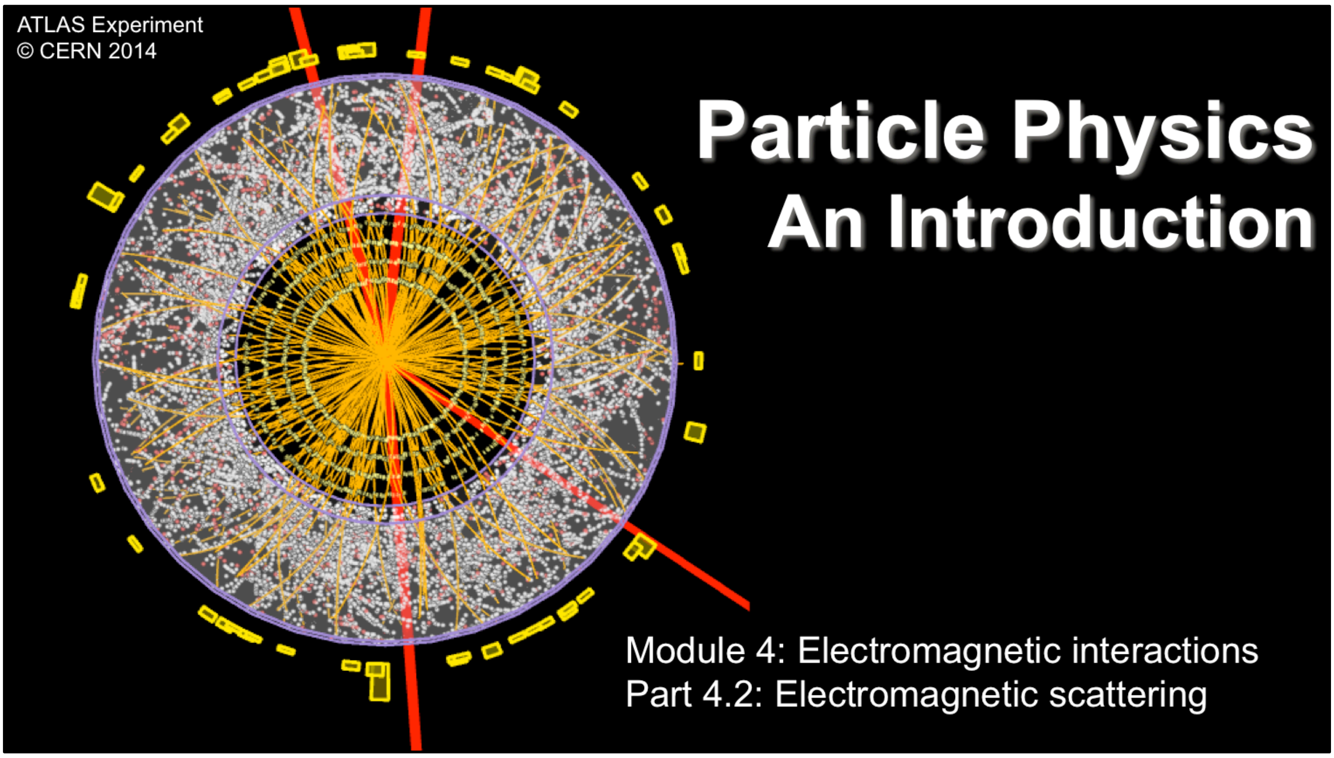
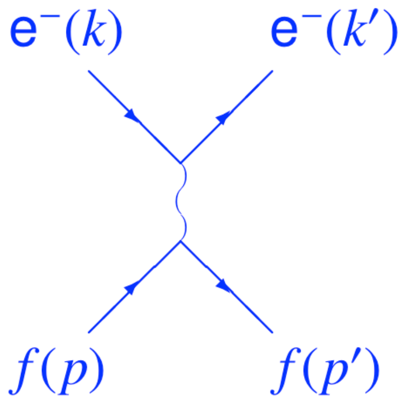


During this fourth module, we go into more details about the properties of electromagnetic interactions.

In this 2nd video, electromagnetic scattering is analyzed in terms of invariant amplitude and cross section.

After following this video you will know:
- Fermi's golden rule;
- The scattering amplitude of a particle in an electromagnetic potential;
- How the target particle generates this potential;
- And the difference between distinguishable and indistinguishable processes.

