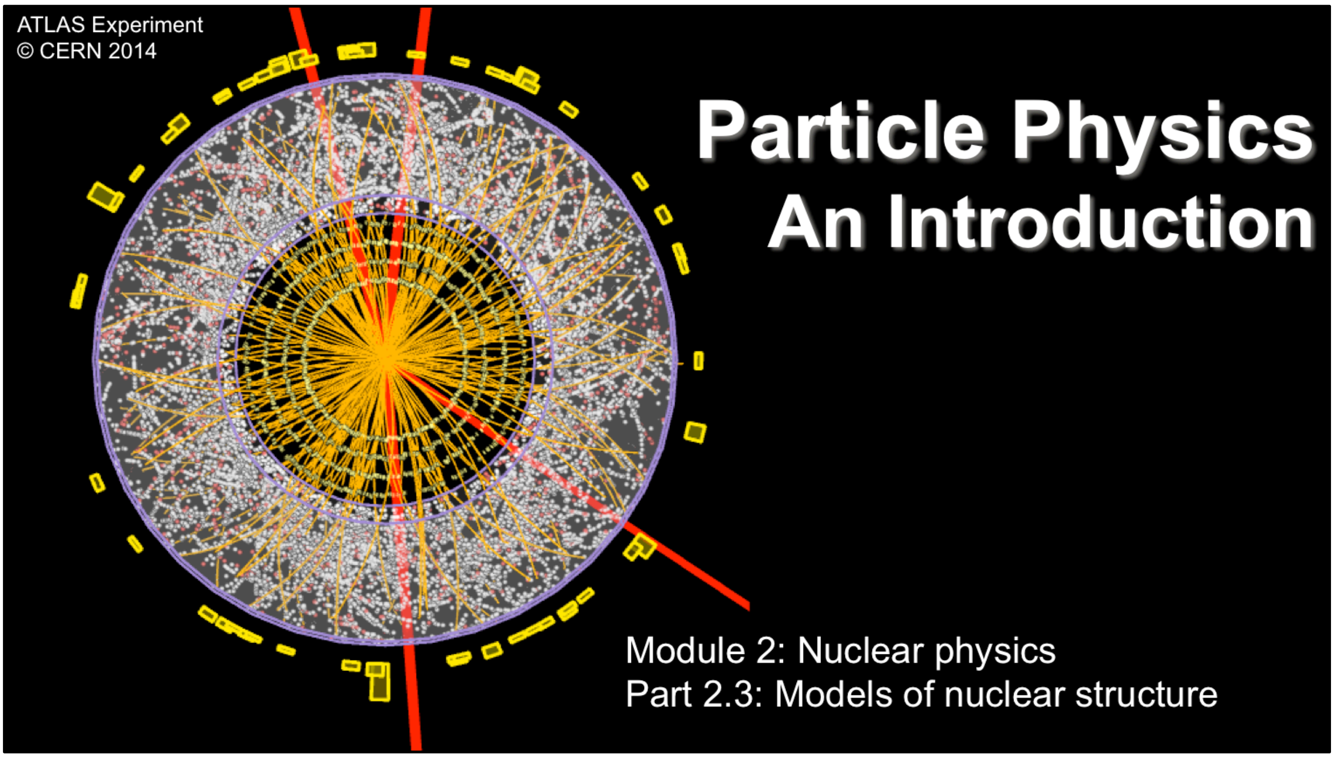
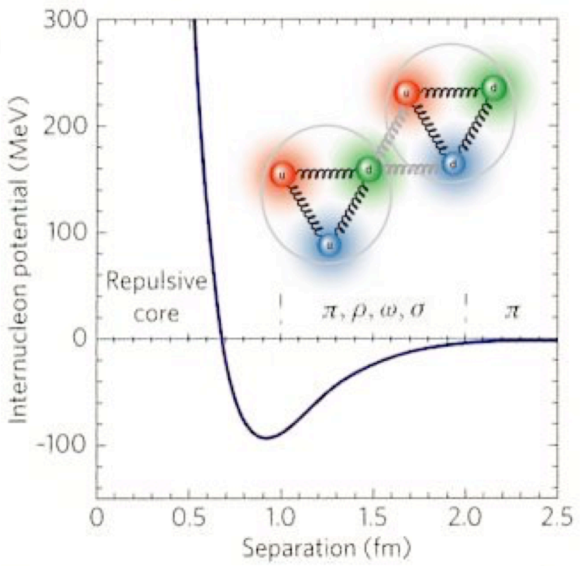


During this second module, we deal with nuclear physics and its applications.

