Linear growth factor of density perturbations in FRW cosmologies.

Andrés Nicolás Ruiz
Instituto de Astronomía Teórica y Experimental and Observatorio Astronómico
CONICET-UNC, Córdoba, Argentina

Abstract

In these notes we show how to derive and solve numerically the second order differential equation for the linear growth factor of density perturbations of first and second order ($D(a)$ and $D_2(a)$, respectively) with an arbitrary dark energy equation of state $w(a)$.

1 The equations

1.1 First order growth factor

According to Peebles (1980), the time evolution for matter perturbations is ruled by the following equation

$$\frac{\partial^2\delta}{\partial t^2}+2H\frac{\partial\delta}{\partial t}-\frac{3}{2}H\Omega_m\delta=0, \tag{1}$$

where $\delta(\vec{x},t)+1=\rho(\vec{x},t)/\bar{\rho}(t)$ is the density contrast, $H(t)$ is the Hubble parameter and $\Omega_m(t)$ is the matter density parameter.

The linear growth factor $D(t)$ is defined in terms of the evolution of the density perturbations as $\delta(\vec{x},t)=D(t)\delta(\vec{x},t_0)$, being t_0 a particular time which defines the normalization, i.e. $D(t_0)=1$. Using this, the differential equation for $D(t)$ is

$$\frac{d^2D}{dt^2}+2H\frac{dD}{dt}-\frac{3}{2}H\Omega_mD=0 \tag{2}$$

At this point it is convenient to change the time variable to the scale factor. If we do the change $t\to a$, the derivate operators change as

$$\begin{aligned}\frac{d}{dt} &= Ha\frac{d}{da}\\ \frac{d^2}{dt^2} &= (Ha)^2\frac{d^2}{da^2}+Ha\left(a\frac{dH}{da}+H\right)\frac{d}{da}\end{aligned}$$

Using this in Eq. (2) we obtain

$$\frac{d^2D}{da^2}+\left(\frac{dH}{da}\frac{1}{H}+\frac{3}{a}\right)\frac{dD}{da}-\frac{3}{2}\frac{\Omega_m}{a^2}D=0 \tag{3}$$

In a matter-dominated Universe $D=a$, so we introduce the variable $G(a)=D(a)/a$ in order to remove that dependency at early times. This quantity is known in the literature as *supression factor*. With this new variable, Eq. (3) takes the form

$$\frac{d^2G}{da^2}+\frac{1}{a}\left(5+\frac{d\ln H}{d\ln a}\right)\frac{dG}{da}+\frac{1}{a^2}\left(3-\frac{3}{2}\Omega_m+\frac{d\ln H}{d\ln a}\right)G=0 \tag{4}$$

In order to avoid the a^{-1} and a^{-2} factors in the second and third terms, respectively, it is convenient to use $\ln a$ as time variable, so

$$\begin{aligned}\frac{d}{da} &= \frac{1}{a}\frac{d}{d\ln a}\\ \frac{d^2}{da^2} &= \frac{1}{a^2}\frac{d^2}{d\ln a^2}-\frac{1}{a^2}\frac{d}{d\ln a}\end{aligned}$$

With this last change, the equation takes the final form

$$\frac{d^2G}{d\ln a^2} + \left(4 + \frac{d\ln H}{d\ln a}\right)\frac{dG}{d\ln a} + \left(3 - \frac{3}{2}\Omega_m + \frac{d\ln H}{d\ln a}\right)G = 0. \tag{5}$$

The Hubble parameter can be expressed in terms of the density parameters of the different components of the Universe. If we consider radiation (γ), matter ($m = dm + b$), curvature (K) and dark energy (DE), we have

$$H = H_0\sqrt{\Omega_\gamma^0 a^{-4} + \Omega_m^0 a^{-3} + \Omega_K^0 a^{-2} + \Omega_{DE}^0 e^{-3\int_0^{\ln a}[1+w(a')]d\ln a'}} \tag{6}$$

where the superscript 0 indicates values today and $w = w(a)$ is an arbitrary dark energy equation of state. With this, the logarithmic derivate of the logarithm of $H(a)$ can be expressed as

$$\frac{d\ln H}{d\ln a} = -\frac{1}{2}\left[4\Omega_\gamma + 3\Omega_m + 2\Omega_K + 3(1+w)\Omega_{DE}\right] \tag{7}$$

where the density parameters as a function of the scale factor are given by

$$\begin{aligned}
\Omega_\gamma &= \frac{\Omega_\gamma^0}{a^4}\frac{H_0^2}{H^2} \\
\Omega_m &= \frac{\Omega_m^0}{a^3}\frac{H_0^2}{H^2} \\
\Omega_K &= \frac{\Omega_K^0}{a^2}\frac{H_0^2}{H^2} \\
\Omega_{DE} &= \frac{\Omega_{DE}^0}{e^{3\int_0^{\ln a}(1+w)d\ln a'}}\frac{H_0^2}{H^2}
\end{aligned}$$

Using Eq. (7) in Eq. (5), and considering that $\sum_{i=\gamma,m,K,DE}\Omega_i = 1$, we obtain the final expression

$$\frac{d^2G}{d\ln a^2} + \left[\frac{5}{2} + \frac{\Omega_K}{2} - \frac{\Omega_\gamma}{2} - \frac{3}{2}w\Omega_{DE})\right]\frac{dG}{d\ln a} + \left[2\Omega_K + \Omega_\gamma + \frac{3}{2}(1-w)\Omega_{DE}\right]G = 0 \tag{8}$$

