Doodles of a Mouse

Aspects of Higher Geometry

Andres Klene

An elephant is a mouse built to
government specifications.

William B. Widnall

Contents

CHAPTER 0

Introduction

checklist:

- treat S-local objects explicitly early on. then we can define sheaves as $\Gamma^{-1}(iso)$-local objects, and do (bousfield) localisation, with the same language
- do proofs in §1.2.
- remove the fundamental infty-groupoid

plan for section 1 is below. we need to talk about sheafification before locality,

- grothendieck topology; site; sheaf; separated presheaf, concrete presheaf. prove that concrete=separated for some specific topology (nLab).

0.1. Stuff

The purpose of this piece is twofold: first, to describe these various directions of generalisation, and secondly to combine them. For example, consider a Lie group G acting on a manifold M. There are various conditions one can impose on the G-action which would make the quotient space M/G a nice space like a manifold or an orbifold. In general, however, the quotient space M/G can be a horrible mess with no surviving smooth structure, which only survives in the geometric world as a topological space. The situation can be rectified in two ways. On one hand, we can view M as a diffeological space, whereby the quotient M/G inherits a canonical diffeology from M. On the other hand, we can consider the groupoid $M//G$, which means we add arrows between points of M when they are related by an element of G, but we do not collapse these related points together. Then $M//G$ is a smooth groupoid. What we study diffeological groupoids in [a section] is the pushout of these ideas.

ADD: the ∞-categorical sheaf condition is just "continuous". really cool? why? answer: this is only the case of the domain is (the op of) a topos: see `https://mathoverflow.net/a/294832/170682`.

The only thing that's both an infinity-topos and a stable infinity-category is the single point category. also, `https://mathoverflow.net/a/10291/170682`.

aside about abelian categories: an additive topos is trivial, and the abelian analogue of a topos ("sheaves of abelian groups on a site", as opposed to sheaves of sets) is a grothendieck abelian category.

reflective subcategories of PSh are sheaf categories...

This writing is a collection of topics in higher geometry. The idea is that ordinary differential geometry—smooth manifolds, principal G-bundles, etc—can be generalised in three directions.

$$\begin{tikzcd}
\text{Derived sheaves} \arrow[drr] & & \text{Sheaves} \arrow[ll, "\text{good intersections}"'] \arrow[rr, "\text{good quotients}"] \arrow[d] & & \infty\text{-stacks} \arrow[dll] \\
& & \text{Derived sheaves} \coprod_{\text{Sheaves}} \infty\text{-stacks} \arrow[d, dashed] & & \\
& & \text{Derived } \infty\text{-stacks} & &
\end{tikzcd}$$

CHAPTER 1

Sheaves

Given a category C of 'atoms', the category of presheaves $\mathsf{Psh}(\mathsf{C})$ can be very roughly viewed as a category of 'molecules'. This idea rests on two important results. First of all, the Yoneda Lemma ensures that we can treat an atom both as a building block for molecules and as a molecule itself, and that it does not matter which perspective we take. Secondly, the fact that 'every presheaf is the colimit of representables' means that every object of $\mathsf{Psh}(\mathsf{C})$ is obtained by gluing together atoms, and therefore deserves (to some extent) to be called a molecule. However, not all combinations of atoms are actually allowed to form molecules in real life, because the laws of physics... so the $\mathsf{Psh}(\mathsf{C})$ might be better seen as a category of 'formal molecules' containing a subcategory of real molecules. That being said, if one simply inserts a formal molecule into the universe and starts the clock, it will have no choice but to obey the laws of nature, eventually (after a possible violent reaction) becoming a genuine molecule. This means that the inclusion

$$\{\text{molecules}\} \hookrightarrow \{\text{formal molecules}\}$$

is a reflective subcategory. (Is this complete nonsense?)

The first part of this chapter is all about reflective subcategories of presheaf categories, which are called <u>topoi</u>.

Emphasise we study generalised smooth spaces as sheaves on some site. this includes diffeological spaces, whose definition is sometimes given less abstractly.

1.1. What is a sheaf?

Recall that a <u>presheaf</u> (of sets) on a topological space M is a functor $X : \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Set}$, where $\mathcal{O}(M)$ is the category of non-empty open subsets with morphisms given by inclusions. A <u>sheaf</u> is a presheaf which satisfies the following 'gluing condition': given any open set $U \subseteq M$ and cover $U = \bigcup_{i \in \mathrm{I}} U_i$, an element $s \in X(U)$ is the same thing as a collection $\{s_i \in X(U_i)\}$ which agree on the overlaps $U_i \cap U_j$. In other words, we have

$$X(U) = \lim \left(\prod_{i \in \mathrm{I}} X(U_i) \rightrightarrows \prod_{i,j \in \mathrm{I}} X(U_i \cap U_j) \right).$$

This property can be viewed as a sort of partial continuity of the functor X, once we identify the colimit

$$U = \operatorname{colim} \left(\coprod_i U_i \leftleftarrows \coprod_{i,j} U_i \cap U_j \right)$$

in the category of opens $\mathcal{O}(M)$, which is equivalently the limit

$$U = \lim \left(\coprod_i U_i \rightrightarrows \coprod_{i,j} U_i \cap U_j \right)$$

in the source of X, the opposite category $\mathcal{O}(M)^{\mathrm{op}}$.

More generally, a presheaf (of sets) on a category C is a functor $X : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. What, then, is the appropriate notion of a sheaf on C? To impose a 'gluing condition' as for sheaves on topological spaces, we require some notion of open cover. This is exactly the purpose of the following definition:

Definition 1.1.1. A Grothendieck topology on C is a class j, whose elements are coverings: collections $\{U_i \to U \mid i \in I\}$ of morphisms with a fixed codomain, satisfying the following axioms:

(isom) if $f : V \to U$ is an isomorphism then $\{f\} \in j$;

(tree) given coverings $\{V_{ij} \to U_i\}$ for each $i \in I$, we have $\{V_{ij} \to U\} \in j$;

$$\begin{tikzcd}
V_{i_0 j_0} \arrow[dr] & V_{i_0 j_1} \arrow[d] & V_{i_0 j_2} \arrow[dl] & V_{i_1 j_0} \arrow[dr] & V_{i_1 j_1} \arrow[d] & V_{i_1 j_2} \arrow[dl] & V_{i_2 j_0} \arrow[dr] & V_{i_2 j_1} \arrow[d] & V_{i_2 j_2} \arrow[dl] \\
& U_{i_0} \arrow[drrr] & & & U_{i_1} \arrow[d] & & & U_{i_2} \arrow[dlll] & \\
& & & & U & & & &
\end{tikzcd}$$

(pull) given a map $f : V \to U$, the pullbacks $\{V \times_U U_i \to V\}$ exist and define a covering of V.

$$\left\{ \begin{tikzcd} V \times_U U_i \arrow[r] \arrow[d] & U_i \arrow[d] \\ V \arrow[r] & U \end{tikzcd} \right\}_{i \in I}$$

(For (pull) to make sense, we require the relevant pullbacks in C to exist.) We can write $j(U)$ for the set of coverings of U; the (pull) axiom actually guarantees that this defines a functor $j : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. A category equipped with a Grothendieck topology is called a site. ⌟

Now fix a site (C, j) and let $U \in \mathsf{C}$. A coverage $\{U_i \to U\}$ leads to the following diagram *in the opposite category* C^{op},

$$\begin{tikzcd}
& \cdots & & U_i \times_U U_j & & \cdots & \\
\cdots & & U_i \arrow[ul] \arrow[ur] & & U_j \arrow[ul] \arrow[ur] & & \cdots \\
& & & U \arrow[ul] \arrow[ur] \arrow[ulll] \arrow[urrr] & & &
\end{tikzcd}$$

and in fact it exhibits U as the limit of the diagram obtained by removing U:

$$U = \lim \left(\coprod_i U_i \rightrightarrows \coprod_{i,j} U_i \times_U U_j \right).$$

In complete analogy with the situation with topological spaces, we make the following definition.

DEFINITION 1.1.2. A presheaf $X : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is a <u>sheaf</u> if it preserves all limits of the form above, for all coverages in j. Sheaves form a full subcategory of $\mathsf{Psh}(\mathsf{C})$, which we denote $\mathsf{Sh}_j(\mathsf{C})$ or $\mathsf{Sh}(\mathsf{C})$. ⌟

REMARK 1.1.3. i.e. preserves certain limits. for example a sheaf must preserve terminal objects. ⌟

So far, we have completely abstracted away from the original context (open subsets of a topological space),

DEFINITION 1.1.4. A category C is <u>concrete</u> if it is equipped with a faithful functor $\Gamma : \mathsf{C} \to \mathsf{Set}$. A site (C, j) is <u>concrete</u> if C is concrete, and for every covering $\{U_i \to U\}$ in C, the morphism

$$\coprod_i \Gamma(U_i) \to \Gamma(U)$$

is a surjective map of sets. ⌟

The objects in a concrete category can be thought of as sets with extra structure, and morphisms as structure-preserving maps. Examples include most categories one first encounters: the category Ring of rings, the category Top of topological spaces, and so on. In each of these cases, the functor Γ is the forgetful functor to Set. However, the forgetful functor $\Gamma : \mathsf{AffSch} \to \mathsf{Set}$ is *not* faithful (consider the spectrum of a field with nontrivial automorphisms) so $(\mathsf{AffSch}, \Gamma)$ is not concrete.

REMARK 1.1.5. The non-concreteness of $(\mathsf{AffSch}, \Gamma)$ is due to the fact that respecting the 'ringed space structure' does not just restrict the set of possible morphisms; instead, it provides more possibilities for morphisms that are invisible at the set-level. Remarkably, despite this, AffSch is actually concretisable (it admits a faithful functor to Set). We construct this functor explicitly[1]. Define the functor $\Lambda : \mathsf{AffSch} \to \mathsf{Set}^{\mathrm{op}} : \operatorname{Spec}(A) \mapsto \{A_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(A)\}$ to be the functor that sends an affine scheme to its set of stalks. The functor

$$\Gamma \amalg \mathscr{P}\Lambda : \mathsf{AffSch} \to \mathsf{Set} : \operatorname{Spec}(A) \mapsto (\Gamma \circ \operatorname{Spec})(A) \coprod (\mathscr{P} \circ \Lambda \circ \operatorname{Spec})(A)$$

is faithful, where $\mathscr{P} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is the (faithful) power-set functor $\mathrm{Mor}_{\mathsf{Set}}(-, \{0,1\})$. When equipped with this functor, the category of affine schemes (or equivalently, $\mathsf{Ring}^{\mathrm{op}}$) is concrete. ⌟

The condition demanded of a concrete site just ensures that a 'covering' actually covers the underlying set of the target, supporting the intuition from open covers of topological spaces.

DEFINITION 1.1.6. Let C be a concrete site. A presheaf $X : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ <u>concrete</u> if for each $U \in \mathsf{C}$ the map

$$X(U) \xrightarrow{\sim} \underset{\mathsf{Psh}(\mathsf{C})}{\mathrm{Mor}}(y_{\mathsf{C}}U, X) \xrightarrow{\Gamma} \underset{\mathsf{Set}}{\mathrm{Mor}}(\Gamma(y_{\mathsf{C}}U), \Gamma(X))$$

is injective. A presheaf $X : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is a <u>concrete sheaf</u> if it is concrete and a sheaf. ⌟

[1] stealing the idea from `https://mathoverflow.net/a/160768`

Knowing that a presheaf X is concrete allows us to treat the set $X(U)$ as 'certain maps from the underlying set of U to the underlying set of X'. In particular, if C has an object which morally qualifies as 'the point', then $X(*)$ should be viewed as the 'set of points' of X. In a reasonable world, this should coincide with the 'underlying set of X', which motivates the following definition.

Definition 1.1.7. A category is <u>terminally concrete</u> if it has a terminal object $* \in \mathsf{C}$ and the functor $\mathrm{Mor}_{\mathsf{C}}(*, -)$ gives C the structure of a concrete category. Terminally concrete sites and terminally concrete (pre)sheaves are defined in the obvious way. ⌟

This terminology is not standard. For example, the nLab uses the phrase `concrete site` for what we call a terminally concrete site, leaving our concrete sites without a name.

1.2. Topoi and quasitopoi

Definition 1.2.1. A <u>topos</u> is a category equivalent to the category of sheaves on a small site. ⌟

Recall that a subcategory is <u>reflective</u> if the inclusion functor admits a left adjoint (which is called the <u>reflector</u>) and that a functor is <u>left/right exact</u> if it preserves finite limits/colimits. Reflectors are automatically right exact (because they are left adjoints) so we will simply write <u>exact reflector</u> for a reflector which preserves finite limits; the target is then an <u>exact reflective subcategory</u>.

We have the following characterisation of topoi as exact reflective

Theorem 1.2.2. *Fix a small site* (C, j)*. The inclusion* $\mathsf{Sh}(\mathsf{C}) \hookrightarrow \mathsf{Psh}(\mathsf{C})$ *admits an exact reflector*

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{C}) \arrow[r, shift left, "\dagger"] \arrow[r, phantom, "\perp"] & \mathsf{Psh}(\mathsf{C}) \arrow[l, hook, shift left] \end{tikzcd}$$

Moreover, every exact reflective subcategory of $\mathsf{Psh}(\mathsf{C})$ *is isomorphic to the subcategory of sheaves with respect to some Grothendieck topology on* C*.*

Proof. can we do this, succinctly, in a way that avoids the double-+ construction? see here[2]. if not, see Lurie[3] instead.

for reverse: write $W = \dagger^{-1}$(isomorphisms). then $f : Y \to X$ is a <u>local epimorphism</u> if $\mathrm{im}(f) \to X$ is in W. (that is, if $\dagger(\mathrm{im}(f) \to X)$ is an isomorphism.) now declare that the covering sieves on $U \in \mathsf{C}$ are those sieves F such that $F \to U$ is a local epimorphism. □

Example 1.2.3. There is a way to "sheafify in a single step" as opposed to applying the +construction twice. The idea[4] is that whereas the + construction glues sections along intersections, sheafifying should really glue them if they locally agree. this takes a while to formalise so we elect to skip this, especially because the + construction leads to interesting discussion. (see https://ncatlab.org/nlab/show/plus+constructio ⌟

Example 1.2.4. Here is an example to show that, while the sheafification functor preserves finite limits, it does not need to preserve all limits. ⌟

[2]https://web.stanford.edu/ dkim04/blog/sheafification/
[3]https://www.math.ias.edu/ lurie/278xnotes/Lecture9-Sheaves.pdf
[4]from https://web.stanford.edu/ dkim04/blog/sheafification/

Definition 1.2.5. A <u>local isomorphism</u> is a morphism in $\mathsf{Psh}(\mathsf{C})$ which is sent to an isomorphism by the sheafification functor. ⌟

Lemma 1.2.6. *A presheaf $X \in \mathsf{Psh}(\mathsf{C})$ is a sheaf if and only if, for all local isomorphisms $A \to B$, the*

Proof. □

Definition 1.2.7. separated presheaf ⌟

Definition 1.2.8. quasitopos ⌟

Example 1.2.9. concrete presheaves are sep for ?? finish proof ⌟

The fact that sheafification preserves finite limits means that, in particular, it preserves the terminal object (which exists, since $\mathsf{Psh}(\mathsf{C})$ is complete). (In fact, the same argument shows that topoi have finite limits.) If C has a terminal object $*$ then, because the Yoneda embedding is continuous, the terminal object of $\mathsf{Sh}(\mathsf{C})$ must coincide with $y_{\mathsf{C}}(*)$.

In fact, we have a functor $\Gamma\colon \mathsf{Sh}(\mathsf{C}) \to \mathsf{Set}$ induced by the unique functor $\mathsf{C} \to \mathbf{1}$. If C has a terminal object, then $\Gamma = \mathrm{Mor}_{\mathsf{C}}(*, -)$ (why?). This has a left adjoint $\mathrm{Disc}\colon \mathsf{Set} \to \mathsf{Sh}(\mathsf{C})$ which sends a set X to the sheafification of the constant presheaf $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}\colon U \mapsto X$. This constitutes a <u>geometric morphism</u> $\mathsf{Sh}(\mathsf{C}) \to \mathsf{Set}$.

Definition 1.2.10. A site (C, j) is <u>local</u> if it has a terminal object $* \in \mathsf{C}$ and $j(*)$ consists of $\{\mathrm{id}\colon * \to *\}$. ⌟

Proposition 1.2.11. *Let (C, j) be a local site. The functor*

$$\Gamma := \operatorname*{Mor}_{\mathsf{Sh}(\mathsf{C})}(y_{\mathsf{C}}(*), -)\colon \mathsf{Sh}(\mathsf{C}) \to \mathsf{Set}$$

has a right adjoint $coDisc\colon \mathsf{Set} \to \mathsf{Sh}(\mathsf{C})$ which is the sheafification of ... what?

Proof. content... □

We will focus our attention on local topoi. The following result provides lots of examples:

Proposition 1.2.12. *Let C be a terminally concrete site. Then the topos $\mathsf{Sh}(\mathsf{C})$ is local.*

Proof. Let $\{U_i \to U\}$ be a covering of $U \in \mathsf{C}$. By definition, the map

$$\coprod_i \operatorname*{Mor}_{\mathsf{C}}(*, U_i) \to \operatorname*{Mor}_{\mathsf{C}}(*, U)$$

is surjective. □

Definition 1.2.13. The composition $\sharp\colon \mathrm{id}_{\mathsf{Sh}(\mathsf{C})} \to coDisc \circ \Gamma$ is the <u>sharp modality</u>. ⌟

Proposition 1.2.14. *Uf*

$$\operatorname*{Mor}_{\mathsf{Sh}(\mathsf{C})}$$

so $(\Gamma \dashv coDisc)\colon \mathsf{Set} \to \mathsf{Sh}(\mathsf{C})$ is the localisation of $\mathsf{Sh}(\mathsf{C})$ at the counits $\{\}$

Remark 1.2.15. If one studies topoi themselves, the definitions presented might seem too 'extrinsic', as they necessarily refer to an external category of presheaves. There are in fact many equivalent definitions of topoi as categories satisfying certain properties; one example is the Giraud axioms. We won't see this perspective $\lrcorner$

1.3. Diffeological spaces

Definition 1.3.1. A Cartesian space is a smooth manifold of the form $\mathbb{R}^n$ for some $n \in \mathbb{N}$ (with its standard smooth structure). Write Cart for the full subcategory of Man on Cartesian spaces. $\lrcorner$

Example 1.3.2. The site Cart has a terminal object $* = \mathbb{R}^0$, and the functor

$$\underset{\mathsf{Cart}}{\mathrm{Mor}}(*, -) : \mathsf{Cart} \to \mathsf{Set}$$

is simply the forgetful functor (which sends a cartesian space to its underlying set of points) so is faithful. The surjective condition is clearly met. Thus Cart is a concrete site. $\lrcorner$

We may wish to focus on smooth sets which have an 'underlying set of points'. In this case, a morphism is just a map between the underlying sets, which preserves the "smooth structure". This idea is encapsulated by concreteness:

Definition 1.3.3. A diffeological space is a concrete sheaf on Cart. A smooth map between diffeological spaces is a morphism of presheaves. Diffeological spaces and smooth maps organise into a category, denoted Dif, which is a full subcategory of $\mathsf{Psh}(\mathsf{Cart})$. $\lrcorner$

Unpacking this, we see that a diffeological space $X : \mathsf{Cart}^{\mathrm{op}} \to \mathsf{Set}$ consists of an underlying set, which we will also call X, together with subsets $X(\mathbb{R}^n) \subseteq \mathrm{Mor}_{\mathsf{Set}}(\mathbb{R}^n, X)$ for all $n \geq 0$ which satisfy the following conditions:

- (cnst) $X(\mathbb{R}^0)$ contains all points: $X(\mathbb{R}^0) = \mathrm{Mor}_{\mathsf{Set}}(\mathbb{R}^0, X)$;
- (pull) $\bigcup_{n\geq 0} X(\mathbb{R}^n)$ is closed under pullbacks in Cart;
- (glue) Suppose we have a good cover $\{f_i : U_i \hookrightarrow \mathbb{R}^n\}$ in Cart and a function $P \in \mathrm{Mor}_{\mathsf{Set}}(\mathbb{R}^n, X^0)$. If $P \circ f_i \in X^n$ for all i. Then $P \in X^n$.