In this 3rd video we will introduce simple models which allow to better understand the structure of nuclei and their properties.
After watching this video your should be able to:

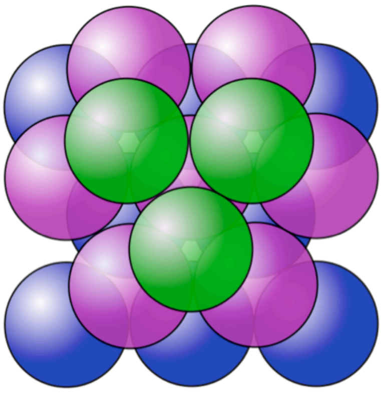
- Describe simple models of nuclear structure;
- Distinguish which features each model describes, and where its limits are;
- In particular know the so-called magic nuclei and their special properties.



Properties of the nuclear force:

- Very short range, limited to nuclear size
- Binding energy per nucleon roughly independent of nuclear size
- Attractive and much stronger than Coulomb repulsion between protons
- Repulsive component at distances comparable to the size of a nucleon ($\simeq 1 \times 10^{-15}$ m $= 1$ fm)

- For reasons discussed in the previous videos, we know quite well the **strong force acting between quarks** inside protons and neutrons, thanks to Quantum Chromodynamics. A complete theory of the **nuclear force** between these, based on first principles of QCD, however does not exist.
- Since we do not know how to calculate the properties of the nuclear force *ab initio*, it is difficult to apprehend **nuclear structure** theoretically.
- Consequently, the models of nuclear structure are rather **qualitative**. Most precede QCD itself, they are thus rarely inspired by quantum field theory.
- However, one can of course base model building on **experimental facts** and on physical intuition.
- Each of the models presented in the following will concentrate on a **particular aspect** of nuclear structure and nuclear properties.



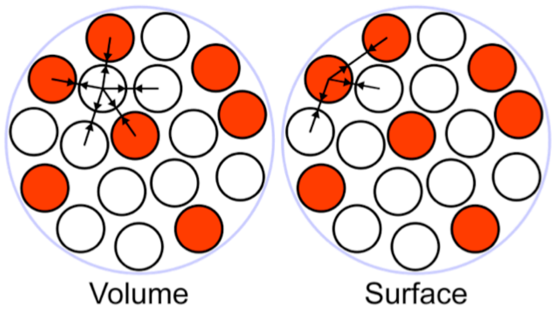
Cubic centered lattice ABC

$$\frac{AV_n}{V_{tot}} = \frac{\sqrt{2}}{6}\pi$$
$$R = r_0 A^{1/3}$$

- The so-called **liquid drop model** has been the first phenomenological model to describe the binding energy and size of nuclei.
- In this model, the **quantum properties** of nucleons are ignored or taken into account ad hoc.
- The model describes the nucleus as a **compact packing** of incompressible nucleons. This gives a total nuclear volume proportional to the number of nuclei, as observed. The mass density of nuclei is approximately independent of *A*.
- These considerations lead to imagine the nucleus like an **incompressible drop of liquid**. The "molecules" of the liquid are the nuclei, the Van der Waals force is replaced by the nuclear force.

Binding energy:

$$\Delta M(A,Z) \;=\; M(A,Z) - Zm_p - (A-Z)m_n < 0$$



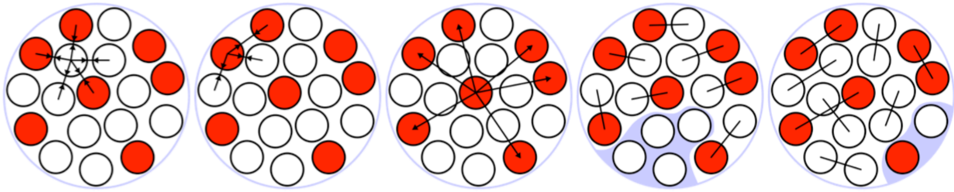
Binding energies BE associated to volume and surface:

$$\mathrm{BE} = \Delta M = -\underbrace{a_1 A}_{\text{Volume}} + \underbrace{a_2 A^{\frac{2}{3}}}_{\text{Surface}}$$

