Simple vehicle model 4dof

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Contents

1 Articulated Vehicle Model

The physical model of the set is illustrated in the figure **??**. In order to characterize the dynamics of this system we use the base $\Omega_{\mathrm{O}} = \{\mathrm{O}\mathbf{ijk}\}$ fixed in the inertial frame .

The base $\Omega_{\mathrm{T}} = \{\mathrm{T}\mathbf{t}_x\mathbf{t}_y\mathbf{t}_z\}$ is attached to the tractor truck and base $\Omega_{\mathrm{S}} = \{\mathrm{S}\mathbf{s}_x\mathbf{s}_y\mathbf{s}_z\}$ is attached to the semi-trailer. The unit vectors $\mathbf{t}_x$ and $\mathbf{s}_x$ point forward in the longitudinal direction of both modules and the verses $\mathbf{t}_y$ e $\mathbf{s}_y$ point to the left.

To assist in the description of the magnitudes on the front axle, the base $\Omega_{\mathrm{F}} = \{\mathrm{F}\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\}$ is defined with respect to the front axle with the vector $\mathbf{e}_x$ pointing forward in the longitudinal direction of the tire and $\mathbf{e}_y$ pointing to the left.

The points T and S define the center of mass of the tractor-truck and the semi-trailer, respectively. F and R locate the front and rear axles, respectively. A is the point of articulation and M is the axis of the semi-trailer.

The point O is the origin of the system and is fixed in the inertial frame. The distance a separates the points F and T and the distance b separates the points T and R . c separates points R and A, d separates points A e S and e separates points S e M.

The velocity vectors $\mathbf{v}$ and the drift angles α receive the subscripts referring to the points to which they are associated.

The modeling of the tractor-trailer and semi-trailer consists of the use of two rigid bodies that move on a horizontal plane and are joined by a point of articulation. In this way, the model has four degrees of freedom. Therefore, generalized coordinates can be given by

$$q_1 = x \tag{1a}$$

$$q_2 = y \tag{1b}$$

$$q_3 = \psi \tag{1c}$$

$$q_4 = \phi, \tag{1d}$$

where x and y are the longitudinal and transverse coordinates of the center of mass of the tractor-truck, respectively. ψ is the absolute orientation angle of the tractor-trailer and ϕ is the relative orientation angle of the semi-trailer.

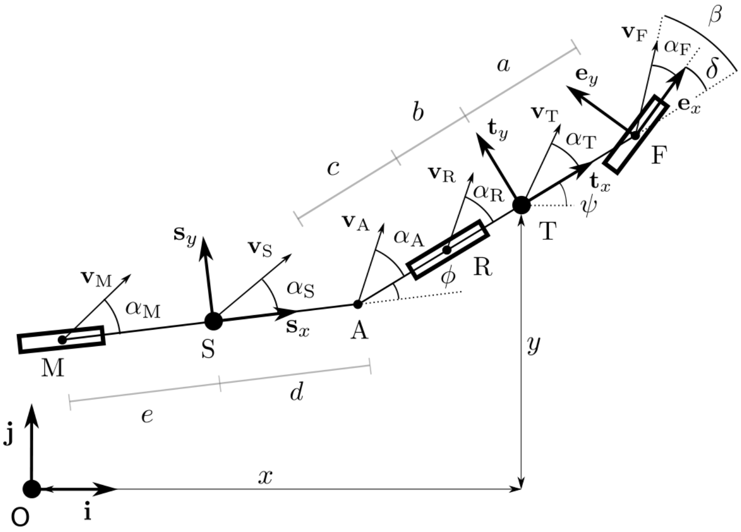

Figure 1: Single track bicycle model.

1.1 Nonlinear Model

The vector position of the tractor-truck center-of-mass in relation to the point O is

$$\mathbf{p}_{\mathrm{T/O}} = x\mathbf{i} + y\mathbf{j}. \tag{2}$$

The position vector of the center of mass of the semi-trailer is

$$\mathbf{p}_{\mathrm{S/O}} = [x - (b+c)\cos\psi - d\cos(\psi - \phi)]\,\mathbf{i} + [y - (b+c)\sin\psi - d\sin(\psi - \phi)]\,\mathbf{j}. \tag{3}$$

Differentiating the equation (2) with respect to time we have

$$\mathbf{v}_\mathrm{T} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}. \tag{4}$$

Deriving the equation (3) with respect to time we have

$$\begin{aligned}\mathbf{v}_{\mathrm{S}} &= \left[\dot{x} + (b+c)\,\dot{\psi}\sin\psi + d\left(\dot{\psi} - \dot{\phi}\right)\sin(\psi - \phi)\right]\mathbf{i} + ...\\ ... &+ \left[\dot{y} - (b+c)\,\dot{\psi}\cos\psi - d\left(\dot{\psi} - \dot{\phi}\right)\cos(\psi - \phi)\right]\mathbf{j}.\end{aligned} \tag{5}$$

The angular velocity vector of the tractor-truck is:

$$\mathbf{w}_T = \dot{\psi}\mathbf{k}. \tag{6}$$

The angular velocity vector of the semi-trailer is

$$\mathbf{w}_S = \left(\dot{\psi} - \dot{\phi}\right)\mathbf{k} \tag{7}$$

The kinetic energy of the system is:

$$T = \frac{1}{2}m_T\mathbf{v}_{\mathrm{T}}\cdot\mathbf{v}_{\mathrm{T}} + \frac{1}{2}m_S\mathbf{v}_{\mathrm{S}}\cdot\mathbf{v}_{\mathrm{S}} + \frac{1}{2}\left\{\mathbf{w}_T\right\}^T\left[\mathbf{J}_T\right]\left\{\mathbf{w}_T\right\} + \frac{1}{2}\left\{\mathbf{w}_S\right\}^T\left[\mathbf{J}_S\right]\left\{\mathbf{w}_S\right\}. \tag{8}$$

Substituting the equations (4), (5), (6), (7) in (8)

$$T = \frac{1}{2}m_T\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}m_S\left(C_1^2 + C_2^2\right) + \frac{1}{2}I_T\dot{\psi}^2 + \frac{1}{2}I_S\left(\dot{\psi} - \dot{\psi}\right)^2, \tag{9}$$

where

$$C_1 = \dot{x} + (b+c)\,\dot{\psi}\sin\psi + d\left(\dot{\psi} - \dot{\phi}\right)\sin(\psi - \phi) \tag{10a}$$

$$C_2 = \dot{y} - (b+c)\,\dot{\psi}\cos\psi - d\left(\dot{\psi} - \dot{\phi}\right)\cos(\psi - \phi). \tag{10b}$$

Deriving the equation (10) we have

$$\dot{C}_1 = \ddot{x} + (b+c)\,\ddot{\psi}\sin\psi + (b+c)\,\dot{\psi}^2\cos\psi + d\left(\ddot{\psi} - \ddot{\phi}\right)\sin(\psi - \phi) + d\left(\dot{\psi} - \dot{\phi}\right)^2\cos(\psi - \phi) \tag{11a}$$

$$\dot{C}_2 = \ddot{y} - (b+c)\,\ddot{\psi}\cos\psi + (b+c)\,\dot{\psi}^2\sin\psi - d\left(\ddot{\psi} - \ddot{\phi}\right)\cos(\psi - \phi) + d\left(\dot{\psi} - \dot{\phi}\right)^2\sin(\psi - \phi) \tag{11b}$$

The partial derivatives of the system's kinetic energy (equation (9)) with respect to the generalized coordinates are:

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial x} = 0 \tag{12a}$$

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial y} = 0 \tag{12b}$$

$$
\begin{aligned}
\frac{\partial T}{\partial q_3} = \frac{\partial T}{\partial \psi} &= m_S C_1 \left[(b+c)\,\dot{\psi} \cos\psi + d\left(\dot{\psi} - \dot{\phi}\right) \cos(\psi - \phi) \right] + ... \\
... &+ m_S C_2 \left[(b+c)\,\dot{\psi} \sin\psi + d\left(\dot{\psi} - \dot{\phi}\right) \sin(\psi - \phi) \right]
\end{aligned}
\tag{12c}
$$

$$
\frac{\partial T}{\partial q_4} = \frac{\partial T}{\partial \phi} = m_S C_1 \left[-d\left(\dot{\psi} - \dot{\phi}\right) \cos(\psi - \phi) \right] + m_S C_2 \left[-d\left(\dot{\psi} - \dot{\phi}\right) \sin(\psi - \phi) \right]. \tag{12d}
$$

