the material of I.2 and I.3 appears in Dubuc (1981), be it with somewhat different proofs. Proposition 1.2 (which is a simple consequence of the fundamental theorem of calculus) explains in some sense the role of ideals of differentiable functions in the classical literature. Proposition 1.6 (which is essentially the implicit function theorem) may be paraphrased as "open in the Euclidean topology = open in the Zariski topology", thus illustrating the analogy with algebraic geometry mentioned in the introduction, and below. The appendix of I.3 is taken from Moerdijk & Reyes (1986).

Near-point determined ideals were explicitly defined in Reyes (1981), in order to formulate the Spectral Theorem of Whitney as "closed = near-point determined" (proposition 4.6). Germ determined ideals, on the other hand, were explicitly defined in Dubuc (1981) in order to reformulate classical results about partitions of unity. Corollary 4.9 appears in Kock (1981). Theorem 4.11 is due to Quê and Reyes (1982), and generalizes an unpublished result of A.P. Caldéron (corollary 4.18). The refinement of 4.13 used in the proof of II.2.4 (vii) is due to O. Bruno (1986). These results are the key to verifying the axioms of integration and preorder ($\leq$) in many of the models.

In connection with the material of Chapter I, we would like to emphasize that C^∞-rings in general, and local C^∞-rings in particular, have hardly been studied in their own right, independently from the construction of models of Synthetic Differential Geometry. As to local C^∞-rings, some steps are taken in Moerdijk, Reyes (1986), and Moerdijk, Reyes, Quê (1987), where localizations of C^∞-rings are studied and the notion of *spectrum* of a C^∞-ring is analyzed. Given such a notion of spectrum, one can then define a *C^∞-scheme* in the obvious way, as something built up from spectra of C^∞-rings. It would be very interesting to see in which respects this deviates from the standard theory of schemes in algebraic geometry, and how it ties in with the classical theory of manifolds, which stand to C^∞-schemes as varieties stand to schemes:

$$\text{manifolds} : C^\infty\text{-schemes} = \text{varieties} : \text{schemes}$$

We should point out here that another notion of "spectrum of a C^∞-ring" is considered in Dubuc (1981). His notion, however, is more restricted than ours, and is suitable for germ-determined C^∞-rings only.

As far as the general theory of C^∞-rings is concerned, some interesting work has been done by Michor, Kriegl, and others; see e.g. Kainz, Kriegl, Michor (1987).

We just mentioned the possibility of pursuing the analogy with the theory of spectra and schemes in algebraic geometry. In our book, it is from a slightly different angle that the analogy with algebraic geometry is brought about, namely via the "functorial approach" to algebraic geometry as exposed in Demazure and Gabriel (1970). Thus, we regard the *duals* of C^∞-rings, or loci, as our basic geometric objects; in fact, passing to this dual category $\mathbb{L}$ of loci, one obtains a *covariant* embedding of the category of manifolds $\mathbb{M}$ into $\mathbb{L}$.

Most of section II.1 is just a restatement of results from chapter I, obtained by formally dualizing. II.2 is expository and tries to motivate the semantics of sheaves for the particular case of $Sets^{\mathbb{L}^{op}}$. As will be apparent, sheaf semantics plays a central role in this monograph; a general exposition of this semantics, due mainly to A. Joyal, appears in Appendix 1. Theorem II.2.4 is due essentially to Lawvere and Kock (cf. Kock's book (1981)), with the exception of the clauses for integration and preorder: 2.4(iv) for $\leq$, and (v) are proved in Quê, Reyes (1982), while 2.4(vii) is due to O. Bruno. The property CMP of section II.3 was introduced by MacLarty (1983); theorems 3.1 and 3.2 are taken from Moerdijk, Reyes (1987); and proposition 3.6, which shows how distributions with compact support can be dealt with in our framework, appears in Quê, Reyes (1982). The notion of path-smoothness is taken from Boman (1967).

