Dictionary Learning & Compressed Sensing

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Overcomplete Dictionaries: Recap

Sparse coding with a complete dictionary:

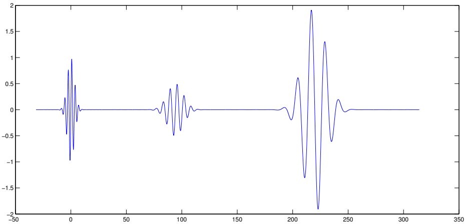
$$\mathbf{x} = \underbrace{\mathbf{U}}_{D \times D} \cdot \mathbf{z}$$

Sparse coding with an over-complete dictionary $(L > D)$:

$$\mathbf{x} = \underbrace{\mathbf{U}}_{D \times L} \cdot \mathbf{z}$$

Why over-complete? 1D example

Consider the following collection of pulses:



We need three bases of $\sin(\alpha t)\exp\left(-(t-\beta)^2/\gamma\right)$ family (Gabor bases) to reconstruct the above signal.

Why over-complete? 2D example

A collection of over-complete Gabor bases obtain a sparse representation for the following image.

Figure: Original Image

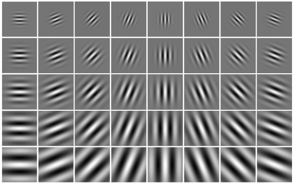


Figure: Gabor Basis

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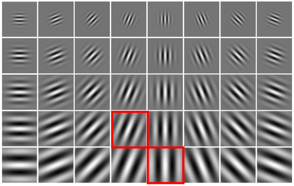

Figure: Gabor Basis

Signal Reconstruction (General Dictionary)

$\mathbf{U} \in \mathbb{R}^{D \times L}$ is overcomplete $(L > D)$:

- Ill-posed problem: more unknowns than equations.
- add constraint: find sparsest $\mathbf{z} \in \Re^L$ such that $\mathbf{x} = \mathbf{U}\mathbf{z}$

Solve mathematical program

$$\begin{aligned} \mathbf{z}^\star \quad &\in \quad \arg\min_{\mathbf{z}} \|\mathbf{z}\|_0 \\ \text{s.t.} \quad & \quad \mathbf{x} = \mathbf{U}\mathbf{z} \end{aligned}$$

- $\|\mathbf{z}\|_0$ counts the number of non-zero elements in $\mathbf{z}$.

Roadmap to Solution

Original Problem is NP-Hard: How to Proceed?

1. Use a greedy algorithm (Matching Pursuit)

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Now we learn the details!

MP Algorithm

Objective:

$$\mathbf{z}^* = \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{x} - \mathbf{U}\mathbf{z}\|_2 \quad \text{s.t.} \quad \|\mathbf{z}\|_0 \leq K$$

Algorithm:

1: $\mathbf{z} \leftarrow \mathbf{0}$, $\mathbf{r} \leftarrow \mathbf{x}$

2: **while** $\|\mathbf{z}\|_0 < K$ **do**

3: Select atom with maximum absolute correlation to residual:

$$d^* \leftarrow \underset{d}{\operatorname{argmax}} \left|\mathbf{u}_d^\top \mathbf{r}\right|$$

4: Update coefficient vector and residual:

$$z_{d^*} \leftarrow z_{d^*} + \mathbf{u}_{d^*}^\top \mathbf{r}$$

$$\mathbf{r} \leftarrow \mathbf{r} - \left(\mathbf{u}_{d^*}^\top \mathbf{r}\right) \mathbf{u}_{d^*}$$

5: **end while**

Matching Pursuit - Minimizing the Residual

Atom selection at iteration t:

$$d^*(t) = \operatorname*{argmax}_d \left|\left\langle \mathbf{r}^t, \mathbf{u}_d \right\rangle\right|$$

Proof for first iteration:

