Differential graded manifolds and shifted symplectic geometry

MTP student reading group

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0 Introduction

0.1 Shifted symplectic geometry in physics

The claim that these notes will attempt to justify is that many areas of study in mathematical physics are cleanly described using the language of shifted symplectic geometry. Here are three results which we prove, which provide examples of this perspective:

Theorem (§4)**.** *The intersection of two Lagrangians in an n-shifted manifold has a $(n-1)$-shifted structure.*

This gives the BV bracket on the critical locus of an action functional.

Theorem (Section §5)**.** *There is an equivalence between Courant algebroids and degree-2 symplectic dg manifolds; Dirac structures with support are Lagrangian submanifolds.*

Theorem (§6)**.** *The mapping stack to an n-shifted manifold has an $(n-d)$-shifted structure.*

This gives that the space of G-bundles with connection on a surface is symplectic, and connections on the bulk 3-manifold is a Lagrangian submanifold.

Some literature on shifted symplectic geometry is often written in the language of derived algebraic geometry. In order to stay in the setting of manifolds and to make the applications to physics more direct, these notes exclusively use the language of differential graded (dg) manifolds. Thankfully, this perspective is not without precedent. As a consequence the contents of these notes can be found in various papers and expository articles. At the beginning of each section we give references to the resource(s) which relates to its contents.

0.2 Derived differential geometry

Remark 0.2.1. One would like to consider differential $\mathbb{Z}$-graded manifolds to capture the full BV formalism.

There are various technical difficulties which arise when one considers $\mathbb{Z}$-graded ("non-connective") differential graded algebras.

0.3 References

- Carchedi, Derived manifolds as differential graded manifolds (§1, §2)
  https://arxiv.org/pdf/2303.11140
- Cattaneo–Schatz: §3, §4
  https://arxiv.org/pdf/1011.3401
- Calaque–Ronchi
  https://arxiv.org/pdf/2510.04625
- Cueca–Maglio–Valencia: Part I
  https://arxiv.org/pdf/2510.09448
- Roytenberg
  https://arxiv.org/pdf/math/0203110
- Ševera
  https://arxiv.org/pdf/1707.00265
- (maybe also) Kotov–Salnikov
  https://arxiv.org/pdf/2108.13496

1 Algebraic preliminaries

Geometry is like algebra, but backwards; a map of smooth manifolds $f : M \to N$ defines a map of rings $f^* : C^\infty(N) \to C^\infty(M)$. The field of modern algebraic geometry fully embraces this perspective: it *defines* 'geometry' as 'algebra, but backwards'. We will embrace it only partially; a differential graded manifold M will be an ordinary smooth manifold M_0, together with an 'enhancement' of its ring of functions $C^\infty(M_0)$ to a differential graded (dg) ring which we denote $C^\infty(M)$. In this section we describe a few important properties of dg rings.

1.1 Graded commutative rings

Definition 1.1.1. Fix some abelian monoid Γ. (In our examples, Γ will only ever be $\mathbb{Z}$, $\mathbb{N}$, or $\mathbb{Z}/2$.) A ring A is *graded by* Γ if there is a decomposition of A into additive subgroups,

$$A = \bigoplus_{i \in \Gamma} A_i,$$

so that if $a \in A_i$ and $b \in A_j$ then $ab \in A_{i+j}$.

For our purposes, Γ will only ever be one of the following abelian monoids:

1. $\Gamma = \mathbb{Z}/2$.

2. $\Gamma = \mathbb{N}$.

3. $\Gamma = \mathbb{Z}$.

4. $\Gamma = 1$.

Definition 1.1.2. Let V be a k-module. The *exterior algebra* generated by V is

$$\bigwedge^{\bullet} V := \frac{\mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \dots}{v \otimes w + w \otimes v = 0}.$$

Proposition 1.1.3. *The exterior algebra $\bigwedge^{\bullet} V$ is graded commutative.*

Proof. It is $\mathbb{N}$-graded (therefore $\mathbb{Z}$-graded) by the tensor degree.