The partial derivatives of the kinetic energy of the system in relation to the temporal derivatives of the generalized coordinates are:

$$
\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{x}} = m_T \dot{x} + m_S C_1 \tag{13a}
$$

$$
\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{y}} = m_T \dot{y} + m_S C_2 \tag{13b}
$$

$$
\begin{aligned}
\frac{\partial T}{\partial q_3} = \frac{\partial T}{\partial \dot{\psi}} &= m_S C_1 \left[(b+c) \sin\psi + d \sin(\psi - \phi) \right] + ... \\
... &+ m_S C_2 \left[-(b+c) \cos\psi - d \cos(\psi - \phi) \right] + I_T \dot{\psi} + I_S \left(\dot{\psi} - \dot{\phi}\right)
\end{aligned}
\tag{13c}
$$

$$
\frac{\partial T}{\partial q_4} = \frac{\partial T}{\partial \dot{\phi}} = m_S C_1 \left[-d \sin(\psi - \phi) \right] + m_S C_2 \left[d \cos(\psi - \phi) \right] - I_S \left(\dot{\psi} - \dot{\phi}\right). \tag{13d}
$$

Deriving the equations (13) with respect to time we have

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m_T \ddot{x} + m_S \dot{C}_1 \tag{14a}
$$

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m_T \ddot{y} + m_S \dot{C}_2 \tag{14b}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) &= m_S \dot{C}_1 \left[(b+c) \sin\psi + d \sin(\psi - \phi) \right] + ... \\
... &+ m_S C_1 \left[(b+c)\,\dot{\psi} \cos\psi + d\left(\dot{\psi} - \dot{\phi}\right) \cos(\psi - \phi) \right] + ... \\
... &+ m_S \dot{C}_2 \left[-(b+c) \cos\psi - d \cos(\psi - \phi) \right] + ... \\
... &+ m_S C_2 \left[(b+c)\,\dot{\psi} \sin\psi + d\left(\dot{\psi} - \dot{\phi}\right) \sin(\psi - \phi) \right] + ... \\
... &+ I_T \ddot{\psi} + I_S \left(\ddot{\psi} - \ddot{\phi}\right)
\end{aligned}
\tag{14c}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_4}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) &= m_S \dot{C}_1 \left[-d \sin\left(\psi - \phi\right)\right] + m_S C_1 \left[-d\left(\dot{\psi} - \dot{\phi}\right)\cos\left(\psi - \phi\right)\right] + ... \\
... &+ m_S \dot{C}_2 \left[d \cos\left(\psi - \phi\right)\right] + m_S C_2 \left[-d\left(\dot{\psi} - \dot{\phi}\right)\sin\left(\psi - \phi\right)\right] - ... \\
... &+ -I_S\left(\ddot{\psi} - \ddot{\phi}\right)
\end{aligned}
\tag{14d}
$$

The force on the front axle is given by:

$$
\mathbf{F}_{\mathrm{F}} = F_{x,\mathrm{F}}\mathbf{e}_x + F_{y,\mathrm{F}}\mathbf{e}_x, \tag{15}
$$

which can be written as

$$
\mathbf{F}_{\mathrm{F}} = \left[F_{x,\mathrm{F}}\cos\left(\psi + \delta\right) - F_{y,\mathrm{F}}\sin\left(\psi + \delta\right)\right]\mathbf{i} + \left[F_{x,\mathrm{F}}\sin\left(\psi + \delta\right) + F_{y,\mathrm{F}}\cos\left(\psi + \delta\right)\right]\mathbf{j}. \tag{16}
$$

The force on the rear axle is:

$$
\mathbf{F}_{\mathrm{R}} = F_{x,\mathrm{R}}\mathbf{t}_x + F_{y,\mathrm{R}}\mathbf{t}_y \tag{17}
$$

or

$$
\mathbf{F}_{\mathrm{R}} = \left[F_{x,\mathrm{R}}\cos\psi - F_{y,\mathrm{R}}\sin\psi\right]\mathbf{i} + \left[F_{x,\mathrm{R}}\sin\psi + F_{y,\mathrm{R}}\cos\psi\right]\mathbf{j}. \tag{18}
$$

The force on the semi-trailer shaft is

$$
\mathbf{F}_{\mathrm{M}} = F_{x,\mathrm{M}}\mathbf{s}_x + F_{y,\mathrm{M}}\mathbf{s}_y \tag{19}
$$

or

$$
\mathbf{F}_{\mathrm{M}} = \left[F_{x,\mathrm{M}}\cos\left(\psi - \phi\right) - F_{y,\mathrm{M}}\sin\left(\psi - \phi\right)\right]\mathbf{i} + \left[F_{x,\mathrm{M}}\sin\left(\psi - \phi\right) + F_{y,\mathrm{M}}\cos\left(\psi - \phi\right)\right]\mathbf{j}. \tag{20}
$$

The generalized forces are

$$
Q_k = \sum_{j=1}^{p} \mathbf{F}_j \cdot \frac{\partial \mathbf{p}_j}{\partial q_k} \qquad \begin{array}{l} k = 1, 2, 3, 4 \\ j = \mathrm{F}, \mathrm{R}, \mathrm{M} \end{array}. \tag{21}
$$

That is equivalent to:

$$
Q_1 = \mathbf{F}_{\mathrm{F}} \cdot \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_1} + \mathbf{F}_{\mathrm{R}} \cdot \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_1} + \mathbf{F}_{\mathrm{M}} \cdot \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_1} \tag{22a}
$$

$$
Q_2 = \mathbf{F}_{\mathrm{F}} \cdot \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_2} + \mathbf{F}_{\mathrm{R}} \cdot \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_2} + \mathbf{F}_{\mathrm{M}} \cdot \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_2} \tag{22b}
$$

$$
Q_3 = \mathbf{F}_{\mathrm{F}} \cdot \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_3} + \mathbf{F}_{\mathrm{R}} \cdot \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_3} + \mathbf{F}_{\mathrm{M}} \cdot \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_3}, \tag{22c}
$$

$$
Q_4 = \mathbf{F}_{\mathrm{F}} \cdot \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_4} + \mathbf{F}_{\mathrm{R}} \cdot \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_4} + \mathbf{F}_{\mathrm{M}} \cdot \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_4}. \tag{22d}
$$

The application points of the forces are located at:

$$\mathbf{p}_{\mathrm{F/O}} = (x + a\cos\psi)\,\mathbf{i} + (y + a\sin\psi)\,\mathbf{j}. \tag{23a}$$

$$\mathbf{p}_{\mathrm{R/O}} = (x - b\cos\psi)\,\mathbf{i} + (y - b\sin\psi)\,\mathbf{j}. \tag{23b}$$

$$\begin{aligned} \mathbf{p}_{\mathrm{M/O}} &= [x - (b+c)\cos\psi - (d+e)\cos(\psi-\phi)]\,\mathbf{i} + ... \\ ... &+ [y - (b+c)\sin\psi - (d+e)\sin(\psi-\phi)]\,\mathbf{j}. \end{aligned} \tag{23c}$$

Therefore, the partial derivatives:

$$\frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_1} = \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial x} = \mathbf{i} \tag{24a}$$

$$\frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_2} = \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial y} = \mathbf{j} \tag{24b}$$

$$\frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_3} = \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial \psi} = -a\sin\psi\mathbf{i} + a\cos\psi\mathbf{j} \tag{24c}$$

$$\frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_4} = \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial \phi} = 0, \tag{24d}$$

$$\frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_1} = \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial x} = \mathbf{i} \tag{25a}$$

$$\frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_2} = \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial y} = \mathbf{j} \tag{25b}$$

$$\frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_3} = \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial \psi} = b\sin\psi\mathbf{i} - b\cos\psi\mathbf{j} \tag{25c}$$

$$\frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_4} = \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial \phi} = 0 \tag{25d}$$

and

$$\frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_1} = \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial x} = \mathbf{i} \tag{26a}$$

$$\frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_2} = \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial y} = \mathbf{j} \tag{26b}$$

$$\begin{aligned} \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_3} = \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial \psi} &= [(b+c)\sin\psi + (d+e)\sin(\psi-\phi)]\,\mathbf{i} + ... \\ ... &+ [-(b+c)\cos\psi - (d+e)\cos(\psi-\phi)]\,\mathbf{j} \end{aligned} \tag{26c}$$

$$\frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial q_4} = \frac{\partial \mathbf{p}_{\mathrm{M/O}}}{\partial \phi} = \left[-\left(d+e\right)\sin\left(\psi-\phi\right)\right]\mathbf{i} + \left[\left(d+e\right)\cos\left(\psi-\phi\right)\right]\mathbf{j} \tag{26d}$$