As we have said, the topos $Sets^{\mathbb{L}^{op}}$ discussed in chapter II really only served as an introduction to the models of chapter III and VI. The first model for the Kock-Lawvere axiom (III.1.10) as well as the algebraic axioms for R (III.1.7(i),(iii),(iv)) was actually constructed by F. W. Lawvere, as far back as 1967. This model was the topos of sheaves on the category $\mathbb{L}_{fp}$ of duals of finitely presented C^∞-rings, equipped with the largest (finest) subcanonical Grothendieck topology, the so-called canonical topology. For many purposes, however, this model was rather unmanageable, since a good description of the canonical topology was, and still is, lacking. (A workable description of this topology seems to depend on hard problems of analysis, cf. Moerdijk, Quê, Reyes (to appear).)

The topos $\mathcal{G}$ was introduced by Dubuc (1981), whereas the topos $\mathcal{F}$ was first described in Bélair (1981). Nevertheless, it was not clear how much analysis and geometry could be developed on the basis of the axioms that hold in these models. In fact, the first paper by Dubuc limits itself to the verification of the Kock-Lawvere axiom and the axioms expressing that R is an archimedean ordered local ring. Besides these axioms, the paper of Bélair also tackles the integration axiom (III.2.10(i)), and the compatibility of $\leq$ with the ring structure of R (III.2.7). Compactness of $[0,1]$, connectedness of R, existence of partitions of unity, etc., which are needed to develop some analysis and geometry, and play such an important role in our book, were not discussed at all in these references.

Let us observe that these topological properties are not discussed in Kock's book (1981) either, perhaps because the emphasis of Kock is rather on the algebraic approach to infinitesimal structures.

Many of the topological properties discussed in section III.3 were first proved for the topos $\mathcal{G}$ in Moerdijk, Reyes (1984a), although the same arguments apply to the topos $\mathcal{F}$. However, there we only considered spaces like R and $[0,1]$, and the treatment in section III.3 is more general and much more systematic. Results of the form of III.3.5, III.3.11 and their corollaries should be familiar to readers with some background in intuitionistic analysis. Chain-connectedness (III.3.8) seems to be the right notion of connectedness in the context of constructive topology. (It was introduced in the more general context of toposes and locales in the appendix of Moerdijk (1986) where a general form of proposition 3.9 appears.)

Chapter IV essentially follows Moerdijk, Reyes (1983), (1984b). The approach to differential forms as maps on the space of infinitesimal n-cubes is quite standard in the works of geometers like Cartan, Lie, etc. In the context of Synthetic Differential Geometry it was explicitly described in Kock, Reyes, Veit (1980); see also Kock (1981). An apology to the reader is due for section 2 on singular homology: we gave rather detailed proofs in order to check that this theory is really constructively valid (many classical sources make free, but unnecessary, use of the excluded middle). With all these preliminaries out of the way, our proof of De Rham's theorem for $\mathcal{G}$ in section 3 is a quite standard Mayer-Vietoris sequence argument (cf. for instance Greub et al. (1972,1973)). Corollaries 4.5 and 4.8, obtained here as consequences of De Rham's theorem in the topos $\mathcal{G}$ (or $\mathcal{F}$), are originally due to van Est (1958), who proved them by the method of "carapaces" of H. Cartan. Corollary 4.5 was also independently proved by J. Glass (1983), who, like us, was unaware of van Est's paper. A final remark about chapter IV: in formulating De Rham's theorem for the topos $\mathcal{G}$, we used the dual of singular homology (which, by the way, is the original formulation of De Rham (1960)), rather than singular cohomology as is often done nowadays. We don't know whether such a De Rham theorem in terms of singular cohomology holds in $\mathcal{F}$ or $\mathcal{G}$ (the trouble with cohomology is that one seems to need the axiom of choice at some place or other).

