Calculus 1

MAST10005

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Semester 1, 2025

(version: April 9, 2025)

Version 1, March 2025

Contents

Preface

We assume some familiarity with the notion of a group (in particular cosets and quotients), modular arithmetic, and the Euclidean algorithm for integers. The book [Hun96] is very good reference for such things. We do not, at least initially, insist that multiplication in a ring should be commutative. For much of the time we shall be considering properties of polynomial rings and their quotients (in which case multiplication is commutative). There are exercises embedded in the text as well as at the end of each section. Some introduce important concepts that are used subsequently.

hello

now we are going to test for even more updates

Some advice

We assume some familiarity with the notion of a group (in particular cosets and quotients), modular arithmetic, and the Euclidean algorithm for integers. The book [Hun96] is very good reference for such things. We do not, at least initially, insist that multiplication in a ring should be commutative. For much of the time we shall be considering properties of polynomial rings and their quotients (in which case multiplication is commutative). There are exercises embedded in the text as well as at the end of each section. Some introduce important concepts that are used subsequently.

This section is not the most important part

Chapter 1

Numbers, Sets and Proofs

Mathematics is built on a foundation of logical reasoning and precise structure. In this chapter, we begin by exploring the basic building blocks of mathematical thought: numbers and sets. From there, we delve into the formal language of proofs, which allows us to establish truth through logical deduction. These fundamental ideas form the core of mathematical rigor and will serve as essential tools throughout your studies.

1.1 Sets

A **set** is a collection of objects called **elements** (or **members**) of that set.[1] The notation $x \in A$ means that x is an element of the set A. The notation $x \notin A$ is used to mean that x is not a member of A.

Let A and B be sets. We say that A is a **subset of** (or is **contained in** B), written $A \subseteq B$, if every element of A is also an element of B (i.e., if $x \in A$, then $x \in B$). Two sets are equal if they have the same members. Thus $A = B$ exactly when both $A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$ and $A \neq B$ then we say that A is a **proper subset** of B and (sometimes) write $A \subset B$.

Sets are often defined either by listing their elements, as in $A = \{0, 2, 3\}$, or by giving a rule or condition which determines membership in the set, as in $A = \{x \in \mathbb{R} \mid x^3 - 5x^2 + 6x = 0\}$.

Here are some familiar (mostly mathematical) sets:

- ▷ natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
- ▷ integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- ▷ rational numbers: $\mathbb{Q} = \left\{ \frac{y}{x} \,\middle|\, x, y \in \mathbb{Z},\ y \neq 0 \right\}$
- ▷ real numbers:[2] $\mathbb{R}$
- ▷ complex numbers: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$
- ▷ $(1, 3) = \{x \in \mathbb{R} \mid 1 < x < 3\}$
- ▷ Greek alphabet (lower case): $\{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi, \pi, \rho, \sigma, \tau, \upsilon, \phi, \chi, \psi, \omega\}$

In these examples we have the following containment relations: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ and $(1, 3) \subset \mathbb{R}$. Note that $(1, 3) \not\subseteq \mathbb{Q}$ because the interval $(1, 3)$ contains real numbers that are not rational. For example, $\sqrt{2} \in (1, 3)$ but $\sqrt{2} \notin \mathbb{Q}$.

As indicated above, the notation $\{\dots\}$ is used for set formation. Sets are themselves mathematical objects and so can be members of other sets. For instance, the set $\{3, 5\}$ consists of two elements, namely the numbers 3 and 5. The set $\{\{3, 5\}, \{3\}, \{7\}, 3\}$ consists of 4 elements, namely the sets $\{3, 5\}, \{3\}, \{7\}$ and the integer 3. Note that $7 \notin \{3, 5\}, \{3\}, \{7\}, 3\}$. Observe that $\{7\}$ is the set whose only element is the number 7, and we have that $7 \in \{7\}$ but $7 \not\subset \{7\}$.

The **empty set**, denoted by $\emptyset$, is the set that has no elements, that is, $x \in \emptyset$ is never true.

[1] In fact, more care is needed in the definition of a set. In general one must place some restriction on set formation. For example, trying to form $\{x \mid x \in x\}$ is a set) or $\{x \mid x \notin x\}$ can lead to logical paradoxes (Russell's paradox). This can be dealt with or excluded in a more formal or axiomatic treatment of set theory. We will be careful to avoid situations where this difficulty arises.

[2] It's a bit more involved to define the real numbers precisely, but one can think of them either as the points on the real line or as (infinite) decimal expansions. In this subject we will be using some standard properties of $\mathbb{R}$, but we will not give a construction.

We have a few ways of describing sets. Descriptive set notation which is referred to as a set builder notation, abbreviated set notation and using cartoons/words.

Setbuilder/Descriptive notation

$$\begin{array}{ccccc} \{ & x \in \mathbb{R} & \mid & x^2+1>37 & \} \\ \uparrow & \uparrow & \uparrow & \uparrow & \\ \text{The set of} & \text{all real} & \text{such that} & \text{this statement} & \\ & \text{numbers} & & \text{is true.} & \end{array}$$

Abbreviated set notation

We can abbreviate descriptive notation. For example the set $A = \{x \in \mathbb{R} \mid \sin(x) = 0\}$ can be written In the form

$$A = k\pi \mid k \in \mathbb{Z}$$

Again, we can read the abbreviated notation by part:

$$\begin{array}{cccccc} A \;\; = & \{ & k\pi & \mid & k \in \mathbb{Z}\} \\ & \uparrow & \uparrow & \uparrow & \uparrow \\ & \text{The set of} & \text{real numbers} & \text{such that} & \text{this statement} \\ & & \text{of this form} & & \text{is true.} \end{array}$$

Finite sets

A set is said to be finite if it only has finitely many elements. For example,

$$\{x \in \mathbb{R} \mid x^2 - 1 = 0\} = \{-1, 1\}$$

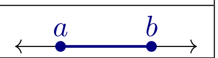
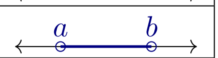
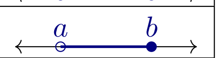
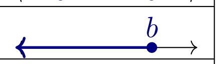
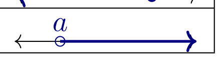
Infinite sets

An infinite set is a set that is not finite. We express infinite sets using list of elements form, by adding some dots.

$$\{(2k+1)\pi \mid k \in \mathbb{Z}\} = \{\ldots, -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi \ldots\}$$

Summary

All the information above is contained and summarised in the table below. You should pay particular attention to the graph and when an open and closed circle aswell as a round and square bracket are used.

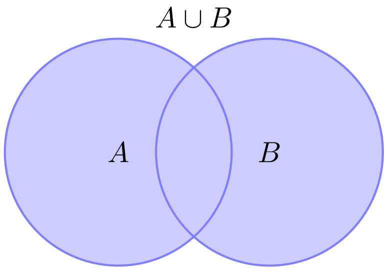
INTERVAL TYPE	SET-BUILDER	INTERVAL	GRAPH
Finite and closed	$\{x \vert a \leq x \leq b\}$	$[a, b]$	
Finite and open	$\{x \vert a < x < b\}$	(a, b)	
Finite and half-open	$\{x \vert x < a \leq b\}$	$(a, b]$	
Infinite and closed	$\{x \vert x \leq b\}$	$(\infty, b]$	
Infinite and open	$\{x \vert x > a\}$	(a, ∞)	

Set operations

There are a few set operations that we need to know. I presume that you are already mostly familiar with them. Either way the best way to understand them is to visualise them through the cartoons below.

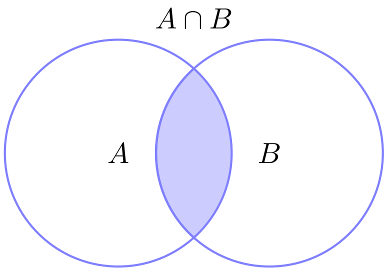
Definition 1.1 (Union of sets)**.** Let A and B be sets. The union of A and B is denoted by $A \cup B$ so that,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



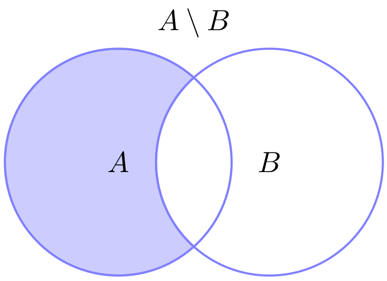
Definition 1.2 (Intersection of sets)**.** Let A and Bbe sets. The intersection of A and B is all the elements that are in both A and B. The intersection of A and B is denoted by $A \cap B$; that is,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$



Definition 1.3 (Complement)**.** If A and B are sets $A \setminus B$ is the set of elements that are in A but not in B.

$$A \setminus B = \{x \in A \mid x \in B\}$$



Cartesian product

For sets A and B, the Cartesian product $A \times B$ is the set of ordered pairs with the first from A and the second from B.

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

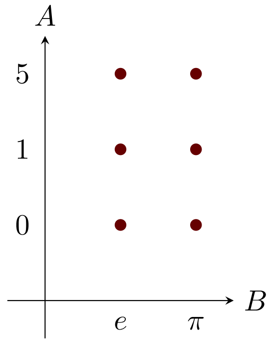
The most well known cartesian product is the *plane*, usually written as $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Example 1. (Cartesian product). List the elemts of $A \times B$ where $A = \{0, 1, 5\}$ and $B = \{e, \pi\}$

If $A = \{0, 1, 5\}$ and $B = \{e, \pi\}$, then

$$A \times B = \left\{ \begin{array}{cc} (0, e), & (0, \pi) \\ (1, e), & (1, \pi) \\ (5, e), & (5, \pi) \end{array} \right\}$$

It may help to see this graphically:



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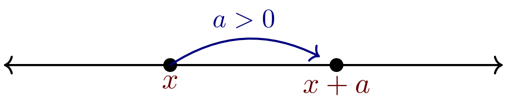
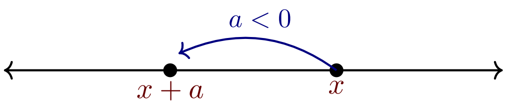
1.2 Inequalities

Although inequalities are often not discussed in elementary mathematics, they play a prominent role in Calculus. We write $a < b$ to say "a is less than b", and $a > b$ to say "a is greater than b". $a < b$ means the same as $b > a$ and are merely two ways of writing the same assertion.

Adding a number

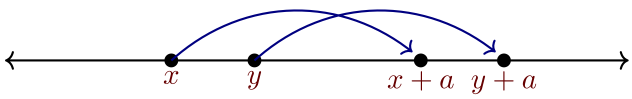
The geometric effect of adding a number a to x is to shifht right if $a > 0$ and left if $a < 0$.





As a result inequalities are therefore preserved when we add the same number on both sides.

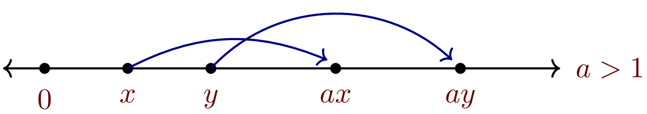
$$x < y \Rightarrow x + a < y + a.$$

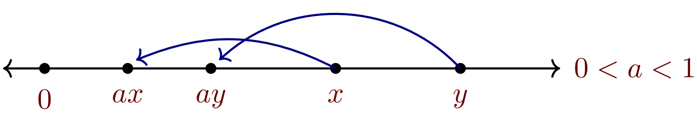


Multiplying by a positive number

Inequalities are also preserved when we multiply both sides by a positive number a. Hence the order is preserved.

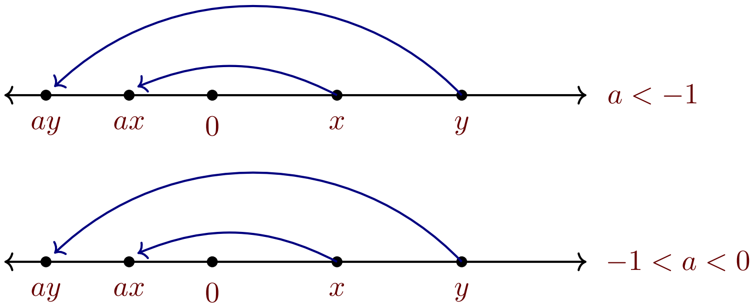
$$x < y \Rightarrow ax < ay$$





Multiplying by a negative number

Inequalities are reversed when we multiply both sides by a negative number a. Hence, the order is *reversed.* Often when people are solving inequality's the main error that pops up is forgetting flip the sign of the inequality when they multiply/divide by a negative number.



We are now equipped with all the skills we need to solving inequalities.

Example 2. (Inequalities). Solve $-2 - \frac{1}{2}x > -4$.

1. Add 2 to both sides:

$$-2 - \frac{1}{2}x + 2 > -4 + 2 \quad \Rightarrow \quad -\frac{1}{2}x > -2$$

2. Multiply both sides by -2 (note: flip the inequality):

$$x < 4$$

3. Final answer: $x < 4$

■

1.3 Quantifiers & Logic

The symbol $\forall$ means 'for all' (or 'for each' or 'for every'). It is called the **universal quantifier**. The general form of a proposition formed using the universal quantifier is

$$\forall x \in A,\ p(x)$$

where, for a given x, $p(x)$ is a statement.

The symbol $\exists$ means 'there exist' (or 'for some'). It is called the **existential quantifier**. The statement

$$\exists x \in A,\ p(x)$$

is true if there is at least one element x in A such that the statement $p(x)$ is true.

Here are some statements constructed using these quantifiers.

1. $\forall x \in \mathbb{R},\, x^2 > 0$ (which is true)
2. $\forall x \in \mathbb{R},\, (x^2 \in \mathbb{Q} \Rightarrow x \in \mathbb{Q})$ (which is false)
3. $\exists x \in \mathbb{R},\, x^2 < 0$ (which is false)
4. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\, x + y = 0$ (which is true)

Logic

Logic is important for proofs. Most of this will just be definitions and will then be applied when we do proofs.

Definition 1.4 (Implication)**.** If p and q is the statement "if p then q" we write

$$p \Rightarrow q$$

- ▷ p is the hypothesis and q is the conclusion
- ▷ if $p \Rightarrow q$ is true and p is true, then q is true

Sometimes we want to switch these statements around and consider q then p.

Definition 1.5. Let p and q be statements. We call the statement $q \Rightarrow p$ the converse of $p \Rightarrow q$.

> **Example 3**. (Logic). Write the converse of if $n \in 100$ is divisible by 4, then n is even.
>
> The converse is:
>
> If $n \in \mathbb{N}$ is even, then n is divisible by 4.
>
> ■

We will now introduce another symbol ($\Leftrightarrow$)

Definition 1.6 (Biconditional)**.** Let p and q be mathematical statements. We call the statement "p if and only if q" the biconditional of p and q, denoted by:

$$p \Leftrightarrow q$$

Chapter 2

Functions

Functions are a fundamental concept of mathematics and a cornerstone of many of its applications. Functions are often used when the value of a quantity of interest is determined by the value of other quantities; e.g. if you know the initial population of bacteria placed in a Petri dish and how long they have been in the dish, you can calculate the current population. Or if you know the mass of a projective as well as the initial velocity and angle at which it was fired, you can calculate how far it will travel.

2.1 Functions

The best way to consider functions is through a practical scenario. Let S be the set of three students A, B, C and M be the marks of available on the test

$$s = \{A, B, C\} \qquad \text{and} \qquad M = \{0, 1, \dots, 100\}$$

We find that the three students then achieve the following scores on the test

$$A \mapsto 51, \quad B \mapsto 32 \quad C \mapsto 73$$

This then has $\Gamma_f = \{(A, 51), (B, 32), (C, 73) \subseteq S \times M\}$. To setup this is in a more conventional order we can introduce the function notation that you are probably used to.

$$f : S \to M$$

Definition 2.1. A function $f : S \to M$ is composed of

- ▷ a set S called the *domain* of f
- ▷ A set M called the *codomain* of f
- ▷ Γ_f is the *graph* of $f : S \to M$

The function $f : S \to M$ is a subset

$$\Gamma_f \subseteq S \times M$$

such that

- ▷ Every student gets a mark. If $s \in S$ then there exits $m \in M$ such that $(s, m) \in \Gamma_f$.

▷ Every student get a unique mark. If $(s_1, m_1) \in \Gamma_f$ and $(s_2, m_2) \in \Gamma_f$ then $m_1 = m_2$.