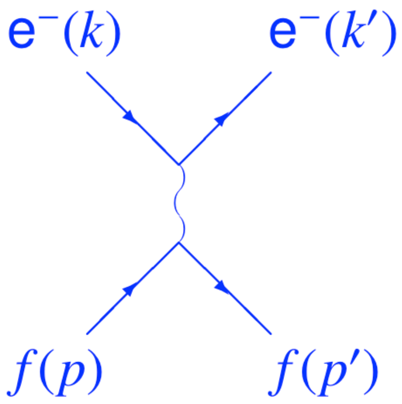


- In the previous video, we have seen how the **cross section** for a scattering process is measured by **normalizing the reaction rate by the luminosity**.
- For one incoming projectile on a single target particle, the cross section is also a **calculable** quantity.
- The main ingredient is again the probability that both interact by **exchanging an intermediary gauge boson**. This probability (or rather its amplitude) is calculable in a quantum field theory, based on a gauge principle.
- One can thus **compare experiment and theory**, with good precision for both. Such a comparison can test the theory, and determine its parameters like masses and coupling constants.

Fermi's Golden Rule:

$$W = |\mathcal{M}|^2 Q$$

- W: reaction rate
- $\mathcal{M}$: invariant amplitude of the process
- Q: density of final states



- The prescription for the calculation of a cross section is called the **Fermi's Golden Rule**.
- In the scattering theory, **the reaction rate *W*** is given as $W = |M|^2Q$ with the **invariant scattering amplitude, *M***, and the density of final states *Q*. The latter is also called the **phase space factor**. It takes into account the number of states - in the quantum sense - that the final state may contain. The more there are, the more probable the reaction becomes.
- For a two-body reaction a + b ➔ c + d, the amplitude *M* can be interpreted as the **probability amplitude** of the process that converts the initial state into the final state. It is therefore a probability amplitude in the same sense as the wave function and its square is a probability density.
- To obtain the **cross section**, we still have to normalize by the **flux of incident projectiles**. This may seem surprising for a single projectile, but the flux is the number of projectiles per second and per unit surface.

Fermi's Golden Rule:

$$d\sigma = \frac{|\mathcal{M}|^2}{F} dQ$$

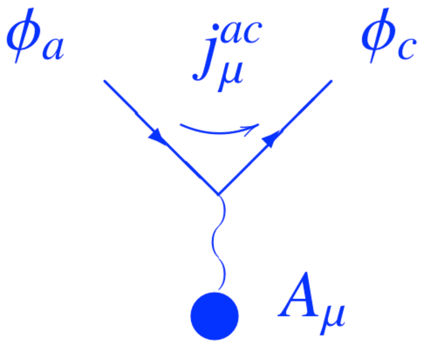
- F: incident flux of initial state particles

$$F = \left|\vec{v}_a - \vec{v}_b\right| 2E_a 2E_b = 4\sqrt{(p_a p_b)^2 + m_a^2 m_b^2}$$

- Q: phase space factor

$$dQ = (2\pi)^4 \delta^{(4)}(p_c + p_d - p_a - p_b) \frac{d^3 p_c}{(2\pi)^3 2E_c} \frac{d^3 p_d}{(2\pi)^3 2E_d}$$

- Here is then **Fermi's Golden Rule** for cross sections. It is obtained by properly normalizing the probability of the reaction and contains both this **squared amplitude $|M|^2$, the incoming flux F and the phase space factor Q**.
- The **incident flux** for the initial state depends on the relative velocity $|v_a - v_b|$, and the volume density of projectile and target. This volume density must be proportional to the energy of the two particles to be relativistically invariant. In our normalization it is simply equal to $2E$ for a single particle per unit volume. The second equation is valid for $v_a // v_b$ and clearly shows the invariance of the flux factor under Lorentz transformations.
- The **phase space factor**, dQ, takes into account the number of states which can be realized by the final state. It contains a δ function that ensures the conservation of the energy-momentum, as well as the number of quantum states for all the particles in the final state. The factors $2E_i$ again come from the way we normalize the wave functions.



Scattering off a fixed potential to first order:

$$\phi_c(x) = \mathbf{V}(x)\phi_a(x)$$

Probability amplitude:

$$\mathcal{M} = -i \int d^4x \; \phi_c^*(x)\mathbf{V}(x)\phi_a(x)$$

How does one proceed to calculate the invariant amplitude? We will outline the argument for the simple case of scalar fields.

- We imagine that to first order, one **local action of the potential** *V(x)* transforms the initial state Φ_a into the final state $\Phi_{c.}$
- The **probability amplitude** of this process – ignoring the normalization for the moment - is given by this integral.
- The argument of the integral is the **probability of finding the state $\boldsymbol{\Phi_c}$** at *x = (x,t)* among all states produced by *V* from the initial state $\Phi_{a.}$ The integration over space-time reflects the fact that this **local interaction can occur wherever** the potential is present.
- To help your intuition, Imagine that the two wave functions Φ_a and Φ_c correspond to free particles everywhere except at the place and time where the potential intervenes. The initial state Φ_a will be a superposition of eigenstates of the free Hamiltonian with a given set of coefficients. The potential will change these coefficients to produce another superposition of eigenstates, with a different set of coefficients. The amplitude *M* is then just the amplitude of the final state Φ_c, which the new mixture contains.