An element of $X(\mathbb{R}^n)$ is called a plot (of dimension n). A smooth map between diffeological spaces $X \to Y$ is then a map of underlying sets, such that post-composition sends plots of X to plots of Y.

Warning 1.3.4 (Pokéball cover). The good coverage on Cart does not satisfy the (tree) axiom of a Grothendieck topology. To see this, let $B^{\mathrm{pre}} = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$ be the unit disc, write $U^{\mathrm{pre}} = \{y \geq 0\} \setminus B$ for the upper half of the complement, and $L^{\mathrm{pre}} = \{y \leqslant 0\} \setminus B$ for the lower half. Thicken each of these sets by a small ε amount, to obtain three open subsets B, T, U which cover $\mathbb{R}^2$. Now we have two good covers:

$$\begin{tikzcd}
U \arrow[dr] & & B \arrow[dl] & L \arrow[d] \\
& U \cup B & & L
\end{tikzcd}$$

but they cannot be "grafted" together; the cover

$$\begin{tikzcd}
U \arrow[dr] & B \arrow[d] & L \arrow[dl] \\
& \mathbb{R}^2 &
\end{tikzcd}$$

is not good. In retrospect, there is indeed no good reason why good coverages should graft. Not so good after all... ⚠

There are two pieces of evidence that Cart should contain the empty set $\varnothing$. the first is that

Let's weaken the definition a bit.

Definition 1.3.5. A quasitopos is a category equivalent to the categories of sheaves on a site (C, j) which are furthermore separated for a finer topology $k \supseteq j$. ⌟

Taking $k = j$, we see that all quasitopoi are topoi.

Example 1.3.6. Manifolds can be viewed as diffeological spaces in a canonical way: plots are smooth maps in the sense of manifolds. In fact $\mathsf{Man} \hookrightarrow \mathsf{Dif}$ is a full subcategory. but colimits are different, as we will see. This is good, because our initial aim was to generalise the idea of a manifold; but Iglesias says this is not the way to think of them. ⌟

Example 1.3.7 (Dtop $\dashv$ Cdif). Any topological space X can be turned into a diffeological space by declaring the plots to be all continuous maps from $\mathbb{R}^n$; this is the continuous diffeology, which assembles into a functor

$$\mathrm{Cdif} : \mathsf{Top} \to \mathsf{Dif}.$$

Conversely, any diffeological space X can be turned into a topological space by declaring $O \subseteq X$ to be open if and only if $P^{-1}(O) \subset U$ is open for each plot $P : U \to X$; this is the D-topology, which also assembles into a functor

$$\mathrm{Dtop} : \mathsf{Dif} \to \mathsf{Top}.$$

These constructions are adjoint. Let $X \in \mathsf{Dif}$ and $Y \in \mathsf{Top}$, and $f : X \to Y$ be a map of sets.

$$\begin{tikzcd}
f \in \operatorname*{Mor}_{\mathsf{Top}}(\mathrm{Dtop}(X), Y) \arrow[rr, dashed, leftrightarrow] \arrow[d, Leftrightarrow] & & f \in \operatorname*{Mor}_{\mathsf{Dif}}(X, \mathrm{Cdif}(Y)) \arrow[d, Leftrightarrow] \\
\begin{array}{c} \forall\, O \text{ open in } Y, \\ f^{-1}(O) \text{ is open in } \mathrm{Dtop}(X) \end{array} \arrow[r, phantom, "\Leftrightarrow"] & \begin{array}{c} \forall\, O \text{ open in } Y \\ \forall\, n \text{ and } P \in X^n \\ (f \circ P)^{-1}(O) \text{ is open in } \mathbb{R}^n \end{array} \arrow[r, phantom, "\Leftrightarrow"] & \begin{array}{c} \forall\, n \text{ and } P \in X^n \\ f \circ P \text{ is continuous} \end{array}
\end{tikzcd}$$

The bijection on homsets is given by the identity map, which is clearly natural. It follows that there is an adjunction Dtop $\dashv$ Cdif. What are the fixed objects of this adjunction? ⌟

Since Dif is a full subcategory of $[\mathsf{Cart}^{\mathrm{op}}, \mathsf{Set}]$, a morphism $: X \to Y$ of diffeological spaces is just a natural transformation of presheaves. The concreteness of diffeological spaces allows us to treat them simply as maps $X^0 \to Y^0$ which send plots to plots. This is made precise by:

Lemma 1.3.8. *Let X, Y be diffeological spaces. Let $f^0 : X^0 \to Y^0$ be a map of sets admitting the following pullback:*

$$\begin{tikzcd} Y^n \arrow[r, phantom, "\subseteq"] & \operatorname*{Mor}_{\mathsf{Set}}(\mathbb{R}^n, Y^0) \\ X^n \arrow[u, dashed] \arrow[r, phantom, "\subseteq"] & \operatorname*{Mor}_{\mathsf{Set}}(\mathbb{R}^n, X^0) \arrow[u, "f^0\circ -"'] \end{tikzcd}$$

(That is, post-composing with f^0 sends plots in X to plots in Y.) Then f^0 extends to a morphism of diffeological spaces. Moreover, every morphism $X \to Y$ arises in this way.

Proof. For any n, define $f^n : X^n \to Y^n$ by $f^n(P) = f^0 \circ P$. Then the naturality square commutes because pre-composition and post-composition commute, so this defines a morphism of diffeological spaces. Moreover, given a morphism $f : X \to Y$, we claim that $f^n(P) = f^0 \circ P$. work out this proof, should be easy □

Definition 1.3.9. Let $f : X \to Y$ be a morphism of diffeological spaces. The pushforward space f_*X and the pullback space f^*Y are defined is defined by

$$(f_*X)^n := f^n(X^n), \qquad (f^*Y)^n := (f^n)^{-1}(Y^n).$$

That is, $(f_*X)^n$ consists of all maps $f^0 \circ P$ where $P \in X^n$, and $(f^*Y)^n$ consists of those $P : \mathbb{R}^n \to X^0$ such that $f^0 \circ P$ is a plot of Y. actually you need to take diffeo generated by this... ⌟

There are always natural inclusion morphisms $f_*X \hookrightarrow Y$ and $X \hookrightarrow f^*Y$. We can ask when these are isomorphisms. The answers provide some of the most important classes of morphisms in diffeology.

Definition 1.3.10. Let $f : X \to Y$ be a morphism of diffeological spaces. We say f is a subduction if $f_*X = Y$; equivalently, if every plot $P : \mathbb{R}^n \to Y^0$ factors through X^0. Dually, f is an induction if f^0 is injective and $X = f^*Y$; equivalently, if every plot of X is the 'restriction' of some plot of Y. ⌟

Remark 1.3.11. There appears to be some asymmetry in our definitions of subduction and induction: an induction must be injective on underlying sets, whereas a subduction is not explicitly declared to be surjective on underlying sets. This is because the condition $f_*X = Y$ already imposes surjectivity, as a diffeology must contain all constant maps. ⌟

Example 1.3.12 (Quotient diffeology). Let X be a diffeological space, let $\sim$ be an equivalence relation on X^0 and consider the quotient map $\pi : X \to X/\sim$. Then π_*X is the quotient diffeological space of this relation. The underlying set of π_*X is the set-theoretic quotient $X/\sim$, and its plots are precisely those which factor through π. Conversely, given a subduction $f : X \to Y$ we can form the equivalence relation $\sim$ on X^0 by

$$x \sim y \iff f^0(x) = f^0(y).$$

Then Y^0 can be identified with $X/\sim$, and the associated subduction $X \to \pi_*X$ is isomorphic to f. We conclude that subductions are the quotients in diffeology. ⌟

Example 1.3.13. Let X be a diffeological space and let $\iota : A \hookrightarrow X^0$ a subset of points. Then A can be promoted to ⌟

Lemma 1.3.14. *Let $\pi : X \to Y$ be a smooth map between diffeological spaces. If π admits a smooth section, then π is a subduction.*

Proof. Let $s : Y \to X$ be a section of π. Given $P \in Y^n$, we want to exhibit some $Q \in X^n$ such that $P = \pi \circ Q$. The construction is as in the following slideshow.

$$
\begin{tikzcd}
Y \arrow[r, "s"] \arrow[rr, bend left, "\mathrm{id}_Y"] & X \arrow[r, "\pi"] & Y \\
 & & \mathbb{R}^n \arrow[u, "P"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "s"] \arrow[rr, bend left, "\mathrm{id}_Y"] & X \arrow[r, "\pi"] & Y \\
\mathbb{R}^n \arrow[u, "P"'] \arrow[ur] \arrow[rr, equal] & & \mathbb{R}^n \arrow[u, "P"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "s"] \arrow[rr, bend left, "\mathrm{id}_Y"] & X \arrow[r, "\pi"] & Y \\
\mathbb{R}^n \arrow[u, "P"'] \arrow[ur] \arrow[rr, equal] & & \mathbb{R}^n \arrow[u, "P"'] \arrow[ul, "Q"]
\end{tikzcd}
$$

That is, bring P to the Y *behind* X, compose with s, then bring $s \circ P$ back to the original Y. □

Example 1.3.15. A smooth manifold M is a diffeological space in a canonical way. Namely, consider $\mathbb{R}^n$ as smooth manifolds and define

$$M^n := \operatorname*{Mor}_{\mathsf{Man}}(\mathbb{R}^n, M).$$

In this way, Man is a full subcategory of Dif. ⌟

Example 1.3.16 (Axes). Consider the axes $X = \mathbb{R} \cup_0 \mathbb{R}$. Can get this space by gluing two lines, and by the subset diffeology of $\mathbb{R}^2$. The former: a plot $\mathbb{R}^n \to X$ is a pair of smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$ which agree at 0. ⌟

Example 1.3.17 (Manifolds with corners). Can do a similar thing; subset diffeo... ⌟

Example 1.3.18 (Products and coproducts). Let $\{X_i \mid i \in I\}$ be a collection of diffeological spaces. Their <u>product</u> is their product in the category of presheaves. A plot of $\prod_{i \in I} X_i$ is therefore a choice of plot for each X_i, because the continuity of the Yoneda embedding authorises the following vertical jump:

$$
\begin{aligned}
\left(\prod_{i \in I} X_i\right)^n = X(\mathbb{R}^n) = \operatorname*{Mor}_{[\mathsf{Cart}^{\mathrm{op}}, \mathsf{Set}]}(\mathbb{Y}\mathbb{R}^n, X) &= \operatorname*{Mor}_{[\mathsf{Cart}^{\mathrm{op}}, \mathsf{Set}]}(\mathbb{Y}\mathbb{R}^n, \prod_{i \in I} X_i) \\
&= \prod_{i \in I} \operatorname*{Mor}_{[\mathsf{Cart}^{\mathrm{op}}, \mathsf{Set}]}(\mathbb{Y}\mathbb{R}^n, X_i) = \prod_{i \in I} X_i^n.
\end{aligned}
$$

The <u>coproduct</u> of $\{X_i \mid i \in I\}$ is their coproduct in the category of presheaves. A plot of $\coprod_{i \in I} X_i$ is therefore ⌟

Combining these examples, we find:

Corollary 1.3.19. Dif *has all limits and colimits.*

Proof. Products and coproducts are clear (ish.. don't check univ property probably)

Equalisers (inductions) and coequalisers () are also clear. so done? □

comment about "there are colimits in manifolds that become colimits in diffeological spaces but not colimits in sheaves" `https://mathoverflow.net/a/49242/170682`

EXAMPLE 1.3.20. Let $X, Y \in \mathsf{Dif}$. The set $C^\infty(X, Y)$ has a canonical diffeology;

$$P : U \to C^\infty(X, Y)$$

is a plot if and only if the map

$$U \times X$$

⌟

EXAMPLE 1.3.21. Irrational torus. ⌟

DEFINITION 1.3.22. The presheaf Ω^k on Cart. it is actually a sheaf, Its value on $*$ is just a singleton set, $\Omega^k(*) = *$. The sheaf Ω^k therefore cannot be concrete, otherwise we would have

$$\Omega^k(\mathbb{R}^n) \subseteq \operatorname*{Mor}_{\mathsf{Set}}(\mathbb{R}^n, *)$$

which would make for a very dull theory of differential forms indeed. There are natural transformations $\mathrm{d} : \Omega^k \Rightarrow \Omega^{k+1}$ induced by the exterior derivative, and pullbacks. ⌟

DEFINITION 1.3.23. Let X be a diffeological space. A <u>differential k-form</u> on X is a natural transformation $X \Rightarrow \Omega^k$. ⌟

Write $\Omega^k(X)$ for the set of k-forms on X. We have a natural map $\mathrm{d}_* : \Omega^k(X) \to \Omega^{k+1}(X)$. Also a morphism $f : X \to Y$ of diffeological spaces induces a pullback $f^* : \Omega^k(Y) \to \Omega^k(X)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, dashed] & Y \arrow[r] & \Omega^k \end{tikzcd}$$

EXAMPLE 1.3.24. do cohomology of a function space maybe? ⌟

PROPOSITION 1.3.25. *homotopy invariance*

PROOF. intro pp 35. very short! □

fact: stationary paths are deformation retract of space of all paths. (obvious.) so we can use that jusitifiably

REMARK 1.3.26. One can define the tangent space by dualising the set of 1-forms $TX = (\Omega^1(X)^*)$, and this is done in [PIZ]. However, this leads to undesirable properties, such as... There are various constructions which we will see shortly. ⌟

PROPOSITION 1.3.27. *A sheaf is concrete iff it is separated for j_{conc}.*

PROOF. Let $W \subseteq \operatorname{Mor}(\mathsf{Psh}(\mathsf{Cart}))$ be the class of all morphisms which are sent to isomorphisms by Γ. For example, all components of

$$\operatorname{Disc} \circ \Gamma \circ y_{\mathsf{Cart}} \to y_{\mathsf{Cart}}$$

are in W, because $\Gamma \circ \operatorname{Disc} = \mathrm{id}_{\mathsf{Set}}$. There is an associated Grothendieck topology to W.

Recall that (i.e. do this up there, in section 1.1) A presheaf $X \in \mathsf{Psh}(\mathsf{Cart})$ is separated for j_{conc} if, for all $f : A \to B$ in W, the map $f^* : \operatorname{Mor}(B, X) \to \operatorname{Mor}(A, X)$ is a monomorphism. (compare with def of projective/injective module?) (some S-local business?) In particular, the induced map

$$\operatorname{Mor}\left(\operatorname*{Mor}_{\mathsf{C}}(-, U), X\right) \to \operatorname{Mor}\left((\mathrm{Disc} \circ \Gamma) \operatorname*{Mor}_{\mathsf{C}}(-, U), X\right)$$

But using def of adjoint, this is just

$$X(U) \to \operatorname{Mor}\left(\Gamma \operatorname*{Mor}_{\mathsf{C}}(-, U), \Gamma(X)\right)$$

That this is a monomorphism is exactly what separated means.

To prove the converse, we need to somehow show that □

1.4. C^∞-rings

Let $\mathsf{Poly}_{\mathbb{R}}$ be the subcategory of Cart with polynomial maps as morphisms. A functor $A : \mathsf{Poly}_{\mathbb{R}} \to \mathsf{Set}$ is the data of sets $A(\mathbb{R}^n)$ for $n \geq 0$, together with functions $p_* : A(\mathbb{R}^n) \to A(\mathbb{R}^m)$ for each polynomial $p : \mathbb{R}^n \to \mathbb{R}^m$. Now assume that A is product-preserving. In this case, p_* takes the form

$$p_* : A^n \to A^m$$

where we abuse notation by writing $A := A(\mathbb{R})$. For example, taking $p : (x, y) \mapsto x + y$ we get an additive structure on the set A:

$$+ : A \times A \xrightarrow{p_*} A,$$

and taking $p : (x, y) \mapsto x \cdot y$ we get a compatible multiplicative structure

$$\cdot : A \times A \xrightarrow{p_*} A.$$

Together with the scaling operations $\{\mathbb{R} \to \mathbb{R} : x \mapsto \lambda x\}_{\lambda \in \mathbb{R}}$, and the unary operations $0, 1 : \mathbb{R}^0 \hookrightarrow \mathbb{R}$ (which serve as the additive and multiplicative identities), this endows the set A with the structure of a commutative $\mathbb{R}$-algebra. Moreover, since any polynomial map $\mathbb{R}^n \to \mathbb{R}^m$ can be built out of the maps we have defined so far (binary addition, binary multiplication, and rescaling), the rest of the functor $A : \mathsf{Poly}_{\mathbb{R}} \to \mathsf{Set}$ is already completely determined by this data. We arrive at the conclusion that a product-preserving functor $\mathsf{Poly}_{\mathbb{R}} \to \mathsf{Set}$ is *the same thing as* a commutative $\mathbb{R}$-algebra. So we have an equivalence of categories

$$[\mathsf{Poly}_{\mathbb{R}}, \mathsf{Set}]^\times \cong \mathsf{CAlg}_{\mathbb{R}}$$

where the $\times$-superscript denotes the (full) subcategory of product-preserving functors, and $\mathsf{CAlg}_{\mathbb{R}}$ denotes the category of commutative $\mathbb{R}$-algebras.

Supported by this idea, we make a more general definition using Cart instead of $\mathsf{Poly}_{\mathbb{R}}$.

