

Parcial 1

Derivación ejercicio 8

8 La aproximación de orden $O(h^2)$ para la derivada progresiva.

Para esto, se escribe el polinomio de interpolación de grado 2 para el conjunto soporte

$$\Omega = \{(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))\}$$

a) Calcular analíticamente el polinomio que interpola el conjunto soporte.

$$\Omega = \{(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))\}$$

$$P_n(x) = \sum_{i=0}^n f(x_i) L_i(x)$$

$$L_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j}$$

Entonces tenemos:

$$- L_0(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right)$$

$$- L_1(x) = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right)$$

$$- L_2(x) = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

Resumiendo tenemos:

$$p(x) = f(x_0) \left(\frac{x-x_1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) + f(x_1) \left(\frac{x-x_0}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) + f(x_2) \left(\frac{x-x_0}{x_2-x_0} \right) \left(\frac{x-x_1}{x_2-x_1} \right)$$

b) Derivar el polinomio interpolador para encontrar la derivada en el punto x_0 :

$$f'(x_0) \approx p'(x_0) = \frac{1}{2h} (-3f(x_0) + 4f(x_1) - f(x_2))$$

* Recordemos que la derivada progresiva de orden 2 $\rightarrow O(h^2)$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \underbrace{\frac{h}{2} f''(x)}_{O(h)} - \underbrace{\frac{h^2}{3!} f'''(x)}_{O(h^2)}$$

- Del enunciado $f'(x_0) \approx p'(x_0)$

$$p'(x) = f(x_0) \left[\left(\frac{1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) + \left(\frac{1}{x_0-x_2} \right) \left(\frac{x-x_1}{x_0-x_1} \right) \right] +$$

$$f(x_1) \left[\left(\frac{1}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) + \left(\frac{1}{x_1-x_2} \right) \left(\frac{x-x_0}{x_1-x_0} \right) \right] +$$

$$f(x_2) \left[\left(\frac{1}{x_2-x_0} \right) \left(\frac{x-x_1}{x_2-x_1} \right) + \left(\frac{1}{x_2-x_1} \right) \left(\frac{x-x_0}{x_2-x_0} \right) \right]$$

$$P''(x_0) = \frac{P(x_0+h) - 2P(x_0) + P(x_0-h)}{h^2}$$

$$P'(x_0) = \frac{P(x_0+h) - P(x_0)}{h} - \frac{h}{2} P''(x_0)$$

$$P'(x_0) = P(x_0+h) - P(x_0) - \frac{h}{2} \left(\frac{P(x_0+h) - 2P(x_0) + P(x_0-h)}{h^2} \right)$$

$$\begin{aligned} P(x_0+h) &= f(x_0) \left(\frac{x_0+h-x_1}{x_0-x_1} \right) \left(\frac{x_0+h-x_2}{x_0-x_2} \right) + f(x_1) \left(\frac{x_0+h-x_0}{x_1-x_0} \right) \left(\frac{x_1+h-x_2}{x_1-x_2} \right) \\ &\quad + f(x_2) \left(\frac{x_0+h-x_0}{x_2-x_0} \right) \left(\frac{x_0+h-x_1}{x_2-x_1} \right) \end{aligned}$$

$$\begin{aligned} &= f(x_0) \left(\frac{x_0+h-x_1}{x_0-x_1} \right) \left(\frac{x_0+h-x_2}{x_0-x_2} \right) + f(x_1) \left(\frac{h}{x_1-x_0} \right) \left(\frac{x_0+h-x_2}{x_1-x_2} \right) \\ &\quad + f(x_2) \left(\frac{h}{x_2-x_0} \right) \left(\frac{x_0+h-x_1}{x_2-x_1} \right) \end{aligned}$$

$$\begin{aligned} P(x_0-h) &= f(x_0) \left(\frac{x_0-h-x_1}{x_0-x_1} \right) \left(\frac{x_0-h-x_2}{x_0-x_2} \right) + f(x_1) \left(\frac{-h}{x_1-x_0} \right) \left(\frac{x_1-h-x_2}{x_1-x_2} \right) \\ &\quad + f(x_2) \left(\frac{-h}{x_2-x_0} \right) \left(\frac{x_0-h-x_1}{x_2-x_1} \right) \end{aligned}$$

Entonces :

$$P'(x_0) = \frac{P(x_0+h)}{h} - \frac{P(x_0)}{h} - \frac{1}{2} \left(\frac{P(x_0+h)}{h} - \frac{2P(x_0)}{h} + \frac{P(x_0-h)}{h} \right)$$

c) Calcule analíticamente la derivada de la función $f(x)$

$$f(x) = \sqrt{\tan(x)}$$

Recordemos que se puede escribir $\sqrt{\quad}$ como $\frac{1}{2}$

$(\tan x)^{1/2}$ con esto:

$$\begin{aligned} f'(x) &= \frac{d(\tan x)^{1/2}}{dx} = \frac{1}{2}(\tan x)^{-1/2} \cdot \frac{d \tan x}{dx} \\ &= \frac{1}{2}(\tan x)^{-1/2} \cdot \sec^2 x \end{aligned}$$