The notion of microlinear space discussed in chapter V is due to F. Bergeron (1980), and is stronger than Demazure's condition (E) (see Demazure (1970)), which had been used in Synthetic Differential Geometry under the name of "infinitesimally linear spaces" (see Kock (1981)), a term long to write and hard to pronounce, especially in French, as was shown conclusively in some seminars in Montréal.

One of the main results of this chapter is our version (V.2.5) of the Ambrose-Palais-Singer theorem, which is proved here for *all* microlinear spaces. This theorem was originally proved for finite dimensional manifolds in Ambrose et al. (1960). Earlier "synthetic" versions of this result were given by Bunge and Sawyer (1983), Kock (1983) and Kock, Lavendhomme (1984). In all of these references, however, additional conditions are imposed on the microlinear space M. In the last paper, for instance, the theorem is proved for those microlinear spaces M which satisfy the condition $T_x(M) \cong R^n$ for all $x \in M$. When this theorem is interpreted in either $\mathcal{F}$ or $\mathcal{G}$, we essentially obtain the classical Ambrose-Palais-Singer theorem for smooth manifolds of finite dimension. However, it turns out that the condition $T_xM \cong R^n$ is redundant, and consequently one can obtain a version of this theorem which includes e.g. function spaces, manifolds with singularities, and algebraic schemes, when interpreted in suitable toposes. Details of these applications can be found in Moerdijk and Reyes (to appear 2).

Proposition 1.4 appears in Lavendhomme (1981). Theorem 1.6 was proved by Reyes, Wraith (1978), and independently by Lavendhomme (1981). Our presentation closely follows Lavendhomme's. The notion of spray used in this chapter is credited to S. Smale by P. Libermann. Much of section V.3 on vectorbundles over microlinear spaces can already be found in Kock and Reyes (1979). The idea of proving the Gauss-Bonnet theorem in dimension 2 by adding infinitesimal angles occurs in Bélair and Reyes (1985), but as it stands, the proof given there is not correct. Curiously enough, we have not been able to find our proof in Cartan (1928), where a different argument using invertible infinitesimals is given. (We don't know how to justify Cartan's argument in our context.) Translation spaces and strong difference, as used in this chapter, are taken from Kock, Lavendhomme (1984). In a classical context, they appear in Kolař (1977); see also White (1982).

The topos $\mathcal{Z}$ of chapter VI first makes its appearance in Quê and Reyes (1982), where it is pointed out that the proofs of the integration axiom and of the compatiblity of the pre-order $\leq$ given for $Sets^{\mathbb{L}^{op}}$ also apply to $\mathcal{Z}$. The first serious study of the topos $\mathcal{Z}$ as a model for analysis, however, was carried out in Moerdijk and Reyes (1987). In order to force the existence of invertible infinitesimals, which are only "partially" present in $\mathcal{Z}$, Moerdijk (1987) introduced the topos $\mathcal{B}$. These two papers constitute the basis of chapter VI.

At first sight, the models $\mathcal{Z}$ and $\mathcal{B}$ look rather unmanageable, since the embedding $\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{Z}$ (or $\mathcal{B}$) seems to preserve *finite* open covers only (rather than countable ones, as in the case of $\mathcal{F}$ and $\mathcal{G}$), thus making the relation with the theory of manifolds harder to spell out. Alternatively, one may say that the consideration of finite open covers only in the site for $\mathcal{Z}$, say, forces a very strong sense of existence upon us, at odds with the usual interpretation as local existence in classical analysis which is captured in $\mathcal{F}$ and $\mathcal{G}$ by the more relaxed notion of countable open cover in the sites. However, this is only superficially so, since countable open covers *are* preserved by the embedding s, provided preservation is correctly formulated! The point is that a family of open subsets of a manifold M, indexed by $\mathbb{N}$, gives rise in a canonical way to an internal family of subobjects of $s(M)$ indexed by the space N of *smooth natural numbers*, and the internal family is an internal cover if the original family covers M. (Dubuc (1981) calls a model "well-adapted" if, among other things, the embedding s preserves countable covers. Unfortunately, he interprets this preservation as requiring that the family $\{s(U_n)\}_{n\in\mathbb{N}}$ indexed by the *external* $\mathbb{N}$ should cover $s(M)$ whenever $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of M. But when the preservation is stated correctly, both $\mathcal{Z}$ and $\mathcal{B}$ turn out to be well-adapted.)