Invariant amplitude:

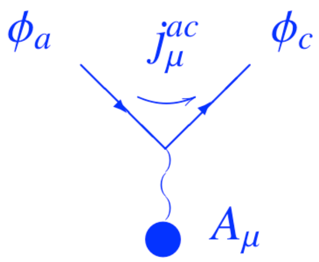
$$\mathcal{M} = -i \int d^4x\, \phi_c^*(x) \mathbf{V}(x) \phi_a(x)$$

Motion inside a potential:

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi = -\mathbf{V} \phi$$

'Minimal substitution" $i\partial_\mu \rightarrow i\partial_\mu + eA_\mu$:

$$\begin{aligned} \mathbf{V} &= -ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right) \\ \mathcal{M} &= -e \int d^4x\, \phi_c^*(x) \left(\partial_\mu A^\mu + A^\mu \partial_\mu\right) \phi_a(x) \\ &= -i \int d^4x\, j_\mu^{ac} A^\mu \end{aligned}$$



- The invariant amplitude M contains all the **dynamics** of a reaction, thus the whole physics of the type of interaction under study.
- It therefore follows from the **potential** operator, V. This is the same operator which changes the homogeneous equations of motion into inhomogeneous ones.
- For electromagnetic interactions, the inhomogeneous equation is derived from the Klein-Gordon equation by the so-called **minimal substitution**. To extract the potential we neglect terms proportional to $e^2 \ll 1$.
- The **invariant amplitude** is given by an integral over the product of the **four-potential A_μ of the target** and the **current density j_μ of the projectile**.

Scattering off a fixed electromagnetic potential:

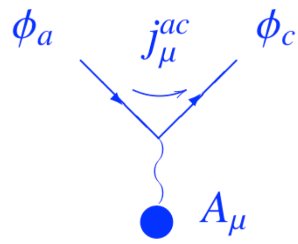
$$\mathcal{M} = -i \int d^4x \, j^{ac}_\mu(x) A^\mu(x)$$

Electromagnetic transition current density:

$$j^\mu_{ac} = -ie\,(\phi_c^* \partial^\mu \phi_a - \phi_a \partial^\mu \phi_c^*)$$

Electromagnetic current density for a free particle:

$$j^\mu = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$



- The **invariant amplitude** of electromagnetic scattering off a fixed target is thus the integral over the product between the **transition current** of the projectile and the **electromagnetic potential** of the target.
- Note that it is a **local interaction**: current and potential are defined at the same point in space-time, but this point may be anywhere. Thus an integral over space-time gives us the total amplitude.
- The **electromagnetic current density** at work here looks like that for a free particle, except that the fields in the initial and final state are not the same, since there was interaction.
- So the electromagnetic current density interacts locally with the potential A_μ, which we identify with the **photon field**. It remains to be understood how this potential is generated, i.e. how the photon is produced.

Homogeneous Maxwell equations:

$$\partial_\mu F^{\mu\nu} = 0 \quad ; \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\partial_\mu \partial^\mu A^\nu + \partial^\nu \partial_\mu A^\mu = 0$$

Invariance under gauge transformations, with arbitrary scalar field $\chi(x)$:

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi$$

Lorenz gauge:

$$\partial_\mu A^\mu = 0 \quad \Longrightarrow \quad \partial_\mu \partial^\mu A^\nu = 0$$

- For electromagnetic interactions, the **Maxwell equations** tell us how the electromagnetic fields are produced by charges and currents.
- Let us start by writing down the **homogeneous Maxwell equations** in four-vector form, using the **electromagnetic field tensor $F_{\mu\nu}$.**
- Translated into equations for the potential, this gives four differential equations for the **four-potential A_μ.**
- Using the invariance of the potential under **gauge transformation** with an arbitrary scalar field *$\chi(x)$*, we can always arrange things such that A_μ has no divergence. We call this arrangement the **Lorenz gauge**. In this gauge, the homogeneous Maxwell equations are nothing else than **four Klein-Gordon equations** for the components of the potential.
- But only three components are independent of each other because of the gauge condition. This justifies that we consider the homogeneous Maxwell equations as the **equations of motion for the free photon**, which is a vector boson and thus has three independent components in its wave function.