The domain and codomain are part of the defining data of a function. The following two functions are *not* the same:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$
$$g : [0, \infty) \to \mathbb{R}, \quad f(x) = x^2$$

2.2 Image and Range

Let $f : A \to B$ be a function and let S be a subset of the domain A. The *iamge* of S under f is the set

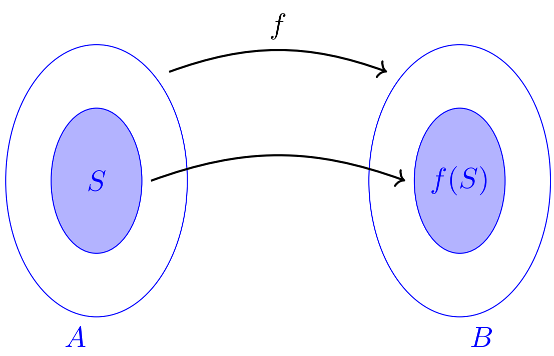
$$\begin{aligned} f(S) &= \{b \in B \mid b = f(s) \text{ for some } s \in S\} \\ &= \{f(s) \mid s \in S\} \end{aligned}$$

As a special case, we can consider the image of the entire domain A under f

$$\begin{aligned} f(A) &= \{b \in B \mid b = f(a) \text{ for some } a \in A\} \\ &= \{f(a) \mid a \in A\} \end{aligned}$$

We call this the range of f. The range is a subset of the codomain ($ranf \subseteq codom$).

The cartoon for this is below.



The range is also considered sometimes as all the values that y or to generalise it the *vertical axis* can take on. It is usually easiest to find the domain and range through graphing whatever the function may be.

We can define addition, subtraction, multiplication and division of real functions. Most of these rules will seem intuitive.

Definition 2.2. Let A be a subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$. For every $x \in A$ we define the following functions with codomain equal to R.

1. $(f + g)(x) = f(x) + g(x)$
2. $(f - g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $(f/g)(x) = f(x)/g(x)$

2.3 Types of functions

The three types of functions that we need to know are *injective, surjective* and *bijective.* We will continue on with the analogy of the students and marks example.

- ▷ Two functions $f : S \to M$ and $g : S \to M$ are *equal* if they satisfy

$$if\ \ s \in S \quad \text{then} \quad f(s) = g(s)$$

- ▷ A function $f : S \to M$ is *injective (one-to-one)* if f satisfies the condition

$$\text{if } s_1, s_2 \in S \text{ and } s_1 \neq s_2 \quad \text{then} \quad f(s_1) = f(s_2)$$

- ▷ A *surjective (onto) function* is a function $f : S \to M$ if f satisfies the condition

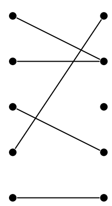
$$\text{if } m \in M \quad \text{then} \quad \text{there exists } s \in S \text{ such that } \ f(s) = m$$

- ▷ A function $f : S \to M$ is *bijective* if it is both injective and surjective.

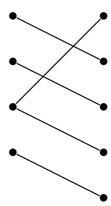
In essence if a function is *injective* then *every student receives a different mark.* If a function is *surjective* then *every mark on the test is achieved.*

Important: When the *codomain equals the range* the function f is *surjective.* To prove that a function is *injective* you need to show that for any $x.y \in \mathbb{R}$ such that $f(x) = f(y)$ then $x = y$.

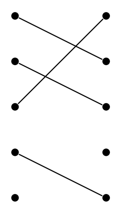
Examples. It is useful to visualize a function $f : S \to M$ as a graph which edges $(s, f(s))$ connecting elements $s \in S$ and $f(s) \in M$. With this in mind the following are examples:



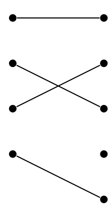
a function



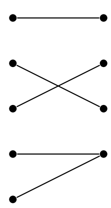
not a function



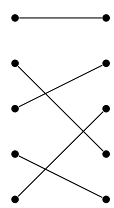
not a function



an injective function



a surjective function



a bijective function

In these pictures the elements of the left column are the elements of the set S and the elements of the right column are the elements of the set M. In order to be a function the graph must have exactly one edge adjacent to each point in S. The function is injective if there is at most one edge adjacent to each point in M. The function is surjective if there is at least one edge adjacent to each point in M.

Example 1. $g : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ defined by $g(x) = \dfrac{2x-7}{1-x}$ is injective, surjective, and bijective.

Injective: Assume $g(x) = g(y)$:

$$\frac{2x-7}{1-x} = \frac{2y-7}{1-y}$$

Cross-multiplying:

$$(2x-7)(1-y) = (2y-7)(1-x)$$

Expanding both sides and simplifying:

$$\frac{2x-7}{1-x} = \frac{2y-7}{1-y} \Rightarrow (2x-7)(1-y) = (2y-7)(1-x)$$

Expanding and simplifying:

$$2x - 7 - 2xy + 7y = 2y - 7 - 2xy + 7x \Rightarrow 2x + 7y = 2y + 7x \Rightarrow x = y$$

$\Rightarrow g$ is injective.

Surjective:
The range of $g(x)$ is

$$\mathbb{R} \setminus \{-2\}$$

. The range is $\mathbb{R} \setminus \{-2\}$ is not a subset of $\mathbb{R} \setminus \{-1\}$. Hence $g(x)$ is not surjective

Conclusion:

- ▷ g is injective
- ▷ g is not surjective
- ▷ $\therefore g$ is not bijective

■

2.4 Composite functions

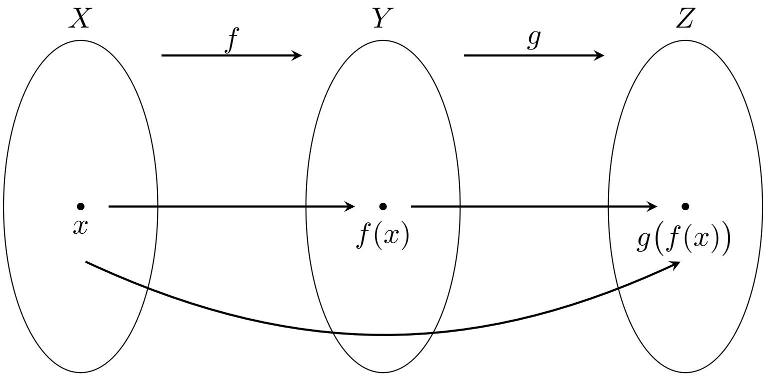
If we have two functions we may want to combine them to form a more complex function. By doing so we can take the values of one function and input them into another function. We call this process *composition* and through it we form *composite functions*.

Let $f : A \to B$ and $g : B \to C$ be functions. The *composition* of g and f is the function

$$g \circ f : A \to A$$

given by

$$(g \circ f)(x) = g(f(x))$$



Composition of functions is *rarely* commutative. So most of the time

$$f \circ g \neq g \circ f$$

For example if we have

$$f(x) = x^2 \text{ and } g(x) = e^x$$

then,

$$f \circ g = e^{2x} \quad \text{and} \quad g \circ f = e^{x^2}$$

which are clearly not the same.

For the composite function $(f \circ g)(x)$ the **domain** of the composite function is given by the domain of $g(x)$

2.5 Inverse functions

Inverse functions are quite easy to understand. If we take a function f then the inverse function is represented by f^{-1}.

Definition 2.3. Let $f : A \to B$ be a function. An inverse of f is a function $g : B \to A$ such that

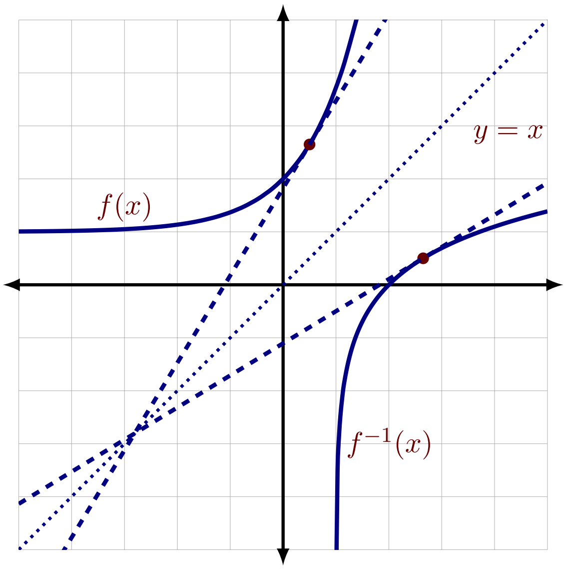
$$(g \circ f)(x) = x \quad \text{and} \quad (f \circ g)(y) = y$$

We can therefore say that f is the inverse function of g and therefore g is the inverse function of f.

Theorem 2.4 (Inverse functions)**.**

1. *A function f has an inverse if and only if f is* ***bijective****.*
2. *If a function has an inverse, that inverse is* ***unique****.*

There are a few other special features about inverse functions. We can draw any inverse function if we have the cartoon for the original function. So if we have f then f^{-1} is simply the *reflection of* f across the line $y = x$.



Chapter 3

Complex numbers

In extending the real number system to solve equations such as $x^2 + 1 = 0$, we are led to the complex numbers: expressions of the form $a + bi$, where $a, b \in \mathbb{R}$ and i satisfies $i^2 = -1$. The set of complex numbers $\mathbb{C}$ allows us to perform all familiar algebraic operations—addition, subtraction, multiplication, and division—in a consistent and complete framework.

3.1 Fundamental definitions

The exponential e^x is the most important expression in mathematics. The expansion of e^x is

$$
\begin{aligned}
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^3 + \frac{1}{5!}x^5 + \cdots \\
&= 1 + x + \frac{1}{2 \cdot 1}x^2 + \frac{1}{3 \cdot 2 \cdot 1}x^3 + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1}x^4 + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}x^5 + \cdots \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots
\end{aligned}
$$

Theorem 3.1 (expontetial)**.** *if $xy = yx$ then $e^{x+y} = e^x e^y$*

Definition 3.2 (Complex number)**.** The set of all complex numbers is given by

$$
\begin{aligned}
\mathbb{C} &= \{x + iy \mid x, y \in \mathbb{R}\} \\
&= \left\{ re^{i\theta} \mid r \in \mathbb{R}_{>0}, \theta \in \mathbb{R} \right\} \cup \{0\}
\end{aligned}
$$

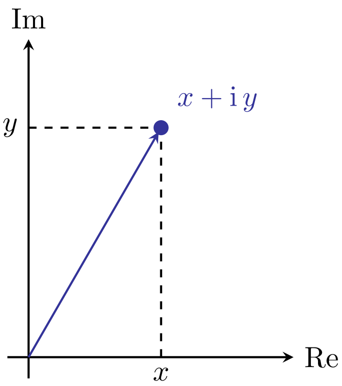
with $e^{i(\theta+2\pi)} = e^{i\theta}$ or $e^{i2\pi} = e^0$

Synonymous with complex numbers is the letter i. The key properties we need to know are $i = \sqrt{-1}$, which leads to $i^2 = -1$. We use this property to help us simplify equations and expressions. Simplifying i^5 then is:

$$
\begin{aligned}
i^5 &= i \times i \times i \times i \times i \\
&= i^2 \times i^2 \times i \\
&= -1 \times -1 \times i \\
&= i
\end{aligned}
$$

3.2 Cartesian form

A complex number given by $z = x + iy$ is in Cartesian form, which was stated on the previous page. x is the real part of z denoted by $Re(z)$ and y is the imaginary part of z and is denoted by $Im(z)$. $z = x + iy$ corresponds to the point (x, y) in the plane



We need to know multiplication, division, addition, and subtraction in Cartesian form. The laws of arithmetic and the relation $i^2 = -1$ then show that

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bd)i$$

Any complex number $a+bi \neq 0$ has a (multiplicative) inverse denoted by $(a+bi)^{-1}$ which is given by:

$$\frac{1}{a+bi} = \frac{1}{a+bi}\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$

Division will be discussed later.

3.3 Complex plane

It is clear that we can view complex numbers $z = a + bi$ as points (a, b) in the plane, where the points $(a, 0)$ on the horizontal axis are called the real axis, and the vertical axis consisting of the points $(0, b)$ is called the imaginary axis. In fact, we can use this point of view to define complex numbers ;

Definition 3.3. For any complex number $z = x+iy$ $(x, y \in \mathbb{R})$, the **conjugate** is defined by

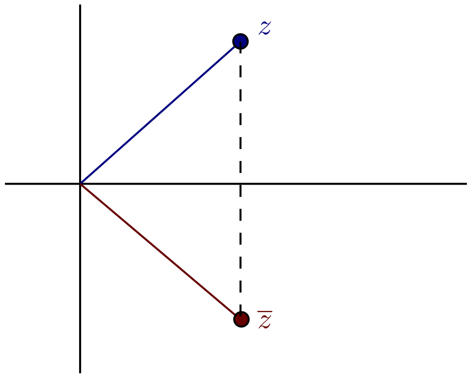
$$\overline{z} = x - iy$$

and the **modulus** (or absolute value) is defined as

$$|z| = \sqrt{x^2 + y^2}$$

The conjugate is clearly a reflection of z over the x axis. For example $\overline{2 + 3i}$ is $2 - 3i$. When we are doing division of complex numbers, as we saw above, we multiply by $a - bi$. This is the same process we would use when rationalising denominators.

To illustrate the point above consider the diagram below



Theorem 3.4 (Properties of conjugates)**.** *Let $z = x + iy$ and $w = a + ib$ be complex numbers. Then:*

1. $z + \overline{z} = 2x = 2Re(z)$
2. $z - \overline{z} = 2yi = 2Im(z)$
3. $z \cdot \overline{z} = x^2 + y^2$
4. $\overline{z + w} = \overline{z} + \overline{w}$
5. $\overline{zw} = \overline{z} \cdot \overline{w}$

We shall provide proofs to (1) and (3)

Proof. To show that $z \cdot \overline{z} = x^2 + y^2$
Assume $z = x + iy$ and $w = a + ib$
To show that the left-hand side equals the right-hand side

$$\begin{aligned}
(x + iy) \cdot (x - iy) &= x^2 - iyx + xiy - i^2y^2 \\
&= x^2 - i^2y^2 \\
&= x^2 - (-1)y^2 \qquad \text{using the fact that } i^2 = -1 \\
&= x^2 + y^2 \\
&= x^2 + y^2 \\
&= \text{right hand side}
\end{aligned}$$

□

Proof. To show that $z + \overline{z} = 2x = 2Re(z)$
Assume $z = x + iy$
So, to show that $(x + iy) + (x - iy) = 2x = 2Re(z)$
We will show that the left hand side is equal to the right hand side

$$\begin{aligned}
(x + iy) + (x - iy) &= (x + x) + (y - y)i \\
&= 2x \\
&= 2Re(z) \\
&= \text{right hand side}
\end{aligned}$$

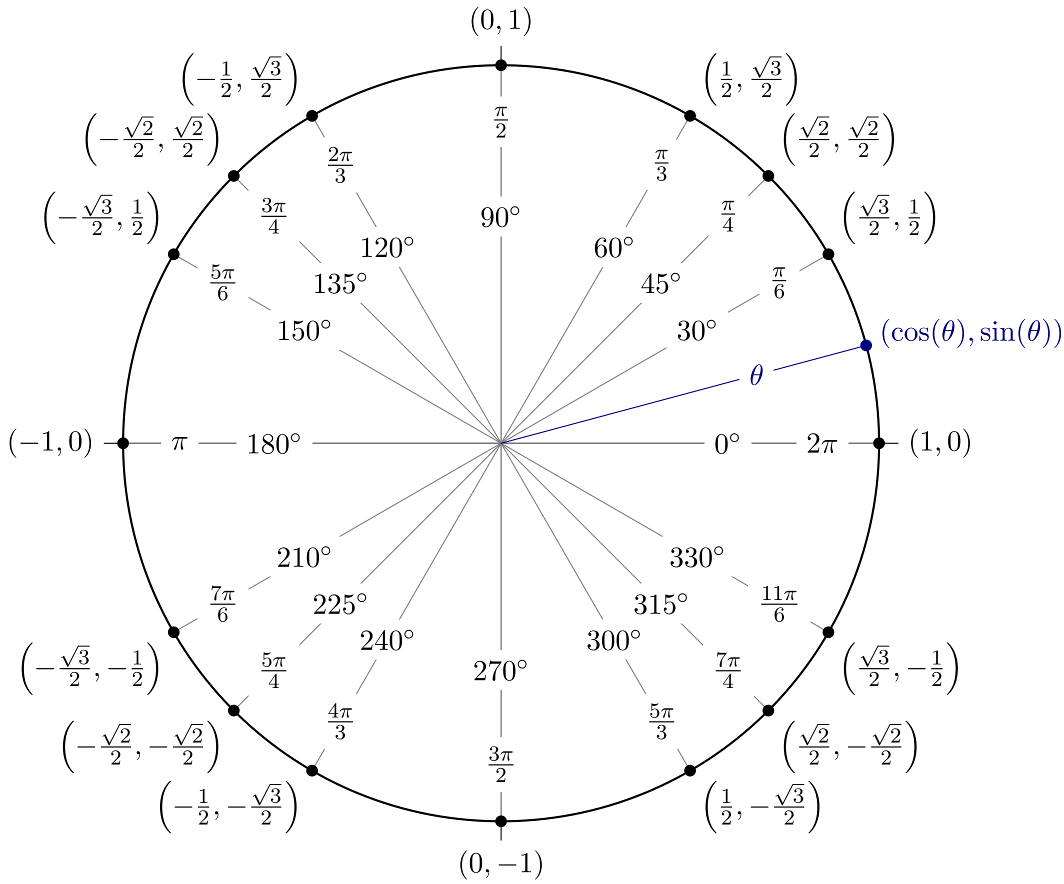
□

3.4 The tattoo

Memories

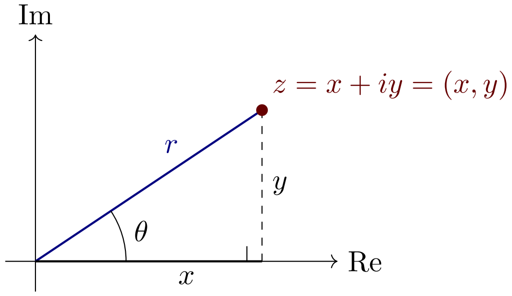
It was in the first semester of my undergraduate education at the University of Melbourne that I first met proofs. The course was MAST10005 Calculus 1. My professor was Arun Ram. He had a great effect on me. "Proof machine", "The tattoo", "Cartoons", and "Just do it" were the four constant pillars that reverberated off the walls of the JH Micallef Theatre during every lecture. These pillars, which seemed so simple, were the pillars of my entire undergraduate and now future maths career. The tattoo is the unit circle. It is one of the most important diagrams that shows us our favourite angles.