Definition 1.4.1. A <u>C^∞-ring</u> is a product-preserving functor $A : \mathsf{Cart} \to \mathsf{Set}$. Write $C^\infty\mathsf{Ring}$ for the category of C^∞-rings. The inclusion $\mathsf{Poly}_{\mathbb{R}} \hookrightarrow \mathsf{Cart}$ induces a forgetful functor $C^\infty\mathsf{Ring} \to \mathsf{CAlg}_{\mathbb{R}}$, which sends a C^∞-ring to its <u>underlying $\mathbb{R}$-algebra</u>. ⌟

Remark 1.4.2. This is a very special case of a more general story: that of Lawvere theories and algebras over them. Specifically, a <u>Lawvere theory</u> is a category with finite products, whose objects are $\mathbf{T} = \{T^n \mid n \in \mathbb{N}\}$ for some object T. An algebra over $\mathbf{T}$ is a product-preserving functor $\mathbf{T} \to \mathsf{Set}$. A ring R defines a Lawvere theory $\{\mathbb{A}^n_R \mid n \in \mathbb{N}\}$ whose algebras are commutative R-algebras. ⌟

This is an established definition, central in derived differential geometry. Cartesian spaces themselves define C^∞-rings under the co-Yoneda embedding $y^{\mathsf{Cart}} : \mathsf{Cart} \to [\mathsf{Cart}, \mathsf{Set}]^{\mathrm{op}}$. Here are two main classes of examples:

Example 1.4.3. Let $X \in \mathsf{Psh}(\mathsf{Cart})$ be a presheaf (for example, a smooth manifold). Then

$$C^\infty(X) := \operatorname*{Mor}_{\mathsf{Psh}(\mathsf{C})}(X, y_{\mathsf{Cart}}(-)) : \mathsf{Cart} \to \mathsf{Set}$$

is continuous (because the Yoneda embedding is), so is a C^∞-ring. ⌟

Example 1.4.4. The forgetful functor $C^\infty\mathsf{Ring} \to \mathsf{CAlg}_{\mathbb{R}}$ admits a left adjoint $F : \mathsf{CAlg}_{\mathbb{R}} \to C^\infty\mathsf{Ring}$. Its essential image consists what we call algebraic C^∞-rings. The class of algebraic C^∞-rings is surprisingly large; for example, the C^∞-ring of a smooth manifold Mv(see steffens 2.3.6 or smth) ⌟

Example 1.4.5. local artinian $\mathbb{R}$-algebras (thickened points). this is why C^∞-rings provide a framework for synthetic differential geometry. relation to (stalks of) jet bundles? ⌟

Remark 1.4.6 (for later?). isbell duality seems to factor through the 'reduction functor' which discards nilpotents. how to sort this out? do example: isbell dual of weil algebras? ⌟

Definition 1.4.7. free C^∞-ring is a coproduct of representable ones. effectively just iterated tensor product of $C^\infty(\mathbb{R})$. ⌟

germs.

Definition 1.4.8. Let A be a C^∞-ring. We say A is finitely generated if it is of the form $C^\infty(\mathbb{R}^n)/I$. We can impose conditions on the ideal I: a finitely generated C^∞-ring $C^\infty(\mathbb{R}^n)/I$ is said to be

- fair if every element $f \in I$ satisfies $[f]_p \in [I]_p$ for all $p \in C^\infty(\mathbb{R}^n)\}$
- finitely presented if I is finitely generated as an ideal.

⌟

Recall that an object K in a category C is called compact if the functor $\mathrm{Mor}_{\mathsf{C}}(K, -)$ preserves filtered colimits; intuitively, a map from K to an ascending union lands in one of the components.

Lemma 1.4.9. *A C^∞-ring is finitely presented if and only if it is a compact object. Also, finitely presented implies fair.*

alfonsi 2.9: Cinf ring has only filtered colimits? why?

Definition 1.4.10. covering of C^∞-rings: D-étale maps? alfonsi around p.22 ⌟

Lemma 1.4.11. *an ordinary C^∞-algebra A is finitely presented precisely if it is compact (which means $\mathrm{Mor}(A, -) : C^\infty\mathsf{Ring} \to \mathsf{Set}$ preserves filtered colimits.) (alfonsi p29) also steffens...*

Definition 1.4.12. Modules over a C^∞-ring. Projective modules and free modules... ⌟

1.5. Isbell duality

Let C be a category of 'test spaces'. Functors from C^{op} are to be thought of as spaces, and from C to be thought of as quantities. There is a duality between these two in the form of an adjunction. This is especially pertinent to us because our spaces are presheaves on Cart satisfying certain properties, and C^∞-rings are copresheaves on Cart satisfying certain properties.

By the Yoneda Lemma, all presheaves $X : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ are naturally isomorphic to the presheaf they represent:

$$X \cong \operatorname*{Mor}_{\mathsf{Psh(C)}} (y_{\mathsf{C}}(-), X).$$

Applying the Lemma to C^{op} instead of C implies that, for all functors (copresheaves) $A : \mathsf{C} \to \mathsf{Set}$, we have a natural isomorphism

$$A \cong \operatorname*{Mor}_{[\mathsf{C},\mathsf{Set}]^{\mathrm{op}}} \left(y^{\mathsf{C}}(-), A\right) = \operatorname*{Mor}_{[\mathsf{C},\mathsf{Set}]} \left(A, y^{\mathsf{C}}(-)\right).$$

We see that presheaves are defined by maps (from representables) *into* them, and copresheaves are defined by maps (to corepresentables) *out* of them. An obvious way to turn a presheaf X into a copresheaf, then, is to treat it as one: define the Isbell dual of X to be

$$X^\vee := \operatorname*{Mor}_{\mathsf{Psh(C)}} (X, y_{\mathsf{C}}(-)).$$

This is visibly a copresheaf

Proposition 1.5.1. *The functors $X \mapsto X^\vee$ and $A \mapsto A^\vee$ define an adjunction.*

Proof. The Isbell functors are (co)continuous in the sense that

$$\begin{aligned}\left(\operatorname*{colim}_i \operatorname*{Mor}_{\mathsf{C}}(-, U_i)\right)^\vee (U) &= \operatorname*{Mor}_{\mathsf{Psh(C)}} \left(\operatorname*{colim}_i \operatorname*{Mor}_{\mathsf{C}}(-, U_i), y_{\mathsf{C}}\right) \\ &= \lim_i \operatorname*{Mor}_{\mathsf{Psh(C)}} \left(\operatorname*{Mor}_{\mathsf{C}}(-, U_i), y_{\mathsf{C}}\right) \\ &= \lim_i \operatorname*{Mor}_{\mathsf{C}}(U_i, -).\end{aligned}$$

The result now follows in the only possible way: if X is a presheaf, then $X \cong \operatorname{colim}_i \operatorname{Mor}_{\mathsf{C}}(-, U_i)$, so we can write

$$\begin{tikzcd}
\operatorname*{Mor}_{[\mathsf{C},\mathsf{Set}]^{\mathrm{op}}} \left(X^\vee, A\right) \arrow[r, dashed, equal] \arrow[d, equal, "\text{def of } X"] & \operatorname*{Mor}_{\mathsf{Psh(C)}} \left(X, A^\vee\right) \arrow[d, equal, "\text{def of } X"'] \\
\operatorname*{Mor}_{[\mathsf{C},\mathsf{Set}]^{\mathrm{op}}} \left(\left(\operatorname*{colim}_i \operatorname*{Mor}_{\mathsf{C}}(-, U_i)\right)^\vee, A\right) \arrow[d, equal, "\text{(above)}"] & \operatorname*{Mor}_{\mathsf{Psh(C)}} \left(\operatorname*{colim}_i \operatorname*{Mor}_{\mathsf{C}}(-, U_i), A^\vee\right) \arrow[d, equal, "\text{continuity}"'] \\
\operatorname*{Mor}_{[\mathsf{C},\mathsf{Set}]^{\mathrm{op}}} \left(\lim_i \operatorname*{Mor}_{\mathsf{C}}(U_i, -), A\right) \arrow[d, equal, "\text{continuity}"] & \lim_{\mathrm{I}} \operatorname*{Mor}_{\mathsf{Psh(C)}} \left(\operatorname*{Mor}_{\mathsf{C}}(-, U_i), A^\vee\right) \arrow[d, equal, "\text{Yoneda}"'] \\
\lim_i \operatorname*{Mor}_{[\mathsf{C},\mathsf{Set}]^{\mathrm{op}}} \left(\operatorname*{Mor}_{\mathsf{C}}(U_i, -), A\right) \arrow[r, equal, "\text{def of } A^\vee"] & \lim_i A^\vee(U_i)
\end{tikzcd}$$

The equalities here denote either literal equalities or natural isomorphisms. □

REMARK 1.5.2. This proof goes through identically in the context of enriched categories; for $\mathcal{V}$ a monoidal category and C a $\mathcal{V}$-enriched category, there is an adjunction

$$[\mathsf{C}^{\mathrm{op}}, \mathcal{V}] \leftrightarrows [\mathsf{C}, \mathcal{V}]^{\mathrm{op}}$$

defined in exactly the same way. Also, the same proof will go through in ∞-category theory, once we have defined the relevant concepts. See Proposition ??. ⌟

list the ways that $X^\vee$ is natural: really comes from yoneda

- kan extensions
- by yoneda, $X(U) = \mathrm{Mor}(h(U), X)$. switch them: $\mathrm{Mor}(X, h(U))$. this is $X^\vee(U)$.

recall from example (...) that the Isbell dual of a presheaf is a C^∞-ring.

LEMMA 1.5.3. *Let $A \in [\mathsf{C}, \mathsf{Set}]^{\mathrm{op}}$. Then $R^\vee \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ is continuous.*

PROOF. Let $\mathrm{I} \to \mathsf{C}$ be a diagram. Then

$$R^\vee\left(\lim_{i\in \mathrm{I}} U_i\right) = \operatorname*{Mor}_{[\mathsf{C},\mathsf{Set}]^{\mathrm{op}}}\left(R, h^{\mathsf{C}}\left(\lim_{i\in \mathrm{I}} U_i\right)\right)$$

□

As a consequence of Lemma **??**, the Isbell dual of any presheaf $X \in \mathsf{Psh}(\mathsf{Cart})$ preserves finite products, and is therefore a C^∞-ring. The converse statement is also true, although the proof is less immediate:

THEOREM 1.5.4. *The Isbell dual of a C^∞-ring is a diffeological space.*

We are grateful to Dmitri Pavlov for providing this fact and its proof via correspondence.

PROOF. First we must show that $A^\vee$ is concrete. We need to check, for all $U \in \mathsf{Cart}$, that the map

$$A^\vee(U) \to \operatorname*{Mor}_{\mathsf{Set}}\left(U, A^\vee(*)\right)$$

is injective. By definition of $A^\vee$, this map is given explicitly by

$$\operatorname*{Mor}_{[\mathsf{Cart},\mathsf{Set}]}(A, C^\infty(U)) \to \operatorname*{Mor}_{\mathsf{Set}}\left(\operatorname*{Mor}_{\mathsf{Cart}}(*, U), \operatorname*{Mor}_{[\mathsf{Cart},\mathsf{Set}]}(A, C^\infty(*))\right)$$
$$f \mapsto (u \mapsto C^\infty(u) \circ f)$$

where $C^\infty(u) : C^\infty(U) \to C^\infty(*)$ is the map induced from a point $u : * \to U$; this is given by

$$(C^\infty(u) \circ f)(a) = f(a) \circ u = f(a)(u(*)).$$

Suppose that two morphisms $f, g : A \to C^\infty(U)$ map to the same element of the right-hand side, which means that, for all $u \in U$ and $a \in A$, we have $f(a)(u) = g(a)(u)$. Since u and a were arbitrary, it follows that $f(a) = g(a)$, then that $f = g$.

Next, we show that $A^\vee$ is a sheaf. Let $U \in \mathsf{Cart}$ and let $\{U_i \to U\}$ be a good cover. We get a diagram like this, in Cart.

$$\begin{tikzcd}
 & \cdots \arrow[dr] & & U_i \times_U U_j \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & \\
\cdots \arrow[drrr] & & U_i \arrow[dr] & & U_j \arrow[dl] & & \cdots \arrow[dlll] \\
 & & & U & & &
\end{tikzcd}$$

We want to show that the sequence

$$A^\vee(U) \longrightarrow \prod_i A^\vee(U_i) \rightrightarrows \prod_{i,j} A^\vee(U_i \times_U U_j)$$

is an equaliser. By definition of $A^\vee$, an element of the middle product which is equalised by the two maps $\rightrightarrows$ is the same thing as a diagram

$$\begin{tikzcd}
 & \cdots & & C^\infty(U_i \times_U U_j) & & \cdots & \\
\cdots & & C^\infty(U_i) \arrow[ul] \arrow[ur] & & C^\infty(U_j) \arrow[ul] \arrow[ur] & & \cdots \\
 & & & A \arrow[ulll] \arrow[ul] \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}$$

Now because $C^\infty(U)$ form a sheaf of rings already, we know that $C^\infty(U)$ is the coproduct of the diagram obtained by deleting A from this. So there exists a unique map as follows:

$$\begin{tikzcd}
 & \cdots & & C^\infty(U_i \times_U U_j) & & \cdots & \\
\cdots & & C^\infty(U_i) \arrow[ul] \arrow[ur] & & C^\infty(U_j) \arrow[ul] \arrow[ur] & & \cdots \\
 & & & C^\infty(U) \arrow[ul] \arrow[ur] & & & \\
 & & & A \arrow[uulll] \arrow[uul] \arrow[uur] \arrow[uurrr] \arrow[u, dashed, "\exists!"'] & & &
\end{tikzcd}$$

Because a map $A \to C^\infty(U)$ is the same thing as an element of $A^\vee(U)$, the candidate equaliser is in fact an equaliser, so $A^\vee$ is a sheaf. □

Let E be the category of closed subspaces of $\mathbb{R}^n$ with smooth maps. Then

$$E^{\mathrm{op}} \to C^\infty\mathsf{Ring} : \ X \mapsto C^\infty(X, -)$$

is full and faithful.

Definition 1.5.5 (tentative). We say a presheaf $X \in \mathsf{Psh}(\mathsf{Cart})$ is <u>finitely generated</u> if the C^∞-ring $X^\vee$ is finitely generated. ⌟

Proposition 1.5.6. *Let X be a finitely generated diffeological space. Then X is Isbell self-dual if and only if it is isomorphic to a closed subset of $\mathbb{R}^n$ with the induced diffeology.*

Proof. Notice that C^∞-homomorphisms $C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$ are in 1-1 correspondence with equivalence classes

$$\frac{\{\varphi : \mathbb{R}^m \to \mathbb{R}^n \mid I \subseteq \varphi_*(J)\}}{\varphi \sim \psi \iff \forall i : \pi_i \circ \varphi - \pi_i \circ \psi \in J}.$$

When $I = m_X$ and $J = m_Y$ the latter is just the class of smooth maps $Y \to X$. It follows that for a finitely generated C^∞-ring $A = C^\infty(R^n)/I$, its Isbell spectrum is a diffeological space whose V-plots are smooth maps $V \to \mathbb{R}^n$ such that all elements of I vanish on the image of V, i.e., maps $V \to Z(I)$ such that the composition $V \to Z(I) \to R^n$ is smooth. That is to say, the Isbell spectrum of $C^\infty(R^n)/I$ simply extracts $Z(I)$ with its smooth structure induced from $\mathbb{R}^n$. Thus, for X to be Isbell self-dual, it must be isomorphic to such a space.

The $\Longleftarrow$ direction is just a matter of category theory: the left adjoint is fully faithful if and only if the unit is a natural isomorphism. □

For the non-finitely-generated case, arbitrary C^∞-rings are the Ind-completion of finitely generated ones, so it seems to me that the category of Isbell-self-dual diffeological spaces could be equivalent to the Pro-completion of (locally) closed subsets of $\mathbb{R}^n$. (Some details would have to be checked here, though.)

Theorem 1.5.7 (potentially false). *The category of Isbell self-dual diffeological spaces is equivalent to the Pro-completion of (locally) closed subsets of $\mathbb{R}^n$.* ⚠

Proof. The idea is that arbitrary C^∞-rings are the Ind-completion of finitely generated ones (Lemma **??**). But who's to say that Isbell duality preserves cofiltered limits? □

One may ask how large this Pro-completion is. For example, does it contain mapping spaces?

Example 1.5.8. Let M, N be manifolds. Then $C^\infty(M, N)$ is the projective limit of ⌟

We want to show that the unit map $\eta : C^\infty(M, N) \to \mathrm{Spec}(Hom(C^\infty(M, N), \mathbb{R}))$ is an isomorphism.

It is injective, since given $f, g \in C^\infty(M, N)$ such that $f \neq g$, we can find $m \in M$ such that $f(m) \neq g(m)$, the $\eta(f)$ and $\eta(g)$ take different values $f(m)$ and $g(m)$ at the point $ev_m \in Hom(C^\infty(M, N), \mathbb{R})$.

For surjectivity, we can reduce to the case $N = \mathbb{R}^n$ (using a locality argument for N) and then to the case $N = \mathbb{R}$. Suppose we are given a C^∞-homomorphism $\alpha : Hom(C^\infty(M, \mathbb{R}), \mathbb{R}) \to \mathbb{R}$. Consider the subobject of the domain of α given by linear maps $C^\infty(M, \mathbb{R}) \to \mathbb{R}$. This subobject is the smooth set $D(M)$ of compactly supported distributions on M (see `https://ncatlab.org/nlab/show/distributions+are+the+smooth+linear+functionals`). Restricting α to $D(M)$ yields a linear map $D(M) \to \mathbb{R}$. Such maps are known (see the cited article) to be in bijective correspondence with smooth functions on M: there is a unique $h \in C^\infty(M)$ such that for every $\psi \in D(M)$ we have $\alpha(\psi) = \int h\psi = \psi(h)$. This gives us a unique potential solution h to the equation $\eta(h) = \alpha$. It remains to verify that indeed $\eta(h) = \alpha$, i.e., α and $\eta(h)$ take the same values on nonlinear elements of $Hom(C^\infty(M,\mathbb{R}),\mathbb{R})$. Roughly speaking, $Hom(C^\infty(M,\mathbb{R}),\mathbb{R})$ looks like (or perhaps literally is) the C^∞-symmetric algebra of $D(M)$, and if two C^∞-homomorphisms coincide on $D(M)$, they must also coincide on the entire symmetric algebra generated by it. (I have an impression that this type of theorem might be in one of Michor's books on infinite-dimensional geometry, this would require digging in the literature.)

The adjunction between diffeological spaces and -rings re- stricts to an equivalence of categories of Isbell-reflexive diffeological spaces and Isbell-reflexive - rings. This adjunction can be seen as the smooth analog of the Zariski adjunction between affine schemes and commutative rings, or the Stein adjunction between Stein spaces and entire functional calculus algebras. Here is a Serre–Swan type theorem to exemplify this duality:

Theorem 1.5.9. *There is an equivalence (the categories of) vector bundles over X and projective $C^\infty(X)$-modules.*

Proof. First, the cartesian closedness of smooth sets implies that the category of free modules over an (Isbell-reflexive) C^∞-ring $C^\infty(X)$ is equivalent to the category of trivial vector bundles over the diffeological space X. Next, finitely generated projective modules are the idempotent completion (= Karoubi completion) of free modules. Thus, finitely generated projective modules over $C^\infty(X)$ are equivalent to the Karoubi completion of the category of trivial vector bundles over X. The latter category is itself equivalent to the category of vector bundles E over X, defined as expected. $\square$

Example 1.5.10. Let M and N be manifolds. The diffeological space $C^\infty(M,N)$ is Isbell-self dual. We show this now (These are Dmitri's words). We

want to show that the unit map $\eta : C^\infty(M,N) \to \mathrm{Spec}(Hom(C^\infty(M,N),\mathbb{R}))$ is an isomorphism.

It is injective, since given $f, g \in C^\infty(M,N)$ such that $f \neq g$, we can find $m \in M$ such that $f(m) \neq g(m)$, the $\eta(f)$ and $\eta(g)$ take different values $f(m)$ and $g(m)$ at the point $ev_m \in Hom(C^\infty(M,N),\mathbb{R})$.