Substituting the equations (16), (18), (20), (24), (25) and (26) in the equations (22) we have:

$$\begin{aligned} Q_1 &= F_{x,\mathrm{F}}\cos\left(\psi+\delta\right) + F_{x,\mathrm{R}}\cos\psi + F_{x,\mathrm{M}}\cos\left(\psi-\phi\right) - \ldots \\ \ldots &- F_{y,\mathrm{F}}\sin\left(\psi+\delta\right) - F_{y,\mathrm{R}}\sin\psi - F_{y,\mathrm{M}}\sin\left(\psi-\phi\right) \end{aligned} \tag{27a}$$

$$\begin{aligned} Q_2 &= F_{x,\mathrm{F}}\sin\left(\psi+\delta\right) + F_{x,\mathrm{R}}\sin\psi + F_{x,\mathrm{M}}\sin\left(\psi-\phi\right) + \ldots \\ \ldots &- F_{y,\mathrm{F}}\cos\left(\psi+\delta\right) + F_{y,\mathrm{R}}\cos\psi + F_{y,\mathrm{M}}\cos\left(\psi-\phi\right) \end{aligned} \tag{27b}$$

$$\begin{aligned} Q_3 &= F_{x,\mathrm{F}}a\sin\delta + F_{x,\mathrm{M}}\left(b+c\right)\sin\phi + \ldots \\ \ldots &+ F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b - F_{y,\mathrm{M}}\left[\left(b+c\right)\cos\phi + \left(d+e\right)\right] \end{aligned} \tag{27c}$$

$$Q_4 = F_{y,\mathrm{M}}\left(d+e\right) \tag{27d}$$

The Euler-Lagrange formulation for this system is given by:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \qquad\qquad k = 1, 2, 3, 4, \tag{28}$$

Substituting the equations (12), (14) and (27) into (28)

$$\begin{aligned} m_T\ddot{x} + m_S\dot{C}_1 &= F_{x,\mathrm{F}}\cos\left(\psi+\delta\right) + F_{x,\mathrm{R}}\cos\psi + F_{x,\mathrm{M}}\cos\left(\psi-\phi\right) - \ldots \\ \ldots &- F_{y,\mathrm{F}}\sin\left(\psi+\delta\right) - F_{y,\mathrm{R}}\sin\psi - F_{y,\mathrm{M}}\sin\left(\psi-\phi\right) \end{aligned} \tag{29a}$$

$$\begin{aligned} m_T\ddot{y} + m_S\dot{C}_2 &= F_{x,\mathrm{F}}\sin\left(\psi+\delta\right) + F_{x,\mathrm{R}}\sin\psi + F_{x,\mathrm{M}}\sin\left(\psi-\phi\right) + \ldots \\ \ldots &- F_{y,\mathrm{F}}\cos\left(\psi+\delta\right) + F_{y,\mathrm{R}}\cos\psi + F_{y,\mathrm{M}}\cos\left(\psi-\phi\right) \end{aligned} \tag{29b}$$

$$\begin{aligned} m_S\dot{C}_1\left[\left(b+c\right)\sin\psi + d\sin\left(\psi-\phi\right)\right] + m_S\dot{C}_2\left[-\left(b+c\right)\cos\psi - d\cos\left(\psi-\phi\right)\right] + \ldots \\ \ldots + I_T\ddot{\psi} + I_S\left(\ddot{\psi}-\ddot{\phi}\right) = \\ F_{x,\mathrm{F}}a\sin\delta + F_{x,\mathrm{M}}\left(b+c\right)\sin\phi + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b - F_{y,\mathrm{M}}\left[\left(b+c\right)\cos\phi + \left(d+e\right)\right] \end{aligned} \tag{29c}$$

$$m_S\dot{C}_1\left[-d\sin\left(\psi-\phi\right)\right] + m_S\dot{C}_2\left[d\cos\left(\psi-\phi\right)\right] - I_S\left(\ddot{\psi}-\ddot{\phi}\right) = F_{y,\mathrm{M}}\left(d+e\right) \tag{29d}$$

Substituting the equations (11) into (29):

$$\begin{aligned}(m_T + m_S)\,\ddot{x} + m_S\left[(b+c)\sin\psi + d\sin(\psi-\phi)\right]\ddot{\psi} - m_S d\sin(\psi-\phi)\,\ddot{\phi} = \\ F_{x,\mathrm{F}}\cos(\psi+\delta) + F_{x,\mathrm{R}}\cos\psi + F_{x,\mathrm{M}}\cos(\psi-\phi) - ... \\ ... - F_{y,\mathrm{F}}\sin(\psi+\delta) - F_{y,\mathrm{R}}\sin\psi - F_{y,\mathrm{M}}\sin(\psi-\phi) \\ -m_S(b+c)\,\dot{\psi}^2\cos\psi - m_S d\left(\dot{\psi}-\dot{\phi}\right)^2\cos(\psi-\phi)\end{aligned} \tag{30a}$$

$$\begin{aligned}(m_T + m_S)\,\ddot{y} - m_S\left[(b+c)\cos\psi + d\cos(\psi-\phi)\right]\ddot{\psi} + m_S d\cos(\psi-\phi)\,\ddot{\phi} = \\ F_{x,\mathrm{F}}\sin(\psi+\delta) + F_{x,\mathrm{R}}\sin\psi + F_{x,\mathrm{M}}\sin(\psi-\phi) + ... \\ ... + F_{y,\mathrm{F}}\cos(\psi+\delta) + F_{y,\mathrm{R}}\cos\psi + F_{y,\mathrm{M}}\cos(\psi-\phi) \\ -m_S(b+c)\,\dot{\psi}^2\sin\psi - m_S d\left(\dot{\psi}-\dot{\phi}\right)^2\sin(\psi-\phi)\end{aligned} \tag{30b}$$

$$\begin{aligned}m_S\left[(b+c)\sin\psi + d\sin(\psi-\phi)\right]\ddot{x} - m_S\left[(b+c)\cos\psi + d\cos(\psi-\phi)\right]\ddot{y} + ... \\ ... + \left\{m_S\left[(b+c)^2 + 2(b+c)\,d\cos\phi + d^2\right] + I_T + I_S\right\}\ddot{\psi} - ... \\ ... - \left\{m_S\left[(b+c)\,d\cos\phi + d^2\right] + I_S\right\}\ddot{\phi} = ... \\ ...F_{x,\mathrm{F}}a\sin\delta + F_{x,\mathrm{M}}(b+c)\sin\phi + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b - F_{y,\mathrm{M}}\left[(b+c)\cos\phi + (d+e)\right] - ... \\ -m_S(b+c)\,d\left(\dot{\psi}-\dot{\phi}\right)^2\sin\phi + m_S(b+c)\,d\dot{\psi}^2\sin\phi\end{aligned} \tag{30c}$$

$$\begin{aligned}-m_S d\sin(\psi-\phi)\,\ddot{x} + m_S d\cos(\psi-\phi)\,\ddot{y} - \left\{m_S\left[d^2 + (b+c)\,d\cos\phi\right] + I_S\right\}\ddot{\psi} + ... \\ \left(m_S d^2 + I_S\right)\ddot{\phi} = ... \\ ...F_{y,\mathrm{M}}(d+e) - m_S(b+c)\,d\dot{\psi}^2\sin\phi\end{aligned} \tag{30d}$$