Someone asked Prof. Ram if he had a tattoo during his infamous "ask me a question time" before class. He answered with a "no". However, if he were to get a tattoo, he said it would be this. I thought that was quite funny.



3.5 Polar form

We have looked at Cartesian form. The alternative way of expressing a complex number is in polar form. There are benefits of each which will be discussed later.



From the diagram

$$\cos(\theta) = \frac{x}{r} \text{ and } \sin(\theta) = \frac{y}{r} \Rightarrow x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$
$$\Rightarrow z = x + iy = r\cos(\theta) + r\sin(\theta)i$$

Definition 3.5. The trigonometric polar form of the complex number z is:

$$z = r\left(\cos(\theta) + i\sin(\theta)\right)$$

Definition 3.6. For any complex number z where $z \neq 0$:

$$z = re^{i\theta}$$

This means that $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$. We call θ the **argument** of z. It is the angle that the complex number z makes with the real axis. "$r = |z|$" which is the *length* or *modulus* of z.

Adding or subtracting $2k\pi$ to a complex number will *not change the argument.* However there is a **special** argument in $(-\pi, \pi]$ found by adding or subtracting multiples of 2π to any argument. The principal argument is denoted by a $\text{Arg}(z)$.

We are usually interested in converting from polar form to Cartesian form and vice versa

Example 1. (Converting from Cartesian to polar form). Convert $z = -3 - 3i$ into polar form.

First we will find the modulus. So,

$$|-3 - 3i| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$

Next step is to find the argument θ of z.

$$\theta = \tan^{-1}\left(\frac{-3}{-3}\right) = \tan^{-1}(1) = \frac{-3\pi}{4}$$

so

$$z = 3\sqrt{2}e^{i\frac{-3\pi}{4}} \quad \text{or} \quad z = 3\sqrt{2}\text{cis}\left(\frac{-3\pi}{4}\right)$$

■

In case it is not clear how we got that the argument θ was $\frac{-3\pi}{4}$ it is because $\tan^{-1}(1)$ is $\frac{\pi}{4}$. However z is in the third quadrant which has the reference angle $\theta + 2\pi$ is $\frac{5\pi}{4}$. Clearly $\frac{5\pi}{4}$ is not between $(-\pi, \pi]$ so we add or subtract multiples of 2π. Subtracting 2π gives $\frac{-3\pi}{4}$.

Theorem 3.7 (Modulus and argument)**.** *Let $z, w \in \mathbb{C}$ with $w \neq 0$*

1. $|zw| = |z||w|$
2. $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$
3. *An argument zw is Arg(z) + Arg(w)*
4. *An argument of $\dfrac{z}{w}$ is Arg(z) - Arg(w)*

These properties are often overlooked but are very helpful when calculating moduli and arguments easier.

Example 2. (Moduli). Evaluate $\left|\dfrac{-2(3-i)(5+2i)}{(1+3i)(7-1)}\right|$

We will apply the property that $|zw| = |z||w|$

$$\begin{aligned}\left|\frac{-2(3-i)(5+2i)}{(1+3i)(7-1)}\right| &= \frac{|-2||3-i||5+2i|}{|1+3i||7-1|} \\ &= \frac{2\sqrt{29}}{5\sqrt{2}} \\ &= \frac{\sqrt{58}}{5}\end{aligned}$$

■

These properties also tell us that when working with complex numbers we can only perform two operations. These are *addition* and *subtraction.*

Theorem 3.8 (Working with e.)**.** *The following properties should be pretty intuitive.*

1. $e^{i0} = 1$
2. $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$
3. $\dfrac{e^{i\theta}}{e^{i\phi}} = e^{i\theta - i\phi} = e^{i(\theta-\phi)}$
4. $e^{i\theta} = e^{i\phi}$ *precisely if $\phi = \theta + 2k\pi$ for some $k \in \mathbb{Z}$*

We can now define the following.

Definition 3.9. if $z_1 = r_1e^{i\theta}$ and $z_2 = r_2e^{i\phi}$ then:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta-\phi)} \qquad \text{and} \qquad z_1 \times z_2 = r_1 \times r_2e^{i(\theta+\phi)}$$

3.6 Solving equations

The fundamental theorem of algebra

Any real polynomial $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$$

with $x \in \mathbb{R}$ and extends to a complex polynomial $q : \mathbb{C} \to \mathbb{C}$ defined by

$$q(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 x + a_0$$

which is defined for all $z \in \mathbb{C}$ has exactly n number of solutions.

Solving $q(z) = 0$ means we may also get solutions to $p(x) = 0$.

Quadratic Equations

If we consider

$$\lambda^2 + a\lambda + b = 0$$

depending on the sign of the discriminat Δ this equation has either real solutions ($\Delta > 0, \text{or } \Delta = 0$) or complex roots ($\Delta < 0$).

Since $\lambda^2 + 1 > 0$ for all $x \in R$ there cannot be a real number that solves $\lambda^2 + 1 = 0$. The name "imaginary number" reflects that at first the "number" i that solves $i^2 + 1 = 0$ was *invented.* While mysterious, this number allows us to solve every quadratic equation:

$$ax^2 + bx + c = 0$$

Formally the solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which seems to make sense even if $\Delta = b^2 - 4ac < 0$. Consider for example the case $a = b = c = 1$,so $x^2 + x + 1 = 0$, then this formula says

$$x = \frac{-1 + \sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \qquad x = \frac{-1 - \sqrt{-3}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

where we have taken $\sqrt{-3} = \sqrt{3i^2} = \sqrt{3}i$.

> **Example 3.** (Solving Quadratics). Solve $p(z) = z^2 - 3iz - 2$ over $\mathbb{C}$.
>
> For this equation $a = 1, b = -3i, c = -2$
>
> So,
> $$z = \frac{-(-3i) \pm \sqrt{(-3i)^2 - 4(1)(-2)}}{2(1)} = \frac{3i \pm \sqrt{-9+8}}{2} = \frac{3i \pm \sqrt{-1}}{2}$$
>
> Therefore,
> $$z = \frac{3i}{2} \pm \frac{i}{2}$$
>
> ■

Example 4. (Solving higher order equations 1). Solve $p(z) = z^3 - 3iz^2 - 2z$

1. Factorise $p(z)$:
$$p(z) = z(z^2 - 3iz - 2) = z(z - 2i)(z - i)$$

2. Apply the null factor law
$$z = 0, 2i, i$$

■

Example 5. (Solving cubics via factorisation). Solve $p(z) = z^3 + z^2 + z + 1$

We need to find a factor.

$$p(-1) = -1 + 1 - 1 + 1 = 0 \quad \text{so} \quad z + 1 \quad \text{is a factor of} \quad p(z)$$

We divide $z^3 + z^2 + z + 1$ by $z + 1$:

$$\begin{array}{c|rrrr}
 & z^2 & +0z & +1 & \\
\hline
z+1 & z^3 & +z^2 & +z & +1 \\
 & \underline{z^3 + z^2} & & & \\
 & 0 & +z & +1 & \\
 & \underline{z + 1} & & & \\
 & 0 & & &
\end{array}$$

So,
$$\frac{z^3 + z^2 + z + 1}{z + 1} = z^2 + 1$$

We now have $(z^2 + 1)(z + 1) = 0$ which we can solve to get

$$z = -1, -i, i$$

■

3.7 Roots

Suppose we wanted to evaluate $(1+i\sqrt{3})^8$. This would be very hard to expand in Cartesian form. Fortunately we can make use of *De Moivre's theorem* which can be used to compute the nth roots of a complex number. De Moivre's theorem only works when a complex number is in **polar form**

Theorem 3.10 (De Moivre's Theorem)**.** *Let $z = re^{i\theta}$. For any integer n:*

$$z^n = r^n e^{in\theta}$$

.

If $z = r\,cis(\theta)$ then,

$$z^n = r^n\,cis(n\theta)$$

We now evaluate $(1+i\sqrt{3})^8$. First, express $1+i\sqrt{3}$ in polar form. Its modulus is 2 and its argument is $\frac{\pi}{3}$, so we write it as $2e^{i\frac{\pi}{3}}$. Raising this to the 8th power gives:

$$(2e^{i\frac{\pi}{3}})^8 = 2^8 e^{i\frac{8\pi}{3}} = 256e^{i\frac{2\pi}{3}}$$

after reducing the angle modulo 2π.

We now convert this back to Cartesian form using Euler's formula:

$$256e^{i\frac{2\pi}{3}} = 256\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= -128 + 128i\sqrt{3}$$

Thus, $(1+i\sqrt{3})^8 = -128 + 128i\sqrt{3}$.

Calculating roots

Calculating roots is best explained through an example. Read the next page for the example. An important point to make beforehand is that when you calculate the roots, the angles are **not necessarily** expressed in principal form, so it will be your job to make sure you express them in principal argument form.

Example 6. (Finding roots). Find the set of cube roots of -8.

1. Set up the equation:
$$z^3 = -8$$

2. Convert -8 to polar form. Note that -8 lies on the negative real axis:
$$|-8| = 8, \quad \arg(-8) = \pi$$
So,
$$z^3 = 8e^{i\pi}$$

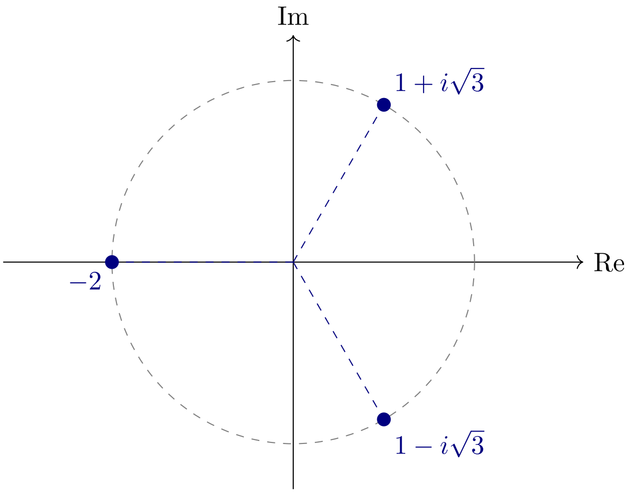
3. Add **multiples** of $2k\pi$, where $k \in \mathbb{Z}$, to the argument:
$$z^3 = 8e^{i(\pi+2k\pi)}$$

4. Apply De Moivre's Theorem to take the cube root:
$$z = 8^{1/3}e^{i(\pi+2k\pi)/3} = 2e^{i(\pi+2k\pi)/3}$$

5. Let $k = 0, 1, 2$ to find the three distinct cube roots:
$$\begin{aligned} k=0: &\quad z_1 = 2e^{i\pi/3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3} \\ k=1: &\quad z_2 = 2e^{i\pi} = 2(-1) = -2 \\ k=2: &\quad z_3 = 2e^{i5\pi/3} = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3} \end{aligned}$$

6. Final answer:
$$\boxed{z = -2, \quad 1+i\sqrt{3}, \quad 1-i\sqrt{3}}$$

We can also plot our solutions on an argand diagram.



Observe how the cube roots of -8 are evenly spaced around a circle of radius 2.

3.8 Regions in the complex plane

You may have equations that look like

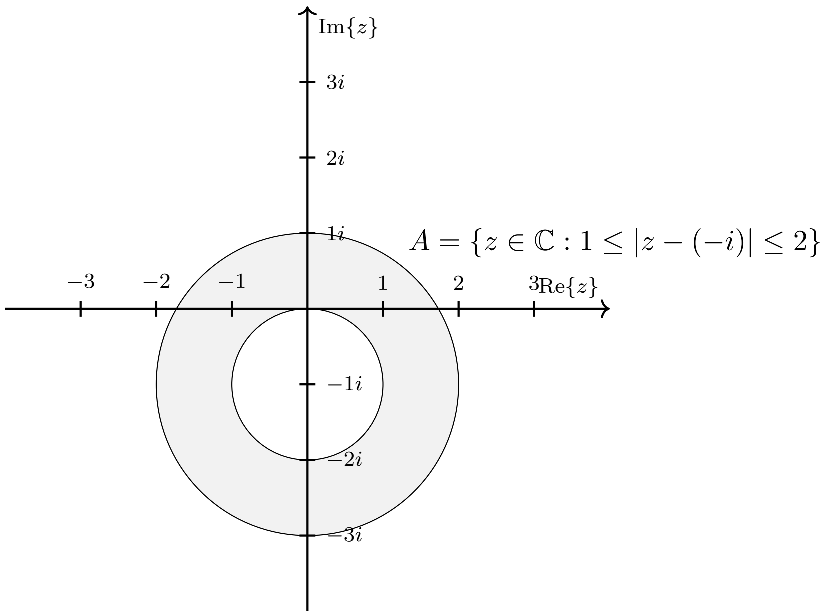
$$A = \{z \in \mathbb{C} : 1 \le |z - (-i)| \le 2\} \qquad B = \{z \in \mathbb{C} : |z + 2| = |z + i|\}$$

It can seem daunting sketching these. You can do this by finding the Cartesian equation. We want an equation in terms of x and y by substituting in $z = x + iy$.

Example 8. (Sketching complex regions). Sketch $A = \{z \in \mathbb{C} : 1 \le |z - (-i)| \le 2\}$

1. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then
$$|z - (-i)| = |x + iy + i| = |x + i(y + 1)| = \sqrt{x^2 + (y + 1)^2}$$
2. The condition becomes:
$$1 \le \sqrt{x^2 + (y + 1)^2} \le 2$$
3. Square all parts to remove the square root:
$$1 \le x^2 + (y + 1)^2 \le 4$$
4. Expand the square:
$$1 \le x^2 + y^2 + 2y + 1 \le 4$$
5. Simplify:
$$0 \le x^2 + y^2 + 2y \le 3$$
6. Geometrically, $x^2 + (y+1)^2 = r^2$ describes a circle centered at $(0, -1)$ with radius r. So the inequality defines all points whose distance from $(0, -1)$ lies between 1 and 2.

■



Chapter 4

Differentiation

4.1 The derivative

In this chapter we shall make precise one of the most important notions in Calculus, namely the *limit* of a function. We would like to say that "a function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but not equal to, a."

Here it is *irrelevant* how or even if f is defined at the point a. For example the functions

$$f(x) = x^2, \quad g(x) = x^2 \quad (x \neq a), \quad h(x) = \begin{cases} x^2 & x \neq a \\ b & x = a \end{cases}$$

should all have the same limit $l = a^2$ at a.

A way to picture what we mean by "we can make $f(x)$ as close as we like to l", is to draw the graph of f, and first choose an interval B around l, which determines two horizontal lines in the plane. Then "by requiring that x be sufficiently close to a" means that we can find an interval A around a, so that the graph of the function f above A lies between the two horizontal lines, except perhaps at a. The idea is that if "f approaches the limit l near a", this should be possible no matter how small we choose the interval B.

Example 4.1. Consider the function $f(x) = 3x$ with $a = 5$. The limit should be $l = 15$. Suppose we want to make that $f(x)$ is within $1/10$ of 15. This means we want

$$15 - \frac{1}{10} < 3x < 15 + \frac{1}{10}$$

which we can also write as

$$-\frac{1}{30} < x - 5 < \frac{1}{30}$$

or simply $|x - 5| < 1/30$. This means that as long as we take x to within $1/30$ of distance from a, $f(x)$ will be within a distance of $1/10$ from l.

Convince yourself that the function $f(x) = x^2$ approaches $l = 9$ near $a = 3$ in this sense. Suppose we would like to make $f(x)$ within distance 1 from $l = 9$. How close does x have to be to $a = 3$?