For surjectivity, we can reduce to the case $N = \mathbb{R}^n$ (using a locality argument for N) and then to the case $N = \mathbb{R}$. Suppose we are given a C^∞-homomorphism $\alpha : Hom(C^\infty(M,\mathbb{R}),\mathbb{R}) \to \mathbb{R}$. Consider the subobject of the domain of α given by linear maps $C^\infty(M,\mathbb{R}) \to \mathbb{R}$. This subobject is the smooth set $D(M)$ of compactly supported distributions on M (see
`https://ncatlab.org/nlab/show/distributions+are+the+smooth+linear+functionals`). Restricting α to $D(M)$ yields a linear map $D(M) \to \mathbb{R}$. Such maps are known (see the cited article) to be in bijective correspondence with smooth functions on M: there is a unique $h \in C^\infty(M)$ such that for every $\psi \in D(M)$ we have $\alpha(\psi) = \int h\psi = \psi(h)$. This gives us a unique potential solution h to the equation $\eta(h) = \alpha$. It remains to verify that indeed $\eta(h) = \alpha$, i.e., α and $\eta(h)$ take the same values on nonlinear elements of $Hom(C^\infty(M,\mathbb{R}),\mathbb{R})$. Roughly speaking, $Hom(C^\infty(M,\mathbb{R}),\mathbb{R})$ looks like (or perhaps literally is) the C^∞-symmetric algebra of $D(M)$, and if two C^∞-homomorphisms coincide on $D(M)$, they must also coincide on the entire symmetric algebra generated by it. (I have an impression that this type of theorem might be in one of Michor's books on infinite-dimensional geometry, this would require digging in the literature.) $\lrcorner$

Proposition 1.5.11 (Potentially false, and missing details). *Suppose we have a finitely presented Isbell self-dual diffeological space. Then the external and internal tangent spaces agree.*

Proof. Pick an Isbell-reflexive algebra $A = C^\infty(\mathbb{R}^n)/(f_i : i \in I)$. Then the external cotangent space of $A^\vee$ can be computed algebraically, as the module of C^∞-derivations of A:

$$E := \frac{C^\infty(\mathbb{R}^n)\langle \mathrm{d}x_1, \dots, \mathrm{d}x_n \rangle}{(f_i = 0, \mathrm{d}f_i = 0)}.$$

The total space of the external tangent bundle can be computed as the C^∞-symmetric algebra of the dual module, which yields

$$\frac{C^\infty(\mathbb{R}^n)[\partial_1, \dots, \partial_n]}{(f_i = 0, df_i = 0)}.$$

This is a diffeological space, whose value on $U \in \mathsf{Cart}$ is of the resulting algebra, we get the set of homomorphisms of C^∞-rings

$$C^\infty(\mathbb{R}^n)[\partial_1, \dots, \partial_n]/(f_i = 0, df_i = 0) \to C^\infty(S).$$

The kinematic tangent space of $A^\vee$ is given by the (possibly derived) internal hom

$$KT(A^\vee) := Hom(\mathrm{Spec}(\mathbb{R}[x]/x^2), A^\vee).$$

The S-points of $KT(A^\vee)$ are morphisms $KT(A^\vee)$ whose S-points are

$$Map(\mathrm{Spec}(\mathbb{R}[x]/x^2) \times S, \mathrm{Spec}\, A),$$

which can be equivalently described as the set of (possibly derived) maps of C^∞-rings:

$$Q(A) \to \mathbb{R}[x]/x^2 \otimes C^\infty(S),$$

where $Q(A)$ is either A or a cofibrant replacement of A, if we want a more refined answer. Sticking to the simpler case $QA = A$, we get maps

$$A \to \mathbb{R}[x]/x^2 \otimes C^\infty(S) = C^\infty(S)[x]/x^2,$$

i.e., derivations of A along some C^∞-homomorphism $A \to C^\infty(S)$, and these connect to the above description of S-points for the algebraic tangent bundle. □

1.6. Cohesive topoi

pullback f^* between sheaves has (under conditions, what are they?) multiple adjoints.

disc has a left adjoint: Π_0 which sends X to connected components. this is a good setting for kinematics, says urs.

Example 1.6.1. This example is completely adjacent to the focus of this chapter, but we include it because it is the result of (rarely observed) original thought on the author's part. A framework for measure theory has been proposed by Gromov, where he defines a generalised probability space to be a functor

$$\mathcal{P} \to \mathsf{Set}$$

where $\mathcal{P}$ is the category of nowhere-vanishing probability distributions on finite sets (which can be identified with points of $\mathrm{int}|\Delta^n|$) where a morphism $f : P \to Q$ is a 'reduction': a set map such that $|f^{-1}(q)| = |q|$ for all $q \in Q$. What is the Isbell dual of a (probability/measure) space? ⌟

CHAPTER 2

∞-categories

The purpose of this chapter is to develop the necessary tools to make sense of the 'homotopical mirror symmetry prophecy' described in §??. Ultimately we will come to the definition of the derived ∞-category associated to an abelian category.

new plan for this section: simplicial sets THEN infinity categories various adjunctions: nerves, homotopy, realisation (need nerve in general and homotopy to define concepts), functors/limits, stability, dold-kan, stability, derived.

The exposition is streamlined where possible; compare the discussion of the homotopy coherent nerve $\mathcal{N}_\mathsf{h}$ here to that given by Lurie in Kerodon.

A topological space X can be viewed as a category as follows: objects are the points of X, and morphisms are homotopy classes of paths between points. This is a groupoid because every path has a homotopy inverse (the reverse path), so it's called the fundamental groupoid of X and denoted $\Pi_1(X)$. (Note that $\mathrm{End}_{\Pi_1(X)}(x)$ recovers the fundamental group $\pi_1(X, x)$!) Taking homotopy classes of paths is necessary for a number of reasons: first, there is no way of naturally composing paths in such a way that triple composition is associative; second, if we accept that identity morphisms should be constant paths (what else could they be?) there can clearly be no notion of genuine inverse.

However, if we remember *how* two paths are homotopic, then we arrive at a sort of 'higher' object called the fundamental 2-groupoid of X, denoted $\Pi_2(X)$. This is an example of a 2-category: objects are points of X, morphisms are paths between points, and 'morphisms between morphisms' are homotopy classes of homotopies between paths. In fact, we can keep going, considering higher and higher homotopies, forever until ∞. We arrive at a object which contains the homotopical information of X. It is called the fundamental ∞-groupoid.

Here is another perspective on $\Pi_\infty(X)$.

Definition 2.0.1. The simplex category Δ is defined by its

- objects: the finite sets $[n] := \{0, \dots, n\}$ for each non-negative integer n;
- morphisms: order-preserving maps.

The objects of Δ can be thought of as labelled simplices, and the morphisms include coface maps $d^i : [n] \to [n+1]$ which miss i, and codegeneracy maps $s^i : [n+1] \to [n]$ which hit i twice. Every order-preserving map $[n] \to [m]$ is actually the composition of these two types of morphism. The topological n-simplex $\{(x_0, \dots, x_n) \in [0, \infty)^{n+1} \mid \sum_i x_i = 1\}$ is denoted by $|\Delta^n|$. ⌟

We claim that $\Pi_\infty(X)$ is described by the functor

$$\mathcal{X} : \Delta^{\mathrm{op}} \to \mathsf{Set} : [n] \mapsto \mathcal{X}_n := \operatorname*{Mor}_{\mathsf{Top}}(|\Delta^n|, X).$$

Then $\mathcal{X}_0$ is the set of points of X, and $\mathcal{X}_1$ is the set of paths in X. An element of $\mathcal{X}_2$ is a map from $|\Delta^2| \to X$; we can restrict to the three edges of the triangle to get a triple of paths

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z \end{tikzcd}$$

A 2-morphism is a continuous $D^2 \to X$ such that $\partial^+ D^2$ maps to f and $\partial^- D^2$ maps to g.

2.1. Simplicial objects

The functor $\mathcal{X}$, which described the homotopy theory of X, is an example of a 'simplicial set'.

Definition 2.1.1. Let C be a category. A simplicial object of C is a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$. These assemble into a functor category $\Delta\mathsf{C} := [\Delta^{\mathrm{op}}, \mathsf{C}]$, the category of simplicial objects of C. ⌟

A simplicial object $X \in \Delta\mathsf{C}$ therefore essentially consists of objects X_n for each non-negative integer n, together with a morphism between them for each corresponding morphim of Δ. These are determined by the images of the cosimplicial and coface maps from Definition 2.0.1, written $s_i := X(s^i)$ and $d_i := X(d^i)$, which must satisfy the simplicial identities:

For $i < j$ we have

$$\begin{tikzcd} X_n \arrow[r, "s_j"] \arrow[d, "d_i"] & X_{n+1} \arrow[r, "d_j"] \arrow[d, "d_i"] & X_n \arrow[d, "d_i"] \\ X_{n-1} \arrow[r, "s_{j-1}"] & X_n \arrow[r, "d_{i-1}"] & X_{n-1} \end{tikzcd}$$

For $i > j + 1$ (equivalently, $j < i - 1$) we have

$$\begin{tikzcd} X_n \arrow[r, "d_{i-1}"] \arrow[d, "s_j"] & X_{n-1} \arrow[r, "s_{i-1}"] \arrow[d, "s_j"] & X_n \arrow[d, "s_j"] \\ X_{n+1} \arrow[r, "d_i"] & X_n \arrow[r, "s_i"] & X_{n+1} \end{tikzcd}$$

Finally, we have

$$\begin{tikzcd} X_{n-1} \arrow[r, "s_j"] \arrow[d, "s_{i-1}"] & X_n \arrow[r, "s_{i+1}"] \arrow[d, "s_i"] \arrow[dr, "\mathrm{id}"] & X_{n+1} \arrow[d, "d_i"] \\ X_n \arrow[r, "s_j"] & X_{n+1} \arrow[r, "d_i"] & X_n \end{tikzcd}$$

and, for

Notation 2.1.2. When there is a name for the objects of C, we refer to objects of $\Delta\mathsf{C}$ by prepending the adjective 'simplicial'. Here are a few examples:

$$\text{An object of } \begin{matrix} \Delta\mathsf{Set} \\ \Delta\mathsf{Ab} \\ \Delta\mathsf{Ring} \\ \Delta\mathsf{Cat} \end{matrix} \text{ is called a } \begin{matrix} \text{simplicial set} \\ \text{simplicial abelian group} \\ \text{simplicial ring} \\ \text{simplicial category} \end{matrix}$$

Simplicial sets are immediately useful, and we will see cause for interest in the other three later. ⌟

Given a simplicial set C, the set $C_n := C([n])$ is naturally isomorphic to $\mathrm{Mor}_{\Delta\mathsf{Set}}(\Delta^n, C)$ by the Yoneda Lemma, where $\Delta^n := \mathrm{Mor}_\Delta(-, [n])$ is the simplicial set represented by $[n]$. The set C_0 are thus the <u>vertices</u> of C, the set C_1 are the <u>edges</u>, and in general the set C are the <u>n-simplices</u> of C.

Definition 2.1.3. Dually, a <u>cosimplicial object</u> of C is a functor $\Delta \to \mathsf{C}$. ⌟

Remark 2.1.4. The classifying space (simplicial complex) $\mathcal{N}(BM)$ of a monoid M is a Kan complex iff M is a group! (HA 4.1.2.4) ⌟

Lemma 2.1.5. *A simplicial group is a Kan complex.*

Proof. Let $\mathcal{G}\colon \Delta^{\mathrm{op}} \to \mathsf{Grp}$. A short, soulless proof is given on the nLab. actually a better one is emilio's: https://drive.google.com/file/d/1VG5DeCVk9kOQoleOC2xyrQ5NH5ZDSp6K/view. We describe the idea without writing down all the simplicial calculations. A map $\Lambda^n_k \to \mathcal{G}$ is determined by n different $(n-1)$-simplices $F_0, \dots, F_{k-1}, F_{k+1}, F_n$. We will explicitly construct an n-simplex $\Sigma \in \mathcal{G}_n$ such that $F_i = d_i\Sigma$ for all i, which will define an extension $\Delta^n \to \mathcal{G}$.

If $k = n$, then we can build Σ as follows. Start with the n-simplex $\Sigma := s_0F_0$. We obviously have $F_0 = d_0\Sigma$, but we do not have $F_i = d_i\Sigma$ for any other $i < n$. This is rectified by replacing Σ by $\Sigma \cdot (s_1d_1\Sigma)^{-1} \cdot s_1F_1$, where $\cdot$ refers to multiplication in the group $\mathcal{G}_n$). Now, as can be verified using the simplicial identities, Σ satisfies $F_i = d_i\Sigma$ for $i \in \{0, 1\}$, but not yet for any other $i < n$. We just incrementally repeat the idea: for each $i \geq 2$, carry out the replacement

$$\Sigma \mapsto \Sigma \cdot (s_id_i\Sigma)^{-1} \cdot s_iF_i. \tag{REPLACEMENT}$$

It can be verified by hand that, for each i, this transformation satisfies $d_i\Sigma = F_i$, and preserves d_ℓ for $\ell < i$ (so that $d_\ell\Sigma = F_\ell$). After repeating this until $i = n-1$, we have (as can be verified by hand) arrived at an n-simplex of the kind described in the first paragraph.

The case of $k = 0$ is almost exactly the same, except that one starts at $\Sigma = s_1F_1$ instead of s_0F_0, and iterates (replacement) until $i = n$ instead of $i = n-1$.

The same thing almost works for inner horns ($0 < k < n$), but one has to 'skip' an $(n-1)$-simplex in this iterative construction, that is, one has to go straight from F_{k-1} to F_{k+1}. More precisely, one starts at $\Sigma := s_0F_0$ as before, and iterates (replacement) until $i = k-1$. Once here, we do a modified transformation (compare with (replacement))

$$\Sigma \mapsto \Sigma \cdot (s_kd_{k+1}\Sigma)^{-1} \cdot s_kF_{k+1}.$$

Only here will we do a calculation:

$$\begin{aligned} d_{k+1}\left(\Sigma \cdot (s_kd_{k+1}\Sigma)^{-1} \cdot s_kF_{k+1}\right) &= d_{k+1}\Sigma \cdot (d_{k+1}s_kd_{k+1}\Sigma)^{-1} \cdot d_{k+1}s_kF_{k+1} \\ &= d_{k+1}\Sigma \cdot (d_{k+1}\Sigma)^{-1} \cdot F_{k+1} \\ &= F_{k+1} \end{aligned}$$

where in the second line we applied $d_{k+1}s_k = \mathrm{id}$ twice. Finally, carry on with (replacement) as before, starting at $i = k+2$ and ending at $i = n$.

This finishes the proof. A fully formal version, complete with all simplicial calculations, is (to the author's knowledge) due to Emilio Minichiello and can be found at
`https://drive.google.com/file/d/1VG5DeCVk9kOQoleOC2xyrQ5NH5ZDSp6K/view`. □

2.2. The Dold–Kan correspondence

see page 14 of https://arxiv.org/pdf/2303.12699

Let $\mathcal{A}$ be an abelian category and let $A \in \Delta\mathcal{A}$ be a simplicial object of $\mathcal{A}$. Define a non-negatively graded chain complex $C_\bullet$ where $C_n = \{\text{non-degenerate } n\text{-simplices of } A\}$:

$$A_0 \xleftarrow{d_0 - d_1} \frac{A_1}{s_0 A_0} \xleftarrow{d_0 - d_1 + d_2} \frac{A_2}{(s_0, s_1)A_1} \longleftarrow \ldots$$

That this is actually a chain complex follows from the simplicial identities:

$$\sum_{i=0}^{n} (-1)^i d_i \circ \sum_{j=0}^{n+1} (-1)^j d_j = \ldots = 0$$

actually do this! for example, to get the computations right before we delete and just go for general case above,

$$(d_0 - d_1) \circ (d_0 - d_1 + d_2) = d_0 d_0 - d_0 d_1 + d_0 d_2 - d_1 d_0 + d_1 d_1 - d_1 d_2 = \ldots = 0.$$

How does one recover A_n, the set of all n-simplices, from the non-degenerate ones? It's quite simple:

$$A_n \cong \bigoplus_{[n] \twoheadrightarrow [k]} C_k$$

which assembles into a simplicial object in the only possible way: the s_i are defined directly using the indexing maps $[n] \twoheadrightarrow [n-1]$, and the d_i are defined.... how?

Theorem 2.2.1 (Dold–Kan correspondence). *This defines an equivalence of categories* $\Delta\mathcal{A} \cong \mathsf{Ch}_{\geq 0}(\mathcal{A})$.

Sketch. The reverse functor is $C_\bullet \mapsto \sigma C$; the latter is a simplicial set given by

$$(\sigma C)_n = \bigoplus_{[n] \twoheadrightarrow [k]} C_k.$$

We show that there is a natural isom $C_\bullet \cong N(\sigma C)$.

If $i < n$, then the map ... (last page of `http://math.uchicago.edu/ amathew/doldkan.pdf`) □

If we look closer, we see that $\Delta\mathcal{A}$ and $\mathsf{Ch}_{\geq 0}(\mathcal{A})$ are symmetric monoidal categories. Indeed, $\mathsf{Ch}_{\geq 0}(\mathcal{A})$ has the tensor product of chain complexes, and $\Delta\mathcal{A}$ has

Remark 2.2.2. When $\mathcal{A} = \mathsf{Ab}$ is the category of abelian groups, this adjunction is the nerve-realisation adjunction of the functor $\Delta \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab}) : [n] \mapsto$ ⌟

Example 2.2.3. Under Dold–Kan type correspondences, constant simplicial objects correspond to discrete chain complexes (i.e. homologically concentrated in degree 0). In derived geometry (more to follow) these are smooth spaces. ⌟

2.3. What is an ∞-category?

Definition 2.3.1. Let C be a small category. The nerve of C is a simplicial set, denoted $\mathcal{N}\mathsf{C}$. Its n-simplices are the set of n composable morphisms in C.

$$\mathcal{N}\mathsf{C}_n = \{x_0 \to \cdots \to x_n\}.$$

The face and degenracy maps are very easy... ⌟

Given $0 \leqslant i \leqslant n$, write Λ^n_i for the simplicial set

Lemma 2.3.2. *Let C be a category. Then, for all $0 \leqslant i \leqslant n$, th*

Proof. Let $\mathcal{S}^n$ denote the simplicial set generated by adjacent edges of Δ^n, called the spine. Then $\mathcal{S}^n \subseteq \Lambda^n_i$ for all $0 < i < n$. ☐

Definition 2.3.3. An ∞-category is a simplicial set C for which every morphism of simplicial sets $\Lambda^n_i \to C$ factors through the inclusion $\Lambda^n_i \hookrightarrow \Delta^n$, for all n and $0 < i < n$. ⌟

Notation 2.3.4. The 0-simplices of an ∞-category are called its objects; its 1-simplices are called its morphisms. ⌟

There is no direct composition of morphisms in an ∞-category. Instead, we can view a pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ as a morphism $\Lambda^2_1 \to \mathsf{C}$, whence 2.3.3 guarantees the existence of a morphism $x \xrightarrow{h} z$...