- Project $\mathbf{r}^0 = \mathbf{x}$ on atom $\mathbf{u}_d$, to get

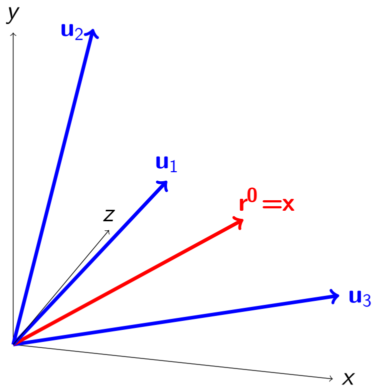
$$\mathbf{x} = \langle \mathbf{x}, \mathbf{u}_d \rangle \mathbf{u}_d + \mathbf{r}^1$$

- Since $\mathbf{r}^1$ is orthogonal to $\mathbf{u}_d$, and $\mathbf{u}_d^\top \mathbf{u}_d = 1$,

$$\|\mathbf{x}\|_2^2 = \left|\langle \mathbf{x}, \mathbf{u}_d \rangle\right|^2 + \left\|\mathbf{r}^1\right\|_2^2$$

- Therefore, $\left\|\mathbf{r}^1\right\|_2^2$ is minimized by maximizing $\left|\left\langle \mathbf{r}^0, \mathbf{u}_d \right\rangle\right|^2$.

MP Example



Bach et al. (2009)

$$\mathbf{z} = (0, 0, 0)$$

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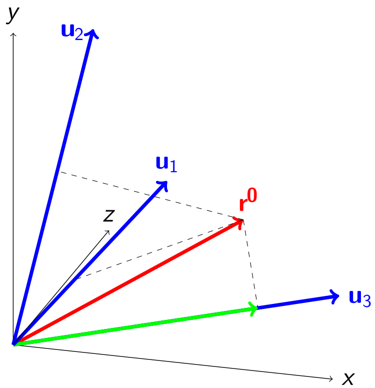

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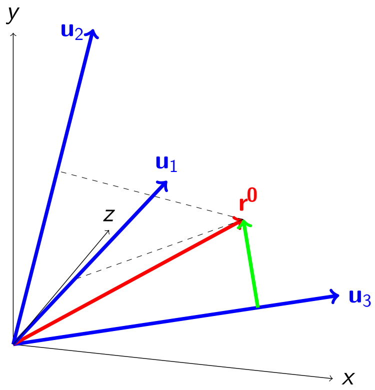

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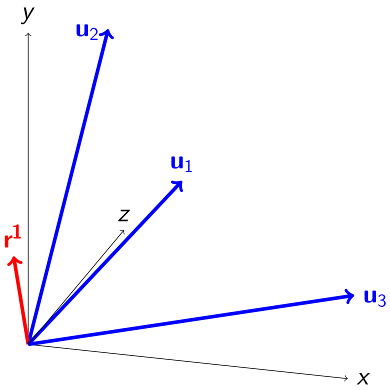

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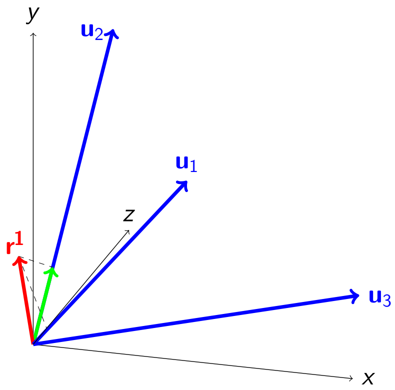

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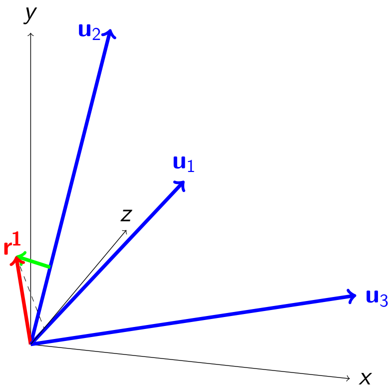

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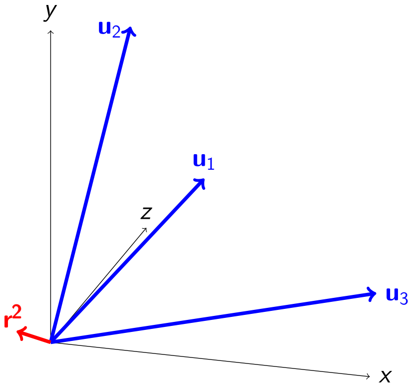
MP Example