Definition 4.2. A function is **differentiable** at a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. In this case the limit is denoted by $f'(a)$ and is called the **derivative of f at a**.

Note that the *difference quotient* $((f(x+h) - f(x))/h)$ is the slope of the line through the points $(x, f(x))$ and $(x+h, f(x+h))$. Therefore define that *tangent line* to the graph of f at $(a, f(a))$ to be the line through the point $(a, f(a))$ with slope $f'(a)$.

We say f is differentiable if f is differentiable at every point on its domain. More generally, we say f is differentiable on say an interval $A = (a, b)$ (or some set of points A) if f is differentiable at every point $a \in A$, and we call the function f' the *derivative of* f on the domain A.

Example 4.3. Compute the derivative of the function $f(x) = x^2$ at $x = a$

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} (2a + h) = 2a$$

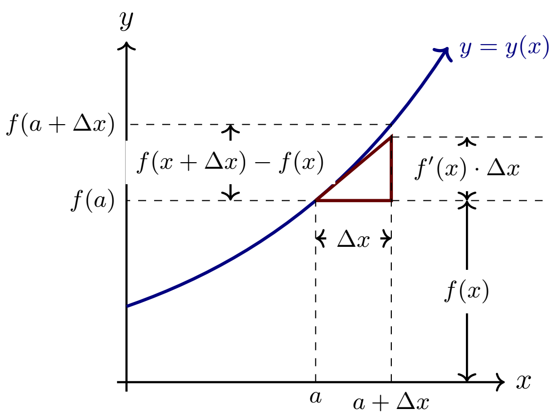


Figure 4.1: Linear approximation of a function

In terms of figure 1.1 it is clear that

$$\begin{aligned} \frac{f(a+\Delta x) - f(a)}{h} &= \frac{\text{change in } f}{\text{change in } x} \\ &= \frac{\text{rise}}{\text{run}} \\ &= \text{slope of the line connecting } (a, f(a)) \textit{ and } (a+\Delta x, f(a+\Delta x)) \end{aligned}$$

This gives that

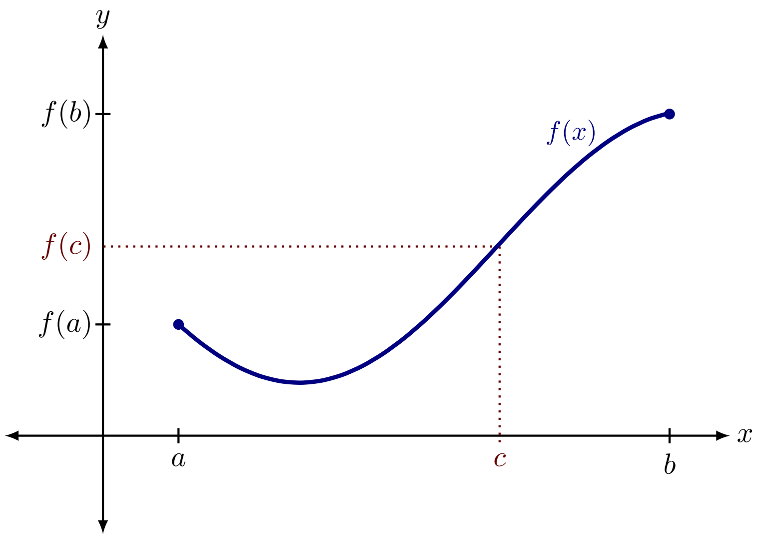
$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = (\text{slope of } f \text{ at the point } x = a)$$

4.2 Continuity and Differentiability

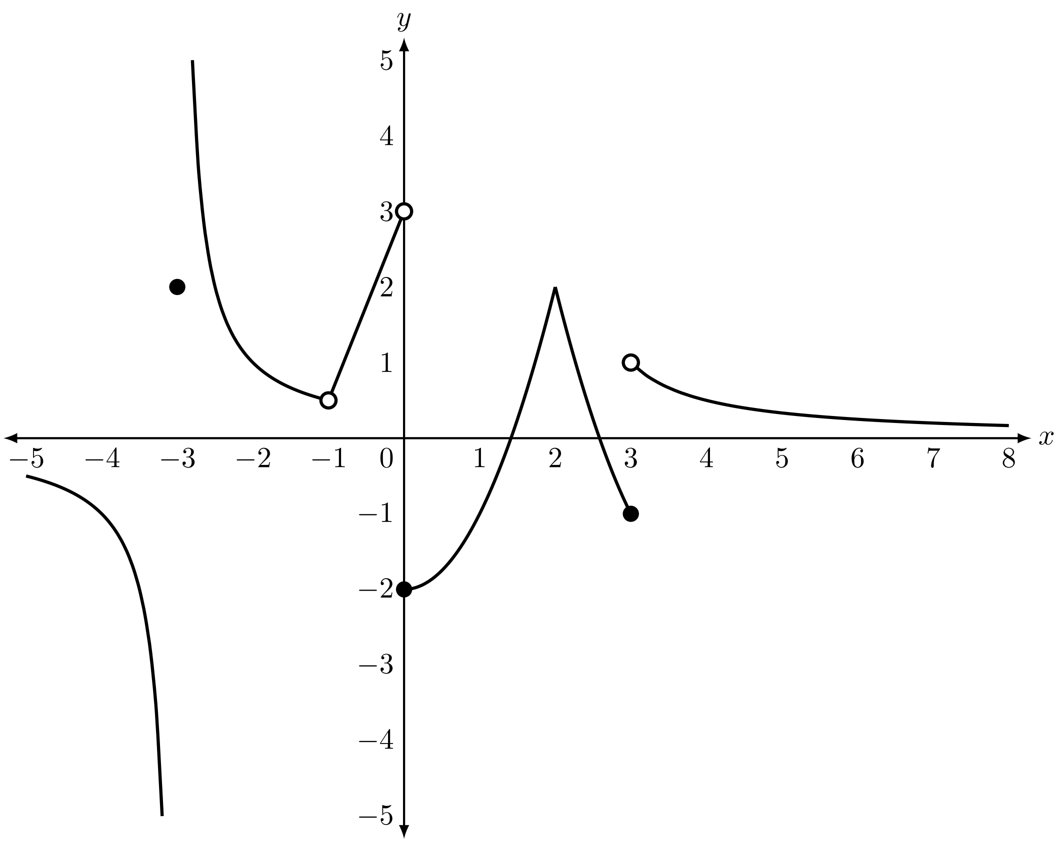
A continuous function is sometimes described, intuitively, as one whose graph can be drawn without lifting your pencil off the paper. This means that there are no breaks, jumps or wild oscillations.

Definition 4.4. A function is **continous at** c if

$$\lim_{x \to c} f(x) = f(c)$$

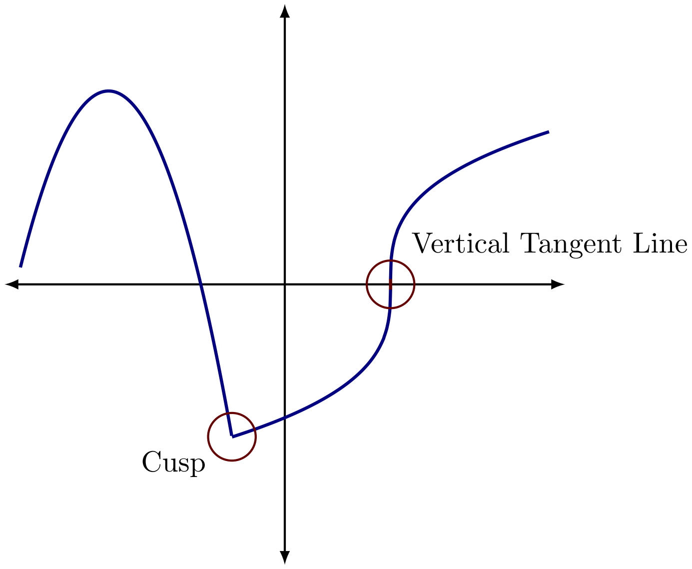


A graph that is not continuous is said to have breaks, jumps, and wild oscillations.



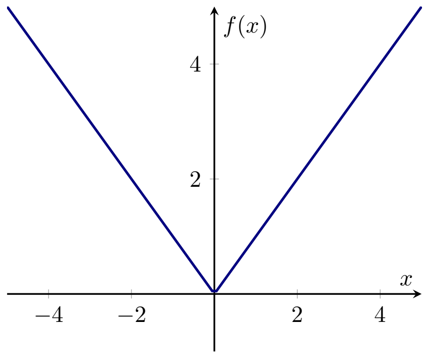
A function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable* at $x = a$ if the slope of the graph at $x = a$ exists (is a real number).

The majority of graphs are differentiable; however, there are 2 main cases for which a graph is not-differentiable. (1) the tangent is vertical and (2) the graph has a 'sharp' point or 'cusp'.



To explain sharp point it is best to consider the graph $f(x) = x$. Remember this can be expressed in piecewise form as,

$$f(x) = |cases - x, x < 0x, x \geq 0$$



Now let us consider $f'(a)$ where a represents the slope of our graph. We will consider the three cases of when $a > 0, a < 0$ and $a = 0$.

$$f(x) = \begin{cases} 1, & if\ a > 0 \\ -1, & if\ a < 0 \\ not\ real & if\ a = 0 \end{cases}$$

As we can see at $x = 0$ we do not have a real number so $f'(0)$ does not exist. Therefore we conclude that $f(x)$ is **not differentiable** at $x = 0$. Hence not all continuous graphs are differentiable

4.3 Differentiation rules

This is a brief list of all the common derivatives that you should know. The chain rule, quotient rule and product rule are discussed in the ensuing pages.

$$\frac{d}{dx}[x^a] = ax^{n-1}$$

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[\log x] = \frac{1}{x}$$

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

Theorem 4.5 (Linearity)**.** *Let f and g be differentiable at the point x. For any constant $c \in \mathbb{R}$ we have:*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \qquad \text{and} \qquad \frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Theorem 4.6 (Product rule)**.** *if f and g are differentiable at x, then so is the product fg, and*

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Theorem 4.7 (Quotient rule)**.** *if f and g are differentiable at x and if $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x and:*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'g(x) - f(x)g'(x)}{(g(x))^2}$$

Theorem 4.8 (Chain rule)**.** *if g is differentiable at x and f is differentiable at $g(x)$, then the composotion $f \circ g$ is differentiable at x and,*

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

if we let $y = f(u)$ where $u = g(x)$, the chain rule can be written

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

For the sake of simplicity take it that the function

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous. However, once again this function is not differentiable at 0: For any $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = \sin(1/h)$$

and this function does not have a limit as $h \to 0$. A very similar function, which *is* differentiable at 0, is

$$f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

However, we will see that for this function the *second derivative* fails to exist at 0.

For any function f, we obtain by taking the derivative another function f' (whose domain may be smaller than the domain of f). Clearly, now starting with the function f', we obtain another function $(f')'$ whose domain we take to be all points where f' is differentiable. This is the **second derivative** f'' of f. In general, we also write

$$\begin{aligned} f^{(0)} &= f \\ f^{(1)} &= f' \\ f^{(2)} &= f'' \\ f^{(k+1)} &= (f^{(k)})', \end{aligned}$$

and we also call $f^{(k)}$, for $k \geq 2$, the **higher order derivatives** of f. The idea is that the more derivatives of a function exist, the more *regular* it is.

4.4 Implicit differentiation

Explicit equations are of the form $y = \textit{the rest}$, for example, $y = x, y = 2x, y = \log x \cos x$. An explicit equation does *not* have the equation in terms of x. For example, $x^4 + y^2 = 1 + x^2y$. This equation would be difficult to rearrange into the form $y = \textit{the rest}$. However, we still need to know how to differentiate this so we can use the technique of **implicit differentiation**.

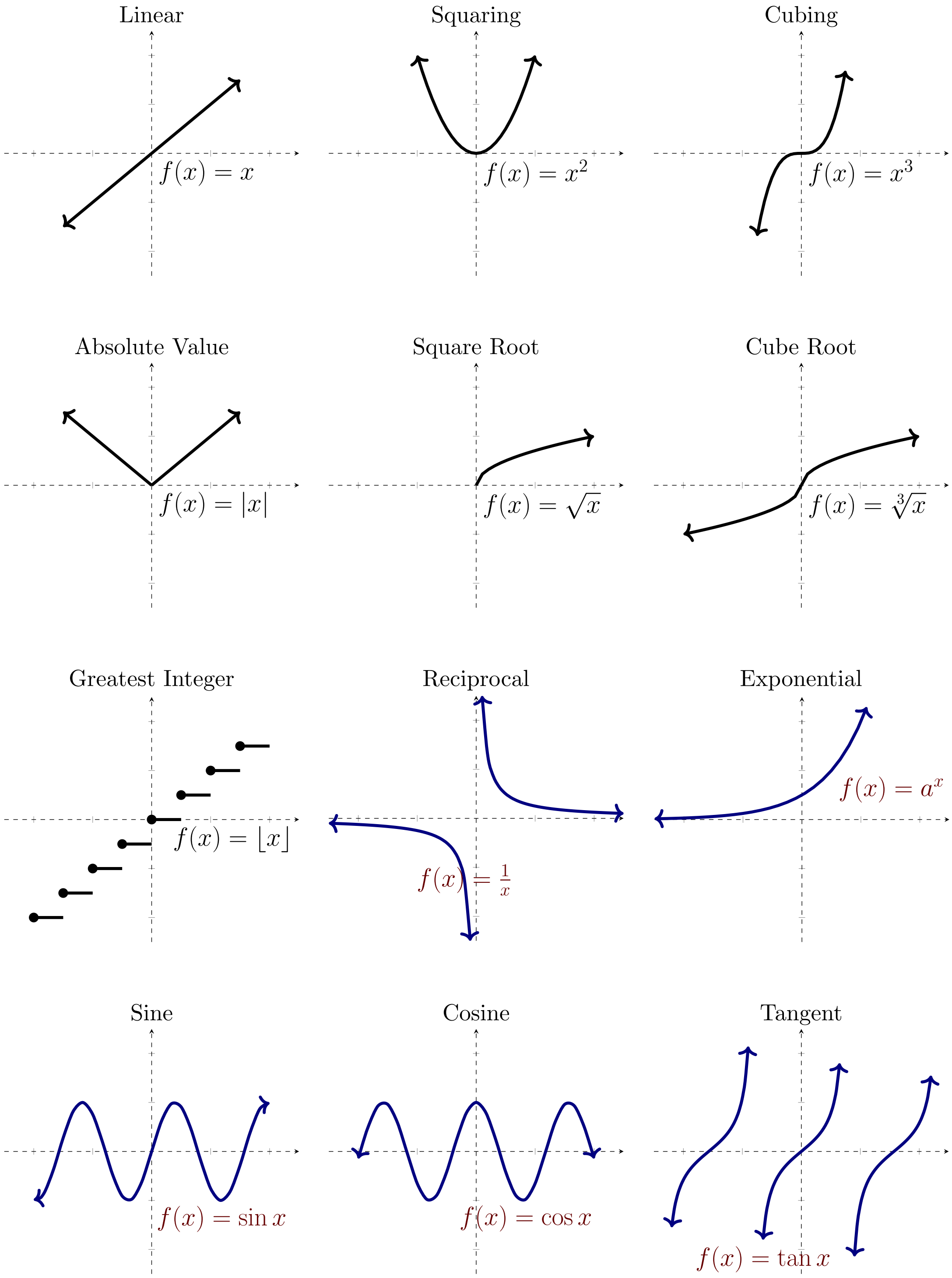
1. Differentiate each side with respect to x.
2. Use the usual differentiation rules to simplify each side.
3. Rearrange for $\frac{dy}{dx}$.

Example. $y^2 = 2x$

$$\frac{d(y^2)}{dx} = \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx}(2y) = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{y}$$

4.5 Graphing

Below is a list of very important graphs that you should know. Having a good idea of these graphs will allow you to



We will now look at some important properties to consider when graphing a function. In particular, we will focus on how to determine whether a function is increasing or decreasing at a specific point.

Increasing and Decreasing Functions

Definition 4.9 (Increasing Function)**.** A function $f(x)$ is said to be *increasing* at a point $x = a$ if the function is rising as x increases near that point. More precisely, this means that for all sufficiently small values of $\Delta x > 0$, we have

$$f(a + \Delta x) > f(a).$$

Equivalently, the function is increasing at $x = a$ if the slope of the function at that point is positive. In calculus terms, this occurs when the derivative satisfies

$$\left.\frac{df}{dx}\right|_{x=a} > 0.$$

This tells us that the tangent line at $x = a$ has a positive gradient and the function is sloping upwards at that point.