The states can be chosen as:

$$\mathrm{z}_1 = x \tag{31a}$$

$$\mathrm{z}_2 = y \tag{31b}$$

$$\mathrm{z}_3 = \psi \tag{31c}$$

$$\mathrm{z}_4 = \phi \tag{31d}$$

$$\mathrm{z}_5 = \dot{x} \tag{31e}$$

$$\mathrm{z}_6 = \dot{y} \tag{31f}$$

$$\mathrm{z}_7 = \dot{\psi} \tag{31g}$$

$$\mathrm{z}_8 = \dot{\phi} \tag{31h}$$

Therefore, the state equations are:

$$\dot{\mathrm{z}}_1 = \mathrm{z}_5 \tag{32a}$$

$$\dot{\mathrm{z}}_2 = \mathrm{z}_6 \tag{32b}$$

$$\dot{\mathrm{z}}_3 = \mathrm{z}_7 \tag{32c}$$

$$\dot{\mathrm{z}}_4 = \mathrm{z}_8 \tag{32d}$$

$$\begin{aligned} (m_T + m_S)\,\dot{\mathrm{z}}_5 + m_S\left[(b+c)\sin \mathrm{z}_3 + d\sin(\mathrm{z}_3 - \mathrm{z}_4)\right]\dot{\mathrm{z}}_7 - m_S d\sin(\mathrm{z}_3 - \mathrm{z}_4)\,\dot{\mathrm{z}}_8 = \\ F_{x,\mathrm{F}}\cos(\mathrm{z}_3 + \delta) + F_{x,\mathrm{R}}\cos \mathrm{z}_3 + F_{x,\mathrm{M}}\cos(\mathrm{z}_3 - \mathrm{z}_4) - ... \\ ... - F_{y,\mathrm{F}}\sin(\mathrm{z}_3 + \delta) - F_{y,\mathrm{R}}\sin \mathrm{z}_3 - F_{y,\mathrm{M}}\sin(\mathrm{z}_3 - \mathrm{z}_4) \\ -m_S(b+c)\,\mathrm{z}_7^2\cos \mathrm{z}_3 - m_S d\,(\mathrm{z}_7 - \mathrm{z}_8)^2\cos(\mathrm{z}_3 - \mathrm{z}_4) \end{aligned} \tag{32e}$$

$$\begin{aligned} (m_T + m_S)\,\dot{\mathrm{z}}_6 - m_S\left[(b+c)\cos \mathrm{z}_3 + d\cos(\mathrm{z}_3 - \mathrm{z}_4)\right]\dot{\mathrm{z}}_7 + m_S d\cos(\mathrm{z}_3 - \mathrm{z}_4)\,\dot{\mathrm{z}}_8 = \\ F_{x,\mathrm{F}}\sin(\mathrm{z}_3 + \delta) + F_{x,\mathrm{R}}\sin \mathrm{z}_3 + F_{x,\mathrm{M}}\sin(\mathrm{z}_3 - \mathrm{z}_4) + ... \\ ... + F_{y,\mathrm{F}}\cos(\mathrm{z}_3 + \delta) + F_{y,\mathrm{R}}\cos \mathrm{z}_3 + F_{y,\mathrm{M}}\cos(\mathrm{z}_3 - \mathrm{z}_4) \\ -m_S(b+c)\,\mathrm{z}_7^2\sin \mathrm{z}_3 - m_S d\,(\mathrm{z}_7 - \mathrm{z}_8)^2\sin(\mathrm{z}_3 - \mathrm{z}_4) \end{aligned} \tag{32f}$$

$$\begin{aligned} m_S\left[(b+c)\sin \mathrm{z}_3 + d\sin(\mathrm{z}_3 - \mathrm{z}_4)\right]\dot{\mathrm{z}}_5 - m_S\left[(b+c)\cos \mathrm{z}_3 + d\cos(\mathrm{z}_3 - \mathrm{z}_4)\right]\dot{\mathrm{z}}_6 + ... \\ ... + \left\{m_S\left[(b+c)^2 + 2(b+c)\,d\cos \mathrm{z}_4 + d^2\right] + I_T + I_S\right\}\dot{\mathrm{z}}_7 - ... \\ ... - \left\{m_S\left[(b+c)\,d\cos \mathrm{z}_4 + d^2\right] + I_S\right\}\dot{\mathrm{z}}_8 = ... \\ ...F_{x,\mathrm{F}}a\sin\delta + F_{x,\mathrm{M}}(b+c)\sin \mathrm{z}_4 + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b - F_{y,\mathrm{M}}\left[(b+c)\cos \mathrm{z}_4 + (d+e)\right] - ... \\ -m_S(b+c)\,d\,(\mathrm{z}_7 - \mathrm{z}_8)^2\sin \mathrm{z}_4 + m_S(b+c)\,d\mathrm{z}_7^2\sin \mathrm{z}_4 \end{aligned} \tag{32g}$$

$$\begin{aligned} -m_S d\sin(\psi - \phi)\,\dot{\mathrm{z}}_5 + m_S d\cos(\psi - \phi)\,\dot{\mathrm{z}}_6 - \left\{m_S\left[d^2 + (b+c)\,d\cos\phi\right] + I_S\right\}\dot{\mathrm{z}}_7 + ... \\ \left(m_S d^2 + I_S\right)\dot{\mathrm{z}}_8 = ... \\ ...F_{y,\mathrm{M}}(d+e) - m_S(b+c)\,d\dot{\psi}^2\sin\phi \end{aligned} \tag{32h}$$

In many instances it is convenient to replace the states $\dot{x}$ and $\dot{y}$ with v_T and a_T. The relationship between these pairs of states is:

$$\dot{x} = v_{\mathrm{T}} \cos\left(\psi + \alpha_{\mathrm{T}}\right) \tag{33a}$$

$$\dot{y} = v_{\mathrm{T}} \sin\left(\psi + \alpha_{\mathrm{T}}\right). \tag{33b}$$

Differentiating with respect to time the equation (33) we have

$$\ddot{x} = \dot{v}_{\mathrm{T}} \cos\left(\psi + \alpha_{\mathrm{T}}\right) - v_{\mathrm{T}} \left(\dot{\psi} + \dot{\alpha}_{\mathrm{T}}\right) \sin\left(\psi + \alpha_{\mathrm{T}}\right) \tag{34a}$$

$$\ddot{y} = \dot{v}_{\mathrm{T}} \sin\left(\psi + \alpha_{\mathrm{T}}\right) + v_{\mathrm{T}} \left(\dot{\psi} + \dot{\alpha}_{\mathrm{T}}\right) \cos\left(\psi + \alpha_{\mathrm{T}}\right). \tag{34b}$$

Thus, by substituting the equations (34) in the equations (30) we have:

$$\begin{aligned}
\left(m_T + m_S\right) \cos\left(\psi + \alpha_{\mathrm{T}}\right) \dot{v}_{\mathrm{T}} - \left(m_T + m_S\right) v_{\mathrm{T}} \sin\left(\psi + \alpha_{\mathrm{T}}\right) \dot{\alpha}_{\mathrm{T}} + \ldots \\
+ m_S \left[\left(b + c\right) \sin\psi + d \sin\left(\psi - \phi\right)\right] \ddot{\psi} - m_S d \sin\left(\psi - \phi\right) \ddot{\phi} = \\
F_{x,\mathrm{F}} \cos\left(\psi + \delta\right) + F_{x,\mathrm{R}} \cos\psi + F_{x,\mathrm{M}} \cos\left(\psi - \phi\right) - \ldots \\
\ldots - F_{y,\mathrm{F}} \sin\left(\psi + \delta\right) - F_{y,\mathrm{R}} \sin\psi - F_{y,\mathrm{M}} \sin\left(\psi - \phi\right) \\
- m_S \left(b + c\right) \dot{\psi}^2 \cos\psi - m_S d \left(\dot{\psi} - \dot{\phi}\right)^2 \cos\left(\psi - \phi\right) + \left(m_T + m_S\right) v_{\mathrm{T}} \sin\left(\psi + \alpha_{\mathrm{T}}\right) \dot{\psi}
\end{aligned} \tag{35a}$$