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

Further, a sequence of morphisms $x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n$ can be

Definition 2.3.5. Let $f, g \colon x \to y$ be edges in a simplicial set C. We say f and g are left homotopic, and write $f \sim_L g$, if there exists a 2-simplex as on the left (below).

$$\begin{tikzcd} x \arrow[r, "g"] & y \\ x \arrow[u, "\mathrm{id}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "g"] \arrow[dr, "f"'] & y \arrow[d, "\mathrm{id}"] \\ & y \end{tikzcd}$$

Dually, f and g are right homotopic if there exists a 2-simplex as on the right (above). ⌟

Lemma 2.3.6. *Suppose C is an ∞-category. Then the following statements hold:*

(1) $f \sim_L f$;

(2) $f \sim_L g$ *if and only if* $f \sim_R g$;

(3) $f \sim g$ *if and only if* $g \sim f$;

(4) *if* $f \sim g$ *and* $g \sim h$ *then* $f \sim h$;

(5) *if* $h_0 \simeq g \circ f$ *and* $h_1 \simeq g \circ f$ *then* $h_0 \sim h_1$.

PROOF. For the first one consider $s_0 f$. For the second, fill in both missing arrows in the following diagram with an f, using the triangles from (1).

$$\begin{tikzcd}
x \arrow[r, "g"] & y \arrow[d, "\mathrm{id}"] \\
x \arrow[u, "\mathrm{id}"] \arrow[ur, "f"] & y
\end{tikzcd}$$

For the third one, fill them in with gs instead. For (2) and (3) you have to use horn filling. For the fourth, fill in the remaining arrow in:

$$\begin{tikzcd}
x \arrow[r, "h"] \arrow[dr, "g"] & y \arrow[d, "\mathrm{id}"] \\
x \arrow[u, "\mathrm{id}"] \arrow[r, "f"'] & y
\end{tikzcd}$$

with a suitable triangle from (1): either an f to conclude $f \sim_L h$ or an h to conclude $f \sim_R h$. For the fifth, we can fill in each 2-simplex in the diagram

$$\begin{tikzcd}
y \arrow[r, "g"] \arrow[dr, "h_2"] & z \arrow[d, "\mathrm{id}"] \\
x \arrow[u, "f"] \arrow[ur, "g"] \arrow[r, "h_1"'] & z
\end{tikzcd}$$

except the (outer??) last one: use horn filling. □

Conversely, given an ∞-category we can get an ordinary category via the following construction.

DEFINITION 2.3.7. Let C be an ∞-category. The homotopy category of C is a category, denoted $\mathsf{h}C$. Its objects are the objects of C, and... ⌟

PROPOSITION 2.3.8. *The functors* $\mathcal{N} : \mathsf{Cat} \rightleftarrows \infty\mathsf{Cat} : \mathsf{h}$ *form an adjunction* $\mathcal{N} \dashv \mathsf{h}$.

PROOF. It is easy to see that the homotopy category of the nerve of C is nothing but C again. Thus $\mathsf{h}\mathcal{N}$ is the identity. □

So h can be seen as a retract of $\infty\mathsf{Cat}$ onto Cat, whose inclusion is $\mathcal{N}$.

There are many similar 'nerve constructions' for various types of category: simplicial categories (categories enriched over $\Delta\mathsf{Set}$), dg categories (categories enriched over $\mathsf{Ch}(\mathsf{Ab})$), A_∞-categories, and so on. We will focus on simplicial categories for now, and see the others later in this chapter. To motivate the definition of such a nerve construction, notice that the ordinary nerve can be viewed as a representable functor:

$$\mathcal{N}\mathsf{C} = \operatorname*{Mor}_{\mathsf{Cat}}(-, \mathsf{C}).$$

To make sense of this, we are implicitly precomposing with the inclusion $\Delta \to \mathsf{Cat}$, viewing the posets $[n]$ as categories themselves. A nerve for simplicial categories should be representable in the same way, but we need to find an analogue of the inclusion $\Delta \to \mathsf{Cat}$; essentially, a good way to turn $[n]$ into a simplicial category. This is the task carried out by the following construction. The idea is that while the ordinary nerve views $0 \to 1 \to 2$ identically to $0 \to 2$, the 'thickened' n-simplex can see they are only homotopic via a 2-simplex.

CONSTRUCTION 2.3.9 (Simplicial thickening). Given $i, j \in [n]$ with $i \leqslant j$, write $[i \rightsquigarrow j]$ for the collection of subsets of $[n]$ with least element i and greatest element j. One can think of $[i \rightsquigarrow j]$ as the set of 'partial compositions' of the edges $i \to i+1 \to \cdots \to j-1 \to j$. Viewing $[i \rightsquigarrow j]$ as a poset with *reverse inclusion* yields a category, which we can turn into a simplicial set by taking the (ordinary) nerve. The thickened n-simplex $\mathfrak{C}[n]$ is a simplicial category with the same objects as $[n]$, and whose morphism simplicial sets are given by the construction just described:

$$\operatorname*{Mor}_{\mathfrak{C}[n]}(i, j) = \mathcal{N}[i \rightsquigarrow j].$$

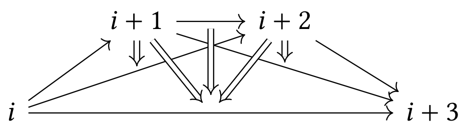
This defines a functor $\mathfrak{C} : \Delta \to \mathsf{Cat}_\Delta$, which left Kan extends to a functor $\mathfrak{C} : \Delta\mathsf{Set} \to \mathsf{Cat}_\Delta$. ⌟

EXAMPLE 2.3.10. For $|i - j| < 2$, there is no difference between $[n]$ and $\mathfrak{C}[n]$. However, for $|i - j| = 2$ we see that $\mathrm{Mor}_{\mathfrak{C}[n]}(i, j)$ has two vertices ($i \to i+1 \to i+2$ and $i \to i+2$) and one non-degenerate edge:

$$\begin{tikzcd} & i+1 & \\ i \arrow[rr] & & i+2 \end{tikzcd} \qquad \begin{tikzcd} & i+1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ i \arrow[ur] \arrow[rr] & {} & i+2 \end{tikzcd}$$

For $|i - j| = 3$, the simplicial set $\mathrm{Mor}_{\mathfrak{C}[n]}(i, j)$ has four vertices, five non-degenerate edges and two 2-simplices (not drawn here).

$$\begin{tikzcd} & i+1 & i+2 & \\ i \arrow[rrr] & & & i+3 \end{tikzcd}$$



For $|i - j| = 4$, the simplicial set has eight vertices. ⌟

DEFINITION 2.3.11. Let C be a simplicial category. The homotopy coherent nerve of C is the simplicial set $\mathcal{N}_{\mathsf{h}}\mathsf{C}$ given by

$$\mathcal{N}_{\mathsf{h}}\mathsf{C}_n = \operatorname*{Mor}_{\mathsf{Cat}_\Delta}(\mathrm{Path}[n], \mathsf{C}).$$

⌟

PROPOSITION 2.3.12. *There is an adjunction* $\mathfrak{C} \dashv \mathcal{N}_{\mathsf{h}}$.

In fact, this is part of a larger story about general nerve constructions. Given any (cocomplete?) category C, a cosimplicial object $F : \Delta \to \mathsf{C}$ determines a 'realisation-nerve adjunction'. More precisely, define the functor $\mathrm{Lan}_{y_\Delta} F : \Delta\mathsf{Set} \to \mathsf{C}$ to be the left Kan extension of F along the Yoneda embedding $y_\Delta : \Delta \to \Delta\mathsf{Set}$, and define the functor $\mathcal{N}_F : \mathsf{C} \to \Delta\mathsf{Set}$ by

$$\mathcal{N}_F : U \mapsto \operatorname*{Mor}_{\mathsf{C}}(F(-), U).$$

THEOREM 2.3.13. *Let* $F : \Delta \to \mathsf{C}$ *be a cosimplicial object of* C. *There is an adjunction* $\mathrm{Lan}_{y_\Delta} \dashv \mathcal{N}_F$.

The category of elements of a simplicial set K has objects $([n], s)$ where $[n] \in \Delta$ and $s \in K_n$, and a morphism $([n], s) \to ([n'], s')$ is a morphism $[n'] \to [n]$ such that $s \mapsto s'$. This comes with a forgetful functor $el(K) \to \Delta$; post-composing F gives a diagram $el(K) \to \mathsf{C}$.

Proof. The functor Lan_{y_Δ} is given explicitly by $\mathrm{Lan}_{y_\Delta} : K \mapsto \mathrm{colim}_{el(K)} F([n])$. We get

$$
\begin{tikzcd}[column sep=tiny]
& \underset{\mathsf{C}}{\mathrm{Mor}}\left(\mathrm{Lan}_{y_\Delta}(K), U\right) \arrow[r, equal, dashed] \arrow[dl, equal, "\text{def}"'] & \underset{\Delta\mathsf{Set}}{\mathrm{Mor}}\left(K, \mathcal{N}_F(U)\right) \arrow[dr, equal, "\text{def}"] & \\
\underset{\mathsf{C}}{\mathrm{Mor}}\left(\underset{el(K)}{\mathrm{colim}}\, F([n]), U\right) \arrow[d, equal, "\text{continuity}"] & & & \underset{\mathsf{C}}{\mathrm{Mor}}\left(\underset{el(K)}{\mathrm{colim}}\, \Delta^n, \mathcal{N}_F(U)\right) \arrow[d, equal, "\text{continuity}"'] \\
\underset{el(K)}{\lim}\, \underset{\mathsf{C}}{\mathrm{Mor}}\left(F([n]), U\right) \arrow[drr, equal, "\text{def of } \mathcal{N}_F"'] & & & \underset{el(K)}{\lim}\, \underset{\Delta\mathsf{Set}}{\mathrm{Mor}}\left(\Delta^n, \mathcal{N}_F(U)\right) \arrow[dl, equal, "\text{Yoneda}"] \\
& & \underset{el(K)}{\lim}\, \mathcal{N}_F(U)_n &
\end{tikzcd}
$$

□

Actually, this gives rise to an equivalence of categories:

Proposition 2.3.14. *The functor* $\mathrm{Lan}_{y_\Delta} : [\Delta, \mathsf{C}] \to [\Delta\mathsf{Set}, \mathsf{C}]$ *is fully faithful. Its essential image consists of cocontinuous functors, which have right adjoints*

Here are some more examples of this phenomenon:

Example 2.3.15. The $N \dashv \mathsf{h}$ and $|-| \dashv Sing$ (and other nerve-homotopy adjunctions) adjunctions have the same structure: they come from a left kan extension of a functor $\Delta \to \mathsf{C}$ for some choice of C: $\mathsf{Top}, \mathsf{Cat}, \mathsf{Cat}_\Delta$, whatever. see bottom of page 3 in `https://emilyriehl.github.io/files/topic.pdf`. ⌟

Later, we will see nerve constructions for dg categories and A_∞-categories.

Remark 2.3.16. One can view a category as a simplicial category by declaring all morphism spaces to be constant simplicial sets. (This should be the right adjoint to the functor which sends a morphism space X to X_0. Is this a functor?) The simplicial nerve of such a constant simplicial category coincides with the ordinary nerve of the ordinary category. ⌟

Example 2.3.17 (should this be a MO question?). The ordinary and simplicial nerve functors are defined, respectively, by

$$N(C)_n = Hom_{Cat}([n], C), \qquad N_\Delta(C)_n = Hom_{Cat_\Delta}(\mathfrak{C}[\Delta^n], C).$$

In fact, every cosimplicial object $F : \Delta \to X$ defines a "nerve construction" $N_F : X \to sSet$ by setting

$$N_F(C) = Hom_X(F(-), C).$$

There are two reasons to hope that a specific nerve construction N is of the form $(\star)$: * the data of F gives an efficient description of N, and * It gives a description of the left adjoint as the left Kan extension of F along the Yoneda embedding $\Delta \to sSet$.

We can

— So, the question: is every nerve construction of the form (⋆)? I am especially interested in ⌟

2.4. Functors, adjoints and limits

Just as a functor between ordinary categories sends morphisms to morphisms in a way that respects composition, a functor between ∞-categories $\mathcal{C} \to \mathcal{D}$ should be a 'map' which sends n-simplices in $\mathcal{C}$ to n-simplices Such an map is compactly defined as a functor of simplicial sets.

Definition 2.4.1. A functor between ∞-categories $\mathcal{C}, \mathcal{D} \in \Delta\mathsf{Set}$ is just a morphism in $\Delta\mathsf{Set}$. Write $\infty\mathsf{Cat}$ for the 1-category of ∞-categories. ⌟

Definition 2.4.2. Let K be a simplicial set. Is the product $- \times K : \mathsf{Cat}_\infty$ of simplicial sets a 1-functor? $\infty\mathcal{C}at$ ⌟

Lemma 2.4.3. *Let K be a simplicial set and $\mathcal{C}$ be an ∞-category. Then the simplicial set $\mathcal{C}^K$ is an ∞-category.* □

Remark 2.4.4. To some extent, the definition of functor between ∞-category is not surprising, because being an ∞-category is a *property* of simplicial sets, so there is no extra structure for functors to preserve. However, Definition 2.4.1 is slightly deeper than that because it reflects properties of the Joyal model structure on $\Delta\mathsf{Set}$—the we will discuss this sort of thing in the next chapter. ⌟

define adjoints. define limit as the adjoint functor to the diagonal, just as before. then say there is another way to think about limits: terminal cones. define slice categories (categories of (co)cones) as efficiently as possible – all the ugliness is there.

A limit of a diagram $F : \mathtt{I} \to \mathsf{C}$ is an initial cone, or more precisely an initial object in the category of cones over F.

Definition 2.4.5. A functor between ∞-categories is a morphism of simplicial sets. ⌟

This is because being an ∞-category is a *property* of simplicial sets, not extra structure.

Definition 2.4.6 (equivalence of ∞-categories). one place says it's if homotopy functor is an equivalence of 1-categories. another place (Groth, 1.35) says it's if the induced simplicial functor is a Dwyer–Kan equivalence. ⌟

the 'grothendieck construction' fails in higher setting? see Groth p.19

There is an internal hom for ∞-categories. The mapping space $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ is the ∞-category defined by

$$\mathrm{Map}(\mathcal{C}, \mathcal{D})_n = \operatorname*{Mor}_{\Delta\mathsf{Set}}(\Delta^n \times \mathcal{C}, \mathcal{D}).$$

Remember that $\mathrm{Map}(\mathcal{C}, \mathcal{D})_n$ is naturally the same thing as $\mathrm{Mor}_{\Delta\mathsf{Set}}(\Delta^n, \mathrm{Map}(\mathcal{C}, \mathcal{D}))$ by the Yoneda Lemma, so we do actually get the internal hom identity

$$\operatorname*{Mor}_{\Delta\mathsf{Set}}(\Delta^n \times \mathcal{C}, \mathcal{D}) \cong \operatorname*{Mor}_{\Delta\mathsf{Set}}(\Delta^n, \mathrm{Map}(\mathcal{C}, \mathcal{D}))$$

We see that $\mathrm{Map}(\mathcal{C},\mathcal{D})_0$ is the set of functors $\mathcal{C} \to \mathcal{D}$.

Recall the situation for ordinary categories. Let $F : \mathsf{I} \to \mathsf{C}$ be a functor between categories. A <u>cone</u> over F consists of an object $\ell \in \mathsf{C}$, together with morphisms $\eta_i : \ell \to F(i)$ commuting with the morphisms in C which come from I. In short, the $\{\eta_i\}$ assemble into a natural transformation $\eta : \underline{\ell} \Rightarrow F$, where $\underline{} : \mathsf{C} \to \mathsf{C}^{\mathsf{I}}$ is the 'constant functor' sending an object $x \in \mathsf{C}$ to the functor $\underline{x} : \mathsf{I} \to \mathsf{C} : (i \to j) \mapsto (x \xrightarrow{\mathrm{id}_x} x)$; the map on morphisms is obvious and fully faithful. Such a cone is a <u>limit cone</u> if, for any other cone (x, ζ) over F, there is a unique morphism $u : x \to \ell$ such that $\eta_i = u \circ \zeta$. The point is that a morphism $x \to \ell$ is the same thing as a natural tranformation $\underline{x} \Rightarrow F$, so the composition

$$\begin{tikzcd} \underset{\mathsf{C}}{\mathrm{Mor}}(x,\ell) \arrow[r, "\underline{}"] \arrow[rr, bend left, dashed] & \underset{\mathsf{C}^{\mathsf{I}}}{\mathrm{Mor}}(\underline{x},\underline{\ell}), \arrow[r, "\eta_*"] & \underset{\mathsf{C}^{\mathsf{I}}}{\mathrm{Mor}}(\underline{x},F), \end{tikzcd}$$

is a bijection. Since the constant functor is fully faithful, this just means that η_* is bijective.

This has a straightforward generalisation to ∞-categories. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor between ∞-categories. Just as above we have a constant functor $\underline{} : \mathsf{C} \to \mathsf{C}^{\mathcal{I}}$ sending $x \in \mathcal{C}$ to the functor

$$\underline{x} : (i \to j) \mapsto (x \xrightarrow{\mathrm{id}_x} x)$$

Definition 2.4.7. Let $F : \mathcal{I} \to \mathcal{C}$ be a map of simplicial sets, where $\mathcal{C}$ is an ∞-category. A <u>cone</u> over F is an object $\ell \in \mathcal{C}$ together with a natural transformation $\eta : \underline{\ell} \Rightarrow F$. It is a <u>limit cone</u> if

$$\begin{tikzcd} \underset{\mathcal{C}}{\mathrm{Mor}}(x,\ell) \arrow[r, "\underline{}"] \arrow[rr, bend left, dashed] & \underset{\mathcal{C}^{\mathcal{I}}}{\mathrm{Mor}}(\underline{x},\underline{\ell}), \arrow[r, "\eta_*"] & \underset{\mathsf{C}^{\mathsf{I}}}{\mathrm{Mor}}(\underline{x},F), \end{tikzcd}$$

is a homotopy equivalence for all $x \in \mathsf{C}$. ⌟

We see that...

Example 2.4.8. An object $* \in \mathcal{C}$ is <u>final</u> if it is final in $\mathrm{h}\mathcal{C}$. This means that, for all $U \in \mathcal{C}$, there is a unique homotopy class of maps from U to $*$. Since this includes the constant maps, we see that $\mathrm{Mor}_{\mathrm{h}\mathcal{C}}(U, *)$ is contractible. ⌟

Definition 2.4.9. Let K be a simplicial set. The <u>left cone</u> $K^{\triangleleft}$ is a simplicial set defined by creating a new vertex $*$, adding a single edge between $*$ and every vertex of K, then filling in all simplices ...

$$(K^{\triangleleft})_n := K_n \cup K_{n-1}$$

and

$$(K^{\triangleright})_n :=$$

with face maps ⌟

Example 2.4.10. pullback square ⌟

The ordinary situation for colimits is pretty much the same, but backwards: a colimit of a diagram $F : \mathsf{I} \to \mathsf{C}$ is an object $c \in \mathsf{C}$ together with a natural transformation $\eta : F \to \underline{c}$ such that $\mathrm{Mor}_{\mathsf{C}^{\mathsf{I}}}(F, ...)$

DEFINITION 2.4.11. colimit ⌟

EXAMPLE 2.4.12. Dually, we have the notion of an initial object. ⌟

EXAMPLE 2.4.13. pushout square ⌟

DEFINITION 2.4.14. A zero object is an object which is both initial and final. An ∞-category with a choice of zero object is called pointed. ⌟

LEMMA 2.4.15. *The subcategory of zero objects is contractible.*

PROOF. content... □

PROPOSITION 2.4.16. *Let C be an ∞-category. Then*

2.5. Enriched ∞-categories

we want to formally say that ∞-categories are enriched over spaces, that the infinity category of chain complexes is enriched over itself, that stable infinity categories are enriched over spectra, ... with as little technology as possible.

haugseng thesis 1.1

DEFINITION 2.5.1. multicategory ⌟

EXAMPLE 2.5.2. A monoidal category V can be viewed as a multicategory by defining

$$\operatorname*{Mor}_{V}((x_1, \dots, x_n), y) := \operatorname*{Mor}_{V}(x_1 \otimes \cdots \otimes x_n, y).$$

A functor from a multicategory to a monoidal category is understood to be a map of multicategories in this way. ⌟

Co classifying multicategory for enrichment: let X be a set of objects. define the multicategory O_X to be the composition multicategory: objects are $X \times X$ and morphisms are

$$\operatorname*{Mor}_{O_X}(((t_0, s_1), (t_1, s_2), \dots, (t_{n-1}, s_n)), (s_0, t_n)).$$

is this the right way round?