Definition 4.10 (Decreasing Function)**.** A function $f(x)$ is said to be *decreasing* at a point $x = a$ if the function is falling as x increases near that point. That is, for all sufficiently small values of $\Delta x > 0$, we have

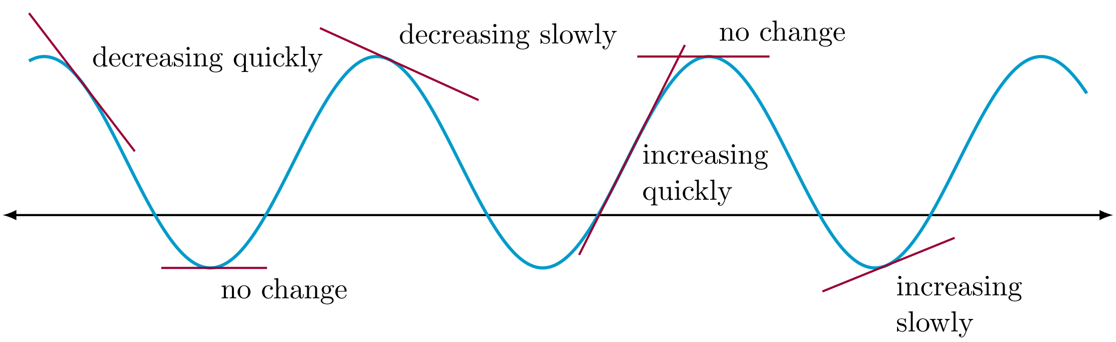
$$f(a + \Delta x) < f(a).$$

Equivalently, the function is decreasing at $x = a$ if the slope of the function at that point is negative. In terms of derivatives, this is expressed as

$$\left.\frac{df}{dx}\right|_{x=a} < 0.$$

This tells us that the tangent line at $x = a$ has a negative gradient and the function is sloping downwards at that point.

These definitions are essential when sketching graphs, as they help us understand the local behaviour of a function based on its derivative.



Stationary points

Stationary points should be though of as they are called.

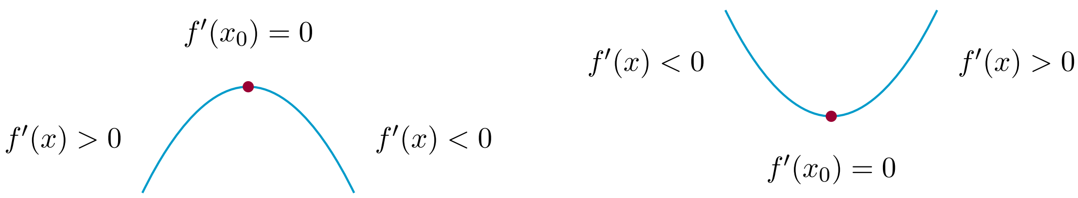
Definition 4.11. A stationary point of f is any point where $(x, f(x))$ where $f'(x) = 0$.

A local maximum is a point $x = a$ where $f(a)$ is bigger than the $f(x)$ around it.

A local minimuim is a point $x = a$ where $f(a)$ is smaller than the $f(x)$ around it.

A stationary point can be:

1. a local maximum:
2. a local minimum:



Theorem 4.12. *If $f(x)$ is continuous and differentiable and $x = a$ is a minimuim then*

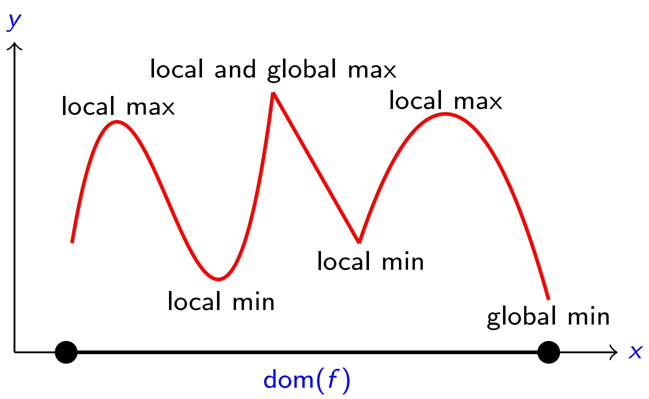
$$f'(a) = 0 \qquad \textit{and} \qquad f''(a) < 0$$

Theorem 4.13. *If $f(x)$ is continuous and differentiable and $x = a$ is a maximuim then*

$$f'(a) = 0 \qquad \textit{and} \qquad f''(a) > 0$$

Definition 4.14. A Critical point of f is a point where a maximum or minimum might occur.

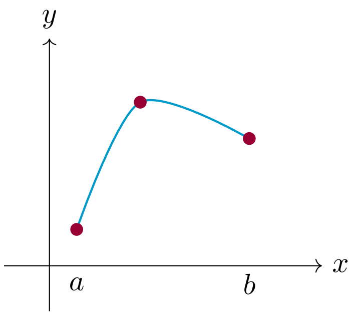
Now that we have looked at stationary points we cabn consider the *global maximum and minimum* points. These are the lowest or highest points on our graph as can be seen below.



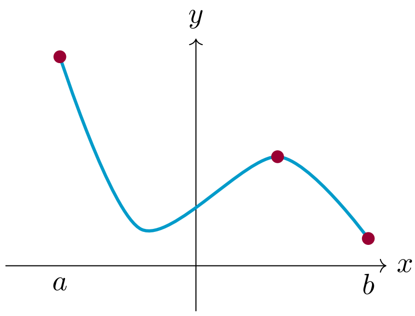
To find the global maximum and minimum you need to consider **both** the value of f at any stationary points and the value of f at any endpoints over the main $[a, b]$. The reason for this is that the global maximum and minimum may occur at the endpoints, not just the stationary points.

Endpoints

Endpoints are as they suggest. The coordinate where a graph concludes. if we consider a function f over the domain $[a, b]$ it would be mistaken to think that $f(b)$ would give the maximum point.



Global max at a stationary point inside $[a, b]$.



Global max is at endpoint a, not a stationary point.

Concavity

In mathematics, concavity refers to the direction in which a curve bends. A function is said to be concave up on an interval if its graph lies above its tangents, resembling the shape of a cup ($\cup$), and concave down if its graph lies below its tangents, resembling a cap ($\cap$)

Theorem 4.15. *Let $f : I \to \mathbb{R}$ be differentiable for I an open interval.*

1. *If f' is increasing on I, then f is concave up on I.*
2. *If f' is decreasing on I, then f is concave down on I.*

As a result, for twice differentiable functions, we can determine concavity using the second derivative.

Corollary 4.16. *Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable at $x = a$.*

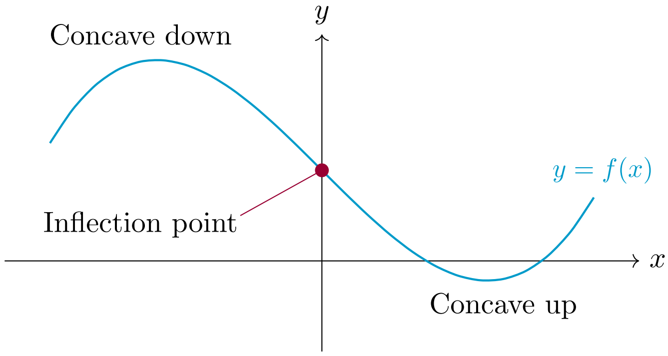
1. *If $f''(a) > 0$, then f is concave up at $x = a$.*
2. *If $f''(a) < 0$, then f is concave down at $x = a$.*

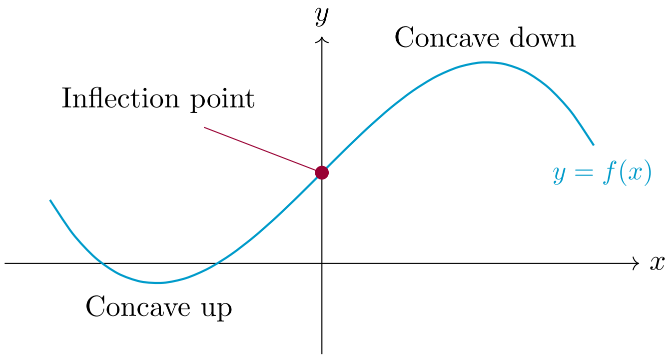
x	$-\pi$	0	π
$\sin(x)$			

At $x = -\pi$ $\sin(x)$ is concave up and at $x = \pi$ $\sin(x)$ is concave down.

Points of inflection

Definition 4.17. A point of inflection is a point in *dom*(f) where f changes between being concave up and concave down.





In order to find a point of inflection you need to find where $f''(x) = 0$. **However** you cannot just solve $f''(x) = 0$ to find a point of inflection because.

- ▷ There are cases where $f''(x) = 0$ but x is not a point of inflection
- ▷ There are cases where concavity chnages at a point missing from the domain of f''.

If you solve $f''(x) = 0$ to determine whether the point $x = a$ is a point of inflection check the value of the sign of the second derivative at either side of $x = a$ and if the sign changes then the point a is a point of inflection.

An example of this is sign in the previous page for $\sin(x)$. A point of inflection that occurs at $f''(x) = 0$ is known as a *stationary point of inflection.*

Asymptotes

Asymptotes are lines that a graph will approach. The three types of asymptotes we are concerned with are vertical, horizontal and oblique asymptotes.

Vertical asymptotes are vertical lines that have the equation $x = \alpha$. They occur as the function f approaches $\pm\infty$ as $x \to \alpha$ from either or both sides of α

Oblique asymptotes are sloping lines with the equation $y = \alpha x + \beta$, that the graph approaches as $x \to \pm\infty$.

Oblique asymptotes only occur when you have a rational function. These are functions of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p, q are polynomials. To find the oblique asymptote, you must do polynomial long division on $f(x)$. The *quotient of this division* would then give you the oblique asymptote.

Consider the function

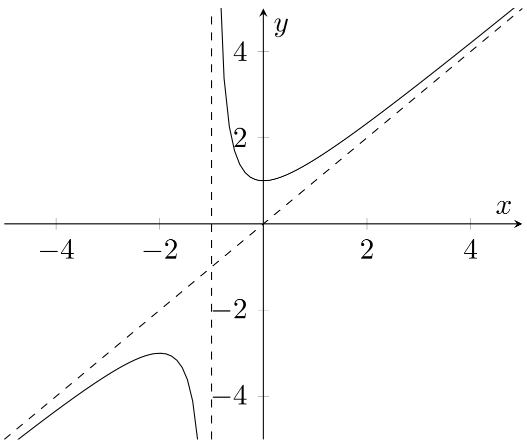
$$f(x) = \frac{g(x)}{h(x)}$$

A vertical asymptote would occur when there is a zero denominator. That is, you would solve $h(x) = 0$. Alternatively for

$$f(x) = e^x + 1$$

as $x \to \pm\infty$ $f(x)$ approaches 1, hence there is a horizontal asymptote at $y = 1$.

The graph below is a rational function composed of both a vertical and oblique asymptote.



Horizontal asymptotes are horizontal lines with the equation $y = \beta$ and occur as f approaches $\pm\infty$ from either side. They are also special versions of oblique asymptotes where $\alpha = 0$.

Chapter 5

Integral Calculus

Integration allows us to compute the area under the graph of many functions. This is useful in applications; e.g. the area under the graph of power input to an appliance versus time is the total energy used by the appliance.

Integration is the reverse of differentiation

5.1 The Fundamental Theorem of Calculus

When it comes to integration there is no bigger definition then the one below. It is what makes integration so special. Let us consider the definition.

Definition 5.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The definite integral of f over the interval $[a, b]$ is

$$\int_a^b f(x)\ dx$$

Evaluating this integral gives the **signed** area between the lines $x = a$ and $x = b$. The *signed* area means that:

- ▷ The areas under the graph above the x-axis are taken to have a positive area.
- ▷ The areas above the graph below the x-axis are taken to have a negative area.

In the integral the values a and b are the *terminals* of the integral whilst the function f is the integrated.

There is a very important relationship between integrals and derivatives. For a function $f : I \to \mathbb{R}$, we call F an **antiderivative** of f if F is differentiable on I. As such

$$\int f(x)\ dx = F(x) \qquad \text{and} \qquad F' = f(x)$$

Theorem 5.2 (Fundamental Theorem of Calculus)**.** *If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\ dx = \int_a^b F'(x)\ dx = F(b) - F(a) = [F(x)]_a^b$$

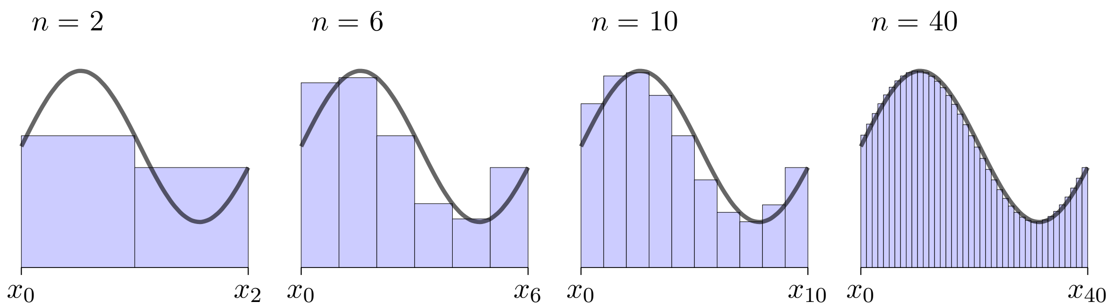
The importance of Theorem 5.2 cannot be overstated. Its implications for mathematics were, are, and will continue to be profound.

5.2 First principles

Before the integral was discovered, mathematicians were still very interested in finding the area under and between curves. The best they could do was approximate the area. One method they could use is to use vertical strips or squares.

This means dividing the area into thin rectangles of width h.

The area under the graph is the limit of this approximation as $h \longrightarrow 0$. Equivalently it is also the number of strips $\longrightarrow \infty$.

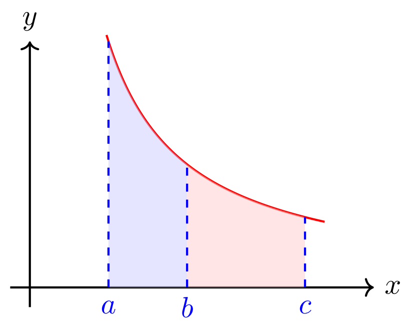


This is clearly tedious and your never going to get an absolutely correct area. This is why we make use of the fundamental theorem of calculus to find the exact area.

5.3 Properties of the integral

Thee following three rules apply to the integral and are clearer when considering the diagram to the right.

1. $\displaystyle\int_a^a f(x)\,dx = 0;$
2. $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx;$
3. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$



Property 2 is known as *additivity*.

Most of these rules are overlooked however knowing them will allow you to manipulate integrals ensuring and solve/evaluate integrals that in their base form would otherwise be tricky to solve.

Theorem 5.3 (Integration is a linear operation)**.** *Let f and g be integratable functions on $[a, b]$ and $k \in \mathbb{R}$. Then*

1. $\displaystyle\int kf(x)\ dx = k\int f(x)\ dx$

2. $\displaystyle\int (f(x) + g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$

Just like with differentiation there are several standard antiderivatives which you need to know.

1. $\displaystyle\int x^a \, dx = \frac{x^{a+1}}{a+1} + C \quad \text{for } a \neq -1$
2. $\displaystyle\int \frac{1}{x} \, dx = \log(|x|) + C$
3. $\displaystyle\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C, \quad \text{for } k \in \mathbb{R}$
4. $\displaystyle\int \sin(kx) \, dx = -\frac{1}{k} \cos(kx) + C, \quad \text{for } k \in \mathbb{R}$
5. $\displaystyle\int \cos(kx) \, dx = \frac{1}{k} \sin(kx) + C, \quad \text{for } k \in \mathbb{R}$
6. $\displaystyle\int \sec^2(kx) \, dx = \frac{1}{k} \tan(kx) + C, \quad \text{for } k \in \mathbb{R}$
7. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C$
8. $\displaystyle\int \frac{-1}{\sqrt{a^2 - x^2}} \, dx = \arccos\left(\frac{x}{a}\right) + C$
9. $\displaystyle\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

Unfortunately, there are no general rules for finding antiderivatives of products or quotients. But in the next section we will learn about integration by substitution which will enable us to find antiderivatives of products and quotients in certain special cases.

Beware: These inequalities will help you avoid common integration errors.