$$\begin{aligned}
\left(m_T + m_S\right) \sin\left(\psi + \alpha_{\mathrm{T}}\right) \dot{v}_{\mathrm{T}} + \left(m_T + m_S\right) v_{\mathrm{T}} \cos\left(\psi + \alpha_{\mathrm{T}}\right) \dot{\alpha}_{\mathrm{T}} + \ldots \\
- m_S \left[\left(b + c\right) \cos\psi + d \cos\left(\psi - \phi\right)\right] \ddot{\psi} + m_S d \cos\left(\psi - \phi\right) \ddot{\phi} = \\
F_{x,\mathrm{F}} \sin\left(\psi + \delta\right) + F_{x,\mathrm{R}} \sin\psi + F_{x,\mathrm{M}} \sin\left(\psi - \phi\right) + \ldots \\
\ldots + F_{y,\mathrm{F}} \cos\left(\psi + \delta\right) + F_{y,\mathrm{R}} \cos\psi + F_{y,\mathrm{M}} \cos\left(\psi - \phi\right) \\
- m_S \left(b + c\right) \dot{\psi}^2 \sin\psi - m_S d \left(\dot{\psi} - \dot{\phi}\right)^2 \sin\left(\psi - \phi\right) - \left(m_T + m_S\right) v_{\mathrm{T}} \cos\left(\psi + \alpha_{\mathrm{T}}\right) \dot{\psi}
\end{aligned} \tag{35b}$$

$$\begin{aligned}
- m_S \left[\left(b + c\right) \sin\alpha_{\mathrm{T}} + d \sin\left(\alpha_{\mathrm{T}} + \phi\right)\right] \dot{v}_{\mathrm{T}} - m_S \left[\left(b + c\right) v_{\mathrm{T}} \cos\alpha_{\mathrm{T}} + d v_{\mathrm{T}} \cos\left(\alpha_{\mathrm{T}} + \phi\right)\right] \dot{\alpha}_{\mathrm{T}} \\
\ldots + \left\{m_S \left[\left(b + c\right)^2 + 2\left(b + c\right) d \cos\phi + d^2\right] + I_T + I_S\right\} \ddot{\psi} - \left\{m_S \left[\left(b + c\right) d \cos\phi + d^2\right] + I_S\right\} \ddot{\phi} = \ldots \\
\ldots F_{x,\mathrm{F}} a \sin\delta + F_{x,\mathrm{M}} \left(b + c\right) \sin\phi + F_{y,\mathrm{F}} a \cos\delta - F_{y,\mathrm{R}} b - F_{y,\mathrm{M}} \left[\left(b + c\right) \cos\phi + \left(d + e\right)\right] - \ldots \\
- m_S \left(b + c\right) d \left(\dot{\psi} - \dot{\phi}\right)^2 \sin\phi + m_S \left(b + c\right) d \dot{\psi}^2 \sin\phi + m_S \left[\left(b + c\right) v_{\mathrm{T}} \cos\alpha_{\mathrm{T}} + d v_{\mathrm{T}} \cos\left(\alpha_{\mathrm{T}} + \phi\right)\right] \dot{\psi}
\end{aligned} \tag{35c}$$

$$\begin{aligned}
m_S d \sin\left(\alpha_{\mathrm{T}} + \phi\right) \dot{v}_{\mathrm{T}} + m_S d v_{\mathrm{T}} \cos\left(\alpha_{\mathrm{T}} + \phi\right) \dot{\alpha}_{\mathrm{T}} - \left\{m_S \left[d^2 + \left(b + c\right) d \cos\phi\right] + I_S\right\} \ddot{\psi} + \ldots \\
\ldots + \left(m_S d^2 + I_S\right) \ddot{\phi} = \ldots \\
\ldots F_{y,\mathrm{M}} \left(d + e\right) - m_S \left(b + c\right) d \dot{\psi}^2 \sin\phi - m_S d v_{\mathrm{T}} \cos\left(\alpha_{\mathrm{T}} + \phi\right) \dot{\psi}
\end{aligned} \tag{35d}$$

The states can be chosen as

$$\mathrm{x}_1 = x \tag{36a}$$

$$\mathrm{x}_2 = y \tag{36b}$$

$$\mathrm{x}_3 = \psi \tag{36c}$$

$$\mathrm{x}_4 = \phi \tag{36d}$$

$$\mathrm{x}_5 = \dot{v}_{\mathrm{T}} \tag{36e}$$

$$\mathrm{x}_6 = \dot{\alpha}_{\mathrm{T}} \tag{36f}$$

$$\mathrm{x}_7 = \dot{\psi} \tag{36g}$$

$$\mathrm{x}_8 = \dot{\phi} \tag{36h}$$

In the system's matrix form of the equation (35) can be written as:

$$\mathbf{M}\left(\mathbf{x}\right)\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right), \tag{37}$$

where the state vector is:

$$\mathbf{x} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \\ \mathrm{x}_5 \\ \mathrm{x}_6 \\ \mathrm{x}_7 \\ \mathrm{x}_8 \end{bmatrix} \tag{38}$$

and the input vector is:

$$\mathbf{u} = \begin{bmatrix} \delta \\ F_{x,\mathrm{F}} \\ F_{x,\mathrm{R}} \\ F_{x,\mathrm{M}} \\ F_{y,\mathrm{F}} \\ F_{y,\mathrm{R}} \\ F_{y,\mathrm{M}} \end{bmatrix}. \tag{39}$$

The matrix bfM is given by

$$\mathbf{M}=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&M_{55}&M_{56}&M_{57}&M_{58}\\0&0&0&0&M_{65}&M_{66}&M_{67}&M_{68}\\0&0&0&0&M_{75}&M_{76}&M_{77}&M_{78}\\0&0&0&0&M_{85}&M_{86}&M_{87}&M_{88}\end{bmatrix}, \tag{40}$$

where the elements are

$$M_{55}=\left(m_T+m_S\right)\cos\left(\psi+\alpha_{\mathrm{T}}\right) \tag{41a}$$

$$M_{56}=-\left(m_T+m_S\right)v_{\mathrm{T}}\sin\left(\psi+\alpha_{\mathrm{T}}\right) \tag{41b}$$

$$M_{57}=m_S\left[\left(b+c\right)\sin\psi+d\sin\left(\psi-\phi\right)\right] \tag{41c}$$

$$M_{58}=-m_Sd\sin\left(\psi-\phi\right) \tag{41d}$$

$$M_{65}=\left(m_T+m_S\right)\sin\left(\psi+\alpha_{\mathrm{T}}\right) \tag{41e}$$

$$M_{66}=\left(m_T+m_S\right)v_{\mathrm{T}}\cos\left(\psi+\alpha_{\mathrm{T}}\right) \tag{41f}$$

$$M_{67}=-m_S\left[\left(b+c\right)\cos\psi+d\cos\left(\psi-\phi\right)\right] \tag{41g}$$

$$M_{68}=m_Sd\cos\left(\psi-\phi\right) \tag{41h}$$

$$M_{75}=-m_S\left[\left(b+c\right)\sin\alpha_{\mathrm{T}}+d\sin\left(\alpha_{\mathrm{T}}+\phi\right)\right] \tag{41i}$$

$$M_{76}=-m_S\left[\left(b+c\right)v_{\mathrm{T}}\cos\alpha_{\mathrm{T}}+dv_{\mathrm{T}}\cos\left(\alpha_{\mathrm{T}}+\phi\right)\right] \tag{41j}$$

$$M_{77}=m_S\left[\left(b+c\right)^2+2\left(b+c\right)d\cos\phi+d^2\right]+I_T+I_S \tag{41k}$$

$$M_{78}=-m_S\left[\left(b+c\right)d\cos\phi+d^2\right]+I_S \tag{41l}$$

$$M_{85}=m_Sd\sin\left(\alpha_{\mathrm{T}}+\phi\right) \tag{41m}$$

$$M_{86}=m_Sdv_{\mathrm{T}}\cos\left(\alpha_{\mathrm{T}}+\phi\right) \tag{41n}$$

$$M_{87} = -m_S \left[d^2 + (b+c)\, d\cos\phi\right] + I_S \tag{41o}$$

$$M_{88} = \left(m_S d^2 + I_S\right) \tag{41p}$$

Functions are given by:

$$\mathbf{f} = \begin{bmatrix} v_{\mathrm{T}} \cos\left(\psi + \alpha_{\mathrm{T}}\right) \\ v_{\mathrm{T}} \sin\left(\psi + \alpha_{\mathrm{T}}\right) \\ \dot{\psi} \\ \dot{\phi} \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}, \tag{42}$$