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_i) \otimes (w_1 \otimes \cdots \otimes w_j) &= (-1)^i w_1 \otimes (v_1 \otimes \cdots \otimes v_i) \otimes (w_2 \otimes \cdots \otimes w_j) \\ &= (-1)^{i+i} (w_1 \otimes w_2) \otimes (v_1 \otimes \cdots \otimes v_i) \otimes (w_3 \otimes \cdots \otimes w_j) \\ &= \dots \\ &= (-1)^{ij} (w_1 \otimes \cdots \otimes w_j) \otimes (v_1 \otimes \cdots \otimes v_i)\end{aligned}$$

□

This construction is a special case of the following. on a graded k-module.

Definition 1.1.4. Let V_* be a graded k-module. The *graded-symmetric algebra* generated by V_* is

$$\operatorname{Sym} V_* := \dots$$

Example 1.1.5. Here are some examples of $\operatorname{Sym} V_*$.

1. If V_* is concentrated in degree 0, $V_* = V_0$, then so is $\operatorname{Sym} V_*$, which is the ordinary symmetric algebra on V_0.

2. If V_* is concentrated in degree 1, $V_* = V_1[1]$, then $\operatorname{Sym} V_* = \bigwedge^{\bullet} V_1$.

Non-example 1.1.6. Here is a physically interesting non-example: If V is equipped with a symmetric bilinear form $\langle -, - \rangle$ then one can use this bilinear form to define the following "deformation" of $\bigwedge^* V$ as follows:

$$\mathrm{Cl}(V) := \frac{\mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \dots}{v \otimes w + v \otimes w = \langle v, w \rangle}.$$

This is the *Clifford algebra* associated to the inner product space $(V, \langle -, -, \rangle)$. It is graded by tensor degree, but it fails to be graded-commutative.

1.2 Differential graded commutative rings

Now suppose that, instead of a graded k-module V_*, we are given a chain complex of k-modules $V_\bullet$.

Definition 1.2.1. Let A be a graded commutative ring. A *degree-n* derivation

Proposition 1.2.2. *A differential graded commutative ring is a graded commutative ring with a degree-1 derivation Q.*

Proof. □

Definition 1.2.3 (Euler derivation)**.** Let A_* be a $\mathbb{Z}$-graded commutative ring. There is a canonical map $E: A \to A$ which acts on homogeneous by $E(a) = |a|a$.

It is a derivation because

$$E(ab) = |ab|ab = |a|a \cdot b + a \cdot |b|b = E(a)b + aE(b)$$

(why isn't it a graded derivation?)

Remark 1.2.4. The space of homogeneous elements of degree n is precisely the n-eigenspace of the Euler derivation.

2 Differential graded manifolds

In this section we develop the necessary theory of differential graded manifolds, which are manifolds equipped with a sheaf of differential graded rings. To familiarise ourselves with this language, and to motivate the definition of dg manifold, we will first describe a definition of ordinary manifolds as spaces with a sheaf of rings.

$$\begin{tikzcd}
\text{manifolds as ringed spaces} \arrow[r] & \text{(differential) graded manifolds as ringed spaces} \\
\text{manifolds} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Stricrly speaking we will not *need* to think about manifolds as ringed spaces, but seeing this perspective breaks down the conceptual jump from manifolds to dg manifolds into two pieces, which can be independently considered. (Also, it is good exposure; you will essentially learn what a scheme is!)

2.1 Manifolds as ringed spaces

A smooth n-dimensional manifold M is a Hausdorff, second-countable topological space equipped equipped with a *smooth atlas*: a collection of homeomorphisms from open sets of M to open sets of $\mathbb{R}^n$, such that transition maps on intersections are diffeomorphisms.

In algebraic geometry, the relevant geometric spaces (such as varieties and schemes) are defined as *ringed spaces*: these are topological spaces with a sheaf of rings. It turns out that manifolds can be defined in a similar way. We will later use this language to define (differential) graded manifolds.

Definition 2.1.1. A *smooth n-dimensional manifold* is a Hausdorff, second-countable topological space M equipped with a sheaf of rings $\mathcal{O}_M$ which is locally isomorphic to $C^\infty(\mathbb{R}^n)$.

To make sense of this, we define the notion of a sheaf of rings on a topological space.