Once this is realized, it is clear that we can only have an adequate model for analysis if we systematically replace the natural number object $\mathbb{N}$ by the object of smooth natural numbers N, and rephrase notions like archimedeanness, finiteness, compactness, etc., accordingly. With these changes, we can transfer the main results for $\mathcal{F}$ and $\mathcal{G}$ to this new context. An unexpected new ingredient in the proofs of these results (such as the preservation of countable open covers by s) is the use of dimension theory in the form of Ostrand's theorem (see Ostrand (1972), or Engelking (1978)). Ostrand's theorem is used in all the results of section VI.3, which are taken from Moerdijk (1987), and also lies at the basis of the transfer principle between $\mathcal{Z}$ and $\mathcal{G}$ discussed in section VII.4.

The axiomatic system for smooth Infinitesimal Analysis (SIA) presented in chapter VII appears in print here for the first time, and a few words may be in order to explain some of its peculiarities. Essentially, we have tried to axiomatize (a part of) the theory of the topos $\mathcal{B}$ i.e., the set of sentences which are true in $\mathcal{B}$. The general set-up as a type theory with variable types roughly follows Feferman (1985). The most peculiar axioms of SIA seem to be the arithmetical ones (1.5), which are supposed to describe the smooth natural numbers $N = s(\mathbb{N})$. Besides limited induction (A13) and recursion (A14), we also have the axiom of bounded search (A15), introduced in Moerdijk (1987). This axiom is a substitute for the axiom of choice for finite sets, (i.e., $\forall x \in [n]\ \exists y \in [m]\varphi(x,y) \to \exists f \in [m]^{[n]}\ \forall x \in [n]\ \varphi(x, fx)$, where $[n] = \{x \in N | 0 \leq x \leq n-1\}$) which is not valid in $\mathcal{B}$ or $\mathcal{Z}$. It is surprising how much mathematics can be developed on the basis of this very weak arithmetical theory of N, provided one reformulates topological notions very carefully to avoid the need of the axiom of finite choice. (For instance, compactness should be defined in terms of finite refinements, rather than subcovers.) We tried to illustrate this in section 2. It should be noticed that although several proofs here are rather delicate (e.g., 2.5), they should be familiar to readers with some experience in Primitive Recursive Arithmetic.

Section 3 on distributions follows the main ideas of Non-Standard Analysis, although there are some important differences: in our context, all functions are smooth, and there are nilpotent infinitesimals besides invertible ones; moreover, global R-linear functionals in our models are automatically continuous. These features simplify some proofs considerably (e.g., proposition 3.4, or the argument from Feynman and Hibbs (1965) discussed in VII.3.16). The proof of theorem 3.7.1 is adapted from De la Vallée-Poussin (1949). This result allows us to give a mathematical proof of corollary 3.7.2, proved by metamathematical means in the context of NSA (cf. Robinson (1966)). Paragraph 3.11 on the square root of δ arose as a solution to a question of A. Royer, which was motivated in turn by his work on foundations of Quantum Mechanics.

The results of section 4 are new.

We shall now mention a few problems and briefly discuss some developments which are closely connected with our subject, but not dealt with in this book.

First of all, there are some straightforward questions that the reader may have asked herself or himself in a number of places. To mention a few: Are there some better, stronger versions of the axiom of integration? In this connection, Kock (1982) has formulated an axiom of integration for Lie groups, and proved its validity in several models. Of course, this matter is tied up with axioms for the existence of solutions for differential equations. MacLarty (1983) and Bunge, Dubuc (1987) make some proposals in this direction.