where

$$\begin{aligned} f_5 &= F_{x,\mathrm{F}} \cos\left(\psi + \delta\right) + F_{x,\mathrm{R}} \cos\psi + F_{x,\mathrm{M}} \cos\left(\psi - \phi\right) - \ldots \\ \ldots &- F_{y,\mathrm{F}} \sin\left(\psi + \delta\right) - F_{y,\mathrm{R}} \sin\psi - F_{y,\mathrm{M}} \sin\left(\psi - \phi\right) - \ldots \\ \ldots &- m_S\left(b+c\right)\dot{\psi}^2 \cos\psi - m_S d\left(\dot{\psi} - \dot{\phi}\right)^2 \cos\left(\psi - \phi\right) + \left(m_T + m_S\right) v_{\mathrm{T}} \sin\left(\psi + \alpha_{\mathrm{T}}\right)\dot{\psi} \end{aligned} \tag{43a}$$

$$\begin{aligned} f_6 &= F_{x,\mathrm{F}} \sin\left(\psi + \delta\right) + F_{x,\mathrm{R}} \sin\psi + F_{x,\mathrm{M}} \sin\left(\psi - \phi\right) + \ldots \\ \ldots &+ F_{y,\mathrm{F}} \cos\left(\psi + \delta\right) + F_{y,\mathrm{R}} \cos\psi + F_{y,\mathrm{M}} \cos\left(\psi - \phi\right) \\ \ldots &- m_S\left(b+c\right)\dot{\psi}^2 \sin\psi - m_S d\left(\dot{\psi} - \dot{\phi}\right)^2 \sin\left(\psi - \phi\right) - \left(m_T + m_S\right) v_{\mathrm{T}} \cos\left(\psi + \alpha_{\mathrm{T}}\right)\dot{\psi} \end{aligned} \tag{43b}$$

$$\begin{aligned} f_7 &= F_{x,\mathrm{F}} a \sin\delta + F_{x,\mathrm{M}}\left(b+c\right)\sin\phi + F_{y,\mathrm{F}} a\cos\delta - \ldots \\ \ldots &- F_{y,\mathrm{R}} b - F_{y,\mathrm{M}}\left[\left(b+c\right)\cos\phi + \left(d+e\right)\right] - \ldots \\ \ldots &- m_S\left(b+c\right) d\left(\dot{\psi} - \dot{\phi}\right)^2 \sin\phi + m_S\left(b+c\right) d\dot{\psi}^2 \sin\phi + \ldots \\ \ldots &+ m_S\left[\left(b+c\right) v_{\mathrm{T}} \cos\alpha_{\mathrm{T}} + d v_{\mathrm{T}} \cos\left(\alpha_{\mathrm{T}} + \phi\right)\right]\dot{\psi} \end{aligned} \tag{43c}$$

$$f_8 = F_{y,\mathrm{M}}\left(d+e\right) - m_S\left(b+c\right) d\dot{\psi}^2 \sin\phi - m_S d v_{\mathrm{T}} \cos\left(\alpha_{\mathrm{T}} + \phi\right)\dot{\psi}. \tag{43d}$$

Therefore, the nonlinear articulated vehicle model is given by the equations (37), (38), (39), (40), (41), (42), and (43).

2 Linearized model

The nonlinear equation (37) can be linearized and written in the matrix form

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}. \tag{44}$$

The linearization of this system can be performed for a vehicle moving in a straight line with a certain velocity $v_\mathrm{T} > 0$. In this case, the states at the point of operation are given by:

$$\mathrm{x}_{1,op} = x_{op} = ? \tag{45a}$$

$$\mathrm{x}_{2,op} = y_{op} = ? \tag{45b}$$

$$\mathrm{x}_{3,op} = \psi_{op} = 0 \tag{45c}$$

$$\mathrm{x}_{4,op} = \phi_{op} = 0 \tag{45d}$$

$$\mathrm{x}_{5,op} = v_{\mathrm{T},op} = v_{\mathrm{T},0} \tag{45e}$$

$$\mathrm{x}_{6,op} = \alpha_{\mathrm{T},op} = 0 \tag{45f}$$

$$\mathrm{x}_{7,op} = \dot{\psi}_{op} = 0, \tag{45g}$$

$$\mathrm{x}_{8,op} = \dot{\phi}_{op} = 0. \tag{45h}$$

NOTE: The states x and y do not influence the system dynamics.

The state operating point vector is:

$$\mathbf{x}_{op} = \begin{bmatrix} \mathrm{x}_{1,op} \\ \mathrm{x}_{2,op} \\ \mathrm{x}_{3,op} \\ \mathrm{x}_{4,op} \\ \mathrm{x}_{5,op} \\ \mathrm{x}_{6,op} \\ \mathrm{x}_{7,op} \\ \mathrm{x}_{8,op} \end{bmatrix}. \tag{46}$$

At this point of operation of the states, the operating point of the states derivative is given by:

$$\dot{\mathrm{x}}_{1,op} = \dot{x}_{op} = v_{\mathrm{T},0} \tag{47a}$$

$$\dot{\mathrm{x}}_{2,op} = \dot{y}_{op} = 0 \tag{47b}$$

$$\dot{\mathrm{x}}_{3,op} = \dot{\psi}_{op} = 0 \tag{47c}$$

$$\dot{\mathrm{x}}_{4,op} = \dot{\phi}_{op} = 0 \tag{47d}$$

$$\dot{\mathrm{x}}_{5,op} = \dot{v}_{\mathrm{T},op} = 0 \tag{47e}$$

$$\dot{\mathrm{x}}_{6,op} = \dot{\alpha}_{\mathrm{T},op} = 0 \tag{47f}$$

$$\dot{\mathrm{x}}_{7,op} = \ddot{\psi}_{op} = 0, \tag{47g}$$

$$\dot{\mathrm{x}}_{8,op} = \ddot{\phi}_{op} = 0. \tag{47h}$$

The vector of the operation point of the states derivative is:

$$\dot{\mathbf{x}}_{op} = \begin{bmatrix} \dot{\mathrm{x}}_{1,op} \\ \dot{\mathrm{x}}_{2,op} \\ \dot{\mathrm{x}}_{3,op} \\ \dot{\mathrm{x}}_{4,op} \\ \dot{\mathrm{x}}_{5,op} \\ \dot{\mathrm{x}}_{6,op} \\ \dot{\mathrm{x}}_{7,op} \\ \dot{\mathrm{x}}_{8,op} \end{bmatrix}. \tag{48}$$

The operating point of the inputs is:

$$\delta_{op} = 0 \tag{49a}$$

$$F_{x,\mathrm{F},op} = 0 \tag{49b}$$

$$F_{x,\mathrm{R},op} = 0 \tag{49c}$$

$$F_{x,\mathrm{M},op} = 0 \tag{49d}$$

$$F_{y,\mathrm{F},op} = 0 \tag{49e}$$

$$F_{y,\mathrm{R},op} = 0. \tag{49f}$$

$$F_{y,\mathrm{M},op} = 0. \tag{49g}$$

The input point vector of inputs is:

$$\mathbf{u}_{op} = \begin{bmatrix} \delta_{op} \\ F_{x,\mathrm{F},op} \\ F_{x,\mathrm{R},op} \\ F_{y,\mathrm{F},op} \\ F_{y,\mathrm{R},op} \end{bmatrix}. \tag{50}$$

Use the Taylor's series to expand the equation (37) and truncating (beyond ?) the first order terms, we obtain:

$$\nabla \mathbf{g}\left(\mathbf{x}_{op}, \dot{\mathbf{x}}_{op}, \mathbf{u}_{op}\right)\begin{bmatrix} \mathbf{x}-\mathbf{x}_{op} \\ \dot{\mathbf{x}}-\dot{\mathbf{x}}_{op} \\ \mathbf{u}-\mathbf{u}_{op} \end{bmatrix} = \nabla \mathbf{f}\left(\mathbf{x}_{op}, \dot{\mathbf{x}}_{op}, \mathbf{u}_{op}\right)\begin{bmatrix} \mathbf{x}-\mathbf{x}_{op} \\ \dot{\mathbf{x}}-\dot{\mathbf{x}}_{op} \\ \mathbf{u}-\mathbf{u}_{op} \end{bmatrix}, \tag{51}$$

where:

$$\mathbf{g} = \begin{bmatrix} \mathrm{g}_1 \\ \mathrm{g}_2 \\ \mathrm{g}_3 \\ \mathrm{g}_4 \\ \mathrm{g}_5 \\ \mathrm{g}_6 \\ \mathrm{g}_7 \\ \mathrm{g}_8 \end{bmatrix}. \tag{52}$$

is the left side of the equation (37), while the right side is given by

$$\mathbf{f} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \\ \mathrm{f}_4 \\ \mathrm{f}_5 \\ \mathrm{f}_6 \\ \mathrm{f}_7 \\ \mathrm{f}_8 \end{bmatrix}. \tag{53}$$