LEMMA 2.5.3. *A V-enriched category with objects X is the same thing as a functor $F : O_X \to V$.*

PROOF. By construction of O_X, the functor F assigns an object $C(x, y)$ to each pair $(x, y) \in X \times X$. The identity $* \to C(x, x)$ comes from the unique map $() \to ((x, x))$. The composition map $C(x, y) \otimes C(y, z) \to C(x, z)$ comes from the unique multimorphism $((x, y), (y, z)) \to (x, z)$. □

An A_∞-space is a monoidal $(\infty, 0)$-category.

2.6. Stable ∞-categories

DEFINITION 2.6.1. Let C be an ∞-category with a zero object. Define the loop functor $\Omega : C \to C$ by the ∞-pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

suspension and loop functors Ω, Σ ⌟

fibre and cofibre diagrams.

DEFINITION 2.6.2. A pointed ∞-category is <u>stable</u> if any of the following (equivalent) conditions hold:

- Ω is an equivalence
- Σ is an equivalence
- a square is a pullback if and only if it is a pushout

⌟

PROPOSITION 2.6.3. *Let C be a stable ∞-category. Then* hC *is triangulated.*

SKETCH. The axioms of a triangulated category are a little complicated, which is why we omit the proof. But the idea is as follows. A morphism $X \xrightarrow{f} Y$ in hC 'lifts' to a morphism $X \xrightarrow{F} Y$ in C, where there is a kernel and a cokernel, with $\operatorname{coker}(f) \cong \Omega \ker(f)$. We get a long exact sequence

which descends to an exact triangle

in hC. □

REMARK 2.6.4. Triangulated categories are additive. The previous result might therefore be surprising, given that a stable ∞-category has no additive structure. It's just one of those things. ⌟

in fact, they are spectra enriched. thus spectra is like Set in the stable ∞-world. very important. stable ∞-topos is trivial, so these are completely orthogonal generalisations of Set (if spectra even is one.)

2.7. Derived ∞-categories

DEFINITION 2.7.1. A category is <u>dg</u> (differential graded) if it is enriched over $\mathsf{Ch}(R\mathrm{Mod})$. ⌟

EXAMPLE 2.7.2. Let $\mathcal{A}$ be an abelian category. Then $\mathsf{Ch}(\mathcal{A})$ is a dg category; given $X_\bullet, Y_\bullet \in \mathsf{Ch}(\mathcal{A})$, we have a chain complex $\mathrm{Mor}_{\mathsf{Ch}(\mathcal{A})}(X_\bullet, Y_\bullet)$, with nth entry

$$\underset{\mathsf{Ch}(\mathcal{A})}{\mathrm{Mor}}(X_\bullet, Y_\bullet)_n = \prod_{i \in \mathbb{Z}} \underset{\mathcal{A}}{\mathrm{Mor}}(X_i, Y_{i+n})$$

and nth differential $\mathrm{d}_n : \operatorname{Mor}_{\mathsf{Ch}(\mathcal{A})}(X_\bullet, Y_\bullet)_n \to \operatorname{Mor}_{\mathsf{Ch}(\mathcal{A})}(X_\bullet, Y_\bullet)_{n-1}$ defined by

$$\mathrm{d}_n f : x \mapsto \mathrm{d}(f(x)) - (-1)^n f(\mathrm{d}x)$$

where we write d for the differentials of both $X_\bullet$ and $Y_\bullet$. $\lrcorner$

Definition 2.7.3. Let C be a dg category. The dg nerve of C is a simplicial set, denoted $\mathcal{N}_{\mathrm{dg}}\mathsf{C}$. Its vertices are the objects of C; the set of edges between any two vertices $X, Y \in \mathsf{C}$ is

$$\left\{ f \in \operatorname*{Mor}_{\mathsf{C}}(X, Y)_0 \ : \ \partial f = 0 \right\};$$

in general, the n-simplices of $\mathcal{N}_{\mathrm{dg}}\mathsf{C}$ are

$$\mathcal{N}_{\mathrm{dg}}\mathsf{C}_n = \{\}$$

$\lrcorner$

Note that $\partial\, \mathrm{id}_X = 0$ by the product rule.

Lemma 2.7.4. *The dg nerve of a dg category is an ∞-category.*

Proof. $\square$

Remark 2.7.5. This diagram commutes.

$$\begin{tikzcd}
& \Delta\mathsf{Ab}\text{-enriched category} \arrow[dl, mapsto, "\text{Forget}"] \arrow[dr, mapsto, "\text{Dold–Kan}"] \arrow[dd, dashed, mapsto] & \\
\text{simplicial category} \arrow[dr, mapsto, "\mathcal{N}_\Delta"'] & & \text{dg category} \arrow[dl, mapsto, "\mathcal{N}_{\mathrm{dg}}"] \\
& \text{stable } \infty\text{-category} &
\end{tikzcd}$$

An author I can't remember said that $\mathcal{N}_{\mathrm{dg}}$ is a 'smaller model' for the dg nerve. But since we have already defined everything else here, we may as well define $\mathcal{N}_{\mathrm{dg}}$ in this way (go backwards along the Dold–Kan arrow). In particular, it makes the proof of the above Lemma unnecessary. See to this later. $\lrcorner$

The ordinary derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is obtained by localising the quasi-isomorphisms in $\mathsf{Ch}(\mathcal{A})$. This yields a triangulated category which is a 'natural' setting for lots of classical homological algebra.

Definition 2.7.6. Let $\mathcal{A}$ be an abelian category. The derived ∞-category of $\mathcal{A}$ is denoted $\mathbb{D}(\mathcal{A})$. It is given by

$$\mathbb{D}(\mathcal{A}) := \mathcal{N}_{\mathrm{dg}}\mathsf{Ch}(\mathcal{A})$$

$\lrcorner$

Proposition 2.7.7. *Let $\mathcal{A}$ be an abelian category. Then $\mathcal{N}_{\mathrm{dg}}\mathsf{Ch}(\mathcal{A})$ is stable.*

Proof. $\square$

The following diagram commutes up to equivalence. (and makes sense because DK is 'lax comonoidal'… define this. for reference: `arXiv:math/0209342`; something with Alexander–Whitney maps)

$$\begin{tikzcd}
& \Delta\mathsf{Ab}\text{-enriched category} \arrow[dl, mapsto, "\text{Forget}"] \arrow[dr, mapsto, "\text{Dold–Kan}"'] \arrow[dd, dashed, mapsto] & \\
\text{simplicial category} \arrow[dr, mapsto, "\mathcal{N}_\Delta"'] & & \text{dg category} \arrow[dl, mapsto, "\mathcal{N}_{\mathrm{dg}}"] \\
& \text{stable } \infty\text{-category} &
\end{tikzcd}$$

in fact, some authors define the ∞-category of chain complexes $\mathcal{CH}(\mathcal{A})$ as $\mathcal{N}_\Delta(DK(\mathsf{Ch}(\mathcal{A})))$ (compare with the diagram: go backwards along the Dold–Kan arrow.)

Example 2.7.8. Let X be a scheme and let $\mathcal{A} = \mathsf{Coh}(X)$ denote the abelian category of coherent sheaves on X. ⌟

(nGpd says we should have something like $\mathbb{D}(\mathcal{A}) \cong \mathcal{N}_{A_\infty} D(\mathcal{A})$)

Definition 2.7.9 (A_∞-nerve). Define a cosimplicial object $\Delta \to A_\infty\mathsf{Cat}_k$ as follows. As in (reference $\mathcal{N}_\Delta$ construction) we will 'freely thicken' Δ^n to an A_∞-category. Define $A_\infty[\Delta^n]$ to be the A_∞-category with objects $\{0, 1, \ldots, n\}$ and

$$\operatorname*{Mor}_{A_\infty[\Delta^n]}(i,j)^\bullet = \begin{cases} k \cdot (ij) & i \leqslant j \\ 0 \text{ (in the paper it says } \emptyset\text{, but surely this is nonsense?)} & i > j \end{cases}$$

where (ij) has degree 0. The multiplication is given by

$$\mu_2 : \operatorname*{Mor}_{A_\infty[\Delta^n]}(j,k)^\bullet \otimes \operatorname*{Mor}_{A_\infty[\Delta^n]}(i,j)^\bullet \to \operatorname*{Mor}_{A_\infty[\Delta^n]}(i,k)^\bullet \ : \ \big((jk),(ij)\big) \mapsto (ik).$$

When $n \geq 2$, the maps μ_n are zero for degree reasons. As usual in this business (see Remark ??) the <u>A_∞-nerve</u> is the functor $\mathcal{N}_{A_\infty} : A_\infty\mathsf{Cat}_k \to \Delta\mathsf{Set}$ given by

$$\mathcal{N}_{A_\infty}(C)_n = \operatorname*{Mor}_{A_\infty\mathsf{Cat}_k}(A_\infty[\Delta^n], C).$$

⌟

This may seem like a strange definition, because the A_∞-categories $A[\Delta^n]$ have no interesting A_∞-structure; they are actually just dg-categories.

Lemma 2.7.10. *content…*

As it stands, the ever-growing family of homological mirror symmetry (HMS) conjectures are phrased as mutual equivalences of triangulated categories, such as

$$D^b\mathsf{Coh}(X) \cong D^\pi\mathsf{Fuk}(\check{X})$$

for mirror pairs of Calabi–Yau's $X, \check{X}$.

However, it has long been suspected (at least since the late 80s) that triangulated categories are morally incorrect fundamental structure to work with. A promising upgrade is the notion of stable ∞-category. These are special types of ∞-category with desirable properties (being stable is a property, not extra structure!) whose 'homotopy category' is canonically triangulated. For example, given an abelian category $\mathcal{A}$, there exists a stable ∞-category $\mathcal{D}^b(\mathcal{A})$ called the derived ∞-category of $\mathcal{A}$, whose homotopy category is the ordinary derived category $D^b(\mathcal{A})$. If HMS describes something deep and fundmantal in nature (it was, after all, discovered by physicists) its statement should ideally involve more fundamental, "correct" concepts.

The reader may find the philosophical argument weak. Aside from wishful thinking, why should HMS admit a lift to the ∞-world? There actually exists a mathematical answer: both sides of mirror symmetry (the A-side and the B-side) come from/are topological quantum field theories, which are examples of ∞-categories by the work of Lurie. So in some sense (that I do not understand), the equivalence 'should' be ∞-categorical in nature anyway.

There are probably various reasons why this has not been seriously pursued. Ordinary HMS is far from well-understood, but it is feasible that ∞-HMS could shed some light on it. The more serious obstacle is the lack of definition of the upgraded A-side. In particular, the ordinary A-side is not naturally the derived category of any abelian category, so there is no obvious choice for the ∞-category that lies above it. Some experts are reportedly thinking about a definition of the upgraded A-side (private communication) but references remain scarce.

Prophecy 2.7.11. *One day, homological mirror symmetry will be understood as a mutual equivalence of stable ∞-categories.*

2.8. Category-theoretic issues

local presentability, accessible functors, whatever.

Remark 2.8.1. sketches and limit sketches, algebras over them ⌟

presenting ∞-categories by relative categories. how does this work, really? motivates model categories.

CHAPTER 3

Model categories

The following explanation essentially comes from `https://mathoverflow.net/a/2198/170682`. Let's briefly describe the idea of homological algebra. We want to work with R-modules, but unfortunately fundamental operations between R-modules (for example, tensor product $\otimes_R$) are not convenient to work with (they are not exact). The solution posed by homological algebra is to enlarge our working category to the category of complexes of R-modules; an R-module is viewed as a complex concentrated in a single degree. The defects of $\otimes_R$ disappear in this setting; we just have to replace a module (or rather, its concentrated complex) with a quasi-isomorphic complex with better properties (a projective resolution) and compute $\otimes_R$ there. This gives a 'better' tensor product, denoted Tor_R^*, which is the 'derived functor' of $\otimes_R$. But remember, all we have done is embed $R\mathrm{Mod}$ into a larger category containing good replacements ($R\mathrm{Mod}$ has enough projectives).

A model category is basically a category with notions of 'quasi-isomorphism', 'projective complex' and 'injective complex' which interact well, in that every object has a quasi-isomorphic 'projective resolution' and 'injective resolution' giving rise to 'better' operations.

What if we want to work with rings instead of R-modules? The category of rings isn't abelian, so we can't embed into a category of complexes. Instead, we use the category of simplicial rings, which comes kitted out with a semi-canonical model structure. The analogue of Tor is the so-called derived tensor product, which provides the language for homotopically correct intersection theory.

In fact, given any category C, the category $\Delta\mathsf{C}$ of simplicial objects comes with the same semi-canonical model structure, which lets us do homological algebra-esque computations. An example we will see later comes when C is something like a category of spaces: the 'injective objects' in the model structure of $\Delta\mathsf{C}$ are things called ∞-stacks.

So, why is the category of simplicial objects a suitable generalisation (to the non-abelian setting) of the category of complexes? The answer lies in the Dold–Kan correspondence: there is an equivalence of categories between non-negatively graded chain complexes of R-modules and simplicial R-modules.

Model categories also provide a way to 'work in coordinates' with ∞-categories. Or provide a framework for abstract homotopy theory. But isn't that the same thing?

John Baez says that model categories are like the non-abelian brethren of derived categories.

in discrete simplicial categories, homotopy pullbacks and pushouts are just normal pullbacks and pushouts

Localising on the nose is easily described – ore localisation, roofs,... too violent. But the problem with ∞-localisation is: difficult to use and describe. we use models.

define $\mathsf{C}[W^{-1}]$.

3.1. What is a model category?

Definition 3.1.1 (Weak factorisation system). Let C be a category and let $J \subseteq \mathrm{Mor}(\mathsf{C})$ be a class of morphisms. A morphism $p : X \to Y$ has the right lifting property (against J) if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\forall\downarrow\in J"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

Write $\mathrm{RL}(J)$ for the class of all morphisms with this property. Dually, a morphism $i : A \to B$ in C has the left lifting property (against J) if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "\forall\downarrow\in J"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

Write $\mathrm{LL}(J)$ for the class of all morphisms with this property. ⌟

Example 3.1.2. Give a simple example (topology, preferably) which exhibits a weak factorisation. ⌟

Definition 3.1.3. A weak factorisation system of C is a pair (L, R) of classes of morphisms such that

$$\mathsf{L} = \mathrm{LL}(\mathsf{R}), \qquad \mathsf{R} = \mathrm{RL}(\mathsf{L}),$$

and every morphism f can be factored functorially as $f = p \circ i$ where $i \in \mathsf{L}$ and $p \in \mathsf{R}$. ⌟

Functoriality here means that for all $X \xrightarrow{f} Y \xrightarrow{g} Z$, the L-component of $g \circ f$ factors through the L-component of f, and the R-component of $g \circ f$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"] \arrow[rrrr, bend left, "g\circ f"] \arrow[ddrr, bend right, "k"'] & & Y \arrow[rr, "g"] \arrow[dr, "j"] & & Z \\ & A \arrow[ur, "p"] \arrow[dr, dashed] & & B \arrow[ur, "q"] & \\ & & C \arrow[ur, dashed] \arrow[uurr, bend right, "r"'] & & \end{tikzcd}$$

where

Proposition 3.1.4. *Let (L, R) be a weak factorisation system of C. Then L and R contain all isomorphisms and are closed under composition.*

Proof. That isomorphisms have both-sided lifting properties is clear. yeah do this. should bd ez □

Definition 3.1.5 (Model category). A model category is a complete and cocomplete category C, together with a triple of morphism classes $W, \mathsf{Cof}, \mathsf{Fib} \subseteq \mathrm{Mor}(\mathsf{C})$ where W satisfies the two-out-of-three property (if two out of $f, g, f \circ g$ are in W, so is the third) and both

$$(\mathsf{Cof} \cap W, \mathsf{Fib}) \qquad \text{and} \qquad (\mathsf{Cof}, \mathsf{Fib} \cap W)$$

are weak factorisation systems.

$$\text{Morphisms of } \begin{array}{c} W \\ \mathsf{Fib} \\ \mathsf{Cof} \end{array} \text{ are called } \begin{array}{c} \underline{\text{weak equivalences}} \\ \underline{\text{fibrations}} \\ \underline{\text{cofibrations}}. \end{array}$$

A (co)fibration which is also a weak equivalence is called a trivial (co)fibration. $\lrcorner$

Remark 3.1.6. If we know W and Cof, then the definition forces $\mathsf{Fib} = \mathrm{RL}(\mathsf{Cof} \cap W)$. If we know W and Fib, the situation is completely similar.

Therefore, any two of $\{W, \mathsf{Cof}, \mathsf{Fib}\}$ determine the third. $\lrcorner$

Definition 3.1.7. Let C be a model category. An object $X \in \mathsf{C}$ is fibrant if the unique map $X \to *$ is a fibration. Dually, X is cofibrant if the unique map $\varnothing \to X$ is a cofibration When X is both fibrant and cofibrant, we say it is fibrant-cofibrant. if it is, you guessed it, both fibrant and cofibrant.

$$\text{Write } \begin{array}{c} \mathsf{C}_{\mathrm{f}} \\ \mathsf{C}_{\mathrm{c}} \\ \mathsf{C}^{\circ} \end{array} \text{ for the full subcategory of } \begin{array}{c} \text{fibrant} \\ \text{cofibrant} \\ \text{bifibrant} \end{array} \text{ objects.}$$

$\lrcorner$

Lemma 3.1.8 (Replacement). *There exist (conditios...) functors*

$$R : \mathsf{C} \to \mathsf{C}_{\mathrm{f}}, \qquad Q : \mathsf{C} \to \mathsf{C}_{\mathrm{c}}$$

called the fibrant replacement functor and the cofibrant replacement functor, respectively.

Proof. Fix an object $X \in \mathsf{C}$. Because $(\mathsf{Cof}, \mathsf{Fib} \cap W)$ is a weak factorisation system, there is a factorisation of the unique map $\varnothing \to X$ as

$$\begin{tikzcd} \varnothing \arrow[rr] \arrow[dr] & & X \\ & RX \arrow[ur] & \end{tikzcd}$$

where RX is cofibrant and $RX \to X$ is a weak equivalence. By definition, this factorisation is functorial, which means that R defines a functor... Dually, since $(\mathsf{Cof} \cap W, \mathsf{Fib})$ is a weak factorisation system, there is a similar functorial factorisation of the unique map $X \to *$, which defines a functor $Q : \mathsf{C} \to \mathsf{C}_{\mathrm{c}}$ just as above. $\square$

3.2. Examples of model categories

Now that we have introduced the basic terminology, we can give some examples.