Definition 2.1.2. A *sheaf of rings* on M is an assignment: For each open set $U \subseteq M$, a ring $\mathcal{F}(U)$. This assignment must be

1. *Functorial*
   - for each inclusion of open sets $U \subseteq V$, a homomorphism $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$;
   - the homomorphism $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity;
   - for each chain of inclusions $U \subseteq V \subseteq W$, an equality $\mathrm{res}_{W,U} = \mathrm{res}_{W,V} \circ \mathrm{res}_{V,U}$.
2. Respect 'gluing'
   - for a family of open sets $\{U_i \mid i \in I\}$ and elements on each open,

     $$f_i \in \mathcal{F}(U_i),$$

     which agree on each overlap $U_i \cap U_j$,

     $$\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j,$$

     then, writing $U = \bigcup_i U_j$, there exists a unique $f \in \mathcal{F}(U)$ such that $f_i = \mathrm{res}_{U,U_i}$.

Example 2.1.3. The example we will be most concerned with is the sheaf of smooth functions on a smooth manifold. Namely, if M is a smooth manifold then the assignment

$$\mathcal{O}_M : U \mapsto C^\infty(U, \mathbb{R})$$

is a sheaf of rings. Moreover, each U has an open subset diffeomorphic to $\mathbb{R}^n$, so this sheaf of rings is locally isomorphic to $C^\infty(\mathbb{R}^n)$.

Here is an example of how the sheaf condition might fail, and how to fix it:

Example 2.1.4 (Constant sheaf)**.** Consider the assignment

$$\mathcal{F} : U \mapsto \{\text{constant functions } U \to \mathbb{R}\}.$$

This is functorial, but crucially not a sheaf. Take two disjoint open subsets U and V. If two constant functions on U and V have different values, then there is no way to glue them to a constant function on $U \sqcup V$; the resulting function will only be *locally constant*. It is then an exercise to check that the assignment

$$\mathcal{F}' : U \mapsto \{\text{locally constant functions } U \to \mathbb{R}\}$$

is a sheaf of rings.

Remark 2.1.5. The definition of sheaf of rings can be compactified: a *sheaf of rings* on M is a "functor $\mathcal{F} : \mathrm{Opens}(M)^{\mathrm{op}} \to \mathsf{Ring}$ which satisfies descent with respect to open covers". This requires some explanation:

- The category $\mathrm{Opens}(M)$ is defined as follows. The objects are the open sets $U \subseteq M$, and morphisms are inclusions of open sets: there is one morphism $U \to V$ if $U \subseteq V$, and no morphisms otherwise.
- The superscript op is short for "opposite", which means all morphisms are reversed. Concretely, the category $\mathrm{Opens}(M)^{\mathrm{op}}$ has the same objects as $\mathrm{Opens}(M)$—open subsets of M—but there is one morphism $V \to U$ if $U \subseteq V$ and no morphisms otherwise.
- "Satisfies descent" comes from an interpretation of the sheaf condition: view the collection of functions $f_i \in \mathcal{F}(U_i)$ as a single element $f \in \mathcal{F}(\coprod_i U_i)$; if the f_i agree on overlaps as indicated above, then f descends to the quotient $U = (\coprod_i U_i)/\sim$.

Now we develop the notion of smooth map between manifolds in this definition.

Definition 2.1.6. Let $(M, \mathcal{O}_M)$ be a smooth manifold and N be a topological space. A continuous map $f : M \to N$ induces an assignment

$$f_*\mathcal{O}_M : \mathrm{Opens}(N)^{\mathrm{op}} \to \mathsf{Ring} : U \mapsto \mathcal{O}_M(f^{-1}(U)).$$

This makes sense because $f^{-1}(U) \subseteq M$ is open for all open $U \subseteq N$; this is what it means for f to be continuous. The assignment respects inclusions; in fact it defines a sheaf of rings on N.

Definition 2.1.7. A smooth map of manifolds $(M, \mathcal{O}_M) \to (M, \mathcal{O}_N)$ is a continuous map of topological spaces $f : M \to N$ together with a map of sheaves

$$\mathcal{O}_N \to f_*\mathcal{O}_M$$

satisfying a technical condition.

Proposition 2.1.8. *This gives the same notion of smooth maps as we are used to. This justifies using the same terminology 'smooth'.*

Proof. Here we only show that a smooth map in the atlas sense defines a smooth map in the sheaf sense; for the reverse direction one really needs the technical condition[1].