The Jacobian of the functions (52) and (53) is:

$$\nabla \mathbf{g} = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \cdots & \frac{\partial g_1}{\partial \dot{x}} & \cdots & \frac{\partial g_1}{\partial \delta} & \cdots & \frac{\partial g_1}{\partial F_{y,\mathrm{R}}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial g_8}{\partial x} & \cdots & \frac{\partial g_8}{\partial \dot{x}} & \cdots & \frac{\partial g_8}{\partial \delta} & \cdots & \frac{\partial g_8}{\partial F_{y,\mathrm{R}}} \end{bmatrix} \tag{54a}$$

$$\nabla \mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \cdots & \frac{\partial f_1}{\partial \dot{x}} & \cdots & \frac{\partial f_1}{\partial \delta} & \cdots & \frac{\partial f_1}{\partial F_{y,\mathrm{R}}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial f_8}{\partial x} & \cdots & \frac{\partial f_8}{\partial \dot{x}} & \cdots & \frac{\partial f_8}{\partial \delta} & \cdots & \frac{\partial f_8}{\partial F_{y,\mathrm{R}}} \end{bmatrix}. \tag{54b}$$

Therefore, linearised equations of motion are given by:

$$(m_T + m_S)\dot{v}_{\mathrm{T}} = F_{x,\mathrm{F}} + F_{x,\mathrm{R}} + F_{x,\mathrm{M}} \tag{55a}$$

$$(m_T + m_S)v_{\mathrm{T},0}\dot{\alpha}_{\mathrm{T}} - m_S(b+c+d)\ddot{\psi} + m_S d\ddot{\phi} = F_{y,\mathrm{F}} + F_{y,\mathrm{R}} + F_{y,\mathrm{M}} - (m_S + m_T)v_{\mathrm{T},0}\dot{\psi} \tag{55b}$$

$$\begin{aligned} -m_S(b+c+c)v_{\mathrm{T},0}\dot{\alpha}_{\mathrm{T}} + \left[I_T + I_S + m_S(b+c+d)^2\right]\ddot{\psi} - \left[I_S + m_S(d^2 + (b+c)d)\right]\ddot{\phi} = \\ F_{y,\mathrm{F}}a - F_{y,\mathrm{R}}b - F_{y,\mathrm{M}}(b+c+d+e) + m_S(b+c+d)v_{\mathrm{T},0}\dot{\psi} \end{aligned} \tag{55c}$$

$$m_S d v_{\mathrm{T},0}\dot{\alpha}_{\mathrm{T}} - (I_S + m_S(d^2 + (b+c)d))\ddot{\psi} + (m_S d^2 + I_S)\ddot{\phi} = F_{y,\mathrm{M}}(d+e) - m_S d v_{\mathrm{T},0}\dot{\psi} \quad (55d)$$

It is noteworthy, that when the sum of the longitudinal forces is zero the velocity v_{T} remains constant.

State equations are given by:

$$\dot{\mathrm{x}}_1 = \mathrm{x}_5 \quad (56a)$$

$$\dot{\mathrm{x}}_2 = (\mathrm{x}_3 + \mathrm{x}_6)v_{\mathrm{T},0} \quad (56b)$$

$$\dot{\mathrm{x}}_3 = \mathrm{x}_7 \quad (56c)$$

$$\dot{\mathrm{x}}_4 = \mathrm{x}_8 \quad (56d)$$

$$(mS + mT)\dot{\mathrm{x}}_5 = FxF + FxM + FxR \quad (56e)$$

$$(m_T + m_S)v_{\mathrm{T},0}\dot{\mathrm{x}}_6 - m_S(b+c+d)\dot{\mathrm{x}}_7 + m_S d\dot{\mathrm{x}}_8 = F_{y,\mathrm{F}} + F_{y,\mathrm{R}} + F_{y,\mathrm{M}} - (m_S + m_T)v_{\mathrm{T},0}\mathrm{x}_7 \quad (56f)$$

$$\begin{aligned} -m_S(b+c+c)v_{\mathrm{T},0}\dot{\mathrm{x}}_6 + \left[I_T + I_S + m_S(b+c+d)^2\right]\dot{\mathrm{x}}_7 - \left[I_S + m_S(d^2 + (b+c)d)\right]\dot{\mathrm{x}}_8 = \\ F_{y,\mathrm{F}}a - F_{y,\mathrm{R}}b - F_{y,\mathrm{M}}(b+c+d+e) + m_S(b+c+d)v_{\mathrm{T},0}\mathrm{x} \end{aligned} \quad (56g)$$

$$m_S d v_{\mathrm{T},0}\dot{\mathrm{x}}_6 - (I_S + m_S(d^2 + (b+c)d))\dot{\mathrm{x}}_7 + (m_S d^2 + I_S)\dot{\mathrm{x}}_8 = F_{y,\mathrm{M}}(d+e) - m_S d v_{\mathrm{T},0}\mathrm{x}_7 \quad (56h)$$

Writing equation (56) in the matrix form of the equation (44), the matrix bfE is given by:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & E_{56} & E_{57} & E_{58} \\ 0 & 0 & 0 & 0 & E_{65} & E_{66} & E_{67} & E_{68} \\ 0 & 0 & 0 & 0 & E_{75} & E_{76} & E_{77} & E_{78} \\ 0 & 0 & 0 & 0 & E_{85} & E_{86} & E_{87} & E_{88} \end{bmatrix}, \quad (57)$$

where the elements are

$$E_{55} = (m_T + m_S) \quad (58a)$$

$$E_{56} = 0 \quad (58b)$$

$$E_{57} = 0 \tag{58c}$$

$$E_{58} = 0 \tag{58d}$$

$$E_{65} = 0 \tag{58e}$$

$$E_{66} = (m_S + m_T)\, v_{\mathrm{T},0} \tag{58f}$$

$$E_{67} = -m_S\,(b + c + d) \tag{58g}$$

$$E_{68} = dm_S \tag{58h}$$

$$E_{75} = 0 \tag{58i}$$

$$E_{76} = -m_S v_{\mathrm{T},0}\,(b + c + d) \tag{58j}$$

$$E_{77} = I_T + I_S + m_S(b + c + d)^2 \tag{58k}$$

$$E_{78} = -I_S - m_S\left[d^2 + (b + c)d\right] \tag{58l}$$

$$E_{85} = 0 \tag{58m}$$

$$E_{86} = dm_S v_{\mathrm{T},0} \tag{58n}$$

$$E_{87} = -I_S - m_S\left[d^2 + (b + c)d\right] \tag{58o}$$

$$E_{88} = m_S d^2 + I_S \tag{58p}$$

The dynamic matrix of the system is given by:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & v_{\mathrm{T},0} & 0 & 0 & v_{\mathrm{T},0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(m_S + m_T)v_{\mathrm{T},0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_S(b + c + d)v_{\mathrm{T},0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -m_S d v_{\mathrm{T},0} & 0 \end{bmatrix} \tag{59}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & a & -b & -(b+c+d+e) \\ 0 & 0 & 0 & 0 & 0 & 0 & d+e \end{bmatrix} \tag{60}$$

Therefore, the linear articulated vehicle model is given by the equation (44) with the matrices given by the equations (57), (59) and (60).