Inhomogeneous Maxwell equations:

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Transition current density of target field:

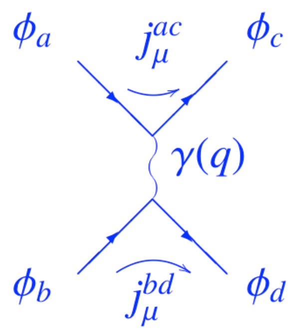
$$j^\nu_{bd} = +ei\sqrt{N_b N_d}\left(p^\nu_b + p^\nu_d\right)e^{-i(p_d - p_b)x}$$

Photon field:

$$A_\mu = -\frac{1}{q^2}j_\mu \quad ; \quad q = p_d - p_b = p_a - p_c$$

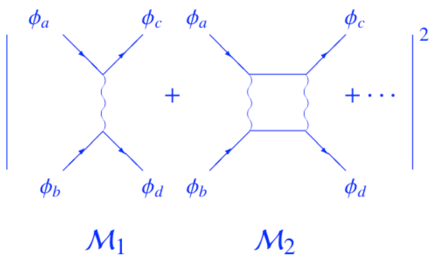
Interaction between current densities of projectile and target:

$$\mathcal{M} = i\int d^4x\, j^{ac}_\mu \frac{1}{q^2} j^\mu_{bd} \propto \frac{e^2}{q^2}$$

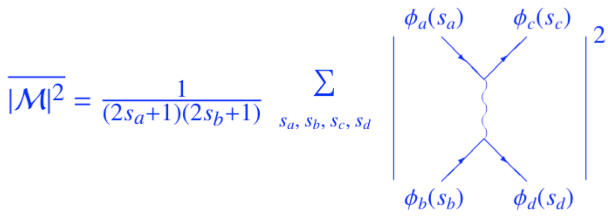


- In the presence of a **current**, the equations become inhomogeneous, there is a **source term** j^ν on the right.
- We construct j^υ from the wave functions of the two other partners in the reaction, the target particle b and its final state d. Their **transition current** is the source of the field.
- The **potential** A_μ that corresponds to this situation is proportional to this current density and inversely proportional to the square of the **momentum transfer** q.
- The invariant amplitude thus corresponds to the **interaction between two currents**.
- The intervening factor $1/q^2$ is called the **photon propagator**. It describes the probability amplitude that a photon with four-momentum q is exchanged between the two currents.
- Since each current density brings in a factor of e as coupling constant, the **amplitude is proportional to** e^2/q^2.
- This process can be represented by a **Feynman diagram**. These diagrams describe elementary processes in a manner that is both elegant and intuitive. If the construction of a Feynman diagram for a given process is still a problem for you, please go back to optional video 4.1a.
- At the same time Feynman diagrams are prescriptions for calculating the corresponding invariant amplitude. The **vertices** are characterized by a coupling constant, the electric charge in the case of electromagnetic interactions, and an operator which describes the properties of the interaction.
- The **lines** correspond to the propagators of the particles, real ones for external lines, virtual ones for internal lines connecting two vertices. The propagator $1/q^2$ we found for the virtual photon is a special case of the more general one, $1/(q^2 - m^2)$, for massive intermediate vector bosons.

Indistinguishable processes



Distinguishable processes



- If **more than one elementary process** can cause a reaction, we must add either the amplitudes or cross sections. Because of the quantum nature of the processes there exist two cases:
    - If initial and final states of the processes are **indistinguishable**, one must **add the amplitudes**.
    - If there is an observable which allows in principle to distinguish different contributions, no matter if actually measured or not, these sub-processes are **distinguishable** and one must **sum the cross sections**.
- The first case occurs for example if one considers **two contributions of different order** in the coupling constant: the exchange of one or two photons can intervene between the same initial and final states. If the two amplitudes M_1 and M_2 describe the two contributions, the squared total amplitude which leads to the cross section is $|M|^2 = |M_1^2 + M_2^2 + 2M_1M_2|$. An **interference term** appears in the probability.
- This is precisely not the case if the processes are **distinguishable in principle**. Take the example of different spin directions of the particles that interact. These are observable in principle.
- An **unpolarized initial state** corresponds to a mixture of all possible spin orientations in equal proportions. We must therefore average the cross section, or $|M|^2$, over the spin directions of the initial particles to calculate the cross section for an unpolarized initial state.
- If one does **not distinguish the polarization** of particles in the **final state** either, one must sum the cross sections corresponding to the spin directions of the final state particles. The square of the amplitude resulting from both operations, **averaging over initial spins and summing over final spins**, is denoted by $|M|^2$-bar.

- In the next video we discuss fermion spin and its consequences in more depth.