Example 3.2.1. model structure on Cat. W is equivalences of categories, cofibrations are injective on objects, fibrations are isofibrations. acyclic fibrations are literally surjective on objects. every object is bifibrant. $\lrcorner$

EXAMPLE 3.2.2. Any complete-cocomplete category admits a 'trivial' model structure, where W consists of all isomorphisms, and Fib and Cof consist of all morphisms. Consequently, all objects are fibrant-cofibrant. This is not very interesting. ⌟

Homological algebra can be viewed through the lens of model categories. Let R be a ring; the category of chain complexes of R-modules has the following model structure:

(WEQ) weak equivalence if it is a quasi-isomorphism;

(FIB)

(COF)

Fibrant objects are projective complexes... the fibrant replacement is 'projective resolution'! view an R-module as a chain complex concentrated in 0th degree.

EXAMPLE 3.2.3 (Classical model structure on $\Delta\mathsf{Set}$). A morphism $f : X \to Y$ in $\Delta\mathsf{Set}$ is a

(WEQ) weak equivalence if $|f| : |X| \to |Y|$ induces isomorphisms on all homotopy groups;

(FIB) fibration if one can fill all horns: for all $n \geq 1$ and $0 \leqslant k \leqslant n$, there is a lift:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

That is, $\mathsf{Fib} = \mathrm{RL}(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1\})$. In general, such a morphism is called a Kan fibration.

(COF) cofibration if it is a monomorphism (levelwise injection).

All objects are cofibrant; the fibrant objects are the Kan complexes. ⌟

EXAMPLE 3.2.4 (Joyal model structure on $\Delta\mathsf{Set}$). A morphism $f : X \to Y$ in $\Delta\mathsf{Set}$ is a

(WEQ) weak equivalence if $|f| : |X| \to |Y|$ induces isomorphisms on all homotopy groups;

(FIB) fibration if one can fill all horns: for all $n \geq 1$ and $0 \leqslant k \leqslant n$, there is a lift:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

That is, $\mathsf{Fib} = \mathrm{RL}(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1\})$. In general, such a morphism is called a Kan fibration.

(COF) cofibration if it is a monomorphism (levelwise injection).

All objects are cofibrant; the fibrant objects are the Kan complexes. ⌟

There are a few model structures on Top.

EXAMPLE 3.2.5 (Classical model structure on Top). ⌟

We will see later that

EXAMPLE 3.2.6 (Kan–Quillen model structure on $\Delta\mathsf{Vect}_{\mathbb{R}}$). Let $\mathsf{Vect}_{\mathbb{R}}$ be the category of real vector spaces. ⌟

EXAMPLE 3.2.7. projective model structure on simplicial presheaves ⌟

EXAMPLE 3.2.8. Injective model structure ⌟

EXAMPLE 3.2.9. Čech model structure ⌟

EXAMPLE 3.2.10. bergner on sCat: equivalences are Dwyer–Kan equivalences. fibrant objects are those with Kan morphism spaces ⌟

EXAMPLE 3.2.11 (Author-name model structure on Dif). ⌟

EXAMPLE 3.2.12. model structure on the category of cosimplicial objects of Cat_Δ, see TT.1.1.5.2. cofibrantly replaces $[n]$ with $\mathfrak{C}[\Delta^n]$. ⌟

EXAMPLE 3.2.13. model structures on functor categories: projective, injective and reedy. ⌟

EXAMPLE 3.2.14 (TT.5.5.9.1). sub model category of normal projective structure? state 5.5.9.2 and explain why 5.5.9.3 follows. ⌟

3.3. Abstract homotopy theory

Suppose we are given two topological spaces X and Y, and two continuous maps $f, g : X \to Y$. A homotopy $f \simeq g$ is a continuous map $H : X \times I \to Y$ which restricts to f at $X \times \{0\}$ and to g at $X \times \{1\}$. Such a homotopy can alternatively be described as a continuous map $X \to Y^I$, namely the one which sends $x \in X$ to $H(x, -) \in Y^I$. (Actually, there are some minor issues here because the category of topological spaces is not Cartesian closed.)

To do abstract homotopy theory, we first and foremost need to abstract the notion of a homotopy. Unfortunately, the natural abstractions of the two notions above ($X \times I \to Y$ and $X \to Y^I$) do not generally coincide, despite the fact that their equivalence was almost tautological in topology. However, we will see that they do coincide on bifibrant objects.

Throughout, fix a model category C.

DEFINITION 3.3.1. Let $X \in \mathsf{C}$. The <u>cylinder functor</u>...

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}_X \times \mathrm{id}_X"] \arrow[dr, "\mathsf{Cof} \cap W \ni"'] & & X \times X \\ & \mathrm{Path}(X) \arrow[ur, "\in \mathsf{Fib}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \amalg X \arrow[rr, "\mathrm{id}_X \amalg \mathrm{id}_X"] \arrow[dr, "\mathsf{Cof} \ni"'] & & X \\ & \mathrm{Cyl}(X) \arrow[ur, "\in \mathsf{Fib} \cap W"'] & \end{tikzcd}$$

The <u>path functor</u>... ⌟

REMARK 3.3.2. What we call cylinders and path objects are usually called very good cylinders and very good path objects. ⌟

DEFINITION 3.3.3. Let $f, g : X \to Y$ be two morphisms in C. A <u>left homotopy</u> between f and g is a factorisation of $f \amalg g : X \amalg X \to Y$ through $X \amalg X \to \mathrm{Cyl}(X)$. A <u>right homotopy</u> between f and g is a factorisation of $f \times g : X \to Y \times Y$ through $\mathrm{Path}(X) \to Y \times Y$. If there exists a left/right homotopy between f and g, we say they are left/right homotopic. ⌟

Lemma 3.3.4. *Let $X, Y \in \mathsf{C}$ and $f, g : X \to Y$. If X is cofibrant and Y is fibrant, then*

$$f \text{ and } g \text{ are left homotopic} \iff f \text{ and } g \text{ are right homotopic.}$$

In this case we say f and g are homotopic and write $f \simeq g$.

Proof. Omitted. □

This immediately implies the following.

Corollary 3.3.5. *When restricted to the full subcategory C°, there is an adjunction* $\mathrm{Cyl} \dashv \mathrm{Path}$. □

The fact that morphisms (or rather, spaces of morphisms) from cofibrant objects to fibrant objects behave so well is very important. There is a general construction called the derived hom where, instead of taking the ordinary morphism space $\mathrm{Mor}_{\mathsf{C}}(-,-)$, we take

$$RMap(-,-) := \mathrm{Mor}_{\mathsf{C}}(Q(-), R(-)).$$

This is the 'homotopically correct' morphism space.

Remark 3.3.6. The reason why it's OK to declare that the simplicial set of morphisms $K \to C$ (where K is a simplicial set and C is an ∞-category) is simply the internal hom C^K is because, in the Joyal model structure, all objects are cofibrant and ∞-categories are fibrant. The (co)fibrant replacement functors in the definition of $RMap$ are thus isomorphismss, so we can pretend they don't exist. ⌟

Definition 3.3.7. The ⌟

Theorem 3.3.8 (Whitehead). *A weak equivalence between bifibrant objects is a homotopy equivalence.*

Proof. Every weak equivalence $X \to Y$ factors (through some object, say Z) as the composition of a trivial fibration with a trivial cofibration. Moreover, if X and Y are bifibrant, then so is Z, so it suffices to prove the statement for trivial (co)fibrations. We choose the case of fibrations; the argument for cofibrations is dual. So, let $p : X \to Y$ be a trivial fibration. Because Y is fibrant, there exists a lift

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

This defines a right inverse of p; we want to show that, up to homotopy, it also defines a left inverse. To see this, consider the diagram

$$\begin{tikzcd} X \amalg X \arrow[rr, "(s \circ p, \mathrm{id})"] \arrow[d] \arrow[dr, "\nabla"] & & X \arrow[d, "p"] \\ \mathrm{Cyl}(X) \arrow[r] & X \arrow[ur, "\mathrm{id}"] \arrow[r, "p"'] & Y \end{tikzcd}$$

which commutes because $p \circ s = \mathrm{id}$. Moreover, the composition $\mathrm{Cyl}(X) \to X \xrightarrow{\mathrm{id}} X$ defines a lift:

$$\begin{tikzcd} X \amalg X \arrow[r, "(s\circ p, \mathrm{id})"] \arrow[d] & X \\ \mathrm{Cyl}(X) \arrow[ur, "h"'] & \end{tikzcd}$$

This constitutes a left homotopy $s \circ p \simeq \mathrm{id}$. It follows that p is indeed a homotopy equivalence. □

Theorem 3.3.9. *There is an equivalence of categories* $\mathrm{Ho}(\mathsf{C}) \cong \mathsf{C}[W^{-1}]$.

Proof. Write Q for the cofibrant replacement functor and R for the fibrant replacement functor. Consider the composite functor

$$\mathsf{C} \xrightarrow{RQ} \mathsf{C}_{\mathrm{cf}} \twoheadrightarrow \mathrm{Ho}(\mathsf{C})$$

We claim that this factors through $\mathsf{C}[W^{-1}]$, which means it sends W to isomorphisms in $\mathrm{Ho}(\mathsf{C})$. This is proved as follows. Let $f : X \to Y$ be a weak equivalence in C_{cf}. We may factorise it functorially as $f = p \circ i$ where $i \in \mathsf{Fib}$ and

By the universal property of $\mathsf{C}[W^{-1}]$, there exists a unique functor F in the following diagram.

$$\begin{tikzcd} \mathsf{C} \arrow[rr] \arrow[dr] & & \mathrm{Ho}(\mathsf{C}) \\ & \mathsf{C}[W^{-1}] \arrow[ur, "F"'] & \end{tikzcd}$$

Now, F is surjective by definition; it is full because the replacement functors are full. It □

Homotopy limits: let I be an index category. Then there is a (projective? what) model structure on C^I. The limit is a functor

$$\lim_I : \mathsf{C}^I \to \mathsf{C}$$

(this is the right adjoint to the constant functor; is this a Quillen pair?) anyway, we get a derived functor

$$\mathrm{holim} := \mathbb{R}\lim_I : \mathrm{Ho}(\mathsf{C}^I) \to \mathrm{Ho}(\mathsf{C}).$$

what this means is that...

3.4. Quillen equivalence

Fix two model categories C and D. What should a ‘functor of model categories’ be?

Lemma 3.4.1. *preserves some stuff...*

Definition 3.4.2. An adjunction $F \dashv U : \mathsf{C} \rightleftarrows \mathsf{D}$ is a Quillen adjunction if F preserves cofibrations and U preserves fibrations. ⌟

Lemma 3.4.3 (Ken Brown’s). *Let $F \dashv U$ be a Quillen adjunction.*

F preserves weak equivalences between cofibrant objects, and
U “ ” fibrant objects.

Proof. This is an application of the Factorisation Lemma, which is omitted. □

Example 3.4.4. simp top are quillen equivalent. ⌟

Example 3.4.5. sSet and sCat (Joyal, bergner) are quillen equiv ⌟

(this means that coherent nerve of locally fibrant is an ∞-category, since right quillen functors preserve fibrant objects)

Example 3.4.6. Dold–Kan is a quillen equivalence ⌟

3.5. Localising model categeories

Definition 3.5.1. A left localisation of C at S is an initial object $\mathsf{C} \to L_S\mathsf{C}$ in the collection

$$\{\text{left Quillen functors } F: \mathsf{C} \to \mathsf{D} \text{ such that } \mathrm{Ho}(F) \text{ sends } S \text{ to isomorphisms.}\}$$

A right localisation $\mathsf{C} \to R_S\mathsf{C}$ is defined analogously, using right Quillen functors. ⌟

Quillen model structure is a bousfield localisation of joyal model structure (`https://emilyriehl.github.io/files/` prop 5.9). also, projective vs cech model structure is the same.

riehl says a Bousfield localisation is another model structure on the same category with more weak equivalences. why is this ok, compared to the whole S-local business? answer: hirschorn.

Throughout, let C be a model category and $S \subseteq \mathrm{Mor}(\mathsf{C})$.

Definition 3.5.2. An object $X \in \mathsf{C}$ is S-local if

$$RMap(-, X): \mathsf{C}^{\circ} \to \Delta\mathsf{Set}$$

sends morphisms in S to weak equivalences. A morphism $f: A \to B$ is an S-local equivalence if

$$f^*: RMap(B, X) \to RMap(A, X)$$

is a weak equivalence for all S-local objects X. ⌟

Example 3.5.3. Čech stuff? T.6.2.2.6: a sheaf is an S-local object (put with bousefield localisation). So in particular, every morphism in S is S-local. Also, every weak equivalence is S-local by hirschorn 3.1.5. so $W \cup S \subseteq \{S\text{-local-equivalences}\}$, and the latter is kind of the best way to create this envelope. ⌟

Here is a

Definition 3.5.4. The left Bousfield localisation of C is a new model category structure on the underlying category of C with

(weq) weak equivalences are S-local equivalences.

(cof) cofibrations are the cofibrations of C. ⌟

Because weak equivalences are automatically S-local, we can think of this as 'adding weak equivalences' in a good way.

PROPOSITION 3.5.5. *The Bousfield localisation is actually a localisation in the sense of 3.5.1.*

PROOF. The identity functor $\mathrm{id}_{\mathsf{C}} : \mathsf{C} \to L_S\mathsf{C}$ send S to weak equivalences by definition, so $\mathrm{Ho}(\mathrm{id}_{\mathsf{C}})$ sends S to isomorphisms. Let $F \dashv U : \mathsf{C} \to \mathsf{D}$ be a Quillen pair such that $\mathrm{Ho}(F)$ sends S to isomorphisms. Then the diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] \arrow[d, "\mathrm{id}_{\mathsf{C}}"'] & \mathsf{D} \\ L_S\mathsf{C} \arrow[ur, dashed] & \end{tikzcd}$$

admits a unique dotted arrow, namely $F : L_S\mathsf{C} \to \mathsf{D}$. We just need to check that this is actually a Quillen functor. □

3.6. Presenting ∞-categories

simplicial model categories. coherent nerve of bifibrant objects. critetion for when such a presentation exists (eg bicomplete...)

EXAMPLE 3.6.1. The <u>∞-category of spaces</u> is given by $\mathcal{S} := \mathcal{N}_\Delta(\mathsf{Kan})$. ⌟

REMARK 3.6.2. This is a model for $\mathcal{S}$. all models are equally valid, and are characterised by 'free cocomplete ∞-category on a single generator. (like Set is free cocompletion of $*$?) (see groth) ⌟

Quillen adjunctions give rise to adjunctions of presented ∞-categories.

in general we have:

$$\{\mathcal{N}(\mathsf{C}) \to \mathcal{S}\} \simeq [\mathsf{C}, \Delta\mathsf{Set}]^{\circ}_{proj}.$$

so the projective model structure is

CHAPTER 4

Stacks

4.1. Moduli problems

see the first chapter of behrend.

the space of all labelled triangles is bigger: the moduli space of triangles is obtained from this by quotienting by S_3. but crucially, this should be the quotient stack.

define ordinary stacks tautologically. stacks are a sort of a pushout of yoneda and symmetries. 'X is the moduli space of points of X' is the Yoneda lemma.

now here's another geometric example: principal G-bundles on a topological space X. isomorphism classes of these are the set $H^1(X, G)$.

4.2. What is an ∞-stack?

Example 4.2.1. Cech groupoid. include triple intersections (2-morphisms): now a Cech 2-groupoid (this is the first example of a 2-groupoid, to inspire $\mathbf{B}^2 G$) ⌟

A principal G-bundle in terms of cocycle data. the cocycle condition looks like functoriality; not a coincidence.

moreover (maybe do this at once?) a map $\tau_{\leqslant n}\tilde{} \to \mathbf{B}^n G$ is a Cech n-cocycle!

Classifying spaces. BG as the classifying space of G; example.

composition in $\mathbf{B}G$ is multiplication.

this is all done more completely (and explained fully) in SCT. here is a summary of the relevant discussion there:

link to line bundles as in https://ncatlab.org/schreiber/files/dcct161227.pdf page 104/105

Remark 4.2.2. For a group G, BG always exists. Can we give BG a group structure? This would be horizontal composition rather than vertical. ⌟

Observe that the data of a principal G-bundle on a manifold X is *the same* as a functor $\check{C}(X) \to \mathbf{B}G$. this motivates left-localising at the class $\{\check{C}(X) \to X\}$.

Theorem 4.2.3. *The left Bousfield localisation of* $\Delta\mathsf{Psh}(\mathsf{Cart})$ *(proj model structure) at the Cech coverings exists; denote it by* $\check{C}^{-1}\Delta\mathsf{Sh}(\mathsf{Cart})$. *All objects are cofibrant; fibrant objects are called* <u>*∞-stacks*</u>. *Denote by*

$$\mathcal{S}\boldsymbol{t} = \mathcal{N}_\Delta\left(\check{C}^{-1}\Delta\mathsf{Sh}(\mathsf{Cart})^\circ\right)$$

the presented ∞-category, called the *<u>∞-category of ∞-stacks</u>.*

Proposition 4.2.4. *If G and H are homotopy equivalent Lie groups, then $\mathbf{B}G$ and $\mathbf{B}H$ are equivalent stacks.*

Proof. is this even true? □

Example 4.2.5. Isomorphism classes of vector bundles over X with fibre k^n are classified as

$$H^1(X, \mathrm{GL}_n(k)) := \pi_0 \mathbf{H}(X, \mathbf{B}\mathrm{GL}_n(k)).$$

⌟

Nonabelian cohomology is not very nice, so we would rather vector bundles were classified in an abelian way. This is what K-theory does.

idea of 2-stack. a sheaf is a 0-stack.

4.3. Higher topos theory

mirror topoi section of sheaves.

see https://mathoverflow.net/questions/345680/whats-an-example-of-an-infty-topos-not-equivalent-to-sheaves-on-a-grothendie

and, to a lesser extent, this https://mathoverflow.net/questions/273085/examples-of-infty-1-topoi-that-are-not-given-as-sheaves-on-a-grothendieck-t

Definition 4.3.1. An <u>∞-topos</u> is... *not* just a category of ∞-sheaves. this is due to accessibility/presentability issues which don't appear in the 1-setting. not sure how much to press this issue, but mention that nobody has proved/found an example of an ∞-topos which is not a category of sheaves. (I think this is in an MO question answered by Lurie?) ⌟

4.4. Higher parallel transport

What is a connection?

Recall the definition of the <u>fundamental groupoid</u> $\Pi_1(X)$, which is

$$\Pi_1(U) = \frac{\text{smooth maps } [0,1] \to U}{\text{relative homotopy}}.$$

Now, let $f : y(U) \to \mathbf{B}G$ be a principal G-bundle on $U \in \mathsf{Cart}$. parallel transport along a flat connection defines a 'holonomy' type functor

$$\Pi_1(U) \to \mathbf{B}G.$$

This is well-defined because, ∇ being flat, the effect of parallel transport depends only on the homotopy type of the path. As discussed above, a connection can be recovered from this functor, too.