1. $\displaystyle\int f(x)g(x) \, dx \neq f(x) \int g(x) \, dx$
2. $\displaystyle\int f(x)g(x) \, dx \neq \int f(x) \, dx \int g(x) \, dx$
3. $\displaystyle\int \frac{f(x)}{g(x)} \, dx \neq \frac{\int f(x) \, dx}{\int g(x) \, dx}$
4. $\displaystyle\int \frac{1}{g(x)} \, dx \neq \log(|g(x)|) + C$

Further to this if we cannot express antiderivatives in terms of well-known functions for example, the integral

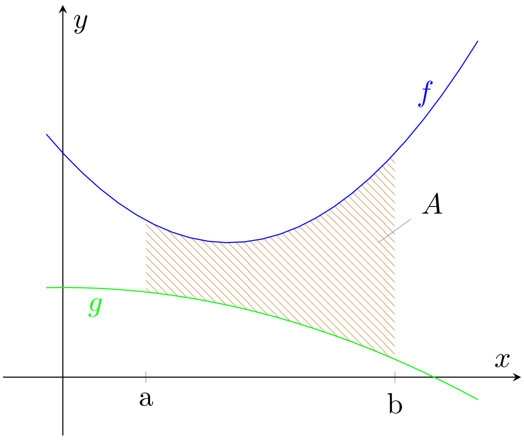
$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \ dx$$

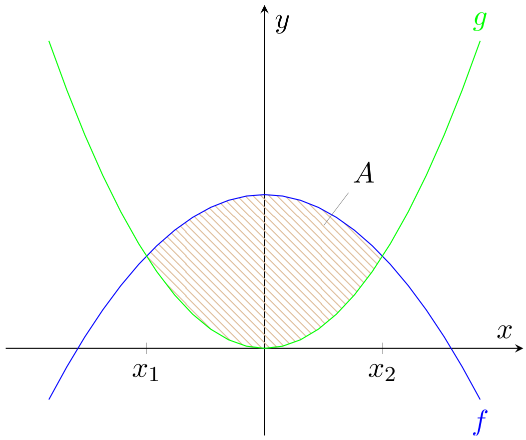
We are very unlikely to antidifferentiate this.

5.4 Area between two curves

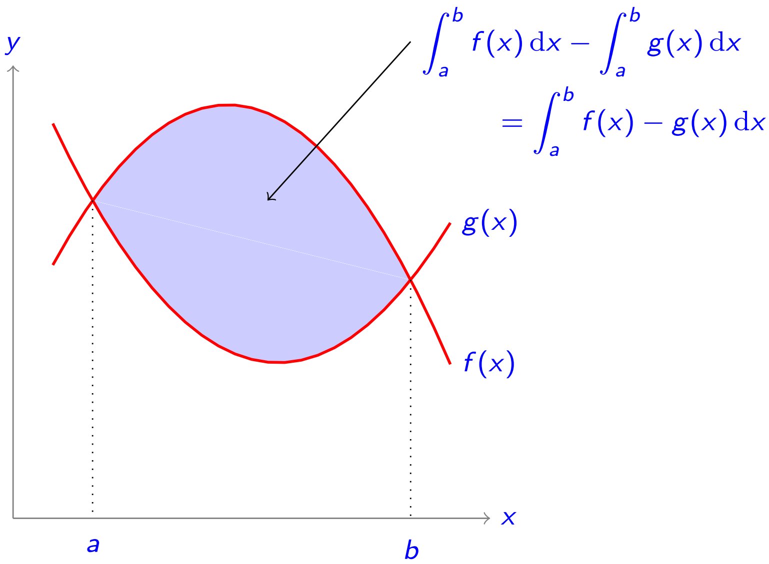
We are presented with two curves and we want to find the area between them. To do so we make use of the rule that

$$\text{Area} = \int_a^b (\text{top curve}) - (\text{bottom curve})\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$





In the case that we have two relations for which x is a function in terms of y, then we can find the area between these two curves by integrating with respect to y.



In order to find the area between two curves.

1. Find the points of intersection between the two curves.
2. Setup your integral and bounds
3. Evaluate your integral and perform the relevant technique.

5.5 Integration by Substitution

Integration by Substitution is the reverse chain rule and can be a difficult process—the topic is based on the following formula.

$$\int f(x)\ dx = \int f(x)\ \frac{dx}{du}du, \quad where\ u =' a\ function\ of\ x'$$

One easy way to understand this equation is to think of a derivative as a fraction (even though it is not a fraction). By thinking in this way, we can see that $\int f(x)\ \frac{dx}{u}du$ is equal to $\int f(x)\ dx$ because the du effectively 'cancels out'.

Example. (Integration by Substitution.) Evaluate $\displaystyle\int 6x^2 e^{3x^3}\ dx$.

1. Set
$$u = 3x^3 \quad \text{and} \quad \frac{du}{dx} = 9x^2$$

2. Subsitute $u = 3x^3$ and $\dfrac{du}{dx}$
$$\int 6x^2 e^{3x^3} = \int 6x^2 e^u \cdot \frac{1}{9x^2}\ du$$

3. Integrate
$$\int \frac{2}{3}e^u\ du = \frac{2}{3}e^u + c$$

4. Sub back in $u = 3x^3$
$$\frac{2}{3}e^{3x^3} + c$$

■

5.6 Linear Substitution

Sometimes substitution must be used in a special way - otherwise, we will be unable to eliminate x, and therefore be unable to integrate our expression. One such way is linear substitution. As implied by the name, this involves the substitution of a linear expression for u. (i.e. $u = ax + b$). Take

$$\int 3x(1-2x)^{\frac{3}{2}}$$

We cannot integrate it directly or perform normal substitution because there is no values for u and $\frac{du}{dx}$ that would simplify our expression and cancel out the remaining x terms.

Despite this if we let $u = 1 - 2x$ which gives $x = \frac{1}{2}(1-u)$ and $\frac{du}{dx} = -2$ we get

$$\int \frac{3}{2}(1-u) \cdot (u)^{\frac{3}{2}} \cdot \frac{-1}{2}\ du$$

which is in a form that we can integrate.

5.7 Integration by Parts

During the previous section we studied integration by substitution, a technique dependent on the chain rule for differentiation.

Here we consider a technique for integration related to the product rule.

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

To see this, note that we can rearrange the equation above to get

$$f(x)g'(x) = \frac{d}{dx}(f(x)g(x)) - f'(x)g(x)$$

Integrating both sides with respect to x yields

$$\begin{aligned}\int f(x)g'(x)\ dx &= \int \frac{d}{dx}(f(x)g(x)) - \int f'(x)g(x)\ dx \\ &= f(x)g(x) - \int f'(x)g(x)\ dx\end{aligned}$$

This technique is known as **integration by parts**.

When we are performing integration by parts we will mostly use the alternative formula which is

$$\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx.$$

It is very important that you pick values for u and $\frac{dv}{dx}$ that allow our working to proceed. Just like integration by substitution, if we pick the wrong values for u and $\frac{dv}{dx}$ we will end up with working that is not necessarily incorrect, but simply unable to proceed.

Example. (Integration by Parts.) Evaluate $\displaystyle\int x\cos(2x)\,dx$

1. Set:

$$\begin{aligned} u &= x & \frac{dv}{dx} &= \cos(2x) \\ \frac{du}{dx} &= 1 & v &= \frac{1}{2}\sin(2x)\end{aligned}$$

2. Apply integration by parts:

$$\int x\cos(2x)\,dx = uv - \int v\,\frac{du}{dx}\,dx = x\cdot\frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\cdot 1\,dx$$

3. Simplify:

$$\int x\cos(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

■

5.8 Integration of Trigonometric Functions

I like to think of the integration of trigonometric functions as an application of integration by substitution.

For an antiderivative like

$$\int \sin^4(x)\cos(x)\,dx$$

we recognise the composite $\sin^4(x) = (\sin(x))^4$ and let $u = \sin(x)$:

$$\begin{aligned} I &= \int \sin^4(x)\cos(x)\,dx = \int u^4 \frac{du}{dx}dx = \int u^4\,du \\ &= \frac{u^5}{5} + c = \frac{1}{5}\sin^5(x) + c. \end{aligned}$$

For a similar antiderivative like

$$\sin^4(x)\cos^3(x)\ dx$$

we can still separate one factor of $\cos(x)$ from $\cos^3(x)$ to act as our $\dfrac{du}{dx}$.

The remaining factor $\cos^2(x)$ can be expressed as

$$\cos^2(x) = 1 - \sin^2(x) \Rightarrow \cos^3(x) = (1 - \sin^2(x))\cos(x).$$

The integral then becomes

$$\int \sin^4(x)(1 - \sin^2(x))\cos(x)\ dx$$

and we can use the substitution $u = \sin(x)$

We can complete the calculation that

$$\begin{aligned} \int \sin^4(x)(1 - \sin^2(x))\cos(x)\ dx &= \int u^4(1 - u^2)\ du \\ &= \int (u^4 - u^6)\ du \\ &= \frac{1}{5}u^5 - \frac{1}{7}u^7 + c \\ &= \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + c \end{aligned}$$

Integrating $f(x) = \sin^m \cos^n(x)\ dx$

More generally this technique makes it easier to find integrals of the form

$$\int \sin^m(x)\cos^n(x)$$

Where at least one of the integers m and n is odd.

If both are odd , we can choose either $u = \sin(x)$ or $u = \cos(x)$ but a bad choice can ake the calculation much harder.

When m and n are both even we can use the complex exponential forms of sin and cos to express $\sin^m(x)\cos^n(x)$ in a form easy to integrate.

Example. Find $\int \sin^{25}(x)\cos^5(x)\,dx$.

We choose $u = \sin(x)$, so that $\frac{du}{dx} = \cos(x)$, hence $du = \cos(x)\,dx$.

$$\int \sin^{25}(x)\cos^5(x)\,dx = \int \sin^{25}(x)\cos^4(x)\cos(x)\,dx$$

Now use the identity $\cos^2(x) = 1 - \sin^2(x)$, so:

$$\cos^4(x) = (\cos^2(x))^2 = (1 - \sin^2(x))^2$$

Substitute $u = \sin(x)$, $du = \cos(x)\,dx$, and rewrite the integral:

$$= \int u^{25}(1-u^2)^2\,du$$

Now expand:

$$(1-u^2)^2 = 1 - 2u^2 + u^4$$

$$\Rightarrow \int u^{25}(1 - 2u^2 + u^4)\,du = \int (u^{25} - 2u^{27} + u^{29})\,du$$

Integrate term by term:

$$= \frac{u^{26}}{26} - \frac{2u^{28}}{28} + \frac{u^{30}}{30} + C$$

Substitute back $u = \sin(x)$:

$$= \frac{\sin^{26}(x)}{26} - \frac{2\sin^{28}(x)}{28} + \frac{\sin^{30}(x)}{30} + C$$

■

5.9 Rational functions with quadratic denominators

When anti-differentiating terms of the form

$$\frac{p(x)}{q(x)} = \frac{p(x)}{quadratic}$$

The method we use depends on how $q(x)$ factorises. There are three possible cases.

1. $q(x)$ is irreducible (does not factorise over the real numbers).
2. $q(x)$ factorises as the square of a linear factor.
3. $q(x)$ factorises as two distinct linear factors.

Cases 1 and 2 can be integrated by substitution. The third case requires a new technique called 'partial fractions' which is discussed later.

Case 1: Irreducible quadratic denominators

Take
$$\int \frac{1}{x^2+2x+2}\,dx$$

We start by examining the denominator:
$$x^2+2x+2$$

This is a quadratic expression. To determine if it can be factored over the reals, compute the discriminant:
$$\Delta = b^2-4ac = 2^2-4(1)(2) = 4-8 = -4$$
Since the discriminant is negative ($\Delta < 0$), the quadratic is irreducible over the real numbers.

We can still simplify it by completing the square:
$$x^2+2x+2 = (x+1)^2+1$$

This transforms the integral into:
$$\int \frac{1}{(x+1)^2+1}\,dx$$

Now let $u = x+1$, so that $du = dx$. The integral becomes:
$$\int \frac{1}{u^2+1}\,du$$

This matches the standard integral formula:
$$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$
In our case, $a=1$, so:
$$\int \frac{1}{u^2+1}\,du = \arctan(u)+C$$

Substituting back $u = x+1$, we conclude:
$$\int \frac{1}{x^2+2x+2}\,dx = \arctan(x+1)+C$$

If the numerator is a constant multiple of the derivative of the denominator, we can still use a simple linear substitution. Consider,
$$I = \int \frac{x+1}{x^2+2x+2}\,dx$$
If we set $u = x^2+2x+2$ then $du = (2x+2)\,dx = 2(x+1)\,dx$. So
$$I = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log(|u|) = \frac{1}{2}\log\left(|x^2+2x+2|\right)+C$$

Case 2: The denominator is the square of a linear factor

This is probably the simplest of the three cases. If we have

$$\int \frac{5}{x^2+4x+4}\,dx$$

then clearly x^2+4x+4 is $(x+2)^2$ which is a perfect square. As a result $u = x+2$ so $du = dx$, which leads to,

$$\int \frac{5}{u^2}\,du = 5\int u^{-2}\,du = -5u^{-1} + C = \frac{-5}{x+2} + C$$

5.10 Partial Fractions

Partial fractions is the opposite of 'common denominator'. It is considerably more difficult to go from a common denominator to partial fractions then it is to go from partial fractions to a common denominator.

It is often hard to integrate a single fraction with a large degree, however much easier to integrate a series of partial fractions. Consider, for example:

$$\frac{6x^4-6x^3-4x^2+12x-14}{(x-1)^3(x+2)(x^2+1)} = \frac{3}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x+2} + \frac{x-4}{x^2+1}$$

The left-hand side is virtually impossible to integrate. However, when we split it up into partial fractions, it is very easy to antidifferentiate.

When integrating a rational function $f(x) = \dfrac{p(x)}{q(x)}$ where p and q are both polynomials we can only convert $f(x)$ into partial fractions if $\deg(p) < \deg(q)$. If not polynomial long division must be performed first.

We then have three cases.

1. Linear Factors

$$\frac{\ldots}{(x+a)(x+b)(x+c)\ldots} \equiv \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$$

2. Irreducible Quadratic Factor

$$\frac{\ldots}{(x+a)(bx^2+cx+d)} \equiv \frac{A}{x+a} + \frac{Bx+C}{bx^2+cx+d}$$

3. Repeated Linear Factor

$$\frac{\ldots}{(x+k)(x+a)^2} \equiv \frac{A}{(x+a)^2} + \frac{B}{x+a} + \frac{K}{x+k}$$

Chapter 6

Differential Equations

A differential equation is a mathematical equation that relates a function with its derivatives. In *Calculus 1*, we study first-order ordinary differential equations (ODEs), especially those that model real-world systems such as population growth or cooling processes. A key technique is separation of variables, where variables are rearranged to integrate both sides. Solving a differential equation involves finding a function that satisfies the given rate of change.

More simply, a differential equation is an equation involving derivatives. For example:

$$\frac{dy}{dx} = \cos(x)$$

This differential equation is more explicitly a *first-order differential equation.* This is because the highest derivative in the differential equation is 1.

The differential equation $\frac{d^{(3)}y}{dt^{(3)}} + t\frac{dy}{dt} + (t^4 - 1)y = \sin(t)$ is a 'third order differential equation'.

Returning to the first order differential equation above solving this gives,

$$y = \sin(x) + C$$

This raises a very important point in that a typical differential equation has *infintely many solutions.* Take $C = \{1, 2, 3\}$ then

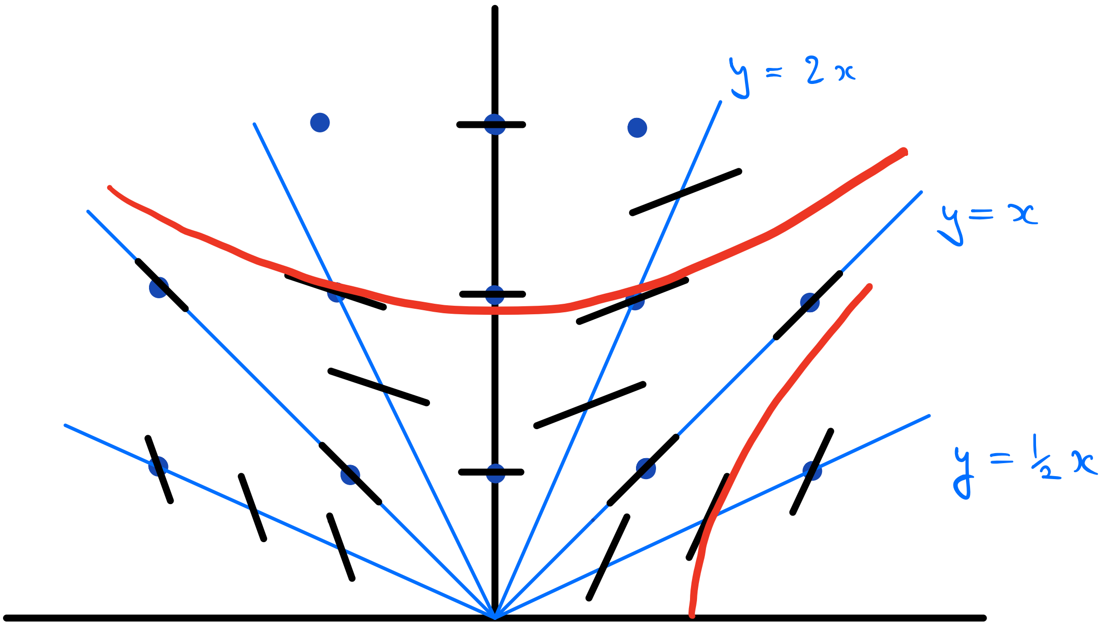
$$y = \sin(x) + 1 \qquad y = \sin(x) + 2 \qquad y = \sin(x) + 3$$

The good news is that the solutions only differ by the presence of some constant(s). The only way that we can find a *particular solution* =is when we are given an *initial condition.* Suppose we were told that $y(\frac{\pi}{2}) = 3$ then,

$$3 = \sin\left(\frac{\pi}{2}\right) + C \Rightarrow 2 = C$$

therefore the particular solution is $y = \sin(x) + 2$.