In the area of differential forms, the reader has probably asked himself whether it is possible to define the wedge product of forms in a direct, synthetic manner, so that the usual properties of this product can be derived. This is indeed the case, and essentially comes down to an infinitesimal version of the usual cap product for (cubical) singular cohomology as M. del C. Minguez (1985) has shown.

To mention another question in this direction, it would be interesting to extend our proof of Gauss-Bonnet (§V.5) to higher dimensions, and see how it relates to a "synthetic" approach to characteristic classes.

Apart from specific questions such as these, there are problems of a more general, programmatic nature, originating from the functorial approach to analysis and differential geometry. As we mentioned already, it would be desirable to develop the theory of C^∞-rings and C^∞-schemes in its own right (following Moerdijk, Reyes (1986) and Moerdijk, Quê, Reyes (1987)), and pursue the analogy

$$\frac{\text{rings}}{\text{algebraic schemes}} = \frac{C^\infty - \text{rings}}{C^\infty - \text{schemes}}$$

An elegant combinatorial approach to differential forms, connections, and the like, which unfortunately hasn't been discussed in our book, has been developed in Kock (1984), (1986). To give some idea of this approach, we recall that a 2-form, say, on a space M is an R-valued map on the space $M^{D\times D}$ of 2-cubes. In the combinatorial version, 2-forms are R-valued functions defined on the space of triples (x_1, x_2, x_3) of points of M which are such that any two of these three points are "infinitely close". (A closely connected notion had already been considered in Algebraic Geometry; the forms there were defined in terms of the "first neighbourhood of the diagonal"). Compared to the approach discussed in our book, the combinatorial approach has the advantage that it can easily be generalized to include G-valued forms, for any group G. On the other hand, however, the relation of two points being "infinitely close" cannot be defined for arbitrary microlinear spaces, and it is not clear how much of the material of chapters IV and V, say, can be developed in the combinatorial context.

Still another approach to differential forms and related concepts has been suggested by Lawvere (1980), and is based on the existence of the "amazing right adjoint" to the functor $(-)^D$ of exponentiation by D (cf. Appendix 4)

$$(-)^D \dashv (-)_D.$$

Using this adjunction, one may regard a 1-form $M^D \to R$, for example, as an ordinary function on M, but with values in a "non-standard" ring R_D. This highly non-classical idea seems promising, but it has not really been pursued yet, as far as we know.

Another entirely non-classical approach was initiated by Penon (1985), to deal with topological and infinitesimal notions. Penon has shown how, by purely logical means, a topology can be defined on all the objects of a topos, which agrees with the topology of manifolds (i.e., objects of the form $s(M)$) that was discussed in sections III.3 and VI.3.

In recent years, several alternative solutions to the problem of generalizing manifolds to include function spaces and spaces with singularities have been proposed in the literature. A particularly appealing one is the theory of convenient vector spaces, cf. Kriegl (1982), Frölicher et. al. (1983). These structures are in a way simpler than the sheaves considered in this book, but one should notice that the theory of convenient vector spaces does not include an attempt to develop an appropriate framework for infinitesimal structures, which is one of the main motivations of our approach, as we emphasized in the Introduction. In this context, it is of interest to compare the category of convenient vector spaces with our models. A first major step was taken by Kock (1986), who showed that there is a full embedding of the category of convenient vector spaces into the Cahier-topos (cf. Appendix 2), with good properties (see also Kock-Reyes (1987)). This result of Kock allows one, for example, to transfer the Ambrose-Palais-Singer theorem of section V.2 immediately to manifolds modelled on convenient vector spaces.

Finally, we would like to draw attention to problems of a more logical nature, connected to our system SIA.

For instance, what is the logical relationship of (version of) SIA to other theories for analysis, such as variants of non-standard analysis and intuitionistic analysis; What is the proof-theoretic strength of SIA? All these questions still seem to be open, in spite of some irresponsible claims to the contrary.

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