Define the path functor $\mathbf{P}_1 : \mathsf{Cart} \to \mathcal{St}$ by

$$\mathbf{P}_1(U) = \frac{\text{standing maps}}{\text{thin homotopy}}$$

Define the stack $\mathbf{B}_\nabla G$ by

$$\mathbf{B}_\nabla G(U) = \operatorname*{Mor}_{\mathcal{St}}(\mathbf{P}_1(U), \mathbf{B}G)$$

4.5. What is a connection, really?

A morphism of C^∞_M-modules $TM \to \mathrm{End}(E)$. A flat connection is actually a morphism of sheaves of Lie algebras. We could consider Lie algebra valued sheaves on some site, or more generally L_∞-algebra valued sheaves on an ∞-site.

Proposition 4.5.1 (CONNECTIONS AND JET FIELDS, Theorem 1). *Let $\pi : E \to M$ be a fibre bundle, and $\pi^1 : J^1\pi \to E$ its (first order) jet bundle. The following structures are equivalent:*

- *a connection on E*
- *a section of π^1.*

Proof. Suppose we are given a section Ψ of π^1; we will construct a connection on E by declaring the horizontal sub-bundle $H \leqslant TE$ to be given at $v \in E$ by the image of $\mathrm{d}s_{\pi(v)} : T_{\pi(v)}E \to T_vM$:

$$H_v := \mathrm{im}(\mathrm{d}s_{\pi(v)})$$

where $s : M \to E$ is any section of π satisfying $\pi^1([s]) = v$. It follows from the definition of $J^1\pi$ that H_v does not depend on the actual representative $s \in [s]$ chose. Conversely, given a splitting $TE = H \oplus \ker(\mathrm{d}\pi)$, we can define a global section of π^1 as follows. For each $v \in E$, choose a local section $s_v : U \to E$ defined on a neighbourhood $U \ni \pi(v)$, such that $s_v : \pi(y) \mapsto y$, and

$$\mathrm{d}(s_v)_{\pi(v)} \text{ is inverse to } \mathrm{d}\pi_v\big|_{H_v}.$$

Now define a global section

$$\Psi : E \to J^1\pi : v \mapsto j^1(s_v)(\pi(v))\}$$

where the s on the right side is the local section chosen for v as above. □

the curvature of a connection is the obstruction to integrability of the horizontal bundle $H \leqslant TE$. Parallel transport exists because connections on bundles over 1-dimensional things are always flat.

Want, for example, parallel transport along surfaces (e.g. complex curves).

4.6. Higher groups and groupoids

We have $H^0(X, A) := \mathrm{Mor}(X, A)$ and $H^n(X, A) = \mathrm{Mor}(X, \mathbf{B}^n A)$.

quotient stack: π_0 recovers the orbit set X/G.

Geometric realisation of a constant simplicial set is that set. what is the geometric realisation of a quotient stack? (generalisation of BG). geometric realisation of cech cover of manifold (how light can this property get? CW complex?)

4.7. Classifying spaces and representability

This is actually a lot deeper than I initially thought.

Cohomology is representable. Given a pointed CW-complex M, we have

$$H^n(M;G) \cong [M, K(G,n)] := \frac{\text{based maps } M \to K(G,n)}{\text{based homotopy}}$$

where $K(G,n)$ is a connected topological space with the property that

$$\pi_i(K(G,n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}.$$

This (insert mild conditions) determines the space up to homotopy equivalence, so we may only speak of a space being "a $K(G,n)$" or "a model for $K(G,n)$", rather than "being $K(G,n)$" on the nose.

Example 4.7.1 ($G = \mathbb{Z}$). We will be primarily interested in cohomology with coefficients in $\mathbb{Z}$. Here are some examples:

- ($n = 1$) The circle S^1 is a $K(\mathbb{Z}, 1)$. The reason it has no nontrivial higher homotopy groups is because it admits a contractible cover; any covering map induces isomorphisms on π_n for all $n \geq 2$ (see Hatcher 4.11).
- ($n = 2$) The infinite complex projective space $\mathbb{P}^\infty$ is a $K(\mathbb{Z}, 2)$.
- A model for $K(\mathbb{Z}, n)$ is the infinite symmetric product of based n-spheres:

  $$K(\mathbb{Z}, n) \simeq colim_i \frac{(S^n)^i}{S_i}$$

  where $(S^n)^i$ embeds into $(S^n)^{i+1}$ by putting the basepoint in the final coordinate. This is a contractible bundle over S^1 in the case $n = 1$, and is straight-up homeomorphic to $\mathbb{P}^\infty$ in the case $n = 2$.

⌟

Should say briefly how this iso works: 'pull back'. put a nice proof here.

The set of isomorphism classes of principal G-bundles is representable; we have

$$\frac{\{\text{principal } G\text{-bundles on } M\}}{\text{isomorphism}} \cong [M, BG]$$

where BG is a topological space, called the <u>classifying space</u> of G. There are a few ways to thinking about it:

quot of contractible total bundle

realisation of the nerve

step functions on interval

Example 4.7.2. ... ⌟

put a nice proof

Associated bundles. Equivalence with line bundles

Observe that $B\mathbb{C}^*$ is a $K(\mathbb{Z},2)$. This observation actually implies that there is an isomorphism

$$H^2(M;\mathbb{Z}) \cong \text{isomorphism classes of line bundles over } M.$$

Also, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \underline{\mathbb{C}} \to \underline{\mathbb{C}}^* \to 0.$$

This gives rise to a LES of sheaf cohomology; because the sheaf $\underline{\mathbb{C}}$ is fine, this means we get a connecting isomorphism $H^1(M;\underline{\mathbb{C}}^*) \cong H^2(M;\mathbb{Z})$. Because $H^1(M;\underline{\mathbb{C}}^*)$ is, by definition, iso classes of $\mathbb{C}^*$-bundles, this gives the same result. what is the explicit connecting morphism?

What about $H^3(M;\mathbb{Z})$? By the same LES this is isomorphic to $H^2(M;\underline{\mathbb{C}}^*)$, but we would prefer an explicit description similar to H^2: isomorphism classes of some bundle-like thing over M. If we had a short exact sequence of sheaves

$$0 \to \underline{\mathbb{C}}^* \to \mathcal{F} \to \mathcal{E} \to 0$$

where $\mathcal{F}$ is fine, then by the same idea as above we would get an isomorphism $H^1(M;\mathcal{E}) \cong H^2(M;\underline{\mathbb{C}}^*)$.

When G is abelian, the groupoid $\mathbf{B}G$ itself has an abelian group structure (what is it?). We can therefore take the $\mathbf{B}$ of it, to get $\mathbf{B}^2G := \mathbf{B}\mathbf{B}G$. Iterating this point, we get $\mathbf{B}^nG$ for all $n \in \mathbb{N}$.

Let H be a seperable complex Hilbert space and let K be the (obviously non-unital) algebra of compact endomorphisms of H. Then H^* acts by automorphisms on K (by conjugation) and this action factors through $\mathbb{C}^* = Z(H^*)$. In fact (Brylinski) all automorphisms of K arise in this way: we have an SES

$$0 \to \mathbb{C}^* \to H^* \to \mathrm{Aut}(K) \to 0$$

Now, the sheaf $\underline{H}^*$ of H^*-valued functions on M is fine, so...

Definition 4.7.3. A stack is a Morita equivalence class of groupoids. ⌟

Example 4.7.4. The stack of F-fibre bundles on Man is presented by $\mathbf{B}\,\mathrm{Aut}(F)$. What is the stack of F-fibre 2-bundles presented by, if it's a 2-stack? ⌟

Example 4.7.5. The de Rham complex $\Omega^\bullet$ is an ∞-stack. ⌟

4.8. L_∞-algebras and dg-algebras

give definitions. big-ass jacobi identities. give example.

BRST. chavelley-einelberg corresondence (interesting).

4.9. Diffeological groupoids and fibre bundles

The problem with Lie groupoids (submersions) doesn't matter here.

fibre bundles in terms of fibrating groupoid. equivalently, locally trivial along plots. (van der, 3.44)

REMARK 4.9.1. https://www.esi.ac.at/uploads/477d50cb-e680-43e4-a3c8-72a50d3870fc.pdf | $LieAlg$ is a full co-reflective subcategory of $LieGrp$ whose objects are the 1-connected Lie groups. ⌟

4.10. Simplicial diffeological spaces

DEFINITION 4.10.1. A simplicial diffeological space is a functor

$$\Delta^{\mathrm{op}} \to \mathsf{Dif}.$$

This is functor, with certain properties, of type

$$\Delta^{\mathrm{op}} \times \mathsf{Cart}^{\mathrm{op}} \to \mathsf{Set}, \qquad \text{or} \qquad \mathsf{Cart}^{\mathrm{op}} \to \Delta\mathsf{Set}.$$

⌟

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \iddots \\ X^1_\bullet & X^1_0 & X^1_1 & \cdots \\ X^0_\bullet & X^0_0 & X^0_1 & \cdots \\ & & & \\ X^\circ_\bullet & X^\circ_0 & X^\circ_1 & \cdots \end{array}$$

There is a constant map— a 'diagonal embedding', if one can call it that—given by $\mathsf{Set} \to \Delta\mathsf{Set} : X \mapsto X_c$ where $X_n = X$ and all maps are the identity. for the functor induced by pushforward; this is the

EXAMPLE 4.10.2. We can view any diffeological space X° as a simplicial one, simply by sending it to the constant presheaf $X^\circ_c : \Delta^{\mathrm{op}} \to \mathsf{Dif} : [n] \mapsto X^\circ$ (where all arrows are the identity). Conversely, we can send a simplicial diffeological space $X^\circ_\bullet$ to its set of vertices X°_0. A morphism $X_\bullet \to {}^cY$... must have the same value on each restricts to a unique morphism $X_0 \to Y$; conversely, any We see that

$$\operatorname*{Mor}_{\Delta\mathsf{Dif}}(X, {}^cY) \cong \operatorname*{Mor}_{\mathsf{Dif}}(X_0, Y)$$

⌟

There is a canonical constant map $\mathsf{Set} \to \mathsf{Dif}$, which sends a set X to the presheaf $\mathsf{Cart}^{\mathrm{op}} \to \mathsf{Set}$ sending everything all objects to X and morphisms to id_X. Write

$$\mathrm{Disc} : \Delta\mathsf{Set} \to \Delta\mathsf{Dif}$$

for the functor induced by pushing forward the constant map. This assigns to each simplicial set $X_\bullet$ the 'constant' simplicial diffeological space.

We have an adjunction $\Pi_\infty \dashv \mathrm{Disc} \dashv \mathrm{ev}_*$

EXAMPLE 4.10.3. We can send a simplicial set $X_\bullet \in \Delta\mathsf{Set}$ to the constant functor $X^c_\bullet : \mathbb{R}^n \mapsto X_\bullet$. ⌟

$$\begin{tikzcd} \mathsf{Dif} \arrow[r] & \Delta\mathsf{Dif} \arrow[l, bend right, ""{name=A}] \arrow[l, bend left, ""{name=B}] \arrow[r, bend left] \arrow[r, bend right] & \Delta\mathsf{Set} \arrow[l] \end{tikzcd}$$

4.11. de Rham cohomology of groupoids

behrend stacks gerbes, page 3. de rham cohomology and equivariant cohomology both special cases of this. Let G be a groupoid...

We get a double complex $\Omega^\bullet(\mathcal{N}G)$, where the horizontal d morphisms are the ordinary exterior derivative, and

$$\partial : \Omega^k(\mathcal{N}G_n) \to \Omega^k(\mathcal{N}G_{n+1}) : \sum_{alt}(-1)^i$$

Write $H^*_{\mathrm{dR}}(\mathcal{N}G)$ for the cohomology of the total complex of $\Omega^\bullet(\mathcal{N}G))$. These are the <u>de Rham cohomology</u> groups of G. This captures

Example 4.11.1. Let M be a smooth manifold with an open cover $\{U_i \mid i \in I\}$. Define a groupoid G whose objects are $Y := \bigsqcup_{i\in I} U_i$, and whose morphisms are . For $n \geq 1$, write

$$Y^{[n]} = Y \times_M \cdots \times_M Y$$

for the fibre product of n copies of Y. Then $Y^{[n]}$ is a union of Let G be the Čech groupoid associated to an open covering of a smooth manifold M: write it out. Then $H^*_{\mathrm{dR}}(\mathcal{N}G)$ recovers the usual de Rham cohomology of M. ⌟

A silly special case of Example 4.11.1 is when the open cover consists of M itself.

Example 4.11.2. Suppose a Lie group A acts smoothly on a smooth manifold M. Let G be the action groupoid... i.e. the smooth quotient stack $[M/A]$.

Then $H^*_{\mathrm{dR}}(\mathcal{N}G)$ recovers the equivariant cohomology $H^*_A(M)$. (expand on this//derive) ⌟

CHAPTER 5

Derived stacks

5.1. Correct intersections

set up definitions and prove bezout holds even for same line.

towards derived morse theory? no need for non-degen critical points

5.2. Stuff

NOTES

first equivalence in C–S' proof of Prop 5.6 is a typo, should have fp.

We have $\mathcal{D}\mathsf{Man} \cong (Alg_{C^\infty}(\mathcal{S})^{fp})^{\mathrm{op}}$. formal smooth loci-type definition. we should apply Isbell duality to $\mathcal{A} \in Alg_{C^\infty}(\mathcal{S})$, i.e.

$$\mathcal{A}^\vee = \underset{[\mathsf{Cart},\mathcal{S}]}{\mathrm{Mor}}(-,\mathcal{A})$$

this defines a functor $\mathcal{A}^\vee : \mathsf{Cart}^{\mathrm{op}} \to \mathcal{S}$ which is... a sheaf? is it? some kind of higher diffeological space? the claim is that, if $\mathcal{A}$ is finitely presented, this should be something like a derived manifold.

dual question: given a presheaf $\mathsf{Cart}^{\mathrm{op}} \to \mathcal{S}$, when is its higher Isbell dual (we need to define higher Isbell duality properly, remember! with $\mathfrak{C} \dashv \mathcal{N}_\Delta$) left exact (I suppose always) and, importantly, finitely presented (compact)? I can actually do this in the Isbell duality section too; a diffeological space is "something" if its Isbell dual is finitely presented / compact?

——-

we want a good way to do intersections.

goals: read nuiten thesis, alfonsi BV paper, and steffens thesis + elliptic rep.

include BV-BRST complex. we already know L_∞-algebras from the previous section!

5.3. Derived manifolds

ON THE UNIVERSAL PROPERTY OF DERIVED MANIFOLDS `https://arxiv.org/pdf/1905.06195` carchiedi and steffens

proof that we need ∞-categories is Proposition 1.10 of Derived Smooth Manifolds https://arxiv.org/pdf/0810.5174 by Spivak.

interesting thing: finite limits $\implies$ idempotent complete in n-categories for all n, but not ∞; see the nLab page for idempotent complete (infinity,1)-category.

compare this universal prop with pregeometries of Lurie. maybe talk about them first? actually better later

prove that Mfd to $C^\infty\mathsf{Ring}$ is FF and preserves transverse pullbacks. bigger question: what limits does Isbell duality preserve?

Towards non-perturbative BV-theory via derived differential geometry: `https://arxiv.org/pdf/2307.15106` alfonsi and young

simplicial approach to derived differential manifolds `https://arxiv.org/pdf/1112.0033` borisov and noel

PROPOSITION 5.3.1. *Let $\mathcal{E}$ be an ∞-topos. There is an equivalence*

$$Alg_T(\mathcal{E}) \cong Geom(\mathcal{E}, [Alg_T^{\mathrm{fp}}(\mathcal{S}), \mathcal{S}])$$

PROOF. A geometric morphism $\mathcal{E} \to [Alg_T^{\mathrm{fp}}(\mathcal{S}), \mathcal{S}]$ is an adjunction... This is determined by the (left exact!) left adjoint, which means the right-side of the supposed equivalence is equivalent to... just see Prop 3.24 in C–S. □

Write $\mathbf{B}T := [Alg_T^{\mathrm{fp}}(\mathcal{S}), \mathcal{S}]$ for the <u>classifying ∞-topos</u> of T.

5.4. Derived C^∞-rings

use ΔSet or Kan or ∞Cat for the codomain...

for derived manifolds via simplicial C^∞-algebra, see https://arxiv.org/pdf/1112.0033

for derived Isbell duality, see page 23 of (commented)

5.5. L_∞-algebras

Let V be a graded vector space.

LEMMA 5.5.1. *An L_∞-algebra structure on V is equivalently a differential $D\colon \bigwedge^\bullet V \to \bigwedge^{\bullet-1} V$ with $D^2 = 0$.*

Probably $D\colon V_2 \to V_1$ should be the bracket.

EXAMPLE 5.5.2. The L_∞-algebra $\mathfrak{b}^n\mathfrak{u}(1)$ (which is really just the abelian Lie n-algebra) ⌟

DEFINITION 5.5.3. the CE algebra is $\bigwedge^\bullet V^*$ with cohomologically graded differential $\mathrm{d} = D^*$. ⌟

EXAMPLE 5.5.4. The *CE* algebra of $\mathfrak{b}^n\mathfrak{u}(1)$ is the cdga with a single generator in degree $n+1$, with $\mathrm{d} = 0$. ⌟

LEMMA 5.5.5. *fibres of $L \to M$ are curved $L_\infty[1]$-algebras $\iff$ $Sym_{\mathcal{O}_M} L^*$ is a sheaf of cdgas.*

PROOF. content... □

5.6. L_∞-algebroids

essentially vector bundle of L_∞-algebras.

EXAMPLE 5.6.1. Tangent Lie algebroid TM. ⌟

PROPOSITION 5.6.2. *The CE-algebra of TM is the de Rham complex $\Omega^\bullet(M)$.*

another model for derived manifolds: https://arxiv.org/pdf/2006.01376

see also `J. Nuiten. Lie algebroids in derived differential topology`

5.7. BV stuff

Let ω be a volume form on d-dimensional M. Then for all $0 \leqslant i \leqslant d$,

$$\iota_\bullet\omega : \Lambda^i TM \xrightarrow{\sim} \Omega^{d-i}T^*M.$$

pull back the cochain differential d_{dR} to get a chain differential on $\Lambda^\bullet TM$. This is the BV complex of M (associated to ω). In fact, this becomes a Poisson 0-algebra with the Schouten bracket.

We should say conceptually what the Schouten bracket is; this may require lots of buildup. For now, let's not. Anyway, the bracket is a degree +1 thing, I think.

EXAMPLE 5.7.1. https://arxiv.org/pdf/math/9911159. try to describe how this works, no proofs. ⌟

5.8. Representing elliptic moduli problems

We discuss https://arxiv.org/pdf/2404.07931

5.9. Derived Morse theory

Let M be a finite-dimensional manifold. A Morse function is a smooth map $S : M \to \mathbb{R}$ satisfying certain transversality conditions. Let's interpret this as an action functional, so that critical points of S are precisely solutions to some "field equation".

Our first task is to exhibit M as the space of smooth sections of some bundle. Since M is finite-dimensional, we could take $E = \{\text{Dirac } \delta\text{-functions on } M\}$, so that a section of E is the same thing as a point of M.

2-point functions (correlators) of coordinate functions in finite-dimensional QFT (bottom of page 4 in https://arxiv.org/pdf/math/0406251) are Hessian matrix elements, which must be invertible in that setting.

CHAPTER 6

Physics

read https://arxiv.org/pdf/1202.1554

connection with feynman integrals.