Coulomb repulsion energy:

$$\mathrm{BE} = -a_1 A + a_2 A^{\frac{2}{3}} + \underbrace{a_3 \frac{Z(Z-1)}{A^{\frac{1}{3}}}}_{\text{Coulomb}}$$

- The **binding energy BE** is equal to the mass difference *ΔM* between the nucleus and the sum of its nucleons. It is the energy won by the bound system and thus negative for stable nuclear states.
- It is principally **proportional to *A***, i.e. to the number of nuclei or the nuclear volume. This is due to the fact that each nucleon mostly interacts with its nearest neighbors.
- But the nuclei on the outer boundary have **less neighbors** and are thus less tightly bound. This is taken into account by a term proportional to the **nuclear surface** $A^{2/3}$. Its role is analogous to the surface tension of a liquid drop. It reduces the binding energy per nucleon, i.e. it makes the binding energy less negative. Hence its positive sign. This surface term is more important for heavy than for light nuclei, which explains their less tight binding.
- The third term takes into account the energy of **Coulomb repulsion** between protons.This term also decreases the absolute value of the binding energy, it thus has a **positive sign.**



Bethe-Weizsäcker formula:

$$\mathrm{BE} = -\underbrace{a_1 A}_{\text{Volume}} + \underbrace{a_2 A^{\frac{2}{3}}}_{\text{Surface}} + \underbrace{a_3 \frac{Z(Z-1)}{A^{\frac{1}{3}}}}_{\text{Coulomb}} + \underbrace{a_4 \frac{(N-Z)^2}{A}}_{\text{Asymmetry}} \pm \underbrace{a_5 A^{-\frac{3}{4}}}_{\text{Pairing}}$$

Experimentally determined parameters (in MeV):

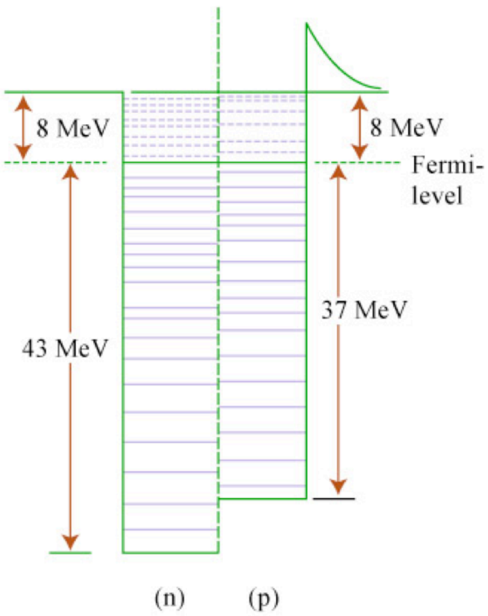
$$a_1 \simeq 15.56 \quad ; \quad a_2 \simeq 17.23 \quad ; \quad a_3 \simeq 0.7 \quad ; \quad a_4 \simeq 23.6 \quad ; \quad a_5 \simeq 34.0$$

https://en.wikipedia.org/wiki/Semi-empirical_mass_formula

Note the different definition of the pairing term!

Up to now, the considerations are purely classical. But **corrections inspired by quantum effects** are important:

- Light nuclei with an **equal number of protons and neutrons** (*Z*=*N*) are particularly stable, i.e. they have a more negative BE.
- Also, there are **more stable even-even nuclei** and **few stable odd-odd nuclei**. Odd and even refers to the number of protons and neutrons.
- One thus adds two empirical terms to obtain the Bethe-Weizsäcker formula:
    - The **asymmetry term** reduces the BE for $N \neq Z$.
    - The **pairing term** has a positive sign for odd-odd nuclei, which are less stable. For even-even nuclei, the sign is negative and the nuclei more stable. For all other combinations, i.e. for odd *A*, this last term is zero.
- Once the coefficients of the terms, a_1 **through** a_5, are adjusted to match measured nuclear masses, the formula describes rather accurately the binding of sufficiently heavy nuclei. It is less accurate for light ones. It fails especially to match the exceptionally large binding energy of the very stable ^{4}He nucleus.
- However, we will use it extensively to qualitatively understand the phenomena of **fusion and fission**.



Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

Boundary conditions:
$\psi(x, y, z) = 0$ for $x = a$, $y = a$ et $z = a$.

Solution:

$$\psi(x, y, z) = A\sin(k_x x)\sin(k_y y)\sin(k_z z)$$

with $k_i a = n_i \pi$.

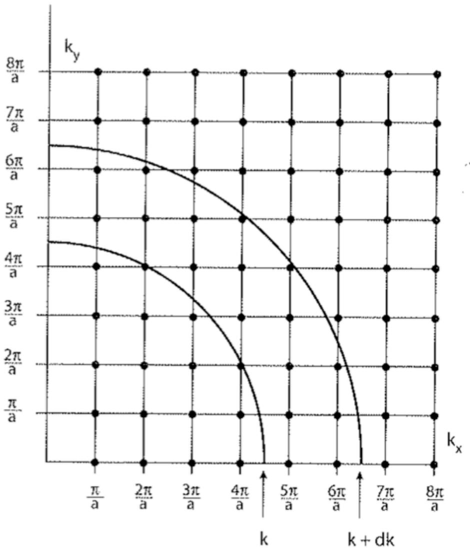
Energy levels:

$$E(n_x, n_y, n_z) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2\pi^2}{2ma^2}n^2$$

with $n^2 = n_x^2 + n_y^2 + n_z^2$.

An approach more compatible with the quantum nature of subatomic particles is the **Fermi gas model**:

- The nucleus is described as a **gas of free protons and neutrons** confined to the nuclear volume by a potential of unspecified origin.
- The nuclei then occupy **quantized energy levels**.
- The potential is modeled as a **spherical well**. Its size represents the size of the nucleus, its depth is adjusted to obtain the right binding energy. The difference between the **ground states of neutrons and protons** is due to the Coulomb potential which is only present for protons and decreases the depth of the potential well.
- Since n and p are fermions, there can be only **two per energy level, with opposite spin**.
- The last filed energy level defines the **Fermi energy,** which refers to the least tightly bound nucleons.
- **How many nucleons** can we store in a given volume? The Schrödinger equation gives the answer:
    - We consider the nucleons to be confined to a square box with side *a*.
    - The solution can be **factorized** with n_x, n_y, n_z positive integer numbers and a normalization factor *A*. The phase space is thus quantified.
    - Setting $E=p^2/2m$ one sees that k_i are the **wave numbers**, equal to the **momentum** along each axis in natural units.



Schrödinger: $k_i a = n_i \pi,\ i = x, y, z$

Number of states dn for nucleons in a volume V and a momentum interval between p and $p + dp$:

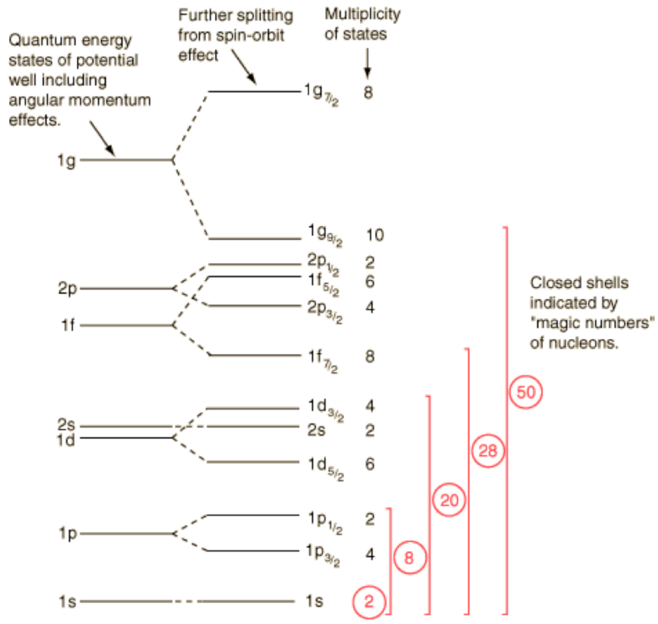
$$dn = \frac{1}{8} 4\pi k^2 dk \frac{1}{(\pi/a)^3} = \frac{4\pi p^2\, dp}{(2\pi\hbar)^3} V$$