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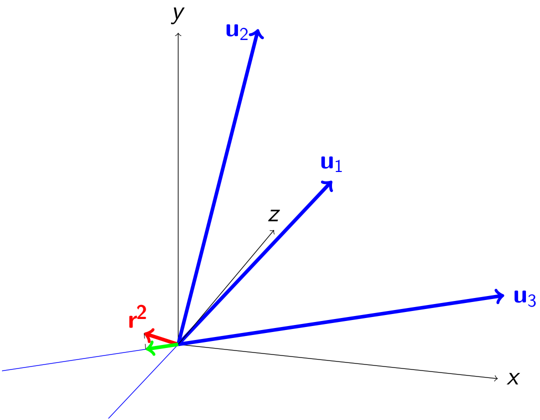

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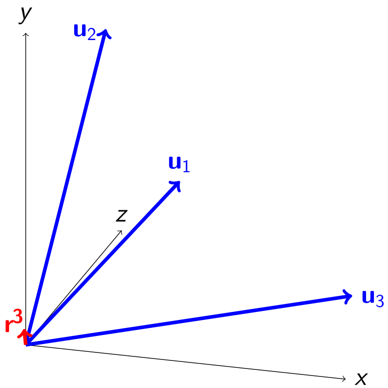

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Pen&Paper - I

$$\mathbf{x} = \left\langle \mathbf{x}, \mathbf{u}_{d(1)} \right\rangle \mathbf{u}_{d(1)} + \mathbf{r}^1$$

$\mathbf{r}^1$ is orthogonal to $\mathbf{u}_{d(1)}$

For the next step we have

$$\mathbf{r}^1 = \left\langle \mathbf{r}^1, \mathbf{u}_{d(2)} \right\rangle \mathbf{u}_{d(2)} + \mathbf{r}^2$$

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Question: Is $\mathbf{r}^2$ orthogonal to $\mathbf{u}_{d(1)}$? When is it true?
Solution:

$$\langle \mathbf{r}^2, \mathbf{u}_{d(1)} \rangle = -\langle \mathbf{r}^1, \mathbf{u}_{d(2)} \rangle \langle \mathbf{u}_{d(2)}, \mathbf{u}_{d(1)} \rangle$$

Pen&Paper - II

This is projection given that $\langle \mathbf{u}_d, \mathbf{u}_d \rangle = 1$ for every d:

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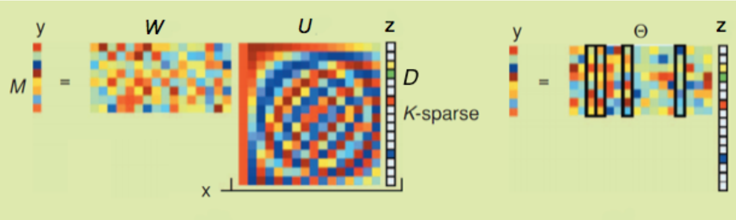
Solution:

$$\begin{aligned} \left\langle \mathbf{u}_{d(1)}, \mathbf{r}^1 \right\rangle &= \left\langle \mathbf{u}_{d(1)}, \mathbf{x} - \left\langle \mathbf{x}, \mathbf{u}_{d(1)} \right\rangle \mathbf{u}_{d(1)} \right\rangle \\ &= \left\langle \mathbf{u}_{d(1)}, \mathbf{x} \right\rangle - \left\langle \mathbf{u}_{d(1)}, \mathbf{x} \right\rangle \left\langle \mathbf{u}_{d(1)}, \mathbf{u}_{d(1)} \right\rangle \end{aligned}$$