□

2.2 Graded manifolds

Here is the traditional definition of a graded manifold:

Definition 2.2.1. A *graded manifold* M consists of

1. a smooth manifold M_0, called the *body* of M
2. a sheaf of graded-commutative rings on M_0, denoted by $\mathcal{O}_M$, whose degree-zero part is the (commutative) ring of smooth functions on M_0:

$$(\mathcal{O}_M)_0 = \mathcal{O}_{M_0}.$$

3. (condition...)

Alternatively, using the definition of smooth manifold from the previous subsection we can phrase the above definition as follows:

Definition 2.2.2. A *graded manifold* M is a Hausdorff, second-countable topological space with a sheaf of graded commutative rings, whose degree-zero part is locally isomorphic to $C^\infty(\mathbb{R}^n)$ for some n.

Every ordinary manifold can be viewed as a graded manifold. There is a more interesting class of examples:

Example 2.2.3. Let A_* be a dg ring with $A_0 = \mathbb{R}$. Then one can upgrade an ordinary manifold M_0 to an interesting dg manifold, equipped with the sheaf of rings

$$\mathrm{Opens}(M_0)^{\mathrm{op}} \to \mathsf{Ring} : \; U \mapsto C^\infty(U) \otimes_{\mathbb{R}} A.$$

Because the

Is every such dg algebra actually bigwedge of a graded vector bundle?

Example 2.2.4. Trivial vector bundles, corresponding to global splittings $C^\infty(M_0) \otimes \bigwedge^* V$.

Example 2.2.5. Let $E \to B$ a vector bundle.

Theorem 2.2.6 (Batchelor's Theorem)**.** *Every graded manifold is (non-canonically) isomorphic to $E[1]$ for some graded vector bundle.*

Proof. □

[1] See `https://math.stackexchange.com/a/2134594/1080586`

2.3 Differential graded manifolds

Definition 2.3.1. A *differential graded manifold* M is a Hausdorff, second-countable topological space with a sheaf of differential graded commutative rings, whose degree-zero part is locally isomorphic to $C^\infty(\mathbb{R}^n)$ for some n.

Proposition 2.3.2. *Every differential graded manifold is a graded manifold with a degree-one cohomological vector field, and vice versa.*

Proof. ... □

2.4 Shifted natural bundles and differential forms

Goal: define the tangent complex, cotangent complex, de Rham complex. we should be careful to define a 'bundle' versus the total space; maybe both?

Definition 2.4.1. The *tangent bundle* of M is... the module of $\mathbb{R}$-linear derivations...

Definition 2.4.2. The *cotangent bundle* of M is...

These definitions agree with the ordinary ones when M is an ordinary manifold. The following "shifted" verions have no ordinary analogue:

Definition 2.4.3. The *n-shifted tangent bundle* of M is...

Definition 2.4.4. The *n-shifted cotangent bundle* of M is...

Definition 2.4.5. The *de Rham complex* of M is the ring of smooth functions $C^\infty(T[1]M)$. equipped with the differential induced by the Lie bracket of vector fields.

Example 2.4.6. Let M be an ordinary manifold. Then

3 Shifted symplectic geometry

3.1 Ordinary symplectic geometry

3.2 Graded symplectic manifolds

Let us first extend the definition of ordinary symplectic manifold to the graded setting; in the next subsection we will define a 'shifted' version.

A differential form $\omega \in \Omega^2(M)$ defines a morphism of differential graded manifolds

$$\omega^? : TM \to T^*M$$

Definition 3.2.1 (Symplectic form)**.** A *symplectic form* on M is a differential 2-form $\omega \in \Omega^2(M)$ which is:

1. closed: $\mathrm{d}\omega = 0$
2. non-degenerate: $\omega^?$ is an equivalence
3. constant along the cohomological vector field: $\mathcal{L}_Q\omega = 0$.

Definition 3.2.2 (Lagrangian)**.** Let (M, ω) be a symplectic dg manifold. A *Lagrangian submanifold* of M is... isotropic structure...

3.3 Shifted symplectic manifolds

The 'internal' grading of the ring of functions $C^\infty(M)$ means...

This time one gets

$$\omega^? : TM \to T^*[n]M.$$

Definition 3.3.1 (n-shifted symplectic form)**.** content...