Now consider the $y' = x/y$. This another differential equation with infinitely many solutions. The way we represent these solutions is through a **direction field.** The direction field for this differential equation is seen on the next page.



The direction field clearly does not show every single solution; however, it provides a visual representation of the general behavior of solutions and helps illustrate the existence of infinitely many integral curves corresponding to different initial conditions.

6.1 Seperable differential equations

A separable differential equation is of the form

$$\frac{dy}{dx} = F(x)G(y)$$

(where F and G are known functions) are called separable. They can often be solved by a technique known as separation of variables, which we now explore.

The first order DE's we solved in the preceding examples were all separable DE's in the special case where $G(y) = 1$. In other words, they were of the (easy to solve) form

$$\frac{dy}{dx} = F(x).$$

To solve a separable differential equation, we first divide both sides by $G(y)$:

$$\frac{dy}{dx} = F(x)G(y) \Rightarrow \frac{1}{G(y)}\frac{dy}{dx} = F(x)$$

This step is often called *separating the variables.* We then attempt to integrate both sides with respect to x:

$$\int \frac{1}{G(y)}\frac{dy}{dx}\,dx = \int F(x)\,dx \Rightarrow \int \frac{1}{G(y)}\,dy = \int F(x)\,dx$$

Notice that the LHS now depends only on y whilst the RHS depends only on x.

But we know how to solve the LHS and RHS! We did this in the two special cases.

If possible we solve the resulting functional equation to obtain a formula for y (containing a constant of integration).

Example. (Seperable differential equations). $\dfrac{dy}{dx} = -6xy^2$

$$\frac{1}{y^2}\frac{dy}{dx} = -6x \Rightarrow \int \frac{1}{y^2}\ dy = \int -6x\ dx$$

Simplifiying leads to

$$\frac{-1}{y} = -3x^2 + C \Rightarrow y = \frac{1}{3x^2 + C}$$

Despite this our method for solving separable differential equations can fail at various steps:

1. We may not be able to find a formula for $\int \frac{1}{G(y)}\ dy$
2. We may not be able to find a formula for $\int f(x)\ dx$
3. After integrating both sides we may not able to solve to obtain a formula for y as a function of x.

Example. Evaluate $\dfrac{dy}{dx} = \cos\bigl(x^2\bigr)e^{-y^2}$

$$ey^2\frac{dy}{dx} = \cos(x) \Rightarrow \int e^{y^2}\ dy = \int \cos^2(x)\ dx$$

Which cannot be integrated so no solution to this DE.

You should also be careful for cases where you have a serpable differential equation in disguise. For example:

$$\frac{dy}{dx} = y^2\sin(x) + y^2 - 4\sin(x) - 4$$

The DE is not in a form that you could seperate the variables nor would you think to. After simple factoriation you are lead to,

$$\frac{dy}{dx} = (y^2 - 4)(\sin(x) + 1)$$

which is now in a form that you can easily recognise as a separable differential equation.

6.2 Autonomous differential equations

We next consider the important special case of separable DE's where $F(x) = 1$ before moving to the general case. These are first-order DE's of the form:

$$\frac{dy}{dx} = G(y)$$

are called autonomous.

We can't simply integrate both sides with respect to x in this case because the RHS is a function of y, not x. However, provided $G(y) \neq 0$, we can divide both sides by $G(y)$:

$$\frac{1}{G(y)}\frac{dy}{dx} = 1 \tag{6.1}$$

and now we can integrate by substitution on the LHS. What happens when $G(y) = 0$ will be discussed later.

Integrating both sides of equation (6.1) with respect to x, using substitution on the LHS:

$$\int \frac{1}{G(y)}\frac{dy}{dx}\,dx = \int 1\,dx \Rightarrow \int \frac{1}{G(y)}\,dy = x + C.$$

Provided we know how to integrate $\dfrac{1}{G(y)}$ with respect to y, this gives a functional equation relating x and y.

Typically, we can then rearrange this functional equation to find the formula for y as a function of x.

Lets consider a very basic yet extremely important example.

Example. (Autonomous differential equations). $\dfrac{dy}{dx} = y$

$$\frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \int \frac{1}{y}\frac{dy}{dx}\ dx = \int 1\ dx \Rightarrow \log(y) = x + C$$

We want to express y in terms of x so simplifying our equation:

$$e^{\log(|y|)} = e^{x+C} \Rightarrow y = e^x e^C \Rightarrow y = Ce^x$$

Note: You must be very careful about when to include constants of integration. Just adding one at the end may give the wrong answer.

6.3 Constant Solutions

In our general method for solving separable differential equations (DEs), the first step typically involves dividing both sides of the equation by $G(y)$. However, this step assumes that $G(y) \neq 0$ for the values of y we are interested in.

But what happens when $G(y) = 0$ for some value of y? Let's suppose $G(y_0) = 0$ for some specific value $y = y_0$. In that case, the original separable DE

$$\frac{dy}{dx} = F(x)G(y)$$

reduces to

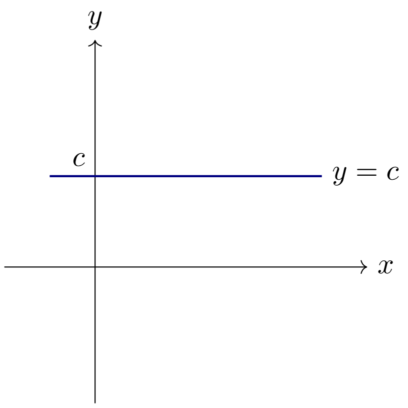
$$\frac{dy}{dx} = F(x) \cdot 0 = 0,$$

when $y = y_0$. This differential equation is trivially simple to solve, since

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad \int \frac{dy}{dx}\,dx = \int 0\,dx \quad \Rightarrow \quad y = C,$$

where C is a constant.

This means that any value y_0 for which $G(y_0) = 0$ gives rise to a **constant solution** to the differential equation. These are often called *equilibrium solutions*, and they correspond to horizontal lines in the direction field of the DE.



In this case, the solution is a horizontal line: a *constant function*. Such solutions often represent steady states in physical systems, where the rate of change is zero. Identifying these is helpful because they provide immediate, exact solutions without any integration required.
To find constant solutions in practice, we solve $G(y) = 0$ directly. This gives the y-values where the right-hand side of the DE vanishes, leading to $\frac{dy}{dx} = 0$.

Example.(Constant solutions). Consider the DE $\dfrac{dy}{dx} = \sin(y)$

$$\frac{dy}{dx} = \sin(y).$$

Constant solutions occur when $\sin(y) = 0$, that is:

$$y = n\pi \quad \text{for all integers } n.$$

This equation has an infinite number of constant solutions: $y = 0$, $y = \pi$, $y = -\pi$, $y = 2\pi$, and so on. Each of these corresponds to an equilibrium where the solution remains fixed in time.

Conclusion. Constant solutions are valuable for quickly identifying stable or unstable behaviors in differential systems. Recognizing when $G(y) = 0$ can help bypass the need for full integration and

6.4 Verifying solutions to differential equations

Consider the equation

$$x^4 + 3x^3 = 5 - x^5$$

If we wanted to check that $x = 1$ is a solution, we would substitute it into the left and right hand side of the equation, so $(1)^4 + 3(1)^3 = 5 - (1)^5$ and so $4 = 4$, so $x = 1$ is a solution.

Therefore, to check a proposed solution to a differential equation, we substitute $f(x)$ and the specified derivatives of f into both sides of the differential equation and see if it works for all x-values.

As a result this process involves calculating some derivatives of $f(x)$.

Example. Verify that $y = e^{3x}$ is a solution of the DE: $\dfrac{d^2y}{dx^2} = 15y - 2\dfrac{dy}{dx}$

1. We begin by computing the first and second derivatives of $y = e^{3x}$:

$$\frac{dy}{dx} = 3e^{3x}, \qquad \frac{d^2y}{dx^2} = 9e^{3x}$$

2. Substitute y, $\dfrac{dy}{dx}$, and $\dfrac{d^2y}{dx^2}$ into the differential equation:

$$\frac{d^2y}{dx^2} = 15y - 2\frac{dy}{dx} \Rightarrow 9e^{3x} = 15e^{3x} - 6e^{3x}$$

$$9e^{3x} = 9e^{3x}$$

Since both sides are equal, we have verified that $y = e^{3x}$ is indeed a solution to the differential equation.

When we are asked to *verify* or *prove* that a given function is a solution of a DE, there is no need to solve the DE. In fact, it is logically incorrect to do so.

Chapter 7

Vectors

There are many different types of quantifiable objects, for example, time, length, speed and height. We can quantify these things with a single (real) number which we call a scalar.

Other quantities, however, require more than just a single magnitude to be completely specified, for example: displacement, velocity and force. These quantities also have a direction. We call these quantities vectors.

In this section, we introduce the notion of vectors, explore their basic properties and operations, learn about some applications of vectors and briefly discuss functions whose values are vectors.

7.1 Vectors in n - dimensional spaces

Recall from chapter 1 that

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a_1, a_2) \mid a_1 \in \mathbb{R} \text{ and } a_2 \in \mathbb{R}\}$$

Definition 7.1. An n-dimensional space over a set is the collection of all n-tuples of elements from that set, written using circular brackets. For example:

$$\begin{aligned}
\mathbb{R}^n &= \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}, \\
\mathbb{C}^n &= \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{C}\}, \\
\mathbb{Z}^n &= \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{Z}\}.
\end{aligned}$$

In this chapter, a vector is simply an element of $\mathbb{R}^n$. For example

- ▷ $(3, 1, 0)$ and $(-1, -1, -1)$ are vectors in $\mathbb{R}^3$;
- ▷ $(0, 0)$ and $(1, 5)$ are vectors in $\mathbb{R}^2$;
- ▷ $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$ is a vector in $\mathbb{R}^{10}$.

It is very important to remember that all vectors have magnitude and direction.

7.2 Vector operations

We will explore the operations of addition, subtraction, scalar multiplication and then introduce the inner product aswell as unit vectors.

Addition

Definition 7.2 (Addition)**.** Addition is the function

$$\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$$

So when you add two vectors together a vector is spurted out. We add vectors component wise.

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

Scalar multiplication

Definition 7.3 (Scalar multiplication)**.** Scalar multiplication is the function

$$\mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$$

So when you multiply a vector by a scalar multiple a vector is spurted out. In principles this is.

$$\lambda(a_1, a_2, \dots, a_n) = (\lambda a_1, \lambda a_2, \dots, \lambda a_n) \qquad [\lambda \in \mathbb{R}]$$

Subtraction is Addition and Scalar multiplication. This is seen in the example below.

Example. If $\vec{v} = (3, -2)$ and $\vec{w} = (1, 5)$ then,

$$\vec{v} + \vec{w} = (3 + 1, -2 + 5)$$

and then,

$$3\vec{w} = (3 \cdot 1, 3 \cdot 5) = (3, 15).$$

Subtraction gives,

$$\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w} = (3, -2) + (-1, -5) = (2, -7)$$

The inner product

The inner product, also commonly referred to as the 'dot product/scalar product', is an operation that takes in a pair of vectors and spits out a scalar. We denote it by using 'inner brackets $\langle\rangle$'.

Definition 7.4 (The inner product)**.** The inner product is the function

$$\langle\rangle : \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R} \qquad \text{given by}$$

$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle = a_1 b_1 + \cdots + a_n b_n$$

The reason that the inner product is also referred to as the 'dot product' is because if we take the vectors $\vec{v}$ and $\vec{w}$ again, then there dot product is simply

$$\vec{v} \cdot \vec{w} = a_1 b_1 + \cdots + a_n b_n$$

Example: Let $\vec{u} = (2, 3, -1)$ and $\vec{v} = (4, 5, 0)$, then find the inner product.

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} &= \langle (2, 3, -1), (4, 5, 0) \rangle \\ &= (2 \cdot 4) + (3 \cdot 5) + (-1 \cdot 0) \\ &= 23 \end{aligned}$$

The Norm

The norm gives us the length of a vector. It returns a scalar. We denote it by using double lines.

Definition 7.5 (The norm)**.** The Norm is the function

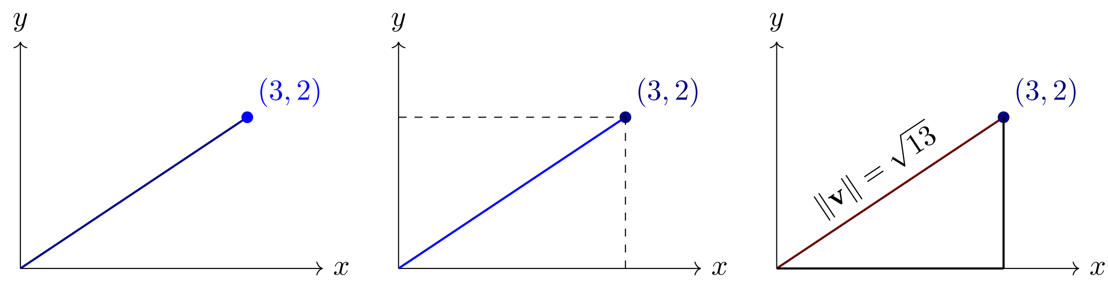
$$\|\| : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0} \qquad \text{given by}$$

$$\|\langle (a_1, \ldots, a_n) \rangle\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Example. Find the norm of $\vec{v} = (1, 2)$.

$$\|\langle (-1, 3, 2) \rangle\| = \sqrt{-1^2 + 3^2 + 2^2} = \sqrt{15}$$

Calculating the norm of vectors is simply an application of Pythagoras' theorem. If we take the vector $(3, 2)$ then we can calculate the norm as follows.



Plot $(3, 2)$ Draw a triangle Calculate the norm.

The Metric

Definition 7.6 (The metric)**.** The *metric* is the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned} d\big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) &= \|(b_1, \ldots, b_n) - (a_1, \ldots, a_n)\| \\ &= \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2} \end{aligned}$$

This is also called the distance between a and b. For vectors $\vec{A}$ and $\vec{B}$ then then the vector $\vec{AB}$ is $\vec{B} - \vec{A}$

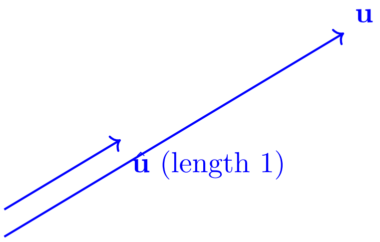
Example. Let $P = (\vec{-}2, 1, 0)$ and $\vec{Q} = (3, -1, 1)$, then

$$d(\vec{P}, \vec{Q}) = \|\vec{PQ}\| = \|Q - P\| = \|(5, -2, 1)\| = \sqrt{30}$$

Unit vectors

A *unit vector* is a vector of length 1. Unit vectors can be constructed in any direction.

$$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$



Example. Let $\vec{v}$ be the vector from $A = (2, 0, -1)$ to $B = (1, 2, -3)$. Find the unit vector of $\vec{v}$.

First, compute the vector $\vec{v}$:

$$\vec{v} = \overrightarrow{AB} = \langle 1-2,\ 2-0,\ -3-(-1)\rangle = \langle -1,\ 2,\ -2\rangle$$

Next, find the magnitude of $\vec{v}$:

$$\|\vec{v}\| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

Now, compute the unit vector $\hat{\vec{v}}$:

$$\hat{\vec{v}} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}\langle -1,\ 2,\ -2\rangle = \left\langle -\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right\rangle$$

We can check that this is true by calculating the magnitude of $\hat{\vec{v}}$:

$$\|\hat{\vec{v}}\| = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = 1$$

So, the unit vector $\hat{\vec{v}}$ is indeed a vector of length 1.

The Zero vector

The zero vector is the vector with all components equal to zero. It is denoted by $\mathbf{0}$ and from a geometric point of view, it is the vector with zero length and no direction.

So in $\mathbb{R}^n$ the zero vector is $\langle 0, 0, \ldots, 0\rangle$.

- In $\mathbb{R}^2$ the zero vector is $\langle 0, 0\rangle$.
- In $\mathbb{R}^3$ the zero vector is $\langle 0, 0, 0\rangle$.
- In $\mathbb{R}^{10}$ the zero vector is $\langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\rangle$.