$\mathbf{r}^1$ is no longer orthogonal to $\mathbf{u}_{d(1)}$

Compressive Sensing



$$\mathbf{y} = \mathbf{W}\mathbf{x} = \mathbf{W}\mathbf{U}\mathbf{z} =: \Theta\mathbf{z}, \text{ with } \Theta = \mathbf{W}\mathbf{U} \in \mathbb{R}^{M \times D}$$

- Surprisingly given any orthonormal basis $\mathbf{U}$ we can obtain a stable reconstruction for any K-sparse, compressible signal.
- This is true under **two conditions**:
  1. All elements $w_{i,j}$ of matrix $\mathbf{W}$ are i.i.d. random variables with a Gaussian distribution with zero mean and variance $\frac{1}{D}$.
  2. $M : M \geq cK \log\left(\frac{D}{K}\right)$, where c is some constant.

Compressive Sensing: Signal Reconstruction

- To recover initial signal $\mathbf{x} \in \mathbb{R}^D$ from measured signal $\mathbf{y} \in \mathbb{R}^M$ we need to find a sparse representation $\mathbf{z}$:

$$\mathbf{y} = \mathbf{W}\mathbf{x} = \mathbf{W}\mathbf{U}\mathbf{z} = \Theta\mathbf{z}, \quad \text{with} \quad \Theta \in \mathbb{R}^{M \times D}$$

- Given $\mathbf{z}$ we can easily reconstruct $\mathbf{x}$ by

$$\mathbf{x} = \mathbf{U}\mathbf{z}$$

- The problem of finding $\mathbf{z}$ appears to be ill-posed as $M \ll D$: many more unknowns than equations.
- Look for the sparsest solution such that equation holds:

$$\mathbf{z}^* \in \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_0, \quad \text{s.t.} \quad \mathbf{y} = \Theta\mathbf{z}$$

- NP hard problem; approximation: Matching Pursuit

Signal Reconstruction using Convex Optimization

- Sparsest solution, under the equality constraint:

$$\mathbf{z}^* \in \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_0, \quad \text{s.t.} \quad \mathbf{y} = \Theta\mathbf{z}$$

  - NP hard problem; approximation: matching Pursuit

- Minimum ℓ_1-norm solution, under the equality constraint:

$$\mathbf{z}^* \in \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1, \quad \text{s.t.} \quad \mathbf{y} = \Theta\mathbf{z}$$

  - convex Optimization Problem

Under suitable conditions on Θ, the solutions of the two problems are equivalent! $\Rightarrow$ can use standard convex optimization methods.

Geometry of Compressive sensing

A signal is k-sparse when it has at most k non zeros $\|\mathbf{z}\|_0 \leq k$. Let

$$\Sigma_k = \{\mathbf{z} : \|\mathbf{z}\|_0 \leq k\}$$

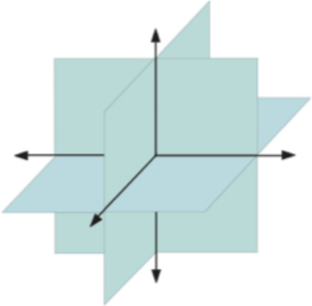


Figure: Union of subspaces defined by $\Sigma_2 \subset \mathbb{R}^3$

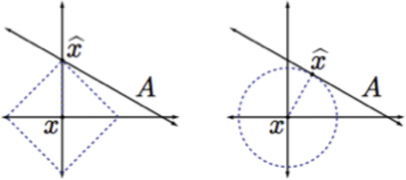
Pen&Paper

What are the geometrical solutions to the following problem for $p = 0, 1, 2$?

$$\underset{\mathbf{x}\in\mathbb{R}^2}{\operatorname{argmin}} \ \|\mathbf{x}\|_p \ \text{ s.t. } \ \langle \mathbf{w}, \mathbf{x} \rangle = 1$$

where $\mathbf{w} = [0.5, 1]$.

Pen&Paper: Answer



$p = 1$ $p = 2$

SC Code

```python
from sklearn.decomposition import sparse_encode
import matplotlib.pyplot as plt

#plot an image
plt.imshow(image)

# sparse coding
z = sparse_encode(x, U, algorithm='lasso_cd', alpha
    =100.0, max_iter=1000)
```