Number of states up to the maximum Fermi momentum p_F:

$$n_F = \frac{V p_F^3}{6\pi^2\hbar^3} \quad ; \quad p_f = \frac{\hbar}{r_0}\left(\frac{9\pi}{8}\right)^{1/3} \simeq 250\text{MeV}$$

$$E_F = p_F^2/2m \simeq 33\text{MeV}$$

- Let us consider **momentum space**. Due to the boundary conditions, for each elementary cube of side *π/a*, there is a **single point** which is compatible with the Schrödinger equation. The **number of viable solutions** with a wave number between *k* and *k+dk* is thus the ratio of the corresponding spherical shell and volume element $(\pi/a)^3$.
- The volume of the box is a^3, but we only consider **one eighth**, since all k_i must be positive.
- In the ground state, all levels up to the maximum momentum are filled. Integrating up to the Fermi momentum, we thus obtain the number of states which can fill the volume. We must of course take into account **the factor of 2 for the two spin** orientations, as well as the **nuclear volume** $V = (4\pi/3)\, r_0^3\, A$.
- The energy corresponding to the maximum momentum is the **Fermi energy E_F,** approximately **33 MeV independent of *A***.
- The result is in fact **independent of the shape** of the confinement volume.
- The value of the Fermi energy means that the nucleons move inside the nucleus with a **considerable velocity**, $\beta \cong 0.4$. The nucleus is thus a very dynamic environment.
- The model also explains in a rather natural way the **asymmetry term** of the Bethe-Weizsäcker equation.
- Given E_F and the binding energy of the last nucleon of roughly -8 MeV, we obtain a **potential depth V_0 of about 40 MeV.**



"Magic" nuclei:

Z or $N = 2, 8, 20, 28, 50, 82, 126 \ldots$

"Doubly magic" nuclei:

$^4He,\ ^{16}O,\ ^{40}Ca,^{48}Ca,^{48}Ni,^{56}Ni,^{208}Pb,\ldots$

In analogy with the atomic shell model, a shell model for the nucleus can be constructed:

- The **orbital quantum number *n*** describes the energy level of the nucleon. For each *n*, there are *n* levels of **orbital angular momentum *l*** and $m_l = 2l+1$ sub-shells of the projection of *l* on an arbitrary axis. These sub-shells are **degenerate in energy** because of the rotational symmetry of the Coulomb potential. There are two spin states per sub-shell with $m_s = \pm 1/2$. A state can thus be described by **4 quantum numbers: *n*, *l*, m_l, and m_s**.
- The nuclei with completely filled shells are particularly stable, which explains the notion of **magic numbers**, ***Z* or *N*** = 2, 8, 20, 28, 50, 82,126…
- Isotopes with shells of both protons and neutrons completely filled, i.e. with Z and N equal to a magic number, are even more stable. Examples of these **doubly magic nuclei** are ^{4}He, ^{16}O, ^{40}Ca, ^{48}Ca, ^{48}Ni, ^{56}Ni, ^{208}Pb etc.
- One of the strong points of the shell model is its prediction of **nuclear spins**. According to the model, p and n fill up the levels independently, two in each sub-shell due to the Pauli principle. In this way, the **last or valence nucleon**, which does not have a pair, will determine the nuclear spin. The immediate consequence is that **even-even nucle**i should have zero spin, in agreement with observation.
- However, the shell model fails to predict the spin of **odd-odd nuclei**, since it does not include interaction between protons and neutrons.
- When the correct spin is predicted, the model also gets the nuclear magnetic moments right, which is important for Nuclear Magnetic Resonance (NMR) phenomena. We will talk about that when we treat applications of nuclear physics.
- It is obvious that there are **more sophisticated nuclear models** than those treated here, but that goes beyond the scope of this introductory course.

In the next video we will rather go over the properties of **unstable nuclei and their radioactive decays**.