Lemma 3.3.2. *If $n \neq 0$ then an n-shifted symplectic form is exact.*

Proof. Euler vector field... □

Proposition 3.3.3. *Every odd-shifted symplectic stack is* 0*-dimensional. (Safronov, introduction)*

Example 3.3.4. The shifted cotangent bundle $T^*[n]M$ of a smooth manifold is an example of an n-shifted symplectic manifold. (In fact, M can be replaced by a derived artin stack locally of finite presentation: D. Calaque. Shifted cotangent stacks are shifted symplectic

3.4 Shifted Lagrangians

Definition 3.4.1 (n-shifted Lagrangian)**.** content...

Proposition 3.4.2. *An n-shifted symplectic structure on X is the same as an $(n+1)$-shifted Lagrangian structure on $X \to *$, where $*$ is viewed as an $(n+1)$-shifted symplectic manifold in the obvious way.*

4 Lagrangian intersections and the BV bracket

Theorem 4.0.1 (Intersections)**.** *The intersection of two Lagrangians in an n-shifted manifold has a $(n-1)$-shifted structure.*

Corollary 4.0.2. *The derived critical locus of an action functional on an ordinary manifold has a (-1)-shifted structure.*

4.1 Lagrangian intersections

Theorem 4.1.1. *The intersection of two Lagrangians in an n-shifted manifold has a $(n-1)$-shifted structure.*

Proof. ... □

4.2 The classical BV formalism

4.3 Infinite-dimensional manifolds

The physical setup is typically the following: one has a space of fields which we will denote X; for the moment suppose that X is a smooth manifold. A point of X is supposed to be a configuration of the physical system being modelled. There is a smooth function $S : X \to \mathbb{R}$ called the *action functional*, and the points $x \in X$ which satisfy the laws of physics are precisely those at which S is minimised, so that $\mathrm{d}_x S = 0$.

In practice, there are complications; for example, the space X is typically infinite-dimensional. In fact, one can make formal sense of infinite-dimensional smooth spaces, and it is not as difficult as one might expect. We mention it briefly here.

First, if X is indeed a finite-dimensional manifold then we know, without any subtlety, what it means to have a smooth map $f : \mathbb{R}^n \to X$. In particular, we have the following formal properties:

- if $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map, then $f \circ \alpha : \mathbb{R}^m \to X$ is also smooth.
- if there is an open cover $\mathbb{R}^n = \bigcup_i U_i$ (where for each i we have $U_i \cong \mathbb{R}^n$, say) then we get smooth maps $f|_{U_i} : U_i \to X$. On the other hand, if we start with the smooth maps f_i which agree on overlaps $U_i \cap U_j$, then we get a smooth

Definition 4.3.1. A *smooth space* X is an assignment: For each Euclidean space $\mathbb{R}^n$, a set $X(\mathbb{R}^n)$; this should be thought of as the set of smooth maps $\mathbb{R}^n \to X$.

5 Courant algebroids and supergravity

Definition 5.0.1. A *Courant algebroid* is ...

Example 5.0.2. When the base is a point, a Courant algebroid is equivalent to a semisimple Lie algebra.

5.1 Generalised geometry

5.2 Equivalence with 2-shifted

Theorem 5.2.1. *There is an equivalence between Courant algebroids and degree-2 symplectic dg manifolds.*

Proof. ... □

6 Mapping spaces and AKSZ field theories

Theorem 6.0.1. *The mapping stack to an n-shifted manifold has an $(n-d)$-shifted structure.*

Theorem 6.0.2. *The stack of G-bundles on a surface is symplectic, because* $\mathrm{B}G$ *is 2-shifted. G-bundles on the bulk form a Lagrangian submanifold.*

7 Quantisation of shifted symplectic manifolds

7.1 Deformation quantisation

If M is an ordinary symplectic manifold, its ring of smooth functions $C^\infty(M)$ gets a Poisson bracket $\{f, g\} := \omega(X_f, X_g)$ where X_f is the Hamiltonian vector field of f, defined by $\mathrm{d}f = \omega(X_f, -)$.

If M is an n-shifted symplectic dg manifold, its ring of functions gets an $(-n)$-shifted Poisson bracket.

7.2 Geometric quantisation

Safronov `https://arxiv.org/pdf/2011.05730`

Related to BV quantisation