By solving Eq. (8), solutions for $G(a)$ and its derivate can be found. These two quantities are easily related with the linear growth factor and its derivate:

$$\begin{aligned}
D(a) &= \frac{G(a)}{a} && (9) \\
\frac{d\ln D}{d\ln a} &= \frac{1}{G}\frac{dG}{d\ln a} + 1 = f(\Omega). && (10)
\end{aligned}$$

1.2 Second order growth factor

Following Bouchet et al. (1995), Bernardeau et al. (2002) and the first chapter of Leclercq (2015), the differential equation for the second order growth factor takes the form

$$\frac{d^2D_2}{da^2} + \left(\frac{dH}{da}\frac{1}{H} + \frac{3}{a}\right)\frac{dD_2}{da} - \frac{3}{2}\frac{\Omega_m}{a^2}\left(D_2 - D^2\right) = 0 \tag{11}$$

and applying the same treatment as for $D(a)$, we obtain

$$\frac{d^2G_2}{d\ln a^2} + \left[\frac{5}{2} + \frac{\Omega_K}{2} - \frac{\Omega_\gamma}{2} - \frac{3}{2}w\ \Omega_{DE})\right]\frac{dG_2}{d\ln a} + \left[2\Omega_K + \Omega_\gamma + \frac{3}{2}(1-w)\Omega_{DE}\right]G_2 = \frac{3}{2}\frac{\Omega_m}{a}D^2 \tag{12}$$

where $D_2(a)$ and $f_2(\Omega) = \frac{d\ln D_2}{d\ln a}$ follow the same relations (9) and (10) in terms of $G_2(a)$.

1.3 Fitting formulas

Analytical expressions can be found in the literature for flat ΛCDM cosmologies who fit exact numerical solutions with reasonable accuracy (see Bernardeau et al. (2002)). These expressions are

$$D(a) = \frac{2}{5}\frac{a\Omega_m}{\Omega_m^{4/7} - \Omega_{DE} + (1+\frac{\Omega_m}{2})(1+\frac{\Omega_{DE}}{70})} \quad , \qquad f = \Omega_m^{5/9} \tag{13}$$

$$D_2(a) = -\frac{3}{7}D^2\Omega_m^{-1/143} \quad , \qquad f_2 = 2\Omega_m^{6/11} = 2f^{54/55} \tag{14}$$

2 The code

The code is written in C and uses some rutines from Numerical Recipies (Press et al., 2002). The package contains the following files:

- `tools.c`: Auxiliar routines.
- `cosmology.c`: Cosmological functions and routines.
- `io.c`: Read input file and set up global variables.
- `main.c`: Main program.
- `numrec.c`: Numerical Recipies routines to solve ODE and to perform integrations.

2.1 Input file

The user must define the following variables/options in the input file:

- `UseTab_wEOS`: Set to 1 for a tabulated time dependent dark energy equation of state $w(a)$ or set to 0 for a linear behaviour.
- `wEOSFile`: File with the tabulated $w(a)$. Ignored if `UseTab_wEOS` = 0.
- `OutputFile`: Name of the file for the output.
- `OmegaRadiation`: Radiation density at $z = 0$.
- `OmegaMatter`: Matter density at $z = 0$.
- `OmegaDarkEnergy`: Dark energy density at $z = 0$.
- `OmegaCurvature`: Curvature density at $z = 0$.
- `Hubble`: Hubble parameter at $z = 0$, in [Mpc km^{-1} s^{-1}].
- `w0EOS` and `waEOS`: Dark energy equation of state of the form $w(a) = w_0 + w_a(1-a)$. Ignored if `UseTab_wEOS` = 1.
- `InitialRedshift`: Initial redshift for the tabulated output.
- `FinalRedshift`: Final redshift for the tabulated output.
- `Nbins`: Number of tabulated values in the output.

If `UseTab_wEOS` = 1, the user must provide an `ASCII` table with the values for the scale factor, a, and the dark matter equation of state, $w(a)$, preferentially equally spaced in $\log(a)$.

2.2 Output file

The output is an `ASCII` table with the following tabulated quantities, equally spaced in $\log(a)$.

- `Column` #1: Cosmological scale factor, a.
- `Column` #2: Redshift, z.
- `Column` #3: Radiation density parameter, $\Omega_\gamma(a)$.
- `Column` #4: Matter density parameter, $\Omega_m(a)$.
- `Column` #5: Curvature density parameter, $\Omega_K(a)$.
- `Column` #6: Dark energy density parameter, $\Omega_{DE}(a)$.
- `Column` #7: Hubble parameter, $H(a)$.
- `Column` #8: First order linear growth factor, $D(a)$.
- `Column` #9: First order cosmological growth rate, $f(\Omega) = d\ln D/d\ln a$.
- `Column` #10: Fit for $D(a)$.
- `Column` #11: Fit for $f(\Omega)$.
- `Column` #12: Second order linear growth factor, $D_2(a)$.
- `Column` #13: Second order cosmological growth rate, $f_2(\Omega) = d\ln D_2/d\ln a$.
- `Column` #14: Fit for $D_2(a)$.
- `Column` #15: Fit for $f_2(\Omega)$.

References

Bernardeau F., Colombi S., Gaztañaga E., Scoccimarro R., 2002, , 367, 1

Bouchet F. R., Colombi S., Hivon E., Juszkiewicz R., 1995, , 296, 575

Leclercq F., 2015, arXiv e-prints, p. arXiv:1512.04985

Peebles P. J. E., 1980, The large-scale structure of the universe

Press W. H., Teukolsky S. A., Vetterling W. T., Flannery B. P., 2002, Numerical recipes in C++ : the art of scientific computing