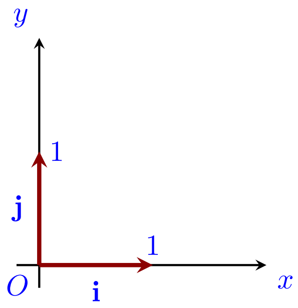
7.3 The standard basis

The standard basis of $\mathbb{R}^2$ is

$$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\} \quad \text{where} \quad \hat{\mathbf{i}} = (1, 0) \quad \text{and} \quad \hat{\mathbf{j}} = (0, 1).$$

Every vector in $\mathbb{R}^2$ can be written in terms of $i - j$ components. For example, the vector $\vec{v} = (3, 2)$ can be expressed as:

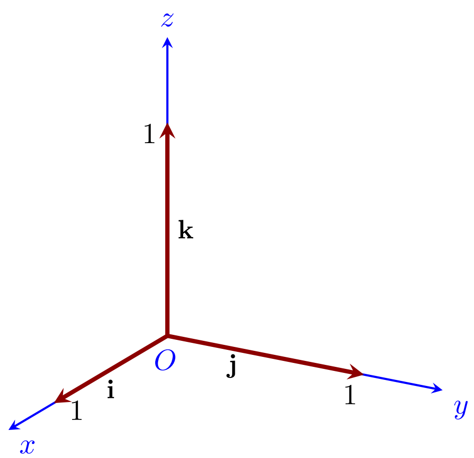
$$\vec{v} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} \quad \text{or} \quad 3i + 2j.$$



$$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\} \quad \text{where} \quad \hat{\mathbf{i}} = (1, 0, 0), \quad \hat{\mathbf{j}} = (0, 1, 0), \quad \hat{\mathbf{k}} = (0, 0, 1).$$

Every vector in $\mathbb{R}^3$ can be written in terms of i-j-k components. For example, the vector $\vec{v} = (3, 2, 1)$ can be expressed as:

$$\vec{v} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \quad \text{or} \quad 3i + 2j + k.$$



7.4 Angles between vectors

Vectors when joined or not form angles with one another. The angle between $\vec{u}$ and $\vec{v}$ is the function

$$\theta : \mathbb{R}^n \times \mathbb{R}^n \mapsto [0, \pi]$$

given by

$$\theta\langle u, v\rangle = \arccos\left(\frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{v}\| \cdot \|\vec{u}\|}\right)$$

Interestingly two vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* if $\langle \vec{u}, \vec{v}\rangle = 0$

Example. Let $\vec{u} = (1, 2)$ and $\vec{v} = (2, -1)$, then find the angle between them.

$$\langle \vec{u}, \vec{v}\rangle = 1 \cdot 2 + 2 \cdot (-1) = 0,$$
$$\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5},$$
$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

So the angle is

$$\theta\langle \vec{u}, \vec{v}\rangle = \arccos\left(\frac{0}{\sqrt{5} \cdot \sqrt{5}}\right) = \arccos 0 = \frac{\pi}{2}$$

7.5 Vector Projections

A vector projection is the projection of one vector onto another. It is a way to express how much of one vector lies in the direction of another vector.

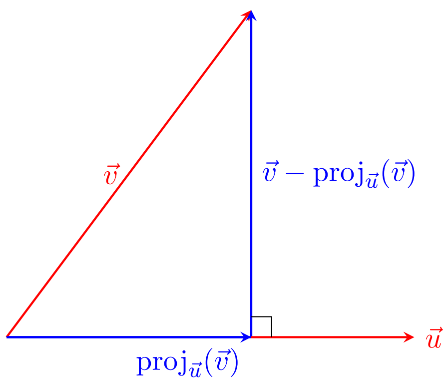
The projection of $\vec{v}$ *onto* $\vec{u}$ *is the function*

$$\text{proj}_{\vec{u}} : \mathbb{R}^n \to \mathbb{R}\vec{u} \quad \text{given by}$$
$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{1}{\|\vec{u}\|^2}\langle \vec{u}, \vec{v}\rangle\vec{u}$$

The lecture slides write

$$\vec{v}_{\|} = [\hat{u} \cdot \vec{v}]\hat{u} = \left(\frac{1}{\|\vec{u}\|}\langle \vec{u}, \vec{v}\rangle\right)\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\|\vec{u}\|^2}\langle \vec{u}, \vec{v}\rangle\vec{u}$$
$$= \text{proj}_{\vec{u}}(\vec{v})$$

and $\quad \vec{v}_{\perp} = \vec{v} - \vec{v}_{\|} = \vec{v} - \text{proj}_{\vec{u}}(\vec{v}).$



Example. Let $\vec{u} = (3, -1, 2)$ and $\vec{v} = (1, 0, 5)$, then find the projection of $\vec{v}$ onto $\vec{u}$.

First, compute the inner product and magnitude needed for the projection formula:

$$\langle \vec{u}, \vec{v}\rangle = 3 \cdot 1 + (-1) \cdot 0 + 2 \cdot 5 = 3 + 0 + 10 = 13$$
$$\|\vec{u}\|^2 = 3^2 + (-1)^2 + 2^2 = 9 + 1 + 4 = 14$$

Now, compute the projection using the formula $\text{proj}_{\vec{u}}(\vec{v}) = \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|^2}\vec{u}$:

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{13}{14}\langle 3, -1, 2\rangle$$
$$= \left\langle \frac{39}{14}, -\frac{13}{14}, \frac{26}{14}\right\rangle$$
$$= \left\langle \frac{39}{14}, -\frac{13}{14}, \frac{13}{7}\right\rangle$$

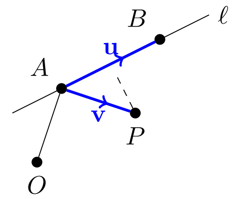
Thus, the projection of $\vec{v}$ onto $\vec{u}$ is $\left\langle \frac{39}{14}, -\frac{13}{14}, \frac{13}{7} \right\rangle$.

An important application of vector projections is finding the distance from a point to a line.

The closest point on a line

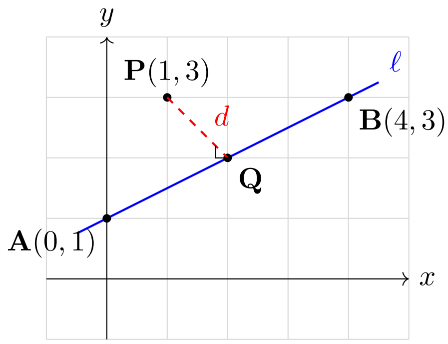
To find the point Q on the line ℓ passing through A and B that is closest to a given point P:

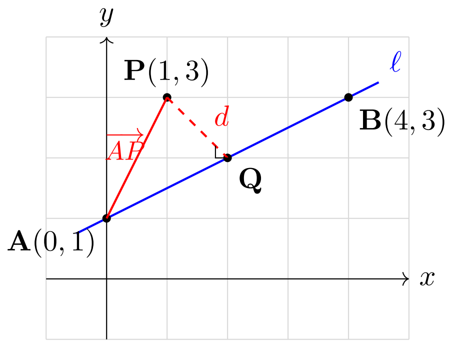
(1) Find the projection $\mathbf{v}_{\parallel}$ of $\mathbf{v} = \overrightarrow{AP}$ onto $\mathbf{u} = \overrightarrow{AB}$.
(2) Use the fact that $\overrightarrow{OQ} = \overrightarrow{OA} + \mathbf{v}_{\parallel}$ to find Q.
(3) The distance from the point P to the line ℓ is $\|\mathbf{v}_{\perp}\|$.



Example. Let $\mathbf{P} = (1, 3)$ and ℓ be the line passing through $\mathbf{A} = (0, 1)$ and $\mathbf{B} = (4, 3)$. Find the closest point Q on the line ℓ to $\mathbf{P}$.

1. Always graph the information you have be given.





2. Find the vector $\vec{AB}$ and the vector $\vec{AP}$.

$$\overrightarrow{AB} = \vec{B} - \vec{A}\langle 4 - 0, 3 - 1\rangle = \langle 4, 2\rangle,$$
$$\overrightarrow{AP} = \vec{P} - \vec{A}\langle 1 - 0, 3 - 1\rangle = \langle 1, 2\rangle.$$

3. Find the projection of $\text{proj}_{\vec{AB}}(\vec{AP})$

$$\frac{1}{||\overrightarrow{AB}||^2} \cdot \langle \overrightarrow{AP}, \overrightarrow{AP} \rangle = \frac{1}{4^2 + 2^2} \cdot (1 \cdot 4, 2 \cdot 2) \cdot \overrightarrow{AB} = \frac{1}{20} \cdot 8 \cdot (4, 2) = \left(\frac{8}{5}, \frac{4}{5}\right)$$

4. Find the point Q

$$Q = \mathbf{A} + \left(\frac{8}{5}, \frac{9}{5}\right) = (0, 1) + \left(\frac{8}{5}, \frac{4}{5}\right) = \left(\frac{8}{5}, \frac{9}{5}\right)$$

5. So the closest point Q on the line ℓ to the point P is $\left(\frac{8}{5}, \frac{9}{5}\right)$.

The shortest distance from the point P to the line ℓ is given by the metric

$$d(\mathbf{P}, \mathbf{Q}) = \sqrt{\left(\frac{8}{5} - 1\right)^2 + \left(\frac{9}{5} - 3\right)^2} = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{6}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{36}{25}} = \sqrt{\frac{45}{25}} = \frac{3\sqrt{5}}{5}.$$

7.6 Parametric equations

Parametric equations are a way to describe curves in space using vectors. They express the coordinates of points on a curve as functions of a parameter, often denoted as t.

A parametric equation of a curve in $\mathbb{R}^2$ is given by a vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

In $\mathbb{R}^3$ the parametric equation of a curve is given by a vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where $x(t)$, $y(t)$, and $z(t)$ are functions of the parameter t.

Sometimes we are required to find the *cartesian equation* of a parametric equation. This is done by eliminating the parameter t from the parametric equations.

Example. Find the cartesian equation of the parametric equation $\mathbf{r}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j}$

To find the cartesian equation, we can use the identity $\sin^2(x) + \cos^2(x) = 1$.

$$\left.\begin{aligned} x(t) = \sin(2t) &\Longrightarrow x^2(t) = \sin^2(2t) \\ y(t) = \cos(2t) &\Longrightarrow y^2(t) = \cos^2(2t) \end{aligned}\right\} \quad \Longrightarrow \quad x^2 + y^2 = 1$$

Collisions

A collision occurs when two objects occupy the same point in at the same time. In the context of parametric equations, we can determine if two objects collide by checking if their position vectors are equal at the same time.

Example. Let $\mathbf{r}_1(t) = (t+1)\mathbf{i} + (t^2 - 4t)\mathbf{j}$ and $\mathbf{r}_2(t) = 2t\mathbf{i} + (6t-9)\mathbf{j}$. Find the time and point of collision.

We set the position vectors equal to each other:

$$t + 1 = 2t \qquad \text{and} \qquad t^2 - 4t = 6t - 9$$

Then,

$$t = 1 \qquad \text{and} \qquad t = 1, 9$$

Therefore, the collision occurs at $t = 1$, with position vector $\mathbf{r}_1(1) = 2\mathbf{i} - 3\mathbf{j}$, which corresponds to the point $(2, -3)$.

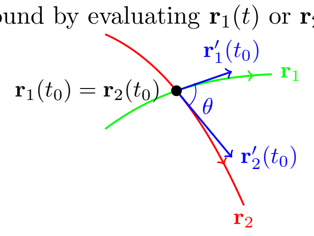
Collisions

For parametric curves decribed by $\mathbf{r}_1 \colon \mathbb{R} \longrightarrow \mathbb{R}^n$ and $\mathbf{r}_2 \colon \mathbb{R} \longrightarrow \mathbb{R}^n$, the set of collision *times* is

$$C = \{t \in \mathbb{R} \mid \mathbf{r}_1(t) = \mathbf{r}_2(t)\}$$

and for $t \in C$, the *place* at which the collision occurs is found by evaluating $\mathbf{r}_1(t)$ or $\mathbf{r}_2(t)$.

As the diagram illustrates, the angle θ between two paths at a collision point $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0)$ can be found by calculating the angle between the velocity vectors of the two curves at the collision point.



7.7 Differentiation vector functions

Differentiation of vector functions involves computing derivatives of vector-valued expressions with respect to a scalar variable, revealing rates of change in both magnitude and direction across dimensions.

Suppose the *position of the fly* is

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} \quad \text{for } t \in \mathbb{R}.$$

The *velocity* is

$$\vec{v} = \vec{r}\,'(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\,\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\,\hat{\jmath}.$$

The *speed* is

$$s = \|\vec{r}\,'(t)\| = \|\vec{v}\|.$$

The *acceleration* is

$$\vec{a} = \vec{r}\,''(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\hat{\imath} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\,\hat{\jmath} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}.$$

Clearly from above for a vector-valued function $\vec{r}(t)$ then $\vec{r}\,'(t)$ gives the *instantenous velocity* at time t. Then the *length* $||\vec{r}\,'(t)||$ of the velocity gives the *speed* of the motion along the curve.

$$\begin{aligned} \text{Speed} &= ||\,\vec{r}\,'(t)\,|| = ||\,x'(t)\hat{\imath} + y'(t)\hat{\jmath}\,|| \\ &= \sqrt{(x'(t))^2 + (y'(t))^2} \end{aligned}$$

Speed measures the distance covered per unit time along the curve. It is important to now that clearly speed is *not a vector*.

Example. For $\vec{r}(t) = 2\cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$

$$\text{Position} = \vec{r}(t) = 2\cos(t)\hat{\imath} + \sin(t)\hat{\jmath} \quad \text{Velocity} = -2\sin(t)\hat{\imath} + \cos(t)\hat{\jmath}$$

Then

$$\text{Acceleration} = -2\cos(t)\hat{\imath} - \sin(t)\hat{\jmath} \quad \text{Speed} = \sqrt{4\sin^2(t) + \cos^2(t)}$$

Appendix A

Binomial expansion

Here is some information on the binomial expansion. This is often overlooked yet is very important especially fo4 differentiation, integration and complex numbers.

We consider expanding expressions of the form $(a+b)^n$ for $n \in \mathbb{N}$. For $n = 2$ we have

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

. For $n = 3$ we have

$$(a+b)^3 = (a+b)(a+b)(a+b) = \cdots = a^3 + 3a^2b + 3ab^2 + b^3$$

In general

$$(a+b)^n = (a+b)(a+b)\dots(a+b)$$

and if all the brackets are expanded there will be 2^n terms.

For a fixed j with $0 \leq j \leq n$, we wonder how many of these terms will be of the form $a^j b^{n-j}$.

We obtain a single term in the expansion of $(a+b)^n$ by choosing a or b from each of the n factors $(a+b)$.

It follows that the coefficient of $a^j b^{n-j}$ is $\binom{n}{j}$, the number of ways of choosing a set of j objects from a set of n objects.

For $n \in \mathbb{N}$ and $0 \leq j \leq n$, a counting argument shows that

$$\binom{n}{j} = \frac{n!}{j!(n-j)!},$$

where $k! = k(k-1)(k-2)\dots 3 \cdot 2 \cdot 1$ and $0! = 1$ by convention.

For example,

$$\binom{4}{2} = \frac{4 \cdot 3 \cdot 2}{2 \cdot 2} = \frac{24}{4} = 6.$$

The numbers $\binom{n}{j}$ are called *binomial coefficients* and $\binom{n}{j}$ is sometimes also called "n choose j".

The reasoning on the information above proves the following result

Theorem A.1 (Binomial Formula)**.**

$$(a+b)^n = \binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{n-2}a^2b^{n-2}+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n,$$

where $\binom{n}{j}$ *is the number of ways of choosing a set of* j *objects from a set of* n *distinct objects.*

▷ The binomial coefficients $\binom{n}{j}$ are very important numbers in mathematics. They can be computed quickly for small values of n and j using ***pascals triangle***, which we discuss below.

Here are the first 9 rows of Pascal's Triangle.

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

Each entry in Pascal's triangle is obtained by summing the two entries above it in the previous row.

It is not too hard to prove that, reading left to right, the jth entry row n in Pascal's triangle is $\binom{n}{j}$.

In particular

$$\binom{n+1}{j+1} = \binom{n}{j} + \binom{n}{j+1}$$

which is clear if you think about choosing a set size $(j+1)$ from a set $(n+1)$ elements, and a fixed element or the larger set.

Appendix B

Trigonometric Identities

Below is all the trigonometric identities that we will use. I would recommend you try to prove them as the semester moves along.

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1, \qquad 1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

Angle Sum and Difference Formulas

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \pm \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Double Angle Formulas

$$\sin(2x) = 2 \sin x \cos x, \qquad \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

Half Angle Formulas

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

Product-to-Sum Identities

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

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