2.1 Drift angles

The speed on the front axle is given by

$$\mathbf{v}_{\mathrm{F}} = \mathbf{v}_{\mathrm{T}} + \mathbf{w}_{T} \wedge \mathbf{r}_{\mathrm{F/T}}, \tag{61}$$

where $\mathbf{r}_{\mathrm{F/T}}$ is the position vector of the point F with respect to the point T. Therefore,

$$\mathbf{v}_{\mathrm{F}} = \left(\dot{x} - a\dot{\psi}\sin\psi\right)\mathbf{i} + \left(\dot{y} + a\dot{\psi}\cos\psi\right)\mathbf{j}. \tag{62}$$

The speed on the rear axle is:

$$\mathbf{v}_{\mathrm{R}} = \mathbf{v}_{\mathrm{T}} + \mathbf{w}_{T} \wedge \mathbf{r}_{\mathrm{R/T}}, \tag{63}$$

where $\mathbf{r}_{\mathrm{R/T}}$ is the position vector of point R with respect to point T. Therefore,

$$\mathbf{v}_{\mathrm{R}} = \left(\dot{x} + b\dot{\psi}\sin\psi\right)\mathbf{i} + \left(\dot{y} - b\dot{\psi}\cos\psi\right)\mathbf{j} \tag{64}$$

In an analogous way, the speed of the semi-trailer axle is:

$$\begin{aligned} \mathbf{v}_{\mathrm{M}} &= \left[\dot{x} + (b+c)\,\dot{\psi}\sin\psi + (d+e)\left(\dot{\psi} - \dot{\phi}\right)\sin(\psi-\phi)\right]\mathbf{i} + \ldots \\ \ldots &+ \left[\dot{y} - (b+c)\,\dot{\psi}\cos\psi - (d+e)\left(\dot{\psi} - \dot{\phi}\right)\cos(\psi-\phi)\right]\mathbf{j}. \end{aligned} \tag{65}$$

Using the equations (62), (64) and (65), the drift angles can be written as

$$\alpha_{\mathrm{F}} = \arctan\left(\frac{\dot{y} + a\dot{\psi}\cos\psi}{\dot{x} - a\dot{\psi}\sin\psi}\right) - (\delta + \psi) \tag{66a}$$

$$\alpha_{\mathrm{R}} = \arctan\left(\frac{\dot{y} - b\dot{\psi}\cos\psi}{\dot{x} + b\dot{\psi}\sin\psi}\right) - \psi \tag{66b}$$

$$\alpha_{\mathrm{R}} = \arctan\left(\frac{\dot{y} - (b+c)\,\dot{\psi}\cos\psi - (d+e)\left(\dot{\psi} - \dot{\phi}\right)\cos(\psi-\phi)}{\dot{x} + (b+c)\,\dot{\psi}\sin\psi + (d+e)\left(\dot{\psi} - \dot{\phi}\right)\sin(\psi-\phi)}\right) - (\psi - \phi) \tag{66c}$$

By performing the change of variables proposed in the equation (33), the drift angles become:

$$\alpha_{\mathrm{F}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\left(\psi+\alpha_{\mathrm{T}}\right)+a\dot{\psi}\cos\psi}{v_{\mathrm{T}}\cos\left(\psi+\alpha_{\mathrm{T}}\right)-a\dot{\psi}\sin\psi}\right)-(\delta+\psi) \tag{67a}$$

$$\alpha_{\mathrm{R}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\left(\psi+\alpha_{\mathrm{T}}\right)-b\dot{\psi}\cos\psi}{v_{\mathrm{T}}\cos\left(\psi+\alpha_{\mathrm{T}}\right)+b\dot{\psi}\sin\psi}\right)-\psi \tag{67b}$$

$$\alpha_{\mathrm{M}} = \tag{67c}$$

Simply put, we obtain:

$$\alpha_{\mathrm{F}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\alpha_{\mathrm{T}}+a\dot{\psi}}{v_{\mathrm{T}}\cos\alpha_{\mathrm{T}}}\right)-\delta \tag{68a}$$

$$\alpha_{\mathrm{R}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\alpha_{\mathrm{T}}-b\dot{\psi}}{v_{\mathrm{T}}\cos\alpha_{\mathrm{T}}}\right) \tag{68b}$$

$$\alpha_{\mathrm{M}} = \tag{68c}$$

Linearising around the point of operation given by the equations (45), (47) and (49) we obtain:

$$\alpha_{\mathrm{F},lin} = \alpha_{\mathrm{T}} + \frac{a}{v_{\mathrm{T},0}}\dot{\psi} - \delta \tag{69a}$$

$$\alpha_{\mathrm{F},lin} = \alpha_{\mathrm{T}} - \frac{b}{v_{\mathrm{T},0}}\dot{\psi}. \tag{69b}$$

$$\alpha_{\mathrm{M},lin} = \tag{69c}$$

3 Simulation

4 Conclusions

5 Linear tire

Linearising the value of the drift angles in (68) at the same operating point we have

Assuming a linear force tire law we obtain:

$$F_{y,\mathrm{F}} = -K_{\mathrm{F}}\alpha_{\mathrm{F}} = -K_{\mathrm{F}}\alpha_{\mathrm{T}} - \frac{aK_{\mathrm{F}}}{v_{\mathrm{T},0}}\dot{\psi} + K_{\mathrm{F}}\delta \tag{70a}$$

$$F_{y,\mathrm{R}} = -K_{\mathrm{R}}\alpha_{\mathrm{R}} = -K_{\mathrm{R}}\alpha_{\mathrm{T}} + \frac{bK_{\mathrm{R}}}{v_{\mathrm{T},0}}\dot{\psi} \tag{70b}$$

Replacing the equations (70) in the linearized equations in (**??**) we have

$$f_{1,lin} = \dot{x} = v_{\mathrm{T}} \tag{71a}$$

$$f_{2,lin} = \dot{y} = v_{\mathrm{T},0}\left(\psi + \alpha_{\mathrm{T}}\right) \tag{71b}$$

$$f_{3,lin} = \dot{\psi} = \dot{\psi} \tag{71c}$$

$$f_{4,lin} = \dot{v}_{\mathrm{T}} = \frac{F_{x,\mathrm{F}} + F_{x,\mathrm{R}}}{m_T} \tag{71d}$$

$$f_{5,lin} = \dot{\alpha}_{\mathrm{T}} = -\frac{K_{\mathrm{F}} + K_{\mathrm{R}}}{m_T v_{\mathrm{T},0}}\alpha_{\mathrm{T}} - \frac{m_T v_{\mathrm{T},0} + \frac{aK_{\mathrm{F}} - bK_{\mathrm{R}}}{v_{\mathrm{T},0}}}{m_T v_{\mathrm{T},0}}\dot{\psi} + \frac{K_{\mathrm{F}}}{m_T v_{\mathrm{T},0}}\delta \tag{71e}$$

$$f_{6,lin} = \ddot{\psi} = -\frac{aK_{\mathrm{F}} - bK_{\mathrm{R}}}{I_T}\alpha_{\mathrm{T}} - \frac{a^2K_{\mathrm{F}} + b^2K_{\mathrm{R}}}{I_T v_{\mathrm{T},0}}\dot{\psi} + \frac{aK_{\mathrm{F}}}{I_T}\delta \tag{71f}$$

In matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\hat{\mathbf{u}} \tag{72}$$

or:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{v}_{\mathrm{T}} \\ \dot{\alpha}_{\mathrm{T}} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & v_{\mathrm{T},0} & 0 & v_{\mathrm{T},0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{K_{\mathrm{F}}+K_{\mathrm{R}}}{m_T v_{\mathrm{T},0}} & -\frac{m_T v_{\mathrm{T},0} + \frac{aK_{\mathrm{F}}-bK_{\mathrm{R}}}{v_{\mathrm{T},0}}}{m_T v_{\mathrm{T},0}} \\ 0 & 0 & 0 & 0 & -\frac{aK_{\mathrm{F}}-bK_{\mathrm{R}}}{I_T} & -\frac{a^2K_{\mathrm{F}}+b^2K_{\mathrm{R}}}{I_T v_{\mathrm{T},0}} \end{bmatrix} \begin{bmatrix} x \\ y \\ \psi \\ v_{\mathrm{T}} \\ \alpha_{\mathrm{T}} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{m_T} & \frac{1}{m_T} \\ \frac{K_{\mathrm{F}}}{m_T v_{\mathrm{T},0}} & 0 & 0 \\ \frac{aK_{\mathrm{F}}}{I_T} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ F_{x,\mathrm{F}} \\ F_{x,\mathrm{R}} \end{bmatrix} \tag{73}$$